Essays on Monetary Policy with Informational Frictions

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ABSTRACT

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This dissertation contains three essays on monetary policy under informational frictions. All three chapters study the situation in which the private sector has imperfect information about the underlying economy and extracts information about the unobserved shocks from the central bank’s interest rate decisions. In this situation, monetary policy has an informational effect, in addition to its direct effect on the nominal budget of the household.

Chapter 1 studies how the equilibrium interest rate of an optimizing discretionary central bank is changed when the interest rate has an informational effect. I build a New Keynesian model in which firms are subject to both nominal frictions and informational frictions. There are two types of aggregate shocks in the private sector: the natural-rate shock, which is mapped from the aggregate component of technology shocks, and the cost-push shock, which is mapped from the aggregate component of wage-markup shocks. The central bank has perfect information on the realization of shocks, and has only one policy instrument which is the nominal interest rate. Private agents do not observe the realization of shocks, and use the interest rate as a public signal to extract information about the shocks. I show that the equilibrium discretionary monetary policy reacts more aggressively to natural-rate shocks and less aggressively to cost-push shocks, relative to the optimal response under perfect information.

Chapter 2 analyzes how the informational effect of interest rates leads to the gains from commitment, and its implications on optimal direct communication strategy. Built upon the model in the previous chapter, I show how commitment to a state-contingent policy rule can change the sensitivity of expected shocks to the interest rate. The key mechanism that yields the gains from commitment is analyzed through the lens of the Phillips curve, which shows the output gap versus
inflation trade-off becomes endogenous to the central bank’s interest-rate decisions. In addition to the informational gains from policy commitment, this chapter also studies the optimal direct communication strategy which interacts with the informational effect through policy rates.

Finally, Chapter 3 explores the optimal strategy for the central bank to conduct monetary policy when both the private sector and the central bank face imperfect information. Forward guidance is modeled as the central bank providing its expectations on monetary policy, conditional on its own imperfect information. I compare three strategies of forward guidance. The first strategy is called instrument-based forward guidance, in which case the central bank announces and commits to its estimate of future policy actions conditional on its information which is currently noisy. The second strategy is called Delphic forward guidance, in which case the central bank only reveals its noisy information, and waits to decide the actual monetary policy when perfect information becomes available. I show that the optimal Delphic forward guidance involves the central bank doing backward induction, by which it takes into account the change in the beliefs in the private sector due to re-optimization in later periods. Lastly, I show the optimal monetary policy is the rule-based Odyssean forward guidance, which is a state-contingent commitment that specifies how the central bank reacts to both the actual shock and the noise in its own information.
Contents

List of Figures ................................................................. v
Acknowledgements ......................................................... vii

1 Monetary Policy with the Informational Effect of Interest Rates 1
   1.1 Introduction ........................................................... 2
   1.2 Private Sector ....................................................... 8
       1.2.1 Informational Frictions ........................................ 9
       1.2.2 Private Sector Optimization Problems ....................... 10
           Household .......................................................... 10
           Firms ................................................................. 11
       1.2.3 Aggregation and Equilibrium in the Private Sector .......... 13
   1.3 Monetary Policy with Serially Un correlated Shocks ............... 17
       1.3.1 The Informational Effect of Interest Rates ................. 17
       1.3.2 The Optimization Problem of the Central Bank .......... 23
           The Phillips Curve ............................................... 24
           The Equilibrium Interest Rate .................................. 28
   1.4 Dynamic Informational Effect ..................................... 32
       1.4.1 States, Beliefs and the Equilibrium in Private Sector .... 32
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4.2</td>
<td>Discretionary Monetary Policy</td>
<td>35</td>
</tr>
<tr>
<td>1.4.3</td>
<td>Quantitative Analysis</td>
<td>40</td>
</tr>
<tr>
<td>1.5</td>
<td>Conclusion</td>
<td>43</td>
</tr>
<tr>
<td>2</td>
<td>The Informational Gains from Policy Commitment</td>
<td>46</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>47</td>
</tr>
<tr>
<td>2.2</td>
<td>Optimal Commitment</td>
<td>52</td>
</tr>
<tr>
<td>2.2.1</td>
<td>The Phillips Curve under Policy Rules</td>
<td>53</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Optimal Policy Rule</td>
<td>58</td>
</tr>
<tr>
<td>2.2.3</td>
<td>Time Inconsistency</td>
<td>63</td>
</tr>
<tr>
<td>2.3</td>
<td>Direct Communication</td>
<td>67</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Interaction between the Informational Effect of Monetary Policy and Central Bank Direct Communication</td>
<td>68</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Value of (External) Information</td>
<td>71</td>
</tr>
<tr>
<td>2.4</td>
<td>Quantitative Assessment</td>
<td>73</td>
</tr>
<tr>
<td>2.4.1</td>
<td>No External Information</td>
<td>76</td>
</tr>
<tr>
<td>2.4.2</td>
<td>Varying Precision of External Information</td>
<td>78</td>
</tr>
<tr>
<td>2.5</td>
<td>Conclusion</td>
<td>82</td>
</tr>
<tr>
<td>3</td>
<td>Monetary Policy Commitment under Imperfect Information</td>
<td>84</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>85</td>
</tr>
<tr>
<td>3.2</td>
<td>The Private Sector</td>
<td>92</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Household</td>
<td>93</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Firms</td>
<td>95</td>
</tr>
<tr>
<td>3.2.3</td>
<td>States and Signals</td>
<td>95</td>
</tr>
</tbody>
</table>
C Appendix for Chapter 3

C.1 Price-setting under Higher Order Belief .......................... 161
C.2 Second Order Approximation to Household’s Welfare .................. 162
C.3 Benchmark Case: No Forward Guidance .......................... 165
C.4 Instrument-based Odyssean Forward Guidance .................. 166
C.5 Delphic Forward Guidance ............................................. 169
  C.5.1 Output Gap Stabilization Policy .................................... 171
  C.5.2 Proof of Lemma 5 .................................................... 173
C.6 Rule-based Odyssean Forward Guidance .......................... 174
## List of Figures

1.1 The Sensitivity of Expected Shocks to Interest Rates and to Actual Shocks . . . . . 22  
1.2 The Phillips Curve under Discretionary Monetary Policy . . . . . . . . . . . . . . . 27  
1.3 Impulse Response of Equilibrium Interest Rate, Output Gap and Inflation . . . . . 42  
2.1 The Phillips Curve under Policy Rule with Unanticipated Deviations . . . . . . . . 55  
2.2 The Phillips Curve under Policy Rule with Anticipated Deviations . . . . . . . . . 57  
2.3 Solution to the Optimal Policy Rule . . . . . . . . . . . . . . . . . . . . . . . . . 61  
2.4 The Phillips Curve after a Natural-rate Shock under Optimal Policy Rule . . . . . . 65  
2.5 The Phillips Curve after a Cost-push Shock under Optimal Policy Rule . . . . . . 67  
2.6 Sensitivity of Beliefs to External Signals . . . . . . . . . . . . . . . . . . . . . . . . 70  
2.7 The Value of Direct Communication . . . . . . . . . . . . . . . . . . . . . . . . . 72  
2.8 Impulse Response with No External Signals . . . . . . . . . . . . . . . . . . . . . 77  
2.9 Impulse Response with Precise External Signals . . . . . . . . . . . . . . . . . . . 80  
3.1 The Equilibrium Price and Output under Fixed Policy Rule without Forward Guid- 
ance . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 103  
3.2 The Equilibrium Price and Output under Optimal Policy Rule without Forward 
Guidance . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 106  
3.3 Optimal Instrument-based Odyssean Forward Guidance . . . . . . . . . . . . . . 111  
3.4 The Equilibrium Price and Output under Optimal Instrument-based Odyssean For- 
ward Guidance . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 112  
3.5 K-th order beliefs after an Aggregate Technology Shock . . . . . . . . . . . . . . 117  
3.6 K-th order beliefs after an Policy Shock . . . . . . . . . . . . . . . . . . . . . . . 119
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7</td>
<td>The Equilibrium Price and Output after Re-optimization</td>
<td>122</td>
</tr>
<tr>
<td>3.8</td>
<td>Delphic Forward Guidance with Backward Induction</td>
<td>124</td>
</tr>
<tr>
<td>3.9</td>
<td>The Equilibrium Price and Output under Delphic Forward Guidance</td>
<td>125</td>
</tr>
<tr>
<td>3.10</td>
<td>Optimal Rule-based Odyssean Forward Guidance</td>
<td>126</td>
</tr>
<tr>
<td>3.11</td>
<td>The Equilibrium Price and Output under Optimal Rule-base Odyssean Forward Guidance</td>
<td>128</td>
</tr>
</tbody>
</table>
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Chapter 1

Monetary Policy with the Informational Effect of Interest Rates
1.1 Introduction

It has become widely accepted that the effect that monetary policy has on the economy depends on the beliefs held by the private sector. While the importance of expectations is well established, the majority of previous literature assumes beliefs are exogenous to monetary policy decisions. In this chapter, I study the case in which the central bank has better information than the private sector about the state of the economy. In this case, private agents find it optimal to use the interest rate as a public signal to extract information about the underlying economy. Consequently, monetary policy has informational effect on the beliefs in the private sector in addition to the direct effect on the nominal budget of the household.

The informational effect of monetary policy builds on the assumption of informational frictions in the private sector. Previous literature has studied both the case in which the central bank is better informed about relevant economic fundamentals than the private sector and the case in which the central bank has less precise information than the private sector does. With few exceptions, the majority of these papers assume that the expectations formed in the private sector about the underlying state of the economy are independent of monetary policy decisions. However, recent empirical papers demonstrate that changes in the interest rate also affect the beliefs in the private sector about economic fundamentals. In this paper, I study how the equilibrium interest rate of an optimizing discretionary central bank is changed by the informational effect of monetary policy.

I build a New Keynesian model with Calvo price rigidity and information frictions in the private sector. There are two types of shocks: natural-rate shocks and cost-push shocks. Due to imperfect information, the equilibrium output gap and inflation depend on both the actual shocks and the beliefs about the shocks, as well as the interest rate decisions by the central bank.

1See Romer and Romer (2000), Romer and Romer (2004), Campbell et al. (2012) and Nakamura and Steinsson (2013) as examples of empirical studies on the informational effect of monetary policy.
The central bank is assumed to have perfect information about both types of shocks. It sets the interest rate conditional on the actual shocks to minimize its loss function given by the weighted sum of squared inflation and the output gap. Private agents with rational expectations correctly understand the best response of the interest rate to different shocks. Therefore, they regard the interest rate as a public signal which simultaneously provides information about the two shocks. In this situation, the interest rate has two effects on the equilibrium in the private sector: the traditionally studied direct effect on the cost of borrowing for consumers and the informational effect on the beliefs in the private sector.

I study a discretionary central bank which sets the interest-rate at any given state of the economy and takes the informational effect of its interest rate decisions to be exogenous. To study how the equilibrium interest rate under imperfect information differs from the one under perfect information, I start with the simple case in which shocks have no serial correlations. Private agents are rational. They correctly understand how interest rates react to both shocks but have imperfect information about the shocks. Private agents form beliefs through a Bayesian updating process, whereby they regard the interest rate set by the central bank as a signal to extract information about the two shocks. When the interest rate reacts positively to both shocks, it becomes one signal that jointly provides information to the two shocks. When the private sector forms expectation about one shock, the prior distribution of the other shock becomes the source of noise in the signal. I demonstrate that beliefs formed through a Bayesian updating process are more sensitive to the shock to which the interest rate responds more aggressively or has a higher ex-ante dispersion.

I start with the situation where shocks have no serial correlation, in which case beliefs about future equilibrium do not play a role in determining current inflation and the output gap. The informational effect applies differently to the equilibrium output gap and inflation. I assume that the consumer is able to observe the current price levels, but that each individual firm does not
observe the aggregate price level. Consequently, the output gap is free from the expectations. However, inflation depends on the beliefs in the private sector, as optimal pricing decisions are strategic complements, where the resetting price of each firm also depends on the firm’s expectation about the aggregate price level. Thus, the interest rate changes the output gap only through the direct effect, but affects inflation through both the direct effect and the informational effect. When the central bank reacts to expansionary shocks by increasing the interest rate, the informational effect dampens the direct effect of the increase in the interest rate, as the private sector updates its beliefs about the expansionary shocks.

To compare how the equilibrium interest rate for a discretionary central bank is changed due to the informational effect, I first examine how the informational effect of the interest rate changes the Phillips curve. The Phillips curve is the constraint that a central bank faces, which captures the co-movement of the output gap and inflation as a result of changes in interest rates. After a marginal increase in the interest rate, the direct effect on a household’s cost of borrowing decreases both the output gap and inflation, which results in a positively sloped Phillips curve under perfect information. However, under imperfect information, as the informational effect dampens the direct effect on inflation, the Phillips curve becomes flatter than that under perfect information.

In addition, the informational effect of monetary policy also changes the intercept of the Phillips curve. Under perfect information, an intercept is only induced by the cost-push shock, as the cost-push shock increases inflation only without changing the natural output level. This positive intercept of the Phillips curve leads to stabilization bias, which is the conflict between the closing the output gap and minimizing inflation. Under perfect information, a central bank increases the interest rate to partially offset the effect of the cost-push shock on inflation, which

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2I use the term “expansionary shocks” to refer to the shocks that cause positive output gap or inflation without the response of interest rates. That is, positive natural-rate shocks (negative current TFP shocks) and positive cost-push shocks.
results in a positive inflation and a negative output gap. Under imperfect information, the Phillips curve has an intercept after both natural-rate shocks and cost-push shocks. This is because private agents always assign a positive possibility to the event that a cost-push shock is realized, once they observe tightening monetary policy. If the realized shock is an actual cost-push shock, the intercept is reduced, because private agents also assign positive positive possibility to the event that a natural-rate shock is realized in which case there is no stabilization bias.

I solve for the Markov perfect equilibrium between the central bank and the private sector. The private sector forms beliefs and makes optimal consumption and pricing decisions while expecting the central bank to play the equilibrium optimizing interest rate at any state of the economy. The central bank optimizes the interest rate to minimize the deviations of inflation and the output gap from their targets, taking as given the informational effect of its interest rate decision. A discretionary central bank does not internalize the change in the informational effect when making interest rate decisions.

The change in the Phillips curve under imperfect information leads to a change in the optimizing discretionary monetary policy in equilibrium. Although the natural-rate shock can be completely offset by discretionary monetary policy under perfect information, this "divine coincidence" cannot be achieved in the presence of informational frictions. This is because even if the actual shock is a natural-rate shock, the private sector still assigns a positive possibility to the event that the interest rate is reacting to a cost-push shock. Consequently, optimizing discretionary policy is "leaning against the wind" after both shocks, seeking a negative correlation between output gap and inflation. I show that the optimizing discretionary interest rate reacts more to natural-rate shocks and less to cost-push shocks than what is optimal under perfect information.

In addition, I extend the analysis to serially correlated shocks to study the dynamic informational effect of the interest rate. In this case, the dynamic informational effect of the current interest
rate comes from the persistent belief-formation process in the private sector. The private agents forms beliefs in the current period by optimally combining current signals and past beliefs. Consequently, the current interest rate has a lagged effect on future equilibrium through its effect on current beliefs. When the central bank considers the dynamic effect of its interest rate decisions, the objective function of a discretionary central bank includes deviations of the output gap and inflation in both current and future periods. The optimal discretionary policy can be characterized as "dynamically leaning against the wind": it is willing to tolerate a positive sum of current inflation and the current output gap if the sum of inflation and the output gap in the future is expected to be negative.

On the quantitative aspects, I compare the equilibrium dynamics using a calibrated model with the case under perfect information. I find that the impulse responses after a natural-rate shock are similar under perfect information and under imperfect information, but the dynamics after a cost-push shock are very different: inflation is largely reduced under imperfect information and the sacrifice in the output gap is also reduced at the same time. This is because as the equilibrium interest rate responds more aggressively to natural-rate shocks and less aggressively to cost-push shocks, the updates in the expected cost push shock is very small under imperfect information, which makes the actual inflation smaller, compared with the response under perfect information. Consequently, the informational effect of interest rate is beneficial, as the central bank does not need to tighten monetary policy to dampen consumption by the amount that it does under perfect information.

**Related Literature**

This chapter connects the theoretical studies on the optimal monetary policy under informational frictions and the empirical studies on the informational effect of interest rates.
On the theoretical side, this field is revived by Woodford (2001), which shows how higher order beliefs lead to a persistent effect of monetary policy, under the assumption of imperfect information which was initially introduced in Phelps (1970) and Lucas (1972).

The majority of papers that study optimal monetary policy under informational frictions assume that beliefs in the private sector are formed independently from monetary policy decisions. Under this assumption, a central bank makes policy decisions every period, taking as given the exogenous beliefs in the private sector. Ball, Mankiw and Reis (2005) assume that information is rigid in the private sector and characterize optimal policy as an elastic price standard. Adam (2007) assumes an endogenous learning process in the private sector and demonstrates that the target of the optimal monetary policy changes from output gap stabilization to price stabilization when information becomes more precise. Angeletos and La’O (2011) solve the Ramsey problem for optimal monetary policy and show that the flexible-price equilibrium is no longer the first-best when information frictions affect real variables.

Recent papers have begun to investigate the situation in which the private sector extracts information about the underlying economy from monetary policy decisions. Baeriswyl and Cornand (2010) note that because monetary policy cannot fully neutralize markup shocks, the central bank alters its policy response to reduce the information revealed about the cost push shock through monetary policy. Berkelmans (2011) demonstrates that with multiple shocks, tightening policy may initially increase inflation. The paper most related to the present work is Tang (2013), which shows that when the private sector has rational expectations, the stabilization bias is reduced when monetary policy has an information effect.

On the empirical side, Romer and Romer (2000) and Romer and Romer (2004) are the first contributions to provide empirical evidence on information asymmetry between the Federal Reserve and the private sector. They show that inflation forecasts by private agents respond to changes in
the policy-rate after FOMC announcements. Faust, Swanson and Wright (2004) further confirm that the private sector revises its forecasts in response to monetary policy surprises. In more recent papers, Campbell et al. (2012) show that unemployment forecasts decrease and CPI inflation forecasts increase after a positive innovation to future federal funds rates. Nakamura and Steinsson (2013) identify the informational effect of the federal funds rate using high-frequency data. In addition, Melosi (2016) captures this empirical pattern using a DSGE model with dispersed information. Garcia-Schmidt (2015) uses Brazilian Survey data to show that inflation forecasts in the private sector increase in the short run after an unexpected tightening policy.

The remainder of the paper is organized as follows. Section 2 characterizes the optimization decisions by the representative household in the private sector, and expresses aggregate output gap and inflation as functions of beliefs. Section 3 analyzes optimizing discretionary policy and gains from commitment to policy rule in the baseline case where shocks are not serially correlated. Section 4 and section 5 discuss two factors that affect the size of gains from commitment: external information and serial correlation in shocks. To quantitatively assess the gains from commitment, I calibrate the full version of my model with serially correlated shocks, external signals and policy implementation error in section 6. Section 7 concludes the paper.

1.2 Private Sector

In this section, I incorporate informational frictions to an otherwise standard New Keynesian model with Calvo-type price rigidity. Fluctuations are driven by two types of shocks: a technology shock (expressed in terms of the "Wicksellian natural rate" in the output gap) and a wage markup shock (expressed in terms of a cost push shock in inflation). I assume that the central bank has perfect information about the two shocks, whereas the private sector cannot directly observe the shocks.
The private sector has rational expectations about the central bank’s behavior. In particular, the private sector correctly understands how the central bank will respond to both shocks and infers information about the shocks from observing the interest rate decision. This section describes the equilibrium level of the aggregate output gap and inflation as functions of beliefs in the private sector.

1.2.1 Informational Frictions

Following Phelps (1970), Woodford (2001), and Angeletos and La’O (2010), I model an "island economy", in which the informational frictions are resulted from geographical isolation. There is a continuum of islands, indexed by \( j \), and a representative household. The household consists of a consumer and a continuum of workers. At the beginning of each period, the household sends one worker to each island, \( j \). There is a continuum of monopolistic firms, each located on one island and indexed by the island. Each firm demands labor in the local labor market in the island and produces a differentiated intermediate good, \( j \). Information is symmetric within an island, as each firm is able to observe its firm-specific shocks. Information is asymmetric across islands, as firms are unable to observe shocks or decisions made by other firms. Consequently, the resetting price of each firm depends on the firm’s expectation of the aggregate price level, which makes aggregate inflation a function of beliefs in the private sector. The consumer of the representative household makes inter-temporal consumption decisions. He is able to observe the current prices of all intermediate goods, but unable to directly observe shocks. Consequently, the inter-temporal consumption decisions are also subject to informational frictions.
1.2.2 Private Sector Optimization Problems

Household

The preferences of the representative household are defined over the aggregate consumption good, \( C_t \), and the labor supplied to each firm, \( N_t(j) \), as

\[
E_t^H \sum_{t=0}^{\infty} \beta^t \left\{ U(C_t) - \int V(N_t(j)) \, d\, j \right\},
\]

where \( E_t^H \) denotes the household’s subjective expectations conditional on its information set, \( \omega_H \). The aggregate good \( C_t \) consists of a continuum of intermediate goods:

\[
C_t = \left( \int_0^1 C_t(j)^{1-\frac{1}{\epsilon}} \, \frac{\epsilon}{\epsilon-1} \right)^{\frac{\epsilon-1}{\epsilon}},
\]

where \( C_t(j) \) is the consumption of intermediate good \( j \) in period \( t \).

The economy is cashless. The household maximizes expected utility subject to the intertemporal budget constraint:

\[
\int P_t(j)C_t(j) \, d\, j + B_{t+1} \leq \int W_t(j)N_t(j) \, d\, j + (1 + i_t)B_t + \Pi_t,
\]

where \( B_t \) is a risk-free bond with nominal interest \( i_t \), which is determined by the central bank. \( \Pi_t \) is the lump-sum component of household income, which includes tax payments and profits from all firms. \( W_t(j) \) and \( N_t(j) \) are the labor wage and labor supply for firm \( j \), respectively.

The household’s optimization problem can be solved in two stages. First, conditional on the level of aggregate consumption, the household allocates intermediate goods consumption to minimize the cost of expenditure conditional on the level of aggregate good consumption. The alloc-
tion of intermediate good consumption that minimizes expenditure yields

\[ C_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\varepsilon} C_t, \quad (1.4) \]

where \( P_t = \left[ \int_0^1 P_t(j)^{1-\varepsilon} d j \right]^{\frac{1}{1-\varepsilon}}. \)

In the second stage, given the aggregate price level, \( P_t \), the household chooses its aggregate consumption, \( C_t \), labor supply to all firms, \( N_t(j) \ \forall \ j \), and savings in the risk-free bond, \( B_{t+1} \). I assume that the utility of aggregate good consumption and the utility of labor supply take the following forms: \( U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \), and \( V(N_{jt}) = \frac{N_{jt}^{1+\varphi}}{1-\varphi} \), where \( \sigma \) is the inverse of the inter-temporal elasticity of substitution and the parameter \( \varphi \) is the inverse of the Frisch elasticity of labor supply.

The inter-temporal consumption decision leads to the following Euler equation:

\[ C_t^{1-\sigma} = \beta (1 + i_t) E_t^H \left( C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} \right). \quad (1.5) \]

Equation (1.5) shows that consumption decisions are forward-looking. Current demand depends the relative cost of consumption today versus consumption tomorrow.

The intra-temporal labor supply decision sets the marginal rate of substitution between leisure and consumption equal to the real wage:

\[ \frac{N_{jt}^{\varphi}(j)}{C_t^{-\sigma}} = \frac{W_t}{P_t}. \quad (1.6) \]

**Firms**

Firms make two decisions to maximize expected profits: the intra-period cost minimization and the optimal pricing decisions. As the cost minimization problem only involves information within
the island and information is symmetric within islands, the intra-period cost minimization problem is free from any informational frictions. The optimal pricing decision, by contrast, is affected by both the Calvo price rigidity and the informational frictions. In each period, a measure $1 - \theta$ of firms get the Calvo lottery to reset their prices. Other firms charge their previous prices. A firm $j$ that resets its price in period $t$ chooses $P_t^*(j)$ to maximize its own expectation of the sum of all discounted profits while $P_t^*(j)$ remains effective. The profit optimization problem can be written as follows:

$$
\max_{P_t^*(j)} \sum_{k=0}^{\infty} \theta^k E_t^j \left\{ Q_{t,t+k} \left[ P_t^*(j) Y_{t+k}(j) - U_{t+k}^w(j) W_{t+k}(j) N_t(j) \right] \right\},
$$

(1.7)

where $E_t^j$ denotes firm $j$’s expectation conditional on its information set, $\omega_j$. $Q_{t,t+k}$ is the stochastic discount factor given by: $Q_{t,t+k} = \beta^k U'(C_{t+k}) \frac{P_{t+k}}{P_{t+k}}$. $U_{t+k}^w(j)$ denotes the wage markup for firm $j$.

Firms face two constraints. The first is the demand for their products, which results from the household’s optimal allocation among intermediate goods. The second constraint is the production technology. Following the tradition of New Keynesian literature, I assume that labor is the only input and each firm produces according to a constant return to scale technology,

$$
Y_t(j) = A_t(j) L_t(j),
$$

(1.8)

where $A_t(j)$ denotes the technology of firm $j$.

There are two sources of uncertainty that affect the pricing decisions of each firm: technology shocks and wage markup shocks. I assume that both shocks have an aggregate component and an idiosyncratic component. The idiosyncratic components are drawn independently in every period,
and are distributed log-normally around their aggregate components.

\[ \log(A_t(j)) \equiv a_t(j) = a_t + s_t^a(j), \quad s_t^a(j) \sim N(0, \sigma_{sa}^2) \]

\[ \log(U_t^w(j)) \equiv u_t^w(j) = u_t^w + s_t^u(j), \quad s_t^u(j) \sim N(0, \sigma_{su}^2) \]

I assume that the aggregate components of both shocks follow AR(1) processes:

\[ a_t = \phi^a a_{t-1} + v_t^a, \quad v_t^a \sim N(0, \sigma_{va}^2) \]

\[ u_t^w = \phi^u u_{t-1}^w + v_t^{uw}, \quad v_t^{uw} \sim N(0, \sigma_{vuw}^2) \]

The first order condition for labor input implies that the nominal marginal cost of production is \( U_t(j)W_t(j)/A_t(j) \). Substituting the marginal cost of production into the optimal pricing decision results in

\[ P_t^*(j) = \frac{\varepsilon}{\varepsilon - 1} \frac{E_t^j(\beta\theta)^k u'(C_{t+k})P_{t+k}^e Y_{t+k}^{u_t(j)w_{t+k}(j)}}{E_t^j(\beta\theta)^k u'(C_{t+k})P_{t+k}^{e-1} Y_{t+k}} \]  (1.9)

Equation (1.9) implies that individual resetting prices are forward-looking and strategic complements. The optimal resetting price of firm \( j \) increases with the expectation of a higher firm-specific marginal cost of production and a higher aggregate price level in both the current and all future periods.

1.2.3 Aggregation and Equilibrium in the Private Sector

Equilibrium variables in the private sector are solved in log deviations from steady state values (i.e., \( x_t \equiv \ln(X_t/X) \)), and denoted by lower-case letters. (See Appendix A. 1 for details.)

The Output Gap

Following the New Keynesian tradition, I express output in terms of the output gap, \( \hat{y}_t \), which is
defined as the difference between $y_t$ and the natural level of output, $y_t^n$. The natural level of output is defined as the output level under flexible prices and perfect information. In this situation, $y_t^n$ becomes a linear function of $a_t$, $y_t^n = \frac{\phi + \sigma}{1 + \phi} a_t$, and follows an AR(1) process, $y_t^n = \phi y_{t-1}^n + \nu_t$, where $\phi = \phi^a$, and $\sigma_{\nu} = \frac{\phi + \sigma}{1 + \phi} \sigma_v$.

The output gap is derived as follows:

$$\hat{y}_t \equiv y_t - y_t^n = E_i^H \hat{y}_{t+1} - \frac{1}{\sigma} \left[ i_t - \left( \frac{1}{1 - \phi} r_t^n - \frac{\phi}{1 - \phi} E_i^H r_t^n \right) - E_i^H \pi_{t+1} \right],$$  \hspace{1cm} (1.10)

where $E_i^H \hat{y}_{t+1} = E_i^H y_{t+1} - E_i^H y_t^n = E_i^H y_{t+1} - \phi E_i^H y_t^n$. $r_t^n$ denotes the natural rate of interest, which is the equilibrium real interest rate that equates output to its natural level under perfect information and flexible prices. It is calculated as $r_t^n \equiv \sigma (E_t y_{t+1} - y_t^n) = \sigma (\phi - 1) y_t^n$.  \hspace{1cm} (3)

If information is perfect, $E_i^H r_t^n = r_t^n$, and expectations about future equilibrium are objective i.e., $E_i^H \hat{y}_{t+1} = E_t \hat{y}_{t+1}$ and $E_i^H \pi_{t+1} = E_t \pi_{t+1}$. Substituting them into the above equation results in the IS curve under perfect information:

$$\hat{y}_t = E_i^H \hat{y}_t - \frac{1}{\sigma} \left[ i_t - r_t^n - E_t \pi_{t+1} \right]$$  \hspace{1cm} (1.11)

The difference between equation (1.10) with equation (1.11) illustrates how the output gap under imperfect information differs from that under perfect information. Specifically, under perfect information, a positive natural-rate shock increases the output gap by $\frac{1}{\sigma} r_t^n$. The positive output gap is caused by the price rigidity, as the adjustments in prices are insufficient, so that the reduction in the equilibrium output is smaller than the reduction in the natural output. In comparison, this

---

3The natural rate shock is mapped from the aggregate component in firm technology shocks in the present model, but it can also be other types of demand shocks as well, for example time preference shocks or government spending shocks. As long as the output target in the next period is not known for the household, the expected natural rate affect the output gap in addition to the actual one.
output gap is enlarged under imperfect information. Absent an interest rate response, the private agents do not update their beliefs about the natural rate. Substituting $E^s_t r^n_t = 0$ into equation (1.10) shows that the output gap becomes $\frac{1}{1-\phi} \frac{1}{\sigma} r^n_t$. Intuitively, as the household does not know about the change in the natural output level in the next period, the household does not reduce current consumption, which is equivalent to a larger positive output gap.

**Inflation**

According to the assumption of Calvo-type price rigidity, the current aggregate price level is the composite of the aggregate price in the previous period and the average resetting prices:

$$p_t = \theta p_{t-1} + (1 - \theta) \int p^*_t (j) d j.$$  \hspace{1cm} (1.12)

The integral of resetting prices potentially leads to the higher order beliefs problem. As equation (1.9) shows, $p^*_t (j)$ includes firm $j$’s expectation about the aggregate price level $P_t$, and, thus, includes other firms’ expectations. This leads to the infinite regress problem, in which each firm uses its firm-specific shock as a private signal, and guesses the private signals observed by other firms. As the focus of my study is on aggregate variables instead of on the distribution of prices across firms, I abstract from this higher order beliefs problem by modeling homogeneous subjective beliefs.\(^4\) This means that when all private agents, including both firms and the household, form expectations about the aggregate variables, all agents use only public signals. Therefore, the information sets are the same across all agents. I denote the homogeneous subjective beliefs in the private sector as $E^s_t$.\(^5\) Mathematically, I assume that the idiosyncratic components of firm-specific shocks have infinite variance. In this case, private signals are completely uninformative, so that

\(^4\)There are many papers that address how higher order beliefs lead to monetary policy to have more persistent effects, for example Woodford (2001) and Angeletos and La’O (2009). For the solution method to the infinite regress problem, see Huo and Takayama (2015), Melosi (2016) and Nimark (2017).

\(^5\)Note that subjective expectations in this paper refer to the rational expectations formed as a result of imperfect information about the state variables.
firms do not use their private signals about firm-specific shocks to form beliefs about aggregate variables.\(^6\)

The aggregation of individual resetting prices leads to the New Keynesian Phillips curve under subjective beliefs: (see Appendix A.2 for the detailed derivation.)

\[
\pi_t = \beta \theta E_t^s \pi_{t+1} + (1- \theta)E_t^s \pi_t + \kappa \theta \hat{y}_t + u_t, \tag{1.13}
\]

where \( \kappa = \frac{(1-\beta \theta)(1-\theta)(\phi+\sigma)}{\theta} \), and \( u_t \) denotes the cost push shock, which is related to the wage markup shock as \( u_t = (1-\theta)(1-\beta \theta)u_w \).

If information is perfect, expected inflation is the same as actual inflation, i.e., \( E_t^s \pi_t = \pi_t \), and expectations about future equilibrium are objective i.e., \( E_t^s \pi_{t+1} = E_t \pi_{t+1} \). Substituting them into equation (1.13) results in the Phillips curve under perfect information:

\[
\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t + \frac{1}{\theta}u_t \tag{1.14}
\]

The difference between equation (1.13) and equation (1.14) shows how the inflation under imperfect information differs from that under perfect information. Under perfect information, a positive cost-push shock increases inflation by \( \frac{1}{\theta}u_t \). As this cost-push shock does not increase the output gap, the central bank faces a conflict between stabilizing inflation and closing the output gap. If it increases the interest rate to dampen inflation, it also creates a negative output gap. When information is imperfect, only a fraction \( \theta \) of the actual cost-push shock is observed by individual firms, as firms only observe their firm-specific shocks. Absent an interest rate response,}

\(^6\)Another way to generate homogeneous beliefs is to assume that firms have the same technology and face the same wage markup but do not observe them when setting prices. This assumption, however, implies that aggregate inflation consists of only the firms’ expectations, and does not consist of actual shocks. Consequently, there will be no trade-off between inflation and the output gap due to the lack of actual cost-push shocks, which makes the optimal monetary policy becomes less interesting.
firms do not update beliefs regarding the aggregate cost-push shock, meaning that the resetting prices change by less than under perfect information. Therefore, imperfect information reduces the stabilization bias under perfect information.

1.3 Monetary Policy with Serially Uncorrelated Shocks

I start the analysis of discretionary monetary policy with informational effect from a simple scenario, in which underlying shocks have no serial correlation. In this case, although private agents are forward-looking, the expectations of future equilibrium variables do not matter for current choices, as future equilibrium variables are expected to be at their steady state levels.

1.3.1 The Informational Effect of Interest Rates

This section uses an arbitrary interest rate response function to illustrate the two effects that interest rates have: the direct effect on the borrowing cost and the informational effect on beliefs in the private sector. It emphasizes how the informational effect on beliefs about different shocks are determined by the interest rate reaction function.

