Toroidal algebra representations and equivariant elliptic surfaces

Samuel DeHority

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2024

Abstract

Toroidal algebra representations and equivariant elliptic surfaces

Samuel DeHority

We study the equivariant cohomology of moduli spaces of objects in the derived category of elliptic surfaces in order to find new examples of infinite dimensional quantum integrable systems and their geometric representation theoretic interpretation in enumerative geometry. This problem is related to a program to understand the cohomological and K-theoretic Hall algebras of holomorphic symplectic surfaces and to understand how it related to the Donaldson-Thomas theory of threefolds fibered in those surfaces.

We use the theory of noncommutative deformations of Poisson surfaces and especially van den Berg's noncommutative $\mathbb{P}^1$ bundles as well as Rains's analysis of moduli theory for quasi-ruled noncommutative surfaces as well as the theory of Bridgeland stability conditions and their relative versions to understand equivariant deformations and birational transformations of Hilbert schemes of points on equivariant elliptic surfaces. The moduli spaces are closely related to elliptic versions of classical integrable systems. We also use these moduli spaces to construct vertex algebra representations of extensions of toroidal extended affine algebras on their equivariant cohomology, building on work of Eswara-Rao–Moody–Yokonuma, of Billig, and of Chen–Li–Tan on vertex representations of toroidal algebras, full toroidal algebras, and toroidal extended affine algebras.

Using Fourier-Mukai transforms and their relative action on families of dg-categories we study the relationship between automorphisms of toroidal extended affine algebras and families of

derived equivalences on dg categories, in particular finding a relativistic (difference) generalization of the Laumon-Rothstein deformation of the Fourier-Mukai duality.
Finally, using the above analysis we extend the construction of Maulik–Okounkov's stable envelopes to moduli of framed torsionfree sheaves on noncommutative surfaces in some cases and use this to study coproducts on associated algebras assigned to elliptic surfaces with applications to understanding new representation theoretic structures in the Donaldson-Thomas theory of local curves.

Table of Contents

List of Figures

List of Tables

Acknowledgements

This work owes a tremendous amount to a great number of people. First and foremost, I would like to thank my advisor Andrei Okounkov without whose constant support, guidance, and patience this would have been impossible. The ideas in this document and my thinking about the subject also owe a great deal to Olivier Schiffmann. I must also acknowledge the crucial input of many others who have taught me mathematics and made my experience in graduate school so memorable and enjoyable.

Chapter 1: Introduction

This thesis is concerned with the geometric representation theory of moduli spaces of objects in the derived category of certain elliptic surfaces admitting actions of an algebraic torus.

The main results are some beginning steps to extending to a new class of examples the narrative which, to certain resolution of conical symplectic singularities, the canonical prototype being Nakajima quiver varieties, assigns 1) algebras which act on the equivariant cohomology or K-theory of the resolution 2) representations of the same algebras on vector spaces underlying equivariant enumerative theories of the resolutions and 3) systems of compatible difference and differential equations.

The specific elliptic surfaces X_R for R a root system in

$$\{A_{-1}, A_0, A_1, A_2, D_4, E_6, E_7, E_8\}$$

have features which make them particularly amenable to analysis in analogy with quiver varieties. They are the natural affine analogue of ADE surfaces, which are themselves conical symplectic resolutions. They also belong to the class of moduli spaces of Higgs bundles and thus have geometric (the Hitchin fibration, nonabelian Hodge theory) and algebraic (Langlands duality) properties constraining and aiding in the analysis of their enumerative geometry.

1.1 Vertex algebras and toroidal algebras

The central algebraic tool for our analysis is the theory of vertex operator algebras. Our narrative centers around vertex algebras and geometric interpretations of their Fourier coefficients. In particular, this work is based on [1] in which a vertex algebra V_{T^*E} was shown to admit a representation of an algebra $\mathfrak{g}_{T^*E}$ which is a supersymmetric extension of the algebra of Hamiltonian vector fields on $\mathbb{C}^{*2}$. There are also close relationships between the geometric representation theory of the surfaces X_R and a family of vertex algebras arising in the 4d/2d duality story in the physics literature. Chapter 2 explains the relevant results from [1] and also carries the narrative slightly further by relating them to the vertex algebras in 4d/2d duality, in particular the small $\mathcal{N} = 4$ superconformal vertex algebra.

This is a promising sign that many of the results of the present work admit generalizations in some form to a broader class of moduli spaces of Higgs bundles, because while the surfaces X_R are quite special and isolated examples, these other vertex algebras form a much broader class.

1.2 Moduli spaces

In [2] families of moduli spaces of objects in dg categories, interpreted as noncommutative ruled surface, were studied in order to find a geometric description of the actions constructed in [1] at the level of vector spaces and algebras, but not cohomology theories. The theory of Bridgeland stability conditions and its relative analogue are used to understand the wall-crossing relevant to the geometric construction of algebra representations on cohomology.

1.3 Stable envelopes and relativistic deformations

Chapter 5 contains the largest technical advances in the present work. The analysis of [2] and in Chapter 4 is carried further, as the family of noncommutative surfaces infinitesimally close to commutative ones are deformed to noncommutative surfaces related to elliptic difference equations.

We describe a family of noncommutative surfaces over a base of the form $E \times E$ where E is an elliptic curve, where moduli spaces of torsionfree sheaves are related to relativistic elliptic integrable systems and also to the geometric representations in cohomology which form the main subject of the paper. We explain the dualities of the dg categories and moduli of torsionfree sheaves covering an action on $\mathrm{GL}(2, \mathbb{Z})$ on $E \times E$.

While the elliptic curve serving as the analogue of the target of the moment map parameter of a Hamiltonian reduction seems to be a new feature of geometric representation theory in holomorphic symplectic geometry, the very same deformation is the necessary ingredient to understand the generalization of stable envelopes the moduli spaces of torsion free sheaves on elliptic surfaces. The stable envelopes are described and they are related to algebra modules relevant to the enumerative geometry of local elliptic curve Calabi-Yau threefolds.

Chapter 2: Vertex algebras

Vertex algebras were introduced in the mathematics literature by Borcherds [3] to formalize algebraic structures arising in two-dimensional quantum field theories with applications to infinite-dimensional Lie algebras. They have since served an indispensable role in the study of the representation theory of infinite dimensional lie algebras and their applications to the enumerative geometry of moduli spaces of sheaves on surfaces and of threefolds.

2.1 Definitions

Given a ring R, the space $R[[z^{\pm}]]$ of formal power series in z and z^{-1} is not a ring, but it has subspaces $R((z)), R[[z]], R[[z^{-1}]]$ of Laurent series, power series in z and those in z^{-1} which are rings.

Basic references for the theory of vertex algebras are [4, 5] which we follow for the definition of vertex algebras.

Let V be a vector space. An element

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(V)[[z^{\pm}]]$$

is called a *field* if for any $v \in V$, the element

$$a(z)v = \sum_{n\in\mathbb{Z}} a_n(v) z^{-n-1}$$

lies in $\mathrm{End}(V)((z))$.

Two fields $a(z), b(w)$ are *weakly local* if

$$\mathrm{Res}_z (z-w)^N [a(z)b(w)] = 0$$

for $N \gg 0$ and are *local* with the stronger condition

$$(z-w)^N [a(z)b(w)] = 0$$

for $N \gg 0$. We use the weak locality condition because we will be interested in fields under the transformation $z \mapsto e^z - 1$ which does not preserve locality. Decompose $a(z) = a_+(z) + a_-(z)$ in the unique ∂_z invariant way, so that $a_+(z) = \sum_{n<0} a_n z^{-n-1}$ and $a_-(z) = \sum_{n\geq 0} a_n z^{-n-1}$. Given two local fields $a(z), b(z)$ their normally ordered product is defined as

$$: a(z)b(z) := a_+(z)b(z) + b(z)a_-(z)$$

and Dong's lemma says that $: ab : (z)$ is local with $a(z)$ and $b(z)$.

The space of all fields on V in the variable z will be denoted $\mathfrak{glf}(V; z)$.

Definition 2.1.1. A *vertex algebra* is

- The *space of states* a vector space V.
- The *vacuum vector* $|0\rangle \in V$.
- The *translation operator* $T \in \mathrm{End}(V)$.
- A linear map

$$Y(-, z) : V \longrightarrow \mathfrak{glf}(V; z)$$

satisfying

- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{Id}_V$. Also for any $a \in V$ we have $Y(a, z)|0\rangle \in V[[z]]$ with constant term a.

- (translation axiom) The commutator $[T, Y(a,z)] = \partial_z Y(a,z)$ and $T|0\rangle = |0\rangle$.

- (locality axiom) The fields $Y(a,z)$ for $a \in V$ are all mutually local.

Vertex algebra modules of a vertex algebra V are similarly defined as vector spaces M with $\mathrm{End}(M)$ valued fields

$$Y_M(a,z) \in \mathrm{End}(M)[[z^{\pm}]]$$

for any $a \in V$, satisfying appropriate axioms.

The formal delta function $\delta(z-w) \in \mathbb{C}[[z^{\pm}, w^{\pm}]]$ defined by $\delta(z-w) = \sum_{k\in\mathbb{Z}} w^n z^{-1-n}$ conveniently expresses commutators of the modes (i.e. coefficients) of fields because if fields $a(z), b(z)$ are weakly local, then

$$[a(z)b(w)] = \sum_{j=0}^{n} c_{j+1}(w)\partial_w^{(j)}\delta(z-w)$$

for some fields $c_j(w)$ where $\partial_w^{(j)} = \frac{1}{j!}\partial_w^j$. The same data is expressed in the singular part of the Operator Product Expansion

$$a(z)b(w) \sim \sum_{j=0}^{n} \frac{c_j(w)}{(z-w)^{j+1}}$$

where $\frac{1}{z-w}$ is expanded in the domain $|z| < |w|$. In this case we denote the field $c_j(w)$ by $a_j b(w)$.

If the fields $a(z), b(w)$ are vertex operators $Y(a,z), Y(b,w)$ then $c_j(w) = Y(a_j b, w)$ motivating the previous terminology. For $j = 0$ we define $a_0 b(z) =: ab : (z)$ and more generally $a_{-n}b(z) =: \partial_z^{(n)} ab : (z)$. The structure theory of vertex algebras then gives the following formulas for their modes

$$[a_n, b_m] = \sum_{j\geq 0} \binom{m}{j} (a_j b)_{m+n-j} \tag{2.1.2}$$

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{n+j}b)_{m+k-j}c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{m+n-j}(b_{k+j}c) - \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{n+k-j}(a_{m+j}c) \tag{2.1.3}$$

the latter identity, called Borcherds identity, holding for any $k, m, n \in \mathbb{Z}$.

Two important special cases are the noncommutative Wick formula

$$a_m : bc : (z) =: a_m b(z)c(z) : + : b(z)a_m c(z) : + \sum_{j=0}^{m-1} \binom{m}{j} (a_j b)_{m-1-j} c(z)$$

which allows the calculation of OPEs of fields built from others and the quasiassociativity of the normal ordered product

$$: a : bc :: (z) =:: ab : c : (z) + \sum_{j \geq 0} a_{-j-2}(b_j c)(z) + b_{-j-2}(a_j c)(z).$$

Definition 2.1.4. A vertex algebra V is said to be *generated* by a collection $\{a^i \mid i \in I\}$ if V is spanned words built from a_i under all products $a_i b$, $i \in \mathbb{Z}$. The set $\{a^i \mid i \in I\}$ *strongly* generates V if V is spanned by words formed with only the negative products $i \in \mathbb{Z}_{\leq 0}$ up to a finite dimensional space of central elements.

If V is strongly generated by $\{a_i \mid i \in I\}$ then V is spanned by the elements $\{a^{i_1}_{-k_1} \cdots a^{i_n}_{-k_n}|0\rangle \mid k_i \geq 0\}$ as a vector space up to a finite dimensional space of central elements.

We will often say that V is generated or strongly generated by the fields $Y(a^i, z)$ if $\{a^i \mid i \in I\}$ generates or strongly generates V.

All of these definitions have extensions to where the underlying vector space V is a superspace $V = V^{\underline{0}} \oplus V^{\underline{1}}$ and one my define a vertex operator superalgebra. We refer to [4] for the required sign modification on the formulas in the preceding.

2.2 Library of examples

In this section we list a number of examples which will be useful for the remainder of the text.

Example 2.2.1. (Free super bosons) Let $(\mathbb{H}, \langle -, - \rangle)$ denote a super space over the field $\mathbb{K}$ with a supersymmetric bilinear form. The Heisenberg-Clifford vertex algebra is the vector space

$$V_{\mathbb{H}} = \mathrm{Sym}^{\bullet}(\mathbb{H}[t^{-1}]) \oplus \mathbb{K}\mathbf{c}.$$

Given $\gamma \in \mathbb{H}$ the fields

$$\alpha(\gamma, z) := Y(\gamma t^{-1}, z) = \sum_{n\in\mathbb{Z}} \alpha_n(\gamma) z^{-n-1} \tag{2.2.2}$$

strongly generate $V_{\underline{H}}$ and satisfy the OPE

$$\alpha(\gamma, z)\alpha(\eta, w) \sim \frac{\langle\gamma, \eta\rangle\mathbf{c}}{(z-w)^2}. \tag{2.2.3}$$

Given a linear functional $\zeta \in \mathbb{H}^\vee$ there is a module called the Fock module

$$\mathcal{F}_{k,\mathbb{H}}(\zeta) \simeq \mathrm{Sym}^\bullet(t^{-1}\mathbb{H}[t^{-1}])$$

with vacuum element $|0\rangle_{k,\zeta} \in \mathcal{F}_{k,\mathbb{H}}(\zeta)$ where

$$\alpha_0(\gamma)|0\rangle_{k,\zeta} = \zeta(\gamma)|0\rangle_{k,\zeta}, \;\; c|0\rangle_{k,\zeta} = k|0\rangle_{k,\zeta}. \tag{2.2.4}$$

Example 2.2.5. (Lattice vertex superalgebras) Let $(\Lambda, \langle, \rangle)$ be an integral lattice with group algebra $\Bbbk[\Lambda]$ spanned by e^λ for $\lambda \in \Lambda$. Let $\mathbb{H} = \Bbbk \otimes \Lambda$ with the induced pairing. The *lattice vertex superalgebra*

$$V_\Lambda := \bigoplus_{\lambda\in\Lambda} \mathcal{F}_{k,\mathbb{H}}(\zeta_\lambda) \otimes e^\lambda \simeq \mathrm{Sym}^\bullet(t^{-1}\mathbb{H}[t^{-1}]) \otimes \mathbb{C}[\Lambda] \tag{2.2.6}$$

contains a copy of the Fock module of the free boson vertex subalgebra for each element of the lattice Λ. The vertex algebra structure depends mildly on an auxiliary cocycle

$$\epsilon : \Lambda \otimes \Lambda \to \mathbb{Z}/2\mathbb{Z}$$

(c.f. [4]) which we omit from the notation. In addition to the fields (2.2.2) there are fields

$$\Gamma_\lambda(z) := Y(1 \otimes e^\lambda, z)$$

with OPE

$$\alpha(\gamma, z)\Gamma_\lambda(w) \sim \frac{(\lambda, \gamma)\Gamma_\lambda(w)}{z - w}. \tag{2.2.7}$$

Example 2.2.8. (Cone vertex algebras) Given a lattice vertex algebra V_Λ and a cone $C \subset \Lambda$ of the lattice (i.e. a saturated submonoid) for example, of a sublattice, then

$$V_{C\subset\Lambda} = \bigoplus_{\lambda\in C} \mathcal{F}_{k,\mathbb{H}}(\zeta_\lambda) \otimes e^\lambda$$

is a vertex subalgebra of V_Λ. These have been studied in relation to mock modular forms [6] and will arise in Chapter 5 in relation to Donaldson-Thomas theory of local surfaces.

Example 2.2.9. (Affine vertex algebra) For a semisimple finite dimensional Lie algebra $\mathfrak{g}$ with invariant form $(-,-)$ the *affine vertex algebra* $V(\mathfrak{g})_k$ at level k is strongly generated by fields $J_x(z)$ for $x \in \mathfrak{g}$. The OPEs are given by

$$J_x(z)J_y(w) \sim \frac{J_{[x,y]}(w)}{z - w} + \frac{k(x, y)}{(z - w)^2}.$$

Example 2.2.10. ((Super)conformal vertex algebras) The Virasoro vertex algebra is strongly generated by a single field

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfying the OPE

$$T(z)T(w) \sim \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial_w T(w)}{z - w}$$

where c is a central element. The modes of $T(z)$ satisfy the commutation relations of the Virasoro algebra which is a central extension of the algebra of polynomial vector fields on $\mathbb{C}^*$.

A conformal structure on a vertex algebra V is an element $\omega \in V$ so that $Y(\omega, z)$ generates a virasoro vertex subalgebra. Then $Y(\omega, z)$ is called the stress tensor.

For example, the free super bosons have a family of conformal structures given by

$$\frac{1}{2}\sum \alpha_{-1}(\gamma_i)\alpha_{-1}(\gamma^i) - \alpha_{-2}(\eta)$$

for $\{\gamma_i\}\{\gamma^i\}$ dual bases of $\mathbb{H}$ and $\eta \in \mathbb{H}$ some element. The central charge is

$$c = \dim(\mathbb{H}) - 12\langle \eta, \eta \rangle k.$$

There are superalgebras containing the algebra of modes of $T(z)$ as finite index subalgebra corresponding to central extensions of algebras of infinitesimal superconformal transformations on supercurves. Among these is the $\mathcal{N} = 4$ small superconformal vertex algebra $V^{\mathcal{N}=4}$ which is strongly generated by fields

$$T(z), \widetilde{G}^{+}(z), \widetilde{G}^{-}(z), G^{+}(z), G^{-}(z), J^{+}(z), J^{0}(z), J^{-}(z)$$

subject to the nonvanishing OPEs

$$\begin{aligned}
J^0(z)J^{\pm}(w) &\sim \frac{\pm 2J^{\pm}(w)}{z-w} & J^0(z)J^0(w) &\sim \frac{c/3}{(z-w)^2}\\
J^{+}(z)J^{-}(w) &\sim \frac{J^0(w)}{z-w} + \frac{c/6}{(z-w)^2} & J^0(z)G^{\pm}(w) &\sim \frac{\pm G^{\pm}(w)}{(z-w)}\\
J^0(z)G^{\pm}(w) &\sim \frac{\pm \widetilde{G}^{\pm}(w)}{(z-w)} & J^{+}(z)G^{-}(w) &\sim \frac{G^{+}(w)}{z-w}\\
J^{-}(z)G^{+}(w) &\sim \frac{G^{-}(w)}{z-w} & J^{+}(z)\widetilde{G}^{-}(w) &\sim \frac{-\widetilde{G}^{+}(w)}{z-w}\\
J^{-}(z)\widetilde{G}^{+}(w) &\sim \frac{-\widetilde{G}^{-}(w)}{z-w} & G^{\pm}(z)\widetilde{G}^{\pm}(w) &\sim \frac{2J^{\pm}(w)}{(z-w)^2} + \frac{\partial J^{\pm}(w)}{z-w}
\end{aligned}$$

$$G^{\pm}(z)\widetilde{G}^{\mp}(w) \sim \frac{T(w) + \partial J^0(w)/2}{z-w} \pm \frac{J^0(w)}{(z-w)^2} + \frac{c/6}{(z-w)^3}$$

in addition to those of the Virasoro vertex algebra.

There are is another $\mathcal{N} = 4$ extension of the Virasoro algebra where the $J^0, J^{\pm}$ currents, which form a an affine $\mathfrak{sl}_2$ algebra, are replaced by a pair of affine $\mathfrak{sl}_2$ currents. The latter version is referred to as the large $\mathcal{N} = 4$ superconformal algebra.

Example 2.2.11. (Symplectic fermion vertex algebra) This example is a special case of Example 2.2.1 where

$$\mathbb{H} \simeq H^{\underline{1}}(E)$$

for E an elliptic curve with the Poincaré pairing. It is referred to by $\mathcal{SF}$ or the *symplectic fermion* vertex algebra [7]. In simple terms, $\mathbb{H}$ has basis σ_+, σ_- of odd vectors with $\langle \sigma_+, \sigma_- \rangle = -\langle \sigma_-, \sigma_+ \rangle = 1$. This vertex algebra is distinguished by the fact that it is among the simplest examples of vertex algebras whose representation category is not semisimple. Its study is facilitated by the following example.

Example 2.2.12. (bc system) The bc-system is a vertex algebra V_{bc} strongly generated by odd fields $b(z), c(z)$ with central element k subject to the OPE

$$b(z)c(w) \sim \frac{k}{z-w}.$$

There is a family of conformal structure on V_{bc} given by

$$T^{\lambda}(z) = (1-\lambda) : \partial c(z) b(z) : +\lambda : \partial b(z) c(z) :$$

with central charge

$$-12\lambda^2 + 12\lambda - 2.$$

Three cases are singled out:

1. When $\lambda = 0$ one gets $c = -2$. Then $T^0(z)$ agrees with the stress tensor of the symplectic fermion model (c.f. Example 2.2.11)generated by the fields $\{\partial c, b\}$ which is a vertex subalgebra of V_{bc}.

2. When $\lambda = 1$ one again finds $c = -2$. Then $T^1(z)$ is the stress tensor of a different symplectic fermion vertex subalgebra generated by fields $\{b, \partial c\}$.

3. At the midpoint of the two previous examples gives a stress tensor $T^{1/2}(z)$ with $c = 1$. In this case V_{bc} is usually referred to as the *free charged fermion* vertex algebra. It is isomorphic to the lattice vertex algebra (Example 2.2.5) with the lattice $\mathbb{Z} \subset \mathbb{R}$ with Gram matrix (1) via the boson-fermion correspondence with the stress tensor on the lattice VOA induced from its underlying free bosonic vertex algebra generated by the field

$$J_{bc}(z) =: c(z)b(z) : . \tag{2.2.13}$$

Example 2.2.14. (Weyl vertex algebra, symplectic bosons, or $\beta\gamma$-system) The Weyl vertex algebra is the vertex algebra strongly generated by two bosonic fields $\beta(z), \gamma(z)$ subject to the OPE

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}.$$

2.2.1 Logarithmic fields

The mathematical formulation of vertex algebras requires that OPEs be expressed using rational functions, but based on the physical context of *logarithmic conformal field theory*, (c.f. [8] for an excellent introduction) there exist conformal field theories with weakly mutually local fields exhibiting logarithmic divergences of correlation functions. A characteristic feature of their study is the non-semisimplicity of the representation category of the resulting algebra. One formalization of the notion of a logarithmic vertex operator is given in [9]. We will instead content ourselves with an analysis of OPEs and fields with logarithmic terms, which have already appeared in a hidden way in Example 2.2.5.

Example 2.2.15. (Logarithmic extension of free boson, real logarithmic modes) Let $\mathbb{H}$ be purely even and let $\gamma \in \mathbb{H}$ be an element. Consider the indefinite integral

$$\phi(\gamma, z) = \int dz\alpha(\gamma, z) = \alpha_{\log}(\gamma) + \alpha_0(\gamma) \log z + \sum_{n\neq 0} \alpha_n(\gamma)\frac{z^{-n}}{-n}. \tag{2.2.16}$$

The logarithmic mode $\alpha_{\log}(\gamma)$ satisfies the commutation relation

$$[\alpha_{\log}(\gamma), \alpha_k(\eta)] = \langle \gamma, \eta \rangle \delta_{n,0}$$

and acts by infinitesimal translation on the space of vacua in (2.2.4), so that

$$\exp(\alpha_{\log}(\gamma))|0\rangle_{1,\zeta} = |0\rangle_{1,\zeta'}$$

where $\zeta' = \zeta + \langle \gamma, - \rangle$. See also [10, §13.1.3].

Example 2.2.17. (Logarithmic extension of bc system, fermionic logarithmic modes) Starting from the symplectic fermions generated by fields

$$\{b, \partial c\}$$

we write the field c as already being an indefinite integral

$$c(z) = \int dz \partial c(z) = \theta^c + c_0 \log(z) + \sum_{n \neq 0} c_n \frac{z^{-n}}{-n}.$$

Then the field

$$\phi_b(z) := \int dz b(z) = \theta^b + b_0 \log(z) + \sum_{n \neq 0} b_n \frac{z^{-n}}{-n}$$

is such that the zero and logarithmic modes satisfy the commutation relations

$$[\theta^c, b_0] = 1, \qquad [\theta^b, c_0] = 1, \qquad [\theta^c, c_0] = [\theta^b, b_0] = 0.$$

2.3 Toroidal superalgebras

In [1] vertex constructions of toroidal superalgebras extended by derivations were provided based on the results of Billig and Chen-Li-Tan [11, 12].

2.3.1 Toroidal algebras

Given a semisimple Lie algebra $\mathfrak{g}$ and the ring of functions A of a smooth affine variety $U = \operatorname{Spec} A$, the Lie algebra

$$\mathfrak{g}[U]$$

with pointwise bracket $[xf, yg] = [x,y]fg$ for $x, y \in \mathfrak{g}$ and $f, h \in A$ defines a Lie algebra. paralleling the case $\operatorname{Spec} A = \mathbb{C}^*$, Kassel [13] calculated the Lie algebra cohomology group to be

$$H^2(\mathfrak{g}, \Bbbk) \simeq \Omega^1_A / d\Omega^0_A$$

of global 1-forms modulo exact global 0-forms. It follows that the universal central extension of $\mathfrak{g}[A]$ is

$$\mathfrak{g}[A] \oplus \Omega^1_A / d\Omega^0_A \tag{2.3.1}$$

with bracket

$$[xf, yg] = [x,y]fg + \langle x, y \rangle f dg.$$

This algebra then has outer derivations given by global sections ξ of $\mathcal{T}^1_A$ which act by the Lie derivative on 1-forms, so that if A has coordinates $a_1, \dots, a_n$ and $\xi = \xi^i \frac{\partial}{\partial a_i}$ then its action on the 1-form $\kappa = \kappa^i da_i$ is

$$[\xi, \kappa] = \left(\xi^i \frac{\partial \kappa_j}{\partial a_i} + \kappa_i \frac{\partial \xi^i}{\partial a_j} \right) da_j$$

so that if $\kappa = df = \frac{\partial f}{\partial a_i} da_i$ then

$$[\xi, df] = d(\xi(f))$$

so that the action of ξ descends to $\Omega^1_A / d\Omega^0_A$.

