Modeling Random Events

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Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2022

Abstract

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In this thesis, we address two types of modeling of random events. The first one, contained in Chapters 2 and 3, is related to the modeling of dependent stopping times. In Chapter 2, we use a modified Cox construction, along with a modification of the bivariate exponential introduced by Marshall & Olkin (1967), to create a family of stopping times, which are not necessarily conditionally independent, allowing for a positive probability for them to be equal. We also present a series of results exploring the special properties of this construction, along with some generalizations and possible applications.

In Chapter 3, we present a detailed application of our model to Credit Risk theory. We propose a new measure of systemic risk that is consistent with the economic theories relating to the causes of financial market failures and can be estimated using existing hazard rate methodologies, and hence, it is simple to estimate and interpret. We do this by characterizing the probability of a market failure which is defined as the default of two or more globally systemically important banks (G-SIBs) in a small interval of time. We derive various theorems related to market failure probabilities, such as the probability of a catastrophic market failure, the impact of increasing the number of G-SIBs in an economy, and the impact of changing the initial conditions of the economy's state variables.

The second type of random events we focus on is the failure of a group in the context of microlending, which is a loan made by a bank to a small group of people without credit histories. Since the creation of this mechanism by Muhammed Yunus, it has received a fair amount of aca-

demic attention. However, one of the issues not yet addressed in full detail is the issue of the size of the group. In Chapter 4, we propose a model with interacting forces to find the optimal group size. We define “optimal" as that group size that minimizes the probability of default of the group. Ultimately, we show that the original choice of Muhammad Yunus, of a group size of five people, is, under the right, and, we believe, reasonable hypotheses, either close to optimal, or even at times exactly optimal, i.e., the optimal group size is indeed five people.

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Acknowledgements

This dissertation is the culmination of many years of work and it would not have come to a successful end without the help and support of multiple people that have been by my side throughout the years. The following is my humble way of thanking them.

First and foremost, Philip, I cannot thank you enough for being my advisor and for always believing in me, even when I did not believe in myself. Thank you for bringing out the mathematician in me and for making me realize how beautiful and rewarding research can be. Our research meetings were a joyful moment that I looked forward to every week. Thank you also for all the professional and life advice you have given me. You have become a true friend to me and I am really going to miss you! Please, extend my gratitude to Diane and Rachel who were also cheering me on and looking out for me through all these years.

Cindy, Xin, Bob, and Sumit, thank you for being on my thesis committee and for the fruitful discussion following my defense. Cindy, thanks for all the professional advice you have given me, particularly this last year; the job market was a bit less stressful with your help. Xin, thanks for being so accommodating even though we have not met each other in person; I really appreciate it! Bob, collaborating with you in one of the chapters of this dissertation was a great learning experience (and enjoyable too!). Also, I will be forever grateful for all your help in the job market process. Sumit, you know I would not have been able to pass the quals without your help. I fondly remember your office hours during those first years of my Ph.D. Thank you for all your patience back then and for all your interesting comments about my research now.

During my time at Columbia, I also had the privilege of being mentored by excellent professors, whose support has been invaluable. I thank Anne, Arian, Bodhi, Cindy, Ioannis, Johannes, Marcel, Marco, Mark, Richard, Sumit, Thibault, Tian, Victor, and Yuqi. Special thanks to Victor for your support at the beginning of my Ph.D. and to Richard for your guidance during this last year of my Ph.D. while I was applying for jobs.

These six years were more manageable thanks to the Department staff, who helped me successfully navigate the paperwork and bureaucracy required by the Ph.D.. Thank you, Dood, Anthony, and Zammy!

At the beginning of my Ph.D., being away from home was really hard, but thanks to all the wonderful friends I have made during these six years, now leaving NYC is not going to be easy. Thank you Agathe, Bridget, Camilo, Enrique, Lula, Missy, Nick, Rachel, Robin, Savi, Shawn, Susanna, and Vale. The moments we shared together filled me with tremendous joy and kept me sane through the gruesome parts of the Ph.D.. Susanna, thanks for reaching out and for showing me grace when I most needed it. Lula, my favorite roomie, thank you for always being there for me. Those two years we lived together are full of good memories that will last a lifetime. I am so lucky to have you in my life!

I was also fortunate to have the support of many people back in Mexico. Thanks to my friends and family for cheering on me even when we cannot meet as often as I would like. My undergraduate professors were undoubtedly a source of inspiration to achieve the dream of getting a Ph.D. Thank you all!

I would never have made it to Columbia, without my close family's support. Gio, thanks for bringing joy to my life since the day you were born. Because of you, I take life a little bit less seriously. As your big sister, growing up together, I used to take care of you, but now it seems that roles are reversed. Thanks for worrying and taking care of me even when we are not physically close. Mom, thanks for showing me, since I was a kid, that nothing is impossible if one works hard enough. Thanks for strongly believing in and supporting my dreams, even when it means being physically far apart. I sincerely thank you for everything you had to give up so that my brother and I could have a better future. I am certain that my dad is so proud of the outstanding job you did raising us up. Mission accomplished!

Last but not least, Edgar, thanks for coming into my life. Meeting you was certainly THE stroke of luck in my life! You are everything I ever wanted in a partner and more. Thanks for your continuous support in all the “adventures” during the last couple of years; from surviving a global pandemic together to moving across the country. Thank you a 11110100001001000000_2 times! I am so looking forward to what the future holds for us. Everything is better with you by my side.

To each and every one of you: “¡muchas veces, muchas gracias!”

Para mamá:

Sin ti, nada de esto sería posible.

Chapter 1: Introduction

Stopping times are used in applications to model random arrivals. A standard assumption in many models is that, given an underlying filtration, they are conditionally independent. This is a widely useful assumption because it is mathematically attractive. However, there are circumstances where it seems to be unnecessarily strong. In this thesis, we go beyond the independence assumption between stopping times. Chapter 2 and 3 are based on two working papers. Chapter 2 presents a mathematical model for dependent stopping times, some interesting consequences of it, and some generalizations. It also sketches some of the possible applications. In Chapter 3, we present an application of the model presented in Chapter 2.

In Chapter 2, we use a modified Cox construction along with an extension of the bivariate exponential introduced by Marshall and Olkin [51] to create a family of stopping times, which are not necessarily conditionally independent, allowing for a positive probability for them to be equal. We provide an interpretation and explore the special properties that this joint distribution has. Not only do we treat the case where the two stopping times have a nonnegative probability of being equal, but more generally we study the case when the two stopping times are “close" to each other, in various metrics. Moreover, we provide some possible generalizations of the model and sketch some applications to modeling COVID-19 contagion (and epidemics in general), Civil Engineering, and to Credit Risk theory.

This approach is attractive because:

1. It is a more realistic model as it will be clear from the applications provided in Chapter 2 and the comprehensive application of Chapter 3.

2. It offers a multivariate distribution that is tractable, facilitating subsequent computations as those obtained in Chapter 2 and 3.

Chapter 3 provides a detailed application of the model presented in Chapter 2 in the context of Credit Risk theory. We propose a new measure of systemic risk by characterizing the probability of a market failure. This probability is defined as the default of two or more globally systemically important banks (G-SIBs) in a small interval of time. The default probabilities of the G-SIBs are correlated through the possible existence of a market-wide stress event. The characterization employs a multivariate Cox process (the one introduced in Chapter 2) across the G-SIBs, which allows us to relate our work to the existing literature on intensity-based models, such as Chava and Jarrow [18], Campbell et al. [17], Shumway [58], and Giesecke [35]. This means, as we show in Chapter 3, that our measure is simple to estimate and interpret. Various theorems related to market failure probabilities are derived, including the probability of a market failure due to two banks defaulting over the next infinitesimal interval, the probability of a catastrophic market failure, the impact of increasing the number of G-SIBs in an economy, and the impact of changing the initial conditions of the economy's state variables. We also show that if there are too many G-SIBs, a market failure is inevitable, i.e., the probability of a market failure tends to one.

Chapter 4 is based on a published paper about microlending which is a loan made by a bank to small group of people without credit histories. It began with the Grameen Bank in Bangladesh, and is widely seen as the creation of Muhammad Yunus (see Yunus [66]), who received the Nobel Peace Prize in 2006 in recognition of his largely successful efforts. Since that time the model of microlending has received a fair amount of academic attention, see Stiglitz [59], Varian [63], Conlin [24], Morduch [52], Chowdhury [22], [23], Tedeschi [62], and Jarrow and Protter [43]. However, one of the issues not yet addressed in full detail is the size of the group. Some attention has nevertheless been paid using an experimental and game theory approach (see Armendariz and Morduch [7], Giné et al. [36], and Ahlin [3], [4]). We, instead, take a mathematical approach where the goal is to minimize the probability of default of the group. To do this, we create a model with interacting forces and make precise the hypotheses of the model. We ultimately show that the original choice of Muhammad Yunus, of a group size of five people, is, under the right, and, we believe, reasonable hypotheses, either close to optimal, or even at times exactly optimal, i.e., the optimal group size is indeed five people.

Let us highlight that Chapter 4 is not directly related to the work presented in Chapters 2 and 3. However, we believe that the unifying topic is the modeling of random events and hence, the title of this thesis.

Chapter 2: Stopping Times Occurring Simultaneously

Probability models are ubiquitous in modern society. The timing of a random event is often crucial to the analysis of the reliability of a system or to the danger of a default within a credit risk context. Such random times are, of course, referred to as stopping times. Some examples of random times of intrinsic interest range from the quotidian banal such as bus arrivals, customers arriving at a restaurant, to the less banal, such as the time a cancer metastasizes, or an individual contracts a contagious disease such as COVID-19. Stopping times appear in Civil Engineering as they model metal fatigue in aircraft, the times of collapse of a bridge, or extreme events such as the recent collapse of the condominium towers in Surfside Florida. A particularly common use of stopping times is in the theory of credit risk, within the discipline of Mathematical Finance.

Indeed, we take the approach pioneered by researchers in Credit Risk, the seminal event being the publication of the book by David Lando in 2004 (see [48]). The "Cox Construction" that Lando uses begins with a filtration of observable events, satisfying the usual hypotheses (see Protter [55]) for the formal definition of the usual hypotheses). To fix notation, let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ denote the underlying filtration of observable events. We will add a random time τ to our model by creating an exponential random variable Z (of parameter 1) that is independent of $\mathcal{F}$ and all of $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We choose an increasing process $A = (A(t))_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and we create the desired random time τ by writing:

$$\tau = \inf_{t\geq 0}\{A(t) \geq Z\}. \tag{2.1}$$

τ is then a totally inaccessible stopping time for an enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t).$$

With τ being a stopping time, the process $1_{\{\tau\geq t\}}$ is adapted and non-decreasing, hence it is a submartingale, whence there exists, via the Doob-Meyer decomposition theorem, a unique, $\mathbb{G}$-predictable, increasing process $C = (C(t))_{t\geq 0}$ such that $1_{\{\tau \geq t\}} - C(t)$ is a martingale. It is easy to show that the process C is in fact the process A of (2.1). The process A is called the compensator of the stopping time τ.

If we repeat this procedure to construct two stopping time τ_1 and τ_2, using two independent exponentials of parameter 1, Z_1 and Z_2, which are independent of each other as well as the underlying filtration $\mathbb{F}$, then we have two totally inaccessible stopping times which are also conditionally independent of each other, and we have $P(\tau_1 = \tau_2) = 0$. This is the approach of Lando and many others, with some notable exceptions such as Bielecki et al. [11] and Jiao and Li [44].

In applications, often the process $A = (A(t))_{t\geq 0}$ is assumed to be of the form

$$A(t) = \int_0^t \alpha(s)ds, \text{ for an adapted process } \alpha(s) \geq 0 \text{ all } s \geq 0. \tag{2.2}$$

Sufficient conditions for $A(t)$ to be of the form (2.2) are known (see, e.g., Ethier and Kurtz [33], Guo and Zeng [39], Zeng [67], and for a general result, Janson et al. [41]).

In this Chapter we are concerned with models where one can have $P(\tau_1 = \tau_2) > 0$, and some of the ramifications of such a model. This model arises in applications when η_1 and η_2 are constructed as Cox processes with independent exponentials Z_1 and Z_2, but with an added complication: there is a third stopping time η_3, and $\tau_1 = \eta_1 \wedge \eta_3$, while $\tau_2 = \eta_2 \wedge \eta_3$. This is a natural situation in Credit Risk for example, as we indicate in Chapter 3, but also in other domains, such as Civil Engineering, and disease contagion which we treat in Section 2.3. The resulting stopping times of interest, τ_1 and τ_2, are no longer conditionally independent, and in simple cases, the bivariate exponential distribution of Marshall and Olkin [51] comes into play. These models do not have densities in $\mathbb{R}_+ \times \mathbb{R}_+$, leading to a two dimensional cumulative distribution function with a singular component. We explore the consequences of such a phenomenon in some detail, and we explain its utility for various kinds of applications caused by the confluence of stopping times that arise naturally in the modeling of random events.

One could argue that it is not of vital importance to have two stopping times with a positive probability of being equal, but rather just to have them be close to each other, even arbitrarily close. We study this situation in Section 2.1.5 and in Chapter 3

In many easy to imagine examples, the times τ_1 and τ_2 are positively correlated. It is possible to imagine, however, situations where they would naturally be negatively correlated. To cover that situation, we slightly modify our constructions, as exhibited in Section 2.2.2. As an example, consider the recent scandal with the Boeing 737 Max airplanes. The two horrifically deadly crashes occurred with the confluence of the airplane's software malfunctioning and the panic of inexperienced and poorly trained pilots. Since airplanes are so carefully constructed, software failures are rare, which made the Boeing 737 Max failures unanticipated; however, unfortunately, poorly trained pilots have recently become rather common, especially in airlines based in poorly regulated countries. (The two examples were Lion Air in Malaysia and Ethiopian Airlines, in Ethiopia). That duality between those “rare" and “common" events is due to their negative correlation.

Our work in this paper falls into a popular thread of prior research. Cox and Lewis [25] studied the case of multiple event types occurring in a continuum, but differently from us, they do not generalize the model proposed by Marshall and Olkin [51]. Moreover, they mostly consider “regular" processes, i.e., where events cannot occur at the same time, which is an important contribution of our work. Diggle and Milne [31] also proposed a multivariate version of the Cox Process but their model does not allow for $P(\tau_1 = \tau_2) > 0$ either. Another example of a multivariate point process is given in Brown et al. [16].

There is existing work on modeling simultaneous defaults, but different from us, under Merton's structural risk model (see Li [49]). Kay Giesecke [35], in a seminal paper concerning Credit Risk published in 2003, was the first (to our knowledge) to consider the Marshall and Olkin model of the bivariate exponential. Later, in 2013, Bielecki et al. [11] also worked in a similar model. In this paper, we develop the ideas present in Giesecke [35] and in Bielecki et al. [11] and go beyond them. Another closely related work is Lindskog and McNeil [50] who consider a Poisson shock model of arbitrary dimension with both fatal and not-necessarily-fatal shocks. Jiao and Li

[44] also allow for simultaneous defaults without using a pure Cox process construction. Inspired by the Jacod Criterion, they use what is known as a density approach, and they can include cases where a stopping time meets another stopping time, specified in advance.

There are also other types of generalizations of Cox Processes (see Gueye and Jeanblanc [37]), where they generalize, not the number of stopping times, but the form of the process $A(t)$. They assume that $A(t)$ is not necessarily of the form given in equation (2.2).

The organization of this Chapter is as follows. Section 2.1, presents the survival function of two conditionally dependent (as opposed to conditionally independent) stopping times, an interpretation of it, a decomposition of it into its singular and absolutely continuous parts, and a series of results exploring the special properties such a modeling approach has. For example, not only do we treat the case where $P(\tau_1 = \tau_2) > 0$, but more generally we study when the two stopping times are 'close' to each other, in various metrics. Section 2.2 provides two generalizations of our model. In the first one, we extend to an arbitrary (but finite) number of such stopping times and in the second one, we allow for a slightly different dependency between the stopping times. Section 2.3 shows the applicability of our results by providing examples in Epidemiology (such as the case of COVID-19 and its variants) and in Civil Engineering (e.g., the recent condo collapse of Champlain Towers in Florida).

2.1 The Survival Function

As a starting point, we will consider the case of having two stopping times, which we will generalize to n stopping times in section 2.2.1. Let us fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ large enough to support an $\mathbb{R}^d$-valued càdlàg stochastic process $X = \{X_t, t \geq 0\}$ and three independent exponential random variables with parameter 1 ($Z_i,\ i = 1, 2, 3$). Then, define:

$$\tau_1 := \min(\eta_1, \eta_3) \quad \text{and} \quad \tau_2 := \min(\eta_2, \eta_3) \tag{2.3}$$

where

$$\eta_i := \inf\{s : A_i(s) \geq Z_i\}, \qquad A_i(s) := \int_0^s \alpha_i(X_r)dr, \qquad Z_i \overset{\text{iid}}{\sim} \text{Exp}(1) \tag{2.4}$$

Theorem 2.1 (Survival function) *Suppose $\alpha_i(\cdot) : \mathbb{R}^d \mapsto [0,\infty)$ for $i = 1,2,3$ are non-random positive continuous functions, which implies that $A_i(s)$ are continuous and strictly increasing. Assume further that* $\lim_{s\to\infty} A(s) = \infty$ *a.s. Then,*

$$\overline{F}_{(\tau_1,\tau_2)}(s,t) := \mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{E}\left[\exp\left[-A_1(s) - A_2(t) - A_3(s \vee t)\right]\right].$$

Proof. Let $\tilde{\mathbb{P}}_{(s,t)}(\cdot) := \mathbb{P}\left(\cdot|(X_u)_{0\leq u\leq(s\vee t)}\right)$. By definition of τ_1 and τ_2, we have,

$$\begin{aligned}
\tilde{\mathbb{P}}_{(s,t)}(\tau_1 > s, \tau_2 > t) &= \tilde{\mathbb{P}}_{(s,t)}(\min(\eta_1,\eta_3) > s, \min(\eta_2,\eta_3) > t) \\
&= \tilde{\mathbb{P}}_{(s,t)}(\eta_1 > s)\,\tilde{\mathbb{P}}_{(s,t)}(\eta_2 > t)\,\tilde{\mathbb{P}}_{(s,t)}(\eta_3 > s, \eta_3 > t) \\
&= \tilde{\mathbb{P}}_{(s,t)}(\eta_1 > s)\,\tilde{\mathbb{P}}_{(s,t)}(\eta_2 > t)\,\tilde{\mathbb{P}}_{(s,t)}(\eta_3 > \max(s,t)) \\
&= \exp\left[-A_1(s) - A_2(t) - A_3(s \vee t)\right] \\
&= \begin{cases} \exp\left[-A_1(s) - A_2(t) - A_3(s)\right] & s > t \\ \exp\left[-A_1(s) - A_2(t) - A_3(t)\right] & s < t. \end{cases}
\end{aligned}$$

The result follows by taking the expectation. □

Remark. From Theorem 2.1, it is clear that τ_1 is not independent of τ_2 and, as we will show in subsection (2.1.2), their joint distribution has an absolutely continuous and a non-trivial singular part, which means that:

$$\mathbb{P}(\tau_1 = \tau_2) > 0. \tag{2.5}$$

Remark. Note that when $X_t = t \in \mathbb{R}^+$ (i.e., there is no randomness coming from X_t), Theorem 2.1 is a generalization of the bivariate exponential (BVE) distribution introduced by Marshall and Olkin [51]. In the specific case where $\alpha_i(X_s) = \alpha_i \in \mathbb{R}$. i.e., the intensity is constant, we recover the Marshall and Olkin BVE with parameter $(\alpha_1, \alpha_2, \alpha_3)$. This is $(\tau_1, \tau_2) \sim \text{BVE}(\alpha_1, \alpha_2, \alpha_3)$.

Remark. The marginal distribution of τ_i is given by:

$$\mathbb{P}(\tau_i > s) = \mathbb{P}(\min(\eta_i, \eta_3) > s) = \mathbb{P}(\eta_i > s)\mathbb{P}(\eta_3 > s) = \mathbb{E}\left[\exp\left(-\int_0^s (\alpha_i + \alpha_3)(r)dr\right)\right] \quad (2.6)$$

which coincides with the marginal distribution in the Cox construction, see Lando [47] [48]. That is, each of the stopping times is a Cox process. However, by construction, they are not independent of each other.

Remark. $\alpha_i()$ is usually known as the intensity of η_i in the $(\mathcal{F}_t \vee \sigma(Z_i))_{t\geq 0}$ filtration where $(\mathcal{F}_t) = \sigma\{X_s, 0 \leq s \leq t\}$.

Remark. The intensity of τ_i for $i = 1, 2$ in the $(\mathcal{F}_t \vee \sigma(Z_i))_{t\geq 0}$ filtration is $(\alpha_i + \alpha_3)(\cdot)$

Caveat. In the rest of the chapter, for easiness of notation, sometimes we write $\alpha_i(s)$ instead of $\alpha_i(X_s)$ for $i = 1, 2, 3$.

2.1.1 Interpretation of Joint Distribution

A way to interpret the distribution given in Theorem 2.1 is the following: Suppose we have a two-component system where each component's life is represented by τ_1 and τ_2 respectively. Any component dies after receiving a shock. Shocks are governed by three independent Cox processes $\Lambda_1(t, \alpha_1(X_t))$, $\Lambda_2(t, \alpha_2(X_t))$ and $\Lambda_3(t, \alpha_3(X_t))$. Events in the process $\Lambda_1(t, \alpha_1(X_t))$ are shocks to component 1, events in the process $\Lambda_2(t, \alpha_2(X_t))$ are shocks to component 2 and events in the process $\Lambda_3(t, \alpha_3(X_t))$ are shocks to both components.

Hence, we get,

$$\begin{aligned}
\mathbb{P}(\tau_1 > s, \tau_2 > t) &= \mathbb{E}\left[\mathbb{P}\left(\tau_1 > s, \tau_2 > t | (X_u)_{0\leq u\leq(s\vee t)}\right)\right] \\
&= \mathbb{E}\left[\tilde{\mathbb{P}}_{(s,t)}\left[\Lambda_1(s, \alpha_1(X)) = 0\right]\tilde{\mathbb{P}}_{(s,t)}\left[\Lambda_2(t, \alpha_1(X)) = 0\right]\tilde{\mathbb{P}}_{(s,t)}\left[\Lambda_3(s \vee t, \alpha_3(X)) = 0\right]\right] \\
&= \mathbb{E}\left[\exp\left(-A_1(s) - A_2(t) - A_3(s \vee t)\right)\right] \quad (2.7)
\end{aligned}$$

Which coincides with Theorem 2.1. In the second line of the previous expression $\tilde{\mathbb{P}}_{(s,t)}(\cdot)$ means $\mathbb{P}\left(\cdot|(X_u)_{0\leq u\leq (s\vee t)}\right)$.