First, since shocks have no correlation, substituting $\phi = 0$ and $E_t^i \hat{y}_{t+1} = E_t^i \pi_{t+1} = 0$ in the IS function and the Phillips curve results in:

$$\hat{y}_t = -\frac{1}{\sigma} (i_t - r^n_t) \quad (1.15)$$

$$\pi_t = (1 - \theta) E_t^i \pi_t + \kappa \theta \hat{y}_t + u_t \quad (1.16)$$

As shown in the IS equation, the output gap is free from subjective beliefs and thus the informa-

---

\footnote{Following the conventional New Keynesian literature, the long-run distortion has been eliminated via Pigouvian tax as an employment subsidy, so that the steady state levels of the output gap and inflation are all zero.}
tional effect of the interest rate does not play a role in determining the output gap. This is because future equilibrium variables are expected to be at steady state levels and the current aggregate price level is observed by the consumer.

In contrast, inflation is affected by subjective beliefs, as individual firms do not observe the aggregate price level when setting optimal prices. Consequently, to express actual inflation in terms of shocks, further substitute the expected aggregate inflation by $E_s \pi_t = \kappa E_s \hat{y}_t + \frac{1}{\theta} E_s u_t$. The expected output gap is different from the actual output gap, as the private sector has imperfect knowledge of the actual $r^n_t$. Specifically, $E_s \hat{y}_t = \hat{y}_t - \frac{1}{\sigma} r^n_t + \frac{1}{\sigma} E_s r^n_t$. As a result, inflation can be expressed in terms of the output gap, the actual shocks and the expected shocks as follows:

$$\pi_t = \kappa \hat{y}_t + (1 - \theta) \frac{\kappa}{\sigma} (E_s r^n_t - r^n_t) + \frac{1 - \theta}{\theta} E_s u_t + u_t.$$ (1.17)

The interest rate has two effects on equilibrium in the private sector. The first one is the direct effect, which is the conventionally studied effect on the borrowing cost for the household. The direct effect of a marginal increase in the interest rate reduces current consumption, as it increases the relative cost of current consumption versus future consumption. In addition, the direct effect of an increase in the interest rate also reduces the aggregate price level, as each firm reduces its resetting price when facing a lower demand. The direct effect of the interest rate on the output gap and inflation are as follows:

$$\frac{\partial \hat{y}_t}{\partial i_t} \bigg|_{direct} = -\frac{1}{\sigma}, \quad \frac{\partial \pi_t}{\partial i_t} \bigg|_{direct} = \frac{\partial \pi_t}{\partial \hat{y}_t} \frac{\partial \hat{y}_t}{\partial i_t} = -\frac{\kappa}{\sigma}.$$ 

The informational effect captures how the interest rate changes the beliefs in the private sector about the two underlying shocks, $E_s r^n_t$ and $E_s u_t$. As the output gap is not affected by the subjective beliefs, it is free from the informational effect of the interest rate. The marginal informational
effect of the interest rate on inflation is the combination of the marginal change in the expected
cost-push shock and in the expected natural-rate shock. The marginal informational effect of the
interest rate on output gap and inflation is

\[
\frac{\partial \hat{y}_t}{\partial i_t} \mid_{\text{informational}} = 0, \quad \frac{\partial \pi_t}{\partial i_t} \mid_{\text{informational}} = \frac{\partial \pi_t}{\partial E^r_t} \frac{\partial E^r_t}{\partial i_t} + \frac{\partial \pi_t}{\partial E^u_t} \frac{\partial E^u_t}{\partial i_t}.
\]

where the partial derivatives of inflation on the expected natural-rate and the expected cost-push
shock are defined in equation (1.17) as:

\[
\frac{\partial E^r_t}{\partial i_t} = (1 - \theta) \frac{\kappa}{\sigma}, \quad \frac{\partial E^u_t}{\partial i_t} = \frac{1 - \theta}{\theta}.
\]

**State and Signals**

To study the informational effect of the interest rate, one first needs to specify the (unobserved)
state variables and the signals about the state variables. As shown in the IS curve and the Phillips
curve, only the aggregate part of the shocks matter in determining the output gap and inflation.
In addition, technology shocks and wage markup shocks can be written in terms of natural-rate
shocks and cost-push shocks, \( r^n_t \) and \( u_t \), respectively.

\[
r^n_t = \phi r^n_{t-1} + \nu_t, \\
u_t = \phi_u u_{t-1} + \nu^u_t,
\]

where the natural-rate shock and the cost-push shock are mapped from the technology shock and
the wage markup shock as \( r^n_t = \frac{\sigma}{\sigma + \phi} \sigma (\phi - 1) a_t \), and \( u_t = (1 - \theta)(1 - \beta \theta) u^\nu_t \).

Denote the auto-coefficients of the natural-rate shock and the cost-push shock as \( \phi \) and \( \phi_u \). By
construction, they are the same as the auto-coefficients of the aggregate technology process and
the wage markup process. In this section, I assume that the two shocks are serially uncorrelated.
(\( \phi = \phi^u = 0 \)) Denote the standard deviation of the natural-rate shock and the cost-push shock as
σ_r and σ_u. By construction, \( \sigma_r = \frac{\phi + \sigma}{1 - \phi} \sigma (\phi - 1) \sigma_v \), and \( \sigma_u = (1 - \theta)(1 - \beta \theta) \sigma_{uv} \).

I assume that private agents have rational expectations regarding the interest rate response function. Under an arbitrary linear interest rate function which responds linearly to the two aggregate shocks, i.e., \( i_t = F_r r^n_t + F_u u_t \), the interest rate becomes one signal that simultaneously provides information about two shocks.

If there is only one shock to which the interest rate responds linearly, the private sector will be able to perfectly infer the actual shock. In this case, the economy becomes identical to the perfect information case. In the case with two shocks, when private agents regard the interest rate as a signal about one shock, the prior distribution of the other shock becomes the source of noise in this signal.

**Belief Formation**

Agents in the private sector are Bayesian, and form best linear forecasts by optimally weighting their prior beliefs (shocks have zero ex-ante mean) and the current signal (the interest rate). Let \( K_r \) and \( K_u \) denote the optimal weights on the two states after observing interest rate changes, which are determined through the optimal filtering process. Beliefs formed about the two states obtained through the Kalman Filtering process are

\[
\begin{bmatrix}
E_t^s r^n_t \\
E_t^s u_t
\end{bmatrix} = \begin{bmatrix}
1 - K_r \\
1 - K_u
\end{bmatrix} \begin{bmatrix}
0 \\
0
\end{bmatrix} + \begin{bmatrix}
K_r \\
K_u
\end{bmatrix} \begin{bmatrix}
K_r F_r & K_r F_u \\
K_u F_r & K_u F_u
\end{bmatrix} \begin{bmatrix}
r_t \\
u_t
\end{bmatrix}, \tag{1.18}
\]

---

8 Another way to maintain imperfect information while having only one state variable is to include an implementation error in the interest rate, meaning the interest rate becomes a noisy signal. In Section 6 where I quantitative assess the gains from commitment, I also incorporate implementation error.
where

\[ K_r F_r = \frac{F_r^2 \sigma_r^2}{F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2}, \]
\[ K_u F_u = \frac{F_u^2 \sigma_u^2}{F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2}. \]

Equation (1.18) shows that in the solution of the Kalman filtering process with an arbitrary interest rate reaction function, the sensitivity of beliefs to the actual shock is the product of the sensitivity of beliefs to the interest rate (\( K_r \) or \( K_u \)) and the sensitivity of the interest rate to the actual shocks, (\( F_r \) or \( F_u \)). The following lemma provides an interpretation of equation (1.18).

**Lemma 1:** Beliefs are more sensitive to the shock (1) to which the interest rate responds more aggressively, and (2) that has higher ex-ante dispersion.

Lemma 1 describes, for a given ex-ante dispersion of the shocks, how the precision of the interest rate as a signal is determined by the interest rate response function of the two shocks. Private agents in the private sector do not know whether a change in interest rate responds to the natural rate shock or to the cost push shock. They believe that the interest rate is more likely to respond to the shock to which it is more sensitive. For example, if the interest rate barely responds to cost-push shocks, then after observing a change in the interest rate, agents in the private sector infer that the change in the interest rate is less likely to be a response to a cost-push shock. Otherwise, provided that \( F_u \) is very small, the change in the interest rate has to come from a large cost-push shock, which is less likely to realize given the prior distribution of the cost-push shock. However, for any given interest rate reaction function, agents in the private sector update more toward the shock that has higher ex-ante dispersion, as the ex-ante mean of the shock has a smaller weight in belief-formation process.

Notice the difference between the sensitivity of beliefs to actual shocks and the sensitivity of
beliefs to the interest rate. I illustrate the difference in the following figure. In this figure, I first hold $\sigma_r = \sigma_u = 0.1$, and illustrate the change in the sensitivity of the expected cost-push shock to the interest rate ($K_u$) and the sensitivity of the expected cost-push shock to the actual shock ($K_r F_r$) while holding $F_r$ fixed at 1.5. Lemma 1 suggests that for a given $F_r$, the sensitivity of the expected cost-push shock to the actual cost-push shock, $\frac{\partial E^*_{1,t}}{\partial u_t} (K_u F_u)$, increases as $F_u$ increases, but it is not necessarily the case for the sensitivity of expected cost push shock to interest rates, $\frac{\partial E^*_{1,t}}{\partial i_t} (K_u)$.

![Figure 1.1: The Sensitivity of Expected Shocks to Interest Rates and to Actual Shocks](image)

In the first row, $F_r$ is fixed at 1.5, and $\sigma_r = \sigma_u = 0.1$. When $F_u$ increases from 0.1 to 3, $K_u F_u$ (right figure) increases monotonically. However, as shown in the left figure, $K_u$ increases first, but then decreases at larger value of $F_u$. In the second row, I hold $F_r = F_u = 2$, and $\sigma_r = 0.1$. Increasing $\sigma_u$ monotonically increases both the sensitivity of beliefs to interest rate and the sensitivity of beliefs (left figure) to the actual shock (right figure).
The above figure shows that when $F_u$ begins to increase from a small value, both the sensitivity of beliefs to interest rate and the sensitivity of beliefs to the actual shock increases. However, as $F_u$ becomes larger, the change in the informational effect is dominated by the interest rate becoming more sensitive to shocks rather than beliefs being more sensitive to interest rate changes. As shown in the left figure in the first row, the sensitivity of $E^t u_t$ to the change in $i_t$ decreases at higher level of $F_u$. Next, in the second row, I fix $F_r = F_u = 2$, and $\sigma_r = 0.1$, and analyze changes in $\sigma_u$ from 0.01 to 1. Both the sensitivity of beliefs to interest rate and to the actual shock increases.

### 1.3.2 The Optimization Problem of the Central Bank

In the previous section, I analyzed the informational effect for a given interest rate rule. Here, I analyze the equilibrium between the private sector and the central bank in which the central bank optimizes in a discretionary way. Specifically, the central bank sets the interest rate to maximize its objective at any given state, taking as given the informational effect of the interest rate. The private sector has rational expectations, in the sense that it perfectly understands the best response function of the interest rate, and extracts information about the current states through the optimal filtering process. Simultaneously, the household chooses consumption and firms optimally set prices.

The optimizing interest rate is an endogenous decision by the central bank, whose objective function consists of equilibrium variables in the private sector. The equilibrium variables in the private sector depend on the beliefs in the private sector, which in turn depend on the equilibrium interest rate reaction function. This introduces circularity into the belief-formation problem. The solution of this problem is discussed by Svensson and Woodford (2003). Following their method, I study the equilibrium interest rate of an optimizing discretionary central bank by first conjecturing an interest rate reaction function, with which private agents form beliefs. Next, I derive the constraint faced by the central bank, which is the Phillips curve under imperfect information, and
solve for the equilibrium interest rate decision under this constraint. Finally, I find the fixed point solution such that the equilibrium response of the interest rate is consistent with the previously conjectured interest rate reaction function.

**The Phillips Curve**

I begin the analysis of the discretionary monetary policy by discussing the constraint faced by the central bank, which is the Phillips curve. The Phillips curve captures the trade-off between output gap stabilization and inflation stabilization, as the interest rate changes both the output gap and inflation. With perfect information, the slope of the Phillips curve is exogenous to the interest rate decision. Moreover, with perfect information, the Phillips curve crosses the origin of the \((\hat{y}_t, \pi_t)\) plane after a natural-rate shock and has a positive intercept after a cost-push shock.

However, with imperfect information, the Phillips curve depends not only on the realization of actual shocks, but also on the expectations about the shocks. As the expectations about the two shocks are determined by the reaction function of the interest rate, I first guess and then verify that in equilibrium, the interest rate is linear in the two aggregate state variables: \(i_t = F_r r^n_t + F_u u_t\). Then, I substitute the expected shocks under this interest rate reaction function, to solve for the Phillips curve with the informational effect of the interest rate rate:

\[
\pi_t = \kappa \hat{y}_t + \left[ (1 - \theta) \frac{\kappa}{\theta} K_r + \frac{1 - \theta}{\theta} K_u \right] i_t - (1 - \theta) \frac{\kappa}{\theta} r^n_t + u_t. \tag{1.19}
\]

To express the trade-off between output gap stabilization and inflation stabilization, I substitute interest rate by its relation with output gap from the IS equation, \(i_t = -\sigma \hat{y}_t + r^n_t\). This results in the
Phillips curve with the informational effect of the interest rate:

$$\pi_t = \left\{ \kappa - \sigma \left[ (1 - \theta) \frac{K_r}{\sigma} + \frac{1 - \theta}{\theta} K_u \right] \right\} \hat{y}_t + \left\{ (1 - \theta) \frac{\kappa}{\sigma} (K_r - 1) + \frac{1 - \theta}{\theta} K_u \right\} r^n_t + u_t, \quad (1.20)$$

where $K_r$ and $K_u$ are determined through the optimal filtering process in equation (1.18).

The following lemma summarizes the differences between the Phillips curve under perfect information and the Phillips curve under imperfect information.

**Lemma 2:** For a given interest rate function that reacts to both shocks in a linear way, the informational effect of the interest rate changes the Phillips curve in three aspects, relative to the Phillips curve under perfect information:

1. The slope of the Phillips curve is flatter than that under perfect information.
2. The intercept after a cost push shock is reduced.
3. There is non-zero intercept after a natural rate shock.

**Proof:** see Appendix A.4

The intuition follows:

For (1), the slope captures the co-movement between the output gap and inflation due to changes in the interest rate. The informational effect of the interest rate on inflation reduces the co-movement between the output gap and inflation. After a tightening monetary policy, the direct effect of the interest rate reduces the output gap, as the higher nominal interest rate increases the real cost of borrowing. Under perfect information, the direct effect on inflation is given by $\kappa$, but under imperfect information, this direct effect is dampened by the informational effect. When observing a higher interest rate, private agents assign a positive possibility to the event that the interest rate is reacting to a positive cost-push shock. This update in the expected cost-push shock leads to an increase in expected inflation. This update of beliefs reduces the direct tightening ef-
fect of the interest rate on inflation, which reduces the co-movement between the output gap and inflation.

For (2), the intercept caused by an actual cost-push shock is reduced because information on the actual cost-push shocks is only partially revealed through the interest rate. Note that although both the actual cost-push shock and the expected cost-push shock induce an increase in inflation, the expected cost-push shock does not cause an intercept. This is because when the interest rate does not change, meaning the output gap stays at zero after in absence of natural rate shock, the private agents do not update expected cost-push shock. In fact, the effect of an expected cost push shock is captured in the slope, rather than the intercept, of the Phillips curve.

For (3), after a positive natural-rate shock, the intercept of the Phillips curve represents the equilibrium output gap and inflation when when the interest rate tracks the natural rate one-to-one, $i_t = r^n_t$. Due to the informational effect of the interest rate, the change in the interest rate makes the private agents simultaneously update beliefs about both shocks. Therefore, inflation changes, with the sign depending on the expected cost-push shock and the difference between the expected and the actual natural-rate shock. First, the expected cost-push shocks increase inflation, because each firm believes the aggregate price level increases, when other firms all have higher wage markups. Second, as the private agents underestimate the realization of the natural-rate shock, each firm expects aggregate demand to be less than the actual level. Consequently, the negative difference between the expected and the actual natural-rate shock decreases inflation. The relative size of the two effects determines the sign of the intercept.

I plot the Phillips curve under imperfect information, in comparison with the Phillips curve under perfect information in Figure 1.2.\footnote{see Section 1.4.3 for parameter values}
In the above figures, I plot the Phillips curve under discretionary policy while fixing the interest rate reaction function to be $F_r = 1$ and $F_u = 1$. The prior distribution of the two shocks are set equal, $\sigma_r = \sigma_u = 0.1$. 
The Equilibrium Interest Rate

When shocks are serially uncorrelated and the interest rate does not respond to lagged variables, the current interest rate does not affect the future output gap or inflation. Thus, when choosing the current interest rate, although the central bank is forward-looking, it only considers the effect on current inflation and the output gap when making current interest rate decision. The optimization problem for the discretionary central bank is given by:

$$\min_{i_t} L(t) = \begin{bmatrix} \pi_t & \hat{y}_t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \pi_t \\ \hat{y}_t \end{bmatrix} + \text{indept. terms}$$  \hspace{1cm} (1.21)$$

subject to

$$\hat{y}_t = -\frac{1}{\sigma} (i_t - r^n_t)$$  \hspace{1cm} (1.22)$$

$$\pi_t = \kappa \hat{y}_t + (1 - \theta) \frac{\kappa}{\sigma} (E^x_t r^n_t - r^n_t) + \frac{1 - \theta}{\theta} u_t$$  \hspace{1cm} (1.23)$$

$$E^x_t r^n_t = K_r i_t$$  \hspace{1cm} (1.24)$$

$$E^x_t u_t = K_u i_t$$  \hspace{1cm} (1.25)$$

where \(\omega\) is a constant that results from the second-order approximation of the household’s utility.\(^{10}\)

**Definition:** A Markov perfect equilibrium between a discretionary central bank and the private sector with rational expectations can be described in aggregate terms in the following way:

(i) Inflation and the output gap result from the household’s optimal consumption choices and firms’ optimal price-setting behavior, which are shown in equations (1.10) and (1.13).

(ii) Beliefs in the private sector about the realization of shocks are formed through the Kalman

\(^{10}\)see Woodford (2011) for general derivation of the second-order approximation of the household’s utility under perfect information, and Adam (2007) for the application to imperfect information. See Appendix for the derivations that apply to the specific assumptions in this paper.
Filtering process as shown in equation (1.18);

(iii) The interest rate is set by the central bank’s constrained optimization problem as specified in (1.21).

To solve for the equilibrium interest rate, we first need to conjecture an interest rate reaction function, $i_t = F_0^0 r_t^n + F_0^0 u_t$, which determines the Phillips curve. Then, the central bank chooses the interest rate to maximize its objective function under the constraint of the Phillips curve. The equilibrium interest rate under rational expectations is found as the fixed point between the conjectured interest rate function and the optimizing interest rate solution. I analyze the characteristics of the optimizing discretionary interest rate in the rest of this section.\textsuperscript{11}

The first-order condition with respect to $i_t$ from equation (1.21) is given by

$$\pi_t = -\left( \frac{\partial \pi_t}{\partial \hat{\gamma}_t} \right)^{-1} \frac{\partial \hat{\gamma}_t}{\partial i_t} \omega \hat{\gamma}_t \equiv -R \hat{\gamma}_t.$$  \hspace{1cm} (1.26)

**Lemma 3:** When shocks are serially uncorrelated, discretionary monetary policy seeks a negative correlation between the current output gap and inflation after both natural-rate shocks and cost-push shocks. The absolute value of the correlation coefficient is greater than that under full information.

The intuition of this result is as follows: As the intercept is generally not zero after both shocks (Lemma 2), the interest rate increases after positive realizations of both shocks. As the optimizing central bank chooses the tangent point between its indifference curve $L(\hat{\gamma}_t, \pi_t)$ and the Phillips curve, the equilibrium $(\hat{\gamma}_t^*, \pi_t^*)$ vector is orthogonal to the Phillips curve. As the Phillips curve has a smaller slope under the informational effect, the resulting vector of $(\pi_t^*, \hat{\gamma}_t^*)$ becomes steeper.

More explicitly, under full information, the absolute value of the correlation between output

\textsuperscript{11}A detailed derivation for solving for the equilibrium optimizing interest rate is provided in Appendix.
gap and inflation is

\[ R_{\text{perfect info}} = \left( \frac{\partial \pi_t}{\partial i^*_t} \right)^{-1} \frac{\partial \hat{\pi}_t}{\partial i^*_t} \omega = \left( -\frac{\kappa}{\sigma} \right)^{-1} \left( -\frac{1}{\sigma} \right) \omega. \]  \hspace{1cm} (1.27)

With information frictions, the marginal effect of the interest rate on inflation is dampened by the informational effect, and thus

\[ R_{\text{imperfect info}} = \left( \frac{\partial \pi_t}{\partial i^*_t} \right)^{-1} \frac{\partial \hat{\pi}_t}{\partial i^*_t} \omega = \left( -\frac{\kappa}{\sigma} + (1 - \theta)\frac{\kappa}{\sigma}K_r + \frac{1 - \theta}{\theta}K_u \right)^{-1} \left( -\frac{1}{\sigma} \right) \omega. \]  \hspace{1cm} (1.28)

Under usual parameter values, the interest rate responds positively to both shocks, i.e., \( F_r > 0 \), \( F_u > 0 \). Therefore, the Kalman gains are positive, \( K_r > 0 \), \( K_u > 0 \), which results in \( R_{\text{imperfect info}} > R_{\text{perfect info}} \).

We now turn to finding the equilibrium interest rate that achieves the target characterized in Lemma 3.

First, recall that the equilibrium interest rate tracks one-to-one with the change in natural rate, as doing so completely closes the output gap and stabilizes inflation. The optimal response to cost-push shock is "leaning against the wind", which results in \( \pi_t = -\frac{\omega}{\kappa} \hat{\pi}_t \).

Denote the equilibrium interest rate under discretionary central bank and perfect information as \( i_t = F^p_r i^*_t + F^p_u u_t \), where

\[ F^p_r = 1, \quad F^p_u = \left( \kappa + \frac{\omega}{\kappa} \right)^{-1} \sigma. \]

Denote the equilibrium interest rate of discretionary monetary policy under imperfect information as \( i_t = F^d_r i^*_t + F^d_u u_t \). The following assumptions help me compare the equilibrium discretionary interest rate under imperfect information and under perfect information.
**Assumption 1:** \((1 - \theta) \frac{K_r}{\sigma} (K_r(F_p^r, F_p^u) - 1) + \frac{1 - \theta}{\theta} K_u(F_p^r, F_p^u) > 1\), where \(K_r\) and \(K_u\) denote the Kalman gains from updating beliefs about the expected natural-rate and cost-push shock as specified in equation (1.18).

**Assumption 2:** \(R^p > \bar{R}\), where \(R^p\) represents the absolute value of the correlation between the output gap and inflation, which is given by equation (1.27). \(^{12}\)

**Proposition 1:** Under Assumptions 1 and 2, the equilibrium discretionary interest rate reacts more aggressively to natural-rate shocks and less aggressively to cost-push shocks under imperfect information, relative to its equilibrium response under perfect information, i.e., \(F_{d}^r > F_{p}^r\) and \(F_{d}^u < \frac{1}{\theta} F_{p}^u\)

**Proof:** see the Appendix A.4

Assumption 1 guarantees that under imperfect information, when the central bank implements \((F_p^r, F_p^u)\), the Phillips curve has a positive intercept after a natural-rate shock. In this situation, as suggested by Lemma 3, the discretionary central bank should increase the interest rate to achieve a negative output gap, which is equivalent to an \(F_{d}^r\) that is greater than \(F_{p}^r\). Two factors drive the change in the optimal response to cost-push shocks under imperfect information, as the informational effect changes both the slope and the intercept of the Phillips curve. First, as suggested by Lemma 2, after a cost-push shock, the intercept decreases from \(\frac{1}{\theta} u_t\) under perfect information to \(u_t\) under imperfect information. Holding the slope constant, this reduction proportionally reduces the decrease in the equilibrium output gap and the equilibrium response of the interest rate. Second, holding the intercept fixed, Assumption 2 dictates that the change in the slope also results in an increase in the equilibrium output gap. Therefore, the two factors result in a smaller response of the interest rate to a cost-push shock, \(F_{d}^u < \frac{1}{\theta} F_{p}^u\).

\(^{12}\)See the Appendix A.4 for specific expression for \(\bar{R}\)
1.4 Dynamic Informational Effect

I extend the analysis to the dynamic informational effect of the interest rate by introducing serially correlated shocks. Since the consumption and pricing decisions are both forward-looking, the expectations about the future states matter for current output gap and inflation. When shocks have serial correlation, current interest rates also affect expectations about future shocks, which leads to the dynamic informational effect of interest rates.

1.4.1 States, Beliefs and the Equilibrium in Private Sector

To analyze the direct informational effect of interest rates, I first study the dynamic learning process in the private sector.

State

Natural-rate shocks and cost push-shocks follow an AR(1) process:

\[
\begin{bmatrix}
  r^n_t \\
u_t
\end{bmatrix} = \begin{bmatrix}
  \phi & 0 \\
  0 & \phi^u
\end{bmatrix} \begin{bmatrix}
  r^n_{t-1} \\
u_{t-1}
\end{bmatrix} + \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  v_r^t \\
v_u^t
\end{bmatrix}.
\] (1.29)

Signals

The information set of the private sector includes the values of all parameters and the entire history of interest rates upon \( t \). I first conjecture and then show that the equilibrium interest rate becomes a function of the state variables in period \( t \), which includes both the actual shocks at time \( t \) and beliefs in period \( t - 1 \).

\[
i_t = F_1 r^n_t + F_2 E^{s}_{t-1} r^n_{t-1} + F_3 u_t + F_4 E^{s}_{t-1} u_{t-1}.
\] (1.30)
The inertial components in the equilibrium interest rate come from the persistent belief updating process. As the private sector optimally weights the signals in the current period and the beliefs in the last period to form current expectations, the current output gap and inflation become functions of past beliefs. Therefore, when a discretionary central bank sets the current interest rate to minimize deviations of the current output gap and inflation, the interest rate in equilibrium also reacts to beliefs in the past period.

As the private agents have perfect memory of their beliefs in the past, they are able to distinguish the fraction of the interest rate that reacts to current shocks from the fraction of the interest rate that reacts to past beliefs. Let \( \hat{\mathfrak{h}}_{i_t} \) denote the fraction of \( i_t \) that reacts to current shocks, which follows:

\[
\hat{\mathfrak{h}}_{i_t} \equiv i_t - F_3 E_{i_{t-1}}^z r_{t-1}^n - F_4 E_{i_{t-1}}^z u_{t-1} = F_1 r_t^n + F_3 u_t.
\]

(1.31)

**Belief Formation**

The private sector forms expectations about current states through the Kalman filtering process. Denote the hidden state variables as

\[
z_t = \Phi z_{t-1} + \nu_t
\]

(1.32)

where \( z_t = [r_t^n, u_t]' \), \( \Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \), and \( \nu_t = [\nu_r^n, \nu_u'^n]' \) with white noise of variance \( Q \).

Denote the observable signal as

\[
s_t = Dz_t
\]

(1.33)

where \( s_t = i_t \), and \( D = [F_1, F_3]' \).

The Kalman filtering process makes beliefs about the current state variables be the optimal combination of beliefs in the last period and signals in the current period:
\[ E_t^s z_t = \Phi E_{t-1}^s z_{t-1}^n + K \left( s_t - D \Phi E_{t-1}^s z_{t-1} \right) \]  

(1.34)

where the optimal weight, \( K \), is determined by Ricatti iteration

\[ K = PD'(DPD')^{-1}, \]  

(1.35)

\[ P = \Phi \left( P - PD'(DPD')^{-1}DP \right) \Phi + Q. \]  

(1.36)

**Solution in the Private Sector under Arbitrary Policy Coefficients**

The equilibrium in the private sector is described by the system of equations summarizing private sector optimization decisions in aggregate variables (equations 1.10 and 1.12), the evolution of shocks (equation 1.29), the interest rate reaction function (equation 1.30), and belief updating process characterized in equation (1.34).

Since the equilibrium involves forward-looking variables, I solve for it by the undetermined coefficients method. I first conjecture that \( \hat{y}_t \) and \( \pi_t \) are linear functions of the state variables in period \( t \), that is, \([r^n_t, u_t, E_{t-1}^s r^n_{t-1}, E_{t-1}^s u_{t-1}]\)

\[
\begin{bmatrix}
\hat{y}_t \\
\pi_t
\end{bmatrix} =
\begin{bmatrix}
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\
\gamma_5 & \gamma_6 & \gamma_7 & \gamma_8
\end{bmatrix}
\begin{bmatrix}
r^n_t \\
E_{t-1}^s r^n_{t-1} \\
u_t \\
E_{t-1}^s u_{t-1}
\end{bmatrix}
\]  

(1.37)

This conjecture allows for the expression of expected future equilibrium variables in terms of
the beliefs about current shocks, $E_t^s r_t^n$ and $E_t^s u_t$:

\[
\begin{bmatrix}
E_t \hat{y}_{t+1} \\
E_t \pi_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\gamma_1 \phi + \gamma_2 & \gamma_3 \phi^u + \gamma_4 \\
\gamma_5 \phi + \gamma_6 & \gamma_7 \phi^u + \gamma_8
\end{bmatrix}
\begin{bmatrix}
E_t^s r_t^n \\
E_t^s u_t
\end{bmatrix}
\]

(1.38)

Substituting these into the IS and the Phillips curve results in expressions of $\hat{y}_t$ and $\pi_t$ as functions of the actual shocks $[r_t^n, u_t]$ and beliefs $[E_t^s r_t^n, E_t^s u_t]$. Applying the belief-updating process yields the expressions as functions that consist only of predetermined states. (See Appendix A.3 for the detailed derivation.)

1.4.2 Discretionary Monetary Policy

A discretionary central bank minimizes the expected output gap and inflation deviations in all periods. The central bank’s optimization problem can be written as follows:

\[
E_t L(t) = E_t [\pi_t^2 + \omega \hat{y}_t^2] + \beta E_t (L(t + 1))
\]

(1.39)

where the output gap follows equation (1.10), inflation follows equation (1.12), the actual shocks evolve following equation (1.29), and beliefs are formed using Kalman filtering process specified in equations (1.34 - 1.36).

$E_t$ denotes the objective expectation. The information set of the central bank at $t$ includes the entire history of natural-rate and cost-push shocks upon $t$ and the beliefs formed in the private sector upon $t - 1$, i.e.,

\[
I_t = \{ r_T^n, E_{T-1}^s r_{T-1}^n, u_T, E_{T-1}^s u_{T-1} \quad \forall T = 0...t \}
\]
\( E_t(L(t+1)) \) includes the deviations of equilibrium inflation and the output gap in all future periods:

\[
E_t(L(t_1)) = \sum_{j=1}^{\infty} \beta^j E_t \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \pi_{t+j} \\ \hat{y}_{t+j} \end{bmatrix} = \sum_{j=1}^{\infty} \beta^j \begin{bmatrix} E_t \pi_{t+j} \\ E_t \hat{y}_{t+j} \end{bmatrix} + \text{indept. terms} \tag{1.40}
\]

When there is serial correlation in shocks, the interest rate has a dynamic informational effect due to the persistent learning process in the private sector. Consequently, this dynamic informational effect changes the objective function of a discretionary central bank.\(^\text{13}\)

**Lemma 4** With dynamic informational effect, the optimizing discretionary monetary policy is dynamically "leaning against the wind" as it targets a negative correlation between current and future deviations of the output gap and inflation.

This can be shown as the first-order condition of the central bank’s objective function:

\[
\left\{ \frac{\partial E_t \pi_t}{\partial i^*_t} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i^*_t} E_t \hat{y}_t \right\} = -\frac{1}{2} \sum_{j=1}^{\infty} \beta^j \left\{ \frac{\partial E_t \pi_{t+j}^*}{\partial i^*_t} E_t \pi_{t+j} + \omega \frac{\partial E_t \hat{y}_{t+j}^*}{\partial i^*_t} E_t \hat{y}_{t+j} \right\} \tag{1.41}
\]

To see that the right-hand side is non-zero, we need to first specify how future equilibrium is affected by current beliefs, and how the current interest rate affects current beliefs. Denote the predetermined state variables at \( t \) as: \( z_t = [r^n_t, E_{t-1}^x r^n_{t-1}, u_t, E_{t-1}^x u_{t-1}]' \). Due to the projected linear relationship, the objective expectation of the inflation and the output gap in \( j \) periods ahead

\(^{13}\)As long as there are shocks that the central bank is unable to completely offset, optimal policy can be described as "leaning against the wind" (or "flexible inflation target policy") - seeking a contemporary negative correlation between the output gap and inflation. For discussion about the conventional within-period "leaning against" policy that is caused by informational frictions, see Adam (2005), Angeletos and La’O (2013), and Tang (2015), among others.
becomes:

\[
\begin{bmatrix}
E_t \pi_{t+j} \\
E_t \hat{y}_{t+j} \\
E_t \hat{y}_{t+j-1} \\
E_t u_{t+j} \\
E_t u_{t+j-1}
\end{bmatrix}
= 
\begin{bmatrix}
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\
\gamma_5 & \gamma_6 & \gamma_7 & \gamma_8
\end{bmatrix} 
\begin{bmatrix}
E_{t+j} z_{t+j} \\
\Gamma E_{t+j} z_{t+j}
\end{bmatrix}
\quad (1.42)
\]

As long as shocks cannot be completely offset by the interest rate, \( \Gamma \) is non-zero.

