Open/Closed Correspondence and Mirror Symmetry

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Abstract

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We develop the mathematical theory of the open/closed correspondence, proposed by Mayr in physics as a class of dualities between open strings on Calabi-Yau 3-folds and closed strings on Calabi-Yau 4-folds. Given an open geometry on a toric Calabi-Yau 3-orbifold relative to a framed Aganagic-Vafa outer brane, we construct a closed geometry on a toric Calabi-Yau 4-orbifold and establish the correspondence between the two geometries on the following levels across both the A- and B-sides of mirror symmetry:

- Numerical Gromov-Witten invariants
- Generating functions of Gromov-Witten invariants
- B-model hypergeometric functions and Givental-style mirror theorems
- Picard-Fuchs systems and solutions
- Integral cycles on Hori-Vafa mirrors and periods
- Mixed Hodge structures

Table of Contents

List of Figures

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To my parents

Published Content and Contributions

This work builds upon the following papers joint with Chiu-Chu Melissa Liu.

[79] C.-C. M. Liu and S. Yu, “Open/closed correspondence via relative/local correspondence,” *Adv. Math.*, vol. 410, no. part A, Paper No. 108696, 43, 2022.

[80] ——, “Orbifold open/closed correspondence and mirror symmetry,” 2022. eprint: 2210.11721.

Chapters 2 – 6 and Appendices A, B are based on [80]. Appendix C is based on [79].

Chapter 1: Introduction

The *open/closed correspondence*, proposed by Mayr [83] as a class of open/closed string dualities and developed by Lerche-Mayr [70], is a conjectural relation between the topological amplitudes at genus zero of an open string geometry on a Calabi-Yau 3-fold with a prescribed Lagrangian boundary condition and a closed string geometry on a corresponding Calabi-Yau 4-fold. The correspondence is expected to hold on multiple levels across both the A- and B-sides of mirror symmetry. The A-side can be interpreted in terms of Gromov-Witten theories. Specifically, the disk invariants of the open 3-fold geometry should correspond to the genus-zero closed Gromov-Witten invariants of the 4-fold geometry, and this could be upgraded to the level of generating functions. The B-side is centered around the observation that both the open and closed geometries define the same Picard-Fuchs system of differential equations. Different forms of representing the solutions to the system, such as hypergeometric functions or period integrals on mirror families, each admit interpretations in terms of both the open and closed geometries, and the two interpretations should match up.

In this work, we offer a mathematical treatment of the above proposal. Given an open geometry on a semi-projective toric Calabi-Yau 3-orbifold $\mathcal{X}$ relative to a Lagrangian suborbifold $\mathcal{L}$ of Aganagic-Vafa type [5], we explicitly construct a semi-projective toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$ as the dual closed geometry (Chapter 2). We establish the following levels of correspondence between the two geometries:

- Numerical Gromov-Witten invariants (Chapters 3 and 4): We show that for corresponding curve classes, the disk invariants of $(\mathcal{X}, \mathcal{L})$, which virtually count stable maps from genus-zero Riemann surfaces with a single boundary component to $(\mathcal{X}, \mathcal{L})$, are identified with the genus-zero closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$.

- Generating functions (Chapter 5): We identify the generating function of disk invariants of $(\mathcal{X}, \mathcal{L})$ and the generating function of genus-zero Gromov-Witten invariants of $\widetilde{\mathcal{X}}$. We further show that the former can be recovered from the equivariant *J-function* of $\widetilde{\mathcal{X}}$.

- B-model hypergeometric functions (Chapter 6): As the B-model analog of the above, we show that the B-model disk function of $(\mathcal{X}, \mathcal{L})$ can be recovered from the equivariant *I-function* of $\widetilde{\mathcal{X}}$. We show the twins are compatible under Givental-style mirror theorems for both the closed [50, 27, 30] and open [5, 56, 41, 42] sectors.

- Picard-Fuchs systems and solutions (Chapter 7): We show that the Picard-Fuchs system of differential equations of $\widetilde{\mathcal{X}}$ is an extension of that of $\mathcal{X}$ by the data of $\mathcal{L}$, and offer a description of the additional solutions that come from open strings. This contextualizes the above correspondence of hypergeometric functions, whose components give solutions to the Picard-Fuchs systems.