2.1.2 Decomposition of the Joint Distribution

As found in Theorem 2.1 and assuming that $X_t = t \in \mathbb{R}^+$, we have that the joint survival function is:

$$\overline{F}_{(\tau_1,\tau_2)}(s,t) = \exp\left[-A_1(s) - A_2(t) - A_3(s \vee t)\right]$$

Theorem 2.2 *Under the conditions of Theorem 2.1 and assuming that $X_t = t \in \mathbb{R}_{,}$, the absolutely continuous and singular parts of $\overline{F}_{(\tau_1,\tau_2)}(s,t)$ are given by:*

$$\overline{F}_{(\tau_1,\tau_2)}(s,t) = \beta\overline{F}_{a.a.}(s,t) + (1-\beta)\overline{F}_{sing}(s,t) \tag{2.8}$$

where

$$\beta := \int_0^\infty (\alpha_1(s) + \alpha_2(s)) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds$$

$$\overline{F}_{a.a.}(s,t) = \frac{1}{\beta}\left[e^{-A_1(s)-A_2(t)-A_3(s\vee t)} - \int_{s\vee t}^{\infty} \alpha_3(x) \exp\left(-\sum_{i=1}^{3} A_i(x)\right) dx\right]$$

$$\overline{F}_{sing}(s,t) = \frac{1}{1-\beta}\left[\int_{s\vee t}^{\infty} \alpha_3(x) \exp\left[-A_1(x) - A_2(x) - A_3(x)\right] dx\right].$$

Proof. Let Λ_i be the waiting time to the first shock in the process $\Lambda_i(t, \alpha_i(t))$ defined in Section 2.1.1. (Recall that we are assuming that $X_t = t$). By the assumptions made in Section 2.1.1, Λ_i are independent and the survival function of each one of them is given by:

$$\mathbb{P}\left(\Lambda_i > t\right) = \exp\left[-A_i(t)\right] \text{ for all } t \geq 0. \tag{2.9}$$

Hence, the density is equal to:

$$f_{\Lambda_i}(t) = \alpha_i(s) \exp\left(-A_i(s)\right) \text{ for all } s \geq 0. \tag{2.10}$$

$\overline{F}_{(\tau_1,\tau_2)}(s,t)$ can be written as,

$$\overline{F}_{(\tau_1,\tau_2)}(s,t) = \mathbb{P}\left(\tau_1 > s, \tau_2 > t|B\right)\mathbb{P}\left(B\right) + \mathbb{P}\left(\tau_1 > s, \tau_2 > t|B^c\right)\mathbb{P}\left(B^c\right) \tag{2.11}$$

where $B := \{\Lambda_3 > \min(\Lambda_1, \Lambda_2)\}$ and $B^c = \{\Lambda_3 \leq \min(\Lambda_1, \Lambda_2)\}$

Now, using the density given in (2.10):

$$\begin{aligned}
\mathbb{P}\left(B^c\right) =& \mathbb{P}\left(\Lambda_3 \leq \Lambda_1,\ \Lambda_3 \leq \Lambda_2\right) \\
=& \mathbb{P}\left(\Lambda_3 \leq \Lambda_1 \leq \Lambda_2\right) + \mathbb{P}\left(\Lambda_3 \leq \Lambda_2 \leq \Lambda_1\right) \\
=& \int_0^\infty \int_z^\infty \int_x^\infty \alpha_1(x)\alpha_2(y)\alpha_3(z) \exp\left[-A_1(x) - A_2(y) - A_3(z)\right] dydxdz \\
&+ \int_0^\infty \int_z^\infty \int_y^\infty \alpha_1(x)\alpha_2(y)\alpha_3(z) \exp\left[-A_1(x) - A_2(y) - A_3(z)\right] dxdydz \\
=& \int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds
\end{aligned} \tag{2.12}$$

From this, $\mathbb{P}\left(B\right) = \int_0^\infty \left(\alpha_1(s) + \alpha_2(s)\right) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds$

Since $\mathbb{P}\left(\min(\Lambda_1, \Lambda_2, \Lambda_3) > t\right) = \prod_{i=1}^3 \mathbb{P}\left(\Lambda_i > t\right) = \exp\left[-\sum_{i=1}^3 A_i(t)\right]$, we get

$$\begin{aligned}
\mathbb{P}\left(\tau_1 > s, \tau_2 > t|B^c\right) &= \mathbb{P}\left[\Lambda_1 > s, \Lambda_2 > t, \Lambda_3 > (s \vee t)|\Lambda_3 \leq \min(\Lambda_1, \Lambda_2)\right] \\
&= \mathbb{P}\left[\Lambda_3 > (s \vee t)|\Lambda_3 \leq \min(\Lambda_1, \Lambda_2)\right] \\
&= \frac{\mathbb{P}\left[(s \vee t) < \Lambda_3 \leq \min(\Lambda_1, \Lambda_2)\right]}{\mathbb{P}\left[\Lambda_3 \leq \min(\Lambda_1, \Lambda_2)\right]} \\
&= \frac{\int_{s\vee t}^\infty \alpha_3(x) \exp\left[-A_1(x) - A_2(x) - A_3(x)\right] dx}{\int_0^\infty \alpha_3(x) \exp\left[-A_1(x) - A_2(x) - A_3(x)\right] dx}
\end{aligned} \tag{2.13}$$

With all these elements, $\mathbb{P}(\tau_1 > s, \tau_2 > t|B)$ is obtained by subtraction.

We can verify that $\mathbb{P}(\tau_1 > s, \tau_2 > t|B^c)$ is a singular distribution since its mixed second partial derivative is zero when $s \neq t$. Conversely, $\mathbb{P}(\tau_1 > s, \tau_2 > t|B)$ is absolutely continuous since its mixed second partial derivative is a density.

□

Remark. The value of $\mathbb{P}(B^c)$ corresponds to $\mathbb{P}(\tau_1 = \tau_2)$. This is,

$$\mathbb{P}(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds. \tag{2.14}$$

Corollary 2.1 *Let $\alpha_i(u) := \alpha_i(X_u)$, where $\alpha_i(\cdot)$ is a positive continuous function and X is an $\mathbb{R}^d$-valued stochastic process adapted to the filtration $\mathbb{F}$ and independent of Z_1, Z_2 and Z_3. Then,*

$$\mathbb{P}(\tau_1 = \tau_2) = \mathbb{E}\left(\int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds\right).$$

Proof. By a similar calculation to get $\mathbb{P}(B^c)$ in the proof of Theorem 3.1, we get

$$\mathbb{P}(\tau_1 = \tau_2 | (X_u)_{u \geq 0}) = \int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds.$$

The result follows by taking the expectation.

□

2.1.3 Estimating the Probability of Equality in Two Stopping Times

Now, we are interested in finding estimates for $\mathbb{P}(\tau_1 = \tau_2)$ (see equation (2.14) in section 2.1.2) under different assumptions for $\alpha_i(\cdot)$ (the intensity of η_i) or $A_i(s)$ (the compensator of η_i). As in Theorem 2.1, we assume $\lim_{s\to\infty} A_i(s) = \infty$ a.a. for every $i = 1, 2, 3$.

Example 1. (*Constant intensity*)

If $\alpha_i(X_s) = \alpha_i \in \mathbb{R}^+$ for all $s \geq 0$, it follows that:

$$\begin{aligned}\mathbb{P}(\tau_1 = \tau_2) &= \int_0^\infty \alpha_3 \exp\left[-(\alpha_1 + \alpha_2 + \alpha_3)s\right] ds \\ &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\end{aligned}$$

Example 2. (*Same intensity*)

If $\alpha_1(X_s) = \alpha_2(X_s) = \alpha_3(X_s) =: \alpha(X_s)$ for all $s \geq 0$, we have that $A_1(s) = A_2(s) = A_3(s) =: A(s)$ for all $s \geq 0$. Moreover we get,

$$\mathbb{P}(\tau_1 = \tau_2) = \frac{1}{3}$$

Proof.

$$\mathbb{P}(\tau_1 = \tau_2) = \mathbb{E}\left[\int_0^\infty \alpha(s) \exp\left[-3A(s)\right] ds\right]$$

Setting $u = A(s)$, then $du = \alpha(s)ds$

$$= \mathbb{E}\left[\int_0^\infty e^{-3u} du\right] = \mathbb{E}\left[\frac{1}{3}\right] = \frac{1}{3}$$

□

Example 3. (*Proportional intensity*)

If $\alpha_i(X_s) = a_i\alpha_3(X_s)$ for $i = 1, 2$, for all $s > 0$, and a_i positive constants, then:

$$\mathbb{P}(\tau_1 = \tau_2) = \frac{1}{a_1 + a_2 + 1}$$

Proof. It follows by a similar proof as Example 2

□

Example 4. (*Bounded intensity*)

Assume $\ell_i\beta(X_s) \leq \alpha_i(X_s) \leq u_i\beta(X_s)$ a.s. for all $s \geq 0$, for $\beta(\cdot)$ positive and integrable with $\int_0^\infty \beta(X_s)ds = \infty$ a.s., where ℓ_i, u_i are positive real random variables such that $0 < \ell_i < u_i < \infty$.

Then, we get

$$\mathbb{E}\left[\frac{\ell_3}{u_1+u_2+u_3}\right] \le \mathbb{P}(\tau_1=\tau_2) \le \mathbb{E}\left[\frac{u_3}{\ell_1+\ell_2+\ell_3}\right]$$

Proof. This follows by a conditioning argument:

$$\begin{aligned}
\mathbb{P}\left(\tau_1=\tau_2 | \left(X_u\right)_{u\ge 0}\right) &= \int_0^\infty \alpha_3(X_s)\exp\left[-A_1(s)-A_2(s)-A_3(s)\right]ds \\
&\le \int_0^\infty u_3\beta(X_s)\exp\left[-\left(\ell_1+\ell_2+\ell_3\right)\int_0^s \beta(X_u)du\right]ds \\
&= \frac{u_3}{\ell_1+\ell_2+\ell_3}
\end{aligned}$$

And then, take the expectation. We obtain the other bound in a similar way.

□

We could analyze different scenarios for the previous two propositions. For example, suppose ℓ_3 is close to u_3 and u_1+u_2 is close to 0 (i.e., $u_1 \approx 0$ and $u_2 \approx 0$). These conditions imply that $\alpha_3(X_s)$ is almost constant and that $\alpha_1(X_s)$ and $\alpha_2(X_s)$ are close to 0. Then $\frac{\ell_3}{u_1+u_2+u_3}$ and $\frac{u_3}{\ell_1+\ell_2+\ell_3}$ are approximately 1. Hence,

$$\mathbb{P}(\tau_1=\tau_2)\approx 1$$

Another way to ensure the previous approximation is by letting ℓ_3 be close to u_3 and $\ell_3 \gg u_1+u_2 > \ell_1+\ell_2$ (where $\gg$ stands for "much greater than").

Example 5. (*Bounded sum compensators*)

If $\ell \le A_1(s)+A_2(s) < u$ for a.s. all $s\ge 0$ where ℓ, u are positive real random variables such that $0<\ell<u<\infty$. Then:

$$\mathbb{E}\left[e^{-u}\right] \le \mathbb{P}(\tau_1=\tau_2) \le \mathbb{E}\left[e^{-\ell}\right]$$

Proof. $\ell \le A_1(s)+A_2(s) < u$ a.s. implies

$$e^{-u} \le \exp\left[-A_1(s)-A_2(s)\right] \le e^{-\ell} \text{ for a.s. all } s\ge 0$$

Then,

$$\mathbb{P}(\tau_1 = \tau_2) = \mathbb{E}\left[\int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds\right]$$
$$\leq \mathbb{E}\left[\int_0^\infty \alpha_3(s) \exp\left[-\ell - A_3(s)\right]\right] ds = \mathbb{E}\left(e^{-\ell}\right)$$

Similarly for the lower bound $\mathbb{E}\left(e^{-u}\right)$

□

Example 6. (*Bounded sum of compensators by another compensator*)

If $\ell A_3(s) \leq A_1(s) + A_2(s) \leq u A_3(s)$ for a.s. all $s \geq 0$ where ℓ, u are positive real random variables such that $0 < \ell < u < \infty$. Then:

$$\mathbb{E}\left[\frac{1}{u+1}\right] \leq \mathbb{P}(\tau_1 = \tau_2) \leq \mathbb{E}\left[\frac{1}{\ell+1}\right]$$

Proof.

$$\mathbb{P}(\tau_1 = \tau_2) = \mathbb{E}\left[\int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds\right]$$
$$\leq \mathbb{E}\left[\int_0^\infty \alpha_3(s) \exp\left[-\left(\ell+1\right) A_3(s)\right] ds\right] = \mathbb{E}\left(\frac{1}{\ell+1}\right)$$

Similarly for the lower bound $\mathbb{E}\left(\frac{1}{u+1}\right)$

□

Example 7. (*Intensity bounded by sum of intensities*)

If $\ell\left[\alpha_1(X_s) + \alpha_2(X_s)\right] \leq \alpha_3(X_s) \leq u\left[\alpha_1(X_s) + \alpha_2(X_s)\right]$ for a.s. all $s \geq 0$ where ℓ, u are positive real random variables such that $0 < \ell < u < \ell + 1 < \infty$. Then:

$$\mathbb{E}\left[\frac{\ell}{u+1}\right] \leq \mathbb{P}(\tau_1 = \tau_2) \leq \mathbb{E}\left[\frac{u}{\ell+1}\right]$$

Proof.

$$\mathbb{P}(\tau_1 = \tau_2) = \mathbb{E}\left[\int_0^\infty \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds\right]$$

$$\leq \mathbb{E}\left[\int_0^\infty u\left[\alpha_1(s)+\alpha_2(s)\right]\exp\left[-\left(\ell+1\right)\left(A_1(s)+A_2(s)\right)\right]ds\right] = \mathbb{E}\left(\frac{u}{\ell+1}\right)$$

Similarly for the lower bound $\mathbb{E}\left(\frac{\ell}{u+1}\right)$ □

2.1.4 Conditional Probabilities

Through this section, let $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot|\,(X_u)_{u\geq 0}\right)$.

Proposition 1.

$$\mathbb{P}\left(\tau_1 = \tau_2, \tau_1 \leq t\right) = \mathbb{E}\left[\int_0^t \alpha_3(s)\exp\left[-A_1(s)-A_2(s)-A_3(s)\right]ds\right]$$

Proof. By the definition of τ_i, we have

$$\begin{aligned}
\tilde{\mathbb{P}}(\tau_1 = \tau_2, \tau_1 > t) =& \tilde{\mathbb{P}}\left(\min(\eta_1,\eta_3) = \min(\eta_2,\eta_3), \min(\eta_1,\eta_3) > t\right)\\
=& \tilde{\mathbb{P}}\left(\min(\eta_1,\eta_3) = \eta_3 = \min(\eta_2,\eta_3), \min(\eta_1,\eta_3) > t\right)\\
=& \tilde{\mathbb{P}}\left(\eta_3 < \eta_1, \eta_3 < \eta_2, \eta_1 > t, \eta_3 > t\right)\\
=& \tilde{\mathbb{P}}\left(t < \eta_3 < \eta_1 < \eta_2\right) + \tilde{\mathbb{P}}\left(t < \eta_3 < \eta_2 < \eta_1\right)
\end{aligned}$$

Recall that, by definition, η_i are independent with density under $\tilde{\mathbb{P}}$ equal to $\alpha_i(s)\exp\left(-A_i(s)\right)$

$$\begin{aligned}
\tilde{\mathbb{P}}(\tau_1 = \tau_2, \tau_1 > t) =& \int_t^\infty\int_z^\infty\int_x^\infty \alpha_1(x)\alpha_2(y)\alpha_3(z)\exp\left[-A_1(x)-A_2(y)-A_3(z)\right]dydxdz\\
&+\int_t^\infty\int_z^\infty\int_y^\infty \alpha_1(x)\alpha_2(y)\alpha_3(z)\exp\left[-A_1(x)-A_2(y)-A_3(z)\right]dxdydz\\
=& \int_t^\infty \alpha_3(s)\exp\left[-A_1(s)-A_2(s)-A_3(s)\right]ds
\end{aligned}$$

This implies,

$$\tilde{\mathbb{P}}(\tau_1 = \tau_2, \tau_1 \leq t) = \int_0^t \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds$$

Take the expectation to get the result. □

Remark. When taking the limit $t \to \infty$ in the previous proposition, we recover, as expected, the result of Corollary 2.1.

Proposition 2.

$$\mathbb{P}\left(\tau_1 = \tau_2 | \tau_1 \leq t\right) = \mathbb{E}\left[\frac{\int_0^t \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds}{1 - \exp[-A_1(t) - A_3(t)]}\right]$$

Proof. Recall that, as established in equation (2.6),

$$\tilde{\mathbb{P}}(\tau_1 \leq t) = 1 - \exp[-A_1(t) - A_3(t)]$$

Using Proposition 1, we get that

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_1 = \tau_2 | \tau_1 \leq t\right) &= \frac{\tilde{\mathbb{P}}\left(\tau_1 = \tau_2, \tau_1 \leq t\right)}{\tilde{\mathbb{P}}\left(\tau_1 \leq t\right)} \\ &= \frac{\int_0^t \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds}{1 - \exp[-A_1(t) - A_3(t)]}\end{aligned}$$

The result follows by taking the expectation. □

Proposition 3.

$$\mathbb{P}\left(\tau_1 = \tau_2 | \tau_1 \leq t, \tau_2 \leq t\right) = \mathbb{E}\left[\frac{\int_0^t \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds}{1 - e^{-A_3(t)}\left(e^{-A_2(t)} + e^{-A_1(t)} - e^{-A_1(t) - A_2(t)}\right)}\right]$$

Proof. Note that:

1. The events $\{\tau_1 = \tau_2, \tau_1 \leq t, \tau_2 \leq t\}$ and $\{\tau_1 = \tau_2, \tau_1 \leq t\}$ are equal

2. $\{\tau_1 \leq t, \tau_2 \leq t\}^C = \{\tau_1 > 0, \tau_2 > t\} \cup \{\tau_1 > t, \tau_2 > 0\} \cup \{\tau_1 > t, \tau_2 > t\}$

Hence we have:

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_1 = \tau_2 | \tau_1 \leq t, \tau_2 \leq t\right) &= \frac{\tilde{\mathbb{P}}\left(\tau_1 = \tau_2, \tau_1 \leq t\right)}{\tilde{\mathbb{P}}\left(\tau_1 \leq t, \tau_2 \leq t\right)} \\ &= \frac{\int_0^t \alpha_3(s) \exp\left[-A_1(s) - A_2(s) - A_3(s)\right] ds}{1 - e^{-A_3(t)}\left(e^{-A_2(t)} + e^{-A_1(t)} - e^{-A_1(t) - A_2(t)}\right)}\end{aligned}$$

Take the expectation to get the result

□

2.1.5 Distance Between Stopping Times

Proposition 4.

For $s < t$,

$$\begin{aligned}\mathbb{P}\left(s < \tau_1 \leq t, s \leq \tau_2 \leq t\right) = \mathbb{E}\Big[&e^{-(A_1+A_2+A_3)(s)} + e^{-(A_1+A_2+A_3)(t)} \\ &-e^{-A_1(s)-A_2(t)-A_3(t)} - e^{-A_1(t)-A_2(s)-A_3(t)}\Big]\end{aligned}$$

In particular, if $s = r - \varepsilon$ and $t = r + \varepsilon$, we get

$$\begin{aligned}\mathbb{P}\left(r - \varepsilon < \tau_1 \leq r + \varepsilon, r - \varepsilon \leq \tau_2 \leq r + \varepsilon\right) = \mathbb{E}\Big[&e^{-(A_1+A_2+A_3)(r-\varepsilon)} \\ &+e^{-(A_1+A_2+A_3)(r+\varepsilon)} - e^{-A_1(r-\varepsilon)-A_2(r+\varepsilon)-A_3(r+\varepsilon)} - e^{-A_1(r+\varepsilon)-A_2(r-\varepsilon)-A_3(r+\varepsilon)}\Big]\end{aligned}$$

Proof. It follows by noticing that:

$$\begin{aligned}\mathbb{P}\left(s < \tau_1 \leq t, s \leq \tau_2 \leq t\right) = \mathbb{P}\left(\tau_1 > s, \tau_2 > s\right) - \mathbb{P}\left(\tau_1 > s, \tau_2 > t\right) \\ - \mathbb{P}\left(\tau_1 > t, \tau_2 > s\right) + \mathbb{P}\left(\tau_1 > t, \tau_2 > t\right)\end{aligned}$$

Then, conditioning on $(X_t)_{t\geq 0}$, use Theorem 2.1, and take the expectation.

□

The next proposition shows that the probability of the two stopping time happening over the next time interval $(t, t+\varepsilon)$ with $\varepsilon \approx 0$ is equal to the expectation of $(\alpha_3\,(X_t))$, which is the common part of the intensities of τ_1 and τ_2 (recall Remark 2.1)

An earlier reader has pointed out a relation of the following proposition to Aven's Lemma. This is, of course, correct, see Aven [8], Ethier and Kurtz [33], and Zeng [67].

Proposition 5.

$$\lim_{\varepsilon\to 0} \frac{\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\tau_1 > t, \tau_2 > t\right)}{\varepsilon} = \mathbb{E}\left(\alpha_3(X_t)\right)$$

Proof. Let $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot|\,(X_u)_{u\geq 0}\right)$ and note that:

$$\tilde{\mathbb{P}}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\tau_1 > t, \tau_2 > t\right) = \frac{\tilde{\mathbb{P}}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon]\right)}{\tilde{\mathbb{P}}\left(\tau_1 > t, \tau_2 > t\right)}$$

For the numerator, using Proposition 4 with t and $t+\varepsilon$,

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon]\right) &= \exp\left[-\left(A_1 + A_2 + A_3\right)(t)\right] \\ &- \exp\left[-A_1(t) - \left(A_2 + A_3\right)(t+\varepsilon)\right] - \exp\left[-\left(A_1 + A_3\right)(t+\varepsilon) - A_2(t)\right] \\ &+ \exp\left[-\left(A_1 + A_2 + A_3\right)(t+\varepsilon)\right]\end{aligned}$$

Dividing by $\tilde{\mathbb{P}}\left(\tau_1 > t, \tau_2 > t\right) = \exp\left[-A_1(t) - A_2(t) - A_3(t)\right]$, we get:

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\tau_1 > t, \tau_2 > t\right) &= 1 - \exp\left[-\int_t^{t+\varepsilon} \left(\alpha_2 + \alpha_3\right)(X_u)du\right] \\ &- \exp\left[-\int_t^{t+\varepsilon} \left(\alpha_1 + \alpha_3\right)(X_u)du\right] + \exp\left[-\int_t^{t+\varepsilon} \left(\alpha_1 + \alpha_2 + \alpha_3\right)(X_u)du\right]\end{aligned}$$

We divide by ε and then use L'Hôpital's rule to get the limit:

$$\lim_{\varepsilon\to 0} \frac{\tilde{\mathbb{P}}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\tau_1 > t, \tau_2 > t\right)}{\varepsilon} = \left(\alpha_2 + \alpha_3\right)(X_t)$$

$$+\left(\alpha_1+\alpha_3\right)\left(X_t\right)-\left(\alpha_1+\alpha_2+\alpha_3\right)\left(X_t\right)=\alpha_3(X_t).$$

Moreover, given $\tilde{\mathbb{P}}\left(\tau_1\in(t,t+\varepsilon],\tau_2\in(t,t+\varepsilon]\,|\tau_1>t,\tau_2>t\right)$ is bounded by 1 and a conditioning argument, we can conclude that:

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\tau_1\in(t,t+\varepsilon],\tau_2\in(t,t+\varepsilon]\,|\tau_1>t,\tau_2>t\right)}{\varepsilon}=\mathbb{E}\left(\alpha_3(X_t)\right).$$

□

Proposition 5 motivates the following one:

Proposition 6.