The evolution of \( E_t z_{t+j} \) includes the auto-correlated actual shocks, and the dynamic process of belief formation. The belief formation yields:

\[
E_s^t r_t^n = K_{11} F_1 r_t^n + \phi (1 - K_{11} F_1) E_s^{t-1} r_{t-1}^n + K_{11} F_3 u_t - \phi^u K_{11} F_3 E_s^{t-1} u_{t-1}
\quad (1.43)
\]

\[
E_s^t u_t = K_{21} F_1 r_t^n - \phi K_{21} F_1 E_s^{t-1} r_{t-1}^n + K_{21} F_3 u_t + \phi^u (1 - K_{21} F_3) E_s^{t-1} u_{t-1}
\quad (1.44)
\]

Thus, the evolution of \( E_t z_{t+j} \) can be summarized as

\[
\begin{bmatrix}
E_t r_{t+j}^n \\
E_t E_{t+j-1} r_{t+j-1}^n \\
E_t u_{t+j} \\
E_t E_{t+j-1} u_{t+j-1}
\end{bmatrix}
= 
\begin{bmatrix}
\phi & 0 & 0 & 0 \\
K_{11} F_1 & \phi - K_{11} F_1 \phi & K_{11} F_3 & -K_{11} F_3 \phi^u \\
0 & 0 & \phi^u & 0 \\
K_{21} F_1 & -K_{21} F_1 \phi & K_{21} F_3 & \phi^u - K_{21} F_3 \phi^u
\end{bmatrix} 
\begin{bmatrix}
E_t r_{t+j-1}^n \\
E_t E_{t+j-2} r_{t+j-2}^n \\
E_t u_{t+j-1} \\
E_t E_{t+j-2} u_{t+j-2}
\end{bmatrix}
\equiv \Lambda E_t z_{t+j-1}
\quad (1.45)
\]

Combine equations (1.42) and (1.45) to express the future equilibrium in terms of current beliefs as follows:

\[
\begin{bmatrix}
E_t \pi_{t+j} \\
E_t \hat{y}_{t+j}
\end{bmatrix}
= \Gamma \Lambda^j E_t z_t
\quad (1.46)
\]

Substituting this expression into the central bank’s objective function transforms the objective function into a weighted sum of current inflation, the current output gap and the persistent state variables which include the current actual shocks and current beliefs. The first-order condition on
\( i_t^* \) results in:

\[
\left\{ \frac{\partial E_t \pi_t}{\partial i_t^*} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i_t^*} E_t \hat{y}_t \right\} + \frac{1}{2} \sum_{j=1}^{\infty} \beta^j \Delta(j - 1) = 0
\] (1.47)

where \( \Delta \) captures how the current interest rate affects future deviations through its informational effect on \([E_t^s r_t^n, E_t^s u_t]^\prime\). (See the Appendix A. 3 for the derivations.)

**Proposition 2:** With serially correlated shocks, the interest rate in the current period affects future equilibrium through the dynamic informational effect. The consideration of the dynamic informational effect makes the equilibrium interest rate target beliefs in addition to targeting the current inflation and the output gap.

The consideration of the dynamic informational effect consists of two parts. The first part is captured by the effect on current equilibrium, because both consumption and pricing decisions are forward-looking. The second part is due to the persistence in the learning process, which is in addition to stabilizing the current economy. This additional beliefs-targeting does not exist with serially uncorrelated shocks.

The effects of discretionary policy on future variables are different from the effects on the actual future variables, as the private sector cannot distinguish the actual shocks from the beliefs.

To see this, use the output gap as an example. First, express the future output gap as the actual shocks, the expected shocks and the interest rate.

\[
\hat{y}_{t+1} = \Xi(1) E_{t+1}^s r_{t+1} + \Xi(2) E_t^s u_{t+1} + \frac{1}{\sigma} r_{t+1} - \frac{1}{\sigma} i_{t+1}
\] (1.48)

Next, express the expected shocks as the beliefs formed with weights assigned on both past
beliefs and signals in this period.\footnote{See Appendix A. 3 for expressions of $\Xi$ and $\Lambda$}

\[ E_{t+1}^sr_{t+1} = \Lambda_1 E_{t}^sr_{t} + \Lambda_2 E_{t}^su_t + K_{11}i_{t+1} \]

\[ E_{t+1}^su_{t+1} = \Lambda_3 E_{t}^sr_{t} + \Lambda_4 E_{t}^su_t + K_{21}i_{t+1} \]

The marginal effect of the interest rate on $\hat{y}_{t+1}$ can then be expressed as the combination of the informational effects on $E_t^sr_{t}$ and $E_t^su_t$.

\[
\frac{\partial \hat{y}_{t+1}}{\partial i_t} = \Xi(1) \left( \Lambda_1 \frac{\partial E_t^sr_{t}}{\partial i_t} + \Lambda_2 \frac{\partial E_t^su_t}{\partial i_t} \right) + \Xi(2) \left( \Lambda_1 \frac{\partial E_t^sr_{t}}{\partial i_t} + \Lambda_2 \frac{\partial E_t^su_t}{\partial i_t} \right)
\]

(1.49)

However, the effect of interest rate on the expected future output gap has an additional term, as the private sector is not able to separate the beliefs from the actual $r_{t+1}$. The effect of discretionary policy on the expected output gap is

\[
\frac{\partial E_t^s\hat{y}_{t+1}}{\partial i_t} = \left( \Xi(1) + \frac{1}{\sigma} \frac{1}{1 - \phi} \right) \left( \Lambda_1 \frac{\partial E_t^sr_{t}}{\partial i_t} + \Lambda_2 \frac{\partial E_t^su_t}{\partial i_t} \right) + \Xi(2) \left( \Lambda_1 \frac{\partial E_t^sr_{t}}{\partial i_t} + \Lambda_2 \frac{\partial E_t^su_t}{\partial i_t} \right)
\]

(1.50)

In addition, the effect of discretionary policy on future variables should be distinguished from the effect of committing to a future interest rate, as the former consists of the informational effect on the current beliefs, and the latter consists only of the direct effect on future borrowing costs. Both of the effects are able to influence the current equilibrium when private agents are forward-looking. The marginal effect of an increase in the future interest rate on $\hat{y}_{t+1}$ is:

\[
\frac{\partial \hat{y}_{t+1}}{\partial i_{t+1}} = -\frac{1}{\sigma}.
\]

(1.51)
To solve for the equilibrium interest rate under discretion, I first propose that the interest rate follows a linear function, \( i_t = F_1 r^n_t + F_2 E^{s}_{t-1} r^n_{t-1} + F_3 u_t + F_4 E^{s}_{t-1} u_{t-1} \), with which the private sector updates beliefs on \( E^{s}_{t} r^n_t \) and \( E^{s}_{t} u_t \). The central bank then chooses the interest rate to minimize the loss function specified in equation (1.39). If the optimizing interest rate is different from the proposed one, the private sector then changes its beliefs about the interest rate reaction function. The optimal interest rate is found as the fixed-point solution in this iteration process. Details of this solution method are provided in Appendix.

The persistence in underlying shocks strengthens the informational effect of interest rate, because it increases the effect of expected future deviations on current consumption and pricing decisions. If the serial correlation is high enough, it may cause optimal discretionary interest rate to fail to exist. The intuition is the following. Suppose that the private sector believes the best response of central bank is to increase the interest rate to the two shocks. If cost push shock is realized to be positive, which makes the inflation positively deviate from steady-state, the nominal effect of the interest rate decreases inflation and the informational effect of the interest rate increases inflation. If the informational effect dominates the direct effect, the inflation increases even further. As a result, a discretionary central bank wants to choose a negative interest rate, which contradicts the beliefs in the private sector that the best response of interest rate is to react positively to the two shocks.

### 1.4.3 Quantitative Analysis

In this section, I analyze the quantitative aspect of the model using a calibrated dynamic model. I calibrate the model parameters in line with the convention in the macroeconomics literature. I set \( \varphi = 1 \) and \( \sigma = 1 \), assuming a unitary Frisch elasticity of labor supply and log utility of consumption. I use \( \beta = 0.99 \), which implies a steady state real return on financial assets of four
percent. For price rigidity, I calibrate $\theta$, the price stickiness parameter, to be 0.5, which is indicated by the average price duration from macro and micro empirical evidences.\(^\text{15}\) For the parameter that governs the elasticity of substitution between intermediate goods, I set $\varepsilon = 4$, which implies a steady state price markup of one-third of revenue.

For the evolution of underlying shocks, I set the auto-correlation of natural-rate shocks to be 0.9, with a standard deviation of 3 percent, as measured by Laubach and Williams (2003). There is less consensus in the persistence and volatility of cost-push shocks, as they stems from a various sources. I set the auto-correlation for cost-push shocks to be 0.3 to avoid informational effect of interest rate being so strong that kills the equilibrium of an optimizing discretionary interest rate. I set the standard deviation of cost-push shocks to be the same as that of natural-rate shocks. In addition, I set the standard deviation of policy implementation error to be the same as the standard deviation of natural rate shock. I assume that there are no external signals apart from the interest rate. A summary of parameter values in the baseline calibration is provided in the Appendix.

Given these parameter values, I compare the equilibrium interest rate under imperfect information and with perfect information.

\[
i_t = 1.1525r^d_t - 0.1372E_{t-1}^s r_{t-1} + 0.1525u_t + 0.3464E_{t-1} u_{t-1},
\]

\[
i_t = 1.0000r^d_t + 2.6490u_t.
\] (1.52)  

\[
i_t = 1.0000r^d_t + 2.6490u_t.
\] (1.53)

In the following figure, I compare the impulse response after different shocks under perfect information and under imperfect information.

Figure 1.3: Impulse Response of Equilibrium Interest Rate, Output Gap and Inflation

In the above figures, parameter values are chosen as $\phi = 0.9$, $\phi^u = 0.3$, unitary Frisch elasticity of labor supply and log utility of consumption, i.e., $\varphi = 1$ and $\sigma = 1$. I use $\theta = 0.5$ for price rigidity and $\varepsilon = 4$ for elasticity of substitution, and discount factor $\beta = 0.99$. 
The above figures show that after a natural rate shock, the equilibrium interest rate, output gap and inflation are very similar under imperfect information and under perfect information. This is because the sensitivity of interest-rate to natural rate shocks is much higher than the sensitivity of interest rate to cost-push shocks, and therefore beliefs after an actual natural-rate shock is close to the actual shocks. In comparison, the equilibrium after cost-push shocks are significantly.

The response of the interest rate is largely reduced when information is imperfect which consequently reduces the expected inflation. As the actual inflation is partially determined by the expected inflation, this informational effect reduces equilibrium inflation without the sacrifice of negative output gap as in the perfect informational case. Lastly, the last row of Figure 1.3 shows that the effect of positive policy shocks is larger under imperfect information. This is because under imperfect information, the informational effect of interest rates increases the expected natural rate which is equivalent to a negative shock to the natural level of output. Therefore, the household reduces consumption and the reduction in aggregate demand brings down inflation.\textsuperscript{16} The combination of the informational effect and the direct effect on the cost of borrowing of the household makes the equilibrium output gap and inflation deviate further away, compared with the case under perfect information.

\section*{1.5 Conclusion}

In this chapter, I studied an economy in which the private sector has imperfect information about the underlying shocks and the central bank has perfect information when making interest rate decisions. In this economy, private agents regard the interest rate as a public signal, and extract information about the underlying economy from the interest rate decisions by the central bank.\textsuperscript{16}

\textsuperscript{16}A positive policy shock also increases the expected cost push shock, which should increases inflation. However this effect is nominated by the effect on the expected natural rate.
Consequently, the interest rate has an informational effect in addition to its direct effect on the cost of borrowing of the household.

I began my analysis by characterizing the equilibrium output gap and inflation in the private sector. I built a New Keynesian model with both nominal frictions and informational frictions. There are two types of shocks: aggregate technology shocks and wage markup shocks, both of which are not directly observed by the private sector.

To study the optimizing response of a discretionary central bank, I started with the simple scenario in which both shocks are serially uncorrelated, which allowed me to concentrate on the within-period informational effect of interest rates. Using an arbitrary interest rate function that responds positively to both shocks, I showed that beliefs in the private sector are more sensitive to the shock to which the interest rate reacts more aggressively or that has higher ex-ante dispersion.

A discretionary central bank sets interest rates to optimize its objective function at any state of the economy, taking as given the informational effect of its interest rate decision. I showed that the informational effect of the interest rate applies differently to the output gap and inflation. Consequently, the informational effect changes the Phillips curve, which captures the co-movements between the output gap and inflation due to changes in the interest rate. I found that in equilibrium, the interest rate reacts more aggressively to natural-rate shocks and less aggressively to cost-push shocks under imperfect information, in comparison to the optimal response of a discretionary central bank under perfect information.

Finally, I analyzed the dynamic aspects of the model by allowing serial correlation in shocks. Due to the serial correlation in shocks, the current interest rate also affects expectations about future equilibrium through its effect on expectations about current state variables. Using a calibrated model, I found that the dynamics after natural-rate shocks are similar and the dynamics after cost-push shocks are very different under imperfect information and under perfect information. This is
because in equilibrium, the interest rate is more sensitive to natural rate shocks and less sensitive to cost push shocks. Consequently, after an actual natural rate shock, the expected natural rate shock is close to the actual one, whereas after a cost push shock, private agents barely update their expectations on cost push shocks. The dynamics after cost-push shocks show that informational frictions can potentially improve welfare, as inflation after cost push shocks is reduced without sacrificing a larger negative output gap. This suggests that if the central bank can control the information conveyed through policy rates, it can potentially improve welfare, which I study in the next chapter.
Chapter 2

The Informational Gains from Policy Commitment
2.1 Introduction

Past literature has demonstrated that if the central bank commits to a policy rule, there are gains from commitment, if beliefs in the private sector are optimally controlled by the central bank. However, previous literature has only studied the case in which commitment changes the expectations regarding the policy itself. In this paper, I study the situation in which monetary policy also conveys information on the realization of shocks in the private sector, and study how gains from commitment can come from the informational effect of monetary policy. I demonstrate that the central bank can change how beliefs about different shocks are formed in the private sector by committing to a state-contingent policy rule, which leads to welfare gains from commitment.

The model of the private agents are the same as the previous chapter: private agents cannot directly observe shocks and use the interest rate as a public signal to extract information. In this economy, interest rates have two effects on the equilibrium output gap and inflation: the traditionally studied direct effect on the borrowing cost of households and the informational effect. In Chapter 1, the discretionary central bank takes as given the informational effect of the interest rate. In Chapter 2, I consider the case in which the central bank with commitment can change how beliefs are formed in the private sector by announcing and committing to a state-contingent rule. When choosing the ex-ante policy rule, the central bank with commitment internalizes the change of the informational effect of its interest rate decisions and balances between the direct effect and the informational effect of the interest rate.

To study how the optimal rule differs from the equilibrium interest rate decision of the discretionary central bank, I start with the simple case in which shocks have no serial correlations. To isolate the within-period informational gains, I focus on the case in which the interest rate only

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1See Kydland and Prescott (1977), Barro and Gordon (1983), Clarida, Gali and Gertler (2000), Woodford (1999), Eggertsson et al. (2003), among others. A more comprehensive review on gains from commitment is provided in the literature review section.
responds to current shocks. This removes the traditionally studied gains from commitment to a delayed response, which comes from the change in expected future equilibrium. I show that even without the traditionally studied effect on the expected future equilibrium, the optimal policy rule can still improve ex-ante welfare compared with the equilibrium under discretion, as the policy rule optimally controls the information revealed about the unobserved shocks.

The informational gains from commitment can be analyzed through the change in the Phillips curve. I demonstrate that relative to the optimizing discretionary interest rate, the optimal interest rate rule responds more aggressively to natural-rate shocks and less aggressively to cost-push shocks. When the private sector believes that the interest rate is less sensitive to the cost-push shocks, beliefs about the cost-push shocks are less sensitive to changes in the interest rate. Consequently, both the slope and the intercept of the Phillips curve are endogenously determined by the policy rule. The optimal policy rule improves ex-ante welfare, because it reduces the sensitivity of expected cost-push shocks to interest rates, which consequently reduces the stabilization bias caused by an actual cost-push shock under perfect information.

The informational effect of monetary policy results in a novel time-inconsistency problem. Different from the traditional time inconsistency, in which the incentives to deviate apply across time periods, the time inconsistency problem in my model applies across states. Once the central bank has committed to a policy rule, it fixes the informational effect of the interest rate, and thus the Phillips curve. Ex-post, the central bank has an incentive to deviate from its committed rule, assuming that such a change in the interest rate response will not change the Phillips curve. Suppose that there is a positive natural-rate shock; then, prior to the realization of the shock, the central bank commits to react more aggressively, relative to the optimizing response under discretion, to reduce the informational effect on the expected cost-push shock. This policy rule reduces the intercept of the Phillips curve. Once the Phillips curve is fixed, the central bank wants to reduce the increase
in the interest rate, as long as such deviation is not anticipated by private agents.

In addition to implementing a state-contingent policy rule, the central bank can also control expectations in the private sector through direct communication. Direct communication is modeled as the central bank providing external signals independently around the actual shocks. Without the informational effect of the interest rate, increasing the precision of the signal about one shock only makes the expected shock closer to the actual shock ex-ante. However, in the presence of an informational effect of the interest rate, the effects of external signals are not independent. Increasing the precision of the external signal about one shock also makes the interest rate a more precise signal about the other shock. Consequently, this interaction effect yields different welfare implications for central bank communication than argued by the conventional wisdom. Providing more precise information about the efficient shock (natural-rate shock) through central bank communication may reduce welfare if the private sector also simultaneously has more precise information about the inefficient shock (cost-push shock) from the interest rate.

Moving to the case where shocks have serial correlation, I calibrate the full version of my model, including external signals, serially correlated shocks, and policy implementation errors. In my calibrated model, I adopt parameter values from previous macroeconomics studies, except for the precision of external information. I allow the policy rule to contingent on expectations formed in the last period. I find that after cost push shocks, the optimal policy rule has inertial components. The interest rate reacts little in the period where the shock is first realized, and reacts more in the following periods to the expectations about the cost-push shock. In this case, the gains from committing to a delayed response reinforce the informational gains from commitment. By reducing the contemporaneous response to the actual cost-push shock, the central bank reduces the sensitivity of expected cost-push shocks to interest rates. At the same time, the central bank commits to a greater tightening monetary policy in following periods, which reduces the expectations in future
inflation.

Varying the precision of external information critically changes the size of the gains from commitment. In the extreme case in which external information is infinitely imprecise, the gains from commitment are negligible. However, when external signals are as precise as actual shocks, the optimal policy rule can improve welfare by 54 percent relative to the equilibrium under optimizing discretionary policy.

**Related Literature**

My paper explores the traditionally studied gains from commitment in the context of informational frictions.

There is a long history of studying the gains from monetary policy commitment. The original treatments can be found in Kydland and Prescott (1977) and Barro and Gordon (1983), who discuss the classical inflationary bias that results from a discretionary central bank having an objective function that contains a positive output gap target. A large literature has developed various methods to overcome the inflationary bias under discretion, including central bank reputation (Barro (1986) and Cukierman and Meltzer (1986) etc) and different central bank preferences (Rogoff (1985), Lohmann (1992) and Svensson (1995) etc).

Another mechanism that leads to gains from commitment is when a discretionary central bank faces stabilization bias. This occurs when there is a trade-off between closing the output gap and minimizing inflation in the current period. By committing to a delayed interest rate response, the central bank is able to decrease current inflation without sacrificing the current output gap; instead it does so through the decrease in expected future inflation. Clarida, Gali and Gertler (2000) study how an ad-hoc cost-push shock introduces a conflict between inflation stabilization and output gap stabilization and describe the optimal commitment to a future interest rate path. Woodford (1999)
studies how an interest rate smoothing objective helps the central bank to commit to a history-dependent policy, to steer private sector expectations about future policy rates. Eggertsson et al. (2003) show that optimal commitment to delayed response can mitigate the distortions created by the zero lower bound on the interest rate.

There are also papers that discuss the gains from policy commitment under imperfect information. Svensson and Woodford (2003) and Svensson and Woodford (2004) assume that the central bank has imperfect information and show that the optimal policy under commitment displays considerable inertia, relative to the discretionary policy, due to the persistence in the learning process. Lorenzoni (2010) and Paciello and Wiederholt (2013) explore the idea that the central bank is able to change the learning process in the private sector if it is able to commit to completely offset inefficient shocks. However, none of the above papers assume that monetary policy has informational effect.

To the best of my knowledge, the only paper that discusses the time inconsistency problem resulting from the informational effect of monetary policy is Stein and Sunderam (2016). The authors use a reduced-form model in which the central bank balances between implementing the optimal target rate and minimizing the information revealed about this target. In their paper, private agents are assumed not to have rational expectations about the central bank’s behaviors. The discretionary central bank always has incentives to deviate from the target interest rate, to reveal less information about its target. In my paper, I assume that private agents have rational expectations about how the central bank would react under both discretionary policy and a policy rule. Relative to the perfect information case, both optimizing discretionary policy and the optimal policy rule exhibit an inertial response to cost-push shocks, but the degree of inertia is higher under commitment.
### 2.2 Optimal Commitment

In this section, I use the same model as described in the previous chapter, and start with the case in which shocks have no serial correlation. With forward-looking agents, expectations about future equilibrium matter for current consumption and pricing decisions. Consequently, even with serially uncorrelated shocks, a committed central bank may choose a policy rule that responds to past shocks, meaning that the expectations about the direct effect of future interest rates also change the current equilibrium, which potentially leads to gains from commitment. The gains from committing to a delayed response still apply under imperfect information. However, to focus on the within-period gains from the informational effect, I study a state-contingent policy rule that only responds to current shocks. Consequently, the IS curve and the Phillips curve are found to be:

\[
\hat{y}_t = -\frac{1}{\sigma} (i_t - r^n_t), \quad (2.1)
\]

\[
\pi_t = (1 - \theta)E^s_t \pi_t + \kappa \theta \hat{y}_t + u_t. \quad (2.2)
\]

As private agents know the structure of the economy, they form expectations on the output gap and inflation accordingly.

\[
E^s_t \pi_t = \kappa E^s_t \hat{y}_t + \frac{1}{\theta} E^s_t u_t, \quad (2.3)
\]

\[
E^s_t \hat{y}_t = -\frac{1}{\sigma} (i_t - E^s_t r^n_t) \quad (2.4)
\]

Private agents are assumed to be unable to observe the realization of shocks directly, and use the interest rate as the only signal to form exceptions about the shocks. When private agents expect the interest rate to react linearly to the two shocks, they optimally form beliefs according to a Kalman
Filtering process:

\[
\begin{bmatrix}
E^s_t r^n_t \\
E^s_t u_t
\end{bmatrix} =
\begin{bmatrix}
1 - K_r \\
1 - K_u
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
K_r \\
K_u
\end{bmatrix}
\begin{bmatrix}
K_r F_r \\
K_u F_u
\end{bmatrix}
\begin{bmatrix}
r_t \\
u_t
\end{bmatrix},
\]

where

\[
K_r F_r = \frac{F_r^2 \sigma_r^2}{F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2},
\]

\[
K_u F_u = \frac{F_u^2 \sigma_u^2}{F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2}.
\]

In contrast to the problem for a discretionary central bank, which takes as given how beliefs will be formed in the private sector, a committed central bank is able to control beliefs by announcing a monetary policy rule prior to the realization of shocks. The central bank with commitment internalizes the change in the Kalman gains when deciding the policy rule to implement. In other words, by choosing a state contingent policy rule, \( i_t = F_c r^n_t + F_u u_t \), the central bank under commitment chooses a direct mapping from the actual shocks to the expected shocks.

2.2.1 The Phillips Curve under Policy Rules

In this section, I show that the Phillips curve becomes endogenous to the choice of the policy rule. A committed central bank internalizes the fact that its policy-rule decisions will change the sensitivity of expected shocks to the interest rate and, consequently, changes the trade-off between inflation and the output gap.

Specifically, the Phillips curve applying to a central bank with commitment that describes the
The available trade-off between the output gap and inflation is given by

\[
\pi_t = \left\{ \kappa - \sigma \left[ (1 - \theta) \frac{\kappa}{\sigma} K^*_r + \frac{1 - \theta}{\theta} K^*_u \right] \right\} y_t^* + \left\{ (1 - \theta) \frac{\kappa}{\sigma} (K^*_r - 1) + \frac{1 - \theta}{\theta} K^*_u \right\} r_t^* + u, \quad (2.6)
\]

The difference with the discretionary case is that the \( K^*_r \) and \( K^*_u \) are no longer constant, but endogenously determined by the choice of policy rule, i.e., \( K^*_r = K(F_r, F_u) \), \( K^*_u = K(F_r, F_u) \) as specified in equation (2.5).

In the following figure, I illustrate how committing to a policy rule changes the Phillips curve through an example such that \( i_t = r_t^* \), i.e., the central bank commits that it tracks the natural rate one-to-one and never responds to cost-push shocks.

In this figure, I plot the available trade-offs of inflation versus the output gap after a natural rate shock, for both the case in which private agents are convinced by the rule and the case in which private agents believe the central bank will be discretionary. As argued in the previous chapter, in the discretionary case, as the interest rate responds to both shocks which makes private agents unable to perfectly tell the realization of shocks, inflation raises as the informational effect of tightening policy brings inflation expectations up. The best available equilibrium under discretion is the tangent point between the Phillips curve and the indifference curve of the central bank, which is approximately \((-0.1, 0.03)\).

However, under the interest rate rule which only responds to natural-rate shocks, private agents assign probability one to the event that a natural-rate shock is realized whenever they observe a change in the interest rate. Consequently, after an actual natural rate shock, the expected natural rate equals the actual natural rate, and the expected cost-push shock equals to the actual cost-push shock which is zero. The economy is no different from the economy under perfect information, in which case the Phillips curve after natural rate shock crosses the origin of the \( (\hat{y}_t, \pi_t) \) plane.
In the above figure, I plot the Phillips curve after a natural rate shock under discretionary policy (black line) and the Phillips curve when private agents are convinced about the rule that $i_t = r^*_t$, i.e., $F_r = 1$ and $F_u = 0$ (blue line). The dotted ellipse denotes the indifference curve of the central bank with the loss function, $L = \pi_t^2 + \omega \bar{y}^2_t$

Under this specific policy rule, the first best equilibrium can be achieved, which shows the important role of commitment. The first best equilibrium is not only determined by the realization of the natural-rate shock and the response of interest rate to the natural-rate shock, but is also determined by how the interest rate would respond if the other shock is realized. It is because the central bank has committed that it will never respond to the cost-push shock, so the interest rate can provide perfect information to the realization of the natural rate shock.

This specific policy rule achieves the optimal informational effect, as it minimizes the expected
cost-push shocks and provides perfect information after natural-rate shocks. However, it is not the optimal commitment, because although it achieves the first best after a natural-rate shock, the central bank also cares about the equilibrium after cost-push shocks. Being completely inelastic to cost-push shocks is not optimal, because the central bank wants to balance the direct effect and the informational effect of interest rates.

Notice that along this Phillips curve, the origin of the \((\hat{y}_t, \pi_t)\) is the only equilibrium point which is consistent with the rule. Other points captures the equilibrium when the informational effect of the interest rate is fixed, (private agents have been convinced that \(i_t = r^*_n\)) but the central bank deviates from this rule. For example, the Phillips curve below \(\pi_t = 0\) is the equilibrium in which the interest rate acts stronger than tracking one-to-one with the natural-rate shock. The Phillips curve is fixed by the informational effect of the interest rate and the realization of the actual shock. The change in the informational effect shifts the Phillips curve, and moving along the Phillips curve captures the direct effect of the interest rate.

In the following figure, I plot the available output gap versus inflation trade-off when changes of the interest rate are expected by private agents. In the first figure, the blue line illustrates the Phillips curve under perfect information, after a natural-rate shock such that \(r^*_n = 1\) and \(u_t = 0\). It crosses the origin and has a positive slope of \(\kappa\). The red curve represents the Phillips curve. Tracing the Phillips curve from a positive output gap to a negative output gap corresponds to an increase in the positive response of the interest rate to the natural-rate shock, which is equivalent to an increasing \(F_r\). Importantly, the effect of the increasing \(F_r\) to inflation also depends on the value of \(F_u\), as \((F_r, F_u)\) jointly determines \((K_r, K_u)\). Therefore, I fix \(F_u = 1\) to illustrate the effect of the change in \(F_r\). When \(F_r\) increases, its marginal effect on the output gap is constant, \(-\frac{1}{\sigma}\), but its marginal effect on inflation changes, because the marginal informational effect of the interest rate changes as \((K_r, K_u)\) changes with respect to \(F_r\).
In the above figures, I plot the Phillips curve under policy rule. The first figure shows the Phillips curve after a natural-rate shock, where I fixed $F_u = 1$ and vary $F_r$. The second figure shows the Phillips curve after a cost-push shock, I fixed $F_r = 1$ and vary $F_u$. Prior distribution of shocks are set equal to each other, such that $\sigma_r = \sigma_u = 0.1$.
In this figure, this change is illustrated by a steeper slope as the output gap decreases. Intuitively, as the private agents expect the interest rate to respond more aggressively to the natural-rate shock, they assign a lower probability to the event that a cost-push shock will be realized. Therefore, the informational effect of the tightening monetary policy leads to a smaller increase in expected inflation, which results in a steeper slope.

In the second figure, the blue line is the Phillips curve under perfect information after a cost-push shock, such that \( r^\pi_t = 0 \) and \( u_t = 1 \). The red curve is the Phillips curve under imperfect information, where I vary the value of \( F_u \) while fixing \( F_r = 1 \). In this figure, the change in the marginal informational effect is more significant than that the first figure. Tracing the Phillips curve from left to right, it represents a decreasing response of the interest rate to a cost-push shock, which is equivalent to a decrease in \( F_u \). As Lemma 1 suggests, a smaller \( F_u \) increases \( K_r \), and has a non-monotonic effect on \( K_u \). The combined effect depends on the value of \( F_u \), together with other parameters. When \( F_u \) is very small, an increase in its value decreases the output gap, but barely decreases inflation. This is because at this level of \( F_u \), an increase in its value increases both \( K_r \) and \( K_u \) (see Figure 1.1, top row). Therefore, the informational effect almost completely offsets the direct effect, which increases the borrowing cost.

### 2.2.2 Optimal Policy Rule

The optimal simple rule is found by choosing the interest rate feedback rule \( i_t = f (r^\pi_t, u_t, \pi_t, \hat{y}_t) \) prior to the realization of shocks, which becomes \( i_t = F_r r^\pi_t + F_u u_t \) in equilibrium. The optimal simple rule is found by choosing \( F_r \) and \( F_u \) to minimize the central bank’s ex-ante loss over the state space:

\[
\min_{\{F_r, F_u\}} \int \int \pi^2_t (r_t, u_t) + \omega \hat{y}^2_t (r_t, u_t) dr_t du_t,
\]

(2.7)
subject to

\[
\hat{y}_t = -\frac{1}{\sigma}[(F_r - 1)r^n_t + F_u u_t], \tag{2.8}
\]

\[
\pi_t = \left\{ -\frac{\kappa}{\sigma}(F_r - 1) + (1 - \theta)\frac{\kappa}{\sigma}(K_r F_r - 1) + \frac{1 - \theta}{\theta} K_u F_r \right\} r^n_t 
+ \left\{ -\frac{\kappa}{\sigma} F_u + (1 - \theta)\frac{\kappa}{\sigma} K_r F_u + \frac{1 - \theta}{\theta} K_u F_u + 1 \right\} u_t \tag{2.9}
\]

\[
E^s_t r^n_t = K_r F_r r^n_t + K_r F_u u_t, \tag{2.10}
\]

\[
E^s_t u_t = K_u F_u r^n_t + K_u F_u u_t. \tag{2.11}
\]

Comparing this problem with the problem for a discretionary central bank (equation 1.21), we find that the available set of combinations of \((\hat{y}_t, \pi_t)\) is expanded due to the additional degree of freedom, i.e. instead of choosing interest rate, the central bank with commitment chooses \(F_r\) and \(F_u\). By committing to a state-contingent rule that is different from \(i_t = F^d_r r^n_t + F^d_u u_t\), the central bank chooses a direct mapping from the actual shocks to the expected shocks. In comparison, even if a discretionary central bank changes its response of interest rate, the private sector still expects that it will follow its equilibrium response, which is described by \((F^d_r, F^d_u)\), meaning that the informational effect of the interest rate cannot be changed when the central bank does not have credible commitment. In other words, \((K_r, K_r)\) are endogenous choice variables only when the central bank has credible commitment. Otherwise, the central bank regards \((K_r, K_r)\) as exogenous to its interest rate decisions.

The key difference between a central bank with commitment and a discretionary central bank is that the central bank with commitment internalizes its policy responses determine the Phillips curve, which is the constraint that it faces. We now turn to comparing the optimal policy rule with the equilibrium interest rate under discretionary central bank using the first-order conditions. The
key factor in this analysis is that the effect of the interest rate after one shock also depends on how it would react to the other shock, because the informational effect, \((K_r, K_u)\) is jointly determined by the response of the interest rate to both shocks, \((F_r, F_u)\). \((F_r, F_u)\) are jointly determined by the first-order condition on \(F_r\) after the \(r_t^R\) shock, and the first-order condition on \(F_u\) after the \(u_t\) shock.

The first-order condition on \(F_r\) after the \(r_t^R\) shock is

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u + \Omega_r \frac{\partial K_r}{\partial F_r} F_r + \Omega_u \frac{\partial K_u}{\partial F_r} F_r = \frac{\omega}{\sigma} \frac{\hat{y}_t}{\pi_t},
\]

where \(\Omega_r = (1 - \theta) \frac{\kappa}{\sigma}\), and \(\Omega_u = \frac{1}{\sigma} - \theta K_u\).

Similarly, the first-order condition on \(F_u\) after the \(u_t\) shock is

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u + \Omega_r \frac{\partial K_r}{\partial F_r} F_r + \Omega_u \frac{\partial K_u}{\partial F_r} F_r = \frac{\omega}{\sigma} \frac{\hat{y}_t}{\pi_t},
\]

To illustrate the difference between the equilibrium interest rate under the discretionary optimizing policy and under the optimal policy rule, I write the first-order condition on the interest rate for a discretionary central bank in terms of \(F_r\) after an \(r_t^R\) shock, and \(F_u\) after a \(u_t\) shock.

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u = \frac{\omega}{\sigma} \frac{\hat{y}_t}{\pi_t}
\]

\[
-\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u = \frac{\omega}{\sigma} \frac{\hat{y}_t}{\pi_t}
\]

where \(\Omega_r = (1 - \theta) \frac{\kappa}{\sigma}\), and \(\Omega_u = \frac{1}{\sigma} - \theta K_u\).

Comparing the first-order conditions under commitment, the discretionary central bank regards the informational effect of the interest rate as exogenous to its interest rate decisions. Specifically, it does not internalize the change in the Kalman gain with respect to a change in the interest rate.
This is because \((K_r, K_u)\) are determined by the private sector’s expectations about the interest rate reaction function. Then, if the private agents believe that the central bank will optimize at any given state, they believe the interest rate will follow \(i_t = F^r_d r^n + F^d_u u_t\).