Example 2.3.2. When $U = (\mathbb{C}^*)^n$ with coordinate ring $A = \Bbbk[s_1^{\pm}, \dots, s_n^{\pm}]$ the Lie algebra (2.3.1) is called then *toroidal algebra* $\widehat{\mathfrak{g}}^{(n)}$. If $n = 1$ then $\Omega^1_A / d\Omega^0_A = \Bbbk \frac{ds}{s}$ is 1-dimensional and we recover the usual description of the affine Lie algebra. After adjoining all global sections of $\mathcal{T}_A^1$ we get the *full toroidal algebra*

$$\widehat{\mathfrak{g}}_{full}^{(n)} = g[s_i^{\pm}] \oplus \Omega_A^1 / d\Omega_A^0 \oplus T_A^1.$$

We restrict from this point to the case of Example 2.3.2. There are various subalgebras of the full toroidal algebra depending on which subalgebra of $\mathcal{T}_A^1$ is adjoined. Pick an A-basis

$$\mathrm{d}_1 = s_1 \frac{\partial}{\partial s_1}, \ldots, \mathrm{d}_n = s_n \frac{\partial}{\partial s_n}$$

of $\mathcal{T}_A^1$ and an A-basis of Ω_A^1 given by

$$\mathrm{k}_i = \frac{ds_i}{s_i}, \quad i = 1, \ldots, n.$$

Adjoining

$$\mathrm{d}_1, \ldots, \mathrm{d}_n$$

gives rise to the *elliptic Lie algebra* when $n = 2$, denoted $\mathfrak{g}^{ell}$. This is a direct generalization of the affine Lie algebra for $n = 1$.

An intermediate example is the *toroidal extended affine algebra*

$$\widehat{\mathfrak{g}}^{(n),ea} = \widehat{\mathfrak{g}}^{(n)} \oplus \ker \operatorname{div}$$

where

$$\begin{aligned} \operatorname{div} : \mathcal{T}_A^1 &\to A \\ \sum \xi_i s_i \frac{\partial}{\partial s_i} &\mapsto \sum_{i=1}^n s_i \frac{\partial \xi_i}{\partial s_i} \end{aligned}$$

is the divergence. The toroidal extended affine algebra is an example of an extended affine algebra, a notion introduced in the physics literature under a different name in [14] c.f. [15] for historical remarks.)

Definition 2.3.3. An *extended affine algebra* is a Lie algebra $\mathfrak{g}$ such that

EA1 There is a symmetric non-degenerate invariant for $(-,-)$ on $\mathfrak{g}$.

EA2 There is a finite dimensional semisimple self-centralizing subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

EA3 Real root elements $x_a \in \mathfrak{g}_\alpha$ act locally nilpotently in the adjoint representation.

EA4 $\mathfrak{g}$ is indecomposable.

EA5 The centralizer of the subalgebra $\mathfrak{g}_c$ generated by real root spaces is contained in $\mathfrak{g}_c$

EA6 The subgroup generated by isotropic roots is free abelian group of finite rank.

The symmetric invariant form on $\widehat{\mathfrak{g}}^{(n),ea}$ is such that

$$(\mathfrak{g}[A], \Omega^1_A/d\Omega^0_A) = (\Omega^1_A/d\Omega^0_A, \Omega^1_A/d\Omega^0_A) = (\ker \operatorname{div}, \ker \operatorname{div}) = (\mathfrak{g}[A], \ker \operatorname{div}) = 0$$

and on elements $(xf, \kappa, \xi), (yg, \lambda, \zeta)$ is given by

$$((xf, \kappa, \xi), (yg, \lambda, \zeta)) = \int_{T_{cpt}} d\sigma \langle x, y\rangle f(\sigma) g(\sigma) + \kappa(\zeta)(\sigma) + \lambda(\xi)(\sigma)$$

where $T_{cpt} \subset U$ is the compact torus and $\langle -, -\rangle$ is the invariant form on $\mathfrak{g}$.

This only descends from Ω^1_A to $\Omega^1_A/d\Omega^0_A$ if the vector fields are required to be divergence free.

There is a further twisting in the bracket on $\widehat{\mathfrak{g}}^{(n)}_{full}$, depending on a parameter $\mu \in \Bbbk$ given by the $\Omega^1_A/d\Omega^0_A$-valued 2-cocycle $(-,-)$ on $\mathcal{T}^1_A$ given by

$$c(f\mathrm{d}_i, g\mathrm{d}_j) = \frac{\mathrm{d}_j f \mathrm{d}_i g}{fg} \left(\sum_{\ell=1}^{n} f \mathrm{d}_\ell g \mathrm{k}_\ell \right)$$

so that

$$\widehat{\mathfrak{g}}^{(n)}_{full}(\mu)$$

with bracket twisted $[-,-]_\mu$ so that

$$[\xi,\zeta]_\mu = [\xi,\zeta] + \mu c(\xi,\zeta)$$

also forms a Lie algebra and its subalgebra $\widehat{\mathfrak{g}}^{(n),ea}$ with the induced bracket does as well.

2.3.2 Hamiltonian vector fields on a super torus

A super extension in the $\mathfrak{g} = \mathfrak{gl}_0$ case for $n = 2$. We specialize to $n = 2$ and rename the variables of the torus s and t. Let $H^*(E) = \Bbbk\langle 1, \sigma_+, \sigma_-, \mathrm{pt}\rangle$ denote the cohomology of an elliptic curve and B the provided basis.

The cohomology of any compact Lie group G is provided with a Hopf algebra structure where the coproduct is given by m^* where $m : G \times G \to G$ is the multiplication. The adjoint of m^* under the Poincaré pairing is m_*. Thus $H^*(E)$ is equipped with the multiplication map $\star$ of the *dual* Hopf algebra, which in this case is the cup product on the dual elliptic curve $E^\vee$.

We define a Lie algebra

$$\mathfrak{g}_{T^*E} = \langle w_\gamma^{a,b}, \boldsymbol{c}_s, \boldsymbol{c}_t\rangle$$

with (super)bracket

$$[w_\gamma^{a,b}, w_\eta^{c,d}] = -\det\begin{pmatrix} a & c \\ b & d\end{pmatrix} w_{\gamma\star\eta}^{a+c,b+d} + \delta_{a+c,0}\delta_{b+c,0}(a\boldsymbol{c}_s + b\boldsymbol{c}_t). \tag{2.3.4}$$

This Lie algebra has an alternative interpretation as a central extension of the algebra of Hamiltonian vector fields on $(\Bbbk^*)^{2|2}$, a $2|2$-dimensional super torus. The Lie algebra $\mathrm{Ham}((\Bbbk^*)^{2|2})$ has elements which are functions $f \in \Bbbk[s^\pm t^\pm, \xi_1, \xi_2]$ where ξ_1, ξ_2 are odd variables and the bracket is given by the the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \log s}\frac{\partial g}{\partial \log t} - \frac{\partial f}{\partial \log t}\frac{\partial g}{\partial \log s}$$

in purely even logarithmic coordinates around the identity.

This description shows that the Lie bracket on $\mathfrak{g}_{T^*E}$ admits a family of deformations over $\mathbb{P}^1$ given over $[X_0 : X_1]$ by

$$X_0\{-,-\}_{ev} + X_1\{-,-\}_{odd}$$

where $\{-,-\}_{ev}$ is the above bracket and

$$\{-,-\}_{odd} = -\frac{\partial f}{\partial \xi_1}\frac{\partial g}{\partial \xi_2} + \frac{\partial f}{\partial \xi_2}\frac{\partial g}{\partial \xi_1}$$

is the (compatible) odd bracket. Over $\infty \in \mathbb{P}^1$ the Lie algebra structure is that of $\mathfrak{p}(0|2)[s^{\pm}, t^{\pm}]$ where $\mathfrak{p}(0|2)$ is the Poisson Lie superalgebra.

Vertex representations

The difficulty in constructing representations of toroidal algebras using vertex operator algebra techniques depends drastically on whether one is interested in constructing the action of the vector fields extending the toroidal algebra.

For the toroidal algebra itself, there is a functor from the category of $\widehat{\mathfrak{g}}$-modules to the category of $\widehat{\mathfrak{g}}^{(n)}$-modules due to Iohara-Saito-Wakimoto [16] and Bergman-Billig [17]. For this, let $V((0)^{\oplus n-1})$ denote the abelian Lattice vertex algebra associated to the lattice with completely degenerate pairing. Then if M is a module for the affine vertex algebra $V(\mathfrak{g})_k$ at level k then

Theorem 2.3.5 ([17], [16]). *1. Modes of the vertex algebra*

$$V(\mathfrak{g})_k \otimes V((0)^{\oplus n-1})$$

satisfy the commutation relations of $\widehat{\mathfrak{g}}^{(n)}$.

2. The map

$$M \mapsto M \otimes V((0)^{(\oplus n-1)})$$

is a functor from $V(\mathfrak{g})_k$-modules to $\widehat{\mathfrak{g}}^{(n)}$ modules.

The generator $xs_1^{k_1}s_2^{k_2}\cdots s_n^{k_n}$ is sent to

$$: J_x(z)\Gamma_{(k_2,\ldots,k_n)}(z) : [z^{-1-k_1}]$$

while the remaining central generators are modes of the fields

$$\Gamma_{(k_2,\ldots,k_n)}(z).$$

The vertex representations of the vector field part of toroidal algebras by contrast is somewhat finicky and generally depends on an auxiliary vertex algebra and a fine-tuning of central charges. The definitive construction was provided in papers of Billig [12, 18].

When $\mathfrak{g}$ is ADE type It is possible to produce a representation of some abelian extension of the universal enveloping algebra of the full toroidal algebra

$$0 \to \mathcal{K} \to \widehat{\mathfrak{g}}^{(n)}_{full,ext} \to \widehat{\mathfrak{g}}^{(n)}_{full} \to 0$$

as modes of the vertex algebra $V(L)$ where

$$L = R \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus n-1}$$

is the direct sum of the root lattice R of $\mathfrak{g}$ and $n-1$ copies of the Lorentzian lattice $II_{1,1}$ [19, 20] where the extension is given by a residue calculation providing the OPEs of some generating fields, but the resulting extension seems somewhat uncontrollable.

To explain Billig's construction, let V_{HV} denote the Heisenberg-Virasoro vertex algebra with cental charges c_V, c_H, c_{HV}, which is generated by a Heisenberg field $I(z) = \sum_{n\in\mathbb{Z}} I_n z^{-n-1}$ of central charge c_H, a Virasoro field $\omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ of central charge c_V so that the OPE between the two is given by

$$\omega(z)I(w) \sim \frac{-c_{HV}/2}{(z-w)^3} + \frac{I(w)}{(z-w)^2} + \frac{\partial I(w)}{z-w}.$$

At the level of modes, the commutator between the Heisenberg and Virasoro fields in

$$[L_n, I_m] = -nI_{n+m} - \delta_{n+m,0}(n^2+n)c_{HV}.$$

It is known that this is the universal central extension of the centerless Virasoro-Heisenberg algebra.

Let u, v denote the basis of $II_{1,1}$ so that the Gram matrix is of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $V^+(II_{1,1}^{\oplus n-1})$ denote the sublattice vertex algebra associated to the lattice spanned by the u_i. Consider the vertex algebra

$$V = V(\mathfrak{g})_{c_\mathfrak{g}} \otimes V(\mathfrak{sl}_{n-1})_{c_{\mathfrak{sl}}} \otimes V_{HV} \otimes V^+(II_{1,1}^{\oplus n-1}).$$

Theorem 2.3.6 (Billig [18]). *For any $\mu \in \mathbb{C}$, At the values of central charges*

$$c_\mathfrak{g} = c \neq 0 \qquad c_{\mathfrak{sl}} = 1 - \mu c$$
$$c_H = (n-1)(1-\mu c) \qquad c_V = 12\mu c - 2(n-1)$$
$$c_{HV} = (n-1)/2$$

The vertex algebra V is a module over $\mathfrak{g}_{full}^{(n)}(\mu)$.

The action of the basis vector fields depend on whether they are of the form $f\mathrm{d}_1$ or d_i for $i > 1$. If $i = 1$ then $s_1^{k_1} \cdots s_n^{k_n}\mathrm{d}_i$ is the z^{-k_1-1} coefficient of the field

$$d_i(k_2, \ldots, k_n, z) = Y(v_{i,-1} \otimes e^{k_2u_2+\cdots+k_nu_n}, z) + \sum_{j=2}^{n} k_p Y(E_{pi,-1} \otimes e^{k_2u_2+\cdots+k_nu_n}, z)$$

where E_{ab} is the ab elementary symmetric matrix in $\mathfrak{gl}_{n-1}$ and we have combined the Heisenberg and $\mathfrak{s}l_{n-1}$ affine vertex algebras into a $\mathfrak{gl}_{n-1}$ vertex algebra.

The action of $s_1^{k_1}\cdots s_n^{k_n}\mathrm{d}_1$ is given by the $-k_1-2$ coefficient of a similar expression.

Arising naturally in the geometry of moduli spaces of framed torsion-free sheaves on $E\times\mathbb{P}^1$ there is a representation of $V_{\mathfrak{sl}_{n-1}}\otimes V_{HV}$ on a n-fold tensor product of the bc system $V_{bc}^{\otimes n-1}$ where

$$J_{E_{ij}}(z)\mapsto :b_i(z)c_j(z):$$

and the stress tensor

$$\omega=:\partial c_1(z)b_1(z)+\cdots+\partial c_n(z)b_{n-1}(z):$$

which by a calculation using Wick's theorem has

$$c_{\mathfrak{sl}_{n-1}},\qquad c_H=n-1,\qquad c_V=-2n-1,\qquad c_{HV}=(n-1)/2.$$

Proposition 2.3.7 ([1])**.** *There is a representation of* $\widehat{\mathfrak{gl}}_{0full}^{(n+1)}$ *on the vertex algebra*

$$(V^+(II_{1,1})\otimes V_{bc})^{\otimes n}.$$

Proof. Follows from the above discussion and the tensor formula

$$V^+(II_{1,1})^{(\otimes n)}=V^+(II_{1,1}^{(\oplus n)})$$

□

By adjoining the fields

$$Y(c_{i,-2}\otimes e^{k_1u_1+\cdots+k_nu_n},z)=:\partial c_i(z)\Gamma_{(k_1,\ldots,k_n)}(z):$$
$$Y(b_{i,-1}\otimes e^{k_1u_1+\cdots+k_nu_n},z)=:b_i(z)\Gamma_{(k_1,\ldots,k_n)}(z):$$

one obtains a superalgebra whose even part is $\widehat{\mathfrak{gl}}_{0full}^{(n+1)}$.

Extended affine subalgebra using coordinated vertex opreators

Billig also provided a construction of the toroidal extended affine subalgebra [12]. The construction was explained in [11] using the language of ϕ-coordinated modules for vertex operator algebras, which is a formalization of the construction in [21] of certain modified vertex operators under the conformal transformation $z \mapsto e^z - 1$.

Precisely, we only consider the case $\phi = ze^z$ which is a special case of a coordinated module construction under a family of conformal transformations [22]. Let V be a vertex algebra.

Definition 2.3.8. A ϕ-coordinated V-module is a space M and a linear map

$$Y_W(-, z) : V \longrightarrow \mathrm{Hom}(W, W((z)))$$

such that $Y_W(|0\rangle, z) = \mathrm{Id}_W$, and for $u, v \in V$ there exists $k \in \mathbb{Z}_{\geq 0}$ such that

$$(z_1 - z_2)^k Y_W(u, z_1) Y_W(v, z_2) \in \mathrm{Hom}(W, W((z_1, z_2)))$$
$$(z_2 e^{z_0} - z_2)^k Y_W(Y(u, z_0)v, z_2) = \left((z_1 - z_2)^k Y_W(u, z_1) Y_W(v, z_2)\right)\Big|_{z_1 = z_2 e^{z_0}}.$$

As a consequence of the definitions, if the original vertex algebra

$$Y(u, z)Y(v, w) \sim \sum_{j \geq 0} \frac{Y(u_{(j)}v, w)}{(z - w)^{j+1}} \tag{2.3.9}$$

corresponding to the bracket

$$[Y(u, z)Y(v, w)] = \sum_{j \geq 0} Y(u_{(j)}v, w) \partial_w^{(j)} \delta(z - w) \tag{2.3.10}$$

then given a ϕ-coordinated module (W, Y_W) the OPE over $\mathbb{C}$ with additive derivatives ∂_w and additive delta function are replaced with their multiplicative versions

$$\partial_w, \delta(z - w) \rightsquigarrow w\partial_w, \delta(z/w)$$

so that

$$[Y_W(u,z), Y_W(v,w)] = \sum_{j\geq 0} Y_W(u_{(j)}v, w)(w\partial_w)^{(j)}\delta\left(\frac{w}{z}\right). \tag{2.3.11}$$

If rather than a vertex algebra V and a module W, we had imposed that V itself had multiplicative OPEs then we would obtain the definition of a *multiplicative vertex algebra*. In simple terms, ϕ-coordinated modules are multiplicative modules over an additive (i.e. ordinary) vertex algebra.

A source of ϕ-coordinated modules was constructed in [21] from a conformal structure ω on V with central charge c, letting $\widetilde{\omega} = \omega - \frac{1}{24}c|0\rangle$ and given $v \in V$ define

$$Y[v,z] = Y(e^{zL(0)}v, e^z - 1)$$

the data $(V, Y[-,z], |0\rangle, \widetilde{\omega})$ defines a vertex algebra such that

$$T(|0\rangle) = |0\rangle, T(\omega) = \widetilde{\omega}, T(a) = a$$

for primary $a \in V$ and for descendants the compatibility of the stress tensor forces

$$T(L_{-k_1}L_{-k_2}\ldots L_{-k_n}a) = \widetilde{L}_{-k_1}\widetilde{L}_{-k_2}\ldots\widetilde{L}_{-k_n}a$$

where

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

and

$$Y[\widetilde{\omega},z] = \sum_{n\in\mathbb{Z}} \widetilde{L}_n z^{-n-2}.$$

This gives a vertex algebra isomorphism from $(V, Y, |0\rangle, \omega)$ to $(V, Y[-,z], |0\rangle, \widetilde{\omega})$.

Given $a \in V$ let

$$Y[a,z] = \sum_{n\in\mathbb{Z}} a[n]z^{-n-1} \tag{2.3.12}$$

and define the bracket nth product by

$$T(a_n b) = T(a)[n]T(b) \tag{2.3.13}$$

using that if a is homogeneous of weight $\operatorname{wt} a$ we have

$$a[n] = \operatorname{Res}_z \left(Y(a,z) \log(1+z)^n (1+z)^{\operatorname{wt} a - 1} \right). \tag{2.3.14}$$

Given an (ordinary) V-module (W, Y_W), in particular for V as a module over itself, one obtains a ϕ-coordinated module.

Proposition 2.3.15 ([11, §6],[23])**.** *Define a* $\operatorname{End}(W)$*-valued field* $X(v,z)$ *by the formula*

$$X^\phi(v,z) := Y_W(z^{L(0)} T(v), z) \tag{2.3.16}$$

The data $(W, X^\phi(-,z))$ *defines a* ϕ*-coordinated* V*-module structure on* W*.*

For the V module structure of V over itself, we have brackets

$$[X^\phi(u,z), X^\phi(v,w)] = \sum_{j \geq 0} X^\phi(u_{(j)}v, w)(w\partial_w)^{(j)} \delta\left(\frac{w}{z}\right). \tag{2.3.17}$$

Example 2.3.18. In the simplest case of a free bosonic field of dimension 1 from Example 2.2.1 the vertex operator

$$Y(a_{-1}, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

has associated ϕ-coordinated vertex operator

$$X^\phi(a_{-1}, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n}$$

and the OPE becomes

$$X^\phi(a_{-1}, z) X^\phi(a_{-1}, w) \sim \frac{czw}{(z-w)^2}.$$

Toroidal extended affine Lie algebras using coordinated modules

We now describe Chen-Li-Tan's interpretation of Billig's construction using ϕ-coordinated modules. Let V be the vertex operator algebra from Theorem 2.3.6 and V^{ϕ} for the ϕ-coordinated vacuum module.

Theorem 2.3.19 ([12], [11]). *1. ([12]) If M is a module for the*

$$V(\mathfrak{g})_{c_{\mathfrak{g}}} \otimes V(\mathfrak{sl}_{n-1}) \otimes Vir \otimes V^{+}(II_{1,1}^{\oplus n-1})$$

factor of V then M inherits a module structure for $\widehat{\mathfrak{g}}^{(n+1),ea}$.

2. When $n = 1$, There exist ϕ-coordinated vertex operators $X^{\phi}(v, z)$ for V^{ϕ} and a set of vectors $\{v\}$ so that Fourier coefficients of the $X^{\phi}(v, z)$ satisfy the commutation relations of $\widehat{\mathfrak{g}}^{(2),ea}$.

In simple terms, one does not need the central part of the affine $\mathfrak{gl}_{n-1}$ factor to produce the representation of the toroidal extended affine algebra. The construction of Theorem 2.3.19 is explicit, but we choose to only write down the result in special cases in the sequel.

Vertex representation of $\mathfrak{g}_{T^*E}$

Consider the Lie algebra $\mathfrak{g}_{T^*E}$. Combine its elements into generating series

$$\Upsilon_m(\gamma, z) = \sum_{n\in\mathbb{Z}} w_{\gamma}^{m,n} z^{-n}. \tag{2.3.20}$$

as in [1]. The defining relation (2.3.4) becomes

$$\begin{aligned}[\Upsilon_m(\gamma, z), \Upsilon_{m'}(\gamma', w)] &= mw\partial_w \Upsilon_{m+m'}(\gamma \star \gamma', w)\delta(w/z) \\ &+ (m + m')\Upsilon_{m+m'}(\gamma \star \gamma', w)w\partial_w\delta(w/z) + \delta_{m+m',0}\langle\gamma, \gamma'\rangle \boldsymbol{c} w\partial_w\delta(w/z).\end{aligned} \tag{2.3.21}$$

or equivalently via the multiplicative-style operator product expansion

$$\Upsilon_m(\gamma, z)\Upsilon_{m'}(\gamma', w) \sim mw\partial_w \Upsilon_{m+m'}(\gamma \star \gamma', w)\frac{w}{z-w}$$
$$+ (m+m')\Upsilon_{m+m'}(\gamma \star \gamma', w)w\partial_w \frac{w}{z-w} + \delta_{m,-m'}\langle \gamma, \gamma' \rangle \boldsymbol{c} w\partial_w \frac{w}{z-w}. \quad (2.3.22)$$

Owing to the multiplicative description of the fields, and the results of Chen-Li-Tan, it is natural to expect a representation in terms of ϕ-coordinated modules. In fact the following ϕ-coordinated fields in the vertex algebra

$$V^+(II_{1,1}) \otimes \mathcal{F}_{H^\perp(E)}$$

defined by

$$\mathrm{d}_m(z) = mz^2 : \omega(z)\Gamma_{mE}(z) : + mz\partial_z \Gamma_{mE}(z) - z\partial_z \left[z : \alpha(\mathrm{pt}, z)\Gamma_{mE}(z) : \right] \quad (2.3.23)$$
$$- mz^2 : \partial_z \alpha(mE, z)\Gamma_{mE}(z) :$$

$$\mathrm{k}_m(z) = \begin{cases} \frac{1}{m}\Gamma_{mE}(z) & m \neq 0 \\ \boldsymbol{\phi}(E, z) - E - \alpha_0(E)\log z & m = 0 \end{cases} \quad (2.3.24)$$

$$\sigma_m^+(z) =: z\alpha(\sigma_+, z)\Gamma_{mE}(z) : \quad (2.3.25)$$

$$\sigma_m^-(z) =: z\alpha(\sigma_-, z)\Gamma_{mE}(z) : . \quad (2.3.26)$$

where

$$\boldsymbol{\phi}(\gamma, z) = \int dz\alpha(\gamma, z) = \gamma + \alpha_0(\gamma)\log z + \sum_{n\neq 0} \frac{\alpha_n(\gamma)z^{-n}}{-n}$$

satisfy the OPEs

$$\mathrm{d}_m(z)\mathrm{d}_{m'}(w) \sim m\frac{w\partial_w \mathrm{d}_{m+m'}(w)w}{z-w} + (m+m')\frac{\mathrm{d}_{m+m'}(w)zw}{(z-w)^2} \tag{2.3.27}$$

$$\mathrm{d}_m(z)\mathrm{k}_{m'}(w) \sim m\frac{w\partial_w \mathrm{k}_{m+m'}(w)w}{z-w} + (m+m')\frac{\mathrm{k}_{m+m'}(w)zw}{(z-w)^2} + \delta_{m+m',0}\frac{zw}{(z-w)^2} \tag{2.3.28}$$

$$\mathrm{d}_m(z)\sigma^{\pm}_{m'}(w) \sim m\frac{w\partial_w \sigma^{\pm}_{m+m'}(w)w}{z-w} + (m+m')\frac{\sigma^{\pm}_{m+m'}(w)zw}{(z-w)^2} \tag{2.3.29}$$

$$\sigma^{+}_m(z)\sigma^{-}_{m'}(w) \sim m\frac{w\partial_w \mathrm{k}_{m+m'}(w)w}{z-w} + (m+m')\frac{\mathrm{k}_{m+m'}(w)zw}{(z-w)^2} + \delta_{m+m',0}\frac{zw}{(z-w)^2}. \tag{2.3.30}$$