If $s<t$

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\tau_1\le s+\varepsilon,\tau_2\le t+\varepsilon|\tau_1>s,\tau_2>t\right)}{\varepsilon^2}=\mathbb{E}\left[\alpha_1(X_s)\left(\alpha_2(X_t)+\alpha_3(X_t)\right)\right]$$

Proof. As $s<t$, we can always find a sufficiently small ε such that $s+\varepsilon<t$. Hence, without loss of generality, we assume $s<s+\varepsilon<t<t+\varepsilon$ and we proceed as in the proof of Proposition 5, but differently from there, we use L'Hôpital's rule twice to get the desired limit. □

The next couple of propositions provide a measure, in two different metrics, of how close the two stopping are from each other.

Proposition 7. (Distance in probability)

$$\mathbb{P}(|\tau_1-\tau_2|\le\varepsilon)=1-\mathbb{E}\left[\int_0^\infty\alpha_1(x)e^{-A_1(x)-(A_2+A_3)(x+\varepsilon)}\,dx+\int_0^\infty\alpha_2(x)e^{-A_2(x)-(A_1+A_3)(x+\varepsilon)}\,dx\right]$$

Proof. Let $\tau_{(1)}:=\min\left(\tau_1,\tau_2\right)$ and $\tau_{(2)}:=\max\left(\tau_1,\tau_2\right)$. Similarly, $\eta_{(1)}:=\min\left(\eta_1,\eta_2,\eta_3\right)$, $\eta_{(3)}:=\max\left(\eta_1,\eta_2,\eta_3\right)$, and $\eta_{(2)}$ be the second largest from $\left(\eta_1,\eta_2,\eta_3\right)$. Also, let $\tilde{\mathbb{P}}(\cdot):=\mathbb{P}\left(\cdot|\left(X_u\right)_{u\ge 0}\right)$. Then,

$$\tilde{\mathbb{P}}\left(\tau_{(2)}>\tau_{(1)}+\varepsilon\right)=\tilde{\mathbb{P}}\left(\eta_{(3)}\ge\eta_{(2)}>\eta_{(1)}+\varepsilon,\eta_3\ge\eta_{(2)}\right)$$

$$= \int_0^\infty f_1(x_1) \int_{x_1+\varepsilon}^\infty f_2(x_2) \int_{x_2}^\infty f_3(x_3)\, dx_3\, dx_2\, dx_1$$
$$+ \int_0^\infty f_2(x_1) \int_{x_1+\varepsilon}^\infty f_1(x_2) \int_{x_2}^\infty f_3(x_3)\, dx_3\, dx_2\, dx_1$$
$$+ \int_0^\infty f_1(x_1) \int_{x_1+\varepsilon}^\infty f_3(x_2) \int_{x_2}^\infty f_2(x_3)\, dx_3\, dx_2\, dx_1$$
$$+ \int_0^\infty f_2(x_1) \int_{x_1+\varepsilon}^\infty f_3(x_2) \int_{x_2}^\infty f_1(x_3)\, dx_3\, dx_2\, dx_1$$

where f_j stands for the density of η_j under $\tilde{\mathbb{P}}$, i.e., $f_j(y) = \alpha_j(y)e^{-A_j(y)} := \alpha_j(X_y)e^{-A_j(y)}$

$$= \int_0^\infty f_1(x_1) \int_{x_1+\varepsilon}^\infty f_2(x_2)e^{-A_3(x_2)}\, dx_2\, dx_1 + \int_0^\infty f_2(x_1) \int_{x_1+\varepsilon}^\infty f_1(x_2)e^{-A_3(x_2)}\, dx_2\, dx_1$$
$$+ \int_0^\infty f_1(x_1) \int_{x_1+\varepsilon}^\infty f_3(x_2)e^{-A_2(x_2)}\, dx_2\, dx_1 + \int_0^\infty f_2(x_1) \int_{x_1+\varepsilon}^\infty f_3(x_2)e^{-A_1(x_2)}\, dx_2\, dx_1$$
$$= \int_0^\infty f_1(x_1) \int_{x_1+\varepsilon}^\infty \left[\alpha_2(x_2) + \alpha_3(x_2)\right] e^{-A_2(x_2)-A_3(x_2)}\, dx_2\, dx_1$$
$$+ \int_0^\infty f_2(x_1) \int_{x_1+\varepsilon}^\infty \left[\alpha_1(x_2) + \alpha_3(x_2)\right] e^{-A_1(x_2)-A_3(x_2)}\, dx_2\, dx_1$$
$$= \int_0^\infty \alpha_1(x_1)e^{-A_1(x_1)-(A_2+A_3)(x_1+\varepsilon)}\, dx_1 + \int_0^\infty \alpha_2(x_1)e^{-A_2(x_1)-(A_1+A_3)(x_1+\varepsilon)}\, dx_1$$

Use this expression, along with the following equality, and then take the expectation to get the desired result

$$\tilde{\mathbb{P}}\left(|\tau_1 - \tau_2| \leq \varepsilon\right) = 1 - \tilde{\mathbb{P}}\left(\tau_{(2)} - \tau_{(1)} > \varepsilon\right)$$

□

Proposition 8.(L^2 distance)

If $\mathbb{E}\left(\tau_i^2\right) < \infty$ and $\lim_{x\to\infty} x^2 e^{-A_i(x)} = 0$ a.s. for $i = 1, 2$ (in other words, $e^{-A_i(x)}$ goes faster to 0 than x^2 goes to infinity when $x \to \infty$), we have

$$\mathbb{E}\left[(\tau_1 - \tau_2)^2\right] = 2\left[\int_0^\infty x e^{-A_1(x)-A_3(x)} dx + \int_0^\infty x e^{-A_2(x)-A_3(x)} dx \right. \\ \left. - \int_0^\infty \int_0^y e^{-A_1(x)-A_2(y)-A_3(y)} dx dy - \int_0^\infty \int_y^\infty e^{-A_1(x)-A_2(y)-A_3(x)} dx dy \right]$$

Proof. Let $\tilde{\mathbb{E}}(\cdot) := \mathbb{E}(\cdot|(X_u)_{u\geq 0})$. F_{τ_i} and $\overline{F}_{\tau_i}$ stand for the cumulative and the survival distribution functions of τ_i given $(X_u)_{u\geq 0}$. Similarly, $F(x, y)$ and $\overline{F}(x, y)$ stand for the cumulative and the survival joint distribution functions of τ_1, τ_2 given $(X_u)_{u\geq 0}$.

We expand the square and handle each term separately. For $\tilde{\mathbb{E}}\left(\tau_i^2\right) < \infty$, we use integration by parts in the following way:

$$\begin{aligned} \tilde{\mathbb{E}}\left(\tau_i^2\right) &= \int_0^\infty x^2 dF_{\tau_i}(x) \\ &= 2\int_0^\infty x\overline{F}_{\tau_i}(x) dx \\ &= 2\int_0^\infty x e^{-A_i(x)-A_3(x)} dx \end{aligned}$$

To find $\tilde{\mathbb{E}}(\tau_1\tau_2)$, we exploit the result of Young [65] on integration by parts in two or more dimensions. If $G(0, y) \equiv 0 \equiv G(y, 0)$ and G is of bounded variation on finite intervals, then:

$$\int_0^\infty \int_0^\infty G(x, y) dF(x, y) = \int_0^\infty \int_0^\infty \overline{F}(x, y) dG(x, y) \tag{2.15}$$

This equality implies that, for $i, j > 0$:

$$\int_0^\infty \int_0^\infty x^i y^j dF(x, y) = \int_0^\infty \int_0^\infty ij x^{i-1} y^{j-1} \overline{F}(x, y) dx dy \tag{2.16}$$

Hence,

$$\begin{aligned}
\tilde{\mathbb{E}}(\tau_1\tau_2) &= \int_0^\infty \int_0^\infty xy dF_{(\tau_1,\tau_2)}(x,y) \\
&= \int_0^\infty \int_0^\infty e^{-A_1(x)-A_2(y)-A_3(x\vee y)} dxdy \\
&= \int_0^\infty \int_0^y e^{-A_1(x)-A_2(y)-A_3(y)} dxdy + \int_0^\infty \int_y^\infty e^{-A_1(x)-A_2(y)-A_3(x)} dxdy
\end{aligned} \tag{2.17}$$

The result follows by taking the expectation.

□

Remark. In the case of independence of τ_1 and τ_2, as $A_3(s) = 0$ for all $s \geq 0$, we get that:

$$\mathbb{E}\left[(\tau_1 - \tau_2)^2\right] = 2\left[\int_0^\infty xe^{-A_1(x)}dx + \int_0^\infty xe^{-A_2(x)}dx - \int_0^\infty e^{-A_1(x)}dx \int_0^\infty e^{-A_2(x)}dx\right] \tag{2.18}$$

2.2 Generalizations

2.2.1 Generalization to K Stopping Times.

Given the interpretation introduced in Section (2.1.1), there is a natural way to extend our model to more than two stopping times. We explicitly motivate and present the case of 3 stopping times. Suppose we have a three-component system where each component's life is represented by τ_1, τ_2, and τ_3 respectively. Any component dies after receiving a shock, which are governed by 6 independent Cox processes:

$$\Lambda_1(t, \alpha_1(X_t)), \Lambda_2(t, \alpha_2(X_t)), \Lambda_3(t, \alpha_3(X_t)), \Lambda_{(1,2)}\left(t, \alpha_{(1,2)}(X_t)\right), \Lambda_{(2,3)}\left(t, \alpha_{(2,3)}(X_t)\right),$$
$$\Lambda_{(1,3)}\left(t, \alpha_{(1,3)}(X_t)\right) \text{ and } \Lambda_{(1,2,3)}\left(t, \alpha_{(1,2,3)}(X_t)\right)$$

Events in the process $\Lambda_i(t, \alpha_i(X_t))$ are shocks to only component i (for $i = 1, 2, 3$), events in the process $\Lambda_{(i,j)}\left(t, \alpha_{(i,j)}(X_t)\right)$ are shocks to component i and j (for $i, j = 1, 2, 3$ and $i \neq j$), and events in the process $\Lambda_{(1,2,3)}\left(t, \alpha_{(1,2,3)}(X_t)\right)$ are shocks to the 3 components.

In this way, considering $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot|(X_u)_{u\geq 0}\right)$ and $A_j(s) = \int_0^s \alpha_j(X_u)du$ we have:

$$\begin{aligned}
\mathbb{P}(\tau_1 > s_1, \tau_2 > s_2, \tau_3 > s_3) &= \mathbb{E}\Big(\tilde{\mathbb{P}}\left[\Lambda_1\left(s_1, \alpha_1(X)\right) = 0\right] \tilde{\mathbb{P}}\left[\Lambda_2\left(s_2, \alpha_2(X)\right) = 0\right] \\
&\tilde{\mathbb{P}}\left[\Lambda_3\left(s_3, \alpha_3(X)\right) = 0\right] \tilde{\mathbb{P}}\left[\Lambda_{(1,2)}\left(s_1 \vee s_2, \alpha_{(1,2)}(X)\right) = 0\right] \tilde{\mathbb{P}}\left[\Lambda_{(1,3)}\left(s_1 \vee s_3, \alpha_{(1,3)}(X)\right) = 0\right] \\
&\tilde{\mathbb{P}}\left[\Lambda_{(2,3)}\left(s_2 \vee s_3, \alpha_{(2,3)}(X)\right) = 0\right] \tilde{\mathbb{P}}\left[\Lambda_{(1,2,3)}\left(s_1 \vee s_2 \vee s_3, \alpha_{(1,2,3)}(X)\right) = 0\right]\Big) \\
&= \mathbb{E}\left(\exp\left[-A_1(s_1) - A_2(s_2) - A_3(s_3) - A_{(1,2)}(s_1 \vee s_2)\right.\right. \\
&\left.\left. -A_{(1,3)}(s_1 \vee s_3) - A_{(2,3)}(s_2 \vee s_3) - A_{(1,2,3)}(s_1 \vee s_2 \vee s_3)\right]\right) \quad (2.19)
\end{aligned}$$

By using a similar technique we can generalize to any number K of stopping times and in section 2.3.1, we will present an application of this natural extension of our model. However, as the number of stopping times K increases, handling the expression presented in equation (2.19) becomes cumbersome. Hence, in Chapter 3, we propose and study in detail a slightly different approach that makes easier to generalize to more than two stopping times.

2.2.2 A More General Distribution

In this section, we generalize the result from Section 2.1 to get a distribution that can allow for τ_1 and τ_2 to have a negative covariance. Recalling equation (2.17) and that $\mathbb{E}(\tau_i) = \int_0^\infty e^{-A_i(s)-A_3(s)}ds$, one can see that the covariance of τ_1 and τ_2 in the previous model is:

$$\begin{aligned}
\text{Cov}\left(\tau_1, \tau_2\right) &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \left(e^{-A_1(x)-A_2(y)-A_3(x\vee y)} - e^{-A_1(x)-A_2(y)-A_3(x)-A_3(y)}\right) dxdy\right] \\
&= \mathbb{E}\left[\int_0^\infty \int_0^\infty e^{-A_1(x)-A_2(y)}\left(e^{-A_3(x\vee y)} - e^{-A_3(x)-A_3(y)}\right) dxdy\right] \quad (2.20)
\end{aligned}$$

As $A_3(x \vee y) < A_3(x) + A_3(y)$ for all $x, y \geq 0$, it is clear that the covariance is always nonnegative.

To get the property of a negative covariance, we stop assuming that the underlying exponential random variables (i.e., Z_1, Z_2, Z_3) are independent (recall equation (2.4) where we use these random variables to define η_i and $A_i(s)$). Now, we assume that (Z_1, Z_2) follow the Gumbel Bivariate

Distribution (see Gumbel [38]). This is, the joint survival function of (Z_1, Z_2) is:

$$\mathbb{P}(Z_1 > s, Z_2 > t) = e^{-s-t-\delta st} \text{ for } 0 \leq \delta \leq 1 \tag{2.21}$$

While $Z_3 \sim \text{Exp}(1)$ and it is independent of (Z_1, Z_2). The rest of the definitions given in (2.4) remain the same. Then, it is easy to check that:

$$\mathbb{P}\left(\eta_1 > s, \eta_2 > t\right) = \mathbb{E}\left[\exp\left(-A_1(s) - A_2(t) - \delta A_1(s)A_2(t)\right)\right] \tag{2.22}$$

$$\mathbb{P}\left(\eta_3 > s\right) = \mathbb{E}\left[\exp\left(-A_3(s)\right)\right] \tag{2.23}$$

Define τ_1 and τ_2 as in equation (2.3). Then, we can get the join distribution of (τ_1, τ_2):

Theorem 2.3 (Survival function) *Suppose $\alpha_i(\cdot) : \mathbb{R}^d \mapsto [0, \infty)$ for $i = 1, 2, 3$ are non-random positive continuous functions, which implies that $A_i(s)$ are continuous and strictly increasing. Assume further that $\lim_{s\to\infty} A(s) = \infty$ a.s. Then,*

$$\overline{F}_{(\tau_1,\tau_2)}(s,t) := \mathbb{P}\left(\tau_1 > s, \tau_2 > t\right) = \mathbb{E}\left[\exp\left[-A_1(s) - A_2(t) - \delta A_1(s)A_2(t) - A_3\left(s \vee t\right)\right]\right].$$

Proof. Let $\tilde{\mathbb{P}}_{(s,t)}(\cdot) := \mathbb{P}\left(\cdot|(X_u)_{0\leq u\leq(s\vee t)}\right)$. By definition of τ_1 and τ_2, we have,

$$\begin{aligned}
\tilde{\mathbb{P}}_{(s,t)}\left(\tau_1 > s, \tau_2 > t\right) &= \tilde{\mathbb{P}}_{(s,t)}\left(\min(\eta_1, \eta_3) > s, \min(\eta_2, \eta_3) > t\right) \\
&= \tilde{\mathbb{P}}_{(s,t)}\left(\eta_1 > s, \eta_2 > t\right)\tilde{\mathbb{P}}_{(s,t)}\left(\eta_3 > s, \eta_3 > t\right) \\
&= \tilde{\mathbb{P}}_{(s,t)}\left(\eta_1 > s, \eta_2 > t\right)\tilde{\mathbb{P}}_{(s,t)}\left(\eta_3 > \max(s,t)\right) \\
&= \exp\left[-A_1(s) - A_2(t) - \delta A_1(s)A_2(t) - A_3(s \vee t)\right] \\
&= \begin{cases} \exp\left[-A_1(s) - A_2(t) - \delta A_1(s)A_2(t) - A_3(s)\right] & s > t \\ \exp\left[-A_1(s) - A_2(t) - \delta A_1(s)A_2(t) - A_3(t)\right] & s < t. \end{cases}
\end{aligned}$$

The result follows by taking the expectation.

□

Remark. From Theorem 2.3, it is clear that τ_1 is not independent of τ_2

Remark. If $\delta = 0$, we have that (τ_1, τ_2) follow the distribution from Theorem 2.1. So this is a generalization of the model presented in Section 2.1.

Remark. The marginal distribution of τ_i is given by:

$$\mathbb{P}(\tau_i > s) = \mathbb{P}(\min(\eta_i, \eta_3) > s) = \mathbb{P}(\eta_i > s)\mathbb{P}(\eta_3 > s) = \mathbb{E}\left[\exp\left(-\int_0^s (\alpha_i + \alpha_3)(r)dr\right)\right] \quad (2.24)$$

which,as in section 2.1 coincides with the marginal distribution in the Cox construction, see Lando [47] [48].

Proposition 9.(Probability of the two stopping times being equal)

$$\mathbb{P}(\tau_1 = \tau_2) = \mathbb{E}\left[\int_0^\infty \alpha_3(t)\exp\left[-(A_1 + A_2 + A_3)(t) - \delta A_1(t)A_2(t)\right] dt\right] \quad (2.25)$$

which is greater than 0 as long as $\alpha_3(t)$ is not identically equal to 0

Proof. Let $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot| (X_u)_{u\geq 0}\right)$. By definition of τ_i, $\{\tau_1 = \tau_2\} = \{\eta_0 < \eta_1, \eta_0 < \eta_2\}$. Hence:

$$\begin{aligned}
\tilde{\mathbb{P}}(\tau_1 = \tau_2) &= \tilde{\mathbb{P}}(\eta_0 < \eta_1, \eta_0 < \eta_2) \\
&= \int_0^\infty \tilde{\mathbb{P}}(t < \eta_1, t < \eta_2 | \eta_0 \leq t)\, d\tilde{\mathbb{P}}(\eta_0 \leq t) \\
&= \int_0^\infty \tilde{\mathbb{P}}(t < \eta_1, t < \eta_2)\, d\tilde{\mathbb{P}}(\eta_0 \leq t) \\
&= \int_0^\infty \alpha_3(t)\exp\left[-(A_1 + A_2 + A_3)(t) - \delta A_1(t)A_2(t)\right] dt \quad (2.26)
\end{aligned}$$

where the third equality follows from the independence of (η_1, η_2) from η_0 and the fourth one follows from the survival function of (η_1, η_2) and the probability density function of η_0 respectively.

The result follows by taking the expectation. □

This is the general set-up. For the rest of this section, we assume that for $i = 1, 2, 3$, we have $\alpha_i(X_t) \equiv \lambda_i \in \mathbb{R}^+$ for all $t \geq 0$ to get tractable computations. Under this assumption, note the following:

1. $\mathbb{P}\left(\eta_1 > s, \eta_2 > t\right) = \exp\left[-\lambda_1 s - \lambda_2 t - \delta\lambda_1\lambda_2 st\right]$

2. $\mathbb{P}\left(\tau_1 > s, \tau_2 > t\right) = \exp\left[-\lambda_1 s - \lambda_2 t - \delta\lambda_1\lambda_2 st - \lambda_3(s \vee t)\right]$. That is (τ_1, τ_2) follow a bivariate exponential (BVE) that is a combination of the Marshall-Olkin BVE (see Marshall and Olkin [51]) and the Gumbel BVE (see [38]). Actually, if $\delta = 0$, we recover the BVE of Marshall and Olkin and if $\lambda_3 = 0$ we recover the BVE of Gumbel.

3. $\mathbb{P}(\tau_1 > t) = \exp\left(-(\lambda_1 + \lambda_3)t\right)$ and $\mathbb{P}(\tau_2 > t) = \exp\left(-(\lambda_2 + \lambda_3)t\right)$, i.e., marginally $\tau_i \sim \mathrm{Exp}(\lambda_i + \lambda_3)$.

Before showing that one can get a negative covariance between the stopping times, we show in the next proposition that, by allowing the possibility of a negative covariance, the probability of the two stopping times being equal is smaller.

Proposition 10.

Fix some values of $\lambda_1, \lambda_2, \lambda_3$. If $\delta \neq 0$, the probability of τ_1 being equal to τ_2 is smaller than the probability of τ_1 being equal to τ_2 for the case of $\delta = 0$

Proof. Let $\mathbb{P}_{\delta\neq0}(A)$ and $\mathbb{P}_{\delta=0}(A)$ stand for the probability of an event A under the law of (τ_1, τ_2) when $\delta \neq 0$ and when $\delta = 0$ respectively

Recall from Example 1 that:

$$\mathbb{P}_{\delta=0}\left(\tau_1 = \tau_2\right) = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \tag{2.27}$$

Also, setting $\alpha_i(t) = \lambda_i$ in (2.26), we get:

$$\begin{aligned}
\mathbb{P}_{\delta\neq0}\left(\tau_1 = \tau_2\right) &= \int_0^\infty \lambda_3 \exp\left[-(\lambda_1 + \lambda_2 + \lambda_3)t - \delta\lambda_1\lambda_2 t^2\right] dt \\
&= \lambda_3 \exp\left[\frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{4\delta\lambda_1\lambda_2}\right] \int_0^\infty \exp\left[-\left(\sqrt{\delta\lambda_1\lambda_2}t + \frac{\lambda_1 + \lambda_2 + \lambda_3}{2\sqrt{\delta\lambda_1\lambda_2}}\right)^2\right] dt \\
&= \frac{\lambda_3}{\sqrt{\delta\lambda_1\lambda_2}} \exp\left[\frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{4\delta\lambda_1\lambda_2}\right] \int_{\frac{\lambda_1+\lambda_2+\lambda_3}{2\sqrt{\delta\lambda_1\lambda_2}}}^\infty e^{-u^2} du
\end{aligned} \tag{2.28}$$

where the second equality follows by completing the square and it is only valid if $\delta \neq 0$. The third equality is just a change of variable.