In Figure 2.3, I draw the optimal \(F^*_r\) at varying values of \(F_u\) which is the solution to equation (2.12), and the optimal \(F^*_u\) for varying values of \(F_r\), which is the solution to equation (2.13). The point where two lines cross is \((F^*_r, F^*_u)\).

![Figure 2.3: Solution to the Optimal Policy Rule](image)

In this figure, each point on the red line represents solution of \(F^*_r(F_u)\) that satisfies the first order condition on \(F_r\) as specified in equation (35), and each point on the blue line represents solution of \(F^*_u(F_r)\) that satisfies the first order condition on \(F_u\) as specified in equation (36). The point where two lines cross defines \((F^*_r, F^*_u)\).

As suggested by Lemma 1, both the sensitivity of the interest rate to shocks, and the prior distribution of shocks matter for the informational effect of the interest rate. Next, I posit assump-
Assumption 3. \( \sigma_r^a = \sigma_u \).

**Proposition 3:** Under Assumptions 1, 2, and 3, i) the optimal policy rule responds more aggressively to a natural-rate shock than the equilibrium response of the interest rate under discretionary policy, for a given response to cost-push shocks, and ii) the optimal policy rule responds less aggressively to a cost-push shock than the equilibrium response of the interest rate under discretionary policy, for a given response to natural-rate shocks.

**Proof:** see the Appendix 2.1.

Under the above assumptions, increasing the response to natural-rate shocks \( (F_r) \) or decreasing the response to cost-push shocks \( (F_u) \) from the equilibrium response under discretion \( (F_r^d, F_u^d) \), decreases the informational effect of the interest rate. Specifically, it means that (1) \( \Omega_r \frac{\partial K_r}{\partial F_r} + \Omega_u \frac{\partial K_u}{\partial F_r} < 0 \), and (2) \( \Omega_r \frac{\partial K_r}{\partial F_u} + \Omega_u \frac{\partial K_u}{\partial F_u} > 0 \).

As the central bank with commitment internalizes the effect of the interest rate decisions on the Phillips curve, it wants to reduce the marginal informational effect of interest rate, making interest rate more "effective" in offsetting the shocks. Assumption (1) guarantees that a higher value of \( F_r \) decreases the marginal informational effect of interest rate after natural-rate shocks, and Assumption (2) guarantees that a lower value of \( F_u \) decreases the marginal informational effect of the interest rate after cost-push shocks.

The gains from commitment come from (a) the increase in the slope of the Phillips curve after both shocks and (b) the decrease in the intercept after natural-rate shocks. As shown in the Phillips curve expressed, (a) and (b) are equivalent, and thus have same implication for the value of \( F_r^c \) and \( F_u^c \). Intuitively, more precise information on natural-rate shocks and less precise information on cost-push shocks reduces the conflict between the direct effect and the informational effect of the
Another way to investigate the comparison between the optimal policy rule and the equilibrium interest rate under discretionary policy is that as a central bank with commitment internalizes the informational effect of the interest rate, it balances between the optimal informational effect and the optimal direct effect on the borrowing cost.

The optimal informational effect is such that the central bank reveals perfect information about the natural-rate shock, and completely withholds information about the cost-push shock. This is because the natural-rate shock is efficient, as it changes the natural level of output together with the price level. Therefore, the natural-rate shock does not cause a conflict between output gap stabilization and inflation stabilization under perfect information. In comparison, the cost-push shock only changes the price level without chancing the natural level of output. Thus, it is inefficient, and leads to a conflict between output gap stabilization and inflation stabilization.\(^2\)

The optimal informational effect of the interest rate can be achieved by either setting \(F_r \to \infty\) or setting \(F_u = 0\). Balancing the optimal informational effect and the optimal direct effect results in the interest rate being more sensitive to the natural-rate shocks and less sensitive to the cost-push shocks, i.e. \(F_r^c > F_r^d\) and \(F_u^c < F_u^d\).

### 2.2.3 Time Inconsistency

In this section, I analyze the time inconsistency problem which refers to the situation in which the central bank has an incentive to deviate from its previously committed policy rule. The conventional wisdom on the time inconsistency problem applies across time periods. For example, as discussed in Eggertsson et al. (2003), when the current interest rate hits the zero lower bound, the

\(^2\)Existing literature has discussed how information on efficient shocks is beneficial. See Morris and Shin (2002), Angeletos and Pavan (2007), Angeletos, Iovino and La'O (2016) as examples.
central bank can encourage current consumption by committing to a lower interest rate in future periods such that \( E_t \pi_{t+1} > 0 \), when consumption decisions are forward-looking. However, the central bank will face a time inconsistency problem at \( t + 1 \), because \( \pi_{t+1} > 0 \) is sub-optimal. In my baseline model, I have shut down this conventional channel of commitment, so that the time inconsistency across time periods does not apply. Instead, I present a novel time inconsistency problem that applies across states. Specifically, the central bank wants to implement a different interest rate conditional on the realization of shocks, rather than according to its previously announced policy rule.

The intuition for the time inconsistency problem is that a discretionary central bank does not take into account the change of the informational effect of interest rates when deviating from the policy rule. Mathematically, if a central bank has convinced the private sector that it will implement \( i_t = F_r^c r^t + F_u^c u_t \), the sensitivity of expected shocks to changes in the interest rate is fixed. At this point, the central bank wants to re-optimize its interest rate decisions. The incentives for deviation are summarized in the following proposition.

**Proposition 4:** After the central bank has committed to a policy rule, there is always a profitable deviation after either natural-rate shocks or cost-push shocks, as long as the deviation is unexpected and thus the informational effect of the interest rate remains unchanged.

**Proof:** see the Appendix 2.1.

The intuition for Proposition 3 is the following. Prior to the realization of shocks, the optimal policy rule has committed to respond more aggressively to natural-rate shocks (Proposition 1), as it optimally weighs between decreasing the combined informational effect of the interest rate and the direct effect on the borrowing cost. If the central bank decides to implement an one-time deviation which is not expected by the private sector, it is able to keep the informational effect fixed and considers only its direct effect. By doing so, the central bank takes the informational
advantage such that the private sector believes the shock is more likely to be a natural-rate shock than a cost-push shock, without actually sacrificing a lower output gap when a natural-rate shock is realized. However, if the private sector anticipates such deviation, the private sector will update the sensitivity of its beliefs to changes in the interest rate, leaving no profitable deviation available for the central bank.

I illustrate the incentives for deviation after a natural-rate shock in the following graph.

![Figure 2.4: The Phillips Curve after a Natural-rate Shock under Optimal Policy Rule](image)

The dotted ellipse is the indifference curve for the central bank whose objective function consists the weighed sum of squared inflation and the squared output gap. The black line is the Phillips curve under discretion, and the red and blue lines are the Phillips curve when the central bank has convinced private agents about the (1) optimal policy rule and (2) \((F_r, F_u) = (1, 0)\), respectively.

As noted above, there is only one point along the Phillips curve under commitment that is consistent with the rule, and such points are denoted by the red circles. The first figure shows that after a natural-rate shock, the central bank has convinced private agents about the optimal rule, it
wants to deviate from the equilibrium under commitment to a point which is closer to the origin by implementing a smaller interest rate than the committed rule. However, a better equilibrium can only be achieved if such deviation is not anticipated, under which case the informational effect of the interest rate is fixed by the commitment, so changing $F_r$ will not change the slope of the Phillips curve. Mathematically, it means that the marginal informational effect of the interest rate, $\Omega_rK_r(F_{cr}, F_{cu}) + \Omega_uK_u(F_{cr}, F_{cu})$, will not change as $F_r$ changes, as the private sector expects the central bank to implement $(F_{cr}, F_{cu})$.

Specifically, after a natural-rate shock, when the central bank deviates to a smaller interest rate response than it had committed to ($F_r < F_{cr}$), the Phillips curve has a lower $\pi_t$ at any level of $\hat{y}_t$. This change in the Phillips curve suggests that by deviating to a smaller interest rate response to a natural-rate shock, the central bank achieves a one-time welfare improvement. However, if such deviation is anticipated, the Phillips curve will shift up, because as private agents expect a lower $\frac{F_c}{F_r}$ ratio, they increase their expected cost push shocks after any change in the interest rate.

The only equilibrium that is consistent with the optimal rule is denoted in the red circle, which locates on an indifference curve that is worse than the one where the equilibrium under discretion is located (the tangent point between the indifference curve and the Phillips curve under discretion). This seems counter-intuitive as there should be gains from commitment. The reason is that the central bank does not optimize at every state, but optimizes across all states. In fact, it sacrifices after a natural-rate shock and gains after a cost-push shock. I illustrate the equilibrium after a cost-push shock in the following figure. Notice that the unit of shock is chosen to be 0.1 instead of 1. This is to make comparable to Figure 2.5 while maintaining the scale of the two figures to be the same.
The dotted ellipse is the indifference curve for the central bank whose objective function consists of the weighted sum of squared inflation and the squared output gap. The black line is the Phillips curve under discretion, and the red and blue lines are the Phillips curve when the central bank has convinced private agents about the (1) optimal policy rule and (2) $(F_r, F_u) = (1,0)$, respectively.

### 2.3 Direct Communication

This section studies the optimal direct communication strategy for the central bank, which is modeled as the central bank providing public signals independently to the informational effect of interest rates. Unlike the informational effect through the interest rate, which is restricted by the signal dimension, central bank direct communication is not bounded by the signal dimension.

The central bank controls for the precision of these external signals. The general consensus on optimal communication strategy for the central bank is that it should provide more precise information about the efficient shocks (natural-rate shocks in this setting) and should not provide
information about the inefficient shocks (cost-push shocks in this setting). In my model, however, the value of central bank communication interacts with the informational effect through policy rate, which should change the optimal direct communication strategy.

2.3.1 Interaction between the Informational Effect of Monetary Policy and Central Bank Direct Communication

Denote the external signals sent through the central bank communications as $m^e_t$ and $m^μ_t$, which are distributed log normally around the actual shocks, $r^n_t$ and $u_t$. I assume that the interest rate does not react to the external signals. However, the existence of external signals changes the informational effect of the interest rate, which consequently changes both the equilibrium interest rate under discretionary central bank and the optimal policy rule.

Signals

The signals consist of both the interest rate and external signals sent through the central bank direct communication, which are summarized as follows:

$$
\begin{bmatrix}
\hat{i}_t \\
m^e_t \\
m^μ_t
\end{bmatrix} = \begin{bmatrix}
F_1 & F_3 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
r^n_t \\
u_t
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
ε^e_t \\
ε^μ_t
\end{bmatrix}
$$

(2.16)

Beliefs

\footnote{see Kramer et al. (2008) for survey of literature on central bank communication.}
The private sector updates beliefs using both the interest rate and the external signals:

\[
\begin{bmatrix}
E^s_t r^n_t \\
E^s_t u_t
\end{bmatrix} = \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23}
\end{bmatrix} \begin{bmatrix}
i_t \\
m^n_t \\
m^n_t
\end{bmatrix}
\]

(2.17)

where the Kalman gains in the \( K \) matrix are determined through the optimal filtering process.

The informational effect of the interest rate interact with central bank communication. When the central bank communicates more precisely about one shock, the private sector assigns a greater weight on the information from direct communication versus the information from the interest rate. At the same time, the interest rate becomes a more precise signal of the other shock. For example, suppose that the central bank precisely communicates about the \( r^n_t \) shock; then after a positive cost-push shock, the private agents know that \( r^n_t = 0 \) through the direct communication by the central bank. In addition, the private agents also observe that the interest rate responds positively, so that they infer precisely that the increase in the interest rate is due to the positive realization of a cost-push shock.

However, the existence of the informational effect of policy rates also changes the effect of central bank communication. \( K_{13} \) and \( K_{22} \) measure how much information is "falsely" updated to beliefs via external signals. Without an informational effect transmitted through the policy rate, \( K_{13} \) and \( K_{22} \) would be equal to zero, as signals are distributed independently, and the signal of one shock does not provide information about the other shock. However, as the interest rate is one signal about the two shocks, the interaction with the informational effect of the interest rate makes the central bank unable to separately convey information. Specifically, both \( K_{13} \) and \( K_{22} \) are negative. Intuitively, suppose that the interest rate does no change and that external signals on natural rate goes up; in this case, the private sector would then back out a negative change in the
cost-push shock.

\( K_{12} \) and \( K_{23} \) measure how much information is "correctly" updated through external signals. In Figure 6, I plot how the sensitivity of beliefs to each signal changes when the interest rate reacts more aggressively to a natural-rate shock, while holding its response to a cost-push shock fixed (varying \( F_r \) from 0.1 to 2 while fixing \( F_u = 1 \)).

![Figure 2.6: Sensitivity of Beliefs to External Signals](image)

In all 6 figures, I set \( F_u = 1 \), and vary \( F_r \) from 0.1 to 2. Ex-ante dispersion of shocks are set to be equal with each other, such that \( \sigma_r = \sigma_u = 0.1 \). The top row is the sensitivity of beliefs about natural-rate shocks with respect to all three signals, the interest rate, the external information about the natural-rate shock, and the external information about the cost-push shock. The second row is the sensitivity of beliefs about cost-push shocks with respect to all three signals, the interest rate, the external information about the natural-rate shock, and the external information about the cost-push shock. As illustrated in the first row, when the sensitivity of \( E_t^x r_t^n \) to the change in the interest rate is
not monotonic, as suggested by Lemma 1. The second figure in the first row shows the weight on $m_r$ decreases as the interest rate becomes a more precise signal about $r^n$. The second row shows that increasing the sensitivity of interest rate to the natural-rate shock also decreases the sensitivity of $E^i u_t$ to interest rate changes, and increases the sensitivity of $E^i u_t$ to central bank communication about the cost-push shock. Intuitively, since the private agents optimally weight these three signals, when the interest rate reacts more aggressively to the natural-rate shock, it becomes a more precise signal than $m_r$. At the same time, it becomes a less precise signal than $m_r^\mu$. For this reason, $K_{12}$ decreases and $K_{23}$ increases.

2.3.2 Value of (External) Information

To assess the value of external information through the direct communication from the central bank, we first need to study how the optimal response of the interest rate changes under discretion and with commitment, as the central bank takes into account the interaction between the informational effect of the interest rate and the direct communication. The Phillips curve with all signals can be obtained as follows:

$$\pi_t = \left\{ \kappa - \sigma \left[ (1 - \theta) \frac{\kappa}{\sigma} K_{11} + \frac{1 - \theta}{\theta} K_{21} \right] \right\} \hat{y}_t$$

$$+ \left\{ (1 - \theta) \frac{\kappa}{\sigma} (K_{11} + K_{12} - 1) + \frac{1 - \theta}{\theta} (K_{21} + K_{22}) \right\} r^n_t + \left\{ (1 - \theta) \frac{\kappa}{\sigma} K_{13} + \frac{1 - \theta}{\theta} K_{23} + 1 \right\} u_t$$

The existence of the informational effect of the interest rate complicates the welfare effect of central bank communication. Without the information effect of interest rate, welfare is maximized when the central bank provides perfectly precise signal about the efficient shock (the natural-rate shock), and completely uninformative signal about the inefficient shock (the cost-push shock). However, with the information effect of the interest rate, if agents in the private agents have precise
information about one shock through communications, they are able to infer precise information about the other shock from the interest rate.

In the following figure, I plot the ex-ante loss of the central bank at varying precision of central bank direct communication.

![Figures plot the indifference curves of the ex-ante central bank loss function, i.e., $EL = \text{var}(\pi_t^2) + \omega \cdot \text{var}(\hat{y}_t)$. The left column is under discretion and the right column is under policy rule. Prior distribution of the shocks are set as $\sigma_r = \sigma_u = 0.1$.](image)

Figure 2.7: The Value of Direct Communication
In the first row of Figure 2.6, I show the contour plot at varying levels of precision of central bank communication under optimizing discretionary policy (left) and under optimal policy rule (right). It shows that when communication about the cost-push shock becomes more imprecise, which is modeled by a lower $\sigma_{eu}$, the ex-ante loss increases. This is consistent with the conventional wisdom that more precise information about the inefficient shock is welfare reducing. However, when the precision of central bank communication about natural-rate shocks increases, which is modeled by a smaller $\sigma_{er}$, the ex-ante loss also increases. This contradicts the conventional wisdom. In summary, when the interest rate is able to provide sufficiently precise information about the efficient shock, additional direct communication about either shocks reduces ex-ante welfare.

Next, I add an implementation error to the interest rate function, such that $i_t = F_r r^n_t + F_u u_t + e_t$. The interest rate becomes a noisier signal of both shocks when the variance of the implementation error increases. I show the contour plot at varying levels of precision of central bank communication, assuming the implementation error of the interest rate has a standard deviation of 0.5. Since interest rate becomes a relatively imprecise signal now, the value of (external) information becomes the same as the conventional wisdom. The loss is minimized when information on natural-rate shock is most precise and information on the cost-push shock is least precise.

### 2.4 Quantitative Assessment

The goal of this section is to quantify the size of the gains from commitment using a dynamic model with varying degrees of information precision. I begin with the case where there are no external signals. In this case, the information precision depends on the prior distribution of the actual shocks. I then consider the case where there are external signals, and in addition, I allow for

---

4For the conventional wisdom on the value of information, see Morris and Shin (2002), Angeletos and Pavan (2007), Angeletos, Iovino and La’O (2016), for examples.
an implementation error in the interest rate. By varying the precision of signals, I quantitatively assess how the gains from commitment depend on the precision of external information.

I use the same dynamic model with parameters calibrated in the same way as in the previous chapter. The actual state variables evolve as

\[ z_t = \Phi z_{t-1} + v_t \]  

(2.19)

where \( z_t = [r^n_t, u_t]' \), \( \Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \), and \( v_t = [v^r_t, v^u_t]' \) with white noise of variance \( Q \).

The private sector forms expectations about current states through the Kalman filtering process. Denote the observable signal as

\[ s_t = Dz_t \]  

(2.20)

where \( s_t = i_t \), and \( D = [F_1, F_3]' \).

The Kalman filtering process makes beliefs about the current state variables be the optimal combination of beliefs in the last period and signals in the current period:

\[ E^s_t z_t = \Phi E^s_{t-1} z^n_{t-1} + K \left( s_t - D \Phi E^s_{t-1} z_{t-1} \right) \]  

(2.21)

where the optimal weight, \( K \), is determined by Ricatti iteration

\[ K = PD'(DPD')^{-1}, \]  

(2.22)

\[ P = \Phi \left( P - PD'(DPD')^{-1}DP \right) \Phi + Q. \]  

(2.23)

The objective function for the committed central bank is the same as the discretionary central bank. I require that the committed central bank can only commit to a rule which responds linearly
to current state variables. Notice that as past beliefs determine current beliefs, they also become current state variables. In equilibrium, the optimal rule follows same functional form as the discretionary interest rate, i.e.,

\[ i_t = F_1 r_t^n + F_2 E^s_{t-1} r^n_{t-1} + F_3 u_t + F_4 E^s_{t-1} u_{t-1}. \]

The coefficients of the optimal rule, [\(F_1, F_2, F_3, F_4\)] are selected to minimize the ex-ante loss from the steady state.\(^5\)

\[
\min_{F_1, F_2, F_3, F_4} E_t L(t) = \int \int \left( \pi_t^2 + \omega \dot{\gamma}_t^2 + \beta E_t L(t+1) \right) dr_t^n du_t \tag{2.24}
\]

where the output gap and inflation are specified as:\(^6\)

\[
\begin{align*}
\dot{y}_t &= E^s_t \dot{y}_{t+1} - \frac{1}{\sigma} \left[ i_t - \left( \frac{1}{1 - \phi} r_t^n - \frac{\phi}{1 - \phi} E^s_t r^n_t - E^s_t \pi_{t+1} \right) \right], \tag{2.25}
\pi_t &= \beta \theta E_t^s \pi_{t+1} + (1 - \theta) E_t^s \pi_t + \kappa \theta \dot{y}_t + u_t. \tag{2.26}
\end{align*}
\]

and the actual shocks and expected shocks follow equations (2.19) - (2.23).

In contrast to the serially uncorrelated case, in which I completely shut down the gains from commitment through delayed response, I allow for such gains in the dynamic case. Potentially, the policy rule can react to current cost-push shocks by a lesser extent and commits to a large response to past beliefs than a discretionary interest rate does. In doing so, not only does interest rate reveal less information about the cost-push shock, it also decreases expected inflation. The gains from committing to a delayed response strengthen the gains from the informational effect.

\(^5\)In steady state, \(E^s_{t-1} r^n_{t-1} = 0\), and \(E^s_{t-1} u_{t-1} = 0\).
\(^6\)See Chapter 1 for derivations.
2.4.1 No External Information

I numerically solve for both the equilibrium interest rate under discretion and the optimal policy rule under the baseline dynamic model, which yields the following results:

\[
i_{\text{discretionary}} = 1.3887r^n_t - 0.3498E_{t-1}^s r^n_{t-1} + 0.1852u_t + 0.3370E_{t-1}^s u_{t-1}
\]

\[
i_{\text{rule}} = 1.3727r^n_t - 0.3374E_{t-1}^s r^n_{t-1} + 0.1830u_t + 0.3332E_{t-1}^s u_{t-1}
\]

Regarding the equilibrium interest rate under discretion, the novelties of the dynamic case are the value of \(F_2\) and \(F_4\). They capture how the interest rate optimally responds to the beliefs in the past period. As beliefs are persistent, reacting to past beliefs leads to inertia in the interest rate. As analyzed in the belief-formation process, this response has no informational effect, as the private sector is able to subtract the part of \(F_2 E_{t-1}^s r^n_{t-1} + F_4 E_{t-1}^s u_{t-1}\) to obtain signals in current period. Specifically, a negative \(F_2\) means central bank counteracts the excess response to the natural-rate shock in the first period. A positive \(F_4\) means that the central bank makes up for the deficient response to the cost-push shock in the first period. The intuition can be found in the output gap equation and inflation equation, which show that an expected natural rate decreases output gap and expected cost push shock increases inflation. As past beliefs contribute positively to current beliefs, the interest rate optimally responds to past beliefs to offset their contribution to current deviations.
Parameters are chosen as $\phi = 0.9$, $\phi^u = 0.3$, unitary Frisch elasticity of labor supply and log utility of consumption. I use $\theta = 0.5$ for price rigidity and $\varepsilon = 4$ for elasticity of substitution, and discount factor $\beta = 0.99$.

Compared with the optimal response of a discretionary central bank, the response coefficients
in the optimal policy rule do not differ substantially. This is because the relative sensitivity of expected cost-push shocks and expected natural-rate shocks to interest rates is already very small when there are no external signals. Consequently, when a central bank with commitment chooses the optimal policy rule to balance the informational effect and the direct effect, it does not differ much from the equilibrium under an optimizing central bank.

This suggests that the welfare gains from commitment is not significant. I calculate the ex-ante welfare loss as the loss function of the central bank, which is 0.0325 for the discretionary case and 0.0324 for the commitment case. In Figure 2.7, I plot the impulse response after a natural-rate shock, a cost-push shock and a policy error, which also show that the difference in equilibrium under discretionary policy and under policy rule is negligible when the interest rate is the only source of information.

### 2.4.2 Varying Precision of External Information

The precision of external signals crucially determines the gains from commitment, as it affects the size of the informational effect of the interest rate. The previous literature does not provide consensus on the degree of information frictions in the private sector. Instead of calibrating the precision of external signals, I investigate how the size of the gains varies with the precision of external signals. I first set the variance of the external signals to be same as the variance of the ex-ante dispersion of the actual shocks. In addition, I also allow for an implementation error in the interest rate.
The interest rate is numerically solved as follows:

\[ i_{\text{discretionary}} = 1.4402 r^n_t - 0.3962 E_{t-1}^s r^n_t + 0.4485 u_t + 0.2578 E_{t-1} u_{t-1}, \]

\[ i_{\text{rule}} = 1.2818 r^n_t - 0.2430 E_{t-1}^s r^n_t + 0.0373 u_t + 0.5184 E_{t-1}^s u_{t-1}. \]

There most significant difference is the response of the interest rate to the cost-push shock. The optimal policy rule responds very little in the period when the cost-push shock is realized, shown as \( F_{3}^{\text{comm}} = 0.0373 \) and in comparison, \( F_{3}^{\text{disc}} = 0.4485 \). In addition, the optimal policy rule responds to a greater extent to the expected cost-push shock in the last period, shown as \( F_{4}^{\text{comm}} = 0.5184 \), and in comparison, \( F_{4}^{\text{disc}} = 0.2578 \). In other words, the interest rate under optimal commitment has an inertial components: it responds little contemporaneously but commits to react more aggressively in future periods.

In this way, the informational gains and the traditionally studied gains from a delayed response reinforce each other. By reacting little to the actual cost-push shock contemporaneously, the central bank reduces the sensitivity of the expected cost-push shock to the interest rate. At the same time, by committing to a greater response in later periods, the central bank reduces expected future inflation.

Figure 2.8 compares the impulse response function after a natural rate shock, a cost push shock and a policy rate shock under discretionary policy and under policy rule.
Parameters are chosen as $\phi = 0.9$, $\phi^u = 0.3$, unitary Frisch elasticity of labor supply and log utility of consumption. I use $\theta = 0.5$ for price rigidity and $\varepsilon = 4$ for elasticity of substitution, and discount factor $\beta = 0.99$. 

Figure 2.9: Impulse Response with Precise External Signals
In the rest of this section, I show how the size of gains depends on the precision of external signals. First of all, holding the standard deviation of $m_r$ fixed at 0.1, I vary the precision of external signal on natural rate shock from 0.01 to 0.1. In the following table I report the welfare gains measured by the ex-ante loss function of the central bank standard deviation of inflation and standard deviation of output gap for both types of central bank. When signal on natural rate changes from relatively imprecise $\sigma_{er} = 0.1$ to very precise $\sigma_{er} = 0.01$, the welfare gains measured by the ex-ante loss function of the central bank increases from 64 percent to 70.4 percent.

<table>
<thead>
<tr>
<th></th>
<th>Discretionary</th>
<th></th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{er} = 0.01$</td>
<td>5.13</td>
<td>$\sigma_{er} = 0.05$</td>
<td>3.78</td>
</tr>
<tr>
<td>Ex-ante Loss</td>
<td>3.76</td>
<td>$\sigma_{er} = 0.1$</td>
<td>1.84</td>
</tr>
<tr>
<td>Inflation</td>
<td>1.45</td>
<td>$\sigma_{er} = 0.01$</td>
<td>1.89</td>
</tr>
<tr>
<td>Output Gap</td>
<td>4.63</td>
<td>$\sigma_{er} = 0.05$</td>
<td>1.10</td>
</tr>
<tr>
<td></td>
<td>4.42</td>
<td>$\sigma_{er} = 0.1$</td>
<td>2.31</td>
</tr>
<tr>
<td></td>
<td>4.46</td>
<td></td>
<td>2.33</td>
</tr>
</tbody>
</table>

Table 2.1: Gains from Commitment at Varying Levels of Precision of External Information on $r_n^e$

The ex-ante loss is calculated as the objective function of the central bank defined in equation (57) $\times 10^2$. The numbers for inflation and output gap are noted in percentage points.

In the following table, I report the size of gains from commitment when holding the standard deviation of $m_r$ fixed at 0.1, and varying the standard deviation of external signal on cost-push shock from 0.01 to 0.1. When signal on natural rate changes from relatively imprecise $\sigma_{er} = 0.1$ to very precise $\sigma_{er} = 0.01$, the welfare gains measured by the ex-ante loss function of the central bank increases from 28 percent to 49 percent.

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7 standard deviation of interest rate implementation error is fixed to be 0.01 in all calibration exercises.

8 This is defined as the objective function for commitment central bank in equation (57), which is the weighted sum of squared deviations from steady state.
Table 2.2: Gains from Commitment at Varying Levels of Precision of External Information on $u_t$

The ex-ante loss is calculated as the objective function of the central bank defined in equation (57) \times 10^2. The numbers for inflation and output gap are noted in percentage points.

2.5 Conclusion

In this chapter, I studied how committing to the optimal policy rule improves ex-ante welfare. To do so, I analyzed how commitment can change the informational effect of interest rates, which consequently changes the Phillips curve. In addition, I also showed how the informational effect through policy rates changes the implications of optimal strategy of central bank direct communication. Quantitatively, using a calibrated dynamic model, I showed that the size of the gains from commitment depends on the degree of informational frictions in the private sector.

I used the same New Keynesian model as the previous chapter. When the central bank has convinced private agents that it would implement a different policy rule than its equilibrium response under discretion, it is able to shift the Phillips curve. Consequently, the informational effect of interest rates makes the Phillips curve endogenous to the central bank’s decision about the optimal policy rule.

The responses of the optimal policy rule to the two shocks are jointly determined. Assuming that the natural-rate shock and the cost-push shock have the same ex-ante dispersion, The optimal
policy rule responds more aggressively to the natural-rate shock and less aggressively to the cost-push shock, relative to the equilibrium optimizing interest rate set by discretionary central bank. By doing so, it achieves an informational advantage as it withholds information about the cost-push shock, which consequently reduces the stabilization bias caused by the actual cost-push shocks under perfect information.

I extended the analysis by studying the interaction between the informational effect of the interest rate and external signals. Central bank direct communication can also be modeled as providing external signals independent to the informational effect of the interest rates. I presented the situation in which providing more precise information about the efficient shocks might reduce welfare. In this case, communicating about the natural-rate shock also makes the interest-rate a more precise signal about the cost-push shock.

Finally, I quantified the size of the gains from commitment by adopting conventionally used parameter values while varying the precision of external signals. I found that when external signals are extremely imprecise, the size of gains from commitment is negligible. However, more precise external information about both shocks increases the size of gains from commitment. Specifically, when the precision of external signals is equal to the prior distribution of actual shocks, committing to the optimal policy rule improves ex-ante welfare by 54 percent relative to the equilibrium under the optimizing discretionary policy.
Chapter 3

Monetary Policy Commitment under Imperfect Information
3.1 Introduction

What should a central bank do when both itself and the private sector have imperfect information about the underlying economy? The empirical relevance of this question can be found in the recent practice of forward guidance policy. Through announcing the forward guidance policy, the central bank provides its forecast on the future path of policy rates, conditional on its own imperfect information in the current period.

Campbell et al. (2012) distinguish two types of forward guidance. The first type is called "Odyssean Forward Guidance", which refers to the situation in which the central bank explicitly commits to a certain policy rate in future periods. The second type is called "Delphic Forward Guidance", which refers to the situation in which the central bank explains their forecasts of economic fundamentals and explains how policy will react if their forecasts are correct, but does not explicitly committing to any policy actions. Their empirical estimations on the effect of FOMC announcements show that although communications in the FOMC statements during the recent financial crisis have Odyssean components, the market interprets it as Delphic communications, which limits the stimulating effect.

The Delphic forward guidance policy has a longer history. For example, the Reserve Bank of New Zealand has been doing this policy for more than two decades. Odyssean forward guidance is advocated by Eggertsson et al. (2003) as a way to boost expected inflation when the target rate hits the zero lower bound. In normal times, however, economists have been questioning the merit of Odyssean forward guidance, as it seems to jeopardize future economy without gaining in current period. In particular, Feroli et al. (2017) argue that forward guidance with commitment\(^1\) has two disadvantages. First, it ignores future macroeconomic news, and second, it reduces flexibility

\(^1\)In Feroli et al. (2017), forward guidance with commitment is referred to as "time-based" forward guidance, whereas forward guidance without commitment (with re-optimization) is referred to as "data-based" forward guidance.
In monetary policy decisions. After all, why should the central bank tie its hands while facing an uncertain future? Feroli et al. (2017) argue that central bank should only communicate what monetary policy would react to future macroeconomic news, but should allow the actual future policy to depend on future information when it becomes available.

In this paper, I analyze different strategies to conduct of monetary policy when both the central bank and the private sector have imperfect information, and find that the optimal strategy to be a state-contingent commitment which I call "rule-based Odyssean forward guidance". I first study the benchmark case of no forward guidance, and demonstrate that when private agents cannot perfectly observe the aggregate technology shocks, they cannot perfectly predict the response in monetary policy which, which breaks the real dichotomy. Then, I demonstrate that an instrument-based Odyssean forward guidance, which is the type of forward guidance under traditional definition, improves ex-ante welfare by providing more precise information about the realization of aggregate technology shocks. Next, I study the time consistent forward guidance: Delphic forward guidance, in which case the central bank re-optimizes to make the actual monetary policy contingent on future information. Lastly, I characterize the optimal monetary policy under imperfect information, which I call rule-based Odyssean forward guidance.

In this paper, I model a flexible-price economy where firms maximize profit in every period. Importantly, I assume that pricing decisions are made before households making consumption decisions. Essentially, price are perfectly flexible across period but completely rigid within period.\(^2\) Technology shock is the only source of uncertainty in the private sector, which composites an idiosyncratic component and an aggregate component. Firms set prices in the beginning of period when they only observe their firm-specific technology, without knowing the aggregate technology or decisions made by other firms. The representative household makes consumption decisions

\(^2\)Most papers on optimal monetary policy with imperfect information use this assumption. Examples include Adam (2007), Woodford (2001).
when all information is revealed in the end of the period.

The equilibrium aggregate price level depends on both the actual and the expected aggregate technology shock, due to the informational frictions. Importantly, due to the monopolistic competitive market, optimal prices are strategically complements. Individual firms increase their price when they expect a higher aggregate price level or a higher nominal aggregate demand. Consequently, firms also care about the expectations held by other firms, which makes higher order beliefs matter in the aggregate price level.

I study four strategies to conduct monetary policy. First of all, I use the case without forward guidance as a benchmark. The central bank decides the nominal aggregate demand in the end of period when perfect information is available. If information is perfect and private agents have rational expectations about the central bank’s behaviors, any change in the nominal demand is anticipated by the private sector. Consequently, the real dichotomy holds under perfect information: prices fully adjust to the change in aggregate nominal demand, and output is determined only by the aggregate technology. This is because there is no monetary policy shock between pricing decisions and consumption decisions, as firms can precisely predict the following aggregate nominal demand.

The real dichotomy breaks down under imperfect information. Individual firms use their estimates about the aggregate technology to form expectations about the aggregate nominal demand, and adjust prices accordingly. After a technology shock, due to informational frictions, private agents underestimate the actual shock and consequently underestimate the change in the nominal aggregate demand. In this situation, a part of the response in the aggregate nominal demand becomes an unanticipated shock to the private sector. Consequently, the aggregate nominal demand policy affects both the price level and also the output. I show that for a discretionary central bank which regards prices are fixed when setting the aggregate nominal demand, it is able to completely
close the output gap. However, when private agents with rational expectations factor the response of policy in their pricing decisions, it leads to higher fluctuations in the price level. Consequently, the central bank with commitment faces the conflict between closing the output gap and stabilizing the price level. The optimal policy rule changes from output gap stabilization to price level stabilization when private signals become more imprecise.