The conclusion is the major part of one of the main theorems of [1]

Theorem 2.3.31 ([1] **).** *The vertex algebra $V_{T^*E} = V^+(II_{1,1} \otimes \mathcal{F}_{H^{\perp}(E)})$ is an irreducible module for $\mathfrak{g}_{T^*E}$.*

We can also combine the generating fields $\Upsilon(\gamma, z_1)$ into a formal generating function *of* fields depending on an additional even parameter z_2 and two odd parameters ψ_+, ψ_-. Denote the generating function $\mathbb{L}(z_1, z_2, \psi_+, \psi_-) \in \mathrm{End}(V_{T^*E})[[z_1^{\pm}, z_2^{\pm}, \psi_+, \psi_-]]$ defined by

$$\mathbb{L}(Z) = \sum_{n\in\mathbb{Z}} \Upsilon_{-m}(1, z_1)\psi_+\psi_- z_2^{-m} + \Upsilon_{-m}(\sigma_-, z_1)\psi_+ z_2^{-m} + \Upsilon_{-m}(\sigma_+, z_1)\psi_- z_2^{-m} + \Upsilon_{-m}(\mathrm{pt}, z_1) z_2^{-m}$$

where $Z = (z_1, z_2, \psi_+, \psi_-)$ is the 2|2-dimensional parameter. Thus

$$w^{a,b}_{\sigma_+^c \sigma_-^d} = \frac{1}{(2\pi i)^2}\int \frac{dz_1}{z_1}\frac{dz_2}{z_2} d\psi_+ d\psi_- \mathbb{L}(Z) z_1^a z_2^b \psi_+^c \psi_-^d.$$

Then owing to the identity of 0|2-dimensional formal functions if

$$f(\phi_+, \phi_-) = f_1\phi_+\phi_- + f_-\phi_+ + f_+\phi_- + f_{+-}1$$

then the equation

$$f(\phi+\xi) = f_1\phi_+\phi_- + f_1\phi_+\psi_- - f_1\phi_-\psi_+ + f_1\psi_+\psi_- + f_-\psi_+ + f_-\phi_+ + f_+\psi_- + f_+\phi_- f_{+-}1$$

is compatible with the generating function for the convolution product on $H^*(E)$ so that if $f_1 = 1, f_+ = \sigma_+, f_- = \sigma_-, f_{+-} = \mathrm{pt}$ then

$$f(\psi) \star f(\phi) = f(\phi + \psi).$$

It follows that the defining relations for $\mathfrak{g}_{T^*E}$ can be expressed via the formula

$$\begin{aligned}
[\mathbb{L}(Z), \mathbb{L}(W)] = w_1 \partial_{w_1} \mathbb{L}(W + \psi) \delta\left(\frac{w_1}{z_1}\right) w_2 \partial_{w_2} \delta\left(\frac{w_2}{z_2}\right) \\
- w_2 \partial_{w_2} \mathbb{L}(W + \psi) w_1 \partial_{w_1} \delta\left(\frac{w_1}{z_1}\right) \delta\left(\frac{w_2}{z_2}\right) \\
+ \boldsymbol{c}_t w_1 \partial_{w_1} \delta\left(\frac{w_1}{z_1}\right) \delta\left(\frac{w_2}{z_2}\right) \delta\left(\frac{\phi_1}{\psi_1}\right) \delta\left(\frac{\phi_2}{\psi_2}\right) \\
+ \boldsymbol{c}_s \delta\left(\frac{w_1}{z_1}\right) w_2 \partial_{w_2} \delta\left(\frac{w_2}{z_2}\right) \delta\left(\frac{\phi_1}{\psi_1}\right) \delta\left(\frac{\phi_2}{\psi_2}\right).
\end{aligned} \tag{2.3.32}$$

2.4 Superconformal and reflection vertex algebras

In this section we describe another family of vertex algebras, closely related to the $\mathcal{N} = 4$ small superconformal algebra. they belong to the more general class of vertex algebras which have been studied under the term 4d/2d duality in the physics literature initiated in [24].

In general, it is expected that to any superconformal field theory in 4d with $\mathcal{N} = 2$ supersymmetries there is a vertex algebra. For various classes of theories, such as those with marginal deformations to weakly coupled gauge theories or for theories of class S there are mathematical descriptions of the resulting vertex algebra. The body of recent literature related to this problem is too vast to properly cite, but some number of references are available in the introduction to [25].

We will be concerned only with the special case when the $4d$ theory is $4d\mathcal{N} = 4$ supersymmetric Yang-Mills theory with type A_k gauge group. The resulting vertex algebra $W_k^{\mathcal{N}=4}$ is expected and/or known to have a number of equivalent descriptions.

1. As a BRST reduction of a system of $T^*\mathfrak{gl}_k$-valued symplectic bosons [24, 26] owing to the weak coupling limit.

2. As the evaluation of a functor predicted in [27]

$$\{\text{complex reflection groups}\} \to \{\text{vertex algebras}\}$$

applied to symmetric group S_k, the Weyl group of $\mathrm{GL}(k)$.

3. As the kernel of a system of screening operators $\mathbb{S}_\alpha$ for α running over the simple roots of the root system A_k where each

$$\mathbb{S}_\alpha : (V_{bc} \otimes V_{\beta\gamma})^{\otimes k} \to (V_{bc} \otimes V_{\beta\gamma})^{\otimes k}$$

is the zero-mode of a screening current $S(z)$.

4. It has a construction as the vertex algebra of global sections of a sheaf of vertex operator algebras over a superscheme whose underlying classical scheme is the hilbert scheme of k points in $\mathbb{A}^2$ [25]. By restricting to a specific open set, one obtains an embedding

$$W_k^{\mathcal{N}=4} \to (V_{bc} \otimes V_{\beta\gamma})^{\otimes k}$$

and this is expected to coincide with the kernel of certain screening operators in general, and known when $k = 2, 3$.

2.4.1 k = 2

When $k = 2$, the algebra $W_k^{\mathcal{N}=4}$ coincides with the small $\mathcal{N} = 4$ superconformal algebra at $c = -9$, and the screening operators coincide with those studied first by Adamović [28].

Realization of Weyl vertex algebra in lattice vertex algebra

First, consider the vertex algebra $V^+(II_{1,1})$. With the fields

$$a(z) = Y(e^u, z) = \Gamma_{(1,0)}(z) \tag{2.4.1}$$

$$a^*(z) = Y(-\frac{1}{2}(u_{-1} + v_{-1})e^{-u}, z) = -\frac{1}{2} : u(z)\Gamma_{(-1,0)}(z) : -\frac{1}{2} : v(z)\Gamma_{(-1,0)}(z) : \tag{2.4.2}$$

the subalgebra generated by the states e^u and $-\frac{1}{2}(u_{-1} + v_{-1})e^{-u}$ is isomorphic to a symplectic boson, or Weyl vertex algebra corresponding to a map

$$\beta(z) \mapsto a(z), \gamma(z) \mapsto a^*(z)$$

of fields. We will also need an extension

$$V^+_{1/2}(II_{1,1})$$

which is as a vector space

$$\mathcal{F}_{H^*(E)} \otimes \mathbb{C}[e^{\pm E/2}] = \mathcal{F}_{H^*(E)} \otimes \mathbb{C}[e^{\pm(1/2,0)}]$$

and contains for example vertex operators $\Gamma_{(1/2,0)}(z) = Y(e^{u/2})$.

From [28] the assignment

$$J^+(z) \mapsto \beta(z) \tag{2.4.3}$$

$$J^-(z) \mapsto -2 : \gamma(z)\beta(z) : +c(z) \tag{2.4.4}$$

$$J^0(z) \mapsto - : \gamma^2(z)\beta(z) : -\frac{3}{2}\partial_z\gamma(z) + : \gamma(z)c(z) : \tag{2.4.5}$$

$$G^+(z) \mapsto c(z) \tag{2.4.6}$$

$$G^-(z) \mapsto : \gamma(z)c(z) : \tag{2.4.7}$$

$$\widetilde{G}^+(z) \mapsto -2 : \beta(z) : \partial_z J_{bc}(z)c(z) : + : \partial_z\beta(z)b(z) : \tag{2.4.8}$$

$$\begin{aligned}\widetilde{G}^-(z) \mapsto \partial_z^2 b(z) + 2 : \gamma(z)\beta(z)\partial b(z) :\\ + : J_{bc}(z)b(z) : + : \gamma(z) : \partial\beta(z)b(z) ::\end{aligned} \tag{2.4.9}$$

$$T(z) =: J^+(z)J^-(z) : + : J^-(z)J^+(z) : +\frac{1}{2} : J^0(z)J^0(z) : \tag{2.4.10}$$

defines a small $\mathcal{N} = 4$ superconformal structure on the vertex algebra $V_{bc} \otimes V_{\beta\gamma}$ at $c = -9$.

Next define the screening current

$$S(z) =: b(z)\Gamma_{(-1/2,0)}(z) :$$

and screening operator $\mathbb{S} = \oint S(z)dz$.

Theorem 2.4.11 (Adamović [28])**.** *There is an identification*

$$\ker_{V_{bc}\otimes V_{\beta\gamma}} \mathbb{S} = Vir^{\mathcal{N}=4}$$

between the algebra generated by fields (2.4.3)-(2.4.10) *and the kernel of the screening operator in* $V_{bc} \otimes V_{\beta\gamma}$.

Define the vertex algebra

$$V^{\text{Jac}}_{T^*E} = V^+(II_{1,1}) \otimes V_{bc}$$

from [1] which differs from V_{T^*E} only by the fermionic zero and logarithmic modes.

Using (2.4.1)-(2.4.2) there is an embedding $V_{\beta\gamma} \to V^+(II_{1,1})$ and hence an embedding

$$V_{\beta\gamma} \otimes V_{bc} \longrightarrow V_{T^*E}^{\mathrm{Jac}} \tag{2.4.12}$$

Corollary 2.4.13. *The vertex algebra $V_{T^*E}^{\mathrm{Jac}}$ admits an $\mathcal{N} = 4$ superconformal structure.*

Under the identification of $\mathfrak{g}_{T^*E}$ with the algebra of Hamiltonian vector fields on the space $\mathrm{Spec}[s^{\pm}, t^{\pm}, \theta_+, \theta_-]$ from section 2.3.2 and of $V_{T^*E}^{\mathrm{Jac}}$ with its module in Theorem 2.3.31, the screening operator is equivalent to the action of the Hamiltonian vector field

$$\{\theta_+\sqrt{t}, -\}.$$

It would be interesting to give a full interpretation of the superconformal algebra analogously using the explicit description in (2.4.3)-(2.4.10).

2.4.2 Infinite limit

We now discuss the infinite limit of the vertex algebra $W_k^{\mathcal{N}=4}$ as studied in [26]. Consider $\mathfrak{gl}_k$-valued field $\beta(X, z)$, $\mathfrak{gl}_k^*$ valued field $\gamma(Y, z)$. Let $\omega(-, -)$ on $\mathfrak{gl}_k \oplus \mathfrak{gl}_k^*$ denote the standard symplectic form so that the fields have OPE

$$\beta(X, z)\gamma(Y, z) \sim \frac{\omega(X, Y)}{z - w}$$

and likewise define odd $\mathfrak{gl}_k$-valued fields $b(X, z)$ and $c(Y, z)$ so that

$$b(M, z)c(N, z) \sim \frac{\omega(M, N)}{z - w}.$$

In particular, up to the action of zero modes of the c-ghost fields, it is expected that the vertex algebra $W_k^{\mathcal{N}=4}$ is written as

$$\ker Q / \operatorname{im} Q$$

where

$$Q = \oint \operatorname{Tr} c(M,z)\beta(X,z)\gamma(Y,z) + \frac{1}{2}\operatorname{Tr} b(N,z)c(M,z)^2$$

is the BRST operator and the trace is taken over the matrix indices.

Furthermore, it is shown in [26] that the resulting vertex algebras stabilize as $k \to \infty$ to an infinitely generated vertex algebra we will denote $W_\infty^{\mathcal{N}=4}$.

Chapter 3: Cohomological Hall algebras

In geometric representation theory, the weight spaces of a representation are the evaluation of some cohomology theory on moduli spaces, or moduli stacks, and the operators are given by correspondences between the spaces.

A rich source of moduli stacks and correspondences is given by the moduli stack $\mathcal{M}$ of objects in an abelian category $\mathcal{A}$. There is then a natural correspondence provided by the stack of extensions

$$\mathcal{M}_{\mathrm{Ext}} = \{0 \to E_1 \to E_2 \to E_2 \to 0 | E_i \in \mathcal{A}\}$$

where we have only written the set of closed points of the stack.

Then the projections $\pi_i : [0 \to E_1 \to E_2 \to E_2 \to 0] \mapsto E_i$ provide a correspondence diagram

$$\begin{array}{ccc} & \mathcal{M}_{\mathrm{Ext}} & \\ {\scriptstyle \pi_1 \times \pi_3} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathcal{M} \times \mathcal{M} & & \mathcal{M} \end{array} .$$

3.1 Cohomological Hall algebras

Depending on the properties of the maps $\pi_1 \times \pi_3$ and of π_2 and the functoriality of the cohomology theory $H^*(-)$ under maps of the corresponding types if one is able to define $(\pi_1 \times \pi_3)^*$ and π_{2*} then the map

$$H^*(\mathcal{M}) \otimes H^*(\mathcal{M}) \to H^*(\mathcal{M})$$
$$x \otimes y \mapsto \pi_{2*}(\pi_1 \times \pi_3^*(x \boxtimes y))$$

defines an algebra structure on $H^*(\mathcal{M})$. In particular, when the abelian category $\mathcal{A}$ has global dimension ≤ 2 then the map $\pi_1 \times \pi_2$ is quasismooth and so in Borel-Moore homology there is a pullback $\pi_1 \times \pi_3^!$ and when the map π_2 is proper there is a pushforward π_{2*}.

3.1.1 Dimension zero sheaves on a surface

The cohomological hall algebra of dimension zero sheaves on a surface was defined in [29] with a related K theoretic construction in [30]. The stack $\mathcal{M}$ is the stack of dimension zero coherent sheaves

$$Coh^0 = \bigsqcup_{n \geq 0} Coh_n$$

on a surface S which is a substack of the stack of all objects in $Coh(S)$. Let $H_*(-)$ denote Borel-Moore homology.

Theorem 3.1.1 ([29, 30])**.** *The CoHA multiplication is well defined on $H_*(Coh^0)$ and defines an associative product $*$.*

Remark 3.1.2. When S has an action of a torus T, the same result holds in T-equivariant Borel-Moore homology.

When S is proper, or equivariantly proper and $H^*(S)$ is pure in the sense that there is a compactification $\overline{S}$ such that ι^* is surjective, then a compete description of the cohomological hall algebra of dimension zero sheaves was provided in [31].

The *tautological classes* are cohomology classes

$$\mathrm{ch}_{k,\gamma} \in H^*(Coh_0), k \geq 0, \gamma \in H^*(\overline{S})$$

given by Kunneth components of the universal bundle $\mathcal{E}_U$ over test schemes $U \to Coh^0 \times \overline{S}$ so that

$$\mathrm{ch}_{k,\gamma} = \int_{\overline{S}} \mathrm{ch}_k \cup \pi_2^* \gamma.$$

In addition there is an auxiliary version of the CoHA called the *extended cohomological Hall algebra* denoted

$$\overline{H}_*(Coh^0) = H_*(Coh^0) \ltimes \Lambda(S)$$

where $H_*(Coh^0)$ and $\Lambda(S)$ is the *universal tautological ring* which is the home for symbols $\mathrm{ch}_{k,\gamma}$ which act on $H_*(Coh^0)$ via the cap product

$$\mathrm{ch}_{k,\gamma} \cap - : H_*(Coh^0) \to H_*(Coh^0).$$

Deformed $W_{1+\infty}$ algebra

The answer is expressed in terms of the *deformed* $W_{1+\infty}$ algebra associated to the surface. This is the algebra $\mathcal{W}_{1+\infty,S}$ generated by elements

$$\psi_n(\gamma), T_n^+(\gamma), T_n^-(\gamma), \qquad n \geq 0, \gamma \in H^*(S)$$

linear in γ subject to the relations

$$[\psi_m(\gamma), \psi_n(\eta)] = 0 \tag{3.1.3}$$

$$[\psi_m(\gamma), T_n^{\pm}(\eta)] = m T^{\pm}_{m+n-1}(\gamma\eta) \tag{3.1.4}$$

$$[T_m^{\pm}(\gamma\eta), T_n^{\pm}(\mu)] = [T_m^{\pm}(\gamma), T_n^{\pm}(\eta\mu)] \tag{3.1.5}$$

and additional relations summarized in [31, §3.1 §3.4]

This is closely related to a vertex algebra $W_{1+\infty}(S)$ which is defined for an equivariantly proper surface with equivariant $c_2(S)^2 = 0$ in [32]. The latter is generated by fields

$$\mathfrak{L}^p(\gamma, z), \qquad p \in \mathbb{Z}_{\geq 0}, \gamma \in H^*(S) \tag{3.1.6}$$

and the modes satisfy the commutation relations

$$[\mathfrak{L}^p_n(\gamma), \mathfrak{L}^q_m(\eta)] = -\det\begin{pmatrix} p & q \\ n & m \end{pmatrix} \mathfrak{L}^{p+q-1}_{n+m}(\gamma\eta) + \text{central} . \tag{3.1.7}$$

There is a homomorphism $\mathcal{W}_{1+\infty,S} \to W_{1+\infty}(S)$ provided in [31] by the map $T^+_n(\gamma) \mapsto \mathfrak{L}^n_{\pm 1}(\gamma)$, $\psi_n(\gamma) \mapsto \mathfrak{L}^n_0(\gamma)$.

Theorem 3.1.8. *[31] The extended CoHA of S is isomorphic to the algebra generated by the $\phi_n(\gamma)$ and $T^+_n(\gamma)$.*

In the case that $S = \mathbb{A}^2$ and the cohomology theory is taken fully equivariantly with respect to the torus $T = (\mathbb{C}^*)^2$ acting on the coordinates with weights t_1 and t_2, which is the basic equivariant local example, the CoHA of dimension zero sheaves had been previously determined by Davison [33]. There the identification

$$H_{*,T}(Coh^0(\mathbb{A}^2)) \simeq \mathbb{Y}(\widehat{\mathfrak{gl}_1})^+$$

was given between the CoHA and the positive half of the *affine Yangian of* $\mathfrak{gl}_1$.

3.1.2 Preprojective algebra

To the knowledge of the author, there is not currently attested in the literature any smooth algebraic surface where the cohomological hall algebra of torsion sheaves (rather than simply dimension zero sheaves) has been explicitly determined excepting the cases where there are no properly supported torsion sheaves of dimension 1, in which case it is handled by Theorem 3.1.8 . On the other hand, for the surface X_{A_n} which is a symplectic resolution of $\mathbb{C}^2/\mathbb{Z}/(n+1)\mathbb{Z}$ then there is a derived equivalence [34]

$$D^b(\mathrm{Coh}_{tors}(X_{A_n})) \xrightarrow{\sim} D^b(\mathrm{Rep}-\Pi^0_Q)$$

between the category of torsion sheaves$\mathrm{Coh}_{tors}(X_{A_n})$ on X_{A_n} and the category of representations of a certain associative algebra Π_Q^0 associated to the cyclic quiver Q with $n+1$.

There is a cohomological Hall algebra for abelian category of representations of the latter, and this has been determined, in localized cohomology in [35] and without localization in [36]. Let $\mathcal{R}ep_{\Pi_Q^0}$ denote the stack of representations of the preprojective algebra of the quiver Q. Let $R_Q = K(\Pi_Q^0)$ denote the Grothendieck group of representations of Π_Q^0 which is the group of connected components of $\mathcal{R}ep_{\Pi_Q^0}$.

It is shown that

$$H_*^T(\mathcal{R}ep_{\Pi_Q^0}) \simeq \mathbb{Y}(\mathfrak{g}_Q)^+$$

where

- The Hopf algebra $\mathbb{Y}(\mathfrak{g}_Q)$ is the Maulik-Okounkov Yangian introduced in [10].
- The positive half $\mathbb{Y}(\mathfrak{g}_Q)^+$ is a tensor factor of a PBW decomposition

$$\mathbb{Y}(\mathfrak{g}_Q) \simeq \mathbb{Y}(\mathfrak{g}_Q)^+ \otimes \mathbb{Y}(\mathfrak{g}_Q)^0 \otimes \mathbb{Y}(\mathfrak{g}_Q)^-. \tag{3.1.9}$$

- The Lie algebra $\mathfrak{g}_Q$ is (as a result of the resolution [37] of a conjecture of Okounkov) is a generalized Kac-Moody moody algebra which is $R_Q \times \mathbb{Z}_{\geq 0}$-graded.

Also as a result of the resolution of the aforementioned conjecture, he root multiplicities

$$\dim \mathfrak{g}_{Q,d,i} = \boldsymbol{a}_Q(d,q)[q^i], \qquad d \in R_Q, i \in \mathbb{Z}_{\geq 0}$$

where $\boldsymbol{a}_Q(d,q)$ is the *Kac polynomial* which counts absolutely indecomposable representations of Q over $\mathbb{F}_q$ of dimension d (c.f. [37]) for the relevant definitions.

3.2 Modules

Modules for the CoHA are provided by the groups $H_*(\mathcal{M}^\circ)$ of moduli spaces (or sometimes stacks) $\mathcal{M}^\circ$ fitting into a diagram

$$\begin{array}{ccc} & \mathcal{M}^\circ_{\text{Ext}} & \\ {\scriptstyle \pi_1\times\pi_3} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathcal{M}\times\mathcal{M}^\circ & & \mathcal{M}^\circ \end{array} \qquad (3.2.1)$$

for an appropriate choice of $\mathcal{M}^\circ$ so that π_3 remains proper. Generally, $\mathcal{M}^\circ$ is a moduli stack of elements of a subcategory of $\mathcal{A}$, a moduli of objects in $\mathcal{A}$ equipped with additional data such as a framing, or an open substack of $\mathcal{M}$ consisting of objects satisfying a stability condition. The stack of extensions M°_{Ext} again parametrizes short exact sequences.

3.2.1 Nakajima quiver varieties

The canonical example is the action is on the Nakajima quiver variety with positive stability condition. Let Q be a quiver with vertex set I and edge set E. Choose dimension vector (v_i) and framing vector (w_i). Let

$$R_{v,w} = T^* \bigoplus_{i\in I} \text{Hom}(W_i, V_i) \oplus T^* \bigoplus_{e:i\to j\in E} \text{Hom}(V_i, V_j).$$

Let $\mu : R_{v,w} \to \oplus_i \mathfrak{Z}(\mathfrak{gl}_{V_i})^*$ denote the moment map. Choose a stability parameter $\theta = (\theta_i)$ corresponding to the equivariant line bundle $\otimes_i \det(g_i)^{-\theta_i}$ where $\det(g_i)$ is the determinant character of $\text{GL}(V_i)$. Let $G_v = \prod_i \text{GL}(V_i)$.

Definition 3.2.2. The *Nakajima quiver variety* $M_{\theta,\lambda}(v, w)$ is the GIT quotient

$$\mu^{-1}(\lambda)//_\theta G_v.$$

By results of Schiffmann-Vasserot [38] and Yang-Zhao [35] for $\theta = (1, \ldots, 1)$ and $\lambda = 0$ there is a representation

$$H_*^T(\mathcal{M}_Q) \to \mathrm{End}(\bigoplus_v H_T^*(M(v,w)))$$

which is known to agree [36] with the action of the positive half of the Yangian action

$$\mathbb{Y}(\mathfrak{g}_Q) \to \mathrm{End}(\bigoplus_v H_T^*(M(v,w)))$$

under an isomorphism $H_*^T(\mathcal{M}_Q) \simeq \mathbb{Y}(\mathfrak{g}_Q)^+$ under the triangular decomposition (3.1.9).

3.2.2 Torsion pairs

A general construction of modules of CoHAs is given by the construction of Diaconescu-Porta-Sala using torsion pairs [39]. In fact, we expect that the positive halves of the geometric actions on moduli spaces of sheaves on surfaces agree with the action of a cohomological hall algebra action on the torsion-free part of a torsion pair, analogous to the identification between the cohomological hall algebra action and the action of the Maulik-Okounkov Yangian.

Their setting gives a context which provides the properness of the map π_2 in the diagram necessary to define the action.

Let $\mathcal{A}$ be an abelian category.

Definition 3.2.3. A *Serre subcategory* $\mathcal{A}' \subset \mathcal{A}$ is a subcategory such that for any short exact sequence

$$0 \to E_1 \to E_3 \to E_2 \to 0$$

we have $E_3 \in \mathcal{A}'$ if and only if $E_1, E_2 \in \mathcal{A}'$.

Definition 3.2.4. A *torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is a pair of full subcategories such that $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$ and every $E \in \mathcal{A}$ fits in to an exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Then T is referred to as the torsion part of E while F is the torsionfree part. This is motivated by the main example of $\mathcal{A} = \mathrm{Coh}(X)$ for a variety X and $\mathcal{T}$ is the subcategory of torsion sheaves while $\mathcal{T}$ is the category of torsionfree sheaves.