Then, using the well-known bound $\int_x^\infty e^{-u^2} du \leq \frac{e^{-x^2}}{2x}$ for all $x > 0$, we have that:

$$\begin{aligned}
&\frac{\lambda_3}{\sqrt{\delta\lambda_1\lambda_2}} \exp\left[\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\delta\lambda_1\lambda_2}\right] \int_{\frac{\lambda_1+\lambda_2+\lambda_3}{2\sqrt{\delta\lambda_1\lambda_2}}}^\infty e^{-u^2} du \\
&\qquad \leq \frac{\lambda_3}{\sqrt{\delta\lambda_1\lambda_2}} \exp\left[\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\delta\lambda_1\lambda_2}\right] \exp\left[-\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\delta\lambda_1\lambda_2}\right] \frac{2\sqrt{\delta\lambda_1\lambda_2}}{2(\lambda_1+\lambda_2+\lambda_3)} \\
&\qquad = \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}
\end{aligned}$$

which shows, as desired:

$$\mathbb{P}_{\delta\neq 0}(\tau_1 = \tau_2) \leq \mathbb{P}_{\delta=0}(\tau_1 = \tau_2) \tag{2.29}$$

□

Proposition 11. (Negative Covariance)

Suppose $\lambda_1 = c_1\lambda_3$, $\lambda_2 = c_2\lambda_3$ and $\delta = 1$. If $5c_1c_2 \geq 4(c_1 + c_2 + 1)$, then $\mathrm{Cov}(\tau_1, \tau_2) < 0$.

Proof. Using the result of Young [65], we have that:

$$\begin{aligned}
\mathbb{E}(\tau_1\tau_2) &= \int_0^\infty \int_0^\infty \exp\left[-\lambda_1 x - \lambda_2 y - \lambda_3(x \vee y) - \lambda_1\lambda_2 xy\right] dxdy \\
&= \int_0^\infty \int_0^y \exp\left[-\lambda_1 x - \lambda_2 y - \lambda_1\lambda_2 xy - \lambda_3 y\right] dxdy \\
&\quad + \int_0^\infty \int_y^\infty \exp\left[-\lambda_1 x - \lambda_2 y - \lambda_1\lambda_2 xy - \lambda_3 x\right] dxdy \\
&= \int_0^\infty \int_x^\infty \exp\left[-\lambda_1 x - \lambda_2 y - \lambda_1\lambda_2 xy - \lambda_3 y\right] dydx \\
&\quad + \int_0^\infty \int_y^\infty \exp\left[-\lambda_1 x - \lambda_2 y - \lambda_1\lambda_2 xy - \lambda_3 x\right] dxdy \\
&= \int_0^\infty \frac{1}{\lambda_2+\lambda_3+\lambda_1\lambda_2 x} \exp\left[-(\lambda_1+\lambda_2+\lambda_3)x - \lambda_1\lambda_2 x^2\Big)\right] dx \\
&\quad + \int_0^\infty \frac{1}{\lambda_1+\lambda_3+\lambda_1\lambda_2 y} \exp\left[-(\lambda_1+\lambda_2+\lambda_3)y - \lambda_1\lambda_2 y^2\Big)\right] dy
\end{aligned} \tag{2.30}$$

We reduce the previous equation even further by completing the square, for instance, take the first summand:

$$\int_0^\infty \frac{1}{\lambda_2+\lambda_3+\lambda_1\lambda_2 x} \exp\left[-(\lambda_1+\lambda_2+\lambda_3)x - \lambda_1\lambda_2 x^2\right)\Big] dx$$
$$= \exp\left[\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\lambda_1\lambda_2}\right] \int_0^\infty \frac{1}{\lambda_2+\lambda_3+\lambda_1\lambda_2 x} \exp\left[-\left(\sqrt{\lambda_1\lambda_2}x + \frac{1}{2}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{\sqrt{\lambda_1\lambda_2}}\right)\right)^2\right] dx$$
$$= \frac{2}{\sqrt{\lambda_1\lambda_2}} \exp\left[\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\lambda_1\lambda_2}\right] \int_{\frac{1}{2}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{\sqrt{\lambda_1\lambda_2}}\right)}^{\infty} \frac{1}{2\sqrt{\lambda_1\lambda_2}u+\lambda_2+\lambda_3-\lambda_1} e^{-u^2} du \quad (2.31)$$

Similarly, the second summand in (2.30) reduces to:

$$\int_0^\infty \frac{1}{\lambda_1+\lambda_3+\lambda_1\lambda_2 y} \exp\left[-(\lambda_1+\lambda_2+\lambda_3)y - \lambda_1\lambda_2 y^2\right)\Big] dy$$
$$= \frac{2}{\sqrt{\lambda_1\lambda_2}} \exp\left[\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\lambda_1\lambda_2}\right] \int_{\frac{1}{2}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{\sqrt{\lambda_1\lambda_2}}\right)}^{\infty} \frac{1}{2\sqrt{\lambda_1\lambda_2}u+\lambda_1+\lambda_3-\lambda_2} e^{-u^2} du \quad (2.32)$$

Using the fact that $\mathbb{E}(\tau_i) = \frac{1}{\lambda_i+\lambda_3}$ and equations (2.31) and (2.32), we get that:

$$\mathrm{Cov}(\tau_1,\tau_2) = \frac{2}{\sqrt{\lambda_1\lambda_2}} \exp\left[\frac{(\lambda_1+\lambda_2+\lambda_3)^2}{4\lambda_1\lambda_2}\right] \int_{\frac{1}{2}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{\sqrt{\lambda_1\lambda_2}}\right)}^{\infty} \left[\frac{1}{2\sqrt{\lambda_1\lambda_2}u+\lambda_2+\lambda_3-\lambda_1} + \frac{1}{2\sqrt{\lambda_1\lambda_2}u+\lambda_1+\lambda_3-\lambda_2}\right] e^{-u^2} du - \left(\frac{1}{\lambda_1+\lambda_3}\right)\left(\frac{1}{\lambda_2+\lambda_3}\right) \quad (2.33)$$

Using the assumption $\lambda_1 = c_1\lambda_3$ and $\lambda_2 = c_2\lambda_3$, we rewrite the previous expression to get:

$$\mathrm{Cov}(\tau_1,\tau_2) = \frac{2}{\lambda_3^2\sqrt{c_1c_2}} \exp\left[\frac{(c_1+c_2+1)^2}{4c_1c_2}\right] \int_{\frac{c_1+c_2+1}{2\sqrt{c_1c_2}}}^{\infty} \left[\frac{1}{2\sqrt{c_1c_2}u+c_2-c_1+1} + \frac{1}{2\sqrt{c_1c_2}u+c_1-c_2+1}\right] e^{-u^2} du - \frac{1}{\lambda_3^2}\left(\frac{1}{c_1+1}\right)\left(\frac{1}{c_2+1}\right) \quad (2.34)$$

As $u \geq \frac{c_1+c_2+1}{2\sqrt{c_1 c_2}}$, we have that $\frac{1}{2\sqrt{c_1 c_2}u+c_2-c_1+1} \leq \frac{1}{2(c_2+1)}$ and $\frac{1}{2\sqrt{c_1 c_2}u+c_1-c_2+1} \leq \frac{1}{2(c_1+1)}$. Then:

$$\begin{aligned}\text{Cov}\left(\tau_1, \tau_2\right) \leq & \frac{1}{\lambda_3^2 \sqrt{c_1 c_2}}\left(\frac{1}{c_2+1}+\frac{1}{c_1+1}\right) \exp\left[\frac{\left(c_1+c_2+1\right)^2}{4 c_1 c_2}\right] \int_{\frac{c_1+c_2+1}{2\sqrt{c_1 c_2}}}^{\infty} e^{-u^2} du \\ & -\frac{1}{\lambda_3^2}\left(\frac{1}{c_1+1}\right)\left(\frac{1}{c_2+1}\right) \\ = & \frac{1}{\lambda_3^2\left(c_1+1\right)\left(c_2+1\right)}\left[\frac{1}{\sqrt{c_1 c_2}}\left(c_1+c_2+2\right) \exp\left[\frac{\left(c_1+c_2+1\right)^2}{4 c_1 c_2}\right] \int_{\frac{c_1+c_2+1}{2\sqrt{c_1 c_2}}}^{\infty} e^{-u^2} du-1\right] \quad (2.35)\end{aligned}$$

Hence, to get a negative covariance, it suffices to show:

$$\frac{1}{\sqrt{c_1 c_2}}\left(c_1+c_2+2\right) \exp\left[\frac{\left(c_1+c_2+1\right)^2}{4 c_1 c_2}\right] \int_{\frac{c_1+c_2+1}{2\sqrt{c_1 c_2}}}^{\infty} e^{-u^2} du<1 \quad (2.36)$$

Set $k := \frac{c_1+c_2+1}{\sqrt{c_1 c_2}}$ and note that our initial assumption implies that $\frac{1}{\sqrt{c_1 c_2}} \leq \frac{5}{4}\left(\frac{\sqrt{c_1 c_2}}{c_1+c_2+1}\right)=\frac{5}{4k}$. Hence:

$$\frac{1}{\sqrt{c_1 c_2}}\left(c_1+c_2+2\right) \exp\left[\frac{\left(c_1+c_2+1\right)^2}{4 c_1 c_2}\right] \int_{\frac{c_1+c_2+1}{2\sqrt{c_1 c_2}}}^{\infty} e^{-u^2} du \leq\left(k+\frac{5}{4k}\right) e^{\frac{1}{4}k^2} \int_{\frac{k}{2}}^{\infty} e^{-u^2} du \quad (2.37)$$

By the proof in the Appendix A, one can see that the previous integral is always smaller than 1 when $k>2$, which is the case here because:

$$\left(\sqrt{c_1}-\sqrt{c_2}\right)^2+1>0 \implies c_1-2\sqrt{c_1 c_2}+c_2+1>0 \implies c_1+c_2+1>2\sqrt{c_1 c_2}$$

Hence:

$$k=\frac{c_1+c_2+1}{\sqrt{c_1 c_2}}>\frac{2\sqrt{c_1 c_2}}{c_1 c_2}>2 \quad (2.38)$$

This shows that equation (2.36) holds and consequently, we have a negative covariance.

□

Remark. If $c := c_1 = c_2$, the condition of Proposition 11 reduces to $5c^2 - 8c - 4 \geq 0$ which is equivalent to $c \geq 2$. The interpretation of this is that to get a negative covariance when $\lambda_1 = \lambda_2$, it suffices to have λ_1, λ_2 sufficiently large compared to λ_3.

2.3 Applications

2.3.1 Application to Epidemiology

Suppose we are interested in knowing what is the probability of n people getting infected with COVID-19 at the exact same time. The time to infection of each person can be modeled as a stopping time. Some current models (see Britton and Pardoux [15]) assume independence of these stopping times and thus the probability of them being equal is 0. However, using our model we can weaken the independence assumption and conclude that:

Theorem 2.4 *If $(\tau_1, \tau_2, \ldots, \tau_n)$ follows the joint distribution described in Section 2.2, then:*

$$\mathbb{P}\left(\tau_1 = \tau_2 = \cdots = \tau_n\right) = \mathbb{E}\left[\int_0^\infty \alpha_{(1,2,\ldots,n)}(u) e^{-\sum_{k=1}^{n} \sum_{j \in C_k^n} A_j(u)} du\right] \tag{2.39}$$

The innermost sum in the exponent is taken over all the possible combinations C_k^n of $\binom{n}{k}$. For example, if $k = 3$, j could be $(1,3,5)$, $(2,3,n)$, etc; if $k = n$, j can only be $(1,2,\ldots,n)$

Remark. To be more specific if $n = 3$, then we get,

$$\begin{aligned}\mathbb{P}\left(\tau_1 = \tau_2 = \tau_3\right) = \mathbb{E}\Bigg[\int_0^\infty \alpha_{(1,2,3)}(u) \exp\big[&-A_1(u) - A_2(u) - A_3(u) \\ &-A_{(1,2)}(u) - A_{(1,3)}(u) - A_{(2,3)}(u) - A_{(1,2,3)}(u)\big] du\Bigg]\end{aligned} \tag{2.40}$$

Proof. Using the notation from Section 2.2, events in the process $\Lambda_{(1,2,\ldots,n)}$ are shocks to the n components of the system. Hence

$$\{\tau_1 = \tau_2 = \cdots = \tau_{n-1} = \tau_n\} =$$

$$\left\{\Lambda_{(1,2,\ldots\ n)} \leq \min\left(\Lambda_1, \ldots, \Lambda_n, \Lambda_{(1,2)}, \ldots, \Lambda_{(n,n)}, \Lambda_{(1,2,3)}, \ldots, \Lambda_{(1,2,\ldots,n-1)}\right)\right\}$$

Then, we find the distribution of:

$$M := \min\left(\Lambda_1, \ldots, \Lambda_n, \Lambda_{(1,2)}, \ldots, \Lambda_{(n,n)}, \Lambda_{(1,2,3)}, \ldots, \Lambda_{(1,2,\ldots,n-1)}\right)$$

under the measure $\tilde{\mathbb{P}}(\cdot)$ which stands for $\mathbb{P}\left(\cdot|(X_s)_{s\geq 0}\right)$

$$\begin{aligned}\tilde{\mathbb{P}}\left[M > t\right] &= \tilde{\mathbb{P}}\left[\Lambda_1 > t, \ldots, \Lambda_n > t, \ldots, \Lambda_{(n,n)} > t, \Lambda_{(1,2,3)} > t, \ldots, \Lambda_{(1,2,\ldots,n-1)} > t\right] \\ &= \exp\left[-A_1(t) - \cdots - A_n(t) - A_{(1,2)(t)} - \cdots - A_{(n,n)(t)} - A_{(1,2,3)(t)} - \cdots - A_{(1,2,\ldots,n-1)(t)}\right]\end{aligned} \tag{2.41}$$

Hence, M, given $(X_s)_{s\geq 0}$, has a continuous distribution with density equal to:

$$f_M(t) = \left[\sum_{k=1}^{n-1} \sum_{j\in C_k^n} \alpha_j(t)\right] e^{-\sum\limits_{k=1}^{n-1} \sum\limits_{j\in C_k^n} \alpha_j(t) A_j(t)} \tag{2.42}$$

The innermost sums are taken over all the possible combinations C_k^n of $\binom{n}{k}$. Then,

$$\begin{aligned}\tilde{\mathbb{P}}\left[\Lambda_{1,2,\ldots,n} < M\right] &= \int_0^\infty \int_x^\infty \alpha_{(1,2,\ldots,n)}(x) e^{-A_{(1,2,\ldots,n)}(x)} f_M(y)\, dy\, dx \\ &= \int_0^\infty \alpha_{(1,2,\ldots,n)}(u) e^{-\sum\limits_{k=1}^{n} \sum\limits_{j\in C_k^n} A_j(u)}\, du\end{aligned} \tag{2.43}$$

The result follows by taking the expectation.

□

Corollary 2.2 *If $\alpha_i(X_t) = \alpha(X_t)$ for all $i = 1, 2, \ldots, n$; $\alpha_{(i,j)}(X_t) = \frac{1}{2}\alpha(X_t)$ for all size 2 combinations in $\binom{n}{2}$; $\alpha_{(i,j,k)}(X_t) = \frac{1}{3}\alpha(X_t)$ for all size 3 combinations in $\binom{n}{3}$; $\ldots$; $\alpha_{(1,2,\ldots,n)}(X_t) = \frac{1}{n}\alpha(X_t)$.*

Then, we have:

$$\mathbb{P}\left(\tau_1 = \tau_2 = \cdots = \tau_n\right) = \frac{1}{n}\left[\frac{1}{\binom{n}{1} + \frac{1}{2}\binom{n}{2} + \cdots + \frac{1}{n}\binom{n}{n}}\right] \tag{2.44}$$

Corollary 2.3 *If $\alpha_i(X_t) = \alpha(X_t)$ for all $i = 1, 2, \ldots, n$; $\alpha_{(i,j)}(X_t) = 2\alpha(X_t)$ for all size 2 combinations in $\binom{n}{2}$; $\alpha_{(i,j,k)}(X_t) = 3\alpha(X_t)$ for all size 3 combinations in $\binom{n}{3}$; $\ldots$; $\alpha_{(1,2,\ldots,n)}(X_t) = n\alpha(X_t)$. Then, we have:*

$$\begin{aligned}\mathbb{P}\left(\tau_1 = \tau_2 = \cdots = \tau_n\right) &= n\left[\frac{1}{\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}}\right] \\ &= \frac{1}{2^{n-1}}\end{aligned} \tag{2.45}$$

2.3.2 Application to Engineering

A classic problem in Operations Research, basically studied in queuing theory, is that of a complicated machine. The machine fails if one of its key parts fails. Knowing this, designers create a certain redundancy by doubling key components, so that if one fails, there is a back-up ready to assume its duties. To save money, however, one can have one back-up for two components. Suppose η_1 is the (random) failure time of one component, and η_2 is the (again, random) failure time of the second component. If they fail at the same time, then the machine itself will fail, since the solitary back-up cannot replace both components simultaneously. One usually considers such a situation to be unlikely, even very unlikely. If it were to happen, however, we would be interested in $P(\eta_1 = \eta_2)$. In the conventional models, $P(\eta_1 = \eta_2)$ is zero, since they are each exponential, and conditionally independent. However if we consider a third time, η_3, and if this third stopping time is the (once again, random) time of an external shock (such as the failure of the air conditioning unit, or a power failure with a surge when the power resumes, etc.), and if we let

$$\tau_1 = \eta_1 \wedge \eta_3 \text{ and } \tau_2 = \eta_2 \wedge \eta_3 \tag{2.46}$$

then we are in the case where $P(\tau_1 = \tau_2) = \alpha > 0$, and in special cases we can calculate the probability α with precision, and in other, more complicated situations, we can give upper and lower bounds, both, for α.

A different kind of example, exemplified by the recent, and quite dramatic example is that of the collapse of the Champlain Towers South, in Surfside, Florida (just north of Miami Beach). The twelve story towers fell at night (1:30AM), and killed 98 people who were in their apartments at the time, and presumably even in their beds. The towers had a pool deck above a covered garage. The towers holding up the pool deck were too thin, and not strong enough to withstand the stresses imposed on them over four decades. Water was seen pouring into the parking garage only minutes before the collapse.

In the main building that collapsed, structural columns were too narrow to accommodate enough rebar, meaning that contractors had to choose between cramming extra steel into a too-small column (which can create air pockets that accelerate corrosion) or inadequately attaching floor slabs to their supports. Our model would have several stopping times, each one representing the failure of a different component of the structure. Choosing two important ones, such as: (1) The corrosion of the rebar supports within the concrete, due to the salt air and massive strains due to violent weather, which plagues the Florida coast during hurricane season; (2) The use of a low quality grade of concrete, violating regulations of the local government in the construction of the towers, leading to concrete integrity decay due to 40 years of seaside weather.

One way to model this is to take a vector of two Cox constructions, using two independent exponentials Z_1 and Z_2 to construct our failure times τ_1 and τ_2. This then gives us that τ_1 and τ_2 are conditionally independent, given the underlying filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. This leads to $P(\tau_1 = \tau_2) = 0$, even if they have the exact same compensators.

Other models have been proposed, such as the general framework presented in the book of Anna Aksamit and Monique Jeanblanc [5], where the random variables Z_1 and Z_2 are multivariate exponentials, but with a joint density describing how they relate to each other. However, even in this more general setting, we have $P(\tau_1 = \tau_2) = 0$.

Assuming it is the stopping times occurring simultaneously that causes the collapse, we want a model that allows us to have $P(\tau_1 = \tau_2) > 0$. Let's assume we have three standard Cox Constructions, with independent exponentials $(Z_1; Z_2; Z_3)$, with different compensators ($\int_0^t \alpha_s^i ds$ for $i = 1, 2, 3$).

Call the three stopping times η_1, η_2, η_3. The time η_3 could be anything, such as a hurricane putting heavy stress on the building, or an earthquake, or some other external factor. (Current forensic analysis, which is ongoing as we write this, suggests that the collapse of the pool-deck created a seismic shock sufficient to precipitate the collapse of the south tower, and weakened the structural integrity of the north tower to such an extent that it was condemned and then deliberately destroyed.) However in this case we can take η_3 to be the time of the flooding of the parking structure under the swimming pool and pool deck in general.

Simulations commissioned by the newspaper *The Washington Post* and done by a team led by Khalid M. Mosalam of the University of California, Berkeley, show how that might have happened and indicate that it is a plausible scenario (Swaine et al. [60]).

As in (2.46), we define

$$\tau_1 = \eta_1 \wedge \eta_3 \text{ and } \tau_2 = \eta_2 \wedge \eta_3, \tag{2.47}$$

where we take the stopping time η_3 to be the time of the collapse of the pool deck. As the noted engineer H. Petroski has pointed out [53] it is often the case that multiple things happen at once in order to precipitate a disaster such as the fall of the Champlain Tower South. Indeed, the forensic engineer R. Leon of Virginia Tech is quoted in *American Society of Civil Engineers*, 2021 [64]: "I think it is way too early to tell,"said Roberto Leon, P.E., F.SEI, Dist.M.ASCE, the D.H. Burrows Professor of Construction Engineering in the Charles Edward Via Jr. Department of Civil and Environmental Engineering at Virginia Tech."It's going to require a very careful forensic approach here, because I don't think the building collapsed just because of one reason. What we tend to find in forensic investigations is that three or four things have to happen for a collapse to occur that is so catastrophic."

Professor Leon is a widely respected authority in forensic civil engineering, and the key insight for us is his last statement that three or four things have to happen simultaneously for a catastrophic collapse.

Less dramatic examples, but quite pertinent to the recent attention being paid to the decay of infrastructure around the US, provide more examples of the utility of this approach. A first example is the Interstate 10 Twin Span Bridge over Lake Pontchartrain north of New Orleans, LA. It was rendered completely unusable by Hurricane Katrina, but the naive explanation was demonstrated false by the fact that several other bridges that had the same structural design remained intact. Upon investigation, it was determined that air trapped beneath the deck of the Interstate 10 bridges was a major contributing factor to the bridge's collapse. While major, it was not the only contributing factor (Chen et al. [19]).

A final example is the derailment of an Amtrak train near Joplin, Montana in September, 2021. 154 people were on board the train, and 44 passengers and crew were taken to area hospitals with injuries. The train was traveling at between 75 and 78 mph, just below the speed limit of 79 mph on that section of track when its emergency brakes were activated. The two locomotives and two railcars remained on the rails and eight cars derailed. Investigations of these types of events take years, but preliminary speculation is that the accident could have been caused by problems with the railroad or track, such as a rail that buckled under high heat, or the track itself giving way when the train passed over. Both might also be possible, leading to the two stopping times τ_1 and τ_2, and a situation where $P(\tau_1 = \tau_2) > 0$. See Hanson and Brown [40].

Chapter 3: Computing the Probability of a Financial Market Failure: A New Measure of Systemic Risk

For regulators, characterizing the probability of a financial market failure, or systemic risk, is important because such a characterization enables them to understand how their regulatory actions affect its magnitude. In this regard, numerous systemic risk measures have been proposed in the literature, each with associated benefits and limitations. For literature reviews of the existing collection of systemic risk measures, see Bisias et al. [12] and Engle [32]. This paper provides another measure of systemic risk, different from the existing set. According to the systemic risk measure taxonomies in Bisias et al. [12], ours is a macroeconomic or macroprudential measure, which is a based on a default intensity model. As such, it is a forward-looking measure, which satisfies the following characteristics:

1. it is consistent with the economic theories relating to the causes of financial market failures (macroeconomic),
2. it uses the existing regulatory designations of globally systemically important banks (G-SIBs), financial institutions that are "too big to fail" (macroeconomic),
3. it can be estimated using existing hazard rate methodologies (default intensity), and
4. it facilitates quantifying the impact of regulatory policy changes on systemic risk (macroprudential).