The second type is an instrument-based Odyssean forward guidance policy. Through forward guidance, the central bank discloses its noisy signals about the aggregate technology, which is used by private agents as a public signal, in addition to their private signals. In addition, the central bank also announces its expectations about the future nominal demand, conditional on this noisy signal. Unlike the benchmark case without forward guidance where firms use their estimates on the aggregate technology to form expectations about the aggregate demand, under instrument-based Odyssean forward guidance, firms get exact information about the aggregate nominal demand. Consequently, there is no monetary policy shock between pricing decisions and consumption decisions, making monetary policy unable to affect real output level.

As the effect of monetary policy is completely absorbed by the aggregate price level, the optimal Odyssean forward guidance policy targets price only. The central bank faces the trade-off between minimizing deviations after technology shocks and minimizing deviations after policy shocks. Policy shocks are resulted from the noise in the central bank’s information which the forward guidance policy is conditional on. I show that when the central bank has imprecise information, it reduces the sensitivity of aggregate nominal demand to its signal, in order to reduce the deviations due to the noise in its information.

The third strategy is Delphic forward guidance. In this case, the central bank reveals its noisy information about the aggregate technology, and explains how nominal aggregate demand will respond accordingly. However, it does not commit to implement such forward guidance policy.
Rather, it let the actual aggregate nominal demand condition on future information. In this situation, private agents need to guess the actual policy, and to do so they have to separately form expectations about the aggregate technology and the noise in the central bank’s information. As optimal prices are strategic complements, higher order beliefs matter in pricing decisions, which make the expected aggregate nominal demand further away from the actual nominal demand.

I first illustrate the incentive for a central bank to be Delphic, which is the time inconsistency problem involves with Odyssean forward guidance: if the central bank has announced the instrument-based Odyssean forward guidance, it wants to deviate if the signal turns out to be a noise instead of an actual technology shock. This is because prices are set at the stage of forward guidance, and the central bank wants to close the output gap by re-optimization. However, closing the output gap makes the aggregate price level more sensitive to aggregate technology shocks, if such deviation is anticipated by the private sector. Such trade-off can be solved by implementing the Delphic forward guidance with backward induction: the central bank decides its response to its noisy signal about the technology while taking into account the change in the expectations in the private sector due to re-optimization.

Lastly, I show the optimal monetary policy is a rule-based Odyssean forward guidance policy, in which case the central bank commits to a state-contingent policy rule instead of committing to a certain policy action. If the information turns out to be an actual technology shock, the central bank commits to increase the aggregate nominal demand higher than under Delphic forward guidance. If the information turns out to be a noise shock, the central bank commits to reduce the aggregate nominal demand than under Delphic forward guidance. Consequently, price deviations can be reduced as the central bank optimally balances between fluctuations in the aggregate price level and the output gap.
Related Literature

Past literature has extensively discussed the effect of forward guidance. Del Negro, Giannoni and Patterson (2012) provide empirical evidence that standard DSGE models tend to overestimate the effect of forward guidance. Angeletos and Lian (2016) answer this puzzle by introducing imperfect information, and argue that imperfect common knowledge predicts that the attenuated effect of forward guidance. Besides Feroli et al. (2017), Campbell et al. (2012) also characterize two types of forward guidance as whether forward guidance is accompanied with or without commitment. In their paper, "Odyssean Forward Guidance" is defined as making explicit commitment to future policy actions, whereas "Delphic Forward Guidance" is defined as forecasting economic conditions without commitment of future policy actions.

In particular, I study the welfare gains from policy commitment in forward guidance. There has been a long history in studying the gains from monetary policy commitment. Classical literature include Kydland and Prescott (1977) and Barro and Gordon (1983), which show the inflationary bias in the central bank’s objective function leads to higher inflation when the private sector has rational expectation. Clarida, Gali and Gertler (2000) study how commitment to a future path of policy rates reduces current stabilization bias between output gap stabilization and inflation stabilization which is induced by ad-hoc cost-push shocks. Woodford (1999) studies how history-dependent policy can be achieved by having interest-smoothing included in the central bank’s objective function. Eggertsson et al. (2003) show optimal commitment to delayed response can mitigate the distortion under zero lower bound of interest rates.

This paper studies how imperfect information leads to gains from monetary policy commitment, which builds on the abundant literature on optimal monetary policy under imperfect information. This field is revived since Woodford (2001), which show how higher order beliefs lead to persistent effect of monetary policy, following the assumption of imperfect information in Phelps...
(1970) and Lucas (1972). Since then there have been many papers studying optimal monetary policy under imperfect information. Papers in this field can be divided by their assumption of whether monetary policy has informational effect.

The majority of papers which characterize optimal monetary policy under information frictions assume monetary policy has no informational effect, and thus assume the beliefs formed in the private sector are independent of policy decisions. Ball, Mankiw and Reis (2005) study optimal monetary policy with sticky information, and conclude that optimal monetary policy should be described as elastic price standard: the central bank should allow price to deviate from target when output deviates from natural rate. Adam (2007) models partial information economy, and allows the precision of private signals to be endogenous. He argue that as signals get more precise, optimal monetary policy changes from output gap stabilization to price level stabilization. Lorenzoni (2010) assumes the central bank has no superior information, and points out announcing monetary policy has trade-off between aggregate stabilization and cross-sectional efficiency. Angeletos and La’O (2011) consider both nominal and real frictions that caused by information friction, and describe optimal monetary policy seeks negative correlation between price level and output.

Recent papers have started to investigate the situation where people extract information on the underlying economy from monetary policy decisions. Baeriswyl and Cornand (2010) emphasize the signaling effect of policy actions, and conclude that the central bank distorts its policy in order to optimally control the information it conveys. Central bank alters optimal policy response in order to reduce information revealed on cost push shock through policy decisions. Berkelmans (2011) demonstrates that with multiple shocks, tightening policy may initially increase inflation. Tang (2013) shows that with rational expected private sector, the stabilization bias is reduced when monetary policy has information effect.

My paper deals with the higher order beliefs problem, and past literature has demonstrated how
higher order beliefs amplify the real effect of monetary policy. (see Woodford (2001) for example)

In addition, many papers have show different solution methods, including Melosi (2016), Huo and Takayama (2015) and Nimark (2017), as examples.

My paper is also motivate by the empirical evidence on the informational effect of monetary policy and the policy discussion on forward guidance. Romer and Romer (2000), Romer and Romer (2004) show the information asymmetry between central bank and the private sector by providing evidence on inflation forecast changes after FOMC announcement. Faust, Faust, Swanson and Wright (2004) further confirm that forecasts by private sector respond to monetary policy changes. Nakamura and Steinsson (2013) use high frequency trading data to identify the informational effect of monetary policy news.

The rest of paper is organized as follows: section 2 sets up the private sector of my model. Section 3 analyzes the three strategies for the central bank to conduct monetary policy under imperfect information, and Section 4 concludes the paper.

3.2 The Private Sector

This section describes a representative agent economy with flexible prices and dispersed, imperfect information. The central bank controls the aggregate nominal demand.

The assumption on informational frictions follows Phelps (1970), Woodford (2001), and Angeletos and La’O (2010), where the informational frictions are results from geographical separation in decisions of firms. There is a continuum of islands, indexed by \(i\), and a continuum of households, each of which consists a representative consumer and a continuum of workers. There is a continuum of firms, each of which is located on one island and indexed by the island. Each firm \(i\) specializes in producing differentiated good \(i\), and sells in a monopolistic competitive market.
Information is symmetric within island, and asymmetric across island, meaning that a firm does not know the shocks or decisions made in other islands.

Each period has three stages. In the first stage, technology shocks are realized in all firms. Each firm $i$ observes its own technology, $A_i$, but not the technology of others, $A_j, j \neq i$. In the second stage, all firms set prices based on its own information set $\omega_i$.\(^3\) In the last stage, all information which was previously dispersed becomes available to everyone. The representative household sends one labor to each of the island. All markets open. Firms demand labors and produce intermediate goods to meet demands. At the same time, the household make consumption and labor supply decisions at all firms.

3.2.1 Household

The preferences of the household are defined over the aggregate consumption good, $C$ and the labor supplied to each firm, $N_i$. As specified in the timing of events, the decisions of the household are made when all information is revealed, so the consumption and labor supply decisions are free from informational frictions. The household hold chooses consumption and labors to maximize its utility:

$$u(C, N) = \frac{C^{1-\sigma}}{1-\sigma} - \int_0^1 \frac{N_i^{1+\psi}}{1+\psi} di,$$  \(3.1\)

where the aggregate good $C$ consists of a continuum of intermediate goods $C_j$ as:

$$C = \left( \int_0^1 C_j^{1-\frac{1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{3.2}$$

\(^3\)Existing literature on informational frictions differs in the assumption of whether firms make pricing or production first. Here, I follow the majority to assume that pricing decisions are made prior to production decisions, and production decisions are made after full information is revealed. For discussion under the alternative assumption where production decisions are made first, see Angeletos and La’O (2010).
The household maximizes utility subject to the nominal budget constraint

\[ PC \leq \int_0^1 W_i N_i + \Pi + T, \]  

(3.3)

where \( \Pi \) stands for all lump-sum income including profits from firms and \( T \) is the nominal transfers from the central bank. \( P \) denotes the aggregate price level which will be derived later.

The household’s optimization problem is solved in two stages. First, conditional on the level of aggregate consumption, the household allocates intermediate goods consumption to minimize the cost of total expenditure.

\[ \min_{\{C_i\}} \int_0^1 P_i C_i \, di. \]  

(3.4)

The result of expenditure minimization problem yields the demand of individual goods as a function of relative price to the aggregate price level and the aggregate consumption:

\[ C_i = \left( \frac{P_i}{P} \right)^{-\varepsilon} C \]  

(3.5)

where \( P \) denotes the aggregate price index after household optimally allocates individual good consumption to minimize expenditure, \( P = \left( \int_0^1 p_j^{1-\varepsilon} d j \right)^{\frac{1}{1-\varepsilon}} \).

The household utility maximization over labor and consumption sets real wage as the marginal rate of substitution between consumption and leisure:

\[ \frac{W_i}{P} = \frac{N_i\psi}{C^{-\sigma}} \]  

(3.6)
3.2.2 Firms

Firms make two decisions: pricing decisions and labor demand decisions. Prices are flexible across periods. In the beginning of each period, each firm sets price to maximize its expected profit conditional on its own information set, $\omega_i$. The optimal pricing decisions can be written as follows:

$$\max_{\{p_i\}} E \{ p_i y_i - w_i n_i | \omega_i \}$$

(3.7)

Firms face two constraints. The first constraint is the demand of individual products, which comes from the demand across individual goods. The second constraint is the production technology, which I assume to be linear in labor, following the tradition in the New Keynesian literature, which is specified as:

$$y_i = a_i n_i$$

(3.8)

where $a_i$ is the firm-specific technology.

3.2.3 States and Signals

The only source of uncertainty in the private sector comes from the technology. I allow for both an aggregate component and an idiosyncratic component. The firm-specific technology becomes a linear sum of the two components after log-linearization. Therefore, the firm-specific technology is a private signal which each firm can use when forming expectations about the aggregate technology. In addition to this private signal, firms can also use monetary policy as a public signal, given that the central bank gets an imperfect signal and release the monetary policy conditional on this signal when information is still imperfect in the private sector (more details in next section).

**Aggregate States**
The only aggregate state variable in the private sector is the aggregate technology shock, which I assume to be i.i.d. with log-normal distribution.

\[ \bar{a} \sim N(0, \sigma_a^2) \]

**Signals**

I assume that information is fully revealed after stage 2, so that the representative household is perfectly informed. Before stage 2, both firms and the central bank are subjected to partial information. Each firm learns its own technology, \( a_i \), which becomes its private signal of \( \bar{a} \).

\[ a_i \equiv \log(A_i) = \bar{a} + s_i \quad s_i \sim N(0, \sigma_s^2) \quad (3.9) \]

The central bank surveys a random sample of firms, and gets a measurement of \( \bar{a} \). If the central bank announces this information to firms through forward guidance, firms get a public signal of the aggregate technology shock. I denote this signal as \( m \). I assume the central bank’s measurement error is normally distributed with mean 0, so the public signal follows:

\[ m = \bar{a} + \nu \quad \nu \sim N(0, \sigma_\nu^2) \quad (3.10) \]

### 3.2.4 Price Setting with Higher Order Beliefs

When firms make pricing decisions, they need to form expectations about the aggregate nominal demand and the aggregate price level. I solve the equilibrium price by backward induction i.e., first substitute the individual good demands and the labor supply function to the optimal prices, and then derive the optimal price for each firm.

Household expenditure minimization problem, together with market clearing condition, deter-
mines individual good demand as a function of aggregate demand:

$$Y_i = \left( \frac{P_i}{P} \right)^{-\varepsilon} Y$$

(3.11)

In stage 2, each firm sets price to maximize expected current period profit:

$$E \{P_i Y_i - W_i N_i | \omega_i \} = E \left\{ \left( \frac{P_i}{P} \right)^{-\varepsilon} Y P_i - P_i^{-\varepsilon(1+\psi)} A_i^{-(1+\psi)} Y^{1+\psi+\sigma} p^{(1+\psi)} | \omega_i \right\}$$

(3.12)

Taking first order condition on $P_i$ results in the optimizing choice of price for each individual firm $i$:

$$P_i^{1+\varepsilon \phi} = \frac{E(1+\phi)}{\varepsilon - 1} E \left\{ P^{1+\varepsilon \phi} Y^{\phi} + \sigma A_i^{-(1+\phi)} | \omega_i \right\}$$

(3.13)

which is approximated in a log-linear form as:

$$p_i = E_i [p + \alpha y] - \beta a_i$$

(3.14)

where $\alpha = \frac{\phi + \sigma}{1+\varepsilon \phi}$, and $\beta = \frac{1+\phi}{1+\varepsilon \phi}$. The operator $E_i$ represents the conditional expectation of firm $i$ on its own information set, $\omega_i$. Equation (3.14) suggests that a firm raises its price when it expects a higher aggregate price level, a higher aggregate demand, or a lower firm-specific technology which is equivalent to a higher cost of production.

**Higher Order Beliefs**

The above equation states the optimal pricing strategy as conditional expectations on aggregate price and output. The aggregate price is defined to be the average of all $p_i$, and thus average over all individual expectation on $p$, which consequently makes $p_i$ depend on others belief, and others belief on others belief, and so on. This creates a higher order beliefs problem. Following
Woodford (2001), I solve the higher order beliefs problem by successively substituting $y$ by the nominal demand equation, $y = n - p$, which yields the following solution. (See Appendix C.1 for detailed derivation.)

$$p_i = \alpha \sum_{j=1}^{\infty} (1 - \alpha)^{j-1} E_i \bar{E}^{j-1} n - \beta \sum_{j=1}^{\infty} (1 - \alpha)^{j} E_i \bar{E}^{j-1} \bar{a} - \beta a_i$$

(3.15)

where $\bar{E} [\cdot]$ denotes the average expectations operator, given by

$$\bar{E} [\cdot] = \int E_i [\cdot] \, di$$

(3.16)

### 3.3 Monetary Policy under Imperfect Information

This section describes optimal monetary policy when both the private sector and the central bank have imperfect information about underlying shocks in the economy. I start with the benchmark case where the central bank only sets the aggregate nominal demand in the last stage, and provides no forward guidance in the beginning of the period. I then study the case of forward guidance, in which the central bank reveals the signals that it has about the aggregate technology shock, and at the same time explains how it will react conditional on its noisy information. I further distinguish three types of forward guidance policy. First is the instrument-based Odyssean forward guidance policy, in which case the central bank commits to the future aggregate nominal demand which is optimal conditional on current noisy signal. The second type is time-consistent Delphic forward guidance, in which case the central bank re-optimizes in the last stage to completely close the output gap. The last type is rule-based Odyssean forward guidance, under which the central bank conditions the actual policy reaction on future information, but commits to policy reaction function instead of committing to any policy response.
3.3.1 Benchmark Case - No Forward Guidance

In this benchmark case, the aggregate nominal demand is set in the last stage of each period when all information is revealed. In this benchmark case, I show both the discretionary monetary policy, in which the central bank considers prices to be fixed before policy decisions, and the case of policy rule, in which the central bank takes into account how its behaviors changes expectations in the private sector, when private agents have rational expectations about the central bank’s behaviors.

In the first stage when firms set optimal prices, the information set of each firm consists only its firm-specific technology, \( \omega_i = \{a_i\} \). Firms weigh their private signals and the prior beliefs to form expectations about the realization of the aggregate technology:

\[
E_i \tilde{a} = \frac{\kappa_s}{\kappa_s + \kappa_a} a_i + \frac{\kappa_a}{\kappa_s + \kappa_a} \bar{a},
\]

where \( \kappa_s = 1/\sigma_s^2 \) denotes the precision of private signals, and \( \kappa_a = 1/\sigma_a^2 \) denotes the precision of the prior.

Optimal prices also depend on the expectations on the response in the aggregate nominal demand. First consider that the policy is linear to the aggregate state variable after log-linearization, i.e.,

\[
n = \gamma_{nfg} \tilde{a}
\]

where \( \gamma_{nfg} \) stands for the sensitivity of nominal aggregate demand to changes in aggregate technology shocks in no forward guidance case.

Under rational expectations, firms form expectations about the aggregate nominal demand con-
ditional on their own expectations about the aggregate technology as:

\[ E_i n = \gamma_{ng} E_i \bar{a} = \frac{\kappa_s}{\kappa_s + \kappa_a} \gamma_{ng} a_i. \]  

(3.19)

As illustrated in the previous section, optimal prices depend on the higher order beliefs, as optimal prices are strategically complements, and firms use their private signals to form expectations about the aggregates. The higher order beliefs on \( \bar{a} \) are solved by first taking average of the individual beliefs,

\[ \bar{E} \bar{a} = \frac{\kappa_s}{\kappa_s + \kappa_a} \bar{a}. \]  

(3.20)

and then taking individual expectation, \( E_i \bar{a} \) on this first order averaged expectation. Continuing this process yields the \( j-th \) order exception as:

\[ E_i \bar{E}^j \bar{a} = \left( \frac{\kappa_s}{\kappa_s + \kappa_a} \right)^{j+1} a_i \]  

(3.21)

Substituting the higher order beliefs on aggregate technology, together with the response of aggregate nominal demand, into the optimal prices yields the equilibrium price and output level:

\[ p = \frac{(\alpha \gamma_{ng} - \beta) \kappa_s - \beta \kappa_a}{\kappa_a + \alpha \kappa_s} \bar{a} \]  

(3.22)

\[ y = \frac{\beta \kappa_s + (\gamma_{ng} + \beta) \kappa_a}{\kappa_a + \alpha \kappa_s} \bar{a} \]  

(3.23)

\textit{Infinite Precision of Private Signals}

When private signals are of infinite precision, the economy approaches to the perfect informational case. Substituting \( \kappa_s = \infty \) into equation (3.23) shows that the output level becomes indepen-
dent of policy responses, as

$$y \rightarrow \frac{\beta}{\alpha} \bar{q} = y^{eff},$$

which defines the efficient level of output.

Substituting $\kappa_s$ into equation (3.22) shows that the price level captures all the effect of monetary policy, as

$$p \rightarrow \alpha \gamma_{nfg} - \frac{\beta}{\alpha}.$$

The real dichotomy holds in this case, i.e., monetary policy only affects nominal variables and does not affect real output. When private agents have rational expectations on the behaviors of the central bank, and perfectly observe the realization in the aggregate technology shock, they can adjust prices completely to absorb changes in the monetary policy. There is no monetary policy shock between pricing decisions and consumption decisions, which makes monetary policy unable to have any effect on the real output.

However, the real dichotomy does not hold under imperfect information. Even though firms have rational expectations on the reaction function of the monetary policy, when they do not know the realization of the aggregate shock, they cannot perfectly foresee the adjustment in the aggregate nominal demand. This unanticipated gap between the expected nominal aggregate demand and the actual aggregate demand makes monetary policy have real effect.

**Zero precision of private signals**

Consider the extreme case where the private signals have zero precision. Firms do not update beliefs on aggregate technology. Consequently, as the aggregate nominal demand only responds to the aggregate technology, firms do not update beliefs on the aggregate nominal demand for any value of $\gamma_{nfg}$. In this situation, firms adjust prices only due to the changes in the cost of production of their own firm, i.e., $\beta a_i$, and do not adjust to the aggregate shocks. As they do not update
beliefs on the aggregate shock, the response in the aggregate nominal demand become a complete unanticipated shock to all the firms, which results in real effect of monetary policy. Specifically, substitute $\kappa_s = 0$ into equations (3.22) and (3.23), we find that the aggregate price level and the aggregate output follow:

$$ p \rightarrow -\beta \bar{a}, \quad y = (\gamma_{ng} + \beta)\bar{a}. $$

Intermediate precision

When private signals have finite precision, both real output and price level depends on $\gamma_{ng}$. I plot the equilibrium aggregate price and output after an aggregate technology shock of size 0.1 in Figure 3.1, to compare the case which $\gamma_{ng} = 1$ and $\gamma_{ng} = 5$.

We find that the output in both figures approaches to the efficient level, suggesting that when information approaches to be perfect, i.e., $\kappa_s \rightarrow \infty$, monetary policy do not matter in determining the equilibrium of real variables. In comparison, the price asymptotically goes to different level depending on the value of $\gamma_{ng}$. Only when $\gamma_{ng} = \frac{\beta}{\alpha}$, price asymptotically goes to zero.

Optimal Stabilization Policy

The optimal monetary policy without forward guidance is chosen to maximize the central bank’s objective function, which is set to be the weighted sum of the squared price deviation and the squared output gap.\(^4\)

\(^4\)The second order approximation of the household’s welfare function yields the weight of the output gap versus price stabilization to be dependent on the precision of private signals. I assume a constant weight to simplify the description of optimal monetary policy. See Appendix C.2 for derivations of the household’s utility function.
Parameter values are chosen to be: $\sigma = 0.2$, $\psi = 0.5$, $\varepsilon = 2$, which makes $\alpha = 0.35$ and $\beta = 0.75$.  

Figure 3.1: The Equilibrium Price and Output under Fixed Policy Rule without Forward Guidance
where $y^{eff}$ denotes the efficient level of output which is the output level when information is perfect. Optimal monetary policy in the benchmark case of no forward guidance is chosen after central bank observes the realization of $\bar{a}$. Therefore, choosing $n$ is equivalent to choosing the value of $\gamma_{nfg}$.

**Discretionary Policy (Output Gap Stabilization Policy)**

The optimal discretionary policy is set when the central bank regards prices as fixed prior to its policy decisions. Therefore, it only minimizes the output gap, i.e., the objective function of the central bank becomes:

$$
\max_n (y - y^{eff})^2
$$

subject to

$$
y = \beta \kappa_s + (\gamma_{nfg} + \beta) \kappa_a \bar{a}
$$

Central bank is able to choose the aggregate nominal demand to completely close the output gap by setting

$$
\gamma_{nfg} = \frac{\beta}{\alpha} - \beta
$$

The output gap stabilization policy is independent to the precision of private signals. This is because when $\gamma_{nfg}$ is at this value, the sensitivity of price is independent of precision of private private signals. Consequently, when the central bank sets $\gamma$ to make the price level absorb all the change in the monetary policy, output can be fixed at the efficient level.

**Policy Rules**

The optimal policy rule is found when the central bank takes into account the effect of policy
decisions on the beliefs in the private sector, which is factored in the pricing decisions. In this case, the central bank does not only consider output gap stabilization, but weighs output gap stabilization with price level stabilization. Central bank chooses the optimal response function, $n = \gamma_{nfg}\bar{a}$ prior to the realization of shocks. The objective function can be written in terms of

$$
max_{\gamma_{nfg}} - E \left[ \left( y - y^{ef} \right)^2 + \tau p^2 \right]
$$

subject to the price and output level specified in equation (3.22) and (3.23).

The optimal monetary policy, is found to be: (see Appendix C.3 for derivation)

$$
\gamma_{nfg}^* = \left( \frac{\kappa_a^2 + \alpha \kappa_s^2}{\kappa + \alpha \kappa_s} \right)^{-1} \left( \frac{\beta (\kappa_a + \kappa_s)(\alpha \kappa_s \tau - \kappa_a)}{\kappa_a + \alpha \kappa_s} + \frac{\beta \kappa_a}{\alpha} \right)
$$

In Figure 3.2, I plot the value of $\gamma_{nfg}$ at varying precision of private signals, comparing with price stabilization policy and output gap minimization policy. If the central bank considers only price stabilization, it achieves so by making $p = \frac{\alpha \gamma_{nfg}^* - \beta \kappa_a}{\kappa_a + \alpha \kappa_s} \bar{a} = 0$, which results in $\gamma_{nfg}^{price \ stab} = \frac{\beta \kappa_a}{\alpha \kappa_s} + \frac{\beta}{\alpha}$. The output gap stabilization policy is the same one as the optimizing discretionary policy, i.e., $\gamma_{nfg} = \frac{\beta}{\alpha} - \beta$.

When the private signals become more precise, the central bank with only price stabilization goal reduces its the sensitivity of aggregate nominal demand to aggregate technology. The reason is that without the response of monetary policy, the marginal change in price to aggregate technology shock is $-\frac{\beta (\kappa_s + \kappa_a)}{\alpha \kappa_s + \kappa_a}$. When $\kappa_s$ increases, the aggregate price becomes more sensitive, so the central bank adjust by making nominal aggregate demand less sensitive to aggregate technology shock.
In the top figure, price stabilization policy is solved as $\gamma_{\text{price stab}} = \frac{\beta k_s}{\alpha k_s} + \frac{\beta}{\alpha}$, and output gap stabilization policy is solved as $\gamma_{\text{ng}} = \frac{\beta}{\alpha} - \beta$. The second figure plots the equilibrium price and output dynamics under optimal policy rule.

Figure 3.2: The Equilibrium Price and Output under Optimal Policy Rule without Forward Guidance
With precise private information, real dichotomy holds and real output are independent with monetary policy. Thus, optimal monetary policy stabilizes price level. In comparison, with imprecise information, monetary policy has little impact on firms pricing decisions, as firms are reluctant to update beliefs on aggregate shocks and the reaction of monetary policy on aggregate shocks. Therefore, optimal monetary policy targets output gap stabilization.

The optimal policy rule without forward guidance can be summarized in the following proposition.

**Proposition 5:** Under imperfect information and without forward guidance, the optimal monetary policy rule shifts from output gap stabilization to price stabilization when the precision of private signals increases.

### 3.3.2 Instrument-based Odyssean Forward Guidance

Forward guidance is modeled as central bank providing its estimate of its monetary policy at the stage in which firms make pricing decisions. Instrument-based Odyssean forward guidance refers to the situation in which after announcing the forward guidance policy in this stage, central bank commits to the policy action in the final stage. In other words, the central bank does not re-optimize when perfect information becomes available in later stage.

At the time when the central bank announces instrument-based Odyssean forward guidance policy, it is also subject to imperfect information. Therefore, the forward guidance can only conditional on the noisy signal that the central bank has, rather than the actual aggregate technology. I study the class of aggregate nominal demand policy that is linear to the central bank’s noisy information.

\[ n = g(m) = \gamma m. \]
There are two pieces of information which are conveyed through the instrument-based Odyssean forward guidance. First is a public signal about the aggregate technology shock, with which private agents form expectations about the aggregate technology shock, together with their private signals. Second, private agents have perfect information on the actual aggregate nominal demand in the last stage, as the central bank commits not to re-optimize.

The second informational effect can be modeled in two ways. First, the central bank can directly announces its imperfect information about the aggregate technology, denoted by $m$, and communicates how monetary policy will respond to it, which is captured by $\gamma_{ofg}$. The second way to model it is that since the aggregate nominal demand policy is linear in only one variable, and private agents have rational expectations on the central bank’s behaviors, private agents are able to back out the informational held by the central bank when it observes $n$. i.e., $m = \frac{n}{\gamma}$.

After an aggregate technology shock, providing forward guidance narrows the gap between the expected and the actual aggregate technology shock. Specifically, Upon receiving the information provided by the central bank, the information set of individual firms becomes $\omega_i = \{m, a_i\}$, with which each firm forms conditional expectation on aggregate technology as:

$$E_i\bar{a} = \frac{\kappa_m}{\kappa_m + \kappa_s + \kappa_a} m + \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_a} a_i$$

(3.30)

As optimal prices are strategic complements, they depend on higher order beliefs on the aggregate technology, which is solved by first taking the average of $E_i\bar{a}$, which yields

$$\bar{E}\bar{a} = \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} \bar{a}_i$$

(3.31)
and then applying $E_i$ on the first order averaged expectations to get the second order:

$$E_i \bar{E} \bar{a} = \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} \left( \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} a_i \right) = \left( \frac{\kappa_m}{K} + \frac{\kappa_s}{K} \right) m + \left( \frac{\kappa_s}{K} \right)^2 a_i \quad (3.32)$$

Iterating the process results in the higher order beliefs on the aggregate technology when firms use both the public signal provided by the forward guidance and their private signals as their firm-specific technologies as:

$$E_i \bar{E}^{j-1} \bar{a} = \left( \frac{\kappa_m}{K} \right)^{k=j} m + \left( \frac{\kappa_s}{K} \right)^{k=1} a_i \quad (3.33)$$

After an aggregate technology shock, providing forward guidance narrows this gap between expected and the actual aggregate technology shock.

Different from the benchmark case in which private agents use their estimates about the aggregate technology to form expectations about the aggregate nominal demand, under instrument-based Odyssean forward guidance, private agents know the aggregate demand exactly. Substitute $E_i \bar{E}^{j-1} n = n$ and the higher order beliefs on the aggregate technology shock to the aggregate price level, we get:

$$p = \left[ \gamma_{ofg} - \beta \frac{1 - \alpha}{\alpha \kappa_m + \kappa_s + \kappa_m} \right] m - \beta \left[ \frac{(1 - \alpha) \kappa_s}{\kappa_m + \kappa_s + \alpha \kappa_s} + 1 \right] a, \quad (3.34)$$

$$y = \left[ \beta \frac{1 - \alpha}{\alpha \kappa_s + \kappa_s + \kappa_m} \right] m + \beta \left[ \frac{(1 - \alpha) \kappa_s}{\kappa_m + \kappa_s + \alpha \kappa_s} + 1 \right] a. \quad (3.35)$$

Equation (3.34) and (3.35) describe the equilibrium price level and output as functions of signal precision. The first thing to notice is that output is independent to $\gamma_{ofg}$. This is because as monetary policy is known by firms make pricing decisions, prices absorb all the effect of monetary policy.
leaving monetary policy unable to affect the real demand of households. As a result, the central bank does not have the trade-off between price level stabilization and output gap stabilization, and optimal monetary policy always stabilizes price level.

However, although there is no longer conflict between price stabilization and output gap stabilization, central bank faces another conflict between minimizing deviations after technology shocks and minimizing deviations after policy shocks, as the nominal demand responds to the noise in the central bank’s information in the same ways as it responds to the actual technology shocks.

Equation (3.35) shows that the output become more sensitive to aggregate technology when either public signals or private signals become more precise. This shows that by providing forward guidance, central bank makes the equilibrium output closer to the efficient level.

At the same time, providing forward guidance also makes the aggregate price level more sensitive to aggregate technology shocks. This seems to be detrimental as price fluctuations are welfare reducing, but this effect can be offset if central bank adjust the response, \( \gamma_{ofg} \) accordingly.

**Optimal Instrument-based Odyssean Forward Guidance**

The central bank commits to implement the nominal aggregate demand that only conditions on its noisy signal, \( m \). I study the class of commitment that is linear to \( m \). Optimal instrument-based Odyssean forward guidance policy can be described as the choice of \( \gamma \) which maximizes the expected objective function conditional on \( m \). In addition, since the choice of \( \gamma \) does not affect output level, the objective function reduces to the minimization of price fluctuations.

\[
\max_{\{\gamma\}} - E[p^2] \tag{3.36}
\]

I plot the solution in the following figure:
Figure 3.3: Optimal Instrument-based Odyssean Forward Guidance

In the above figure, parameter values are chosen as $\sigma = 0.2$, $\varphi = 0.5$, $\varepsilon = 2$, which makes $\alpha = 0.35$ and $\beta = 0.75$. Public signal refers to the central bank’s signal about the aggregate technology with precision $\kappa_m = \frac{1}{\sigma_m^2}$.

The expression of $\gamma^*_{ofg}$ is found to be:

$$\gamma^* = \frac{1}{\sigma_a^2 + \sigma_v^2}$$

$$\left\{ \beta \frac{1 - \alpha}{\alpha} \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} + \beta \left( \frac{1 - \alpha}{\kappa_m + \kappa_a + \alpha \kappa_s} + 1 \right) \right\} \sigma_a^2 + \left[ \beta \frac{1 - \alpha}{\alpha} \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} \right] \sigma_v^2$$

The central bank balances between minimizing deviations after actual aggregate technology shocks and after noise shocks. when choosing the optimal $\gamma$. As shown in this figure, when the precision of public signals increases, $\gamma^*$ increases. The intuition is that when precision of public signal is less precise, the central bank are less confident on its measurement and thus reacts less aggressively to its own measurement.
Figure 3.4: The Equilibrium Price and Output under Optimal Instrument-based Odyssean Forward Guidance

In the above figure, parameter values are chosen as $\sigma = 0.2$, $\varphi = 0.5$, $\epsilon = 2$, which makes $\alpha = 0.35$ and $\beta = 0.75$. Both the aggregate technology shock and the noise shock are of size 0.1.
In Figure 3.4, I plot the equilibrium price level and the output gap after a technology shock and a noise shock. Comparing with Figure 3.2 shows that instrument-based Odyssean forward guidance improves ex-post welfare after the technology shock, as both the equilibrium output and the price level become closer to the target levels. On the other hand, instrument-based Odyssean forward guidance introduces fluctuations due to noise shock in the central bank’s information. This suggests that there exists time inconsistency problem, as the central bank wants to re-optimize after noise shocks, which I analyze in the following section.