The cohomological hall algebra multiplication of an abelian category restricts to a product on the stack of objects $\mathcal{M}_{\mathcal{A}'\subset\mathcal{A}}$ in a Serre subcategory $\mathcal{A}'$ of $\mathcal{A}$ as long as $\mathcal{M}_{\mathcal{A}'\subset\mathcal{A}}$ is an open substack. Let S be a proper surface. This property is satisfied for the Serre subcategory of torsion sheaves on a surface $\mathrm{Coh}_{tors}(S)$. Likewise the substack of torsionfree sheaves is also open. Furthermore, as special case of the results in [39] we have the following result. Let

$$Coh_{tf}(S) \subset Coh(S), \qquad Coh_{tors}(S) \subset Coh(S)$$

denote the open substacks of torsion-free or torsion sheaves respectively.

Theorem 3.2.5 ([39])**.** *The torsion CoHA*

$$H_*(Coh_{tors}(S))$$

acts via multiplication on the left on the module

$$H_*(\mathrm{Coh}_{tf}(S)).$$

3.2.3 Nakajima operators

Because the stack of torsion sheaves contains the stack of dimension zero sheaves, the dimension zero CoHA is a subalgebra of the torsion CoHA. Furthermore, the module $H_*(Coh_{tors})$ splits into a direct sum of modules for torsion-free coherent sheaves of a fixed rank $Coh_{tf,r}$. When $r = 1$, the stack of torsion-free coherent sheaves is representable by a space

$$\mathrm{Hilb}_S = \bigsqcup_{n\geq 0} S^{[n]}$$

where $S^{[n]}$ is the moduli space of ideal sheaves of length n subschemes of S, the *Hilbert scheme of points* of S. There is an identification

$$Coh_{tf,1} \simeq \mathrm{Hilb}_S \times B\mathbb{G}_m$$

and the CoHA of dimension zero sheaves acts on the Hilbert scheme [31].

In particular, the negative modes of the currents $\mathfrak{L}^0(\gamma)$ from (3.1.6) act on

$$\mathcal{F}_S := \bigoplus_{n\geq 0} H^*(S^{[n]})$$

and moreover letting $\mathbb{H} =^* (S)$ with its Poicaré pairing $\langle , \rangle$ we have

Theorem 3.2.6 ([40, 41])**.** *There is an isomorphism*

$$\mathcal{F}_S \simeq \mathcal{F}_{\mathbb{H}}$$

between the cohomology of the Hilbert schemes and the vacuum module for the free super bosons valued in $\mathbb{H}$*.*

Under the isomorphism, the modes of the generating fields $\alpha_-k(\gamma)$ for $k > 0$ act via the correspondences $\pi_{12*}(P_k \cup \pi_3^*\gamma)$ where $P_k(\gamma) \subset S^{[n]} \times S^{[n+k]} \times S$ is the cycle

$$P_k = \{I_1, I_2, x \mid I_2 \subset I_1, \mathrm{supp}(I_1/I_2) = x\}.$$

The negative modes $\alpha_{-k}(\gamma)$ for $k < 0$ act via the $\alpha_k(\gamma)^\tau$ where for an operator $\phi : H^*(X) \to H^*(Y)$ its adjoint ϕ^τ is defined so that

$$\langle a, \phi b\rangle_Y = (-1)^{(\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y)/2} \langle \phi^\tau a, b\rangle_X. \tag{3.2.7}$$

Remark 3.2.8. When S is not proper but has a T-action with proper fixed points, the same result of Theorem 3.2.6 holds with the induced equivariant cohomology on $S^{[n]}$ and with $\mathbb{H} = H^*_T(S)$.

The preceding narrative has proceeded in almost the exact opposite of the proper historical order of discovery, in which the pioneering results of Nakajima and Grojnowski inspired a tremendous amount of activity in trying to understand cohomological Hall algebras of surfaces and related abelian categories.

The result of Theorem 3.2.6 together with results in [40] connecting lattice vertex algebras to other rank 1 torsionfree sheaves provide a vertex operator algebra language to study the representations of cohomological hall algebras of surfaces. In particular, if

$$\mathrm{NS}(S) = \mathrm{Pic}(S)/\mathrm{Pic}_0(S)$$

is the Neron-Severì lattice of S then consider the vertex algebra

$$V_S = \mathcal{F}_{H^*(S)/H^2_{alg}(S)} \otimes V(\mathrm{NS}(S))$$

which is the tensor product of the lattice vertex algebra modelled on $\mathrm{NS}(S)$, $H^2_{alg}(S) = \mathbb{k} \otimes_{\mathbb{Z}} \mathrm{NS}(S)$ and $\mathcal{F}_{H^*(S)/H^2_{alg}(S)}$ a free bosonic vertex algebra modelled on the remaining cohomology classes of S. Then there is an isomorphism

$$V_S = \bigoplus_{d \in \mathbb{Z}, \lambda \in \mathrm{NS}(S)} H^*(M^0((1,0,0) + d[\mathrm{pt}] + \lambda))$$

between the cohomology of the moduli spaces

$$M^0(v) = \{E \in \mathrm{Coh}(S) \mid \mathrm{ch}(E) = v, \det(E) = \mathcal{O}_S\}$$

of rank 1 torsion-free sheaves with trivial determinant and the vertex algebra.

3.2.4 Torsion sheaves on T^*C

Here we outline some slightly imprecise predictions for the action of the cohomological hall algebra of torsion sheaves on the equivariant holomorphic symplectic surface T^*C where C is a smooth genus g curve.

Action of zero modes of bc-system

The moduli space of rank 1 torsion free sheaves on T^*C admits a map

$$M(1, c_1, d) \xrightarrow{\det} \mathrm{Pic}^{c_1}(T^*C) \tag{3.2.9}$$

with target

$$\mathrm{Pic}^{c_1}(T^*C) \simeq Pic^0(T^*C) \simeq \mathrm{Jac}(C)$$

by the usual determinant construction [42]. Here $\mathrm{Pic}^{c_1}(T^*C)$ is the moduli space of line bundles with first chern class c_1.

Given a basis

$$\sigma_{+,1}, \dots, \sigma_{+,g}, \sigma_{-,1}, \dots, \sigma_{-,g} \tag{3.2.10}$$

for $H^{\underline{1}}(C)$ of A and B cycles so that $\langle \sigma_{+,i}, \sigma_{-,j} \rangle = \delta_{ij}$. Fix a point $p \in C$. The map $\iota : C \to \mathrm{Jac}(C)$ given by $x \mapsto O_C(x - p)$ induces a map $\iota^* : H^*(\mathrm{Jac}\, C) \to H^*(C)$ which is an isomorphism on $H^1(-)$ and $H^*(\mathrm{Jac}\, C)$ is freely generated as a supercommutative ring by $H^1(\mathrm{Jac}\, C)$. Denote by $\xi_{\pm,i}$ the class such that $\iota^*(\xi_{\pm,i}) = \sigma_{\pm,i}$.

There is an action

$$\mathrm{Pic}^0(T^*C) \times M(1, c_1, d) \xrightarrow{m_\otimes} M(1, c_1, d) \tag{3.2.11}$$

$$(\mathcal{L}, E) \mapsto \mathcal{L} \otimes E \tag{3.2.12}$$

which coves the multiplication map $\mathrm{Jac}(C) \times \mathrm{Jac}(C) \xrightarrow{m} \mathrm{Jac}(C)$ on the base of (3.2.9).

Given $\gamma \in H^*(\mathrm{Jac}(C))$ let

$$\gamma \star_{\otimes} - : H^*(M(1, c_1, d)) \to H^*(M(1, c_1, d))$$

denote $m_{\otimes *}(\gamma \boxtimes -)$.

We have an decomposition of the vertex algebra

$$V_{\mathbb{H}} \simeq V_{\mathbb{H}^0(C)} \otimes \mathcal{SF}^{\otimes g}$$

so that the fields

$$\phi(\sigma_{\pm,i}, z) = \int \alpha(\sigma_{\pm,i}, z), \qquad i = 1, \ldots, g$$

are identified with the fields of the logarithmic extension of the bc system in Example 2.2.17. By assigning

$$\alpha_0(\sigma_{\pm,i}) \mapsto \xi_{\pm,i} \cup - \tag{3.2.13}$$

$$\alpha_{\log}(\sigma_{\pm,i}) \mapsto \xi_{\mp,i} \star_{\otimes} - \tag{3.2.14}$$

In the notation of the bc-system, the modes $\alpha_{\log}(\sigma_{\pm,i})$ are denoted $\theta^{b/c,i}$.

Chapter 4: Moduli spaces of objects on surfaces

In this chapter, we explain the main classes of example of moduli spaces whose cohomology groups provide weight spaces of representations of vertex algebras and of quantum groups.

There are a number of reasons that moduli spaces of objects in the derived category arise in the study of geometric representation theory, even if one is only concerned with moduli spaces of sheaves. Among these reasons we highlight two. Firstly, automorphisms of geometrically defined algebras associated to a variety X may arise via derived equivalences which do not preserve the standard heart $\mathrm{Coh} \subset D^b(\mathrm{Coh}(X))$. Secondly, birational models of moduli spaces of sheaves often admit descriptions as moduli space of objects in $D^b(\mathrm{Coh}(X))$.

4.1 Elliptic surfaces

An elliptic surface S is (for us) a smooth quasiprojective surface over with a map $\pi : S \to B$ whose generic fiber is a smooth elliptic curve and a section $\sigma : B \to S$. A minimal elliptic surface is one not admitting a contraction $S \to S'$ to a smooth S' which also admits an elliptic fibration. Over the complex numbers, the singular fibers of minimal elliptic surfaces were classified by Kodaira and admit an affine ADE classification. The ADE classification is not a bijection and they are labelled by indices

$$I_k, k \geq 0, \quad I_k^*, k \geq 0, \quad II, III, IV, II^*, III^*, IV^*.$$

The intersection pairing on the 1-cycles contracted by π is even on account of minimality and negative semidefinite. The dual graph of a basis of irreducible 1-cycles for the curves contracted by π is the Dynkin diagram of an ADE root system of type given by Table 4.1.

I_k	I_k^*	II	III	IV	II^*	III^*	IV^*
$\widehat{A}_{k-1}$	$\widehat{D}_{k+4}$	$\widehat{A}_0$	$\widehat{A}_1$	$\widehat{A}_2$	$\widehat{E}_8$	$\widehat{E}_7$	$\widehat{E}_6$

Table 4.1: Singular fibers and corresponding affine root systems

4.1.1 Equivariant elliptic surfaces

The elliptic surfaces which serve as the main examples of this text are those admitting a $\mathbb{C}^*_\hbar$ action such that the elliptic fibration $\pi : S \to \mathbb{A}^1$ is equivariant with respect to this map where $\mathbb{C}^*_\hbar$ scales $\mathbb{A}^1$ with a positive weight.

Two consequences immediately follow: there is at most one singular fiber which can only be located over $0 \in \mathbb{A}^1$, and all other fibers are mutually isomorphic. In particular, the j-invariant map $\mathbb{A}^1 \xrightarrow{j} \mathbb{P}^1$ must be constant and because type $I_k, k > 1$ and $I_k^*, k > 1$ singular fibers only appear at poles of $j(t)$ the singular fiber can only be of type in

$$\Sigma \in \{I_0, I_0^*, II, III, IV, II^*, III^*, IV^*\}$$

. If τ_Σ is the elliptic parameter associated to any fiber of an equivariant elliptic surface with a singular fiber of type Σ at 0 then after acting by $\mathrm{SL}(2,\mathbb{Z})$ so that τ_Σ lies in the fundamental domain D in the upper half plane, we have that $\tau_{I_0^*}\tau_{I_0}$ may be anything but the remaining values must lie on the boundary of the domain as indicated in Figure 4.1.

We can uniquely identify the type of singular fiber a minimal equivariant elliptic surface over $\mathbb{A}^1$ by the finite Cartan type of the affine Cartan type associated with the singular fiber by Table 4.1.

Proposition 4.1.1. *There are two infinite families $X_{A_{-1},j} A_{D_4,j} j \in \mathbb{A}^1$ with arbitrary j invariant and 6 isolated equivariant elliptic surfaces $X_{A_0}, X_{A_1}, X_{A_2}, X_{E_6}, X_{E_7}, X_{E_8}$ with the indicated singular fiber.*

We will omit the j index when it is irrelevant. We give two constructions of these surfaces. For the surfaces A_R for $R \in \{A_{-1}, D_4, E_6, E_7, E_8\}$ let $\Gamma \in \{\mathbb{Z}/1\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}\}$ be a

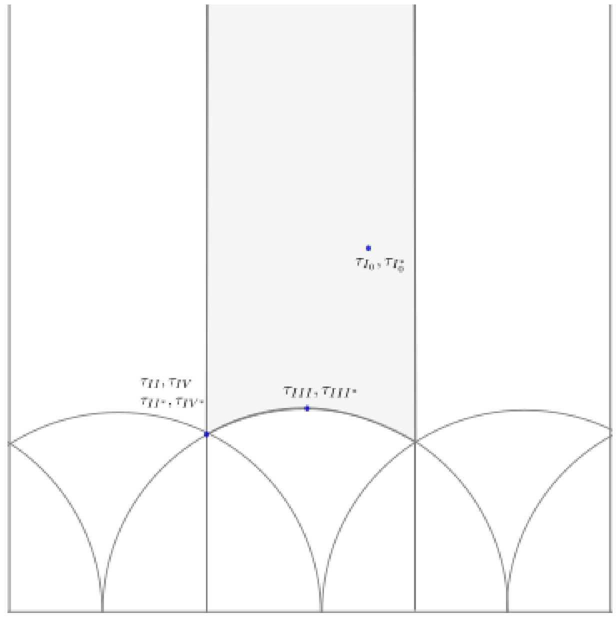


Figure 4.1: The values of τ_Σ in the upper half plane

group of automorphisms of an elliptic curve E. Then there is a resolution

$$X_R \to T^*E/\Gamma$$

where $g \in \Gamma$ acts on $T^*E = E \times \mathbb{A}^1$ by $g \cdot (E, z) \mapsto (g \cdot E, \rho(g)z)$ for a character of Γ. The resulting surface is minimal because it is a resolution of symplectic singularities on a symplectic quotient.

For the type A_0, A_1, A_2 surfaces, construct an auxiliary surface S_R as a resolution

$$S_R \to TE/\Gamma \tag{4.1.2}$$

where Γ acts on $TE = E \times \mathbb{A}^1$ by $g(E, z) \mapsto (E, \rho^{-1}z)$. Then X_R is obtained as a contraction

$$S_R \to X_R \tag{4.1.3}$$

to a minimal elliptic surface X_R.

The resolution of such non-Gorenstein finite quotient singularities with respect to groups

$$\Gamma \subset \mathrm{GL}(2, \mathbb{C})$$

admits a quiver description in terms of the Wemyss reconstruction algebras, generalizing the ADE classification of the McKay correspondence [43, 44] for which we only need the cyclic case. Let $\Gamma = \langle g \rangle \subset \mathrm{GL}(2, \mathbb{C})$ be a cyclic group denoted $\frac{1}{r}(1, a)$ generated by

$$g = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix}$$

with r/a in reduced terms and ϵ an rth root of unity. Consider the Jung-Hirzebruch continued fraction

$$r/a = \alpha_1 - \frac{1}{\alpha_2 - \frac{1}{\alpha_3 - \cdots}}$$

with each $\alpha_i \geq 2$. Denote the resulting sequence $[\alpha_1, \ldots, \alpha_n]$. The following is classical, c.f. [45] for references.

Theorem 4.1.4. *The exceptional fiber of the resolution of $\mathbb{A}^2/\Gamma$ has a chain $E_1, \ldots, E_n$ of rational curves of self intersections*

$$\langle C_i \cdot C_j \rangle = -\alpha_i \delta_{i+j,0} + 1\delta_{i,j+1} + 1\delta_{j,i+1}.$$

When $\gcd(r, a) = \ell > 1$ the action of $\mathbb{Z}/\ell\mathbb{Z}$ induced by g^ℓ produces no singularity and the action of $\mathbb{Z}/\ell\mathbb{Z}$ is of the form $\frac{1}{\ell}(b, 1)$ so eventually the theorem applies.

Remark 4.1.5. The resolution is no longer derived equivalent to the stack quotient but we may instead use semiorthogonal decomposition provided in (4.1.18) below in order to understand the derived category of the relatively minimal elliptic surfaces in terms of those on the stack quotient.

Example 4.1.6. (X_{A_2} surface) The quotient TE/Γ has three singularities equivalent to the action of

$\mathbb{Z}/3\mathbb{Z}$ on $\mathbb{A}^2$ of the form $(x, y) \mapsto (\chi(\gamma)x, \chi(\gamma)x)$. Let E_i, $i = 1, 2, 3$ denote the three exceptional fibres with $E_i^2 = -3$ by Theorem 4.1.4. Let E_c be the reduced class induced by the strict transform of the zero section of TE. The elliptic fiber $E_1 + E_2 + E_3 + 3E_c$ has self intersection zero and thus we calculate the intersection matrix is of the form

$$\begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

After blowing down E_c we are left with an affine A_2 collection of -2 curves.

Example 4.1.7. (X_{A_1} surface) Here $\Gamma = \mathbb{Z}/4\mathbb{Z}$ and the quotient TE/Γ has has two singularities isomorphic to quotients by $\frac{1}{4}(1, 1)$ and one isomorphic to that by $\frac{1}{4}(1, 2)$. The resulting resolution has exceptional curves E_1, E_1' of self-intersection -4 and a curve E_2 of self intersection -2, and $1/4$ the strict transform of the zero section E_c with self intersection -1. Contracting E_c followed by E_2 gives a type III singular fiber.

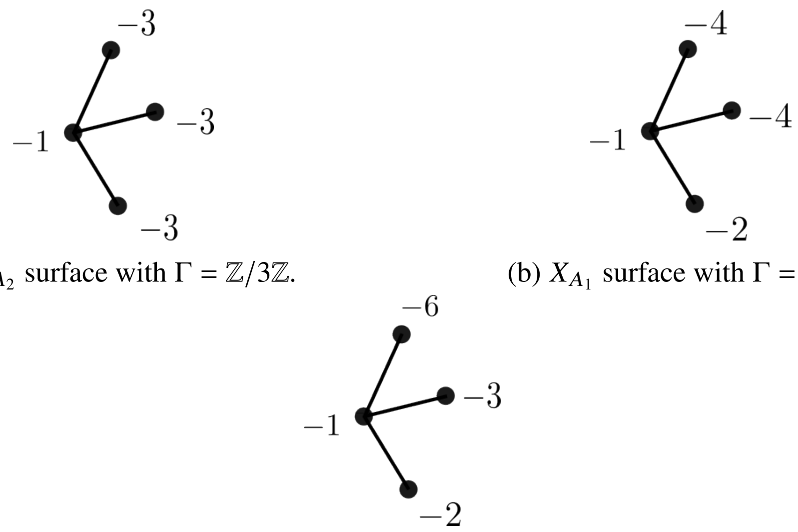
Example 4.1.8. (X_{A_0} surface) The quotient TE/Γ has singularities of type $\frac{1}{6}(1, 1)$, $\frac{1}{6}(1, 2)$, and $\frac{1}{6}(1, 3)$. The resolution has curves E_6, E_3, E_2, E_c of self intersection $-6, -3, -2, -1$ and blowing down E_c followed by E_2 followed by E_3 results in a type II cuspidal singular elliptic fiber.

To study the equivariant cohomology of the resulting surface, we use an alternate construction as a resolution of a singular Weierstrass model. We are only concerned with the special case producing rational elliptic surfaces.

Consider the $\mathbb{P}^2$ bundle

$$Y = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$$

with fiber homogeneous coordinates $[X : Y : Z]$ and coordinate $z \in \mathbb{P}^1$. Given sections $a(t) \in H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ and $b(t) \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ then the twisted family of plane cubics S_0 is the zero set of



(a) X_{A_2} surface with $\Gamma = \mathbb{Z}/3\mathbb{Z}$. (b) X_{A_1} surface with $\Gamma = \mathbb{Z}/4\mathbb{Z}$.

(c) X_{A_0} surface with $\Gamma = \mathbb{Z}/6\mathbb{Z}$.

Figure 4.2: Intersection diagrams after resolving cyclic quotient singularities.

the equation

$$Y^2 Z = X^3 + a(t)XZ^2 + b(t)Z^3. \tag{4.1.9}$$

Because of the equality

$$j(t) = 1728 \frac{4a(t)^3}{4a(t)^3 + 27b(t)^2} \tag{4.1.10}$$

we can only obtain isotrivial surfaces when $a(t) = 0$ or $b(t) = 0$. In this case there is an additional $\mathbb{C}^*$ action coming from coordinate actions on (t, X, Y, Z) which preserves (4.1.9) and this gives rise to an action on S_0. There are options depending on the order of the zero of $a(t)$ or $b(t)$. For the following also recall that if we blowup a point p on a curve C with $C^2 = k$ which is smooth on C and the surface then the strict transform $\widetilde{C}$ satisfies $C^2 = k - 1$.

$a(t) = t, b(t) = 0$: In this case at $(0, 0, 0)$ in the $Z \neq 0$ chart there is a singularity

$$y^2 = x^3 + tx$$

which is an A_1 singularity. This is the only singularity of the surface and its resolution has two rational curves meeting at a double point, giving the surface X_{A_1}. The group action in coordinates descents the one of $\mathbb{A}^3$ with coordinates (x, y, t) with $\mathbb{C}^*_\hbar$-weights

$$(2\hbar, 3\hbar, 4\hbar).$$

$a(t) = 0, b(t) = t$: In this case there is no singularity and we immediately recover the X_{A_0} surface. The weights at 0 are

$$(2\hbar, 3\hbar, 6\hbar).$$

$a(t) = 0, b(t) = t^2$: The singularity at $(0, 0, 0)$ is an A_2 singularity whose blowup gives the type IV singular fiber. The weights of the action at zero before the blowup are

$$(2\hbar, 3\hbar, 3\hbar).$$

The weights at the torus fixed points on the blownup charts and at infinity are recorded in Table 2 in [1].

4.1.2 Tubular canonical algebras

Given E an elliptic curve and $\Gamma \subset \mathrm{Aut}(E)$ there is an alternative description of $D^b(\mathrm{Coh}[E/\Gamma])$ as the derived category of modules for a path algebra of a quiver with relations. See [46, 47] More generally, given a weighted projective line C with points x_i, $i = 1, n$ of order ℓ_i so that in $K_num(C)$ we have $\ell_i[x_i] = [\mathrm{pt}]$ where pt is the class of a non-orbifold point, there is a set of projective generators

$$\{O_C, O_C(\mathrm{pt}), O_C(kx_i) \mid i = 1, \ldots n, k = 1, \ldots, \ell_i - 1\}$$

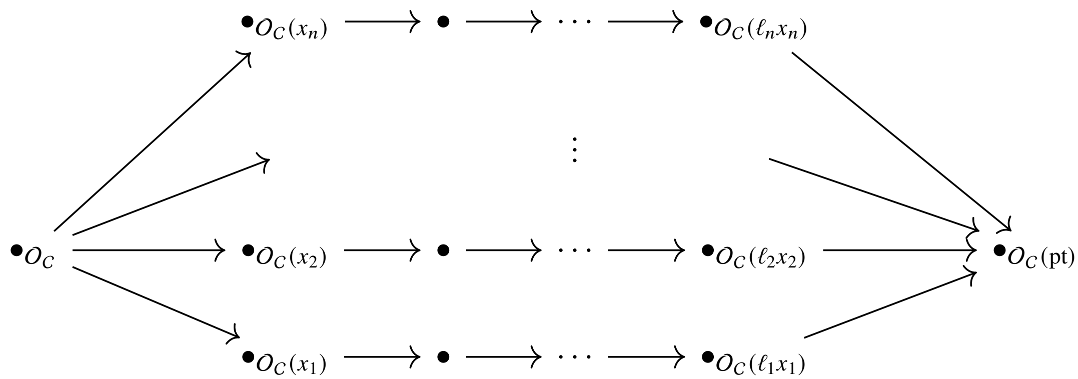
so that

$$\mathcal{A} = O_C \oplus O_C(\mathrm{pt}) \oplus \bigoplus_{\substack{1 \le i \le n \\ 1 \le k \le \ell_i - 1}} O_C(kx_i) \tag{4.1.11}$$

satisfies

$$D^b(\mathrm{Coh}[E/\Gamma]) \simeq D^b(\mathrm{Rep}(\mathrm{End}(\mathcal{A}))). \tag{4.1.12}$$

The endomorphisms End($\mathcal{A}$) are described as the path algebra of the quiver



Letting $\iota_i^{(j)} : \mathcal{O}_C((i-1)x_j) \to \mathcal{O}_C(ix_j)$ the relations depend on a $n-2$ tuple $(\lambda_3, \dots, \lambda_n)$ of pairwise distinct non-zero elements of $\mathbb{k}$ and are described by the equation

$$\iota_{\ell_j}^{(j)} \iota_{\ell_j - 1}^{(j)} \cdots \iota_1^{(j)} = \iota_{\ell_1}^{(1)} \iota_{\ell_1 - 1}^{(1)} \cdots \iota_1^{(1)} + \lambda_j \iota_{\ell_2}^{(2)} \iota_{\ell_2 - 1}^{(2)} \cdots \iota_2^{(2)}$$

where by rescaling we can $\lambda_3 = 1$ and more generally we view λ_j as the position of the jth point of $x_1 = 0, x_2 = \infty$. Let $\Lambda = \Lambda(x, \lambda) = \text{End}(\mathcal{A})$ denote the resulting algebra.