Our measure of systemic risk is the *probability that any two G-SIBs default at the "same time."* A G-SIB is any financial institution that has been designated by the Financial Stability Board (FSB) as large enough such that if it fails, its failure affects the health of the financial system. Operationally, a bank is designated as a G-SIB if various indicators of its financial health, in aggregate, exceed some threshold (see the report of the FSB [34] and the methodology of the Bank for International Settlements [9]). There were 30 such G-SIBs designated by the FSB in 2020. By the "same time" we mean within a short time period of each other, say 1 week.

The idea underlying our measure is that if one G-SIBs fails, regulators can manage the resulting crisis to ensure that a market wide failure does not occur. Examples of such past episodes include the failure of Long Term Capital Management in 1998 and Lehman Brothers, together with Bear Sterns, in 2008. For both of these episodes, regulators were able to manage the crisis and prevent a market-wide failure. However, if two (or more) G-SIBs fail within a short time period of each other, then our measure asserts that the crisis is uncontrollable by regulators and the market fails.

Our measure is consistent with economic theories of market failures because, in a reduced form fashion, it implicitly includes the causes for the failure, e.g. the "drying-up" of short-term funding, the bursting of an asset price bubble, or the propagation of defaults in a network of banks due to inter-linked funding (see Allen and Carletti [6], Acemoglu et al. [1], Jarrow and Lamichhane [42]). And, it also explicitly incorporates the marginal impact of a G-SIBs's default on the probability of a financial market failure (see Acharya et al. [2] for related discussion).

By its definition, our measure builds upon the fundamental analysis already done by regulators in identifying financial institutions that are G-SIBs. As such, it explicitly includes as part of its inputs, the expert judgement and analysis of regulators based on public and non-public information (see the methodology of the Bank for International Settlements [9]). The inclusion of this non-public information into our systemic risk measure is a benefit of using the G-SIB designations.

Our measure can also be estimated due to its construction, because the probability of a market failure incorporates the existing marginal probabilities of a G-SIB defaulting. These probabilities can be obtained as in the existing hazard rate estimation literature, see Chava and Jarrow [18], Campbell et al. [17], and Shumway [58]. Finally, given the analytic representation of our systemic risk measure, it is easy to compute the impact of a regulatory policy change on the probability of a market failure, e.g., such regulatory actions might be the breaking-up of a G-SIB or the increase in a G-SIB's capital. These regulatory changes correspond to modifying various input variables underlying the market failure probabilities and determining their impact on the resulting value.

For some previous work on intensity-based models for correlated default times, see Giesecke [35] who proposed a model that is similar, but less general than ours. He only treats the case of constant default intensity, i.e, $\alpha_s \equiv \alpha$ for all $s \geq 0$. Another closely related work is Lindskog and McNeil [50] who consider a Poisson shock model of arbitrary dimension with both fatal and not-necessarily-fatal shocks.

An outline for this chapter is as follows. Section 3.1 presents the model, while Section 3.2 contains the key theorems. Section 3.3 provides comparative statics and concludes.

3.1 The Model

The following model is based on the work presented in the previous chapter. Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfying the usual conditions and large enough to support a $\mathbb{R}^d$ -valued right continuous with left limits existing stochastic process $X = \{X_t, t \geq 0\}$ and $K + 1$ independent exponential random variables each with parameter 1, i.e. $(Z_i, i = 0, ..., K)$.

Consider a financial market that contains $i = 1, ..., K$ financial institutions that are classified as G-SIBs, i.e. too big to fail. There can be numerous other financial institutions in the market, but their existence will not be explicitly included in our systemic risk measure. However, these non-G-SIBs are implicitly included as will be subsequently noted.

The stochastic process X represents a vector of state variables characterizing the health of the economy and the K G-SIBs. It includes macro variables such as the inflation rate, the unemployment rate, the level of interest rates, and G-SIB specific balance sheet quantities such as their capital ratios.

3.1.1 The G-SIBs' Default Times due to Idiosyncratic Events

Define the default time for the i^{th} G-SIB due to *idiosyncratic events* as

$$\eta_i := \inf\{s : A_i(s) \geq Z_i\}$$

where $A_i(s) = \int_0^s \alpha_i(X_r)dr$ and $Z_i \sim \text{Exp}(1)$. The process $\alpha_i(\cdot) : \mathbb{R}^d \to [0, \infty)$ is the default intensity of the i^{th} G-SIB dependent upon the state variable process X. The default intensity is assumed to be a non-random, positive, continuous function. This implies that $A_i(s)$ are continuous and strictly increasing for any $s \geq 0$. We note that the default intensity of the i^{th} G-SIB can be estimated using standard hazard rate estimation techniques as done in Chava and Jarrow [18], Campbell et al. [17], and Shumway [58].

An idiosyncratic event causing default for a G-SIB is one that is unique to the bank, after conditioning on the state variable process X. For example, it could be due to fraudulent trades by a rogue trader or incompetent management. As defined, by construction, the idiosyncratic event default times of the G-SIBs are Cox processes, conditionally independent across G-SIBs given the filtration generated by X over $[0, \infty)$ denoted as $\mathbb{F}_X = \sigma(X_t : t \in [0, \infty))$. This implies that $\mathbb{P}(\eta_i = \eta_j) = 0$ for $i \neq j$. More explicitly, by taking an expectation on the following expression,

$$\begin{aligned}\mathbb{P}\left(\eta_i = \eta_j | \mathbb{F}_X\right) &= \int_0^\infty \int_0^\infty 1_{\{x=y\}} f_{\eta_i}(x) f_{\eta_j}(y) dy dx \\ &= \int_0^\infty \mathbb{P}\left(\eta_j = x | \mathbb{F}_X\right) f_{\eta_i}(x) dx = 0\end{aligned}$$

where $f_{\eta_i}(x)$ and $f_{\eta_j}(y)$ are the continuous distributions of η_i and η_j given $\mathbb{F}_X$.

Example 1. (*Destructive Competition*)

A useful example of an idiosyncratic default intensity is one that depends on the number of G-SIBs, i.e. $A_i(t, K) := \int_0^t \alpha_i(X_u, K)du$, where the marginal probability of a default increases with K for each $i = 1, 2, \ldots, K$.

The interpretation is that as the number of G-SIBs increase, the banks compete more aggressively with each other to maintain market share and profitability. In doing so, they take on riskier investments to increase expected returns, which in turn, increases idiosyncratic default risk. This is in fact what occurred prior to the credit crisis of 2007 when financial institutions invested in riskier AAA rated collateralized debt obligations (CDOs) instead of the riskless AAA rated U.S. Treasuries to obtain increased yields (see Crouhy et al. [26] or Protter [54] for a more detailed explanation).

A special case of this intensity is when $\alpha_i(X_t, K) = \ln(K) + b(X_t)$ for an appropriate measurable and integrable $b(\cdot) : \mathbb{R}^d \to [0, \infty)$. Then,

$$A_i(t, K) := \int_0^t \alpha_i(X_u, K) du = t \ln(K) + \int_0^t b(X_u) du = t \ln(K) + B(t).$$

3.1.2 Market-Wide Stress Events

Next, define η_0 to be the occurrence of a market-wide stress event, as distinct from an idiosyncratic default event specific to a single G-SIB. For example, it could be the drying up of short term funding markets, the bursting of an asset price bubble (as in the housing market prior to the 2007 credit crisis), or a large collection of non G-SIBs defaulting in a short period of time. This market-wide stress event implicitly includes the influence of the remaining non-G-SIBs in the market, and their inter-relationships among themselves and the G-SIBs.

Define the first time that a market-wide stress event occurs as

$$\eta_0 \coloneqq \inf\{s : A_0(s) \geq Z_0\}$$

where $A_0(s) = \int_0^s \alpha_0(X_r) dr$, $Z_0 \sim \text{Exp}(1)$, and $\alpha_0(\cdot) : \mathbb{R}^d \to [0, \infty)$ is a non-random, positive, continuous function. Note that, because of the conditional independence assumption given $\mathbb{F}_X$, we have $\mathbb{P}(\eta_i = \eta_0) = 0$ for all i. Here the intensity process of a market-wide stress event, $\alpha_0(X)$, probably cannot be estimated using historical time series data given the infrequency of their occurrence. However, a financial institution or regulator can use expert judgement to facilitate the practical computation of this quantity.

3.1.3 The G-SIBs' Default Times

Finally, we define the default time of the i^{th} G-SIB as

$$\tau_i = \min(\eta_0, \eta_i)$$

for $i = 1, ..., K$. This is the first time that either an i^{th} G-SIB defaults due to an idiosyncratic event or a market-wide stress event occurs. Note that this definition implicitly characterizes the market-wide stress event as one which is catastrophic enough to cause G-SIBs to default on their obligations.

Given the above structure, we have that the survival probability of the i^{th} G-SIB is

$$\mathbb{P}(\tau_i > t) = \mathbb{E}\left[\exp\left(-A_0(t) - A_i(t)\right)\right] \tag{3.1}$$

Remark 1. (*Destructive Competition*)

For the special case of destructive competition (see Example 1), we have that the i^{th} G-SIB's survival probability $\mathbb{P}(\tau_i > t) = \mathbb{E}\left[e^{-A_0(t)-A_i(t,K)}\right]$ is decreasing as K increases. This implies, of course, that as the number of G-SIB's increases, the probability of any single G-SIB defaulting increases.

3.1.4 The Market Failure Time

We now can define a financial market failure. To fix the intuition, as an initial attempt, we first define a financial market failure as the event

$$\left\{\omega \in \Omega : \tau_i = \tau_j \text{ for some } (i, j) \in (1, ..., K) \times (1, ..., K), i \neq j\right\},$$

and the probability of a financial market failure, our systemic risk measure, as

$$\mathbb{P}\left(\tau_i = \tau_j \text{ for some } (i, j) \in (1, ..., K) \times (1, ..., K), i \neq j\right).$$

This is the probability that two G-SIBs default at the exactly the same time. The idea underlying this market-failure probability is that if one G-SIB fails, regulators can manage the resulting failure to ensure that a market-wide failure does not occur. However, if two (or more) G-SIBs fail at the same time, then such an event is uncontrollable by the regulators, resulting in a market-wide failure.

Unfortunately, there is a problem with this initial systemic risk measure. Given the definition of the i^{th} G-SIB's default time τ_i and the conditional independence assumption given $\mathbb{F}_X$ across η_i for $i = 0, 1, ..., K$, a market failure occurs under this definition with probability one if and only if $\eta_0 \leq \eta_i$ and $\eta_0 \leq \eta_j$ for some pair $i \neq j$. This is because $\mathbb{P}\left(\eta_i = \eta_j\right) = 0$ for $i \neq j$, $(i, j) \in (1, ..., K) \times (1, ..., K)$. In essence, a market failure only occurs under this definition, in probability, when a market-wide stress event occurs. In probability, the existence of G-SIBs is irrelevant to this initial systemic risk measure. To remove this problem, we generalize the definition of a financial market failure event to be

$$\left\{\omega \in \Omega : \left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i, j) \in (1, ..., K) \times (1, ..., K), i \neq j\right\}$$

for a given $\varepsilon > 0$, and our (final) systemic risk measure as

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i, j) \in (1, ..., K) \times (1, ..., K), i \neq j\right).$$

Under this systemic risk measure, a market failure can occur for two reasons: a market-wide stress event occurs, or two G-SIBs experience idiosyncratic default events within an ε time period of each other. The next section characterizes this market-wide default probability.

3.2 Theorems

This section provides the key theorems characterizing the probability distribution of defaults times for the various G-SIBs ("banks") and the probability of a market failure.

3.2.1 The Joint Distribution of Banks' Default Times

Our first theorem characterizes the joint probability distribution of the banks' default times $(\tau_1, \tau_2, \ldots, \tau_K)$.

Theorem 3.1 *(Joint Distribution $(\tau_1, \tau_2, \ldots, \tau_K)$).*

$$\mathbb{P}\left(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_K > t_K\right) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{K} A_i(t_i) - A_0\left(\max\left(t_1, t_2, \ldots, t_K\right)\right)\right)\right] \tag{3.2}$$

Proof. Let $M := \max\left(t_1, t_2, \ldots, t_K\right)$ and $X_M := (X_u)_{0 \le u \le M}$. Then,

$$\begin{aligned}\mathbb{P}\left(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_K > t_K | X_M\right) &= \mathbb{P}\left(\bigcap_{i=1}^{K} \eta_i \wedge \eta_0 > t_i | X_M\right)\\ &= \mathbb{P}\left(\eta_1 > t_1, \ldots, \eta_K > t_K, \eta_0 > M | X_M\right) = \exp\left(-\sum_{i=1}^{K} A_i(t_i) - A_0\left(M\right)\right)\end{aligned}$$

The last equality follows from the mutual independence of $\eta_1, \eta_2, \ldots, \eta_K, \eta_0$. We can conclude the theorem by taking an additional expectation.

□

This distribution is a multivariate version of the Cox process, that is, marginally each bank's default time is a Cox process. However, the default times across the banks are not independent. The difference in the joint distribution, relative to a standard Cox process, is due to the last term in the exponent, $A_0\left(\max\left(t_1, t_2, \ldots, t_K\right)\right)$, which depends on the distribution of the first K default times exceeding the given times $t_1, t_2, \ldots, t_K$. The form of this multivariate distribution is tractable, facilitating subsequent computations.

Remark 2. (*Destructive Competition*)

When there is destructive competition (see Example 1), the joint survival probability of the K banks is decreasing in the number of G-SIBs, i.e.

$$\mathbb{P}\left(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_K > t_K\right) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{K} A_i(t_i, K) - A_0\left(\max\left(t_1, t_2, \ldots, t_K\right)\right)\right)\right]$$

is decreasing as K increases.

Corollary 3.1 *(Constant Default Intensities)*

If $\alpha_i(X_t) = \alpha_i$ for all $t \geq 0$, all $i = 1, 2, \ldots, K$, and where $\alpha_i \in \mathbb{R}$, then:

$$\mathbb{P}\left(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_K > t_K\right) = \exp\left[-\sum_{i=1}^{K} \alpha_i t_i - \alpha_0 \max\left(t_1, t_2, \ldots, t_K\right)\right] \tag{3.3}$$

This constant default intensity case clarifies expression (3.2).

Next, we can deduce the probability of a market failure.

3.2.2 The Market Failure Probability

This is the key theorem in our paper.

Theorem 3.2 *(Market Failure Probability)*

$$\begin{aligned}
&\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i, j) \in (1, \ldots, K) \times (1, \ldots, K), i \neq j\right) = \\
&\quad 1 - \mathbb{E}\Bigg[\sum_{j \in P} \int_0^{\infty} f_{j_1}(x_1) \int_{x_1+\varepsilon}^{\infty} f_{j_2}(x_2) \int_{x_2+\varepsilon}^{\infty} f_{j_3}(x_3) \cdots \int_{x_{K-3}+\varepsilon}^{\infty} f_{j_{K-2}}(x_{K-2}) \\
&\quad \int_{x_{K-2}+\varepsilon}^{\infty} f_{j_{K-1}}(x_{K-1}) \exp\left[-A_{j_K}(x_{K-1}+\varepsilon) - A_0(x_{K-1}+\varepsilon)\right] dx_{K-1}\, dx_{K-2} \ldots dx_2\, dx_1\Bigg]
\end{aligned} \tag{3.4}$$

where the sum is taken over all possible permutations P of K, i.e.

$$j = (j_1 = \sigma(1), j_2 = \sigma(2), \ldots, j_K = \sigma(K))$$

and $f_{j_k}(x_k)$ is the density of η_{j_k} given $\mathbb{F}_X$, i.e, $f_{j_k}(x) = \alpha_{j_k}(X_x) \exp\left[-A_{j_k}(x)\right]$.

Proof. For $k = 1, \ldots, K$, let $\tau_{(k)}$ be the k^{th} order statistic of $(\tau_1, \tau_2, \ldots, \tau_K)$. For example, $\tau_{(1)} = \min(\tau_1, \tau_2, \ldots, \tau_K)$ and $\tau_{(K)} = \max(\tau_1, \tau_2, \ldots, \tau_K)$. Similarly, for $k = 1, 2, \ldots, K, K+1$, let $\eta_{(k)}$

be the k^{th} order statistic of $(\eta_0, \eta_1, \eta_2, \ldots, \eta_K)$. For example, $\eta_{(1)} = \min(\eta_0, \eta_1, \eta_2, \ldots, \eta_K)$ and $\eta_{(K+1)} = \max(\eta_0, \eta_1, \eta_2, \ldots, \eta_K)$.

Let $f_k(x)$ be the density of η_k given $\mathbb{F}_X$, which is $f_k(x) = \alpha_k(X_x)e^{-A_k(x)}$. Note that by construction $\eta_0, \eta_1, \eta_2, \ldots, \eta_K$ are mutually independent.

Define $\tilde{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|\mathbb{F}_X)$. Hence, $\tilde{\mathbb{P}}(\eta_j > t) = \exp\left(-A_j(t)\right)$ and note that

$$\left\{\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j),\ i \neq j\right\}^C = \left\{\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j\right\}$$

Hence,

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j\right) \\
&\qquad\qquad = \tilde{\mathbb{P}}\left(\tau_{(K)} \geq \tau_{(K-1)} + \varepsilon, \tau_{(K-1)} \geq \tau_{(K-2)} + \varepsilon, \ldots, \tau_{(2)} \geq \tau_{(1)} + \varepsilon\right) \\
&\qquad\qquad = \tilde{\mathbb{P}}\left(\eta_{(K+1)} \geq \eta_{(K)} \geq \eta_{(K-1)} + \varepsilon, \eta_{(K-1)} \geq \eta_{(K-2)} + \varepsilon, \ldots, \geq \eta_{(1)} + \varepsilon, \eta_0 \geq \eta_{(K)}\right)
\end{aligned}$$

If η_0 was not greater or equal to $\eta_{(K)}$, then there exists some $k < K$ such that $\eta_{(k)} = \eta_0$. This implies that $\tau_{(k)} = \tau_{(k+1)} = \cdots = \tau_{(K)}$, which is the situation we wish to avoid.

Now, we can work with the $K+1$ independent random variables, i.e., $(\eta_0, \eta_1, \eta_2, \ldots, \eta_K)$. We just need to compute the distribution of the order statistics, which follows by the usual construction of taking the sum over all possible permutations. Let us start by finding the number of possible permutations. Because of the restriction $\eta_0 \geq \eta_{(K)}$, there are $2\,(K!)$ permutations and then,

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\eta_{(K+1)} \geq \eta_{(K)} \geq \eta_{(K-1)} + \varepsilon, \eta_{(K-1)} \geq \eta_{(K-2)} + \varepsilon, \ldots, \eta_{(2)} \geq \eta_{(1)} + \varepsilon, \eta_0 \geq \eta_{(K)}\right) \\
&\qquad = \sum_{j \in \tilde{P}} \int_0^\infty f_{j_1}(x_1) \int_{x_1+\varepsilon}^\infty f_{j_2}(x_2) \cdots \int_{x_{K-1}+\varepsilon}^\infty f_{j_K}(x_K) \int_{x_K}^\infty f_{j_{K+1}}(x_{K+1})\, dx_{K+1}\, dx_K \ldots dx_2\, dx_1 \quad (3.5)
\end{aligned}$$

where the sum is taken over all $2(K!)$ possible permutations such that

$$j = (j_1 = \sigma(1), j_2 = \sigma(2), \ldots, j_K = \sigma(K), j_{K+1} = 0)$$

or

$$j = (j_1 = \sigma(1), j_2 = \sigma(2), \dots j_K = 0, j_{K+1} = \sigma(K))$$

We can simplify (3.5). Let us fix one of the permutations $j = (j_1, j_2, \dots, j_{K+1})$ of K such that $j_1 = \sigma(1), j_2 = \sigma(2), \dots, j_K = \sigma(K), j_{K+1} = 0$ and let us focus on the 2 innermost integrals,

$$\begin{aligned}
&\int_{x_{K-1}+\varepsilon}^{\infty} f_{j_K}(x_K) \int_{x_K}^{\infty} f_{j_{K+1}}(x_{K+1})\, dx_{K+1}\, dx_K \\
&\qquad = \int_{x_{K-1}+\varepsilon}^{\infty} f_{j_K}(x_K) \int_{x_K}^{\infty} \alpha_{j_{K+1}} \left(X_{x_{K+1}}\right) e^{-A_{j_{K+1}}(x_{K+1})}\, dx_{K+1}\, dx_K \\
&\qquad = \int_{x_{K-1}+\varepsilon}^{\infty} \alpha_{j_K} \left(X_{x_K}\right) e^{-A_{j_K}(x_K) - A_{j_{K+1}}(x_K)}\, dx_K \\
&\qquad = \int_{x_{K-1}+\varepsilon}^{\infty} \alpha_{\sigma(K)} \left(X_{x_K}\right) e^{-A_{\sigma(K)}(x_K) - A_0(x_K)}\, dx_K. \qquad (3.6)
\end{aligned}$$

Now, fix a similar permutation of j with the only difference being that $j_K = 0$ and $j_{K+1} = \sigma(K)$. By a similar calculation as in (3.6), we get,

$$\int_{x_{K-1}+\varepsilon}^{\infty} f_{j_K}(x_K) \int_{x_K}^{\infty} f_{j_{K+1}}(x_{K+1})\, dx_{K+1}\, dx_K = \int_{x_{K-1}+\varepsilon}^{\infty} \alpha_0 \left(X_{x_K}\right) e^{-A_0(x_K) - A_{\sigma(K)}(x_K)}\, dx_K \qquad (3.7)$$

After noting that the $K-1$ outer integrals of the 2 fixed permutations are the same, we can join (3.6) and (3.7) to get,

$$\begin{aligned}
&\int_0^{\infty} f_{j_1}(x_1) \int_{x_1+\varepsilon}^{\infty} f_{j_2}(x_2) \cdots \int_{x_{K-1}+\varepsilon}^{\infty} \left[\alpha_0 \left(X_{x_K}\right) + \alpha_{\sigma(K)} \left(X_{x_K}\right)\right] e^{-A_0(x_K) - A_{\sigma(K)}(x_K)}\, dx_K \dots dx_1 \\
&\quad = \int_0^{\infty} f_{j_1}(x_1) \int_{x_1+\varepsilon}^{\infty} f_{j_2}(x_2) \cdots \int_{x_{K-2}+\varepsilon}^{\infty} f_{\sigma(K)}(x_{K-1}) e^{-A_0(x_{K-1}+\varepsilon) - A_{\sigma(K)}(x_{K-1}+\varepsilon)}\, dx_{K-1} \dots dx_1 \qquad (3.8)
\end{aligned}$$

Now, we only need to take the sum of terms like (3.8) over all the possible $K!$ permutations of $(1, 2, \ldots, K)$ and the result follows by taking a further expectation.