- Integral cycles on Hori-Vafa mirrors and periods (Chapter 8): We give an identification of integral relative 3-cycles on the Hori-Vafa mirror of $\mathcal{X}$ with integral 4-cycles on the Hori-Vafa mirror $\widetilde{\mathcal{X}}$. We show in addition that the periods over corresponding cycles are equal (up to a multiplicative constant), thereby offering an alternative perspective of the correspondence of solutions to Picard-Fuchs systems.

- Mixed Hodge structures (Chapter 9): Finally, as a dual statement to the correspondence of cycles, we show that the mixed Hodge structure governing middle-dimensional forms on mirror families of $\widetilde{\mathcal{X}}$ is an extension of that for $\mathcal{X}$.

We now provide additional details and discussions for our results.

1.1 Numerical correspondence

Let $\mathcal{X}$ be a toric Calabi-Yau 3-orbifold with semi-projective coarse moduli space X, $\mathcal{L} \subset \mathcal{X}$ be a Lagrangian brane of Aganagic-Vafa type, and $f \in \mathbb{Z}$ be an additional parameter called the *framing* of the brane $\mathcal{L}$. The coarse moduli space L of $\mathcal{L}$ intersects a unique torus-fixed line l in X, corresponding to a 2-cone in the fan. The corresponding torus-invariant substack $\mathfrak{l} \subset \mathcal{X}$ has a cyclic generic stablizer group $\mu_{\mathfrak{m}}$ for some $\mathfrak{m} \in \mathbb{Z}_{>0}$. The Lagrangian $\mathcal{L}$ interesects $\mathfrak{l}$ along a stacky circle: $\mathcal{L} \cap \mathfrak{l} \cong S^1 \times B\mu_{\mathfrak{m}}$. We assume that $\mathcal{L}$ is *outer*, i.e. $l \cong \mathbb{C}^1$.

Open Gromov-Witten invariants of $(\mathcal{X}, \mathcal{L}, f)$ [28, 41, 42, 64] give virtual counts of twisted stable maps from bordered orbifold Riemann surfaces

$$u : (\mathcal{C}, \partial\mathcal{C}) \to (\mathcal{X}, \mathcal{L}).$$

We focus on *disk invariants*, which concern the case where the domain $\mathcal{C}$ has arithmetic genus zero and one boundary component. Let

$$\beta' = \beta + d[B] \in H_2(X, L; \mathbb{Z}),$$

be an effective class, where $\beta \in H_2(X; \mathbb{Z})$ is effective, B is the disk in l bounded by the intersection $L \cap l$, and $d \in \mathbb{Z}_{>0}$, and let $\lambda \in \mu_{\mathfrak{m}}$ be a monodromy profile. Then, the disk invariant

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{\beta', (d,\lambda)}^{\mathcal{X}, (\mathcal{L}, f)} \in \mathbb{Q}$$

virtually counts degree-β' maps u with boundary profile $u_*[\partial\mathcal{C}] = (d, \lambda) \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times \mu_{\mathfrak{m}}$. Here, $\gamma_1, \ldots, \gamma_n \in H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$ are any second Chen-Ruan orbifold cohomology [25] classes of $\mathcal{X}$ given as insertions to the interior of the domain. We refer to Chapter 3 for formal definitions, noting for now that the disk invariants are defined by equivariant localization with respect to a 2-dimensional subtorus T' of the algebraic 3-torus of $\mathcal{X}$ specified by the Calabi-Yau condition, and further depend on a circle action on the pair $(\mathcal{X}, \mathcal{L})$ specified by the framing f. Moreover, the definition adopts suitable T'-equivariant lifts of the insertions γ_i.

Our first main result is the explicit construction of a toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$ whose genus-zero Gromov-Witten invariants coincide with the disk invariants of $(\mathcal{X}, \mathcal{L}, f)$. This establishes the *numerical* open/closed correspondence. We note in advance that similar to above, the closed invariants are defined by equivariant localization with respect to a 3-dimensional subtorus $\widetilde{T}'$ of the algebraic 4-torus of $\widetilde{\mathcal{X}}$ specified by the Calabi-Yau condition, and further depend the action of a 1-dimensional subtorus T_f specified by f.