### 3.3.3 Delphic Forward Guidance

This section studies the time-consistent forward guidance policy, which I refer to as Delphic forward guidance. Under this strategy, the central bank only reveals its estimation about the aggregate technology shock through forward guidance, and does not commit to any policy actions. Private agents with rational expectations anticipate that the central bank will implement the actual nominal aggregate demand to close the output gap in the last stage, taking prices as given. I characterize the Delphic forward guidance with backward induction, which internalizes the expectations in the private sector about the re-optimization of monetary policy.

As the actual policy under Delphic forward guidance depends on two state variables: the aggregate technology and the noise in the central bank’s information, private agents need to separately form higher order beliefs on the aggregate technology shock and the policy error.
Higher Order Beliefs on the Aggregate Nominal Demand

The actual nominal demand policy now depends on two state variables, the actual technology shock, \( \bar{a} \), and the noise in the central bank’s signal, \( \nu \):

\[
n = \gamma^a \bar{a} + \gamma^\nu \nu
\]  
(3.38)

When central bank announces the forward guidance policy, it describes its estimate of the actual shock, \( m \), and how it is going to react to it, \( \gamma^a \text{fg} \). This is equivalent to announcing the estimate of the aggregate nominal demand to be:

\[
E[n|m] = \gamma^a m = \gamma^a (\bar{a} + \nu)
\]  
(3.39)

The difference between the announced Delphic forward guidance policy and the actual policy is:

\[
E[n|m] - n = (\gamma^a - \gamma^\nu) \nu
\]  
(3.40)

The case of instrument-based Odyssean forward guidance in the previous section can then be described by restricting \( \gamma^a = \gamma^\nu \).

Private agents need to form expectations about the noise in the central bank’s information. In addition, as optimal prices are strategic complements, both the higher order beliefs about the aggregate technology shock and about the policy error play a role in determining aggregate price level. In the first stage when price-setting takes places, all firms need to form expectations on both \( \bar{a} \) and \( n \). Firm \( i \)'s conditional expectation on \( \bar{a} \) is same as in the case under forward guidance with commitment, where the firm uses both the private and the public signal as:
\[
E_i\bar{a} = \frac{\kappa_m}{\kappa_m + \kappa_s + \kappa_a} m + \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_a} a_i
\]  
(3.41)

To guess \( n \), firms separately form expectations about the two state variables to which \( n \) reacts: the aggregate technology and the policy error:

\[
E_in = \gamma^a E_i\bar{a} + \gamma^\nu E_i\nu
\]  
(3.42)

The expectation on \( \nu \) is formed as the difference between \( m \) and \( E_i\bar{a} \). In other words, firms treat the difference between their estimation and the central bank’s estimation as the noise in the central bank’s information.

\[
E_i\nu = E_im - E_i\bar{a} = m - E_i\bar{a}
\]  
(3.43)

Consequently, as long as information is not perfect, private firms under-estimate (over-estimate) the aggregate technology shock after the aggregate technology shock (noise shock).

To form expectations on the aggregate nominal demand after re-optimization, substitute \( E_i\bar{a} \) and \( E_i\nu \) into equation (3.42) and get:

\[
E_in = \left( \gamma^a \frac{\kappa_m}{\kappa_m + \kappa_s + \kappa_a} + \gamma^\nu \frac{\kappa_s + \kappa_a}{\kappa_m + \kappa_s + \kappa_a} \right) m + \left( \gamma^a - \gamma^\nu \right) \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_a} a_i
\]  
(3.44)

This expression shows that when the central bank re-optimizes monetary policy, which can be captured by the difference between \( \gamma^a \) and \( \gamma^\nu \), private agents use both the public signal and their private signals to form expectations on the aggregate demand. This is due to the fact that private agents use their private signals on aggregate technology to guess the noise in the public signal.
In the following passage, I simplify the expression of equation (3.44) as:

\[ E_i n = \rho_m m + \rho_a a_i \]  

(3.45)

where \( \rho_m = \gamma^\mu \frac{K_m}{K_m + K_s + K_a} + \gamma^\nu \frac{K_s}{K_m + K_s + K_a} \), and \( \rho_a = (\gamma^\mu - \gamma^\nu) \frac{K_s}{K_m + K_s + K_a} \).

As optimal prices are strategic complements, higher order beliefs about the aggregate nominal demand affect the aggregate price level. To calculate the higher order beliefs, start with expressing first order average expectations on aggregate demand as:

\[ \bar{E}n = (\rho_m + \rho_a)\bar{a} + \rho_m v. \]  

(3.46)

Applying individual expectation, \( E_i \) to the above equation results in:

\[ E_i \bar{E}n = \rho_m m + \rho_a E_i \bar{a} = \rho_m m + \rho_a \left( \frac{K_m}{K} m + \frac{K_s}{K} a_i \right), \]  

(3.47)

which shows that the weight on public signal increases as beliefs are taken to a higher order.

Continuing this process to get the \( j - th \) order beliefs on the nominal aggregate demand:

\[ E_i \bar{E}^j n = \rho_m m + \rho_a E_i \bar{E}^{j-1} \bar{a} \]  

(3.48)

In the following figures, I plot the evolution of higher order beliefs after an aggregate technology shock (Figure 3.6) and after a policy shock (Figure 3.7). I compare the higher order beliefs on monetary policy when central bank re-optimize strongly (\( \gamma^\mu - \gamma^\nu = 0.5 \)) and when central bank re-optimize less strongly (\( \gamma^\mu - \gamma^\nu = 0.1 \)).
Figure 3.5: K-th order beliefs after an Aggregate Technology Shock

Figure 3.6 shows that while the actual aggregate nominal demand is 1, private agents under-
estimate the aggregate nominal demand, because they cannot distinguish whether the shock is an aggregate technology shock (in which case they expect \( n \) raises by 1 unit) or a noise shock (in which case they expect \( n \) raises by 0.5 unit). The gap between the actual and the expected nominal demand increases when beliefs are taken to a higher order. In addition, the gap increases when the central bank re-optimizes to a larger extent, which is captured by a larger difference between \( \gamma^a \) and \( \gamma^u \).

Figure 3.6 shows that after a noise shock in the central bank’s information, private agents over-estimate the aggregate nominal demand. This is because agents assign a positive probability to the event that the realized shock is an aggregate technology shock in which case aggregate nominal demand increases by 1 unit rather than 0.5 unit. This gap between the expected and the actual aggregate nominal demand increases when beliefs are taken to higher orders and when the difference between \( \gamma^a_{dfg} \) and \( \gamma^u_{dfg} \) is larger.

The difference between the actual re-optimized monetary policy, \( n \), and the average expectation on it depends on the precision of public and private signals. First, if information is perfect in the private sector, i.e., \( \kappa_s = \infty \), then \( \bar{E}n = n \), meaning that the private sector can perfectly predicts the aggregate nominal demand after re-optimization. When the central bank has perfect information, \( \kappa_m = \infty \), then \( \bar{E}n = \gamma^a (\bar{a} + \nu) \), suggesting that the private sector believes the actual policy after re-optimization is the same as the forward guidance with commitment.

Another interesting scenario is when the information of the central bank has zero precision. In this situation, after an aggregate technology shock, the average expectation on policy error is \( \bar{E}\nu = m - \frac{\kappa_s}{\kappa_s + \kappa_a} a \), suggesting that even if the error in the central bank’s signal is realized to be zero, the private sector believes that the policy error is between \((0, 1)\), because the private agents underestimate the realization in the technology shock due to their imperfect information.
Equilibrium price level is found by substituting the higher order beliefs on aggregate technol-
logy shocks and on noise shocks. (derivations in Appendix C.5):

\[ p = (\phi_m + \phi_a)\bar{a} + \phi_m\nu \]  
\[ y = (\gamma^u - \phi_m - \phi_u)\bar{a} + (\gamma^u - \phi_m)\nu \]

where

\[ \phi_m = \rho_m + \left[ \rho_a(1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right] \frac{\kappa_m}{\alpha \kappa_a + \kappa_m + \kappa_a} \]  
\[ \phi_u = [\alpha \rho_a - \beta] \frac{(1 - \alpha)\kappa_s}{\kappa_m + \kappa_a + \alpha \kappa_s} + \alpha \rho_a - \beta \]  
\[ \rho_m = \gamma^u \frac{\kappa_m}{\kappa_m + \kappa_a + \kappa_a} + \gamma^v \frac{\kappa_s + \kappa_a}{\kappa_m + \kappa_s + \kappa_a} \]  
\[ \rho_u = (\gamma^u - \gamma^v) \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_a} \]

**Time Inconsistency Problem**

This section shows the time inconsistency problem for instrument-based Odyssean forward guidance policy, which can be shown as the incentive for central bank to choose \( \gamma^u \) different from \( \gamma^v \). The time inconsistency problem raises due to the change in the central bank’s objective function: the instrument-based Odyssean forward guidance policy is chosen to minimize the price fluctuations, but once prices are fixed, the central bank’s objective changes to output gap stabilization. As the household has perfect information, the central bank wants to make the actual nominal demand to contingent on the actual shocks.

Mathematically, after the instrument-based Odyssean forward guidance policy is announced and prices are determined, the central bank wants to re-optimize in a way which makes \( \frac{\partial y}{\partial a} = 0 \) and
\[ \frac{\partial p}{\partial \nu} = 0, \text{ which is equivalent to:} \]

\[ \gamma^\alpha - \phi_m - \phi_a = \frac{\beta}{\alpha}, \quad (3.55) \]

\[ \gamma^\nu - \phi_m = 0 \quad (3.56) \]

The solution of the above equations requires that the difference between \(\gamma^\alpha\) and \(\gamma^\nu\) to be:

\[ \gamma^\alpha - \gamma^\nu = \frac{\beta}{\alpha} - \beta. \quad (3.57) \]

at which the output gap is completely closed after both technology shocks and noise shocks.

In the following figure, I show the equilibrium price and output level after a technology shock and after a noise shock when the central bank announces the optimal Odyssean forward guidance, but actually deviate to re-optimize in the last stage, i.e., I take \(\gamma^\alpha\) the same value as the optimizing policy under forward guidance with commitment, and then take \(\gamma^\nu\) to close the output gap. As shown in the figures, firms set prices after taking into account policy re-optimization.

Comparing this figure with Figure 3.5 shows that after a positive technology shock, the aggregate price level decreases to a larger extent. In both of the two cases, the actual aggregate nominal demand is the same, the realized shock is an aggregate technology shock and the sensitivity of monetary policy to technology shocks are the same in both cases. However, when private agents expect the central bank to re-optimize, they understand that the central bank reduces the aggregate nominal demand after a noise shock is realized. Consequently, the expected aggregate nominal demand is lower than the actual one.
Figure 3.7: The Equilibrium Price and Output after Re-optimization

The figure compares instrument-based Odyssean forward guidance with re-optimization. When re-optimization is expected, the aggregate price level is lower after both actual technology shocks and noise shocks.
Comparing the sensitivity of aggregate price level to noise shocks shows that after a positive noise shock, the increase in price is smaller after re-optimization. This is because under re-optimization, private agents know that with positive possibility, the realized shock is a noise shock, in which case the aggregate nominal demand after re-optimization is smaller.

**Lemma 5:** *If the central bank announces the optimal instrument-based Odyssean forward guidance policy but re-optimizes, it is able to completely close the output gap. However, if such deviation is anticipated by the private sector, the aggregate price becomes more sensitive to aggregate technology shocks.*

*Proof* see Appendix.

**Delphic Forward Guidance Policy with Backward Induction**

The policy trade-off described in Lemma 5 can be solved by a discretionary Delphic forward guidance with backward induction. In the second step, the central bank chooses how it reacts to the noise shock to close the output gap. In the first step, the central bank chooses how it reacts to its signal about the aggregate technology shock. In the following figure, I plot the Delphic forward guidance policy with backward induction at different precision of public signals. Comparing this with Figure 7 shows that when the central bank receives a positive signal of aggregate technology shock, it should increases nominal demand more if it expects to re-optimize after a noise shock. The reason is that it will reduce the actual aggregate nominal demand if the signal it receives is a positive noise, and such re-optimization is anticipated by private agents with positive possibility.
In the following figures, I plot the aggregate price and output after technology shocks and noise shocks when the central bank does forward guidance with re-optimization. Comparing this with the equilibrium price and the output gap under commitment, it shows that the price dynamics are the same whereas the output gap is completely closed under re-optimization.
Figure 3.9: The Equilibrium Price and Output under Delphic Forward Guidance
3.3.4 Rule-base Odyssean Forward Guidance

In this section, I study the optimal state-contingent policy rule, which I refer to as rule-based Odyssean forward guidance. There is Delphic component, in the sense that the actual monetary policy conditions on future information, but the central bank is not discretionary, as it does not regard the aggregate price as fixed.

Mathematically, the optimal state-contingent policy rule is chosen by choosing $\gamma^a$ and $\gamma^v$ in the beginning of period to maximize the ex-ante loss:

$$\max\{\gamma^a, \gamma^v\} - E\left[\left(y - y^{eff}\right)^2 + \tau p^2\right]$$

(3.58)

Figure 3.10: Optimal Rule-based Odyssean Forward Guidance

In other words, the central bank relaxes the restriction that (1) being an instrument-based
Odyssean forward guidance in which case $\gamma^a = \gamma^b$ and (2) being Delphic in which case $\gamma^a - \gamma^b = \frac{\beta}{\alpha} - \beta$. The solution of optimal rule-base Odyssean forward guidance policy is provided in Appendix C.6.

I plot the value of $\{\gamma^a, \gamma^b\}$ at varying precision of public information in Figure 3.10. Comparing this figure with Figure 3.8, we find that when public signals are very imprecise, the optimal rule for Delphic forward guidance policy is to be more sensitive to the signal of actual shock, and reduce the aggregate nominal demand if the signal turns out to be a noise instead of an actual shock.

Intuitively, when public information about the aggregate technology is very imprecise, firms want to decrease their prices after receiving their private signals, because for a given sensitivity of aggregate nominal demand to technology shocks, firms under-estimate the positive response in the aggregate nominal demand. Consequently, the central bank increases its sensitivity to aggregate nominal demand shocks.

At the same time, the central bank commits to reduce the actual nominal aggregate demand after noise shocks. Consequently, firms increase their prices by a less amount after noise shocks. This leads to the time-inconsistency problem: firms have increased prices after receiving the public signal from the forward guidance, and a reduction in the actual nominal aggregate demand will decrease the real output.

I plot the equilibrium aggregate price level and the output level in the following figure. It shows that the central bank reduces the sensitive of the aggregate price level after technology shocks at the cost of output gap fluctuations. At the same time, after a positive policy shock, the increase in the aggregate price level is reduced, at the cost of a negative output gap.
Figure 3.11: The Equilibrium Price and Output under Optimal Rule-base Odyssean Forward Guidance
3.4 Conclusion

In this paper, I analyzed the optimal conduct of monetary policy when both the private sector and the central bank have imperfect information about the underlying economic fundamentals. Forward guidance is defined as the central bank providing its forecast about the monetary policy conditional on its imperfect information. Literature has distinguished two strategies to conduct forward guidance, which are instrument-based Odyssean forward guidance and Delphic forward guidance. This paper finds that the optimal way to conduct monetary policy is making state-contingent commitment, which I call the rule-based Odyssean forward guidance policy.

To study the effect of monetary policy under imperfect information, I built a New Keynesian economy where prices are perfectly flexible across periods, but completely rigid within a period. Firms set prices under imperfect information, and households make consumption decisions after perfect information is revealed. The central bank decides the aggregate nominal demand to affect the consumption decisions of households. Consequently, the expectations of firms about the aggregate nominal demand becomes the center of the discussion.

If information is perfect and private agents have rational expectations on the central bank’s behavior, firms are able to perfectly foresee the aggregate nominal demand. Consequently, the real dichotomy holds. However, when firms have imperfect information about the realization of the aggregate technology, even they perfectly understand the reaction function of the central bank to the aggregate technology shocks, they still have imperfect information on the actual nominal aggregate demand. Consequently, the real dichotomy breaks down.

With instrument-based Odyssean forward guidance, the central bank announces its noisy information about the aggregate technology, and commits to implement the aggregate nominal demand which is optimal conditional on its noisy information. In comparison, under Delphic forward guid-
ance, the central bank only reveals its information on the economic fundamentals without committing to any policy response. As private agents have rational expectations, they form expectation on the actual nominal aggregate demand by combining the public signal from the Delphic forward guidance and their private signals about the aggregate technology shocks.

Time inconsistency problem arises after the central bank announces the instrument-based Odyssean forward guidance: if a noise shock is realized, the central bank wants to re-optimize to close the output gap. However, if such deviation is anticipated by private agents, the optimization makes the aggregate price more sensitive to aggregate technology shocks. To solve this policy trade-off, the optimal Delphic forward guidance is found by backward induction. Lastly, I analyzed the optimal strategy which is a rule-based Odyssean forward guidance. By making state-contingent commitment, the central bank can achieve the optimal balance between output gap stabilization and price stabilization.
References


134


135
Appendix A

Appendix for Chapter 1

A.1 Log-Linearization and Aggregation

From the household first order conditions, we first do log-linear approximation to the Euler equation by

\[ y_t = E_t y_{t+1} - \frac{1}{1-\sigma} (i_t - E_t \pi_{t+1}) \]  

(A.1.1)

The log-linear approximation to the labor supply is \( \varphi n_t(j) + \sigma y_t = w_t(j) \) where \( w_t \) denotes the log approximated real wage, \( \log(W_t/P_t) \). Recall that resource constraint implies that \( c_t^j = y_t^j \forall j \), which further implies \( c_t = y_t \). We can then write the labor supply as follows:

\[ \varphi n_t(j) + \sigma y_t = w_t(j) \]  

(A.1.2)

Next, we want to relate individual firm’s real marginal cost of production to aggregate output.
To to this, first integrate equation A.1.2

$$\int w_t(j) = \varphi \int n_t(j) dj + \sigma y_t \quad (A.1.3)$$

Then, substitute the log-linear approximation of the individual good demand, i.e., \( y_t(j) - y_t = -\varepsilon (p_t(j) - p_t) \), which results in:

$$\int n_t(j) dj = y_t + \int (-\varepsilon)(p_t(j) - p_t) - \int a_t(j) = y_t - a_t \quad (A.1.4)$$

Substitute this into \( \int w_t(j) \), and then deduct \( a_t \) from both sides:

$$\int w_t(j) - a_t(j) = (\phi + \sigma)y_t - (1 + \phi)a_t \quad (A.1.5)$$

Define natural level of output as the equilibrium output level without price rigidity and under perfect information, which makes \( y^n_t \) as a linear function of aggregate technology. Then, write the above equation in terms of output gap:

$$\int w_t(j) - a_t(j) = (\phi + \sigma)(y_t - y^n_t) \quad (A.1.6)$$

We know move on to the firm’s side. Taking log-linear approximation of individual firm’s optimal resetting prices:

$$p^*_t(j) = (1 - \beta \theta)E_t^l \left\{ \Sigma(\beta \theta)^k [p_{t+k} + u_{t+k}(j) + w_{t+k}(j) - a_{t+k}(j)] \right\} \quad (A.1.7)$$

The Calvo assumption implies that the aggregate price index is an average of the price charged by the fraction of \( 1 - \theta \) of firms which reset their prices at \( t \), and the fraction of \( \theta \) of firms whose
prices remain as the last period prices. Thus, the log-linear approximation of the aggregate price in period $t$ becomes:

$$p_t = \theta p_{t-1} + (1 - \theta) \int p_t^*(j) d j$$  \hspace{1cm} (A.1.8)

Subtract $p_{t-1}$ from both sides to express in terms of inflation:

$$\pi_t = (1 - \theta) \left( \int p_t^*(j) - p_{t-1} \right)$$  \hspace{1cm} (A.1.9)

As explained in Section 2.3 of Chapter 1, the subjective beliefs are assumed to be homogeneous in order to abstract from the higher order beliefs problem in aggregating prices. This assumption allows me to write individual resetting prices as:

$$p_t^*(j) = (1 - \beta \theta)(E_t^s p_t + u_t(j) + w_t(j) - a_t(j)) + (1 - \beta \theta) \Sigma_{k=1}^{\infty} (\beta \theta)^k E_t^s (p_{t+k} + u_{t+k}(j) + w_{t+k} - a_{t+k})$$  

(A.1.10)

Integrate over $j$:

$$\int p_t^*(j) d j = (1 - \beta \theta)(E_t^s p_t + u_t + w_t - a_t) + (1 - \beta \theta) \Sigma_{k=1}^{\infty} (\beta \theta)^k E_t^s (p_{t+k} + w_{t+k} - a_{t+k})$$  \hspace{1cm} (A.1.11)

To write in difference equation, first calculate:

$$\beta \theta \int E_t^s p_{t+1}^*(j) d j = (1 - \beta \theta) \Sigma_{k=1}^{\infty} E_t^s (p_{t+k} + u_{t+k} + w_{t+k} - a_{t+k}) = \beta \theta E_t^s p_{t+k}^*$$  \hspace{1cm} (A.1.12)

The second equation holds due to homogeneous beliefs.
Subtract equation A.1.12 from equation A.1.11

\[
\int p^*_t(j) d j - \beta \theta E^*_t p_{t+1} = (1 - \beta \theta) E^*_t p_t + (1 - \beta \theta) u_t + (1 - \beta \theta)(\varphi + \sigma) \hat{y}_t
\]

(A.1.13)

\[
\int p^*_t(j) d j - p_t - 1 = \beta \theta (E^*_t p^*_{t+1} - E^*_t p_t) + E^*_t p_t - p_{t-1} + (1 - \beta \theta) u_t + (1 - \beta \theta)(\varphi + \sigma) \hat{y}_t
\]

\[
\pi_t = \beta \theta E^*_t \pi_{t+1} + (1 - \theta) E^*_t \pi_t + (1 - \theta)(1 - \beta \theta) u_t
\]

\[
+ (1 - \beta \theta)(1 - \theta)(\varphi + \sigma) \hat{y}_t
\]

In the last equation, I assume that aggregate price is observable after one period, i.e., \( p_{t-1} = E^*_t p_{t-1} \)

Write inflation as:

\[
\pi_t = \beta \theta E^*_t \pi_{t+1} + (1 - \theta) E^*_t \pi_t + \kappa \theta \hat{y}_t + u_t
\]

(A.1.14)

where \( \kappa = \frac{(1 - \beta \theta)(1 - \theta)(\varphi + \sigma)}{\theta} \), and \( u_t = (1 - \theta)(1 - \beta \theta) u_t \)

### A.2 Solution to the Markov Perfect Equilibrium under Discretionary Monetary Policy

In this section, I first solve the model with serially uncorrelated shocks and then solve the model with serially correlated shocks. For both cases, I solve for the fixed point where the beliefs by people in the private sector on the best response of interest rate at any state match the optimizing discretionary interest rate. This means that in equilibrium people have rational expectation.

**Equilibrium Optimizing Discretionary Policy with Serially Uncorrelated Shocks** The solution takes the following steps:

1. I conjecture that interest rate reacts linear to both shocks, i.e., \( i_t = F^0 r^n_t + F^0 u_t \).

2. With this interest rate, I solve for the beliefs formed about natural-rate shock and cost-push...
shock in the private sector as functions of interest rate.

3. With beliefs formed in private sector, $E_t^s r_t^n$ and $E_t^s u_t$, the actual shocks, $r_t^n$ and $u_t$, I solve for $\hat{\gamma}_t$ and $\pi_t$ as a function of $i_t$.

4. Solve for $i_t$ that minimizes the loss function, $L_t = \pi_t^2 + \omega \hat{\gamma}_t$, and express interest rate as actual shocks, $i_t = F_r r_t^n + F_u u_t$.

5. Iterate the process until convergence.

Specifically, in step 1, $i_t = F^0_r r_t^n + F^0_u u_t$.

In step 2, beliefs about underlying shocks follow:

$$E_t^s r_t^n = K_r i_t \quad (A.2.1)$$

$$E_t^s u_t = K_u i_t \quad (A.2.2)$$

where $K_r F_r^0 = \frac{F_r^{02} \sigma_r^2}{F_r^{02} \sigma_r^2 + F_u^{02} \sigma_u^2}$, and $K_u F_u^0 = \frac{F_u^{02} \sigma_u^2}{F_r^{02} \sigma_r^2 + F_u^{02} \sigma_u^2}$.

In step 3, write out the expression of output gap and inflation as function of interest rate:

$$\hat{\gamma}_t = -\frac{1}{\sigma} (i_t - r_t^n) \quad (A.2.3)$$

$$\pi_t = \kappa \hat{\gamma}_t + (1 - \theta) \frac{\kappa}{\sigma} (E_t^s r_t^n (i_t) - r_t^n) + \frac{1 - \theta}{\theta} E_t^s u_t (i_t) + u_t \quad (A.2.4)$$

In step 4, I first write out the first order condition of interest rate:

$$\pi_t \frac{\partial \pi_t}{\partial i_t} + \omega \hat{\gamma}_t \frac{\partial \hat{\gamma}_t}{\partial i_t} = 0 \quad (A.2.5)$$
Substitute $\hat{y}$ and $\pi_t$ by equation A.2.3 and A.2.4

$$\left\{ (1 - \theta) \frac{\kappa}{\sigma} (E_t^s r^n_t - r^n_t) + \frac{1 - \theta}{\theta} E_t^s u_t + u_t \right\} \frac{\partial \pi_t}{\partial i_t} + \left( \frac{\omega}{\theta} \frac{\partial \hat{y}_t}{\partial i_t} + \kappa \frac{\partial \pi_t}{\partial i_t} \right) \left\{ - \frac{1}{\sigma} (i_t - r^n_t) \right\} = 0$$

(A.2.6)

Substituting $E_t^s r^n_t$ and $E_t^s u_t$ as functions of $i_t$ leads to:

$$\lambda_1 r^n_t + \lambda_2 u_t + \lambda_3 i_t = 0$$

(A.2.7)

where $\frac{\partial \hat{y}_t}{\partial i_t} = -\frac{1}{\sigma}$, and $\frac{\partial \pi_t}{\partial i_t} = -\frac{\kappa}{\sigma} + (1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u$, and

$$\lambda_1 = \left\{ \left( \frac{\kappa}{\sigma} \frac{\partial \pi_t}{\partial i_t} + \frac{\omega}{\theta} \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma} - \frac{\partial \pi_t}{\partial i_t} (1 - \theta) \frac{\kappa}{\sigma} \right\}$$

$$\lambda_2 = \frac{\partial \pi_t}{\partial i_t}$$

$$\lambda_3 = \frac{\partial \pi_t}{\partial i_t} (1 - \theta) \frac{\kappa}{\sigma} K_{11} + \frac{\partial \pi_t}{\partial i_t} \frac{1 - \theta}{\theta} K_{21} - \left( \frac{\kappa}{\sigma} \frac{\partial \pi_t}{\partial i_t} + \frac{\omega}{\theta} \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma}$$

Rearranging the above equation to get:

$$i_t = F_1 r^n_t + F_3 u_t$$

(A.2.8)

where $F_1 = -\frac{\lambda_1}{\lambda_3}$, and $F_3 = -\frac{\lambda_3}{\lambda_3}$.

In step 5, I iterate the above process until $F_r = F^0_r$ and $F_u = F^0_u$.  

141
A.3 Equilibrium Optimizing Discretionary Policy with Serially Correlated Shocks

In this section, I solve for the general version of the dynamic information case where I have serially correlated shocks, external signals which captures central bank direct communication, and implementation error.

The solution method is similar to the case with serially uncorrelated shocks, as solving for optimizing interest rate in equilibrium involves conjecture of interest rate response function. In addition to this conjecture, solving equilibrium variables in the private sector also requires additional step of undetermined coefficient to deal with the subjective expectation of future equilibrium variables.

1. I conjecture that interest rate reacts linear to both current shocks and past beliefs, i.e.,
   \[ i_t = F_1 r^n_t + F_2 E^{s}_{t-1} r^n_{t-1} + F_3 u_t + F_4 E^{s}_{t-1} u_{t-1}. \]

2. With this interest rate, I solve for the beliefs formed about natural-rate shock and cost-push shock in the private sector as functions of current signals (interest rate and central bank communication) plus past beliefs.

3. (Undetermined Coefficient) I conjecture that output gap and inflation are linear functions of current state variables which include actual shocks and past beliefs. As a result, I am able to express the forward-looking output gap and inflation as functions of current actual shocks and current beliefs.

4. With beliefs formed in private sector, \( E^{s}_t r^n_t \) and \( E^{s}_t u_t \), the actual shocks, \( r^n_t \) and \( u_t \), I solve for \( \hat{y}_t \) and \( \pi_t \) as a function of \( i_t \).
5. Solve for $i_t$ that minimizes the loss function, $L_t = \pi_t^2 + \omega \hat{\gamma}_t$, and express interest rate as actual shocks, $i_t = F_r r_{t}^n + F_u u_{t}$.

6. Iterate the process until convergence.

Specifically, in step 1, I conjecture that $i_t = F_1 r_{t}^n + F_3 E^{s}_{t-1} r_{t-1}^n + F_3 u_{t} + F_4 E^{s}_{t-1} u_{t-1}$.

In Step 2, to solve the beliefs formed in the private sector, I first specify the evolution of actual shocks:

**State:**

\[
\begin{bmatrix}
    r_{t}^n \\
    u_{t}
\end{bmatrix} = \begin{bmatrix}
    \phi & 0 \\
    0 & \phi^u u_{t}
\end{bmatrix} + \begin{bmatrix}
    v_t \\
    v^u_t
\end{bmatrix}
\]

which I denote as $z_t = \Phi z_{t-1} + v_t$, where $\Phi = \begin{bmatrix}
    \phi & 0 \\
    0 & \phi^u
\end{bmatrix}$ and $v_t = [v_t, v^u_t]$ with the white noise of variance $Q$.