There is always a finite extension of the group $- \otimes \mathcal{L}$ for $\mathcal{L} \in \text{Pic}(C)$ which acts by derived autoequivalences on Λ. In particular there is the *Auslander-Reiten transition* $\tau(-) = - \otimes \omega$ There is an enhanced group of derived autoequivalences of $D^b(\Lambda)$ exactly when the weighted projective line arises as $[E/\Gamma]$ for Γ a finite group [47, 48, 49]. There are generated given by the mutation functors associated to a simple sheaf $\mathcal{S} \in D^b(\text{Coh}([E/\Gamma]))$ and defined by $\text{tw}_{\mathcal{S}}(E)$ as the cone

$$\bigoplus_{\substack{i \in \mathbb{Z} \\ 1 \leq j \leq p(\mathcal{S})}} \text{Hom}(\mathcal{S} \otimes \omega^j[-i], E) \otimes (\mathcal{S} \otimes \omega^j)[-i] \xrightarrow{\text{ev}} E \to \text{tw}_{\mathcal{S}}(E) \to \tag{4.1.13}$$

where p is the order of the action of $\tau(-)$ on $\mathcal{S}$, which is finite because we started with an elliptic curve.

The group descends to an action on $K_{num}(C)$ and together with the group generated by $\mathbb{D} =$

$\mathcal{H}om(-, O_C)$ and $-\otimes\mathcal{L}$, $\mathcal{L}\in \mathrm{Pic}(C)$ preserves the space spanned by [pt] and $[O_C]$ and the action on this space surjects onto $\mathrm{GL}(2,\mathbb{Z})$.

4.1.3 Fourier-Mukai transform

The equivariant elliptic surfaces X_R or certain compactifications of them admit enhanced groups of autoequivalences. These are based on relative versions, worked out in [50] c.f. [51] of the equivalences for abelian varieties discovered by Mukai [52].

We distinguish two cases where different descriptions of the derived equivalences are more suitable.

For the equivariant elliptic surfaces $X_{A_0}, X_{A_1}, X_{A_2}$, the fibers of the relative compactified Jacobian

$$\overline{J}(X)/\mathbb{A}^1$$

parametrizes torsion-free sheaves on the fibers of the map $X \to \mathbb{A}^1$ and we have an isomorphism

$$X \simeq \overline{J}(X)/\mathbb{A}^1 \tag{4.1.14}$$

induced by the section class as well as a relative Poincaré sheaf

$$\mathcal{P} \in D^b(\mathrm{Coh}(X \times \overline{J}(X)/\mathbb{A}^1))$$

inducing a Fourier-Mukai transform

$$\pi_{2*}(\pi_1^*(-) \otimes \mathcal{P}) : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(\overline{J}(X)))$$

which when composing with the isomorphism (4.1.14) gives an autoequivalence of $D^b(\mathrm{Coh}(X))$.

For the cases of X_R when $R \in \{D_4, E_6, E_7, E_8\}$ the relative compactified Jacobian is somewhat harder to completely describe explicitly.

thus instead we use the fact that for the surface $\overline{X}_R = \mathbb{P}_{[E/\Gamma]}(\mathcal{T} \oplus O)$ which is a projective line over the elliptic orbifold, we have a semiorthogonal decomposition

$$D^b(\mathrm{Coh}(\overline{X_R})) = \langle D^b(\mathrm{Coh}([E/\Gamma])), D^b(\mathrm{Coh}([E/\Gamma]))\rangle \tag{4.1.15}$$

which is induced by a relative Beilinson resolution over $[E/\Gamma]$. Furthermore, it implies a triangulated equivalence between $D^b(\mathrm{Coh}(\overline{X_R}))$ and a derived category $\mathrm{kos}_{[E/\Gamma]}(\mathcal{T} \oplus \mathcal{O})$ of *Koszul data* consisting of triples (a, b, η) with $a, b \in D^b(\mathrm{Coh}(\overline{X_R}))$ and $\eta : a \otimes (\mathcal{T} \oplus \mathcal{O}) \to b$ a morphism. Maps between Koszul data (a_1, b_1, η_1) to (a_2, b_2, η_2) are pairs of maps $\phi_a : a_1 \to a_2$ and $\phi_b : b_1 \to b_2$ with $\eta_2 \circ \phi_a = \phi_b \circ \eta_1$.

The derived equivalence with $\mathrm{kos}_{[E/\Gamma]}(\mathcal{T} \oplus \mathcal{O})$ is a relative version on the equivalence (4.1.12) in the special case where there are no points, and where the category of vector spaces over the base field $\Bbbk$ is replaced with $\mathrm{Coh}([E/\Gamma])$.

While this is a nontrivial fact [53], the autoequivalences of $D^b(\mathrm{Coh}([E/\Gamma]))$ do lift to the derived category $D^b(\mathrm{Coh}(\overline{X_R}))$.

Other derived equivalences are given by *spherical twists* introduced by Seidel and Thomas [54]. Let D be a smooth proper triangulated category.

Definition 4.1.16. A *spherical object* $S \in D$ is an object such that $\mathrm{End}(S) \simeq \Bbbk \oplus \Bbbk[2]$.

Then the twist functor $T_S(-)$ defined by the (functorial) cone

$$\mathrm{Hom}(S, E) \otimes S \xrightarrow{\mathrm{ev}}\to E \to T_S(E) \to \tag{4.1.17}$$

gives an autoequivalence of D.

Semiorthogonal decompositions

For the stacky surfaces $\overline{X_R}$ consider surfaces $\overline{S_R}'$ is the minimal resolution obtained by blowing up all cyclic quotient singularities and $\overline{S_R}$ which is the relatively minimal elliptic surface obtained as a blowdown

$$\overline{S_R}' \to \overline{S_R}$$

in (-1) curves in the fiber over $\infty \in \mathbb{P}^1$.

Ishii and Ueda [55] have shown in the general situation of projective surfaces with cyclic quotient singularities and their canonical stacks that the derived category of $\overline{S_R}'$ embeds as a semiorthogonal factor in that of $\overline{X_R}$ so that

$$D^b(\mathrm{Coh}(\overline{X}_R)) = \langle D^b(\mathrm{Coh}(\overline{S_R}')), E_1, \ldots, E_k\rangle$$

for exceptional objects $E_1, \ldots, E_r$.

Moreover, by the blowup formula [56] there is a semiorthogonal decomposition

$$D^b(\mathrm{Coh}(\overline{S_R}')) = \langle D^b(\mathrm{Coh}(\overline{S_R})), F_1, \ldots, F_\ell\rangle$$

combining these decompositions demonstrates a semiorthogonal decomposition

$$D^b(\mathrm{Coh}(\overline{X_R})) = \langle D^b(\mathrm{Coh}(\overline{S_R})), \mathcal{A}\rangle \tag{4.1.18}$$

where $\mathcal{A}$ is generated by $\mathcal{E}_1, \ldots, \mathcal{E}_k, \mathcal{F}_1, \ldots, \mathcal{F}_\ell$. We may thus study moduli of objects in the derived category $D^b(\mathrm{Coh}(\overline{X_R}))$ which lie in the first factor of (4.1.18) in order to understand moduli of objects on $\overline{S_R}$.

Elliptic Weyl groups

One result of [2] is an identification of the action on numerical K theory of the group generated by spherical twists, derived duality, and the $\mathrm{SL}(2, \mathbb{Z})$ group of autoequivalences induced by relative Fourier-Mukai transforms. It is expressed in terms of elliptic root systems. An elliptic root system [57, 58] is a set of roots $R = R^{re} \sqcup R^{im}$ in a Euclidean space E with a degenerate symmetric pairing $\langle -, - \rangle$ with 2-dimensional radical satisfying certain axioms. We take the definition from [1] which differs in the(quite degenerate) case relevant to the surface T^*E. The elliptic Weyl group W_R^{ell} is generated by reflections through the real roots $\alpha \in R^{re}$.

As lattice we have we have that the lattice Q spanned by R is of the form

$$Q_{fin} \oplus K_{num}(E) \simeq Q_{fin} \oplus \mathbb{Z}^{\oplus 2}$$

where Q_{fin} is the lattice spanned by a finite root system R_{fin}. A choice of identification

$$\mathrm{rad}(\langle -, - \rangle) \cap R \xrightarrow{\sim} K_{num}(E)$$

gives splittings

$$0 \to Q_{fin} \to W_R^{ell} \to W_R^{aff} \to 0$$
$$0 \to Q_{fin} \to W_R^{aff} \to W_R^{fin} \to 0.$$

We can extend this definition to the *isotropic extended elliptic Weyl group*

$$IW_R^{ell} \subset \mathrm{Aut}(R)$$

from [2] by adjoining elements which act on rad(⟨-, -⟩) by $\mathrm{GL}(2, \mathbb{Z})$.

4.2 Non commutative surfaces

A deformation of the derived categories of the surfaces $\overline{X_R}$ from (4.1.15) is provided by a family of *noncommutative* $\mathbb{P}^1$*-bundles* over the elliptic orbifold curve $[E/\Gamma]$.

These are generalizations of the construction of the projectivization $\mathbb{P}_X(\mathcal{E})$ of a rank two locally O_X-module when the commutative O_X is replaced with a rank two sheaf bimodule. They were introduced by van den Bergh in [59].

Consider a pair X, Y of n-dimensional smooth projective varieties, or orbifolds $X = [\mathfrak{X}/G], Y =$

$[\mathfrak{Y}/H]$.

Definition 4.2.1. The category of *sheaf bimodules*

$$\text{shbimod}(X, Y) \subset \text{Coh}(X \times Y)$$

is the additive subcategory of sheaves $\mathcal{E}$ so that $\pi_{i*}\mathcal{E}$ is locally free for $i = 1, 2$.

The *rank* of the sheaf bimodule is the rank of $\pi_{i*}\mathcal{E}$.

Remark 4.2.2. When X and Y are orbifolds, the pushforward map $\text{Coh}(X \times Y) \to \text{Coh}(Y)$ generalizes the pushforward map

$$\text{Coh}^G(\mathfrak{X}) \to \text{Coh}(\text{pt})$$
$$\mathcal{E} \mapsto H^0(\mathcal{E})^G.$$

Similar constructions apply to derived functors. See [60] for details on equivariant derived categories.

Given a sheaf bimodule $\mathcal{E} \in \text{shbimod}(X, Y)$ define

$$a \otimes_X \mathcal{E} = \pi_{2*}(\pi_1^* a \otimes \mathcal{E})$$

and likewise given $\mathcal{E} \in \text{shbimod}(X, Y)$ and $\mathcal{F} \in\in \text{shbimod}(Y, Z)$ define $\mathcal{E} \otimes \mathcal{F} \in \text{shbimod}(X, Z)$ by

$$\mathcal{E} \otimes \mathcal{F} = \pi_{13*}(\pi_{12}^*\mathcal{E} \otimes \pi_{23}^*\mathcal{F}).$$

Given $\mathcal{E}$ a sheaf bimodule it has left and right adjoints ${}^*\mathcal{E}$ and $\mathcal{E}^*$ such that

$$\text{Hom}_Y(a \otimes_X \mathcal{E}, b) \simeq \text{Hom}_Y(a, b \otimes_Y \mathcal{E}^*)$$
$$\text{Hom}_Y(a \otimes_X {}^*E, b) \simeq \text{Hom}_Y(a, b \otimes_Y \mathcal{E}).$$

A sheaf $\mathbb{Z}$-algebra over X is a category enriched over shbimod(X) with objects $\mathbb{Z}$ and a single arrow $\mathcal{A}_{ij} : [i] \to [j]$ for each $i \leq j$. Given a $\mathbb{Z}$-graded algebra $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k$ one can construct a $\mathbb{Z}$-algebra so that $\mathcal{A}_{ij} = \mathcal{A}_{j-i}$, but $\mathbb{Z}$-algebras relax the condition that these agree up to translation.

Given a rank 2 sheaf bimodule $\mathcal{E}$, van den Bergh constructs a *sheaf $\mathbb{Z}$-algebra* denoted $\mathcal{A}^{\mathcal{E}}$. This algebra is freely generated by the ith iterated adjoint $\mathcal{A}^{\mathcal{E}}_{i,i+1} = \mathcal{E}^{*i}$ where $\mathcal{E}^{*-k} = {}^{*k}\mathcal{E}$ for $k > 0$, modulo the $\mathbb{Z}$-ideal generated by the images

$$Q_i := \mathrm{im}(\mathcal{O}_{\Delta_X} \to \mathcal{E}^{*i} \otimes_X \mathcal{E}^{*(i+1)})$$

generated by the adjunction. When $\mathcal{E} = \mathcal{O}_X \otimes V_{x,y}$ is a rank 2 bundle on the diagonal with local fiber sections x, y, each $Q_i \subset \mathcal{E} \otimes_X \mathcal{E}$ is generated by the section $x \otimes y - y \otimes x$ and so this construction generalizes the symmetric algebra on a locally free sheaf.

An $\mathcal{A}^{\mathcal{E}}$-module is a collection $(M_i)_{i \in \mathbb{Z}}$ of $\mathcal{O}_X$-modules with multiplication maps $M_i \otimes_X \mathcal{A}_{ij} \to M_j$. Denote by $\mathrm{Gr}(\mathcal{A})$ the category of modules and $\mathrm{gr}(\mathcal{A})$ the full subcategory of Noetherian modules. There is a Serre subcategory $\mathrm{Tors}(\mathcal{A}) \subset \mathrm{Gr}(\mathcal{A})$ of direct limits of right bounded modules and its intersection with $\mathrm{gr}(\mathcal{A})$ denoted $\mathrm{tors}(\mathcal{A})$. Then the categories

$$\mathrm{QCoh}(\mathbb{P}^1_X(\mathcal{E})) := \mathrm{Gr}(\mathcal{A})/\mathrm{Tors}(\mathcal{A})$$
$$\mathrm{Coh}(\mathbb{P}^1_X(\mathcal{E})) := \mathrm{gr}(\mathcal{A})/\mathrm{tors}(\mathcal{A})$$

are noncommutative analogues of their usual definitions when $\mathcal{E}$ is a vector bundle supported on the diagonal.

All of the above construction work over a Noetherian base scheme S.

The derived category $D^b(\mathrm{Coh}(\mathbb{P}^1_X(\mathcal{E})))$ admits an S-linear semiorthogonal decomposition

$$D^b(\mathrm{Coh}(\mathbb{P}^1_X(\mathcal{E}))) = \langle D^b(\mathrm{Coh}(X)), D^b(\mathrm{Coh}(X)) \rangle \tag{4.2.3}$$

and likewise there is a triangulated equivalence between $D^b(\mathrm{Coh}(\mathbb{P}^1_X(\mathcal{E})))$ and a category

$$\mathrm{kos}_X(\mathcal{E}) := \{(a, b, \eta) \mid a, b \in \mathrm{Coh}(X), \eta : a \otimes_X \mathcal{E} \to b\} \tag{4.2.4}$$

with morphism as in the undeformed case.

Our example is almost exclusively the case when $X = \mathscr{C} = [E/\Gamma]$ and $\mathcal{E}$ is a deformation of $\mathcal{T}_{\mathscr{C}} \oplus \mathcal{O}_{\mathscr{C}}$ so that $\mathbb{P}^1_{\mathscr{C}}(\mathcal{E})$ deforms $\mathbb{P}^1_{\mathscr{C}}(\mathcal{T}_{\mathscr{C}} \oplus \mathcal{O}_{\mathscr{C}}) = \mathrm{Tot}(\omega_{\mathscr{C}})$.

The stack of sheaf bimodules

$$shbimod(X, X) \subset Coh(X \times X)$$

is an open substack of the stack of coherent sheaves. The tangent complex at $\mathcal{E} \in shbimod(X, X)$ is

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})[1]$$

and so the tangent space to $\mathcal{T}_{\mathscr{C}} \oplus \mathcal{O}_{\mathscr{C}}$ in the moduli of sheaf bimodules is the group

$$T_{[\mathcal{E}]} shbimod(X, X) = \mathrm{Ext}^1(T_{\mathscr{C}} \oplus \mathcal{O}_{\mathscr{C}}, T_{\mathscr{C}} \oplus \mathcal{O}_{\mathscr{C}}). \tag{4.2.5}$$

If the group $\mathbb{C}^*_\hbar$ scales $\mathcal{T}_{\mathscr{C}}$ with weight $-\hbar$ then the subspace of $T_{[\mathcal{E}]} shbimod(X, X)$ of positive weight is

$$B = \mathrm{Ext}^1_{X \times X}(\Delta_* \mathcal{T}_{\mathscr{C}}, \mathcal{O}_\Delta).$$

The Hochschild homology module of a derived category of coherent sheaves $D^b(\mathrm{Coh}\, X)$ are the groups

$$HH_i(X) := \mathrm{Hom}(\Delta_! \mathcal{O}_X[i], \mathcal{O}_\Delta)$$

and so we have an identification

$$B = HH_0([E/\Gamma]). \tag{4.2.6}$$

The construction $HH_i(X)$ is functorial for autoequivalences of dg-enhanced derived categories. Thus given $\Phi \in \mathrm{Aut}(D^b(\mathrm{Coh}([E/\Gamma])))$ there is a map $\Phi^{HH} : HH_0([E/\Gamma]) \to HH_0([E/\Gamma])$. This action lifts to an action on the family of derived categories of noncommutative $\mathbb{P}^1$-bundles

$$D^b(\mathrm{Coh}(\mathbb{P}^1_{[E/\Gamma]}(\Xi))/B) \tag{4.2.7}$$

where $\Xi \in \mathrm{shbimod}\,[E/\Gamma]_B$, $[E/\Gamma]_B$ is the universal extension

$$0 \to \mathcal{O}_B \boxtimes \Delta_* \mathcal{T}_{[E/\Gamma]} \to \Xi \to \mathcal{O}_B \boxtimes \mathcal{O}_\Delta \to 0.$$

The action of $\Phi \in \mathrm{Aut}(D^b(\mathrm{Coh}([E/\Gamma])))$ on $D^b(\mathrm{Coh}(\mathbb{P}^1_{[E/\Gamma]}(\Xi))/B)$ is B-linear after base change along Φ^{HH}. There is also a deformation of the functor $\mathbb{D}(-)$ over the commutative $\mathbb{P}^1$-bundle which acts by a reflection in the base, sending the bimodule extension $\mathcal{E}$ to $\mathcal{E}^\tau$, the transposed sheaf bimodule without taking an adjoint, and on Koszul data acting via

$$(a, b, \eta) \mapsto (\mathbb{D}_{[E/\Gamma]}(b), \mathbb{D}_{[E/\Gamma]}(a), \eta^*) \tag{4.2.8}$$

where η^* is defined via the adjunction

$$\begin{aligned} \eta \in \mathrm{Hom}(a \otimes_{[E/\Gamma]} \mathcal{E}, b) &\simeq \mathrm{Hom}(a, b \otimes \mathcal{E}^*) \\ &\simeq \mathrm{Hom}(\mathbb{D}_{[E/\Gamma]}(b \otimes \mathcal{E}^*), \mathbb{D}_{[E/\Gamma]}(a)) \\ &\simeq \mathrm{Hom}(\mathbb{D}_{[E/\Gamma]}(b) \otimes \mathcal{E}^\tau, \mathbb{D}_{[E/\Gamma]}(a)) \ni \eta^*. \end{aligned}$$

Futhermore, we can calculate $HH_0([E/\Gamma])$ via the orbifold Hochschild-Konstant-Rosenberg isomorphism [61, 62]

$$HH_*([X/\Gamma]) = \left(\bigoplus_{g \in \Gamma} HH_*(X^g) \right)^\Gamma .$$

Calculating the term $* = 0$ we get an three-way isomorphism

$$HH_0([E/\Gamma]) \simeq K_num([E/\Gamma]) \simeq \widehat{\widehat{\mathfrak{h}}} \tag{4.2.9}$$

where $\mathfrak{h} \simeq \mathfrak{h}_{fin} \oplus K_{num}(E)$ where $\widehat{\widehat{\mathfrak{h}}}$ is the Cartan subalgebra of the toroidal algebra of type $A_{-1}, D_4, E_6, E_7, E_8$ depending on Γ. The second identification is the same as the one from [49].

We summarize the above construction in Theorem 4.2.10 together with an explanation of how (part of, and conjecturally all of) the group IW_R^{ell} acts on the family. In particular we have an action of IW_R^{ell} on $HH_0([E/\Gamma])$ under the isomorphism (4.2.9).

Theorem 4.2.10 ([2])**.** *There exits a family of dg categories $D^b(\mathrm{Coh}(\mathbb{P}^1_{[E/\Gamma]}(\Xi)))$ over*

$$B = HH_0([E/\Gamma]).$$

The subgroup of IW_R^{ell} generated by anti autoequivalences of $[E/\Gamma]$ acts on $D^b(\mathrm{Coh}(\mathbb{P}^1_{[E/\Gamma]}(\Xi)))$ by relative anti autoequivalences covering its action on $HH_0([E/\Gamma])$.

4.3 Relative Bridgeland stability conditions

Definition 4.3.1. ([63]) A (numerical) *Bridgeland stability condition* on a dg category $\mathcal{D}$ is a heart $\mathcal{A}$ of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on the homotopy category of $\mathcal{D}$ and a stability function $Z : K_{num}(\mathcal{A}) \to \mathbb{C}$ satisfying the *support* and *Harder-Narasimhan* properties.

A relative notion of Definition 4.3.1 was introduced in [64], or properly speaking a number of notions of various versions of stability conditions in families, with conditions sufficient to produce good relative moduli spaces of objects in $\mathcal{D}$.

Given a Noetherian base B and an B-linear dg category $\mathcal{D}$, a family of t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}_b$ for every point $b \in B$, not necessarily closed. The t-structures satisfies *universal openness of flatness* is lying in the heart $\mathcal{A}$ is universally an open condition. Likewise a family of stability conditions is a stability condition on each $\mathcal{D}_b$ satisfying a number of axioms, 1) having universally locally constant central charges, 2) universally satisfying openness of geometric stability, 3) universal openness of slicing and 3) the support property. These are Definition 20.1 and Definition 21.15 in [64] for the precise meaning of these terms.

They then prove in particular for families of smooth projective surfaces S/B satisfying the Bogomolov-Gieseker inequality that

Theorem 4.3.2. *[64] Given $\omega, \beta \in N^1(S)_{\mathbb{Q}}$ with ω relatively ample*

1. *There exist families of stability conditions $\underline{\sigma} = (\sigma_{\omega_b,\beta_b})$ on $D^b(\mathrm{Coh}(S)/B)$ agreeing on closed points with those in [65].*

2. *For any relative Mukai vector v there exists a stack $\mathcal{M}_{\underline{\sigma}}(v)$ over B of $\underline{\sigma}$-semistable objects with good moduli space $M_{\underline{\sigma}}(v)$ which is an algebraic space proper over B parametrizing S-equivalence classes of semistable objects.*

3. *There is a relatively strictly nef line bundle $\ell_{\underline{\sigma}}$ on $M_{\underline{\sigma}}(v)$.*

In the above theorem, relatively strictly nef implies that on any effective curve the line bundle has strictly positive degree. Its existence is not enough to prove that $M_{\underline{\sigma}}(v)$ is projective because one needs to prove that it has enough sections to demonstrate ampleness. This has been done in some specific situations such as for K3 surfaces [66, 64] and for a specific wall governing the Bridgeland stability model of the Uhlenbeck compactification [67].

4.3.1 Noncommutative surfaces

While Theorem 4.3.2 does not apply directly to the families of noncommutative surfaces in Theorem 4.2.10, in the $X_{A_{-1}}$ case and subject to technical hypotheses on the assumptions going in to the proof of the existence of relative stability conditions, we were able to show that the results of Theorem 4.3.2 extended to the case of the families of noncommutative $\mathbb{P}^1$ bundles.

In particular, if we define $\mathrm{Pic}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ as a subquotient of $K_{num}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ and the ample cone $\mathrm{Amp}(\mathbb{P}^1_{\mathscr{C}}(\mathcal{E}))$ as in [68] to be number classes given $\omega, \beta \in \mathrm{Pic}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ with ω relatively ample, of the form

$$\omega = NE + f, N \gg 0, \beta \in \mathrm{Pic}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$$

where E is the class equivalent on the central surface to a smooth elliptic fiber.

In [2] we proposed a family $\mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ of stability conditions so that for

$$\underline{\sigma}_{\omega,\beta} \in \mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi)) \subset \mathrm{Stab}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$$

the family deforms a stability condition on the central fiber in which the moduli spaces of framed stable objects with specific framing and Mukai vector admit Lagrangian fibrations.

Theorem 4.3.3. *[2]In type A_{-1} the family $\underline{\sigma}_{\omega,\beta}$ is a family of stability conditions on $\mathbb{P}^1_{\mathscr{C}}(\Xi)$. There exist good moduli spaces $M_{\underline{\sigma}}(v)$ projective over $HH_0(\mathscr{C})$ for the algebraic stacks $\mathcal{M}_{\underline{\sigma}}(v)$.*

4.3.2 Moduli spaces

A relatively quasiprojective open subspace of the moduli spaces of Theorem 4.3.3 is most relevant to geometric representation theory.

Over the entire space $HH_0(\mathscr{C})$ the divisor at infinity remains somehow commutative, in the sense that there exist torsion free sheaves supported at points there. Relatedly, there is a restriction map

$$M_{pug}(\mathbb{P}^1_{\mathscr{C}}(\Xi)) \xrightarrow{(-)|_{D_\infty}} M_{pug}(\mathscr{C})$$

where $M_{pug}(X)$ represents the algebraic stack of perfect universally glueable complexes, constructed in [69] and more or less extended to the noncommutative surfaces in [68], c.f. [2]. At the level of Koszul data the map is the cone of the composition

$$(a, b, \eta) \mapsto \mathrm{Cone}\left(a \to a \otimes_{\mathscr{C}} \mathcal{E} \xrightarrow{\eta} b\right)$$

where $a \to a \otimes_{\mathscr{C}} \mathcal{E}$ is induced by the inclusion $O_\Delta \to \mathcal{E}$.