□

This theorem shows that the market failure time is computable given the banks' and the market-wide stress event's intensity processes. In this form, it is quite abstract. To understand this probability better, we consider two special cases in the subsequent remarks.

Corollary 3.2 *(Identically Distributed Default Times)*

If we assume $\alpha(X_x) = \alpha_1(X_x) = \alpha_2(X_x) = \cdots = \alpha_K(X_x)$ a.s., i.e. $\eta_1, \eta_2, \ldots, \eta_K$ are identically distributed given $\mathbb{F}_X$, then in Theorem 3.2 we can replace the sum over all the permutations by $K!$. More specifically,

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \; for \; some \; (i,j) \in (1,...,K) \times (1,...,K), i \neq j\right)$$
$$= 1 - \mathbb{E}\left[K! \int\limits_0^\infty f(x_1) \int\limits_{x_1+\varepsilon}^\infty f(x_2) \cdots \int\limits_{x_{K-3}+\varepsilon}^\infty f(x_{K-2}) \right.$$
$$\left. \int\limits_{x_{K-2}+\varepsilon}^\infty \alpha(X_{x_{K-1}}) e^{-A(x_{K-1}) - A(x_{K-1}+\varepsilon) - A_0(x_{K-1}+\varepsilon)} \, dx_{K-1} \, dx_{K-2} \ldots dx_1 \right] \quad (3.9)$$

where $f(x) = \alpha(X_x) e^{-A(x)}$.

Corollary 3.3 *(Constant Default Intensities)*

If $\alpha_i(X_t) = \alpha_i$ for all $t \geq 0$, all $i = 1, 2, \ldots, K$, and where $\alpha_i \in \mathbb{R}$, then the market failure probability is

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \; for \; some \; (i,j) \in (1,...,K) \times (1,...,K), i \neq j\right) =$$
$$1 - \sum_{j \in P} \prod_{i=1}^{K-1} \left[\frac{\alpha_{j_i}}{\alpha_0 + \sum\limits_{k=i}^{K} \alpha_{j_k}} e^{-\varepsilon\left(\alpha_0 + \sum\limits_{k=i+1}^{K} \alpha_{j_k}\right)} \right] \quad (3.10)$$

where the sum is taken over all possible permutations P of K, i.e.

$$j = (j_1 = \sigma(1), j_2 = \sigma(2), \ldots, j_K = \sigma(K))$$

Here, we see that the market failure probability is easily computed given estimates of the default intensities.

Proof. (Expression (3.10)). Use Theorem (3.2) with $\alpha_k(x) \equiv \alpha_k$ which implies $f_k(x) = \alpha_k e^{-\alpha_k x}$ for all k. Solve the integrals to get

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j\right) = \left[\frac{\alpha_{j_1}}{\alpha_0 + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_K}}\right]\left[\frac{\alpha_{j_2}}{\alpha_0 + \alpha_{j_2} + \cdots + \alpha_{j_K}}\right]\ldots$$
$$\ldots\left[\frac{\alpha_{j_{K-2}}}{\alpha_0 + \alpha_{j_{K-2}} + \alpha_{j_{K-1}} + \alpha_{j_K}}\right]\left[\frac{\alpha_{j_{K-1}}}{\alpha_0 + \alpha_{j_{K-1}} + \alpha_{j_K}}\right]\exp\left[-\varepsilon\left(\alpha_0 + \alpha_{j_2} + \alpha_{j_3} \cdots + \alpha_{j_K}\right)\right] \times$$
$$\exp\left[-\varepsilon\left(\alpha_0 + \alpha_{j_3} + \alpha_{j_4} + \cdots + \alpha_{j_K}\right)\right] \times \cdots \times \exp\left[-\varepsilon\left(\alpha_0 + \alpha_{j_{K-1}} + \alpha_{j_K}\right)\right]\exp\left[-\varepsilon\left(\alpha_0 + \alpha_{j_K}\right)\right]$$

□

The following corollary related to the market failure probability as $\varepsilon \to 0$ follows easily from Theorem 3.1.

Corollary 3.4 *(Market Failure when $\varepsilon \to 0$)*

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\tau_1 > t, \tau_2 > t,\right)}{\varepsilon} = \mathbb{E}\left(\alpha_0(X_t)\right) \tag{3.11}$$

Proof. Note that

$$\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\tau_1 > t, \tau_2 > t, (X_u)_{u \leq t+\varepsilon}\right)$$
$$= \frac{\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\, (X_u)_{u \leq t+\varepsilon}\right)}{\mathbb{P}\left(\tau_1 > t, \tau_2 > t |\, (X_u)_{u \leq t+\varepsilon}\right)}$$

For the numerator, using Theorem 3.1 with $K = 2$,

$$\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] \,|\, (X_u)_{u \leq t+\varepsilon}\right) =$$

$$\mathbb{P}\left(\tau_1 > t, \tau_2 > t | \left(X_u\right)_{u \le t+\varepsilon}\right) - \mathbb{P}\left(\tau_1 > t, \tau_2 > t+\varepsilon | \left(X_u\right)_{u \le t+\varepsilon}\right)$$

$$- \mathbb{P}\left(\tau_1 > t+\varepsilon, \tau_2 > t | \left(X_u\right)_{u \le t+\varepsilon}\right) + \mathbb{P}\left(\tau_1 > t+\varepsilon, \tau_2 > t+\varepsilon | \left(X_u\right)_{u \le t+\varepsilon}\right) =$$

$$\exp\left[-\left(A_1 + A_2 + A_0\right)(t)\right] - \exp\left[-A_1(t) - \left(A_2 + A_0\right)(t+\varepsilon)\right]$$

$$- \exp\left[-\left(A_1 + A_0\right)(t+\varepsilon) - A_2(t)\right] + \exp\left[-\left(A_1 + A_2 + A_0\right)(t+\varepsilon)\right]$$

Dividing by $\mathbb{P}\left(\tau_1 > t, \tau_2 > t | \left(X_u\right)_{u \le t+\varepsilon}\right) = \exp\left[-A_1(t) - A_2(t) - A_0(t)\right]$, we get:

$$\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] | \tau_1 > t, \tau_2 > t, \left(X_u\right)_{u \le t+\varepsilon}\right) = 1 - \exp\left[-\int_t^{t+\varepsilon} \left(\alpha_2 + \alpha_0\right)(X_u) du\right]$$

$$- \exp\left[-\int_t^{t+\varepsilon} \left(\alpha_1 + \alpha_0\right)(X_u) du\right] + \exp\left[-\int_t^{t+\varepsilon} \left(\alpha_1 + \alpha_2 + \alpha_0\right)(X_u) du\right]$$

Hence, by using L'Hôpital's rule, we get:

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] | \tau_1 > t, \tau_2 > t, \left(X_u\right)_{u \le t+\varepsilon}\right)}{\varepsilon} =$$

$$he \left(\alpha_2 + \alpha_0\right)(X_t) + \left(\alpha_1 + \alpha_0\right)(X_t) - \left(\alpha_1 + \alpha_2 + \alpha_0\right)(X_t) = \alpha_0(X_t).$$

Moreover, given $\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] | \tau_1 > t, \tau_2 > t, \left(X_u\right)_{u \le t+\varepsilon}\right)$ is bounded by 1 and a conditioning argument, we can conclude that

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\tau_1 \in (t, t+\varepsilon], \tau_2 \in (t, t+\varepsilon] | \tau_1 > t, \tau_2 > t, \right)}{\varepsilon} = \mathbb{E}\left(\alpha_0(X_t)\right). \tag{3.12}$$

□

That is, the probability of a market failure due to two banks defaulting over the next time interval $(t, t+\varepsilon]$ with $\varepsilon \approx 0$ is equal to the probability of a market-wide stress event occurring, as noted before.

3.2.3 Catastrophic Market Failure

It is of interest to determine the probability that all banks will default within a small interval of time. This would be a catastrophic market failure.

Theorem 3.3 *(Occurrence of all the Default Times). Let $\tau_{(K)} = max\{\tau_1, ..., \tau_K\}$.*

$$\mathbb{P}\left(\tau_{(K)} \leq \varepsilon\right) = 1 + \mathbb{E}\left[e^{-A_0(\varepsilon)}\left(\prod_{i=1}^{K}\left(1 - e^{-A_i(\varepsilon)}\right) - 1\right)\right] \tag{3.13}$$

Proof. As in the proof of Theorem 3.2, define $\tilde{\mathbb{P}}(\cdot) = \mathbb{P}\left(\cdot|\mathbb{F}_X\right)$. Hence, $\tilde{\mathbb{P}}(\eta_j > t) = \exp\left(-A_j(t)\right)$. Then, for any K, we have:

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_{(K)} \leq \varepsilon\right) =& \tilde{\mathbb{P}}\left(\tau_1 \leq \varepsilon, \tau_2 \leq \varepsilon, \ldots, \tau_K \leq \varepsilon\right) \\ =& \tilde{\mathbb{P}}\left(\tau_1 \leq \varepsilon, \tau_2 \leq \varepsilon, \ldots, \tau_K \leq \varepsilon, \eta_0 > \varepsilon\right) + \tilde{\mathbb{P}}\left(\tau_1 \leq \varepsilon, \tau_2 \leq \varepsilon, \ldots, \tau_K \leq \varepsilon, \eta_0 \leq \varepsilon\right)\end{aligned}$$

Because $\tau_i = \min\left(\eta_i, \eta_0\right)$, the following events are equal:

$$\{\tau_1 \leq \varepsilon, \tau_2 \leq \varepsilon, \ldots, \tau_K \leq \varepsilon, \eta_0 > \varepsilon\} = \{\eta_1 \leq \varepsilon, \eta_2 \leq \varepsilon, \ldots, \eta_K \leq \varepsilon, \eta_0 > \varepsilon\}$$

$$\{\tau_1 \leq \varepsilon, \tau_2 \leq \varepsilon, \ldots, \tau_K \leq \varepsilon, \eta_0 \leq \varepsilon\} = \{\eta_0 \leq \epsilon\}$$

Hence,

$$\tilde{\mathbb{P}}\left(\tau_{(K)} \leq \varepsilon\right) = \tilde{\mathbb{P}}\left(\eta_1 \leq \varepsilon, \eta_2 \leq \varepsilon, \ldots, \eta_K \leq \varepsilon, \eta_0 > \varepsilon\right) + \tilde{\mathbb{P}}\left(\eta_0 \leq \varepsilon\right)$$

Using the independence of η_i

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_{(K)} \leq \varepsilon\right) &= \tilde{\mathbb{P}}\left(\eta_0 > \varepsilon\right)\tilde{\mathbb{P}}\left(\eta_1 \leq \varepsilon\right)\tilde{\mathbb{P}}\left(\eta_2 \leq \varepsilon\right)\ldots\tilde{\mathbb{P}}\left(\eta_K \leq \varepsilon\right) + \tilde{\mathbb{P}}\left(\eta_0 \leq \varepsilon\right) \\ &= e^{-A_0(\varepsilon)}\left[\prod_{i=1}^{K}\left(1 - e^{-A_i(\varepsilon)}\right)\right] + 1 - e^{-A_0(\varepsilon)} \\ &= 1 + e^{-A_0(\varepsilon)}\left[\prod_{i=1}^{K}\left(1 - e^{-A_i(\varepsilon)}\right) - 1\right]\end{aligned}$$

The result follows by taking an additional expectation. □

Remark 3. (*Catastrophic Market Failure with Identical Distributions*)

Let $A_i(\varepsilon)$ for $i = 1, 2, \ldots, K$ be the same for all banks and independent of K, say $A(\varepsilon)$, implying that the G-SIBs' default times are identically distributed. Then, $\lim_{K\to\infty} \prod_{i=1}^{K} \left(1 - e^{-A(\varepsilon)}\right) = 0$, which implies that

$$\lim_{K\to\infty} \mathbb{P}\left(\tau_{(K)} < \varepsilon\right) = 1 - \mathbb{E}\left(e^{-A_0(\varepsilon)}\right) = \mathbb{P}\left(\eta_0 < \varepsilon\right). \tag{3.14}$$

As the number of G-SIBs approaches infinity, the probability of a catastrophic market failure is equal to the probability of a market-wide stress event occurring. This makes sense since, except for the market-wide stress event, the probability of idiosyncratic defaults are independent across G-SIBs.

Corollary 3.5 *(Catastrophic Market Failure with Constant Default Intensities)*

If $\alpha_i(X_t) = \alpha_i$ for all $t \geq 0$, all $i = 1, 2, \ldots, K$, and where $\alpha_i \in \mathbb{R}$, then:

$$\mathbb{P}\left(\tau_{(K)} \leq \varepsilon\right) = 1 + e^{-\alpha_0 \varepsilon}\left(\prod_{i=1}^{K}\left(1 - e^{\alpha_i \varepsilon}\right) - 1\right) \tag{3.15}$$

Remark 4. (*Destructive Competition*)

With destructive competition (see Example 1) and assuming that for all $i = 1, 2, \ldots, K$, $A_i(t, K) = \ln(K) + B(t)$, as the number of G-SIBs approach infinity,

$$\lim_{K\to\infty} \mathbb{P}\left(\tau_{(K)} < \varepsilon\right) = 1 - \mathbb{E}\left[e^{-A_0(\varepsilon)}\left(1 - e^{-e^{-B(\varepsilon)}}\right)\right]. \tag{3.16}$$

If $\alpha_i(X_t, K) = \ln(K) + b(X_t)$ and $\varepsilon < 1$, then

$$\lim_{K\to\infty} \mathbb{P}\left(\tau_{(K)} < \varepsilon\right) = \mathbb{P}\left(\eta_0 < \varepsilon\right). \tag{3.17}$$

As documented, under destructive competition, the probability of a catastrophic market failure increases to the indicated limit as the number of GSIBs approaches infinity.

Proof. (Destructive Competition)

Recall that

$$\mathbb{P}\left(\tau_{(K)} \le \varepsilon\right) = 1 - \mathbb{E}\left[e^{-A_0(\varepsilon)}\left(1 - \prod_{i=1}^{K}\left(1 - e^{-A_i(\varepsilon,K)}\right)\right)\right] \tag{3.18}$$

Given $A_i(\varepsilon, K)$ are equal to $\ln(K) + B(t)$ we get that

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\tau_{(K)} \le \varepsilon\right) &= 1 - e^{-A_0(\varepsilon)}\left(1 - \left(1 - e^{-A(\varepsilon,K)}\right)^K\right) \\
&= 1 - e^{-A_0(\varepsilon)}\left(1 - \left(1 - e^{-\ln(K) - B(\varepsilon)}\right)^K\right) \\
&= 1 - e^{-A_0(\varepsilon)}\left(1 - \left(1 - \frac{e^{-B(\varepsilon)}}{K}\right)^K\right) \\
&\overset{K\to\infty}{\longrightarrow} 1 - e^{-A_0(\varepsilon)}\left(1 - e^{-e^{-B(\varepsilon)}}\right)
\end{aligned} \tag{3.19}$$

If $\alpha_i(X_t, K) = \ln(K) + b(X_t)$, then

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\tau_{(K)} \le \varepsilon\right) &= 1 - e^{-A_0(\varepsilon)}\left(1 - \left(1 - e^{-A(\varepsilon,K)}\right)^K\right) \\
&= 1 - e^{-A_0(\varepsilon)}\left(1 - \left(1 - e^{-\varepsilon \ln(K) - B(\varepsilon)}\right)^K\right) \\
&= 1 - e^{-A_0(\varepsilon)}\left(1 - \left(1 - \frac{e^{-B(\varepsilon)}}{K^{\varepsilon}}\right)^K\right) \\
&\overset{K\to\infty}{\longrightarrow} 1 - e^{-A_0(\varepsilon)}
\end{aligned} \tag{3.20}$$

The results follow by taking an expectation and interchanging limits, which we can do because $\tilde{\mathbb{P}}(\cdot)$ is bounded by 1. □

3.2.4 Bounds on the Market Failure Probability

For some empirical applications, it may prove useful to obtain bounds on the market failure probability as in the subsequent theorem.

Theorem 3.4 *(Bounds on the Market Failure Probability)*

$$\mathbb{P}\left(\left|\tau_i-\tau_j\right|<\varepsilon \, for\ some\ (i,j)\right) \leq \binom{K}{2} - \sum_{i=1}^{K}\sum_{j\neq i}^{K} \mathbb{E}\left[\int_0^{\infty} \alpha_i(X_x) e^{-A_i(x)-A_j(x+\varepsilon)-A_0(x+\varepsilon)} dx\right] \quad (3.21)$$

$$\mathbb{P}\left(\left|\tau_i-\tau_j\right|<\varepsilon \, for\ some\ (i,j)\right) \geq 1-\mathbb{E}\left[e^{-A_0((K-1)\varepsilon)} \sum_{j\in P} \exp\left(-\sum_{i=1}^{K-1} A_{j_{i+1}}(i\varepsilon)\right)\right] \quad (3.22)$$

where the sum is taken over all possible permutations P of K, i.e.

$$j=(j_1=\sigma(1), j_2=\sigma(2), \ldots, j_K=\sigma(K))$$

Proof. As in the proof of Theorem 3.2, let $f_k(x)$ to be the density of η_k given $\mathbb{F}_X$, which is $f_k(x)=\alpha_k(X_x)\exp\left[-A_k(x)\right]$. Note that by construction $\eta_0, \eta_1, \eta_2, \ldots, \eta_K$ are mutually independent. Moreover, define $\tilde{\mathbb{P}}(\cdot)=\mathbb{P}\left(\cdot|\mathbb{F}_X\right)$. Hence, $\tilde{\mathbb{P}}(\eta_j>t)=\exp\left(-A_j(t)\right)$.

For the first inequality, the upper bound, note the following:

$$\begin{aligned}&\tilde{\mathbb{P}}\left(\left|\tau_i-\tau_j\right|<\varepsilon \ for\ some\ (i,j)\in(1,...,K)\times(1,...,K), i\neq j\right)\\ &=\tilde{\mathbb{P}}\left(\bigcup_{i=1}^{K}\bigcup_{j>i}|\tau_i-\tau_j|\leq\varepsilon\right)\leq\sum_{i=1}^{K}\sum_{j>i}\tilde{\mathbb{P}}\left(|\tau_i-\tau_j|\leq\varepsilon\right)\end{aligned} \quad (\star)$$

For any i, j, we have:

$$\tilde{\mathbb{P}}\left(|\tau_i-\tau_j|\leq\varepsilon\right)=1-\tilde{\mathbb{P}}\left(\tau_i-\tau_j>\varepsilon, \tau_i>\tau_j\right)-\tilde{\mathbb{P}}\left(\tau_j-\tau_i>\varepsilon, \tau_j>\tau_i\right)$$

Let us focus on $\tilde{\mathbb{P}}\left(\tau_i-\tau_j>\varepsilon, \tau_i>\tau_j\right)$ and note that in the next equality we do not consider the event $\{\eta_i>\eta_j>\eta_0\}$ because it implies $\tau_i=\tau_j$ and so it is impossible to have $\tau_i-\tau_j>\varepsilon$

$$\begin{aligned}\tilde{\mathbb{P}}\left(\tau_i-\tau_j>\varepsilon, \tau_i>\tau_j\right)&=\tilde{\mathbb{P}}\left(\tau_i-\tau_j>\varepsilon, \eta_0>\eta_i>\eta_j\right)+\tilde{\mathbb{P}}\left(\tau_i-\tau_j>\varepsilon, \eta_i>\eta_0>\eta_j\right)\\ &=\int_0^{\infty}\int_{y+\varepsilon}^{\infty}\int_x^{\infty} f_0(z)f_i(x)f_j(y)\,dz\,dx\,dy+\int_0^{\infty}\int_{y+\varepsilon}^{\infty}\int_z^{\infty} f_i(x)f_0(z)f_j(y)\,dx\,dz\,dy\end{aligned}$$

$$= \int_0^\infty f_j(y) \int_{y+\varepsilon}^\infty \alpha_i(X_x) e^{-A_i(x)-A_0(x)}\, dx\, dy + \int_0^\infty f_j(y) \int_{y+\varepsilon}^\infty \alpha_0(X_z) e^{-A_i(z)-A_0(z)}\, dz\, dy$$

$$= \int_0^\infty f_j(y) \int_{y+\varepsilon}^\infty \left[\alpha_i(X_x) + \alpha_0(X_x)\right] e^{-A_i(x)-A_0(x)}\, dx\, dy = \int_0^\infty \alpha_j(y) e^{-A_j(y)-A_i(y+\varepsilon)-A_0(y+\varepsilon)}\, dy$$

$\tilde{\mathbb{P}}\left(\tau_i - \tau_j > \varepsilon, \tau_j > \tau_i\right)$ follows by an analogous argument:

$$\tilde{\mathbb{P}}\left(\tau_i - \tau_j > \varepsilon, \tau_j > \tau_i\right) = \int_0^\infty \alpha_i(y) e^{-A_i(y)-A_j(y+\varepsilon)-A_0(y+\varepsilon)}\, dy$$

Joining these 2 probabilities, we get:

$$\tilde{\mathbb{P}}\left(|\tau_i - \tau_j| \le \varepsilon\right) = 1 - \int_0^\infty \alpha_j(y) e^{-A_j(y)-A_i(y+\varepsilon)-A_0(y+\varepsilon)} dy - \int_0^\infty \alpha_i(y) e^{-A_i(y)-A_j(y+\varepsilon)-A_0(y+\varepsilon)} dy$$

Using this along with $(\star)$, we get:

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon\ for\ some\ (i,j) \in (1,...,K) \times (1,...,K), i \neq j\right) \\
&\quad \le \sum_{i=1}^K \sum_{j>i} \left[1 - \int_0^\infty \alpha_j(y) e^{-A_j(y)-A_i(y+\varepsilon)-A_0(y+\varepsilon)} dy - \int_0^\infty \alpha_i(y) e^{-A_i(y)-A_j(y+\varepsilon)-A_0(y+\varepsilon)} dy\right] \\
&\qquad = \binom{K}{2} - \sum_{i=1}^K \sum_{j \neq i}^K \int_0^\infty \alpha_i(y) e^{-A_i(y)-A_j(y+\varepsilon)-A_0(y+\varepsilon)} dy
\end{aligned}$$

Finally, the desired upper bound follows by taking an additional expectation.