**Signals**

As people in private sector have perfect memory of beliefs they have in the past, they are able to back out the part of interest rate that reacts to current shocks, which I denote as

\[
\hat{i}_t \equiv i_t - F_3 E^{s}_{t-1} r_{t-1}^n - F_4 E^{s}_{t-1} u_{t-1}
\]

All signals are summarized as.

\[
\begin{bmatrix}
    \hat{i}_t \\
    m_r^t \\
    m_u^t
\end{bmatrix} = \begin{bmatrix}
    F_1 & F_3 \\
    1 & 0 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    r_{t}^n \\
    u_{t}
\end{bmatrix} + \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    e_{t} \\
    e^r_{t} \\
    e^{u}_{t}
\end{bmatrix}
\]

143
which I denote as \( s_t = Dz_t + R_t \)

**Beliefs**

People in private sector are Bayesian, and update beliefs through the Kalman Filtering process, in which they optimally weigh between all current signals and past beliefs by their variances. The beliefs follow:

\[
\begin{bmatrix} E_s^t \ r_t^n \\ E_s^t \ u_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \begin{bmatrix} E_s^{t-1} \ r_{t-1}^n \\ E_s^{t-1} \ u_{t-1} \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} \begin{bmatrix} \hat{t}_t \\ m_t^r \\ m_t^u \end{bmatrix} - \begin{bmatrix} F_1 & F_3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi & 0 \\ 0 & \phi^u \end{bmatrix} \begin{bmatrix} E_s^{t-1} \ r_{t-1}^n \\ E_s^{t-1} \ u_{t-1} \end{bmatrix}
\]

(A.3.4)

Write out the expression for \( \hat{t}_t \) and collect terms:

\[
E_s^t \ r_t^n = (K_{11}F_1 + K_{12}) r_t^n + \phi (1 - K_{11}F_1 - K_{12}) E_s^{t-1} r_{t-1}^n
\]

(A.3.5)

\[
+ (K_{11}F_3 + K_{13}) u_t + \phi^u (-K_{11}F_3 - K_{13}) E_s^{t-1} u_{t-1} + K_{12} \epsilon_t^r + K_{13} \epsilon_t^u + K_{11} \epsilon_t
\]

\[
E_s^t \ u_t = (K_{21}F_1 + K_{22}) r_t^n + \phi (-K_{21}F_1 - K_{22}) E_s^{t-1} r_{t-1}^n
\]

(A.3.6)

\[
+ (K_{21}F_3 + K_{23}) u_t + \phi^u (1 - K_{21}F_3 - K_{23}) E_s^{t-1} u_{t-1} + K_{22} \epsilon_t^r + K_{23} \epsilon_t^u + K_{21} \epsilon_t
\]

Denote the above equations as \( E_s^t \ r_t^n = \Psi(1) r_t^n + \Psi(2) E_s^{t-1} r_{t-1}^n + \Psi(3) u_t + \Psi(4) E_s^{t-1} u_{t-1} + \Psi(5) \epsilon_t^r + \Psi(6) \epsilon_t^u + \Psi(7) \epsilon_t \), and \( E_s^t \ u_t = \Psi(8) r_t^n + \Psi(9) E_s^{t-1} r_{t-1}^n + \Psi(10) u_t + \Psi(11) E_s^{t-1} u_{t-1} + \Psi(12) \epsilon_t^r + \Psi(13) \epsilon_t^u + \Psi(14) \epsilon_t \). I will use this notation in solving equilibrium in the private sector by the method of undetermined coefficients.
In step 3, the first write out the the forward-looking output gap and inflation as:

\[
\hat{y}_t = E_t^s \hat{y}_{t+1} - \frac{1}{\sigma} \left[ i_t - \left( \frac{1}{1 - \phi} r^n_t - \frac{\phi}{1 - \phi} E_t^s r^n_t \right) \right] - E_t^s \pi_{t+1}
\]  
(A.3.7)

\[
\pi_t = \beta \theta E_t^s \pi_{t+1} + (1 - \theta) E_t^s \pi_t + \kappa \theta \hat{y}_t + u_t
\]  
(A.3.8)

Following the method of undetermined coefficients, I first need to conjecture that equilibrium variables are linear functions to current state variables, which include current actual shocks \((r^n_t, u_t)\), past beliefs, \((E_{t-1}^s r^n_{t-1}, E_{t-1}^s u_{t-1})\), and noise in current signals, \((\varepsilon_t^r, \varepsilon_t^u, e_t)\).

\[
\begin{bmatrix}
\hat{y}_t \\
\pi_t
\end{bmatrix}
= 
\begin{bmatrix}
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\
\gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11}
\end{bmatrix}
\begin{bmatrix}
r^n_t \\
E_{t-1}^s r^n_{t-1} \\
u_t \\
E_{t-1}^s u_{t-1}
\end{bmatrix}
+ 
\begin{bmatrix}
\gamma_5 & \gamma_6 & \gamma_7 \\
\gamma_{12} & \gamma_{13} & \gamma_{14}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_t^r \\
\varepsilon_t^u \\
e_t
\end{bmatrix}  

(A.3.9)

Next, substitute this conjecture into the forward-looking variables, \(E_t^s \hat{y}_{t+1}\) and \(E_t^s \pi_{t+1}\). Notice that noise of all signals are temporary, which are expected to be zero in future period.

\[
\begin{bmatrix}
E_t^s \hat{y}_{t+1} \\
E_t^s \pi_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
\gamma_1 \phi + \gamma_2 & \gamma_3 \phi^u + \gamma_4 \\
\gamma_8 \phi + \gamma_9 & \gamma_{10} \phi^u + \gamma_{11}
\end{bmatrix}
\begin{bmatrix}
E_t^s r^n_t \\
E_t^s u_t
\end{bmatrix}  

(A.3.10)

First substitute this into the output gap expression:

\[
\hat{y}_t = \left( \gamma_1 \phi + \gamma_2 \right) + \frac{1}{\sigma} \left( \gamma_8 \phi + \gamma_9 \right) - \frac{1}{\sigma} \left( \frac{\phi}{1 - \phi} E_t^s r^n_t \right)
\]  
(A.3.11)

\[
+ \left( \gamma_3 \phi^u + \gamma_4 + \frac{1}{\sigma} \left( \gamma_{10} \phi^u + \gamma_{11} \right) \right) E_t^s u_t - \frac{1}{\sigma} i_t + \frac{1}{\sigma} \left( \frac{1}{1 - \phi} r^n_t \right)
\]

Next work on \(\pi_t\), as the actual inflation also includes the expected current inflation, and ex-
expected current inflation includes expected current output gap, I first need to calculate:

\[
E^t_y = E^t_y + \frac{1}{\sigma} [i_t - E^t_y - E^t_{\pi+1}]
\]  
(A.3.12)

\[
E^t_{\pi} = \beta E^t_{\pi+1} + \kappa \left\{ E^t_y - \frac{1}{\sigma} [i_t - E^t_y - E^t_{\pi+1}] \right\} + \frac{1}{\theta} E^t_i u_t
\]  
(A.3.13)

Substitute \( E_t \pi_t \) into \( \pi_t \):

\[
\pi_t = \beta \theta E^t_{\pi+1} + (1 - \theta) \left\{ \beta E^t_{\pi+1} + \kappa E^t_y + \frac{1}{\theta} E^t_i u_t \right\} + \kappa \theta \hat{y}_t + u_t
\]  
(A.3.14)

\[
= \beta E^t_{\pi+1} + (1 - \theta) \kappa \left\{ (\gamma_1 \phi + \gamma_2) E^t_i r^n_t + (\gamma_3 \phi^u + \gamma_4) E^t_i u_t \right\} - (1 - \theta) \frac{\kappa}{\sigma} i_t + (1 - \theta) \frac{\kappa}{\sigma} E^t_i r^n_t
\]

\[
+ (1 - \theta) \frac{\kappa}{\sigma} \left\{ (\gamma_8 \phi + \gamma_9) E^t_i r^n_t + (\gamma_{10} \phi^u + \gamma_{11}) E^t_i u_t \right\} + \frac{1 - \theta}{\theta} E^t_i u_t + \kappa \theta \hat{y}_t + u_t
\]

\[
= \left\{ (1 - \theta) \kappa (\gamma_1 \phi + \gamma_2) + (1 - \theta) \frac{\kappa}{\sigma} \left( \beta + (1 - \theta) \frac{\kappa}{\sigma} \right) (\gamma_8 \phi + \gamma_9) \right\} E^t_i r^n_t
\]

\[
+ \left\{ (1 - \theta) \kappa (\gamma_3 \phi^u + \gamma_4) + \frac{1 - \theta}{\theta} + \left( \beta + (1 - \theta) \frac{\kappa}{\sigma} \right) (\gamma_{10} \phi^u + \gamma_{11}) \right\} E^t_i u_t
\]

\[-(1 - \theta) \frac{\kappa}{\sigma} i_t + \kappa \theta \hat{y}_t + u_t
\]

The values of \( \gamma \) can be solved in the following matrix:
\[ \gamma = M\psi + c \]

where

\[ \gamma = [\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8]^{T} \]

\[ M = \begin{bmatrix}
\phi \psi_1 & \psi_1 & \phi \psi_6 & \psi_8 & 0 & 0 & 0 & 0 \\
\phi \psi_2 & \psi_2 & \phi \psi_9 & \psi_9 & 0 & 0 & 0 & 0 \\
\phi \psi_3 & \psi_3 & \phi \psi_{10} & \psi_{10} & 0 & 0 & 0 & 0 \\
\phi \psi_4 & \psi_4 & \phi \psi_{11} & \psi_{11} & 0 & 0 & 0 & 0 \\
\phi \psi_5 & \psi_5 & \phi \psi_{12} & \psi_{12} & 0 & 0 & 0 & 0 \\
\phi \psi_6 & \psi_6 & \phi \psi_{13} & \psi_{13} & 0 & 0 & 0 & 0 \\
\phi \psi_7 & \psi_7 & \phi \psi_{14} & \psi_{14} & 0 & 0 & 0 & 0 \\
(1 - \theta) \phi \psi_1 + \kappa \theta & (1 - \theta) \phi \psi_2 & (1 - \theta) \phi \psi_{3} + \kappa \theta & (1 - \theta) \phi \psi_{8} & 0 & 0 & 0 & 0 \\
(1 - \theta) \phi \psi_2 & (1 - \theta) \phi \psi_3 + \kappa \theta & (1 - \theta) \phi \psi_{9} & (1 - \theta) \phi \psi_{8} & 0 & 0 & 0 & 0 \\
(1 - \theta) \phi \psi_3 & (1 - \theta) \phi \psi_8 & (1 - \theta) \phi \psi_{10} + \kappa \theta & (1 - \theta) \phi \psi_{9} & 0 & 0 & 0 & 0 \\
(1 - \theta) \phi \psi_8 & (1 - \theta) \phi \psi_{10} + \kappa \theta & (1 - \theta) \phi \psi_{11} & (1 - \theta) \phi \psi_{12} + \kappa \theta & 0 & 0 & 0 & 0 \\
(1 - \theta) \phi \psi_{10} & (1 - \theta) \phi \psi_{11} & (1 - \theta) \phi \psi_{12} + \kappa \theta & (1 - \theta) \phi \psi_{13} & 0 & 0 & 0 & 0 \\
(1 - \theta) \phi \psi_{11} & (1 - \theta) \phi \psi_{13} & (1 - \theta) \phi \psi_{14} & (1 - \theta) \phi \psi_{13} & 0 & 0 & + \kappa \theta & 0 \\
(1 - \theta) \phi \psi_{12} & (1 - \theta) \phi \psi_{14} & (1 - \theta) \phi \psi_{14} & (1 - \theta) \phi \psi_{14} & 0 & 0 & + \kappa \theta & 0 \\
\end{bmatrix} \]

\[ c = \begin{bmatrix}
-\frac{1}{3} \phi \psi_1 - \frac{1}{3} \psi_1 + \frac{1}{3} \phi \psi_2 + \frac{1}{3} \psi_2 \\
-\frac{1}{3} \phi \psi_2 - \frac{1}{3} \psi_2 + \frac{1}{3} \phi \psi_3 + \frac{1}{3} \psi_3 \\
-\frac{1}{3} \phi \psi_3 - \frac{1}{3} \psi_3 + \frac{1}{3} \phi \psi_4 + \frac{1}{3} \psi_4 \\
-\frac{1}{3} \phi \psi_4 - \frac{1}{3} \psi_4 + \frac{1}{3} \phi \psi_5 + \frac{1}{3} \psi_5 \\
-\frac{1}{3} \phi \psi_5 - \frac{1}{3} \psi_5 + \frac{1}{3} \phi \psi_6 + \frac{1}{3} \psi_6 \\
-\frac{1}{3} \phi \psi_6 - \frac{1}{3} \psi_6 + \frac{1}{3} \phi \psi_7 + \frac{1}{3} \psi_7 \\
(1 - \theta) \phi \psi_1 + \frac{1}{3} \psi_1 + \frac{1}{3} \phi \psi_9 + \frac{1}{3} \psi_9 \\
(1 - \theta) \phi \psi_2 + \frac{1}{3} \psi_2 + \frac{1}{3} \phi \psi_9 + \frac{1}{3} \psi_9 \\
(1 - \theta) \phi \psi_3 + \frac{1}{3} \psi_3 + \frac{1}{3} \phi \psi_{10} + \frac{1}{3} \psi_{10} + 1 \\
(1 - \theta) \phi \psi_4 + \frac{1}{3} \psi_4 + \frac{1}{3} \phi \psi_{11} + \frac{1}{3} \psi_{11} + 1 \\
(1 - \theta) \phi \psi_5 + \frac{1}{3} \psi_5 + \frac{1}{3} \phi \psi_{12} + \frac{1}{3} \psi_{12} + 1 \\
(1 - \theta) \phi \psi_6 + \frac{1}{3} \psi_6 + \frac{1}{3} \phi \psi_{13} + \frac{1}{3} \psi_{13} + 1 \\
(1 - \theta) \phi \psi_7 + \frac{1}{3} \psi_7 + \frac{1}{3} \phi \psi_{14} + \frac{1}{3} \psi_{14} + 1 \\
\end{bmatrix} \]

\( \gamma \) can be uniquely pinned down by the above linear system.
In step 5, in order to solve for the optimizing interest rate, I first need to specify central bank’s objective function.

**Central Bank Objective Function**

As current interest rate has persistent effect through the dynamic learning process, central bank also considers how current interest rate affect future equilibrium. Consequently, the loss function includes output gap and inflation of current and all future periods.

\[
E_t L(t) = [\pi_t^2 + \omega \hat{\pi}_t^2] + \beta E_t(L(t + 1))
\]  
(A.3.15)

where the \(E_t(L(t + 1))\) is:

\[
\sum_{j=1}^{\infty} \beta^j E_t \left\{ \begin{bmatrix} \pi_{t+1} & \hat{\pi}_{t+1} \\ \hat{y}_{t+1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \pi_{t+j} \\ \hat{y}_{t+j} \end{bmatrix} \right\} 
\]
(A.3.16)

The central banks expectation is **objective**, denoted by \(E_t\), in the sense that it observes all past shocks, and expects all future shocks to be zero. The information set of central bank at period \(t\) is:

\[ I_t = \{ r^n_T, u_T, \forall T = 0...t \} \]

Let \( z_t = [r^n_T, E_t^{s} r^n_{t-1}, u_t, E_t^{s} u_{t-1}]' \) denote the persistent state variables. So the central bank’s...
objective expectation of future period output gap and inflation becomes a linear function of $E_tz_{t+j}$:

$$
\begin{bmatrix}
E_t\pi_{t+j} \\
E_t\hat{\gamma}_{t+j}
\end{bmatrix} =
\begin{bmatrix}
\gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4
\end{bmatrix}
E_t z_{t+j} = \Gamma E_t z_{t+j}
$$

(A.3.17)

$E_t z_{t+j}$ follows:

$$
\begin{bmatrix}
E_t r^n_{t+j} \\
E_t E^s_{t+j-1} r^n_{t+j-1} \\
E_t u_{t+j} \\
E_t E^s_{t+j-1} u_{t+j-1}
\end{bmatrix}
= \Lambda
\begin{bmatrix}
E_t r^n_{t+j-1} \\
E_t E^s_{t+j-2} r^n_{t+j-2} \\
E_t u_{t+j-1} \\
E_t E^s_{t+j-2} u_{t+j-2}
\end{bmatrix}
$$

(A.3.18)

where

$$
\Lambda =
\begin{bmatrix}
\phi & 0 & 0 & 0 & 0 \\
K_{11}F_1 + K_{12} & \phi(1 - K_{11}F_1 - K_{12}) & K_{11}F_3 + K_{13} & -\phi''(K_{11}F_3 + K_{13}) \\
0 & 0 & \phi'' & 0 \\
K_{21}F_1 + K_{22} & -\phi(K_{21}F_1 + K_{22}) & K_{21}F_3 + K_{23} & \phi''(1 - K_{21}F_3 - K_{23})
\end{bmatrix}
$$

Combine it with A.3.17

$$
\begin{bmatrix}
E_t\pi_{t+j} \\
E_t\hat{\gamma}_{t+j}
\end{bmatrix} = \Gamma \Lambda^{j-1} E_t z_{t+1}
$$

(A.3.19)

Substitute into the $E_t(L(t+1))$:

$$
\Sigma \beta^j E_t z'_{t+1} (\Lambda^{j-1})^{\theta} \Omega \Gamma^{j-1} E_t z_{t+1} = \Sigma \beta^j E_t z'_{t+1} \Theta^{j-1} E_t z_{t+1}
$$

(A.3.20)

Take the first order condition on $i^*_i$ of $E_tL(t+1)$:
\[
\left\{ \frac{\partial E_t \pi_t}{\partial i_t} E_t \pi_t + \omega \frac{\partial E_t \hat{y}_t}{\partial i_t} E_t \hat{y}_t \right\} + \frac{1}{2} \sum_{j=1}^{\infty} \beta^j \Delta(j - 1) = 0 \tag{A.3.21}
\]

where

\[
\Delta_{j-1} = \left( \Theta_{j-1}^{21} + \Theta_{j-1}^{12} \right) \phi r^n \frac{\partial E_t^s r^n}{\partial i_t} + \left( \Theta_{j-1}^{32} + \Theta_{j-1}^{23} \right) \phi_u u_t \frac{\partial E_t^s r^n}{\partial i_t} + \left( \Theta_{j-1}^{42} + \Theta_{j-1}^{24} \right) E_t^s u_t \frac{\partial E_t^s r^n}{\partial i_t}
\]

\[
\quad + \Theta_{j-1}^{22} \cdot 2E_t^s r^n \frac{\partial E_t^s r^n}{\partial i_t} + \left( \Theta_{j-1}^{34} + \Theta_{j-1}^{43} \right) \phi^u u_t \frac{\partial E_t^s r^n}{\partial i_t} + \left( \Theta_{j-1}^{42} \right)
\]

\[
\quad + \Theta_{j-1}^{24} E_t^s r^n \frac{\partial E_t^s u_t}{\partial i_t} + \Theta_{j-1}^{44} \cdot 2E_t^s u_t \frac{\partial E_t^s u_t}{\partial i_t}
\]

\[
\equiv \Delta_{j-1}(1) r^n + \Delta_{j-1}(2) u_t + \Delta_{j-1}(3) E_t^s u_t + \Delta_{j-1}(4) E_t^s r^n
\]

\[
+ \Delta_{j-1}(5) r^n + \Delta_{j-1}(6) u_t + \Delta_{j-1}(7) E_t^s r^n + \Delta_{j-1}(8) E_t^s u_t
\]

To solve for the first order condition on interest rate, first write equilibrium variables in terms of \(i_t\):

**Beliefs:**

\[
E_t^s r^n = \phi (1 - K_{11} F_1 - K_{12}) - K_{11} F_2 \left( E_{t-1}^s r^n \right) \tag{A.3.22}
\]

\[
- (K_{11} F_4 + \phi u (K_{11} F_3 + K_{13})) E_{t-1}^s u_{t-1} + K_{12} r^n + K_{13} u_t + K_{11} i_t
\]

\[
E_t^s u_t = \phi u (1 - K_{21} F_3 - K_{23}) - K_{21} F_4 \left( E_{t-1}^s u_{t-1} \right) \tag{A.3.23}
\]

\[
- (\phi (K_{21} F_1 + K_{22}) + K_{21} F_2) E_{t-1}^s r^n + K_{22} r^n + K_{23} u_t + K_{21} i_t
\]

**Output gap:**

\[
\hat{y}_t = \Xi (1) E_t^s r^n + \Xi (2) E_t^s u_t - \frac{1}{\sigma} i_t + \frac{1}{\sigma} \frac{1}{1 - \phi} r^n \tag{A.3.24}
\]
Inflation:

$$\pi_t = \kappa \theta \hat{y}_t + \Xi(3)E^s_t r^n_t + \Xi(4)E^s_t u_t - (1 - \theta) \frac{\kappa}{\sigma} \hat{y}_t + u_t$$  \hspace{1cm} (A.3.25)

Substitute the above endogenous variables into the first order condition on $i_t$:

$$\lambda_1 E^s_t r^n_t + \lambda_2 E^s_t u_t + \lambda_3 r^n_t + \lambda_4 u_t + \lambda_5 i_t = 0$$  \hspace{1cm} (A.3.26)

where

$$\lambda_1 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \Xi(1) + \frac{\partial \pi_t}{\partial i_t} \Xi(3) + \frac{1}{2} \Sigma \beta^j (\Delta_{j-1}(4) + \Delta(7))$$  \hspace{1cm} (A.3.27)

$$\lambda_2 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \Xi(2) + \frac{\partial \pi_t}{\partial i_t} \Xi(4) + \frac{1}{2} \Sigma \beta^j (\Delta_{j-1}(3) + \Delta(8))$$  \hspace{1cm} (A.3.28)

$$\lambda_3 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \frac{1}{\sigma} \frac{1}{1 - \phi} + \frac{1}{2} \Sigma \beta^j (\Delta_{j-1}(1) + \Delta(5))$$  \hspace{1cm} (A.3.29)

$$\lambda_4 = \frac{\partial \pi_t}{\partial i_t} + \frac{1}{2} \Sigma \beta^j (\Delta(2) + \Delta(6))$$  \hspace{1cm} (A.3.30)

$$\lambda_5 = \left( \kappa \theta \frac{\partial \pi_t}{\partial i_t} + \omega \frac{\partial \hat{y}_t}{\partial i_t} \right) \left( -\frac{1}{\sigma} \right) + \frac{\partial \pi_t}{\partial i_t} \left( -(1 - \theta) \frac{\kappa}{\sigma} \right)$$  \hspace{1cm} (A.3.31)

and partial derivatives are derived as:

$$\frac{\partial E^s_t r^n_t}{\partial i_t} = K_{11}$$  \hspace{1cm} (A.3.32)

$$\frac{\partial E^s_t u_t}{\partial i_t} = K_{21}$$  \hspace{1cm} (A.3.33)

$$\frac{\partial \hat{y}_t}{\partial i_t} = \Xi(1) \frac{\partial E^s_t r^n_t}{\partial i_t} + \Xi(2) \frac{\partial E^s_t u_t}{\partial i_t} - \frac{1}{\sigma}$$  \hspace{1cm} (A.3.34)

$$\frac{\partial \pi_t}{\partial i_t} = \kappa \theta \frac{\partial \hat{y}_t}{\partial i_t} + \Xi(3) \frac{\partial E^s_t r^n_t}{\partial i_t} + \Xi(4) \frac{\partial E^s_t u_t}{\partial i_t} - (1 - \theta) \frac{\kappa}{\sigma}$$  \hspace{1cm} (A.3.35)
Further substitute $E_t^s r^p_t$ and $E_t^s u_t$:

\[
0 = \lambda_1 \{(\phi (1 - K_{11} F_1 - K_{12}) - K_{11} F_2) E_{t-1}^s r^p_{t-1} - (K_{11} F_4 + \phi u (K_{11} F_3 + K_{13})) E_{t-1}^s u_{t-1}
\]
\[
+ K_{12} r^p_t + K_{13} u_t + K_{11} i_t \} + \lambda_2 \{(\phi u (1 - K_{21} F_3 - K_{23}) - K_{21} F_4) E_{t-1}^s u_{t-1}
\]
\[
- (\phi (K_{21} F_1 + K_{22}) + K_{21} F_2) E_{t-1}^s r^p_{t-1} + K_{22} r^p_t + K_{23} u_t + K_{21} i_t \} + \lambda_3 r^p_t + \lambda_4 u_t + \lambda_5 i_t
\]

The above equation solves the optimal nominal interest rate. Comparing with the guessed form yields the solution of $[F_1, F_2, F_3, F_4]$

\[
F_1 = -\frac{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_3}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \quad (A.3.36)
\]

\[
F_2 = -\frac{\lambda_1 \phi (1 - K_{11} F_1 - K_{12}) - K_{11} F_2 - \lambda_2 (\phi (K_{21} F_1 + K_{22}) + K_{21} F_2)}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \quad (A.3.37)
\]

\[
F_3 = -\frac{\lambda_1 K_{13} + \lambda_2 K_{23} + \lambda_4}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \quad (A.3.38)
\]

\[
F_4 = -\frac{-\lambda_1 (K_{11} F_4 + \phi u (K_{11} F_3 + K_{13})) + \lambda_2 (\phi u (1 - K_{21} F_3 - K_{23}) - K_{21} F_4)}{\lambda_1 K_{11} + \lambda_2 K_{21} + \lambda_5} \quad (A.3.39)
\]

I iterate the process until the conjectured interest rate function matches the above solution.

### A.4 Proofs

#### A.4.1 Proof of Lemma 2

The first (change of slope) and second point (intercept after cost push shock) are obvious. The following shows the proof for the their point (the sign of intercept after natural rate shock). The sign depends on the combination of parameter values and expectation on interest rate reaction
function. Specifically, from Kalman Filtering, we know that $K_{11}F_1 + K_{21}F_3 = 1$, which indicates $0 < \frac{\partial E_i r_i}{\partial r_i} < 1$ and $0 < \frac{\partial E_i u_i}{\partial u_i} < 1$ as long as both $F_1$ and $F_3$ are non-zero. However, $K_{11} = \frac{\partial E_i r_i}{\partial r_i}$ may be greater or less than one, depending on whether $F_1$ is greater or smaller than 1. Consequently, the sign of the impact of $r_i$ on the intercept of Phillips Curve under imperfect information, $\{ \frac{(1 - \theta)}{\sigma} (K_{11} - 1) + \frac{1-\theta}{\sigma} K_{21} \}$ is ambiguous.

A.4.2 Second Order Approximation of Household’s Utility Function

Follow Woodford (2003), Gali (2010), Walsh (2010) to prove that maximizing the utility of household is equivalent, up to second order approximation, to

$$W = -\frac{1}{2} E_0 \Sigma \beta' \left( (\epsilon^{-1} + \phi) \epsilon^2 \text{var}(p_t(j)) + (\sigma + \phi) \hat{\gamma}_t^2 \right) \quad (A.4.1)$$

The next step is to prove the relationship between \text{var}(p_t(j)) with \text{var}(\pi_t). Denote $\Delta_t = \text{var}_j[log p_{jt}]$. Since $\text{var}_j \bar{P}_{t-1} = 0$, we have

$$\Delta_t = \text{var}_j[log p_{jt} - \bar{P}_{t-1}] \quad (A.4.2)$$

$$= E_j[log p_{jt} - \bar{P}_{t-1}]^2 - [E_j log p_{jt} - \bar{P}_{t-1}]^2$$

$$= E_j[log p_{jt-1} - \bar{P}_{t-1}]^2 + (1 - \theta)(\int p_{ij}^* - \bar{P}_{t-1})^2 - (\bar{P}_t - \bar{P}_{t-1})^2$$

As noted in Appendix A.1, $\bar{P}_t = (1 - \theta) \int p_{ij}^* + \theta \bar{P}_{t-1}$, we have $(1 - \theta) \int log p_{ij}^* + \theta \bar{P}_{t-1}$, which implies that $(1 - \theta) \int log p_{ij}^* - (1 - \theta)p_{t-1} = \bar{p} - \bar{p}_{t-1}$. So, we have:

$$\int log p_{ij}^* = \left( \frac{1}{1 - \theta} \right) (\bar{p}_t - \bar{p}_{t-1}) \quad (A.4.3)$$
Substitute this into equation A.4.2 and get
\[ \Delta_t = \theta \omega - 1 + (\theta - \bar{\rho} - \bar{p}_{t-1})^2. \]
Applying the definition of inflation results in:
\[ E_t \Sigma \beta' \Delta_t = \frac{\theta}{(1 - \theta)(1 - \theta \beta)} E_t \Sigma \beta' \pi_t^2 + t.i.p. \]  
(A.4.4)

### A.4.3 Proof of Lemma 4

To prove the optimal response for a discretionary rate, express out the inflation expression in Lemma 3, \( \pi_t = -R\hat{y}_t \) as follows:

\[ \kappa \hat{y}_t + \Omega_r (K_r - 1) r''_t + \Omega_u K_u + u_t = -R\hat{y}_t \]  
(A.4.5)

Then, substitute \( \hat{y}_t \) and \( \pi_t \) by interest rate, which results in:

\[ \left[ (R + \kappa) \frac{1}{\sigma} - \Omega_r \right] r''_t + u_t = \left[ (R + \kappa) \frac{1}{\sigma} - (\Omega_r K_r + \Omega_u K_u) \right] i_t \]  
(A.4.6)

To prove that \( F_r > 1 \) is equivalent to prove
\[ \Omega_r K_r + \Omega_u K_u > \Omega_r \] which is just Assumption 1.

To prove that \( F_u < \theta F_u^p \) is equivalent to prove:

\[ (R^p + \kappa) \frac{1}{\sigma} < (R + \kappa) \frac{1}{\sigma} - [\Omega_r K_r + \Omega_u K_u] \]

\[ \iff \Omega_r K_r + \Omega_u K_u < \frac{1}{\sigma} (R - R^p) \]

As \( (\hat{y}_t, \pi_t) \) is chosen optimally by the central bank, we know that \( R \cdot \text{slope} = \omega \) where \( \omega \) is in the CB’s objective function:

\[ \pi_t^2 + \omega \hat{y}_t^2 \]
So,

\[ \frac{1}{\sigma} \left( \frac{\omega}{R^p} - \frac{\omega}{R} \right) = (1 - \theta) \frac{\kappa}{\sigma} K_r + \frac{1 - \theta}{\theta} K_u \]  

(A.4.7)

Importantly, the RHS is just the difference between \( R_p \) and \( R \), as it measures the sensitivity inflation to beliefs.

Substitute this into the above equation leads to:

\[ \frac{\omega}{R^p} - \frac{\omega}{R} < R - R^p \]

\[ \Leftrightarrow \omega < R^p \cdot R \]

As Lemma 3 suggests that \( R > R^p \), the sufficient condition for the above inequality to hold is that \( R > \bar{R} \equiv \sqrt{\omega} \), where \( \omega \) is defined as the second order approximation of household utility.
Appendix B

Appendix for Chapter 2

B.1 Proof of Proposition 3

The proof of Proposition 1 take two steps. First I show that evaluated at the equilibrium optimizing discretionary interest rate, the partial derivative of combined Kalman gains weighted by their effects on inflation is negative with respect to $F_r$ and is positive with respect to $F_u$.

**Step 1:** Under the assumption that $\sigma_r = \sigma_u$, I prove the following result.

Result 1. $\Omega_r \frac{\partial K_r}{\partial F_r} + \Omega_u \frac{\partial K_u}{\partial F_r} < 0$

Result 2. $\Omega_r \frac{\partial K_r}{\partial F_u} + \Omega_u \frac{\partial K_u}{\partial F_u} > 0$

The partial derivatives of Kalman gains on $F$:

$$\frac{\partial K_r}{\partial F_r} = D^{-1} \left( F_r^2 \sigma_r^2 \sigma_u^2 - F_r^2 \sigma_r^2 \sigma_r^2 \right)$$

$$\frac{\partial K_u}{\partial F_u} = D^{-1} \left( F_r^2 \sigma_r^2 \sigma_u^2 - F_u^2 \sigma_u^2 \sigma_u^2 \right)$$

$$\frac{\partial K_u}{\partial F_r} = D^{-1} \left( 2F_rF_u \sigma_r^2 \sigma_u^2 \right)$$

$$\frac{\partial K_r}{\partial F_u} = D^{-1} \left( 2F_rF_u \sigma_r^2 \sigma_u^2 \right)$$
where $D = F_r^2 \sigma_r^2 + F_u^2 \sigma_u^2$.

To show Result 1, notice that $\frac{\partial K_r}{\partial F_r} < 0$, so it becomes sufficient to show $\frac{\partial K_r}{\partial F_r}$. This inequality holds when $F_u < F_r$ under discretion. From Lemma 4, we know that $F_r > F_r^p = 1$, and $F_u < \theta F_u^n$.

In addition, we can derive the optimizing discretionary interest rate under perfect information to be: $F_u^p = \frac{\kappa^2}{1 + \kappa^2},$ which leads to $F_u < 1$. Under $\sigma_r = \sigma_u$, we have $\frac{\partial K_r}{\partial F_r} < 0$.

To show Result 2 is involves more steps. First, we know that $\frac{\partial K_r}{\partial F_u} = 0$, and under parameter specifications, $0 < \Omega_r < \Omega_u$. Thus, the sufficient condition for Result 2 to be hold is $\Omega_u \frac{\partial K_r}{\partial F_u} + \Omega_u \frac{\partial K_u}{\partial F_u} > 0$, which is equivalent to prove that $\frac{\partial K_r}{\partial F_u} + \frac{\partial K_u}{\partial F_u} > 0$.

From the partial derivatives of $K$ to $F$, and with the assumption $\sigma_r = \sigma_u$, we have

$$\frac{\partial K_r}{\partial F_u} + \frac{\partial K_u}{\partial F_u} = D^{-1} \left( F_r^2 \sigma_r^2 \sigma_u^2 - F_u^2 \sigma_u^2 \sigma_u^2 - 2F_r F_u \sigma_r^2 \sigma_u^2 \right)$$

(B.1.1)

Holding $F_r$ fixed, the quadratic function $f(F_u) = F_r^2 \sigma_r^2 \sigma_u^2 - F_u^2 \sigma_u^2 \sigma_u^2 - 2F_r F_u \sigma_r^2 \sigma_u^2$ reaches maximum at negative value of $F_u$. The range of $F_u$ is $[0, F_r]$ (Lemma 4). So, I only need to show $f(0) > 0$ and $f(F_r) < 0$.

$$f(0) = F_r^2 \sigma_r^2 \sigma_u^2 > 0$$

(B.1.2)

$$f(F_r) = -2F_r F_u \sigma_r^2 \sigma_u^2 < 0$$

(B.1.3)

This completes the prove of Result 2.

**Step 2:** I prove that Result 1 and Result 2 makes Proposition 2 to hold.

**Part 1, $F_r^c > F_r^d$**

Suppose on the contrary that $F_r^c = F_r^d$. Then, express out the first order condition on $F_r$ after $r^n$ shock for both discretionary central bank and central bank with commitment, which is $\pi_i \frac{\partial \pi}{\partial F_r} +$
\( \hat{\omega} \hat{y}_t \frac{\partial \hat{\omega}}{\partial F_r} = 0 \). As \( \Omega_u > 0 \) and \( D > 0 \)

\[
- \frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_r \frac{\partial K_r}{\partial F_r} F_r + \Omega_u K_u + \Omega_u \frac{\partial K_u}{\partial F_r} F_r|_{comm} = \frac{\omega}{\sigma} \frac{\hat{\omega}}{\sigma} \frac{\partial \hat{\omega}}{\partial F_r}|_{comm}
\]

(B.1.4)

\[
- \frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u|_{disc} = \frac{\omega}{\sigma} \frac{\hat{\omega}}{\sigma} \frac{\partial \hat{\omega}}{\partial F_r}|_{disc}
\]

(B.1.5)

Under Assumption 1, The LHS of E.1 is smaller than the LHS of E.2, which makes \( \frac{\hat{\omega}}{\pi_t} \) to be smaller for commitment central bank. Substitute \( \hat{\omega}_t = -\frac{1}{\sigma} (F_r - 1) r_t^n \), we have

\[
\frac{F_r^c}{\pi_t^c} > \frac{F_r^d}{\pi_t^d}
\]

(B.1.6)

The necessary condition for this inequality to hold is 1: \( F_r^c > F_r^d \), or \( \pi_t^c < \pi_t^d \).

Next, write out the expression for \( \pi_t^c < \pi_t^d \):

\[
- \frac{1}{\sigma} (F_r - 1) + \Omega_r K_r F_r + \Omega_u K_u F_r|_{comm} < - \frac{1}{\sigma} (F_r - 1) + \Omega_r K_r F_r + \Omega_u K_u F_r|_{disc}
\]

(B.1.7)

The necessary condition for this inequality to hold is: either \( F_r^c > F_r^d \), or \( \Omega_r K_r F_r + \Omega_u K_u F_r|_{comm} < \Omega_r K_r F_r + \Omega_u K_u F_r|_{disc} \). Under Assumption, this implies \( F_r^c > F_r^d \). In conclusion, we have \( F_r^c > F_r^d \).

**Part 2, \( F_u^c < F_u^d \)**

Suppose on the contrary that \( F_u^c = F_u^d \). Then, express out the first order condition on \( F_u \) after \( u_t \) shock for both discretionary central bank and central bank with commitment, which is \( \pi_t \frac{\partial \pi_t}{\partial F_u} + \omega \hat{\gamma} \frac{\partial \omega}{\partial F_u} = 0 \)
\[ -\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_r \frac{\partial K_r}{\partial F_u} F_u + \Omega_u K_u + \Omega_u \frac{\partial K_u}{\partial F_u} F_u |_{\text{comm}} = \frac{\omega \hat{\pi}_t}{\sigma \hat{\pi}_t} |_{\text{comm}} \quad (B.1.8) \]

\[ -\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u |_{\text{disc}} = \frac{\omega \hat{\pi}_t}{\sigma \hat{\pi}_t} |_{\text{disc}} \quad (B.1.9) \]

Under Assumption 2, the LHS of A.4.2 is smaller than the LHS of B.1.8, which makes \( \hat{\pi}_t \) to be bigger for commitment central bank. Substitute \( \hat{\pi}_t = -\frac{1}{\sigma} F_u \), we have

\[ \frac{F_u^c}{\pi_i^c} < \frac{F_u^d}{\pi_i^d} \quad (B.1.10) \]

The necessary condition for this inequality to hold is: \( F_u^c < F_u^d \), or \( \pi_i^c > \pi_i^d \).