Given $\mathcal{F}$ in the image of the structure sheaf O_{D_∞} under some derived equivalence of $\mathscr{C}$ define relative framed moduli spaces

$$M_{\underline{\sigma}}(v;\mathcal{F}) \tag{4.3.4}$$

by the pullback

$$\begin{array}{ccc} M_{\underline{\sigma}}(v;\mathcal{F}) & \longrightarrow & [\mathcal{F}] \\ \downarrow & & \downarrow \\ \mathcal{M}_{\underline{\sigma}}(v) & \xrightarrow{(-)|_{D_\infty}} & \mathcal{M}_{pug}(\mathscr{C}). \end{array} \tag{4.3.5}$$

where we denote by the same letter $\mathcal{F}$ the pullback of the sheaf $\mathcal{F}$ to $\mathscr{C}_B$.

The moduli space $M_{\underline{\sigma}}(v;\mathcal{F})$ admits an action of $\mathbb{C}^*_\hbar$ and the structure morphism to B is equivariant with respect to the action on $B \simeq HH_0(\mathscr{C})$.

The main conjectures for the structure of the moduli space $M_{\underline{\sigma}}(v;\mathcal{F})$ describe their relationship with the roots of the toroidal extended affine Lie algebra.

There is a wall-and-chamber structure on $\mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ such that for points in the chambers all semistable sheaves are stable. For stability conditions $\underline{\sigma}^0$ on a wall W and $\underline{\sigma}$ in an adjacent chamber, $\underline{\sigma}$-stability implies $\underline{\sigma}^0$-semistability. Thus given $W \subset \mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ a wall there exist a contraction morphism on $M_{\underline{\sigma}}(v)$ over B induced by the composition

$$M_{\underline{\sigma}}(v) \to \mathcal{M}_{\underline{\sigma}^0}(v) \to M_{\underline{\sigma}^0}(v).$$

The fibers consist of families of $\underline{\sigma}$-stable objects which become strictly semistable and S-equivalent for $\underline{\sigma}^0$.

Given a Mukai vector v and a primitive class α of a torsion sheaf on $\mathbb{P}^1_{\mathscr{C}}(\Xi)$ which we identify with a root of R^{ell}, define a wall

$$W^{\mathrm{Stab}}_\alpha(v) := \{\underline{\sigma} \in \mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi)) \mid \exists \mathcal{E} \in M_{\underline{\sigma}}(v)^{ss} \setminus M_{\underline{\sigma}}(v)^s, \alpha \in \mathrm{JH}_{\underline{\sigma}}(\mathcal{E})\}$$

where $\mathrm{JH}_{\underline{\sigma}}(\mathcal{E})$ is the set of Mukai vectors of Jordan-Holder factors of $\mathcal{E}$.

Also consider the wall

$$W_{\alpha}^{HH_0(\mathscr{C})}(v) := \{\xi \subset HH_0(\mathscr{C}) \mid M_{\underline{\sigma}}(v)_{\xi} \to M_{\underline{\sigma}^0}(v)_{\xi} \text{ not biregular } \}$$

for σ^0 in $W_{\alpha}^{\mathrm{Stab}}(v)$.

Finally, under the identification $\widehat{\widehat{\mathfrak{h}}} \simeq HH_0(\mathscr{C})$ consider the wall $\alpha^{\perp}$.

Conjecture 4.3.6. *The morphism $M_{\underline{\sigma}}(v;\mathcal{F}) \to HH_0(\mathscr{C})$ is semiprojective under the $\mathbb{C}_{\hbar}^*$ action. There is a 1-1 correspondence between the set of non-empty walls $W_{\alpha}^{\mathrm{Stab}}(v)$ in $\mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$, the set of nonempty walls $W_{\alpha}^{HH_0(\mathscr{C})}(v)$ and the set of primitive roots α such that $\langle \alpha, v\rangle \leq v^2/2$.*

Theorem 4.3.7 ([2])**.** *Conjecture 4.3.6 is true for $R = A_{-1}$. The definition and semiprojectivity of the central fiber bijection between walls $W_{\alpha}^{\mathrm{Stab}}(v)$ and roots holds in the D_4, E_6, E_7, E_8 cases.*

The remaining results would follow from a careful proof that the proposed relative stability conditions actually satisfy the hypotheses in [64]. Towards Conjecture 4.3.6 in the other cases, [2] showed that there was an equivalence between the set of roots in R^{ell} and the set of hyperplanes in $HH_0(\mathscr{C})$ where there exist $a \in \mathrm{Coh}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ of primitive class α such that $a|_{D_\infty} = 0$.

The wall

$$W_{\alpha}^{HH_0(\mathscr{C})}(v)$$

, if nonempty for some particular v, corresponds to the locus of $b \in HH_0(\mathscr{C})$ in the deformation where there exit torsion sheaves of class α on $\mathbb{P}^1_{\mathscr{C}}(\Xi_b)$ avoiding D_∞. In general for a weighted projective line $\mathscr{C}$ with points $\underline{x}$ of stabilizers of order $\underline{\lambda}$ corresponding to a star shaped Dynkin diagram D corresponding to an Kac-Moody algebra $\mathfrak{g}_D$ with root system $R \subset \mathfrak{h}^*$, there is an identification

$$HH_0(\mathscr{C}) \simeq \widehat{\mathfrak{h}} \tag{4.3.8}$$

with the Cartan subalgebra of the affinized star shaped Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm}] \oplus \Bbbk c$. This agreed with an identification of $K_{num}(\mathrm{Coh}_{tors}(\mathscr{C}))$ with R. The identification (**??**) identifies root hyperplanes

$\alpha^{\perp} \subset HH_(\mathscr{C})$ with the locus of $b \in HH_(\mathscr{C})$ where there are torsion sheaves in $\mathbb{P}^1_{\mathscr{C}}(\Xi_b)$. See [2] for more details.

4.4 Birational models

Given $\mathcal{F} \in \mathrm{Coh}(D_\infty)$ in the image of $\mathcal{O}_{D_\infty}$ under $\mathrm{Aut}(D^b(\mathrm{Coh}[E/\Gamma]))$.

As a consequence of the analysis of the stability conditions in $\mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ under derived equivalence we can obtain complete information about the holomorphic symplectic birational geometry of the moduli spaces $M_{\underline{\sigma}_0}(v;\mathcal{F})$, and more generally of the family if we assume the integrability of the stability conditions in $\mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$.

Theorem 4.4.1. *[2]*

1. *Assuming the integrability of the stability conditions* $\mathrm{Stab}^{lf}(\mathbb{P}^1_{\mathscr{C}}(\Xi))$ *the moduli spaces*

$$M_{\underline{\sigma}}(v;\mathcal{F})$$

are all semiprojective and birational over $HH_0([E/\Gamma])$.

2. *Every K-trivial birational model of the central fiber* $M_{\underline{\sigma}_0}(v;\mathcal{F})$ *is of the form* $M_{\underline{\sigma}'_0}(v;\mathcal{F})$ *for some* $\underline{\sigma}'_0$.

3. *Each birational model* $M_{\underline{\sigma}}(v;\mathcal{F})$ *is equivalent under a derived autoequivalence to a moduli space* $M_H(v;\mathcal{F})$ *of Gieseker semistable framed torsionfree sheaves.*

4. *The moduli space* $M_H(v;\mathcal{F})$ *represents the stack of framed torsionfree sheaves.*

As a consequence, all of the $M_{\underline{\sigma}}(v;\mathcal{O}_{D_\infty})$ and all of the $M_{\underline{\sigma}}(v;\mathcal{F})$ are birational to $X_R^{[n]}$ for some n.

The birational transformations between various $M_{\underline{\sigma}}(v;\mathcal{O}_{D_\infty})$ may be explicitly described. The morphism factors into a finite sequence of Gieseker-Uhlenbeck flops and stratified Mukai flops. This would have followed in the analytic category and non-equivariantly from the relevant story on K3 surface [70] but Theorem 4.4.1 demonstrates the compatibility with the $\mathbb{C}^*_\hbar$ action and with the equivariant deformations.

4.5 Higher rank framing

The natural generalization of the framed moduli spaces $M_{\underline{\sigma}}(v;\mathcal{F})$ when the framing

$$\mathcal{F} = \bigoplus_{i=1}^{k} \mathcal{F}_k \tag{4.5.1}$$

is not stable, is the fiber of the diagram

$$\begin{array}{ccc} \mathcal{M}_{tf}(v;\mathcal{F}) & \longrightarrow & [\mathcal{F}] \\ \downarrow & & \downarrow \\ M_{tf}(v) & \xrightarrow{(-)|_{D_\infty}} & \mathcal{M}_{tf}(\mathscr{C}). \end{array} \tag{4.5.2}$$

giving the stack $\mathcal{M}_{tf}(v;\mathcal{F})$ of torsionfree sheaves framed at D_∞. Here $M_{tf}(v)$ a component of the stack of torsionfree sheaves on $\mathbb{P}^1_{\mathscr{C}}(\Xi)$, and we denote by $\mathcal{F}$ its basechange to $\mathscr{C}_B$ as usual. The stack is equivalently described by the moduli functor which assigns to a scheme S the set

$$\{(\mathcal{E},\phi) \mid \mathcal{E} \in S \times \overline{X_R} \text{ torsionfree flat over } S, \phi : \mathcal{E}|_{D_\infty \times S} \xrightarrow{\sim} \mathcal{F} \boxtimes \mathcal{O}_S\}. \tag{4.5.3}$$

We will see in Section 5.1.4 some analysis of the stability conditions on the stack $\mathcal{M}_{tf}(v;\mathcal{F})$, which are not irrelevant to geometric representation theory or enumerative geometry. Nevertheless, they are the natural home for higher framing rank enumerative invariants, shift operators, for stable envelopes and tensor structures on categories of modules for three-loop algebras. On account of Theorem 4.4.1 this is a generalization of the stably framed case.

Brosius filtration

Many properties of $\mathcal{M}_{tf}(v;\mathcal{F})$ depend on the *Brosius filtration*, in particular stabilizer of points.

Let $X = \mathbb{P}_{\mathscr{C}}(\mathcal{E})$ be a geometrically ruled surface over a potentially orbifold curve $\mathscr{C} = [C/\Gamma]$ where $\mathcal{E}$ is rank 2 and admits a quotient $\mathcal{O}_{\mathscr{C}}$.

Let $(\mathcal{E}, \phi) \in M_{tf}(v; \mathcal{F})$ be a torsion-free sheaf. Let $\mathfrak{P}$ denote the generic fiber of $p : X \to C$. Let

$$\mathcal{E}_{\mathfrak{P}} \simeq \mathcal{O}_{\mathfrak{P}}(d_1)^{\oplus u_1} \oplus \cdots \oplus \mathcal{O}_{\mathfrak{P}}(d_\ell)^{\oplus u_\ell}$$

with be the restriction of $\mathcal{E}$ to the generic fiber.

Definition 4.5.4. The splitting type of $\mathcal{E}$ is the symbol $(d_1^{u_1}, \ldots, d_\ell^{u_\ell})$ where $d_1 \gneq \ldots \gneq d_\ell$.

If $\ell = 1$ the sheaf is said to have equal splitting type.

The *Brosius filtration* of $\mathcal{E}$ (which doesn't use or depend on the framing) is the filtration

$$[\pi^*(\pi_*(\mathcal{E}(-d_1 D_\infty)))](d_1 D_\infty) \subset \cdots \subset [\pi^*(\pi_*(\mathcal{E}(-d_\ell D_\infty)))](d_\ell D_\infty) \subset \mathcal{E}.$$

Denote by

$$\mathcal{B}_k := [\pi^*(\pi_*(\mathcal{E}(-d_k D_\infty)))](d_k D_\infty) \tag{4.5.5}$$

the k-th step in the filtration with $\mathcal{B}_{\ell+1} = \mathcal{E}$ and $\mathcal{B}_0 = 0$ and denote by $\mathcal{Q}_k := \mathcal{E}/\mathcal{B}_k$ the associated quotient and $\mathcal{T}_k \subset \mathcal{Q}_k$ the torsion subsheaf which is $\mathcal{O}_{\mathbb{P}^1}(d_k) \otimes \mathcal{I}_{Z \subset X}$ for X supported on some $\mathbb{P}^1$ fibers with $\mathcal{Z}$ non-empty on each component of X [71, §3.1.2].

Proposition 4.5.6. *The stabilizer of any* $(\mathcal{E}, \phi) \in M(v; \mathcal{F})$ *is unipotent.*

Proof. The proof is by induction on the length ℓ of the canonical filtration. If $\ell = 1$ and $\alpha \in \mathrm{Aut}(\mathcal{E})$ preserves the framing ϕ then $\alpha - \mathrm{Id}$ acts by zero on the generic fiber of π, hence has torsion image and is zero because $\mathcal{E}$ is torsionfree.

If $\ell > 1$ then $\alpha \in \mathrm{Aut}(\mathcal{E})$ preserves $\mathcal{B}_1$ because of the long exact sequence

$$\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_1) \to \mathrm{Hom}(\mathcal{B}_1, \mathcal{E}) \to \mathrm{Hom}(\mathcal{B}_1, \mathcal{Q}_1) = 0$$

associated to the sequence defining $\mathcal{Q}_1$. The latter vanishing follows from the restriction of $\mathcal{B}_1$ to the fibres along which $\mathcal{T}_k$ are supported and the non-emptiness of Z on these fibers, together with the vanishing of $\mathrm{Hom}(\mathcal{B}_1, Q_1/\mathcal{T}_1)$ by the same argument. Thus α is induced by the identity on $\mathcal{B}_1$, a unipotent element $\alpha' \in \mathrm{Aut}(Q_1, Q_1)$ and a map $\alpha'' \in \mathrm{Hom}(Q_1, \mathcal{B}_1)$ and thus α is unipotent. □

For the elliptic surface $\overline{X}_{A_{-1}}$ we have another criterion that guarantees trivial stabilizers.

Proposition 4.5.7. *Suppose $\mathcal{F}$ is polystable and all $\mathcal{F}_i$ are mutually nonisomorphic. Then any $\mathcal{F}$-framed torsion free sheaf $(\mathcal{E}, \phi)$ has trivial stabilizer.*

Proof. The torsion-free sheaf is generically a vector bundle, and because $\overline{X}_{A_{-1}}$ is a product $E \times \mathbb{P}^1$ this is equivalent to a rational map

$$f : \mathbb{P}^1 \to \mathrm{Bun}_{[\mathcal{F}]}(E)$$

satisfying $f(\infty) = \mathcal{F}$. Thus the generic point is sent to a polystable bundle with mutually nonisomorphic factors and hence $\mathrm{Aut}((\mathcal{E}, \phi))$ is contained in the torus $\mathrm{Aut}(\mathcal{F})$ and is hence trivial by Proposition 4.5.6. □

Corollary 4.5.8. *If $\mathcal{F}$ is polystable and all of the $\mathcal{F}_i$ are mutually nonisomorphic then the stack $\mathcal{M}^{tf}(v; \mathcal{F})$ is representable by an algebraic space locally of finite presentation.*

We expect to find a similar criterion for the other surfaces; in particular the result of Proposition 4.5.7 and Corollary 4.5.8 should hold when the framing sheaves are chosen appropriately.

Chapter 5: Geometric representations and tensor products

5.1 Elliptic Ruijsenaars-Schneider system and stable envelopes

The classical elliptic Ruijsenaars-Schneider systems [72, 73] are families of commuting elliptic difference equations and are deformations of the elliptic Calogero-Moser systems. The families of moduli of framed objects studied in [2] are generalizations of the phase spaces of the elliptic spin Calogero-Moser system based on the relation between noncommutative $\mathbb{P}^1$ bundles and Calogero-Moser systems studied by Ben-Zvi and Nevins [74, 75]. The identification of special cases of the difference analogue of the systems with the elliptic Ruijsenaars system was shown by Penciak [76].

Stable envelopes [10] are perhaps the main geometric tool for the geometric construction of quantum groups.

It turns out that the relativistic deformations of these systems behave much more nicely with higher framing torus rank. Thus we study what we expect to be a generalization of the phase spaces of the elliptic spin Ruijsenaars-Schneider system and show how they are related to the construction of stable envelopes for these geometries.

5.1.1 Line bundle deformation

We restrict ot the XA_{-1} surface.

There is another deformation of $\Delta_* \mathcal{T}_E \oplus O_\Delta$ in the space of sheaf bimodules corresponding to one of the $\mathbb{C}^*_\hbar$-fixed directions of the tangent space (4.2.5). This integrates to a family

$$\mathbb{P}^1_E(\mathcal{P}^\vee \oplus O_E \boxtimes O_{\mathrm{Jac}(E)})/\mathrm{Jac}(E)$$

where $\mathcal{P}$ is the Picard line bundle, so that the surface over $\mathcal{L} \in \mathrm{Jac}(E)$ is

$$\mathbb{P}^1(\mathcal{L}^\vee \oplus \mathcal{O}_E) = \mathrm{Tot}_E(\mathcal{L}) \bigsqcup D_\infty \tag{5.1.1}$$

and is smooth a projective over $\mathrm{Jac}(E)$.

Define the relative stack of framed torsion-free sheaves

$$\mathcal{M}^{\mathcal{P}}_{tf}(v; \mathcal{F}) \tag{5.1.2}$$

for a framing $\mathcal{F}$ which we assume to be formed from base change from E using the analogue of the diagram (4.5.2). Given $\mathcal{L} \in \mathrm{Jac}(E)$ a closed point let $\mathcal{M}^{\mathcal{L}}_{tf}(v; \mathcal{F})$ denote the fiber over L.

The results of Propositions 4.5.6 and 4.5.7 still hold although a different proof is needed for 4.5.6 because there is no longer a fibration over $\mathbb{P}^1$. Fortunately the argument of [71, Thm 5.4] carry over to this more general situation exactly to show triviality of the stabilizer. In the polystable mutually nonisomorphic case we denote the algebraic space with the symbol $M^{\mathcal{L}}_{tf}(v; \mathcal{F})$. Likewise, we expect the following.

Conjecture 5.1.3. *For very general $\mathcal{L}$ and any polystable $\mathcal{F}$ with $\mathcal{F}_i$ mutually nonisomorphic the algebraic space $M^{\mathcal{L}}_{tf}(v; \mathcal{F})$ is representable by a scheme.*

Remark 5.1.4. This is a special case of Conjecture 5.1.11

The scheme will of course not be of finite type. This is a generalization, motivated by Theorem 4.4.1 (4) of the following result of Nevins.

Theorem 5.1.5 ([71])**.** *Conjecture 5.1.3 is true when $\mathcal{F} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is a sum of nonisomorphic line bundles of the same degree.*

The results of Theorem 5.1.5 seems to fail over the central surface $E \times \mathbb{P}^1$ where the separatedness of the algebraic space $M_{tf}(v; \mathcal{F})$ seems to fail.

The deformation (5.1.1) breaks most of the symmetry present on the central surface which is related to the Lagrangian fibration structure and the associated derived equivalences. In particular the deformation is distinct from the deformation along the commutative hyperplane $W_\alpha^{HH_0([E])}$ for $\alpha = \delta_{\text{pt}}$ the class of a point. The latter deformation is the family of $\mathbb{P}^1$ bundles associated to the universal extension over $\text{Ext}^1(\mathcal{T}_E, \mathcal{O}_E)$, and is a union of an *affine* bundle over E with D_∞. In particular, the relative derived equivalences of Theorem 4.2.10 exchange the latter family with noncommutative deformations infinitesimally close to $\mathbb{P}^1(\mathcal{T}_E \oplus \mathcal{O}_E)$ related to differential operators and twisted differential operators on E.

The deformation of (5.1.1) instead exchange the family (5.1.1) with a family of finitely deformed noncommutative $\mathbb{P}^1$ bundles. A derived autoequivalence $\Phi \in \text{Aut}(D^b(\text{Coh}\, E))$ das usual lifts to a family of *dg* categories $D_\Phi(\mathcal{L})$ with semiorthogonal decompositions

$$D_\Phi(\mathcal{L}) = \langle D^b(\text{Coh}(E)), D^b(\text{Coh}(E))\rangle$$

the Koszul data for objects in this dg category consists of triples (a, b, η) where $a, b \in \text{Coh}(E)$ and

$$\eta : \Phi(\Phi^{-1}(a) \otimes (\mathcal{L} \oplus \mathcal{O})) \to b. \tag{5.1.6}$$

When Φ is the usual Fourier-Mukai transform acting via $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $K_{num}(E)$ with $\Phi([\text{pt}]) = [\mathcal{O}_E]$ and $\Phi([\mathcal{O}_E]) = -[\text{pt}]$ The bimodule is of the form

$$\begin{aligned} \mathcal{E} &= \mathcal{O}_\Delta \oplus \mathcal{P}^* \circ \Delta_* \mathcal{L} \circ \mathcal{P} \\ &= \mathcal{O}_\Delta \oplus \mathcal{O}_{\Gamma_{(-)+s}} \end{aligned}$$

where $\circ$ is convolution of coherent kernels, $(-) + s : E \to E$ is translation by $s \in E$ with graph $\Gamma_{(-)+s}$. Here $s \in E$ corresponds to $\mathcal{L}$ under the chosen identification $E \simeq \text{Jac}(E)$.

Consider the family over $E_\tau \simeq E$

$$\mathbb{P}^1_{E_{E_\tau}}(\Gamma) \tag{5.1.7}$$

of noncommutative $\mathbb{P}^1$ bundles over E with sheaf bimodule

$$\Gamma := \mathcal{O}_\Delta \oplus \Gamma_{E_\tau} \tag{5.1.8}$$

where $\Gamma_{E_\tau} \subset E \times E \times E = \mathcal{O}_{m_{132}}$ is the graph of multiplication from the first and third factor to the second factor.

In summary we have the following, which is very likely able to be found somewhere in [68]

Proposition 5.1.9. *The Fourier-Mukai transform Φ gives a family of equivalences of noncommutative $\mathbb{P}^1$ bundles*

$$D^b(\mathrm{Coh}(\mathbb{P}^1_E(\Gamma)))/E_\tau \xrightarrow{\Phi} D^b(\mathrm{Coh}(\mathbb{P}^1_E(\mathcal{P}^\vee \oplus \mathcal{O})))/\mathrm{Jac}(E)$$

covering the identification $D^b(\mathrm{Coh}(E)) \to D^b(\mathrm{Coh}(\mathrm{Jac}(E)))$ given by the identity.

Remark 5.1.10. The moduli spaces $M_{\underline{\sigma}}(v; \mathcal{F})$ from (4.5.2) are birational models for deformations of the elliptic Calogero-Moser systems. Conjecture 5.1.3 thus becomes after Proposition 5.1.9 a conjecture for the existence of a tower of infinite but finite dimensional generalizations of generalizations of the classical elliptic Ruijsenaars-Schneider models related to the enumerative geometry problems studied in the sequel. The Fourier transform over closed in W_{pt} was studied in [76] to identify tweaking flows on spectral sheaves with the flows of the elliptic RS system.

We expect the following

Conjecture 5.1.11. *For very general $b \in E_\tau \times \mathrm{Jac}(E)$ and $\mathcal{F}$ polystable with $\mathcal{F}_i$ mutually nonisomorphic the algebraic space $M_{tf}(v; \mathcal{F})_b$ is representable by a scheme.*

5.1.2 Spin generalization

The goal of this section is to describe a generalization of the identification in Theorem 4.3.7 between roots and walls in the difference deformation. We will need of course a larger space containing both of the deformations exchanged by the Fourier transform in Proposition 5.1.9.

One can of course study the entire stack $shbimod(E, E)$, or rather its substack denoted

$$shbimod(E, E)^{\Delta}$$

consisting of sheaf bimodules with a sub-bimodule

$$0 \to O_{\Delta} \to \mathcal{E}$$

but for simplicity we restrict ourself to the good moduli space B isomorphic to

$$E_{\tau} \times \mathrm{Jac}(E)$$

where the fiber over the point $(s, \mathcal{L})$ is the bimodule of the form

$$O_{\Delta} \oplus \Gamma^{\mathcal{L}}_{(-)+s}$$

where

$$\Gamma^{\mathcal{L}}_{(-)+s} = \gamma_* \mathcal{L}$$

where $\gamma : \mathcal{E} \to \Gamma^{\mathcal{L}}_{(-)+s}$ is the isomorphism with the third factor. The universal sheaf bimodule on $E \times E \times E_{\tau} \times \mathrm{Jac}(E)$ therefore has the form

$$\Lambda = \pi_{14}^* \mathcal{P} \otimes \pi_{123}^* \Gamma$$

with Γ from (5.1.8).

We get a family of noncommutative $\mathbb{P}^1$ bundles

$$\mathbb{P}^1_{E_{E \times \mathrm{Jac}(E)}}(\Lambda) \tag{5.1.12}$$

over B. Consider $\alpha \in K_{num}(E)$ primitive. Recall that torsion sheaves of class α are though of class

$(\alpha, \alpha) \in K_{num}(\mathbb{P}^1_E(\mathcal{E})) = K_{num}(E) \oplus K_{num}(E)$.

Consider a torsion sheaf t with Koszul data (a, a, η). Recall that torsion sheaves lie in the heart $\mathrm{Coh}(\mathbb{P}^1_E(\mathcal{E}))$ and also in the heart $\mathrm{kos}(\mathcal{E})$.

Definition 5.1.13. A torsion sheaf t of primitive class α is *supported at the zero section* if $t|_{D_\infty} = 0$ and the composition

$$\eta|_{a \otimes_E \Gamma^{\mathcal{L}}_{(-)+s}} = 0.$$

Definition 5.1.14. The elliptic wall $W_{\alpha^\perp} \subset E_\tau \times \mathrm{Jac}(E)$ is the subvariety of $E \times \mathrm{Jac}(E)$ where for $b \in B$ there exist stable torsion sheaves t on $\mathbb{P}^1_E(\Lambda_b)$ of class α with $t|_{D_\infty} = 0$ which are not supported at the zero section.

Let $[n] : E \to E$ be the multiplication by n map.