For the lower bound, it suffices to bound $\mathbb{P}\left(|\tau_i - \tau_j| > \varepsilon \text{ for all } (i,j)\right)$ from above. This is because:

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right) = 1 - \mathbb{P}\left(|\tau_i - \tau_j| > \varepsilon \text{ for all } (i,j)\right)$$

From Theorem 3.2, we know that,

$$\tilde{\mathbb{P}}\left(|\tau_i-\tau_j|>\varepsilon \text{ for all } (i,j)\right)=\sum_{j\in P}\int_0^\infty f_{j_1}(x_1)\int_{x_1+\varepsilon}^\infty f_{j_2}(x_2)\int_{x_2+\varepsilon}^\infty f_{j_3}(x_3)\cdots\int_{x_{K-3}+\varepsilon}^\infty f_{j_{K-2}}(x_{K-2})$$
$$\int_{x_{K-2}+\varepsilon}^\infty f_{j_{K-1}}(x_{K-1})\exp\left[-A_{j_K}(x_{K-1}+\varepsilon)-A_0(x_{K-1}+\varepsilon)\right]\,dx_{K-1}\,dx_{K-2}\ldots\,dx_3\,dx_2\,dx_1$$

As $x_{K-1}\geq x_{K-2}+\varepsilon$ in the innermost integral in each one of the terms of the previous sum, then $e^{-\left(A_{j_K}+A_0\right)(x_{K-1}+\varepsilon)}\leq e^{-\left(A_{j_K}+A_0\right)(x_{K-2}+2\varepsilon)}$ and hence:

$$\int_0^\infty f_{j_1}(x_1)\int_{x_1+\varepsilon}^\infty f_{j_2}(x_2)\cdots\int_{x_{K-2}+\varepsilon}^\infty f_{j_{K-1}}(x_{K-1})e^{-\left(A_{j_K}+A_0\right)(x_{K-1}+\varepsilon)}dx_{K-1}\ldots\,dx_1$$
$$\leq\int_0^\infty f_{j_1}(x_1)\int_{x_1+\varepsilon}^\infty f_{j_2}(x_2)\ldots e^{-\left(A_{j_K}+A_0\right)(x_{K-2}+2\varepsilon)}\int_{x_{K-2}+\varepsilon}^\infty f_{j_{K-1}}(x_{K-1})\,dx_{K-1}\ldots\,dx_2\,dx_1$$
$$=\int_0^\infty f_{j_1}(x_1)\int_{x_1+\varepsilon}^\infty f_{j_2}(x_2)\cdots\int_{x_{K-3}+\varepsilon}^\infty f_{j_{K-2}}(x_{K-2})e^{-\left(A_{j_K}+A_0\right)(x_{K-2}+2\varepsilon)-A_{j_{K-1}}(x_{K-2}+\varepsilon)}\,dx_{K-2}\ldots\,dx_1$$

As $x_{K-2}\geq x_{K-3}+\varepsilon$ in the innermost integral, we have

$$e^{-\left(A_{j_K}+A_0\right)(x_{K-2}+2\varepsilon)-A_{j_{K-1}}(x_{K-2}+\varepsilon)}\leq e^{-\left(A_{j_K}+A_0\right)(x_{K-3}+3\varepsilon)-A_{j_{K-1}}(x_{K-3}+2\varepsilon)}$$

Then, we solve the integral with respect to x_{K-2}. We keep going in a similar fashion to conclude that:

$$\int_0^\infty f_{j_1}(x_1)\int_{x_1+\varepsilon}^\infty f_{j_2}(x_2)\cdots\int_{x_{K-2}+\varepsilon}^\infty f_{j_{K-1}}(x_{K-1})e^{-\left(A_{j_K}+A_0\right)(x_{K-1}+\varepsilon)}\,dx_{K-1}\ldots\,dx_2\,dx_1$$
$$\leq\exp\left[-\left(A_{j_K}+A_0\right)((K-1)\varepsilon)-A_{j_{K-1}}((K-2)\varepsilon)-\cdots-A_{j_3}(2\varepsilon)-A_{j_2}(\varepsilon)\right]$$
$$=\exp\left[-A_0\left((K-1)\varepsilon\right)-\sum_{i=1}^{K-1}A_{j_{i+1}}\left(i\varepsilon\right)\right]$$

Then we take the sum over all possible permutations P of K to get:

$$\tilde{\mathbb{P}}\left(|\tau_i - \tau_j| > \varepsilon \text{ for all } (i,j)\right) \leq \sum_{j \in P} \exp\left[-A_0\left((K-1)\varepsilon\right) - \sum_{i=1}^{K-1} A_{j_{i+1}}(i\varepsilon)\right]$$

Recall that the sub index j in the sum stands for any of the possible permutations. This is $j = (j_1 = \sigma(1), j_2 = \sigma(2), \ldots, j_K = \sigma(K))$ Using the complement of the previous probability, we have

$$\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right) \geq 1 - e^{-A_0((K-1)\varepsilon)} \sum_{j \in P} \exp\left(-\sum_{i=1}^{K-1} A_{j_{i+1}}(i\varepsilon)\right)$$

Finally, the desired lower bound follows by taking an additional expectation. □

Corollary 3.6 *(Constant Default Intensities) If $\alpha_i(X_t) = \alpha_i$ for all $t \geq 0$, all $i = 1, 2, \ldots, K$, and where $\alpha_i \in \mathbb{R}$, then:*

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \textit{ for some } (i,j)\right) \leq \binom{K}{2} - e^{-\alpha_o \varepsilon} \sum_{i=1}^{K} \alpha_i \sum_{j \neq i}^{K} e^{\alpha_j \varepsilon}\left(\frac{1}{\alpha_i + \alpha_j + \alpha_0}\right) \tag{3.23}$$

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \textit{ for some } (i,j)\right) \geq 1 - e^{\alpha_0 \varepsilon (K-1)} \sum_{j \in P} \exp\left(-\varepsilon \sum_{i=1}^{K-1} i \alpha_{j_{i+1}}\right) \tag{3.24}$$

3.3 Comparative Statics

This section explores how the market failure probability changes when the initial conditions in the economy change. We consider both changing the number of G-SIBs and the initial state of the economy.

3.3.1 The Number of G-SIBs

For regulatory purposes, it is important to understand how the probability of a market failure changes with the inclusion of another G-SIB. This relates to macro-prudential policy regarding whether the number of banks in the economy being "too large to fail" or designated as G-SIBs should be reduced (by breaking them up into smaller institutions) to decrease the market failure probability (see Berndt et al. [10], Schich and Toader [57] for issues related to G-SIBs designation).

Computing the market failure probability for K banks versus $K+1$ banks in Theorem 3.2 and taking the difference yields the marginal impact of adding another G-SIB to the economy. It is easy to show that this probability increases as more G-SIBs enter the market.

Theorem 3.5 *(Increasing the Number of G-SIBs)*

$$\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \; for \; some \; (i,j) \in (1,\ldots,K) \times (1,\ldots,K), i \neq j\right) < \mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \; for \; some \; (i,j) \in (1,\ldots,K+1) \times (1,\ldots,K+1), i \neq j\right) \quad (3.25)$$

provided that $\alpha_i(\cdot)$ for $i = 0, 1, 2, \ldots, K$ and the underlying process, i.e., $(X_t)_{t\geq 0}$ remain fixed.

Proof. Let

$$A := \{\left|\tau_i - \tau_j\right| < \varepsilon \; for \; some \; (i,j) \in (1,...,K)^2, i \neq j\} \quad (3.26)$$

$$B := \{\left|\tau_i - \tau_j\right| < \varepsilon \; for \; some \; (i,j) \in (1,\ldots,K,K+1)^2, i \neq j\} \quad (3.27)$$

Note that the event B can be decomposed in the following way:

$$B = A \cup \{\left|\tau_{K+1} - \tau_j\right| < \varepsilon \; for \; some \; j \in (1,...,K)\} \quad (3.28)$$

It is clear that $A \not\subseteq B$ and hence the result follows .

□

This result suggests that as the number of G-SIBs goes to infinity, the market failure probability converges to one, as the following theorem documents.

Theorem 3.6 *(Limit as $K \to \infty$)*

$$\lim_{K\to\infty} \mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \ for\ some\ (i,j) \in (1,...,K)\times(1,...,K), i \neq j\right) = 1. \tag{3.29}$$

Proof. Similar to the proof of Theorem 3, for $k = 1, 2, \ldots, K$, let $\tau_{(k)}$ and $\eta_{(k)}$ be the k^{th} order statistic of $(\tau_1, \tau_2, \ldots, \tau_K)$ and $(\eta_1, \eta_2, \ldots, \eta_K)$ respectively. For example, $\tau_{(1)} = \min(\tau_1, \ldots, \tau_K)$, $\eta_{(1)} = \min(\eta_1, \ldots, \eta_K)$, $\tau_{(K)} = \max(\tau_1, \ldots, \tau_K)$ and $\eta_{(K)} = \max(\eta_1, \eta_2, \ldots, \eta_K)$.

Define $\tilde{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|\mathbb{F}_X)$. Hence, $\tilde{\mathbb{P}}(\eta_j > t) = \exp\left(-A_j(t)\right)$ and note the following equality of events:

$$\left\{\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j),\ i \neq j\right\}^C = \left\{\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j\right\}$$

Then, we get:

$$\begin{aligned}\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j\right) &= \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_0 \geq \eta_{(K)}\right) \\ &\quad + \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_{(k-1)} \leq \eta_0 < \eta_{(K)}\right) \\ &\quad + \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_0 < \eta_{(K-1)}\right)\end{aligned} \tag{3.30}$$

If $\eta_0 < \eta_{(K-1)}$, then at least there exists one pair of (i,j), $i \neq j$ such that $\tau_i = \tau_j = \eta_0$ and hence, $\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_0 < \eta_{(K-1)}\right) = 0$

If $\left|\tau_i - \tau_j\right| \geq \varepsilon$ for all$(i,j), i \neq j$, then $\tau_{(2)} \geq \tau_{(1)}+\varepsilon, \tau_{(3)} \geq \tau_{(2)}+\varepsilon, \ldots, \tau_{(K)} \geq \tau_{(K-1)}+\varepsilon$, which implies $\tau_{(K)} \geq \tau_{(1)} + (K-1)\varepsilon$. Moreover, on the event $\{\eta_0 \geq \eta_{(K)}\}$, we have that $\tau_{(K)} = \eta_{(K)}$ and $\tau_{(1)} = \eta_{(1)}$. Hence,

$$\begin{aligned}\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_0 \geq \eta_{(K)}\right) &\leq \tilde{\mathbb{P}}\left(\tau_{(K)} \geq \tau_{(1)} + (K-1)\varepsilon, \eta_0 \geq \eta_{(K)}\right) \\ &= \tilde{\mathbb{P}}\left(\eta_{(K)} \geq \eta_{(1)} + (K-1)\varepsilon, \eta_0 \geq \eta_{(K)}\right) \\ &= \tilde{\mathbb{P}}\left(\eta_{(1)} + (K-1)\varepsilon \leq \eta_{(K)} \leq \eta_0\right) \\ &\leq \tilde{\mathbb{P}}\left(\eta_{(1)} + (K-1)\varepsilon \leq \eta_0\right)\end{aligned} \tag{3.31}$$

$$
\begin{aligned}
\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_0 \geq \eta_{(K)}\right) &\leq \tilde{\mathbb{P}}\left(\tau_{(K)} \geq \tau_{(1)} + (K-1)\,\varepsilon, \eta_0 \geq \eta_{(K)}\right) \\
&= \tilde{\mathbb{P}}\left(\eta_{(K)} \geq \eta_{(1)} + (K-1)\,\varepsilon, \eta_0 \geq \eta_{(K)}\right) \\
&= \tilde{\mathbb{P}}\left(\eta_{(1)} + (K-1)\,\varepsilon \leq \eta_{(K)} \leq \eta_0\right) \\
&\leq \tilde{\mathbb{P}}\left(\eta_{(1)} + (K-1)\,\varepsilon \leq \eta_0\right) \qquad (3.32)
\end{aligned}
$$

When taking the limit $K \to \infty$, as η_i for $i = 0, 1, \ldots, K$ under $\tilde{\mathbb{P}}$, is a.s. finite, we get:

$$
\lim_{K\to\infty} \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_0 \geq \eta_{(K)}\right) \leq \lim_{K\to\infty} \tilde{\mathbb{P}}\left(\eta_{(1)} + (K-1)\,\varepsilon \leq \eta_0\right) = 0 \quad (3.33)
$$

Now, on the event $\{\eta_{(k-1)} \leq \eta_0 < \eta_{(K)}\}$, we have that $\tau_{(K)} = \eta_0$ and $\tau_{(1)} = \eta_{(1)}$. By a similar reasoning as above,

$$
\begin{aligned}
&\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j, \eta_{(k-1)} \leq \eta_0 < \eta_{(K)}\right) \\
&\qquad \leq \tilde{\mathbb{P}}\left(\tau_{(K)} \geq \tau_{(1)} + (K-1)\,\varepsilon, \eta_{(k-1)} \leq \eta_0 < \eta_{(K)}\right) \\
&\qquad = \tilde{\mathbb{P}}\left(\eta_0 \geq \eta_{(1)} + (K-1)\,\varepsilon, \eta_{(k-1)} \leq \eta_0 < \eta_{(K)}\right) \\
&\qquad \leq \tilde{\mathbb{P}}\left(\eta_{(1)} + (K-1)\,\varepsilon \leq \eta_0\right) \stackrel{K\to\infty}{\longrightarrow} 0 \qquad (3.34)
\end{aligned}
$$

The result follows after taking an expectation. The interchange in the expectation and limit as $K \to \infty$ is justified as $\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j),\ i \neq j\right)$ is bounded by 1. □

Consequently, the number of G-SIBs in the economy needs to be restricted by regulators to ensure that the probability of a market failure is at an acceptable level. If the probability of a market failure as implied by the number of existing G-SIBs is too high, then Theorem 3.2 enables the regulators to select the number of G-SIBs such that the market failure probability is below some given threshold. This implies, of course, that the excess G-SIBs must be broken-up into smaller banks.

Instead of breaking-up the G-SIBs, regulators can alternatively control the probability of a market failure by requiring the existing G-SIBs to change their asset/liability structures. This tool is discussed in the next subsection.

3.3.2 Changing the State of the Economy and Banks' Balance Sheets

This section explores the impact of changing the state of economy vector X_r on the market failure probability. The idea, of course, is that some of the inputs are under the control of the regulators, e.g. required capital of a G-SIB. We investigate the impact of changes in the initial conditions on the market failure probability. To facilitate the exposition, let $X_r = (x_1(r), ..., x_d(r))$, so that $\alpha_i(X_r) = \alpha_i(x_1(r), ..., x_d(r))$.

We redefine the default time for the i^{th} G-SIB due to *idiosyncratic events* and the first time that a market-wide stress event occurs in the following way:

$$\eta_i := \inf \{s : \alpha_i (X_0) + A_i(s) \geq Z_i\} \text{ for } i = 1, \dots, K$$
$$\eta_0 := \inf \{s : \alpha_0 (X_0) + A_0(s) \geq Z_0\} .$$

Just as before, the default time of the i^{th} G-SIB is

$$\tau_i = \min (\eta_0, \eta_i) .$$

Then, it is easy to check that

$$\mathbb{P} \left(\eta_i > t | (X_u)_{0 \leq u \leq t}\right) = \exp (-\alpha_i (X_0) - A_i(t))$$
$$\mathbb{P} \left(\tau_i > t | (X_u)_{0 \leq u \leq t}\right) = \exp (-\alpha_i (X_0) - \alpha_0 (X_0) - A_i(t) - A_0(t)) .$$

To ensure that the probability distributions of η_i and τ_i are correctly defined, we assign a positive probability to the event $\{\eta_i = 0\}$ such that

$$\mathbb{P} (\eta_i = 0 | X_0) = 1 - \exp (-\alpha_i (X_0)) .$$

This implies that

$$\mathbb{P}(\tau_i = 0|X_0) = 1 - \exp(-\alpha_i(X_0) - \alpha_0(X_0)).$$

The interpretation is that there is a positive probability of an "instantaneous" default at $t = 0$. Under these modifications, we have the following result.

Theorem 3.7 *(Comparative Statics)*

$$\frac{\partial}{\partial x_\ell(0)}\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \textit{ for some } (i,j) \in (1,...,K)\times(1,\ldots,K), i \neq j\right)$$
$$= \mathbb{E}\left[\left(\sum_{i=0}^{K}\frac{\partial\alpha_i(X_0)}{\partial x_\ell(0)}\right)\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \textit{ for all } (i,j), i \neq j\right)\right]$$
$$= \mathbb{E}\Bigg[\left(\sum_{i=0}^{K}\frac{\partial\alpha_i(X_0)}{\partial x_\ell(0)}\exp\left(-\sum_{i=0}^{K}\alpha_i(X_0)\right)\right)\sum_{j\in P}\int_0^\infty \hat{f}_{j_1}(x_1)\int_{x_1+\varepsilon}^\infty \hat{f}_{j_2}(x_2)\int_{x_2+\varepsilon}^\infty \hat{f}_{j_3}(x_3)\ldots$$
$$\cdots\int_{x_{K-2}+\varepsilon}^\infty \hat{f}_{j_{K-1}}(x_{K-1})\exp\left[-A_{j_K}(x_{K-1}+\varepsilon) - A_0(x_{K-1}+\varepsilon)\right]\,dx_{K-1}\ldots\,dx_3\,dx_2\,dx_1\Bigg] \quad (3.35)$$

where $\tilde{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|\mathbb{F}_X)$ *and the sum in the second equality is taken over all possible permutations P of K, i.e.,* $j = (j_1 = \sigma(1), j_2 = \sigma(2), \ldots, j_K = \sigma(K))$ *and* $\hat{f}_{j_k}(x) = \alpha_{j_k}(X_x)\exp\left[-A_{j_k}(x)\right]$.

Proof. First note that

$$\frac{\partial}{\partial x_\ell(0)}\mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right) = \frac{\partial}{\partial x_\ell(0)}\mathbb{E}\left[\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right)\right]$$

As $\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right) \leq 1$, we can interchange expectation and derivative and so, it suffices to find

$$\frac{\partial}{\partial x_\ell(0)}\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right)$$

and then take an expectation.

Now, with the change of definition of η_i and η_0, by a similar fashion as in Theorem 3.2, we get that

$$\tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right) = 1 - \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j), i \neq j\right) =$$

$$1 - \exp\left(-\sum_{i=0}^{K} \alpha_i(X_0)\right) \sum_{j \in P} \int_0^\infty \hat{f}_{j_1}(x_1) \int_{x_1+\varepsilon}^\infty \hat{f}_{j_2}(x_2) \int_{x_2+\varepsilon}^\infty \hat{f}_{j_3}(x_3) \ldots$$

$$\cdots \int_{x_{K-2}+\varepsilon}^\infty \hat{f}_{j_{K-1}}(x_{K-1}) \exp\left[-A_{j_K}(x_{K-1}+\varepsilon) - A_0(x_{K-1}+\varepsilon)\right] dx_{K-1} \ldots dx_1$$

where the sum in the second equality is taken over all possible permutations P of K. Hence, $j = (j_1 = \sigma(1), j_2 = \sigma(2), \ldots, j_K = \sigma(K))$ and $\hat{f}_{j_k}(x) = \alpha_{j_k}(X_x) \exp\left[-A_{j_k}(x)\right]$.

Differentiating the previous equation with respect to $x_\ell(0)$, we obtain

$$\frac{\partial}{\partial x_\ell(0)} \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j)\right) = -\frac{\partial}{\partial x_\ell(0)} \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j), i \neq j\right)$$

$$= \left(\sum_{i=0}^{K} \frac{\partial \alpha_i(X_0)}{\partial x_\ell(0)}\right) \tilde{\mathbb{P}}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j), i \neq j\right)$$

$$= \left(\sum_{i=0}^{K} \frac{\partial \alpha_i(X_0)}{\partial x_\ell(0)} \exp\left(-\sum_{i=0}^{K} \alpha_i(X_0)\right)\right) \sum_{j \in P} \int_0^\infty \hat{f}_{j_1}(x_1) \int_{x_1+\varepsilon}^\infty \hat{f}_{j_2}(x_2) \ldots$$

$$\cdots \int_{x_{K-2}+\varepsilon}^\infty \hat{f}_{j_{K-1}}(x_{K-1}) \exp\left[-A_{j_K}(x_{K-1}+\varepsilon) - A_0(x_{K-1}+\varepsilon)\right] dx_{K-1} \ldots dx_1$$

Given estimates of the relevant intensities, these partial derivatives are easily computed and they provide the information that regulators can use to determine the impact of their regulatory restrictions on the probability of a market failure.

□

Remark 5. (*Linear Approximation*)

For some simpler calculations, if $\alpha_i(X_r) = \sum_{j=1}^{d} \beta_{ij} x_j(r)$ for $\beta_{ij} \in \mathbb{R}$, then

$$\frac{\partial}{\partial x_\ell(0)} \mathbb{P}\left(\left|\tau_i - \tau_j\right| < \varepsilon \text{ for some } (i,j) \in (1,...,K) \times (1,\ldots,K), i \neq j\right) =$$

$$\left(\sum_{i=0}^{K} \beta_{i\ell}\right) \mathbb{P}\left(\left|\tau_i - \tau_j\right| \geq \varepsilon \text{ for all } (i,j), i \neq j\right) \quad (3.36)$$

Chapter 4: Optimal Group Size in Microlending

Microfinance analyzes the lending mechanisms for people without access to traditional credit systems because of their low income, lack of collateral, or credit history. There are many lending mechanisms studied in the literature, but we will focus on group lending, which was introduced by Muhammad Yunus of the Grameen bank in Bangledesh. He won the Nobel Peace Prize (2006) for his efforts. In this mechanism, a loan is made to all members of the group for a fixed period of time (often less than a year).

One of the crucial components of group lending is known as contingent renewal, which is a penalty that eliminates or reduces access to future loans to all members in a group if any of them defaults. This is to say, the default of at least one individual provokes the default of the group. Note that by default of a member we mean that she stops paying her share of the loan. See Diener et al. [28], Diener and Mauk [29], and Diener and Santos [30], for a study of the consequences of implementing various penalties for default.

The group lending mechanism is thought to be useful because it induces: (i) peer selection of members in a group since they are better informed than the lender about other potential borrowers, (ii) peer pressure to help enforce payments by other members in the group, and (iii) peer monitoring between the group members, but particularly by a leader, to ensure continued performance on the loan. The structure of having a group leader is known in the literature as having an intermediary. From the lender's point of view, it involves a delegation of the task of monitoring the loan. These ideas have, of course, been well studied in the Economics literature, and we rely on the seminal work of Philip Bond ([13]). In his paper, Bond shows that the joint liability of intermediary borrowers arises naturally in models of a financial intermediary as a delegated moderator. Bond shows that intermediation with joint liability is Pareto superior to intermediation without joint liability. As such, Bond builds on prior work of Morduch ([52]) and Krasa and Villamil ([46]).

The existing academic literature primarily focuses on understanding why the group lending mechanism is successful in reducing defaults. Both static and dynamic models have been analyzed (see Stiglitz [59], Varian [63], Conlin [24], Morduch [52], Chowdhury [22], [23], and Tedeschi [62]). A related and somewhat unexplored issue is to determine an optimal group size. We define optimal size as the one that maximizes the probability of no default of the group. While cultural and other non-economic factors influence default, this paper focuses solely on group size.

The problem of an optimal group size has been analyzed by economists in the past (see Armendariz and Morduch [7], Giné et al. [36], and Ahlin [3], [4]). Most of them take an intuitive, experimental and, often verbal approach. In contrast, ours is more probabilistic in nature, and thereby quantifiable. A first approach to what we present here is the article of Jarrow and Protter [43]. We also note that the problem of group size has been tackled using tools from Game Theory (see Rezaei et al. [56]).