Theorem 1.1 (See Theorem 4.1)**.** *With $\mathcal{X}, \mathcal{L}, f$ as above, there is a toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$ satisfying that:*

- *The coarse moduli space $\widetilde{X}$ of $\widetilde{\mathcal{X}}$ is semi-projective.*
- *There is an inclusion $\iota : \mathcal{X} \to \widetilde{\mathcal{X}}$ which induces an inclusion $\iota_* : H_2(X, L; \mathbb{Z}) \to H_2(\widetilde{X}; \mathbb{Z})$.*
- *Given any effective class $\beta' = \beta + d[B] \in H_2(X, L; \mathbb{Z})$, monodromy $\lambda \in \mu_{\mathfrak{m}}$, and insertions $\gamma_1, \dots, \gamma_n \in H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$, we have*

  $$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X}, (\mathcal{L}, f)}_{\beta', (d, \lambda)} = \langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}, T_f}_{\widetilde{\beta}}$$

  where $\widetilde{\beta} = \iota_*(\beta') \in H_2(\widetilde{X}; \mathbb{Z})$ and $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}, T_f}_{\widetilde{\beta}}$ is a genus-zero, degree-$\widetilde{\beta}$ closed Gromov-Witten invariant of $\widetilde{\mathcal{X}}$. Here as insertions, each $\widetilde{\gamma}_i \in H^2_{\mathrm{CR}, \widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{Q})$ is a suitable lift of γ_i under $H^*_{\mathrm{CR}, \widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{Q}) \to H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$, and $\widetilde{\gamma}_\lambda \in H^4_{\mathrm{CR}, \widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{Q})$ is a fixed class depending on λ only.

Informally, the 4-orbifold $\widetilde{\mathcal{X}}$ is constructed as follows: We first add a new irreducible toric divisor $\mathcal{D}$ to $\mathcal{X}$ whose position depends on the position of $\mathcal{L}$, obtaining a log Calabi-Yau pair $(\mathcal{X} \sqcup \mathcal{D}, \mathcal{D})$. This is done so that $\mathcal{D}$ contains a new torus-fixed point which, at the level of coarse moduli, compactifies the line $l \cong \mathbb{C}^1$ into a $\mathbb{P}^1$. Then, we consider the toric Calabi-Yau 4-orbifold $\mathrm{Tot}(\mathcal{O}_{\mathcal{X} \sqcup \mathcal{D}}(-\mathcal{D}))$ and take $\widetilde{\mathcal{X}}$ to be the "minimal" toric partial compactification of $\mathrm{Tot}(\mathcal{O}_{\mathcal{X} \sqcup \mathcal{D}}(-\mathcal{D}))$ with a semi-projective coarse moduli space, in the sense that no additional toric divisors are added. We refer to Chapter 2 for the formal construction and specifically Section 2.6 for explicit examples.

We note that Theorem 1.1 covers a family of examples for the open/closed correspondence obtained by Bousseau-Brini-van Garrel [16] from constructions and enumerative theories based on smooth *Looijenga pairs*. Such examples are expected to extend to the orbifold setting [16], as exemplified in [17], and we expect Theorem 1.1 to cover this more general family of orbifold examples as well.[1]

Theorem 1.1 is proven by a direct and careful comparison of the localization computations of the open and closed Gromov-Witten invariants. One challenge is that, as we take additional steps in partially compactifying the 4-orbifold into one with semi-projective coarse moduli space, we also introduce new connected components to the torus-fixed loci of the moduli spaces of twisted stable maps. We address this issue by arguing that these additional components have no contribution to the closed invariants.

We note that the open and closed invariants can be bridged by the *relative* invariants [71, 72] of the log Calabi-Yau pair $(\mathcal{X} \sqcup \mathcal{D}, \mathcal{D})$. As an intermediate step of the open/closed correpondence, a version of the *log-local principle* of van Garrel-Graber-Ruddat [48] in the non-compact setting may be obtained (see also Conjecture 1.1 of [16]). See Appendix C for a discussion in the smooth case.