Proposition 5.1.15. *The elliptic wall $W_{\alpha^\perp}$ for $\alpha = m[\mathrm{pt}] + n[\mathcal{O}_E]$ primitive is the image of the map* $[n] \times [m] : E \to E \times \mathrm{Jac}(E)$.

Proof. This is exactly the locus of $(s, \mathcal{L})$ where there exists a non-zero map

$$\tau_s^*(a \otimes \mathcal{L}) \to a$$

for a a simple sheaf of class $m[\mathrm{pt}] + n[\mathcal{O}_E]$. □

Let $\mathcal{O}(\mathfrak{o})$ be the degree 1 line polarization identifying $\mathrm{Jac}(E)$ and E.

Consider the action of $\mathrm{GL}(2, \mathbb{Z})$ on $E_\tau \times \mathrm{Jac}(E)$ given on generators by

$$d = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \kappa \times \mathrm{Id}$$

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto [(s, \mathcal{O}(x - \mathfrak{o})) \mapsto (x, \mathcal{O}(\kappa(s) - \mathfrak{o}))]$$

$$f = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto [(s, \mathcal{O}(x - \mathfrak{o})) \mapsto (s, \mathcal{O}(x + s - 2\mathfrak{o}))]$$

where $\kappa : E \to E$ is the Kummer involution $s \mapsto -s$.

Consider the assignment where d is sent to the deformed Verdier duality transformation (4.2.8), s is sent to conjugation of the bimodule by by $\Phi_{\mathcal{P}}$ and f is sent to conjugation of the bimodule by $(-) \otimes \mathcal{O}(\mathfrak{o})$.

The following is an immediate generalization of Theorem 4.2.10.

Theorem 5.1.16. *There is an action of* $\mathrm{GL}(2, \mathbb{Z})$ *be relative antiequivalence on* $\mathbb{P}^1_E(\Lambda)$ *covering the action on* $E_\tau \times \mathrm{Jac}(E)$. *Under this action the elliptic wall* $W_{\alpha^\perp}$ *is sent to* $W_{\gamma\alpha^\perp}$ *for* $\gamma \in \mathrm{GL}(2, \mathbb{Z})$.

Then given $\mathcal{F} \in \mathrm{Coh}(D_\infty)$ we define the relative moduli stack of framed torsion free sheaves

$$M_{tf}(v; \mathcal{F})$$

over $E_\tau \times \mathrm{Jac}(E)$.

We expect results similar to Theorem 4.4.1 to hold for the moduli space $M_{tf}(v; \mathcal{F})$.

5.1.3 Fixed points

For $\mathcal{F}$ simple, $\mathbb{C}^*_\hbar$-fixed $\mathcal{F}$-framed torsionfree sheaves on $E \times \mathbb{P}^1$ are equivalent by the Rees construction to chains $(\mathcal{A}_*, \phi_*)$

$$[\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \to \ldots \to \mathcal{F} \to \mathcal{F} \to \cdots]$$

such that

1. Each $\mathcal{A}_i \in \mathrm{Coh}(E)$
2. Each ϕ_i is injective
3. For $i \gg 0$ $\mathcal{A}_i \simeq \mathcal{F}$ and $\phi_i : \mathcal{F} \to \mathcal{F}$ is the identity.

Definition 5.1.17. A $\mathcal{F}$-terminating $\mathrm{Coh}(E)$-chain is a chain $(\mathcal{A}_*, \phi_*)$ satisfying the above conditions. More generally define an $\mathcal{F}$-terminating $\mathrm{Coh}(C)$ chain for a smooth curve C.

Given a dimension vector $\alpha_* = ([\mathcal{A}_1], [\mathcal{A}_2] \ldots, [\mathcal{A}_\infty] = [\mathcal{F}])$ the moduli space $\mathrm{Filt}(E, \alpha_*)$ is a smooth connected projective variety [77].

For the other surfaces $\overline{X}_R$ where the central fiber we only give the definition in rank 1. Let $(x_1, \ldots, x_n)$ be the n orbifold points of $[C/\Gamma]$ and suppose that that X_R has r_k fixed points coming from the resolution of the singularity at the k-th orbifold point. Letting E_0 denote the central fiber of the fibration we define

Definition 5.1.18. An ($\mathcal{O}_{E_0}$-terminating) $\mathrm{Coh}(E_0)$-chain is a pair $((\mathcal{A}_*, \phi_*), \underline{\lambda})$ where where $(\mathcal{A}_*, \phi_*)$ is an $\mathcal{O}_{\mathbb{P}^1}$-terminating $\mathrm{Coh}(\mathbb{P}^1)$-chain if there is a $\mathbb{P}^1$ $\mathbb{C}^*_\hbar$-fixed component of E_0 and the $\underline{\lambda} = (\lambda_{i,k})_{i=1,\ldots n}, k = 1, \ldots, r_{n-1}$ is a tuple such that the $\lambda_{i,k}$ are partitions.

The fixed points of ideal sheaves on $\overline{X_R}$ with the singularities avoiding D_∞ are given by $\mathrm{Coh}(E_0)$-chain and again their moduli are smooth projective varieties.

We now have descriptions of $A \times \mathbb{C}^*_\hbar$-fixed points on the algebraic spaces $M_{tf}(v; \mathcal{F})$ assuming polystable mutually nonisomorphic $\mathcal{F}_i$, as well as the A-fixed points of the difference deformation along the $W_{[\mathrm{pt}]^\perp}$ elliptic wall where A scales the framing isomorphism under the torus acting as automorphisms of $\mathcal{F}$.

Recall $\mathcal{M}^{\mathcal{P}}_{tf}(v; \mathcal{F})$ from (5.1.2) which we now recognize as the base change of $M_{tf}(v; \mathcal{F})$ to $W_{[\mathrm{pt}]}$.

Proposition 5.1.19. *Let $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{F}_i$ be polystable with $\mathcal{F}_i$ mutually nonisomorphic.*

1. The $A \times \mathbb{C}^*_\hbar$-fixed points are isomorphic to the base change to $W_{[\mathrm{pt}]}$ of

$$\prod_{i=1}^{n} \mathrm{Filt}(E, \alpha_*^{(i)})$$

for some dimension vectors $\alpha_*^{(i)}$ such that $\alpha_\infty^{(i)} = [\mathcal{F}_i]$.

2. For a very general point $p \in W_{[\mathrm{pt}]}$ the A-fixed points of $M_{tf}(v; \mathcal{F})$ are isomorphic to

$$\bigsqcup_{v_1 + \ldots + v_n = v} \prod_{i=1}^{n} M_{tf}(v_i; \mathcal{F}_i).$$

Proof. (1) Given a $\oplus_{i=1}^n \mathcal{F}_i$-terminating $\mathrm{Coh}(E)$-chain the kernel in $\mathcal{A}_i$ of the map to $\oplus_{j \neq i} \mathcal{F}_j$ gives the decomposition into a direct sum of $\mathcal{F}_i$-terminating $\mathrm{Coh}(E)$ chains.

For part (2) the argument of [71] carries over exactly. □

5.1.4 δ-stable framed objects

The notion of (ω, δ)-stability where $\delta \in \mathbb{R}[t]$ is a polynomial for framed torsion-free sheaves $(\mathcal{E}, \phi)$ was introduced by Huybrechts and Lehn in [78]. While moduli spaces of ω-Gieseker semistable torsion-free sheaves with Poincare polynomial P on a projective variety X arise as GIT quotients of a quot scheme

$$\mathrm{Quot}_P(V \otimes \mathcal{O}_X(-m))$$

with respect to a line bundle $\mathcal{L}_{\omega,m}$, the moduli spaces of of stable sheaves framed at a module $\mathcal{F}$ supported along a subvariety D arise as GIT quotients of a closed subscheme of

$$\mathrm{Quot}_P(V \otimes \mathcal{O}_X(-m)) \times \mathbb{P}(\mathrm{Hom}(V, H^0(\iota_* \mathcal{F}(m))))$$

with respect to line bundles $\mathcal{L}_{\omega,m} \boxtimes \mathcal{L}_\delta$.

When $X = S$ is a projective surface and the framing sheaf is supported along a divisor, we choose $\delta(m)$ to be a linear polynomial.

Suppose $\delta(m) = \delta_1 m + \delta_0$ and a polarization $\omega \in \mathrm{Amp}(S)$. Given a framed torsion-free sheaf $(\mathcal{E}, \phi)$ define its Hilbert polynomial to be

$$P_{(\mathcal{E},\phi)}(m) = P_{\mathcal{E}}(m) - \epsilon\delta(m)$$

where

$$\epsilon = \begin{cases} 1 & \text{if } \phi \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The reduced Hilbert polynomial of the pair is

$$p_{(\mathcal{E},\phi)} := P_{(\mathcal{E},\phi)}(m)/\mathrm{rk}(E).$$

Definition 5.1.20. The pair $(\mathcal{E}, \phi)$ is said to be (ω, δ)-(semi)stable if for every nontrivial submodule $\mathcal{E}'$ we have

$$p_{(\mathcal{E}',\phi|_{\mathcal{E}'})}(\leq) p_{(\mathcal{E},\phi)}.$$

In the case that we are only concerned with strictly stable sheaves it is common to let δ be a real number, and then the relevant polynomial is $\delta(m) = \delta m$. In the case a framed torsion-free sheaf $(\mathcal{E}, \phi)$ is (ω, δ)-stable if for any subsheaf $\mathcal{G}$ of smaller rank have we have

$$\frac{c_1(\mathcal{G}) \cdot \omega}{\mathrm{rk}\,\mathcal{G}} < \begin{cases} \frac{c_1(\mathcal{E})\cdot\omega-\delta}{\mathrm{rk}\,\mathcal{E}} & \mathcal{G} \subset \ker\phi \\ \frac{c_1(\mathcal{E})\cdot\omega-\delta}{\mathrm{rk}\,\mathcal{E}} + \frac{\delta}{\mathrm{rk}\,\mathcal{G}} & \text{otherwise.} \end{cases} \tag{5.1.21}$$

Let P be the class of the $\mathbb{P}^1$ fiber in $E \times \mathbb{P}^1$ Consider polarizations of the form

$$\omega = E + 1/NP$$

for $N > 0$. The equations (5.1.21) for there to be a (ω, δ)-stable direct sum of the form

$$(\mathcal{E}, \phi) = (\mathcal{G}_1 \oplus \mathcal{G}_2, \phi_1 \oplus \phi_2)$$

include those with

$$\mathcal{G} \in \{(\mathcal{G}_1, \phi_1), (\mathcal{G}_2, \phi_2), (\mathcal{G}_1 \cap \ker(\phi_1), 0), (\mathcal{G}_2 \cap \ker(\phi_2), 0)\}.$$

The resulting four equations define a region of the $(\delta, 1/N)$ plane. We have plotted those regions for varying $c_1(\mathcal{G}_1)$ when $v(\mathcal{E}) = (2, P, -1)$ in Figure 5.1.

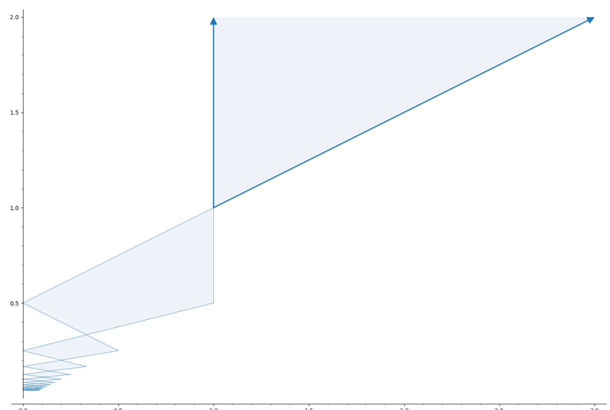


Figure 5.1: Region plot of (ω, δ) stability of summands of a framed sheaf of $v(\mathcal{E}) = (2, P, -1)$.

For a stable summand (G_i) with $\phi_i : \mathcal{G}_i|_{D_\infty} \xrightarrow{\sim} \mathcal{F}_i$ the degree of $\mathcal{F}_i$ determines the P-component of the degree of (G_i). We have the following structural result for the regions. Pick a direct summand $\mathcal{F}_i$ of $\mathcal{F}$. Assume for simplicity that $c_1(\mathcal{E}) = \ell P$ for some ℓ. Define $C_{v(\mathcal{G}_i)}$ to be the region in the $(\delta, 1/N)$-plane where there is a (ω, δ)-stable direct sum

$$(\mathcal{E}, \phi) = (\mathcal{G}_i \oplus \mathcal{G}', \phi_i \oplus \phi').$$

We have the following structural statement. Let $v(\mathcal{G}) = (\mathrm{rk}(\mathcal{G}), v(\mathcal{G})_E E + v(\mathcal{G})_P P, d)$.

Proposition 5.1.22. *The region $C_{v(\mathcal{G}_i)}$ is bounded unless $v(\mathcal{G})_E = 0$. For any two different values of $v(\mathcal{G})_E$ the regions are disjoint. The region $C_{v(\mathcal{G}_i)}$ is empty unless there are no directs sums with* *nontrivial stabilizer.*

Proof. Immediate consequence of (5.1.21). □

Thus as in Figure 5.2, for each $\mathcal{F}_i$ we get a tower of regions $C_{i,n}$ as n varies over the possible values of $v(\mathcal{G})_E$. Furthermore, for a fixed i, the regions become translates of the same quadrilateral after the change of coordinates $\rho = \delta N, M = 1/1/N = N$.

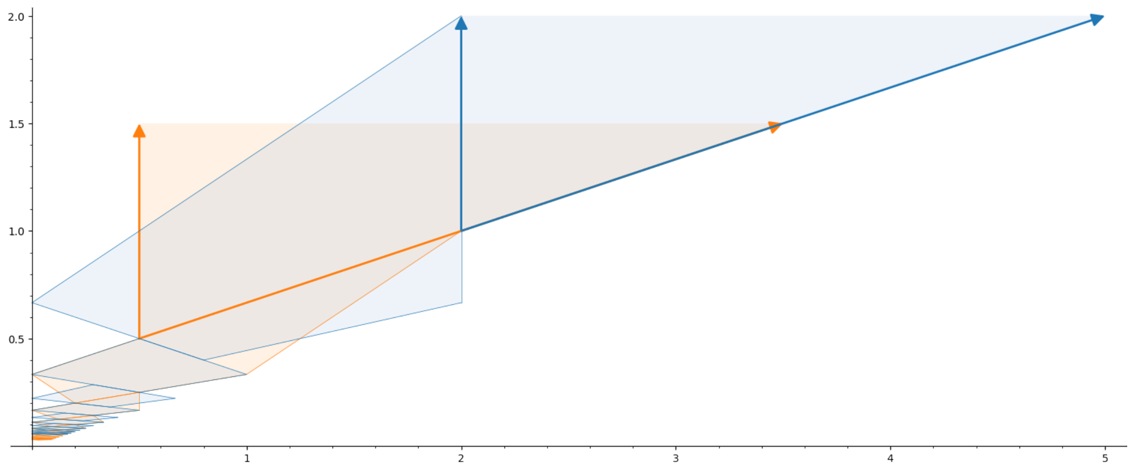


Figure 5.2: Region plot in the $(\delta, 1/N)$ plane of stable decompositions of $v(E) = (3, 2P, -1)$ into stable direct summands of type $c(G) = (1, 0P + \theta E, *)$ for varying θ in one color and those of type $c(G) = (1, P + \theta E, *)$ in another.

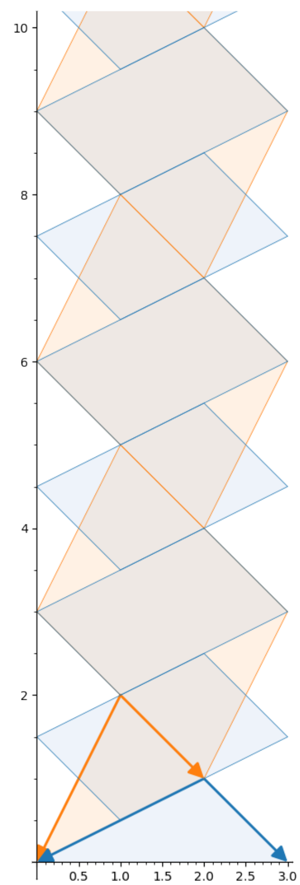


Figure 5.3: The same diagram as Figure 5.2 under the coordinates (ρ, N) where $\rho = \delta/(1/N)$, $N = 1/(1/N)$

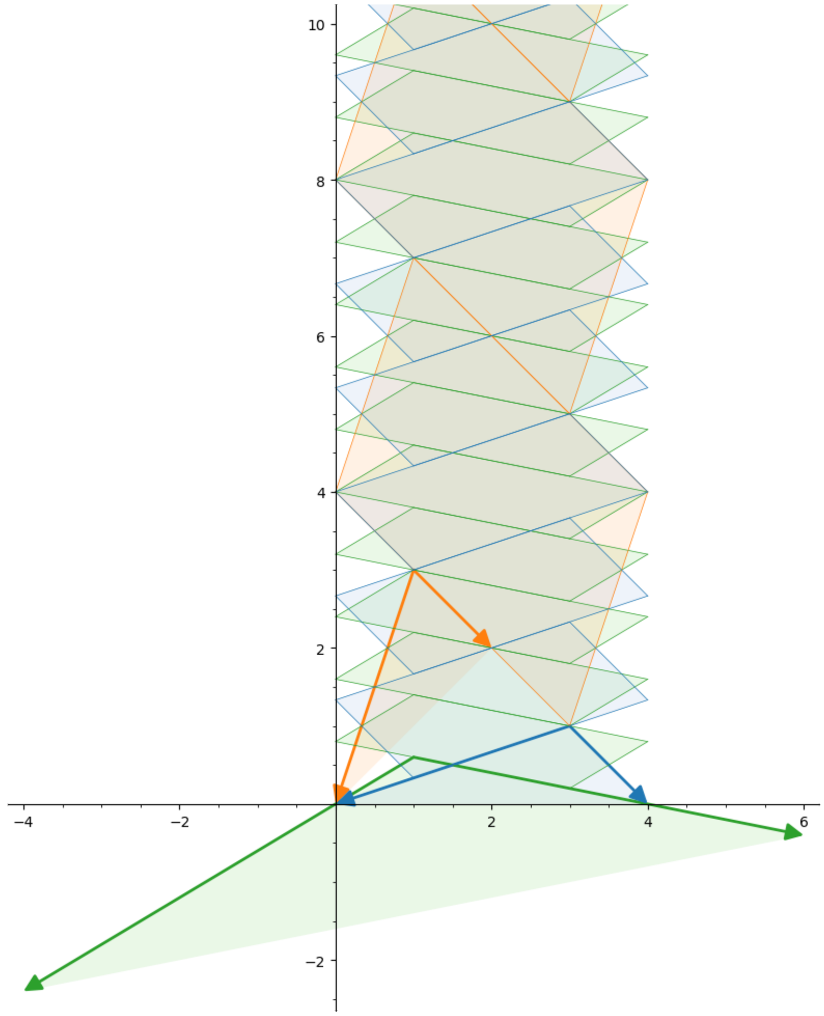


Figure 5.4: A higher rank example for $v(\mathcal{E}) = (4, 3P, -1)$ in the (ρ, N)-plane.

5.2 Stable envelopes

When the equivariant cohomology group $H_T^*(X)$ of some variety X is interpreted as a space of states of a quantum integrable system where the Hamiltonian acts by cup product, the classes supported at a given fixed point are eigenfunction if isolated, and some sort of generalized eigenfunction otherwise. The existence of a basis in which Hamiltonians are diagonalizable is not a unique feature of quantum integrable systems: every unitary matrix is diagonalizable. Instead, it is the existence of other bases which govern integrability. The R-matrix of a spin chain for example is seen as an operator exchanging two different orderings on the spins and the Yang-Baxter equation satisfied by the R-matrix describes the compatibility of the reorderings. In the framework of [10] the spin bases are generalized in geometry to the image of a stable envelope

$$\mathrm{Stab}_{\mathfrak{C}} : H_T^*(X^A) \to H_T^*(X)$$

which depends on a choice of chamber $\mathfrak{C} \subset \mathfrak{a}_{\mathbb{R}}$ the Lie algebra of A, a torus preserving the symplectic form on X, and a polarization ϵ. The chamber lies in the wall-and-chamber structure induced by the walls where there are more than generic fixed points. The polarization ϵ is a choice of square root of the Euler class $e(N_Z)$ of the normal bundle to Z in X in $K_T(Z)$ for every component $Z \in X^A$.

The main result of this section is the existence of stable envelopes under the condition that the statement of Conjecture 5.1.3 is satisfied for some choice of $\mathcal{L}$ and framing $\mathcal{F}$. For simplicity of presentation and the relativistic deformations having been written down, we restrict to the case $R = A_{-1}$.

In our case, $M_{tf}(v; mcF)$ is not holomorphic symplectic but satisfies the weaker assumption that there is a non-degenerate isomorphism $T_{M_{tf}(v;mcF)} \to \hbar \otimes T^{tw}_{M_{tf}(v;mcF)}$ where $T^{tw}_{M_{tf}(v;mcF)}$ is a vector bundle and a topological isomorphism $T^{tw}_{M_{tf}(v;mcF)} \simeq T^{\vee}_{M_{tf}(v;mcF)}$ induced by Serre duality and deformation over a base $J \simeq \mathrm{Jac}(E)$ from a bundle with fiber $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}(-D_\infty) \otimes \mathcal{L})$ at $\mathcal{L} \in J$ to a bundle with fiber $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}(-D_\infty))$. We can thus define Lagrangians, Lagrangian residues as in the usual story of holomoprhic symplectic geometry.

It is possible to formulate the stable envelopes purely on the algebraic space $M_{tf}(v; \mathcal{F})$ over the central surface $E \times \mathbb{P}^1$ but the standard construction applies almost verbatim after the deformation. In particular, the topological isomorphism induces equalities of all equivariant chern characters of bundles.

Theorem 5.2.1 ([10])**.** *Given $\mathfrak{C}$ and ϵ there exists a unique T-equivariant Lagrangian correspon-*

[0]We are confident that nobody will confuse the notation for a stable envelope with that of the space of stability conditions on a triangulated category. The latter will always be written with an argument.

dence

$$L_{\mathfrak{C}} \in H_T^*(X^A \times X)$$

which is proper over X satisfying the conditions in [10, Thm 3.3.4].

Assuming the result of Conjecture 5.1.3 is satisfied, so in particular in the rank 2 case of split line bundles, $M_{tf}(v;\mathcal{F})$ is an infinite union of varieties X_N under open embeddings so that for any $x \in X$ we have $x \in X_N$ for $N \gg 0$. We therefore obtain stable envelopes

$$\mathrm{Stab}_{\mathfrak{C},N} : H_T^*(X_N^A) \to H_T^*(X_N).$$

In the limit we obtain a limit stable envelope which is a map

$$\mathrm{Stab}_{\mathfrak{C}} : H_T^*(X^A) \to \prod_{Z \in \pi_0(X^A)} H_T^*(Z) \tag{5.2.2}$$

obtained by restricting the stable envelope $\mathrm{Stab}_{\mathfrak{C},N}$ for large enough N to a different fixed point.

Remark 5.2.3. It is reasonable to expect that there is a global choice of polarization would arise from the construction of [2, §3.5.1].

The map (5.2.2) satisfies the usual upper triangularity with respect to the attracting order depending on the chamber which now has analytic consequences.

In particular, the inverse operator

$$\mathrm{Stab}_{\mathfrak{C}}^{-1} : H_T^*(X^A) \to \prod_{Z \in \pi_0(X^A)} H_T^*(Z)_{loc}$$

makes sense in a completion in localized equivariant cohomology.

Remark 5.2.4. The attracting partial ordering on $\pi_0(X^A)$ for $X = M_{tf}(v;\mathcal{F})$ is in general neither bounded above or below. However, expressions such as

$$\mathrm{Stab}_{-\mathfrak{C}}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}}$$

do not make sense a priori because even $H_T^*(\mathrm{pt})$ coefficients of this expression contain infinite sums. Likewise, the notation $\mathrm{Stab}_{\mathfrak{C}}^{-1}$ should only be seen as suggestive because the source is only a subspace of the target of $\mathrm{Stab}_{\mathfrak{C}}$.

5.3 Geometric actions

5.3.1 Elliptic surfaces

The identification of the cohomology for an equivariantly proper surface S of

$$V_{\mathrm{NS}(S)} := \bigoplus_{m,\lambda\in\mathrm{NS}(S)} H_T^*(M_S(1,\lambda,m))$$

with the Lattice vertex algebra $V_{NS(S)}$ tensor an auxiliary free bosonic field provides a large number of operators with representation theoretic significance on this module.

In [79] it was shown that when S is an A_n surface, the Fourier coefficients of vertex operators

$$Y(e^{E_i}, z)$$

provided the formula for the action of an algebra action defined by Nakajima in [80], which agrees with operators from the Yangian action from [10] which are primitive under the coproduct Δ.

In [81] this was extended to the Lie algebra generated by real roots acting on $V_{\mathrm{NS}(S)}$ when S is a K3 surface and $\mathrm{NS}(S)$ satisfies some the property that it is generated by (-2) curves which intersect pairwise transversally at at most one point.

Finally, this was used and extended in [2] to prove an identification of operators defined by natural correspondences on the moduli space $M_{\underline{\sigma}}(v;\mathcal{F})$ for simple $\mathcal{F}$ associated to the surface $\overline{X_R}$.

Theorem 5.3.1 ([2])**.** *Under the identification of V_{T^*E} with the vertex algebra of Theorem 2.3.31, the generators $w_\gamma^{a,b}$ of $\mathfrak{g}_{T^*E}$ act via the specialization of Nakajima-Baranovsky operators on the root hyperplane $W_{\alpha^\perp}^{HH_0(E)}$ where $\alpha = a[E] + b[\mathrm{pt}]$.*

Analogous claims were conjectured, and some proven, for the other surfaces.