Next, write out the expression for \( \pi_i^c < \pi_i^d \) after \( u_t \) shock:

\[ -\frac{1}{\sigma} F_u + \Omega_r K_r F_u + \Omega_u K_u F_u |_{\text{comm}} < -\frac{1}{\sigma} F_u + \Omega_r K_r F_u + \Omega_u K_u F_u |_{\text{disc}} \quad (B.1.11) \]

The necessary condition for this inequality to hold is: either \( F_u^c < F_u^d \), or \( \Omega_r K_r F_u + \Omega_u K_u F_u |_{\text{comm}} < \Omega_r K_r F_u + \Omega_u K_u F_u |_{\text{disc}} \). Under Assumption 2, this implies \( F_u^c < F_u^d \). In conclusion, we have \( F_u^c > F_u^d \).
B.2 Proof of Proposition 4

The optimal polity rule satisfies:

\[ -\frac{\kappa}{\sigma} + \Omega_r \left( K_r + \frac{\partial K_r}{\partial F_r} F_r \right) + \Omega_u \left( K_u + \frac{\partial K_u}{\partial F_u} F_r \right) = \frac{\omega}{\sigma} \hat{y}_t \]  
(B.2.1)

\[ -\frac{\kappa}{\sigma} + \Omega_r \left( K_r + \frac{\partial K_r}{\partial F_r} F_r \right) + \Omega_u \left( K_u + \frac{\partial K_u}{\partial F_u} F_r \right) = \frac{\omega}{\sigma} \hat{y}_t \]  
(B.2.2)

After \( K_r \) and \( K_u \) are determined, the optimal one-time discretionary interest rate aims to minimize the ex-post loss:

\[ \pi_t^2 + \omega \hat{y}_t^2 \]  
(B.2.3)

subjected to the constraint that \( \pi_t = \{ \kappa - \sigma (\Omega_r K_r + \Omega_u K_u) \} \hat{y}_t + (\Omega_r (K_r - 1) + \Omega_u K_u) r_t + u \). The first order condition of interest rate satisfies:

\[ -\frac{\kappa}{\sigma} + \Omega_r K_r + \Omega_u K_u = \frac{\omega}{\sigma} \hat{y}_t \]  
(B.2.4)

As long as \( F_r \neq 0 \) and \( F_u \neq 0 \), B.2.4 is different from equation B.2.1 and B.2.2, which implies that one-time discretionary interest rate is different from the response of interest rate under optimal policy rule. Since the one-time discretionary interest rate is derived from the ex-post loss minimization problem, it implies that there is always a profitable deviation.
Appendix C

Appendix for Chapter 3

C.1 Price-setting under Higher Order Belief

This section derives the optimal prices set by individual firms as functions their higher order beliefs on the aggregate technology shocks and the aggregate price level.

From $p_i = E_i [p + \alpha y] - \beta a_i$, substitute the real output by the aggregate nominal demand as $y$ by $y = n - p$:

$$p_i = E_i [p + \alpha (n - p)] - \beta a_i$$

(C.1.1)

$$= (1 - \alpha) E_i p + \alpha E_i n - \beta a_i$$

(C.1.2)

Next, to deal with the aggregate price level in log-linear form, I first take log-linear approximation of the aggregate price, $P^{1-\varepsilon} = \int_0^1 P_i^{1-\varepsilon} di$, which yields $p = \int_0^1 p_i di$. Substitute the aggregate price level as integral of individual prices results in the following expression:

$$p_i = (1 - \alpha) E_i \int_0^1 [(1 - \alpha) E_j p + \alpha E_j n - \beta a_j] d j + \alpha E_i n - \beta a_i,$$

(C.1.3)
which can be simplified as:

\[ p_i = (1 - \alpha)^2 E_i \bar{E} p + \alpha (1 - \alpha) E_i \bar{E} n + \alpha E_i n - (1 - \alpha) \beta E_i \bar{a} - \beta a_i \]  

(C.1.4)

where I use \( \bar{E} [\cdot] \) to be average expectation operator:

\[ \int_0^1 E_j (\cdot) \, dj = \bar{E} (\cdot) \]  

(C.1.5)

\[ \bar{E}^j [\cdot] = \int E_i \bar{E}^{j-1} [\cdot] \, di = \bar{E} \bar{E}^{j-1} \]  

(C.1.6)

Iterating this substitution process leads to the optimal individual price with higher order beliefs:

\[ p_i = (1 - \alpha)^\infty E_i \bar{E}^\infty p + \alpha \Sigma_{j=0}^\infty (1 - \alpha)^j E_i \bar{E}^j n - \beta \Sigma_{j=0}^\infty (1 - \alpha)^j+1 E_i \bar{E}^j \bar{a} - \beta a_i \]  

(C.1.7)

C.2 Second Order Approximation to Household’s Welfare

The second order Taylor expansion of \( U \) around a steady state \((\bar{C}, \bar{N})\) yields

\[ U - \bar{U} \approx U_c \bar{C} \left( \frac{C - \bar{C}}{C} \right) - U_n \bar{N} \left( \frac{N - \bar{N}}{N} \right) + \frac{1}{2} U_{cc} \bar{C}^2 \left( \frac{C - \bar{C}}{C} \right)^2 + \frac{1}{2} U_{nn} \bar{N}^2 \left( \frac{N - \bar{N}}{N} \right)^2 \]  

(C.2.1)

From market clearing condition, \( C = Y \). I use \( \hat{y} \) to denote log deviation from steady state level output, i.e., \( \hat{y} = \log \left( \frac{Y}{\bar{Y}} \right) \), and take second order approximation by using the Taylor series expansion \( \frac{Y}{\bar{Y}} = 1 + \hat{y} + \frac{1}{2} \hat{y}^2 \)

Equation C.2.1 becomes

\[ U - \bar{U} \approx U_c \bar{C} \left( \hat{y} + \frac{1 - \sigma}{2} \hat{y}^2 \right) - U_n \bar{N} \left( \hat{n} + \frac{1 + \psi}{2} \hat{n}^2 \right) \]  

(C.2.2)
Now apply the individual good demand function to the aggregation of labor,

\[ N = Y \int_0^1 \left( \frac{P_i}{P} \right)^{-\varepsilon} \frac{1}{A_i} \, di, \text{ and we have } \hat{n} = \hat{\delta} + \log \left[ \int_0^1 \left( \frac{P_i}{P} \right)^{-\varepsilon} \frac{1}{A_i} \, di \right] \]  

(C.2.3)

Denote \( d \equiv \log \left[ \int_0^1 \left( \frac{P_i}{P} \right)^{-\varepsilon} \frac{1}{A_i} \, di \right] \) as the price dispersion adjusted by idiosyncratic shocks.

Lemma 1 \( d \simeq \frac{\varepsilon}{2} \text{var}_i(p_i) + \frac{1}{2} \text{var}_i(a_i) + \varepsilon \text{cov}(p_i a_i) \)

Proof:

\[
\left( \frac{P_i}{P} \right)^{-\varepsilon} \frac{1}{A_i} = \exp(-a_i) \cdot \exp[-\varepsilon p_i] = 1 + (-\varepsilon) \hat{p}_i + (-1) \hat{a}_i + \frac{\varepsilon^2}{2} \hat{p}_i^2 + \frac{1}{2} \hat{a}_i^2 + \varepsilon \hat{p}_i \hat{a}_i
\]

(C.2.4)

Note that from the definition of \( P \), we have

\[ 1 = \int_0^1 \left( \frac{P_i}{P} \right)^{1-\varepsilon} \, di. \]

(C.2.5)

Apply a second order approximation to get

\[ E_i(\hat{p}_i) = \frac{\varepsilon - 1}{2} E_i(\hat{p}_i^2) \]

(C.2.6)

Now, substitute it into equation C.2.4 and take integral, we have:

\[
\int_0^1 \left( \frac{P_i}{P} \right)^{-\varepsilon} \frac{1}{A_i} \, di = 1 + \frac{1}{2} \varepsilon E_i(\hat{p}_i^2) + \frac{1}{2} E_i(\hat{a}_i^2) + \varepsilon E_i(\hat{p}_i \hat{a}_i)
\]

(C.2.7)
Notice that \( \int_0^1 \hat{p}_t^2 \simeq \int_0^1 (p_i - E_i p_t)^2 di \), we rewrite it as

\[
\int_0^1 \left( \frac{p_i}{\bar{p}} \right)^{-\varepsilon} \frac{1}{A_i} di = 1 + \frac{1}{2} \varepsilon \text{var}_i(p_i) + \frac{1}{2} \varepsilon \text{cov}(p_i, a_i) \]

\[
\simeq \frac{\varepsilon}{2} \text{var}_i(p_i) + \frac{1}{2} \text{var}_i(a_i) + \varepsilon \text{cov}(p_i a_i) \tag{C.2.8}
\]

Now, substitute the expression for \( d \) into equation C.2.4.

\[
U - \hat{U} \tag{C.2.9}
\]

\[
= U_c C \left( \hat{y} + \frac{1 - \sigma}{2} \hat{y}^2 \right) - U_n N \left( \hat{y} + d + \frac{1 + \psi}{2} (\hat{y} + d)^2 \right) U - \hat{U}
\]

\[
= U_c C \left( \hat{y} - \frac{\sigma}{2} \hat{y}^2 \right) - U_n N \left[ \hat{y} + \varepsilon \text{var}_i(p_i) + \frac{1}{2} \varepsilon \text{var}_i(a_i) + \varepsilon \text{cov}(p_i a_i) \right]
\]

\[
- U_n N \frac{\psi}{2} \left[ \hat{y}^2 + \frac{\varepsilon}{2} \text{var}_i(p_i)^2 + \frac{1}{2} \varepsilon \text{var}_i(a_i)^2 + \varepsilon^2 \text{cov}(p_i a_i) + \hat{y} \varepsilon \text{var}_i(p_i) + \hat{y} \text{var}_i(a_i) + 2 \hat{y} \varepsilon \text{cov}(p_i a_i) \right]
\]

\[
- U_n N \frac{\psi}{2} \left[ \frac{\varepsilon}{2} \text{var}_i(p_i) \text{var}_i(a_i) + \varepsilon^2 \text{var}_i(p_i) \text{cov}(p_i a_i) + \varepsilon \text{var}_i(a_i) \text{cov}(p_i a_i) \right]
\]

\[
- \frac{1}{2} \left\{ (\sigma U_c C - \psi U_n N) (\hat{y} - \hat{y}^n)^2 + \varepsilon \text{var}_i(p_i) \right\} \tag{C.2.10}
\]

Private signals make \( \text{cov}(p_i, a_i) \) to be dependent on the precision of signals, which makes the weight between the price level and the output gap stabilization dependent on the precision of signals. For simplicity, I assume a constant weight when discussing optimal monetary policy in this paper. i.e., the maximization problem writes as:

\[
- \left\{ (\hat{y} - \hat{y}^n)^2 + \frac{\varepsilon}{\sigma U_c C - \psi U_n N} \text{var}_i(p_i) \right\} \tag{C.2.11}
\]
C.3 Benchmark Case: No Forward Guidance

This section derives the aggregate price level and the output level in the benchmark case in which the central bank sets the nominal aggregate demand in the last stage and does not give forward guidance. In this situation, firms use their private information on the aggregate technology shock to form expectations about the monetary policy.

I study the class of optimal policy which is linear to the aggregate technology shock, \( n = \gamma \alpha \), which makes individual expectations about the aggregate nominal demand take the form: \( E_i n = \gamma E_i \alpha \). Firm \( i \) sees only its own technology \( \alpha_i \) as the private signal, and thus the conditional expectation becomes

\[
E_i \bar{\alpha} = \frac{\kappa_s}{\kappa_s + \kappa_a} \alpha_i + \frac{\kappa_a}{\kappa_s + \kappa_a} \mu_a = \frac{\kappa_s}{\kappa_s + \kappa_a} \alpha_i
\]  

(C.3.1)

Average over \( i \) and get

\[
\bar{E} \bar{\alpha} = \frac{\kappa_s}{\kappa_s + \kappa_a} \bar{\alpha}
\]  

(C.3.2)

Apply \( E_i \) to this first-order averaged expectation:

\[
E_i \bar{E} \bar{\alpha} = \frac{\kappa_s}{\kappa_s + \kappa_a} E_i \bar{\alpha} = \left( \frac{\kappa_s}{\kappa_s + \kappa_a} \right)^2 \alpha_i
\]  

(C.3.3)

Continuously iterate the substitution process and get:

\[
E_i \bar{E}^j \bar{\alpha} = \left( \frac{\kappa_s}{\kappa_s + \kappa_a} \right)^{j+1} \alpha_i
\]  

(C.3.4)
Apply equation C.3.4 into equation C.1.7

\[ p_i = (1 - \alpha)^\infty E_i E^\infty p + \alpha \gamma \sum_{j=0}^{\infty} (1 - \alpha)^j \left( \frac{\kappa_s}{\kappa_s + \kappa_a} \right)^{j+1} a_i - \beta \sum_{j=0}^{\infty} (1 - \alpha)^{j+1} \left( \frac{\kappa_s}{\kappa_s + \kappa_a} \right)^{j+1} a_i - \beta a_i \]

(C.3.5)

I guess and verify that the higher order expectation on \( p \) is less than \( \frac{1}{1 - \alpha} \), which makes \( (1 - \alpha)^\infty E_i E^\infty p \rightarrow 0 \). This leads to

\[ p_i = \frac{(\alpha \gamma - \beta) \kappa_s - \beta \kappa_a}{\kappa_a + \alpha \kappa_s} a_i \]

(C.3.6)

Integrate over \( i \) and apply \( y = n - p = \gamma \bar{a} - p \) to get the equilibrium aggregate price level and output:

\[ p = \frac{(\alpha \gamma - \beta) \kappa_s - \beta \kappa_a}{\kappa_a + \alpha \kappa_s} \bar{a} \]

(C.3.7)

\[ y = \frac{\beta \kappa_s + (\gamma + \beta) \kappa_a}{\kappa_a + \alpha \kappa_s} \bar{a} \]

(C.3.8)

**C.4 Instrument-based Odyssean Forward Guidance**

Instrument-based Odyssean forward guidance changes the pricing behaviors in two aspects. (1) on \( E_i \bar{a} \): firms use both their private signals of firm-specific technologies and the public signal which they get from the forward guidance to form expectations about the aggregate technology. (2) on \( E_i n \): instead of using expectations about the aggregate technology to form expectations about the aggregate demand, the Odyssean forward guidance gives the exact information on the aggregate nominal demand to private agents, so \( E_i n = n \).
Specifically, the expectations about the aggregate technology are formed as:

\[
E_i \tilde{a} = \frac{\kappa_m}{\kappa_m + \kappa_s + \kappa_\xi} m + \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_\xi} a_i + \frac{\kappa_\xi}{\kappa_s + \kappa_\xi} \mu_a
\]  

(C.4.1)

\[
E_i \tilde{a} = \frac{\kappa_m}{\kappa_m + \kappa_s + \kappa_\xi} m + \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_\xi} a_i
\]

Integrate over all \(i\) yields:

\[
\tilde{E} \tilde{a} = \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} \tilde{a}
\]  

(C.4.2)

where I denote \(K = \kappa_m + \kappa_s + \kappa_\xi\).

Apply \(E_i\) to the first-order averaged expectations:

\[
E_i \tilde{E} \tilde{a} = \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} E_i \tilde{a} = \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} \left[ \frac{\kappa_m}{K} m + \frac{\kappa_s}{K} a_i \right] = \left[ \frac{\kappa_m + \kappa_s}{K} \right] m + \left( \frac{\kappa_s}{K} \right)^2 a_i
\]  

(C.4.3)

Iterating this process leads to the higher order beliefs on the aggregate technology:

\[
E_i \tilde{E}^{j-1} \tilde{a} = \left( \frac{\kappa_m}{K} \right) \sum_{k=1}^{j-1} \left( \frac{\kappa_s}{K} \right)^{k-1} m + \left( \frac{\kappa_s}{K} \right)^j a_i
\]  

(C.4.4)

In addition, firms do not need to form expectation on \(n\), since they observe \(n\) perfectly. Similarly as the benchmark case, I guess and verify that higher order expectation on \(p\) does not explode, and thus equation C.1.7 now becomes

\[
p_i = \alpha \sum_{j=1}^{\infty} (1 - \alpha)^{j-1} n - \beta \sum_{j=1}^{\infty} (1 - \alpha)^j E_i \tilde{E}^{j-1} \tilde{a} - \beta a_i
\]  

(C.4.5)

where \(E_in\) is substituted by the actual aggregate nominal demand, \(n\). Now, substitute the higher
order expectations on the aggregate technology by equation C.4.4

\[ p_i = \alpha \Sigma_{j=1}^{\infty} (1 - \alpha)^j n - \beta \Sigma_{j=1}^{\infty} (1 - \alpha)^j \left\{ \left( \frac{K_m}{K} \right) \Sigma_{k=1}^{j} \left( \frac{K_s}{K} \right)^{k-1} m + \left( \frac{K_s}{K} \right)^{j} a_i \right\} - \beta a_i \] (C.4.6)

To simplify this equation, first work on the second term:

\[ \Sigma_{j=1}^{\infty} (1 - \alpha)^j \left\{ \left( \frac{K_m}{K} \right) \Sigma_{k=1}^{j} \left( \frac{K_s}{K} \right)^{j-1} \right\} = (C.4.7) \]

\[ (1 - \alpha)^j \frac{K_m}{K} + (1 - \alpha)^{2j} \frac{K_m}{K} + (1 - \alpha)^{3j} \frac{K_m}{K} + \cdots \]

Collecting terms gives:

\[ (1 - \alpha) \frac{K_m}{K} \left\{ \Lambda_1 \cdot 1 + \Lambda_2 \frac{K_s}{K} + \Lambda_3 \left( \frac{K_s}{K} \right)^2 + \cdots \right\} \] (C.4.8)

where \( \Lambda_1 = \frac{1}{\alpha}, \Lambda_2 = \frac{1 - \alpha}{\alpha}, \Lambda_3 = \frac{(1 - \alpha)^2}{\alpha} \). Thus, equation C.4.7 becomes

\[ \Sigma_{j=1}^{\infty} (1 - \alpha)^j \left\{ \left( \frac{K_m}{K} \right) \Sigma_{k=1}^{j} \left( \frac{K_s}{K} \right)^{j-1} \right\} = \Sigma_{j=1}^{\infty} (1 - \alpha)^j \left\{ \left( \frac{K_m}{K} \right) \Sigma_{k=1}^{j} \left( \frac{K_s}{K} \right)^{k-1} \right\} \]

\[ = \frac{1 - \alpha}{\alpha} \frac{K_m}{\alpha K_s + \kappa_z + K_m} \]

Substituting into equation C.4.4 leads to:

\[ p_i = n - \beta \left[ \frac{1 - \alpha}{\alpha} \frac{K_m}{\alpha K_s + \kappa_z + K_m} m + \frac{(1 - \alpha) K_s}{K_m + \kappa_z + \alpha K_s} a_i \right] - \beta a_i \] (C.4.10)

\[ p = \left[ \gamma_{fc} - \beta \frac{1 - \alpha}{\alpha} \frac{K_m}{\alpha K_s + \kappa_z + K_m} \right] m - \beta \left[ \frac{(1 - \alpha) K_s}{K_m + \kappa_z + \alpha K_s} + 1 \right] \bar{a} \] (C.4.11)

\[ y = \left[ \beta \frac{1 - \alpha}{\alpha} \frac{K_m}{\alpha K_s + \kappa_z + K_m} \right] m + \beta \left[ \frac{(1 - \alpha) K_s}{K_m + \kappa_z + \alpha K_s} + 1 \right] \bar{a} \] (C.4.12)
C.5 Delphic Forward Guidance

Comparing with the case of instrument-based Odyssean forward guidance, firms under Delphic forward guidance need to form expectations about the aggregate nominal demand after re-optimization. That is, firms need to form conditional expectation on both \( n \) and \( \bar{a} \). The conditional expectation on \( \bar{a} \) is the same as in the case under instrument-based Odyssean forward guidance. As explained in Chapter 3, the expectations on the aggregate demand is formed using both the private signals and the public signal about the aggregate technology shock.

\[
E_i n = \rho_m m + \rho_a a_i \tag{C.5.1}
\]

where \( \rho_m = \gamma'' \frac{k_m}{k_m + k_t + k_a} + \gamma_i \frac{k_t + k_a}{k_m + k_t + k_a}, \rho_a = (\gamma'' - \gamma_i) \frac{k_a}{k_m + k_t + k_a} \)

To solve for the higher order beliefs, first integrate individual expectations over \( i \), and the apply \( \tilde{E}_i \):

\[
\int_0^i E_i n = \bar{E}n = \rho_m m + \rho_a \bar{a} \tag{C.5.2}
\]

\[
E_i \tilde{E}n = \rho_m m + \rho_a E_i \bar{a} \tag{C.5.3}
\]

\[
\tilde{E}^2 n = \rho_m m + \rho_a \tilde{E} \bar{a} \tag{C.5.4}
\]

Iterating this process yields:

\[
E_i \tilde{E}^j n = \rho_m m + \rho_a E_i \tilde{E}^{j-1} \bar{a} \tag{C.5.5}
\]

Re-write equation C.1.7 as:

\[
p_i = \alpha \left[ \sum_{j=1}^\infty (1 - \alpha)^j E_i \tilde{E}^j n + E_i n \right] - \beta \sum_{j=1}^\infty (1 - \alpha)^j E_i \tilde{E}^{j-1} \bar{a} - \beta a_i \tag{C.5.6}
\]

169
The second term, \(-\beta \sum_{j=1}^{\infty} (1 - \alpha)^j E_i \bar{E}^{j-1} \bar{a} - \beta a_i\) has already been solved in the case of forward guidance with commitment, so that we only need to work on the first term here. Substitute \(E_i \bar{E}^{j} n\) and get

\[
\alpha \left[ \sum_{j=1}^{\infty} (1 - \alpha)^j E_i \bar{E}^{j} n + E_i n \right] = \alpha \left\{ \sum_{j=1}^{\infty} (1 - \alpha)^j \left[ \rho_m m + \rho_a E_i \bar{E}^{j-1} \bar{a} \right] + \rho_m m + \rho_a a_i \right\} \quad (C.5.7)
\]

Substitute the expression for \(E_i \bar{E}^{j-1} \bar{a}\) as in the case of forward guidance with commitment,

\[
\alpha \left[ \sum_{j=1}^{\infty} (1 - \alpha)^j \rho_m m + \rho_a \sum_{j=1}^{\infty} (1 - \alpha)^j E_i \bar{E}^{j-1} \bar{a} + \rho_a a_i \right] = \alpha \left\{ \frac{\rho_m}{\alpha} + \frac{1}{\alpha} \sum_{j=1}^{\infty} \left( \frac{\kappa_m}{K} \right)^{j-1} \left( \frac{\kappa_s}{K} \right)^{j-1} m + \frac{(1 - \alpha) \kappa_s}{\kappa_m + \kappa_a + \kappa_m} a_i + \rho_a a_i \right\} \quad (C.5.8)
\]

Substitute this as the first term in equation C.5.6

\[
p_i = m \left\{ \rho_m + \left[ \rho_a (1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right] \frac{\kappa_m}{\alpha \kappa_s + \kappa_m} \right\} + a_i \left\{ [\alpha \rho_a - \beta] \frac{(1 - \alpha) \kappa_s}{\kappa_m + \kappa_a + \kappa_s} + \rho_a a_i - \beta \right\} \quad (C.5.9)
\]

Aggregate over \(i\) to get the aggregate price level. Then, take the difference between the aggregate nominal demand and the price level to get the real output.

\[
p = \phi_m m + \phi_a \bar{a} = (\phi_m + \phi_a) \bar{a} + \phi_m \nu \quad (C.5.10)
\]

\[
y = n - p = (\gamma_a^c - \phi_m - \phi_a) \bar{a} + (\gamma_u^c - \phi_m) \nu \quad (C.5.11)
\]
where

\[ \phi_m = \rho_m + \left( \rho_a (1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right) \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} \quad (C.5.12) \]

\[ \phi_a = \left[ \alpha \rho_a - \beta \right] \frac{(1 - \alpha) \kappa_s}{\kappa_m + \kappa_s + \alpha \kappa_s} + \alpha \rho_a - \beta \quad (C.5.13) \]

\[ \rho_m = \gamma^\rho_{fnc} \frac{\kappa_m}{\kappa_m + \kappa_s + \kappa_a} + \gamma^\rho_{fnc} \frac{\kappa_s + \kappa_a}{\kappa_m + \kappa_s + \kappa_a} \quad (C.5.14) \]

\[ \rho_a = \left( \gamma^\rho_{fnc} - \gamma^\rho_{fnc} \right) \frac{\kappa_s}{\kappa_m + \kappa_s + \kappa_a} \quad (C.5.15) \]

C.5.1 Output Gap Stabilization Policy

This section derives the necessary conditions for the central bank to close the output gap. This can be used for two purpose: (1) to illustrate the time inconsistency which is involved with Odyssean forward guidance, and (2) as the first step in solving the Delphic forward guidance with backward induction.

As shown in the previous derivations, the output gap is contingent on two aggregate state variables. To close the output gap is equivalent to make the sensitivity of the aggregate output to both shocks to be zero.

\[ \phi_m = \rho_m + \left( \rho_a (1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right) \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} = \gamma^\rho_{fnc} \quad (C.5.16) \]

\[ \phi_a = \gamma^\rho_{fnc} - \gamma^\rho_{fnc} - \frac{\beta}{\alpha} \equiv \gamma^\rho_{fnc} - \frac{\beta}{\alpha} \quad (C.5.17) \]

Substitute the above equations to the expression of \( \phi_m \) and \( \phi_a \), and solve for \( \rho_m \) and \( \rho_a \) as a function of \( \gamma^\rho_{fnc} \) and \( \gamma^\rho_{fnc} \):
\[
\begin{align*}
\gamma_{fnc} - \gamma_{fnc}^\rho - \frac{\beta}{\alpha} &= (\alpha \rho_a - \beta) \left( \frac{1 - \alpha}{\kappa_m + \kappa_a + \alpha \kappa_s} + 1 \right) \quad (C.5.18) \\
\gamma_{fnc}^\rho &= \rho_m + \left[ \rho_a (1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right] \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} \quad (C.5.19)
\end{align*}
\]

This result leads to the solution of \( \rho_m \) as a function of \( \rho_a \):

\[
\rho_m = \gamma_{fnc}^\rho - \left[ \rho_a (1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right] \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} \quad (C.5.20)
\]

Use this result to solve the expression of \( \rho_a \)

\[
\rho_a = \frac{\left( \gamma_{fnc}^\rho - \gamma_{fnc}^\rho \right) \alpha \kappa_s + \kappa_a + \kappa_m}{\kappa_s K} + \beta \quad (C.5.21)
\]

Use this result to solve for \( \gamma_{fnc}^\rho - \gamma_{fnc}^\rho \) by equating it with the definition of \( \rho_a \):

\[
\left( \gamma_{fnc}^\rho - \gamma_{fnc}^\rho \right) \frac{\kappa_s}{K} = \left( \gamma_{fnc}^\rho - \gamma_{fnc}^\rho \right) \frac{\alpha \kappa_s + \kappa_a + \kappa_m}{\kappa_s K} + \beta \quad (C.5.22)
\]

\[
\left( \gamma_{fnc}^\rho - \gamma_{fnc}^\rho \right) = \frac{\beta}{\alpha} (1 - \alpha) \quad (C.5.23)
\]

Plug the expression of \( \left( \gamma_{fnc}^\rho - \gamma_{fnc}^\rho \right) \) in the above equation to \( \rho_a \) to solve for \( \rho_a \)

\[
\rho_a = \frac{\beta}{\alpha} (1 - \alpha) - \frac{\beta}{\alpha} \frac{\alpha \kappa_s + \kappa_a + \kappa_m}{\kappa_s K} + \beta = \frac{1 - \alpha \kappa_s}{\alpha \kappa_s K} \quad (C.5.24)
\]

Plug this to equation C.5.20 to solve for \( \rho_m \), which we use to equate with the definition of \( \rho_m \) in equation C.5.14
\[ \rho_m = \gamma_a \frac{K_m}{K} + \gamma^\nu \frac{K_a + K_s}{K} = \gamma^\nu - \left[ \beta \frac{1 - \alpha}{\alpha} \frac{K_s}{K} (1 - \alpha) - \beta \frac{1 - \alpha}{\alpha} \right] \frac{K_m}{\alpha (K_a + K_s + K_m)} \quad (C.5.25) \]

which results in

\[ \gamma^\nu - \gamma^\nu = \beta \frac{1 - \alpha}{\alpha} \quad (C.5.26) \]

**C.5.2 Proof of Lemma 5**

This section prove Lemma 5 by calculating the sensitivity of the aggregate price level under re-optimization to (1) the aggregate technology shock and (2) the noise shock, and comparing them with the ones without re-optimization.

\[ \frac{\partial p}{\partial \nu} = \gamma^\nu \frac{K_m}{K} + \gamma^\nu \frac{K_a + K_s}{K} + \frac{(1 - \alpha) K_s}{K} (\gamma^\mu - \gamma^\nu) \frac{K_m}{K} - \beta \frac{1 - \alpha}{\alpha} \frac{K_m}{K'} \quad (C.5.27) \]

\[ \frac{\partial p}{\partial \nu} = \gamma^\mu - \beta \frac{1 - \alpha}{\alpha} \frac{K_m}{K'} \quad (C.5.28) \]

So the difference:

\[ \frac{\partial p}{\partial \nu} |_{\nu c} - \frac{\partial p}{\partial \nu} |_{c} = (\gamma^\mu - \gamma^\nu) \left\{ \frac{K_m}{K} (1 - \alpha) \frac{K_s}{K} - \frac{K_a + K_s}{K} \right\} \quad (C.5.29) \]

This is less than zero, because

\[ \frac{K_m}{K} (1 - \alpha) \frac{K_s}{K} < \frac{(1 - \alpha) K_s}{K} < \frac{K_s}{K} < \frac{K_a + K_s}{K} \quad (C.5.30) \]
The sensitivity of price to technology shocks

\[
\frac{\partial p}{\partial a} = \left( \gamma^a \frac{\kappa_m}{K} + \gamma^v \frac{\kappa_s + \kappa_a}{K} \right) + (\gamma^a - \gamma^v) \frac{\kappa_s}{K} \left( \frac{(1 - \alpha)\kappa_m}{K'} + \alpha \frac{K}{K'} \right) - \left( \beta \frac{1 - \alpha}{\alpha} \frac{\kappa_m}{K'} + \beta \frac{K}{K'} \right)
\]

(C.5.31)

\[
\frac{\partial p}{\partial a} = \gamma^a - \beta \frac{1 - \alpha}{\alpha} \frac{\kappa_m}{K'} - \beta (1 - \alpha) \frac{\kappa_s}{K'} - \beta
\]

(C.5.32)

So the difference is

\[
\frac{\partial p}{\partial a} \bigg|_{nc} - \frac{\partial p}{\partial a} \bigg|_c = (\gamma^a - \gamma^v) \left\{ \frac{\kappa_s}{K} \left( \alpha \kappa_a + \alpha_s + \kappa_m \right) - \frac{\kappa_m + \kappa_s}{K} \right\}
\]

(C.5.33)

which is less than zero.

C.6 Rule-based Odyssean Forward Guidance

In this section, I solve for the optimal state-contingent policy rule which minimizes the weighted sum of the variance of the output gap and the variance of inflation. The objective function of the central bank is written as:

\[
(\gamma^a - \phi_m - \phi_a)^2 \sigma_a^2 + (\gamma^v - \phi_m)^2 \sigma_v^2 + \tau \left[ (\phi_m + \phi_a)^2 \sigma_a^2 + \phi_m^2 \sigma_v^2 \right]
\]

The first order condition on \( \gamma^a \) results in:

\[
\sigma_a^2 \left[ (\gamma^a - \phi_m - \phi_a) \frac{d (\gamma^a - \phi_m - \phi_a)}{d \gamma^a} + \tau (\phi_m + \phi_a) \frac{d (\phi_m + \phi_a)}{d \gamma_{fnc}} \right] + \sigma_v^2 \left[ (\gamma^v - \phi_m) \frac{d (\gamma^v - \phi_m)}{d \gamma^a} + \tau \phi_m \frac{d \phi_m}{d \gamma_{fnc}} \right] = 0
\]

(C.6.1)
Re-arrange the above equation to get:

\[
\sigma_a^2 \left\{ (\gamma^a - \phi_m - \phi_a) + (-\gamma^a + (1 + \tau)(\phi_m + \phi_a)) \frac{d(\phi_m + \phi_a)}{d\gamma^a} \right\} \quad \text{(C.6.2)}
\]

\[
+ \sigma_v^2 \left\{ (\gamma^v - \phi_m) + (-\gamma^v + (1 + \tau)\phi_m) \frac{d\phi_m}{d\gamma^v} \right\} = 0 \quad \text{(C.6.3)}
\]

where

\[
\frac{d(\phi_m + \phi_a)}{d\gamma^a} = \frac{\partial \phi_m}{\partial \rho_m} \frac{\partial \rho_m}{\partial \gamma^a} + \frac{\partial \phi_m}{\partial \rho_a} \frac{\partial \rho_a}{\partial \gamma^a} + \frac{\partial \phi_a}{\partial \rho_m} \frac{\partial \rho_m}{\partial \gamma^a} + \frac{\partial \phi_a}{\partial \rho_a} \frac{\partial \rho_a}{\partial \gamma^a} \quad \text{(C.6.4)}
\]

\[
\frac{d\phi_m}{d\gamma^a} = \frac{\partial \phi_m}{\partial \rho_m} \frac{\partial \rho_m}{\partial \gamma^a} + \frac{\partial \phi_m}{\partial \rho_a} \frac{\partial \rho_a}{\partial \gamma^a} \quad \text{(C.6.5)}
\]

where the partial derivatives are calculated are:

\[
\frac{\partial \phi_m}{\partial \rho_m} = 1 \quad \text{(C.6.6)}
\]

\[
\frac{\partial \phi_a}{\partial \rho_m} = 0 \quad \text{(C.6.7)}
\]

\[
\frac{\partial \phi_m}{\partial \rho_a} = (1 - \alpha) \frac{\kappa_m}{\alpha \kappa_s + \kappa_a + \kappa_m} \quad \text{(C.6.8)}
\]

\[
\frac{\partial \phi_a}{\partial \rho_a} = \frac{\alpha (1 - \alpha) \kappa_s}{\kappa_m + \kappa_a + \alpha \kappa_s} \quad \text{(C.6.9)}
\]

\[
\frac{\partial \rho_m}{\partial \gamma^a} = \frac{\kappa_m}{K} \quad \text{(C.6.10)}
\]

\[
\frac{\partial \rho_a}{\partial \gamma^a} = \frac{\kappa_s}{K} \quad \text{(C.6.11)}
\]

\[
\frac{\partial \rho_m}{\partial \gamma^v} = \frac{\kappa_m + \kappa_a}{K} \quad \text{(C.6.12)}
\]

\[
\frac{\partial \rho_a}{\partial \gamma^v} = \frac{\kappa_s}{K} \quad \text{(C.6.13)}
\]