1.2 Correspondence of generating functions

Our next main result promotes the numerical open/closed correspondence to the level of *generating functions*. For $\lambda \in \mu_{\mathfrak{m}}$, we define the following generating function of disk invariants of $(\mathcal{X}, \mathcal{L}, f)$, following Fang-Liu-Tseng [42]:

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) := \sum_{\beta \in E(X)} \sum_{d \in \mathbb{Z}_{>0}} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{\langle \boldsymbol{\tau}_2^l \rangle_{\beta + d[B],(d,\lambda)}^{\mathcal{X},(\mathcal{L},f)}}{l!} \mathsf{X}^d$$

[1] The construction of the examples in [17] needed to consider *fractional* framings $f \in \mathbb{Q}$ on the brane $\mathcal{L}$. Open Gromov-Witten invariants in this setting can be similarly defined (see e.g. [44]) and our Theorem 1.1 can be extended to this setting in a straightforward way.

which takes value in $\mathbb{C}$. Here, we take

$$\boldsymbol{\tau}_2 = \sum_{a=1}^{K} \tau_a u_a$$

where $\{u_1, \ldots, u_K\}$ is a suitable basis for $H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$ and $\tau_1, \ldots, \tau_K$ are complex variables. Moreover, $E(X)$ is the semigroup $\mathrm{NE}(X) \cap H_2(X; \mathbb{Z})$ and X is an additional variable for the open sector.

On the other hand, for $\lambda \in \mu_{\mathfrak{m}}$, we consider the following generating function of genus-zero closed Gromov-Witten invariants of $\mathcal{X}$:

$$\langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}}, T_f}(\widetilde{\boldsymbol{\tau}}_2) := \sum_{\widetilde{\beta} \in E(\widetilde{X})} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle \widetilde{\boldsymbol{\tau}}_2^l, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}}, T_f},$$

which also takes value in $\mathbb{C}$. Here, we take

$$\widetilde{\boldsymbol{\tau}}_2 = \sum_{a=1}^{K+1} \widetilde{\tau}_a \widetilde{u}_a$$

where $\{\widetilde{u}_1, \ldots, \widetilde{u}_{K+1}\}$ is a suitable basis for $H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}}; \mathbb{Q})$, with $\widetilde{u}_a$ an appropriate lift of u_a for $a = 1, \ldots, K$, and $\widetilde{\tau}_1, \ldots, \widetilde{\tau}_{K+1}$ are complex variables. The class $\widetilde{\gamma}_\lambda$ is as in Theorem 1.1. Moreover, $E(\widetilde{X})$ is the semigroup $\mathrm{NE}(\widetilde{X}) \cap H_2(\widetilde{X}; \mathbb{Z})$.

Using the numerical correspondence (Theorem 1.1), we give the following correspondence between the two generating functions.

Theorem 1.2 (See Theorem 5.4)**.** *For any $\lambda \in \mu_{\mathfrak{m}}$, the correspondence*

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) = \langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}}, T_f}(\widetilde{\boldsymbol{\tau}}_2)$$

holds under the relation $\widetilde{\tau}_a = \tau_a$ for $a = 1, \ldots, K$ and $\widetilde{\tau}_{K+1} = \log \mathsf{X}$.

A notable feature of Theorem 1.2 is that the extra closed variable $\widetilde{\tau}_{K+1}$ is identified with (the logarithm) of the open variable X, which was predicted by the original proposal [83, 70].

The generating function $\langle\langle \widetilde{\gamma}_\lambda \rangle\rangle^{\widetilde{\mathcal{X}},T_f}$ is related in a standard way to another generating function of genus-zero closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$ known as the *J-function* [92, 31, 50]. As an application of Theorem 1.2, we show that the generating function $F_\lambda^{\mathcal{X},(\mathcal{L},f)}$ of disk invariants of $(\mathcal{X},\mathcal{L},f)$ can be recovered from the $\widetilde{T}'$-equivariant small J-function of $\widetilde{\mathcal{X}}$, denoted $J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z)$. It is a power series in the inverse of an extra variable z with coefficients valued in the $\widetilde{T}'$-equivariant Chen-Ruan cohomology of $\widetilde{\mathcal{X}}$.