5.4 Coproduct

5.4.1 Tensor product of vertex algebras

Owing to Proposition 5.1.19 there is an isomorphism of vector spaces

$$V_{T^*E} \otimes V_{T^*E} \simeq \bigoplus_v H_T^*(M_{tf}(v; \mathcal{F}_1 \oplus \mathcal{F}_2))_{loc} \tag{5.4.1}$$

when $\mathcal{F}_1$ and $\mathcal{F}_2$ are non isomorphic degree zero line bundles. More generally, we get an isomorphism

$$\bigoplus_v H_T^*(M_{tf}(v; \bigoplus_i \mathcal{F}_i))_{loc} \simeq \bigotimes_i \gamma_* V_{T^*E} \tag{5.4.2}$$

where $\gamma \in \mathrm{GL}(2, \mathbb{Z})$ is the operator sending the class of $\mathcal{F}_i$ to the class of $\mathcal{O}_E$ in $K_{num}(E)$, which acts via algebra automorphisms on $\mathfrak{g}_{T^*E}$ and then pushforward the action on an irreducible module.

Given a chamber $\mathfrak{C}$ and in the polystable degree zero case we fix an identification of $V_{T^*E}^{\otimes k}$ with a subspace of the completion of $\bigoplus_v H_T^*(M_{tf}(v; \mathcal{F}))$ using the stable envelope via

$$V_{T^*E}^{\otimes k} \simeq H_T^*\left(\bigsqcup_v M_{tf}(v; \mathcal{F})^A\right) \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} H_T^*\left(\bigsqcup_v M_{tf}(v; \mathcal{F})^A\right)^{\frown}$$

where $()^{\frown}$ denotes the completion from (5.2.2).

Explicitly, we then expect that the action of the operator of taking

$$c_1(\lambda) \cup -$$

where λ is a determinant line bundle in the tensor product of two factors is given by the formula

$$\Delta(c_1(\lambda) \cup -) = 1 \otimes c_1(\lambda) \cup + c_1(\lambda) \cup \otimes 1 \\ - \hbar \sum_{(m,n) \in \mathbb{Z}_{\geq 0} \oplus \mathbb{Z} \setminus \{0\} \times \mathbb{Z}_{\leq 0}} \alpha(\lambda) w_{\gamma_i}^{-m,-n} \otimes w_{\gamma^i}^{m,n} \tag{5.4.3}$$

exactly matching [10, §10.1] and its algebraic analysis in affine Yangians in [82].

Note in particular that this operation only makes sense after taking the completion required to define the inverse of the stable envelope owing to the fact that the weights of the representation of $\mathfrak{g}_{T^*E}$ do not lie in a convex cone.

Let $M(\mathcal{F})$ denote the completion of $\otimes_i \gamma_{i*} V_{T^*E}$ required to define the iterated coproduct of (5.4.3). Let $A_{\mathcal{F}}$ denote the framing torus for framing sheaf $\mathcal{F}$ so that V_{T^*E} is a module over the ring $H^*_{\mathbb{C}^*_\hbar \times A_{\mathcal{F}}}(\mathrm{pt})$.

We conclude with a definition.

Definition 5.4.4. The algebra $\mathcal{A}_\hbar(\mathfrak{g}_{T^*E})$ is defined to be the image of the action of $c_1(\lambda)\cup$ and Δx for $x \in \mathfrak{g}_{T^*E}$ in

$$\prod_{\mathcal{F}} \mathrm{End}(M(\mathcal{F})).$$

5.5 Donaldson-Thomas module

We now state a conjecture concerning the the module structure of the vector space underlying the Donaldson-Thomas invariants on local elliptic curves.

Many of the claims have immediate generalizations the the other equivariant elliptic surfaces but we present only the T^*E case for simplicity.

Consider some some class α_* of asymptotic $\mathrm{Coh}(E)$-chains $(\mathcal{A}_*, \phi_*)$ parametrizing a connected component of the Filt space $\mathrm{Filt}(E, \alpha_*)$.

Definition 5.5.1. An α_* terminal $\mathrm{Coh}(E)$-bichain is an commutative array $(\mathcal{A}_{i,j}, \phi^{\downarrow}_{i,j}, \phi^{\rightarrow}_{i,j})_{i,j\geq 1}$ where each $\mathcal{A}_{i,j} \in \mathrm{Coh}(E)$ and maps

$$\phi^{\downarrow}_{i,j} : \mathcal{A}_{i,j} \to \mathcal{A}_{i+1,j}$$
$$\phi^{\rightarrow}_{i,j} : \mathcal{A}_{i,j} \to \mathcal{A}_{i,j+1}$$

such that

1. Each $\phi_{i,j}^{\downarrow/\rightarrow}$ is injective.

2. $(\mathcal{A}_{i,N}, \phi_{i,N}^{\downarrow})$ stabilizes to a fixed $\mathcal{F}$-terminal $\mathrm{Coh}(E)$-chain $(\mathcal{A}_{i,\infty}, \phi_{i,\infty})$ for large N.

3. $(\mathcal{A}_{N,j}, \phi_{N,j}^{\rightarrow})$ stabilizes to the trivial $\mathcal{A}_{\infty,\infty}$-terminal $\mathrm{Coh}(E)$-chain.

For example, the following gives a $[\mathcal{A}_{1,\infty} \to \mathcal{A}_{2,\infty} \to \cdots \to \mathcal{F}]$-terminal $\mathrm{Coh}(E)$-bichain if all of the axioms on the maps are satisfied.

$$
\begin{array}{ccccccccccc}
\mathcal{A}_{1,1} & \longrightarrow & \mathcal{A}_{1,2} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}_{1,\infty} & \longrightarrow & \mathcal{A}_{1,\infty} & \longrightarrow & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{A}_{2,1} & \longrightarrow & \mathcal{A}_{2,2} & \xrightarrow{\phi^{\rightarrow}} & \cdots & \longrightarrow & \mathcal{A}_{2,\infty} & \longrightarrow & \mathcal{A}_{2,\infty} & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle \phi^{\downarrow}} & & & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \cdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & &
\end{array}
.
$$

5.5.1 1-leg vertex

The moduli stack $M_{\alpha_{*,*}}$ of $\alpha_{*,\infty}$-terminal $\mathrm{Coh}(E)$-bichains of dimension vector $\alpha_{*,*}$ are smooth and admit proper maps to $\mathrm{Filt}(\alpha_*)$. We assume that $\alpha_\infty = \mathcal{F}$ which satisfies the consequence of Conjecture 5.1.11.

By the Rees construction, $\mathrm{Coh}(E)$-bichains correspond to $\mathbb{C}^{*2}_{t_1,t_2}$-fixed torsion-free sheaves on $E \times \mathbb{A}^2$. We instead interpret them as $\mathbb{C}^{*2}_{t_1,t_2}$ fixed torsion-free sheaves on the total space of $\mathcal{L} \oplus \mathcal{L}^\vee$ where $\mathcal{L} \in \mathrm{Jac}(E)$ is very general so that the vertical slicing maps are maps to smooth varieties.

The *1-leg DT vertex* of $\mathrm{Tot}_E(\mathcal{L} \oplus \mathcal{L}^\vee)$ is the generating series

$$V_{\alpha_*}(z,p)=\sum_{\alpha_{*,*}} z^{\chi}p^d\int_{[M_{\alpha_{*,*}}]^{vir}/\mathrm{Filt}(\alpha_*)}\frac{1}{e(\mathcal{N}^{vir})}\in H_T^*(M_{tf}(v;\mathcal{F}))_{loc}[[p,z]] \tag{5.5.2}$$

where

1. The terms χ and d are the euler characteristic and curve class of the $\mathbb{C}^{*2}_{t_1,t_2}$-fixed torsion-free sheaf.
2. The K-theory class $\mathcal{N}^{vir}$ is the virtual normal bundle to $M_{\alpha_{*,*}}$ in the moduli stack of torsion-free sheaves.

5.5.2 DT module

In this section we conjecture a relation between the vector space underlying the DT-vertices to representations of $\mathcal{A}_\hbar(\mathfrak{g}_{T^*E})$.

It is based on an analogy with the case of the DT vertex of $\mathbb{C}^3$. In that case the affine Yangian

$$\mathbb{Y}_{t_1,t_2}(\widehat{\mathfrak{gl}_1}) \tag{5.5.3}$$

has a module

$$\mathbb{M}=\bigoplus_{\lambda\in\Lambda}\Bbbk|\lambda\rangle \tag{5.5.4}$$

called the MacMahon module [83, 84]. Here Λ is the set of plane partitions, in bijection with $\mathbb{C}^{*3}_{t_1,t_2,t_3}$-fixed ideal sheaves in $\mathbb{C}^3$.

The affine Yangian also has its Fock module $\mathcal{F}(a)$ which we identify with the vertex algebra $V_{\mathbb{A}^2}$ where the zero mode of the generating field acts by a. The MacMahon module admits an embedding

$$\mathbb{M}\to\mathcal{F}(a)\otimes\mathcal{F}(\hbar+a)\otimes\mathcal{F}(2\hbar+a)\otimes\cdots \tag{5.5.5}$$

obtained by projecting a plane partition onto the i partition sliced in the z direction [85] and it is the submodule generated by the action of $\mathbb{Y}_{t_1,t_2}(\widehat{\mathfrak{gl}_1})$ vacuum vector in this representation.

In the present situation, let

$$V_{T^*E}(a)$$

denote the $\mathcal{A}_\hbar(\mathfrak{g}_{T^*E})$ module where the universal bundle over $M_{tf}(v; \mathcal{O}_E)$ is given a weight of a. Let

$$\mathbb{V}_0 = \bigoplus_{\substack{\alpha \in \{\alpha_{*,*}| \\ \alpha_{*,\infty} = [\mathcal{O}]\}}} H_T^*(M_{\alpha_{*,*}}) \tag{5.5.6}$$

denote the vector space underlying the 0-leg DT vertex of $\mathrm{Tot}_E(\mathcal{L} \oplus \mathcal{L}^\vee)$.

We have an embedding

$$\mathbb{V}_0 \to V_{T^*E}(a) \otimes V_{T^*E}(\hbar + a) \otimes V_{T^*E}(2\hbar + a) \otimes \cdots \tag{5.5.7}$$

Assuming (5.4.3) defines a completed coproduct on $\mathcal{A}_\hbar(\mathfrak{g}_{T^*E})$ we can let $\mathcal{A}_\hbar(\mathfrak{g}_{T^*E})$ act on $\mathbb{V}_0$.

Conjecture 5.5.8. *The orbit of the vacuum in the target of* (5.5.7) *under the action of* $\mathcal{A}_\hbar(\mathfrak{g}_{T^*E})$ *is* $\mathbb{V}_0$.

In particular, this implies that $\mathbb{V}_0$ embeds into an infinite tensor product of *cone vertex algebras* from Example 2.2.8.

Remark 5.5.9. If Conjecture 5.5.8 is true, then many formulas, in particular (5.4.3) make sense without requiring a completion in the module $\mathbb{V}_0$ because the weight spaces in the tensor product of cone vertex algebras lie in a convex come.

References

[1] S. DeHority, "Elliptic surfaces and toroidal superalgebras," *arXiv preprint arXiv:2311.00349*, 2023.

[2] S. DeHority, "Toroidal analogues of the grothendieck-springer map," *arXiv preprint arXiv:2311.00355*, 2023.

[3] R. E. Borcherds, "Vertex algebras, kac-moody algebras, and the monster," *Proceedings of the National Academy of Sciences*, vol. 83, no. 10, 3068–3071, 1986.

[4] V. G. Kac, *Vertex algebras for beginners*. American Mathematical Soc., 1998, ISBN: 0-8218-1396-X.

[5] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves, Second edition* (Mathematical Surveys and Monographs). American Mathematical Society, Aug. 2004, vol. 88, DOI: http://dx.doi.org.ezproxy.cul.columbia.edu/10.1090/surv/088, ISBN: 978-0-8218-3674-3.

[6] M. C. Cheng and G. Sgroi, "Cone vertex algebras, mock theta functions, and umbral moonshine modules," *arXiv preprint arXiv:2203.03052*, 2022.

[7] H. G. Kausch, "Symplectic fermions," *Nuclear Physics B*, vol. 583, no. 3, 513–541, 2000.

[8] T. Creutzig and D. Ridout, "Logarithmic conformal field theory: Beyond an introduction," *Journal of Physics A: Mathematical and Theoretical*, vol. 46, no. 49, p. 494 006, 2013.

[9] B. N. Bakalov and J. J. Villarreal, "Logarithmic vertex algebras," *Transformation Groups*, 1–63, 2022.

[10] D. Maulik and A. Okounkov, "Quantum groups and quantum cohomology," *arXiv:1211.1287 [math-ph]*, Jun. 2018, arXiv: 1211.1287.

[11] F. Chen, H. Li, and S. Tan, "Toroidal extended affine lie algebras and vertex algebras," *arXiv preprint arXiv:2102.10968*, 2021.

[12] Y. Billig, "Representations of toroidal extended affine lie algebras," *Journal of Algebra*, vol. 308, no. 1, 252–269, 2007.

[13] C. Kassel, "Kähler differentials and coverings of complex simple lie algebras extended over a commutative algebra," *Journal of Pure and Applied Algebra*, vol. 34, no. 2–3, 265–275, 1984.

[14] R. Høegh-Krohn and B. Torrésani, “Classification and construction of quasisimple lie algebras,” *Journal of functional analysis*, vol. 89, no. 1, 106–136, 1990.

[15] E. Neher, *Lectures on extended affine lie algebras*, 2010. arXiv: `1003.2352 [math.RA]`.

[16] K. Iohara, Y. Saito, and M. Wakimoto, “Hirota bilinear forms with 2-toroidal symmetry,” *arXiv preprint solv-int/9811001*, 1998.

[17] S. Berman and Y. Billig, “Irreducible representations for toroidal lie algebras,” *Journal of Algebra*, vol. 221, no. 1, 188–231, 1999.

[18] Y. Billig, “A category of modules for the full toroidal lie algebra,” *International Mathematics Research Notices*, vol. 2006, no. 9, 68395–68395, 2006.

[19] S Eswara Rao and R. Moody, “Vertex representations for n-toroidal lie algebras and a generalization of the virasoro algebra,” *Communications in mathematical physics*, vol. 159, no. 2, 239–264, 1994.

[20] R. V. Moody, S. E. Rao, and T. Yokonuma, “Toroidal lie algebras and vertex representations,” *Geometriae dedicata*, vol. 35, no. 1–3, 283–307, 1990.

[21] Y. Zhu, *Vertex operator algebras, elliptic functions and modular forms*. Yale University, 1990, ISBN: 9798207589855.

[22] H. Li, “-coordinated quasi-modules for quantum vertex algebras,” *Communications in mathematical physics*, vol. 308, no. 3, 703–741, 2011.

[23] J. Lepowsky, “Application of a ‘jacobi identity’for vertex operator algebras to zeta values and differential operators,” *Letters in Mathematical Physics*, vol. 53, 87–103, 2000.

[24] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. Van Rees, “Infinite chiral symmetry in four dimensions,” *Communications in Mathematical Physics*, vol. 336, 1359–1433, 2015.

[25] T. Arakawa, T. Kuwabara, and S. Möller, *Hilbert schemes of points in the plane and quasi-lisse vertex algebras with $\mathcal{N} = 4$ symmetry*, 2023. arXiv: `2309.17308 [math.RT]`.

[26] K. Costello and D. Gaiotto, *Twisted holography*, 2021. arXiv: `1812.09257 [hep-th]`.

[27] F. Bonetti, C. Meneghelli, and L. Rastelli, “Voas labelled by complex reflection groups and 4d scfts,” *Journal of High Energy Physics*, vol. 2019, no. 5, 1–68, 2019.

[28] D. Adamović, “A realization of certain modules for the n= 4 superconformal algebra and the affine lie algebra a 2 (1),” *Transformation groups*, vol. 21, 299–327, 2016.

[29] M. Kapranov and E. Vasserot, "The cohomological hall algebra of a surface and factorization cohomology," *Journal of the European Mathematical Society*, vol. 25, no. 11, 4221–4289, 2022.

[30] Y. Zhao, "On the k-theoretic hall algebra of a surface," *International Mathematics Research Notices*, vol. 2021, no. 6, 4445–4486, 2021.

[31] A. Mellit, A. Minets, O. Schiffmann, and E. Vasserot, "Coherent sheaves on surfaces, cohas and deformed $W_{1+\infty}$-algebras," *arXiv preprint arXiv:2311.13415*, 2023.

[32] W.-P. Li, Z. Qin, and W. Wang, "Hilbert schemes and w-algebras," *International Mathematics Research Notices*, vol. 2002, no. 27, 1427–1456, 2002.

[33] B. Davison, *Affine bps algebras, w algebras, and the cohomological hall algebra of* $\mathbb{A}^2$, 2022. arXiv: `2209.05971 [math.RT]`.

[34] T. Bridgeland, A. King, and M. Reid, "Mukai implies mckay: The mckay correspondence as an equivalence of derived categories," *arXiv preprint math/9908027*, 1999.

[35] Y. Yang and G. Zhao, "The cohomological hall algebra of a preprojective algebra," *Proceedings of the London Mathematical Society*, vol. 116, no. 5, 1029–1074, 2018.

[36] O. Schiffmann and E. Vasserot, *Cohomological hall algebras of quivers and yangians*, 2023. arXiv: `2312.15803 [math.RT]`.

[37] T. M. Botta and B. Davison, *Okounkov's conjecture via bps lie algebras*, 2023. arXiv: `2312.14008 [math.RT]`.

[38] O. Schiffmann and E. Vasserot, "On cohomological hall algebras of quivers: Yangians," *arXiv preprint arXiv:1705.07491*, 2017.

[39] D.-E. Diaconescu, M. Porta, and F. Sala, "Cohomological hall algebras and their representations via torsion pairs," *arXiv preprint arXiv:2207.08926*, 2022.

[40] H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces* (University Lecture Series). Providence, Rhode Island: American Mathematical Society, Sep. 1999, vol. 18, ISBN: 978-0-8218-1956-2.

[41] I. Grojnowski, "Instantons and affine algebras i: The hilbert scheme and vertex operators," *Mathematical Research Letters*, vol. 3, no. 2, 275–291, 1996.

[42] D. Huybrechts and M. Lehn, "The geometry of moduli spaces of sheaves," p. 281,

[43] M. Wemyss, "Reconstruction algebras of type a," *Transactions of the American Mathematical Society*, vol. 363, no. 6, 3101–3132, 2011.

[44] M. Wemyss, “The mckay correspondence,” *Mathematische annalen*, vol. 350, no. 3, 631–659, 2011.

[45] M. Reid, “Surface cyclic quotient singularities and hirzebruch-jung resolutions,” *manuscript available at http://www. warwick. ac. uk/masda/surf*, 2012.

[46] W. Geigle and H. Lenzing, “A class of weighted projective curves arising in representation theory of finite dimensional algebras,” Springer, 2006, 265–297.

[47] D. Happel and C. M. Ringel, “The derived category of a tubular algebra,” in *Representation Theory I Finite Dimensional Algebras* (Lecture Notes in Mathematics), V. Dlab, P. Gabriel, and G. Michler, Eds., Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986, vol. 1177, 156–180, ISBN: 978-3-540-16432-6.

[48] I. Burban and B. Kreußler, “Derived categories of irreducible projective curves of arithmetic genus one,” *Compositio Mathematica*, vol. 142, no. 5, 1231–1262, Sep. 2006.

[49] I. Burban and O. Schiffmann, “The composition hall algebra of a weighted projective line,” *Journal für die reine und angewandte Mathematik (Crelles Journal)*, vol. 2013, no. 679, 75–124, 2013.

[50] T. Bridgeland, “Fourier-mukai transforms for elliptic surfaces,” *Journal für die reine und angewandte Mathematik (Crelles Journal)*, vol. 1998, no. 498, 115–133, 1998.

[51] D. Arinkin, “Autoduality of compactified jacobians for curves with plane singularities,” *arXiv preprint arXiv:1001.3868*, 2010.

[52] S. Mukai, “Duality between d (x) and with its application to picard sheaves,” *Nagoya Mathematical Journal*, vol. 81, 153–175, 1981.

[53] D Kaledin, “How to glue derived categories,” *Bulletin of Mathematical Sciences*, vol. 8, no. 3, 477–602, 2018.

[54] P. Seidel and R. Thomas, “Braid group actions on derived categories of coherent sheaves,” 2001.

[55] A. Ishii and K. Ueda, “The special mckay correspondence and exceptional collections,” *Tohoku Mathematical Journal, Second Series*, vol. 67, no. 4, 585–609, 2015.

[56] A. Bondal and D. Orlov, “Semiorthogonal decomposition for algebraic varieties,” *arXiv preprint alg-geom/9506012*, 1995.

[57] K. Saito and T. Takebayashi, “Extended affine root systems iii (elliptic weyl groups),” *Publications of the Research Institute for Mathematical Sciences*, vol. 33, no. 2, 301–329, 1997.

[58] K. Saito, "Extended affine root systems i (coxeter transformations)," *Publications of the Research Institute for Mathematical Sciences*, vol. 21, no. 1, 75–179, 1985.

[59] M. Van den Bergh, "Non-commutative P1-bundles over commutative schemes," *Transactions of the American Mathematical Society*, vol. 364, no. 12, 6279–6313, 2012.

[60] A. Elagin, *On equivariant triangulated categories*, 2014.

[61] D. Arinkin, A. Căldăraru, and M. Hablicsek, "Formality of derived intersections and the orbifold hkr isomorphism," *Journal of Algebra*, vol. 540, pp. 100–120, 2019.

[62] V. Baranovsky, "Orbifold cohomology as periodic cyclic homology," *International Journal of Mathematics*, vol. 14, no. 08, pp. 791–812, 2003.

[63] T. Bridgeland, "Stability conditions on triangulated categories," *Annals of Mathematics*, vol. 166, no. 2, 317–345, Sep. 2007.

[64] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, and P. Stellari, *Publications mathématiques de l'IHÉS*, vol. 133, no. 1, 157–325, Jun. 2021, arXiv:1902.08184 [math].

[65] A. Bayer, E. Macrì, and P. Stellari, "The space of stability conditions on abelian threefolds, and on some calabi-yau threefolds," *Inventiones mathematicae*, vol. 206, 869–933, 2016.

[66] A. Bayer and E. Macrì, "Projectivity and birational geometry of bridgeland moduli spaces," *Journal of the American Mathematical Society*, vol. 27, no. 3, 707–752, 2014.

[67] T. Tajakka, "Uhlenbeck compactification as a bridgeland moduli space," *International Mathematics Research Notices*, vol. 2023, no. 6, 4952–4997, 2023.

[68] E. M. Rains, "The birational geometry of noncommutative surfaces," no. arXiv:1907.11301, Jul. 2019, arXiv:1907.11301 [nlin].

[69] M. Lieblich, "Moduli of complexes on a proper morphism," *arXiv preprint math/0502198*, 2005.

[70] A. Bayer and E. Macrì, "Mmp for moduli of sheaves on k3s via wall-crossing: Nef and movable cones, lagrangian fibrations," *Inventiones mathematicae*, vol. 198, no. 3, 505–590, 2014.

[71] T. A. Nevins, "Moduli spaces of framed sheaves on ruled surfaces," Ph.D. The University of Chicago, United States – Illinois, ISBN: 9780599696396.

[72] S. Ruijsenaars, "Complete integrability of relativistic calogero-moser systems and elliptic function identities," *Communications in Mathematical Physics*, vol. 110, 191–213, 1987.

[73] S. N. Ruijsenaars and H. Schneider, “A new class of integrable systems and its relation to solitons,” *Annals of Physics*, vol. 170, no. 2, 370–405, 1986.

[74] D. Ben-Zvi and T. Nevins, “Flows of calogero-moser systems,” *International Mathematics Research Notices*, vol. 2007, rnm105, 2007.

[75] D. Ben-Zvi and T. Nevins, “Perverse bundles and calogero–moser spaces,” *Compositio Mathematica*, vol. 144, no. 6, 1403–1428, 2008.

[76] M. Penciak, “Spectral description of the spin ruijsenaars-schneider system,” *arXiv preprint arXiv:1909.08107*, 2019.

[77] T. Mochizuki, *The structure of the cohomology ring of the filt schemes*, 2003. arXiv: `math/0301184 [math.AG]`.

[78] D. Huybrechts and M. Lehn, “Framed modules and their moduli,” *International Journal of Mathematics*, vol. 06, no. 02, 297–324, Apr. 1995.

[79] K. Nagao, “Quiver varieties and frenkel-kac construction,” *arXiv:math/0703107*, Mar. 2007, arXiv: math/0703107.

[80] H. Nakajima, “Quiver varieties and kac-moody algebras,” *Duke Mathematical Journal*, vol. 91, no. 3, 515–560, Feb. 1998.

[81] S. DeHority, “Affinizations of lorentzian kac-moody algebras and hilbert schemes of points on k3 surfaces,” *arXiv preprint arXiv:2007.04953*, 2020.

[82] N. Guay, H. Nakajima, and C. Wendlandt, “Coproduct for yangians of affine kac–moody algebras,” *Advances in Mathematics*, vol. 338, 865–911, Nov. 2018.

[83] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, “Quantum toroidal gl_1-algebra: Plane partitions,” 2012.

[84] H. Awata, M. Fukuma, Y. Matsuo, and S. Odake, “Representation theory of the $W_{1+\infty}$ algebra,” *Progress of Theoretical Physics Supplement*, vol. 118, 343–373, 1995.

[85] A. Tsymbaliuk, “The affine yangian of gl1 revisited,” *Advances in Mathematics*, vol. 304, 583–645, 2017.