An outline for this chapter is as follows. Section 4.1 presents the model and our main theorem, while Section 4.2 outlines an interpretation for it. Section 4.3 shows an example and discusses its intuitiveness, Section 4.4 analyzes the example, and Section 4.5 concludes.

4.1 The Model

Caveats: We begin with a few caveats. Different scenarios of Microlending have been considered in the academic literature. For example, there is the issue of the size of the loan affecting the group in group lending, and in particular its size (with larger loans leading to somewhat larger groups, see Rezaei et al. [56]). In this paper, the loan size is fixed, and we will not consider its influence on the group size. Some researchers take as their point of view the maximization of the profits of the lending institution (see, e.g., Bourjade and Schindele [14]). This runs counter to the spirit embraced in Yunus [66], and again, while a valid consideration, it is not our concern in this paper. We are concerned only with minimizing the possibility of default on the loan; admittedly this is related to maximizing the profits of the lender, along with the interest rates charged (see Jarrow and Protter [43]). Finally, we mention that we do not discuss the transaction costs of banks, and what the effect of group lending is on them. We also implicitly assume that the members of a given group form a fairly homogeneous collection of people (see Devereux and Fish [27], later echoed in Bourjade and Schindele [14].) This homogeneity assumption is reflected in our assumption of identical distributions, within a group of a given size, allowing the distributions to change with the group size.

We now introduce the notation for our model. Let N_i be the event of no default of member i in a group of size k $(k \in \mathbb{Z}^+,\ k \geq 2)$, $\mathcal{N}_k$ be the event that the group of size k does not default and, $\varphi(k) := 1 - \mathbb{P}_k\left(N_1\right)$, i.e., the probability of default of member 1 in a group of size k. Recall that, as explained in the introduction, the group lending mechanism implies that if at least one member of the group defaults, then the whole group defaults. This is what we call *default of a group*.

We make the following assumptions:

1. For fixed size k, the group members are independent and identically distributed.

2. The probability of no default of one person depends on the size of the group. We make this explicit by writing: $\mathbb{P}_k\left(N_1\right)$.

3. $\mathbb{P}_k(N_1) > 0$. Otherwise the problem is trivial as the members will default for sure.

We are interested in finding an optimal group size, that is, finding the number of people k^* that maximizes the probability of no default of the group. Using our assumptions, along with our definition of default of a group, this translates into maximizing:

$$\mathbb{P}\left(\mathcal{N}_k\right) = \mathbb{P}\left(\bigcap_{i=1}^{k} N_i\right) = \left[\mathbb{P}_k\left(N_1\right)\right]^k = (1 - \varphi(k))^k \tag{4.1}$$

For a moment, suppose that $\varphi(\cdot)$ is constant in k, hence as $\varphi(\cdot) < 1$, $(1 - \varphi(\cdot))^k$ decreases as k increases. So, in order to have a maxima in (4.1), it makes sense to require that $1 - \varphi(k)$ increases with k, which means that $\varphi(k)$ needs to decrease with k. The question is then, at what speed? This motivates the following theorem.

Theorem 4.1 *Let* $\varphi(x) = \frac{1}{f(x)}$, *for all* $x \in \mathbb{R}^+$ *If:*

1. $f(x) > 1$ *for all* $x \geq 2$

2. $f(x) \in C^2$

3. $f'(x) > 0$ *for all* $x \geq 2$

4. There exist $a, b \in \mathbb{R}$ $(a < b,\ a \geq 2)$ *such that either:*

   (a) $f(a) - af'(a) = \frac{1}{2}$ *and* $f(b) - bf'(b) = 1$

   and

   (b) $f''(x) < 0$ *for all* $x \in (a, b)$

   or

   (c) $f(a) - af'(a) = 1$ *and* $f(b) - bf'(b) = \frac{1}{2}$

   and

   (d) $f''(x) > 0$ *for all* $x \in (a, b)$

Then $(1 - \varphi(x))^x$ *has a unique maximizer* x^* *in* (a, b)*. Moreover, if* a*,* b *are unique, then* x^* *is the unique maximizer.*

Proof. Let $\mathcal{S}(x) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right)\left(\frac{1}{f(x)}\right)^n$ and note the following:

- $\mathcal{S}(x)$ is a decreasing function in x.
- $\mathcal{S}(x) \in \left(\frac{1}{2}, 1\right)$, for all $x \in \mathbb{R}$ because:

$$\frac{1}{2} < \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right)\left(\frac{1}{f(x)}\right)^n = \mathcal{S}(x) < \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right) = 1$$

Set $h(x) := f(x) - xf'(x)$ and note that in (a, b), $h(x)$ is a monotone function because of condition (4b) or (4d). More explicitly:

$$h'(x) = f'(x) - [f'(x) + xf''(x)] = -xf''(x) > 0 \text{ (or } < 0) \qquad \text{for all } x \in (a, b)$$

Moreover, the monotonicity of $h(x)$ and condition (4a) or (4c) imply $\frac{1}{2} < h(x) < 1$ for all $x \in (a, b)$

In this way for all $x \in (a, b)$:

- Both $\mathcal{S}(x)$ and $h(x)$ are continuous and monotone.
- $\mathcal{S}(x)$ is bounded between $\left(\frac{1}{2}, 1\right)$. This actually holds for all $x \in \mathbb{R}^{+}$.
- $h(x)$ increases from $\frac{1}{2}$ to 1 (or decreases from 1 to $\frac{1}{2}$).

Then, there exists a unique $x^{*} \in (a, b)$ such that

$$h(x^{*}) = \mathcal{S}(x^{*}) \tag{4.2}$$

We shall see that this x^{*} is actually the unique maximizer. Thanks to equation (4.2), we have:

$$\begin{aligned}
f(x^{*}) - x^{*} f'(x^{*}) &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right)\left(\frac{1}{f(x^{*})}\right)^{n} \\
\iff 0 &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{n+2}\right)\left(\frac{1}{f(x^{*})}\right)^{n} + x^{*} f'(x^{*}) - f(x^{*}) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{f(x^{*})}\right)^{n} - \sum_{n=0}^{\infty} \left(\frac{1}{n+2}\right)\left(\frac{1}{f(x^{*})}\right)^{n} + x^{*} f'(x^{*}) - f(x^{*}) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{f(x^{*})}\right)^{n-1} - \sum_{n=2}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{f(x^{*})}\right)^{n-2} - f(x^{*}) + x^{*} f'(x^{*}) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{f(x^{*})}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{f(x^{*})}\right)^{n-2} + x^{*} f'(x^{*}) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{f(x^{*})}\right)^{n+1} - \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{f(x^{*})}\right)^{n} + \frac{x^{*} f'(x^{*})}{(f(x^{*}))^{2}} \qquad (4.3)
\end{aligned}$$

Now, recall we want to find a maxima for $(1 - \varphi(x))^{x}$. This is equivalent to maximizing:

$$\mathcal{U}(x) := x \ln(1 - \varphi(x))$$

Note that $\mathcal{U}'(x) = \ln(1 - \varphi(x)) - \frac{x}{1-\varphi(x)} \varphi'(x)$. So, it suffices to find x^{*} (the maximizer) such that

$\mathcal{U}'(x^*) = 0$, which is equivalent to $g(x^*) = 0$ where:

$$g(x) := [1 - \varphi(x)] \ln (1 - \varphi(x)) - x\varphi'(x)$$

Recall the Taylor expansion of $\ln(1 - y)$ for $|y| < 1$:

$$\ln(1 - y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}$$

Then:

$$\begin{aligned} g(x) &= [1 - \varphi(x)] \left[-\sum_{n=1}^{\infty} \frac{\varphi^n(x)}{n} \right] - x\varphi'(x) \\ &= \sum_{n=1}^{\infty} \frac{\varphi^{n+1}(x)}{n} - \sum_{n=1}^{\infty} \frac{\varphi^n(x)}{n} - x\varphi'(x) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \left(\frac{1}{f(x)}\right)^{n+1} - \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \left(\frac{1}{f(x)}\right)^{n} + x\frac{f'(x)}{(f(x))^2} \end{aligned}$$

Finally, it is easy to see that this last line and (4.3) imply $g(x^*) = 0$ and we can conclude x^* is the unique maximizer in (a, b). If a, b are unique, it is clear that the maximizer is unique. □

Remark. The conditions (4a) and (4c), which may seem mysterious at first glance, are inspired by Taylor's Theorem, from calculus.

4.2 Interpretation of the Theorem

Recall formula (4.1) $\mathbb{P}(\mathcal{N}_k) = \mathbb{P}\left(\bigcap_{i=1}^{k} N_i\right) = [\mathbb{P}_k(N_1)]^k = (1 - \varphi(k))^k$. As we briefly discussed in section (II), because of our independence and identical distribution assumptions, there are 2 interacting forces affecting $\mathbb{P}(\mathcal{N}_k)$. On the one hand, $\mathbb{P}(\mathcal{N}_k) = (1 - \varphi(k))^k$ decreases as k increases because $0 < \varphi(k) < 1$. On the other hand, we set *a fortiori* $\varphi(k)$ to decrease as k increases with the hope to find a maximizer k^*. Lending to a group has advantages over lending to an individual, but as the size of the group increases, the advantages diminish and tend to zero.

There should, therefore, be some happy (and optimal) compromise of a group size being big, but not too big!

There are two opposing forces here. On the one hand, as the group size increases, the responsibility for performing one's tasks becomes dispersed, increasing the likelihood that one or more member of the group may default. Typically, there will be a leader or primary organizer, the force behind the loan, and she will need to ride herd on the other members, keeping them in line, if need be. The larger the group, the more diffused her efforts will be, and therefore the less effective. Mathematically, as there are more people in the group, because of our assumptions, there are more independent chances of failure as it is riskier to have $k+1$ possible defaults than k. This causes $\mathbb{P}(\mathcal{N}_k)$ to decrease as k increases. Note that the effect of this force is free of the choice of $\varphi(k)$.

On the other hand, as the group size increases, there are more collective resources (material and non material) which decrease the likelihood of default. In our model, this is captured by $\varphi(k)$ as different functions correspond to different contribution of the resources and hence different k^* (optimal group size). Moreover, as the group size contracts, each person becomes more important and there are less resources available, making it harder to recover from a mistake, or a temporary period of misfortune. In the limit case of only one borrower, lenders in Ghana (for example) have found that, there being no peer pressure at all, the borrower has a serious probability of simply absconding with the money. [1]

Now, the issue is to find the right speed of decay of $\varphi(k)$. This is addressed by our theorem. It is important to note that our theorem is useful because not all $\varphi(k)$ work. For example, the seemingly natural choice of $\varphi(k) = \frac{1}{k}$ does not satisfy the assumptions of our theorem and one can check that it does not have a finite maximizer greater or equal to 2. Other examples of functions that do not satisfy the assumptions of our theorem are $\varphi(k) = 1/k^r$ for $r \geq 1$ or, more generally, $\varphi(k) = 1/e^{rk}$ for $r \geq 1$

[1] Personal conversation of my advisor (Philip Protter), Accra, Ghana, August 22, 2018; with Prof. Dr. Olivier Menouken Pamen, of AIMS, Ghana

4.3 Example

In this section, we provide a function that satisfies our theorem and whose maximizer is close to 5, i.e. $x^* \approx 5$. As explained in Yunus [66], this is the group size proposed by Muhammad Yunus. Let us consider the following choice $f(x)$, note that it is a function of the size x of the group:

$$f(x) = x^\alpha + (\ln x)^\beta \tag{4.4}$$

This example captures two different forces at play to avoid default. The part of $f(x)$ given by x^α represents the contribution of the material resources available to the group such as the amount of land they possess or the collective financial resources. Meanwhile, the component $(\ln x)^\beta$ represents the contribution of the non material resources of the group; for example, the quality of the group, the peer pressure, or the information available to the group. As the group size increases, there are more collective resources available and thus, the probability of default decreases. We chose $\ln x$, which has a distinctly slower growth rate than x, for non material resources of the group because we think that this is less relevant than the material resources.

Let us consider $\alpha = p$ and, for simplicity, its reciprocal, i.e., $\beta = 1/p$, for all $p \in \left[\frac{1}{2}, 1\right]$. The exponents of x and of $\ln x$ are chosen in this way because we want to have countervailing forces for the interaction of the material and non-material resources of the group. That is, the less contribution of material resources, the more contribution of non material resources we need. Of course, there is no need to consider $\beta = 1/p$, but, as done in Section 4.4, this is chosen for mathematical convenience. Other choices that work are $\alpha = p$, $\beta = 1/p^2$ or $\beta = 1/p^3$ for all $p \in \left[\frac{1}{2}, 1\right]$, among others.

In Section 4.4, we show that the example given in (4.4) satisfies the conditions of the theorem, but let us now note that the cases $p = \frac{1}{2}$ and $p = 1$ are relevant as calculations show they lead to $x^* = 5.13$ and $x^* = 4.62$ respectively. Therefore, in the extreme cases, i.e., a great contribution of either material ($p = 1$) or non material collective resources ($p = 1/2$), the optimal group size is 5, which coincides with the maximizer proposed by Muhammad Yunus. More generally, we can see that when $p \in [0.5, 0.539]$ or $p \in [0.993, 1]$, $x^* \in [4.5, 5.5)$, giving an integer maximizer of 5.

We wish to note that, although $f(x) = x^p$, for all $p \in [1/2, 1)$ works with our theorem, we believe this function does not capture the complexity of the situation we are trying to model. For this function, when p is close to 1, eg. $p = 0.999$, the maximizer is $x^* = 503.45$. This should not be surprising as $x^{0.999}$ is close to x, which as previously discussed does not have a finite maximizer. Moreover, when p is close to 1/2, e.g. $p = 0.501$, $x^* = 1.956$. A similar, but less drastic situation occurs with $f(x) = (\ln x)^{1/p}$, for all $p \in [1/2, 1]$

Other examples of functions that work with our theorem are the following:

- $f(x) = (x \ln(x))^p$, $p \in [1/2, 3/4]$. By numerical calculations, the maximizer x^* is between 5.17 and 25.52, depending on the value of p
- $f(x) = (\ln(\ln(x)))^p$, $p > 0$ where we necessarily need to consider $x \geq e^e$ to satisfy $f(x) \geq 1$, which is a modified version of condition 1 of our theorem. Then, for example, if $p = 0.1$, $x^* = 18.23$, if $p = 1$, $x^* = 22.28$, and if $p = 10$, $x^* = 309.77$
- $f(x) = (x \ln(x))^p + (\ln(\ln(x)))^{1/p}$, $p \in [1/2, 3/4]$. By numerical calculations, the maximizer x^* is between 6.56 and 18.67, depending on the value of p

4.4 Analysis of Example

Now, we show that the example given in (4.4) satisfies the conditions of Theorem 4.1. Conditions 1 and 2 are immediate.

Condition (3): $f'(x) > 0$ for all $x \geq 2$

Proof. $f'(x) = px^{p-1} + \frac{1}{p}\left(\frac{1}{x}\right)(\ln x)^{\frac{1}{p}-1}$ As $x \geq 2$, it is clear $f'(x) > 0$ □

Condition (4a): There exist a and b such that $f(a) - af'(a) = \frac{1}{2}$ and $f(b) - bf'(b) = 1$

Proof. Set

$$h_p(x) := f(x) - xf'(x) = x^p + (\ln x)^{\frac{1}{p}} - px^p - \frac{1}{p}(\ln x)^{\frac{1}{p}-1} = (1-p)x^p + (\ln x)^{\frac{1}{p}-1}\left(\ln x - \frac{1}{p}\right)$$

We will show that $h_p(x)$ is increasing in x by taking the derivative with respect to x and showing it is positive.

$$\frac{\partial}{\partial x} h_p(x) = (1-p)px^{p-1} + \left(\frac{1}{p} - 1\right)(\ln x)^{\frac{1}{p}-2}\left[\ln x - \frac{1}{p}\right]\frac{1}{x} + \frac{1}{x}(\ln x)^{\frac{1}{p}-1}$$

It is clear $(1-p)\,px^{p-1} > 0$. So, it suffices to show

$$\left(\frac{1}{p} - 1\right)(\ln x)^{\frac{1}{p}-2}\left[\ln x - \frac{1}{p}\right]\frac{1}{x} + \frac{1}{x}(\ln x)^{\frac{1}{p}-1} \geq 0 \tag{4.5}$$

For reasons that will become clear later, we only consider $x \geq e$. As $\ln x + 1 \geq 2$ and $1 \leq \frac{1}{p} \leq 2$, it follows that

$$\begin{aligned} \ln x + 1 &\geq \frac{1}{p} \\ \frac{1}{p}\left(\ln x + 1 - \frac{1}{p}\right) &\geq 0 \\ \frac{1}{p}\left(\ln x - \frac{1}{p}\right) - \left(\ln x - \frac{1}{p}\right) + \ln x &\geq 0 \\ \left(\frac{1}{p} - 1\right)\left(\ln x - \frac{1}{p}\right) + \ln x &\geq 0 \end{aligned}$$

This shows (4.5) and thus that $h_p(x)$ is increasing for $x \geq e$

We now show that $h_p(e) = (1-p)e^p + 1 - \frac{1}{p}$ is concave in p and hence there exists a local maxima, namely p^*. (Recall $\frac{1}{2} \leq p \leq 1$).

$$\begin{aligned} \frac{\partial}{\partial p} h_p(e) &= -e^p + (1-p)e^p + \frac{1}{p^2} = -pe^p + \frac{1}{p^2} \\ \frac{\partial^2}{\partial p^2} h_p(e) &= -e^p - pe^p - \frac{2}{p^3} < 0 \Longrightarrow \quad h_p(e) \text{ is concave} \end{aligned}$$

Now, to find the maxima, we set the derivative equal to 0, i.e. $\frac{\partial}{\partial p} h_p(e) = 0$

$$0 = -pe^p + \frac{1}{p^2}$$

$$1 = p^3 e^p$$

$$1 = p e^{\frac{1}{3}p}$$

Set $u = \frac{1}{3}p$, we need to solve $ue^u = \frac{1}{3}$, which we do by using the product logarithm. Hence, by numerical approximation, $p^* \approx 0.772883 \Longrightarrow h_p(e)|_{p=0.773} \approx 0.1981 < \frac{1}{2}$ $\square$

4.5 Conclusion

In this chapter we construct a theoretical model for the determination of the optimal number of people in a group loan. As these loans are intended for low-income borrowers with little or no collateral, and with no credit history, one of the starting points to maximize the repayment rate is to determine the best possible size of the group. We discuss a theorem that provides sufficient conditions for the optimal group size to be finite, greater than 1, and unique. We also provide examples of functions that satisfy our theorem and we analyze in detail the one that we believe has a more natural interpretation and whose associated optimal group size is approximately 5, the number chosen by Muhammad Yunus. An empirical study of the proposed model awaits subsequent research.

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Appendix A: Bound for Complementary Error Function

Showing that equation (2.37) is smaller than 1 for any $k > 2$ is equivalent to showing the following inequality:

$$\int_x^\infty e^{-u^2} du \leq \frac{8x}{16x^2+5} e^{-x^2} \text{ for all } x \geq 1 \tag{A.1}$$

This is equivalent to the problem of finding an upper bound for the complementary error (erfc) function (recall that $\text{ercf}(x) = \frac{2}{\pi}\int_x^\infty e^{-u^2} du$), which has been widely studied in the literature. A classical result is the Chernoff-Rubin bound (see Chernoff [20]). More recent work can be found in Chiani et al [21], Karagiannidis and Lioumpas [45], and in Tanash and Riihonen [61]. However, to our knowledge, there is no bound of the type we need here.

We will show a slightly tighter bound than the one in (2.37), namely, we would like to optimize the following bound in ℓ

$$\int_x^\infty e^{-u^2} du \leq \frac{2x}{4x^2+\ell} e^{-x^2} \text{ for all } x \geq 1 \tag{A.2}$$

When we say optimize, we mean that $\frac{2x}{4x^2+\ell} e^{-x^2}$ is as close as possible (yet bigger) to $\int_x^\infty e^{-u^2} du$ for all $x \geq 1$.

In other words, define

$$h(x,\ell) := \frac{2x}{4x^2+\ell} e^{-x^2} - \int_x^\infty e^{-u^2} du \tag{A.3}$$

We want to find ℓ^* such that:

1. $h(x,\ell^*) \leq h(x,\ell)$ for all $x \geq 1$ and $\ell \neq \ell^*$.

2. $h(x,\ell^*) \geq 0$ for all $x \geq 1$

To get the first item in the previous list, we need ℓ to be as large as possible. This is because $h(x,\ell)$ is decreasing in ℓ. However, if we wish to have $h(x,\ell) \geq 0$, ℓ cannot be indiscriminately large.

To find ℓ^*, let us first look at $\frac{\partial}{\partial x} h(x,\ell)$.

$$\frac{\partial}{\partial x} h(x,\ell) = e^{-x^2}\left[-\frac{2\left(4x^2-\ell\right)}{\left(4x^2+\ell\right)^2} - \frac{4x^2}{4x^2+\ell}\right] + 1 \tag{A.4}$$

We can find that:

$$\frac{\partial}{\partial x} h(x,\ell) < 0 \text{ when } x > \sqrt{\frac{\ell^2+2\ell}{8-4\ell}} \tag{A.5}$$

Hence $h(x,\ell)$ is increasing when $x \in \left[1, \sqrt{\frac{\ell^2+2\ell}{8-4\ell}}\right)$, $h(x,\ell)$ is maximized when $x = \sqrt{\frac{\ell^2+2\ell}{8-4\ell}}$ and it is decreasing when $x \in \left(\sqrt{\frac{\ell^2+2\ell}{8-4\ell}}, \infty\right)$.

Also note that

$$\lim_{x\to\infty} h(x,\ell) = 0 \text{ for any } \ell \text{ positive} \tag{A.6}$$

Hence, to satisfy the second item in the list above (i.e., $h(x,\ell^*) \geq 0$ for all $x \geq 1$), it suffices to pick ℓ^* such that $h(1,\ell^*) = 0$, i.e.,

$$h(1,\ell^*) = \frac{2}{4+\ell^*}e^{-1} - \int_1^\infty e^{-u^2}du = 0 \tag{A.7}$$

Solving for ℓ^*, we find that:

$$\ell^* = \frac{2}{e\int_1^\infty e^{-u^2}du} - 4 \approx 1.27935 \tag{A.8}$$

We can then conclude that the maximum of $h(x,\ell^*)$ occurs when

$$x = \sqrt{\frac{(\ell^*)^2+2\ell^*}{8-4\ell^*}} \approx 1.2043 \quad \text{Recall that } \ell^* \approx 1.27935 \tag{A.9}$$

We can then numerically calculate that

$$h(1.2043, \ell^*) \approx 0.00131266 \tag{A.10}$$

This last equation means that the maximum difference between $\frac{2x}{4x^2+\ell} e^{-x^2}$ and $\int_x^\infty e^{-u^2} du$ is 0.00131266

As $h(x, \ell)$ is decreasing as ℓ increases, $h(x, \ell) \geq 0$ for all $x \geq 1$ as long as $\ell < \ell^* \approx 1.27935$. In (A.1), we set $\ell = \frac{5}{4} < \ell^*$ and hence the work above shows that equation (2.37) is indeed smaller than 1.