Theorem 1.3 (See Theorem 5.8)**.** *For any $\lambda \in \mu_{\mathfrak{m}}$,*

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) = [z^{-2}] \left(J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Big|_{T_f}$$

under the relation $\widetilde{\tau}_a = \tau_a$ for $a = 1, \dots, K$ and $\widetilde{\tau}_{K+1} = \log \mathsf{X}$.

Here, $(-,-)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}$ denotes the $\widetilde{T}'$-equivariant orbifold Poincaré pairing of $\widetilde{\mathcal{X}}$, the notation $\big|_{T_f}$ stands for taking weight restriction to the 1-torus T_f, and the notation $[z^{-2}]$ stands for taking the coefficient of z^{-2} in the power series expansion.

1.3 B-model hypergeometric functions and mirror symmetry

Under the well-known *mirror theorem* [50, 27, 30], the J-function of $\widetilde{\mathcal{X}}$ as the A-model side is identified with the *I-function* of $\widetilde{\mathcal{X}}$ on the B-model side, which is an explicit hypergeometric function defined by the toric data. Let $I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z)$ denote the $\widetilde{T}'$-equivariant I-function of $\widetilde{\mathcal{X}}$, which depends on variables $\widetilde{q} = (\widetilde{q}_1, \dots, \widetilde{q}_{K+1})$ that are related to $\widetilde{\tau}_1, \dots, \widetilde{\tau}_{K+1}$ under the *closed mirror map* $\widetilde{\boldsymbol{\tau}}_2 = \widetilde{\boldsymbol{\tau}}_2(\widetilde{q})$.

Similarly, on the open side, the *open mirror theorem* of [5, 56, 41, 42] identifies the generating function $F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X})$ of disk invariants of $(\mathcal{X},\mathcal{L},f)$ with an explicit hypergeometric function $W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x)$ which we refer to as the *B-model disk function*. It depends on closed-sector variables $q = (q_1, \dots, q_K)$ and an open-sector variable x that are related to $\tau_1, \dots, \tau_K, \mathsf{X}$ under the closed mirror map $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_2(q)$ and the open mirror map $\mathsf{X} = \mathsf{X}(q,x)$.

Now consider the web of relations in Figure 1.1, where the horizontal arrows represent the mirror theorems mentioned above. The vertical arrow on the left represents our Theorem 1.3 as a version of the open/closed correspondence on the A-model side. On the B-model side, we prove the parallel statement that the B-model disk function can be recovered from the $\widetilde{T}'$-equivariant I-function of $\widetilde{\mathcal{X}}$, filling in the vertical arrow on the right.

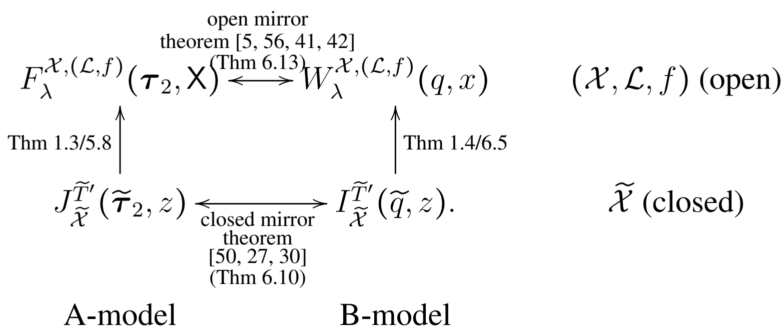

Figure 1.1: Compatibility of correspondences with mirror symmetry.

Theorem 1.4 (See Theorem 6.5). *For each* $\lambda \in \mu_{\mathfrak{m}}$,

$$W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x) = [z^{-2}] \left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Big|_{T_f}$$

under the relation $\widetilde{q}_a = q_a$ *for* $a = 1, \dots, K$ *and* $\widetilde{q}_{K+1} = x$.

Our proof is by an explicit comparison of the hypergeometric functions and is indenpendent of mirror symmetry or the A-model correspondence. Again, the extra closed variable $\widetilde{q}_{K+1}$ is identified with the open variable x. In fact, we show that the closed mirror map of $\widetilde{\mathcal{X}}$ can be identified with the open-closed mirror map of $(\mathcal{X}, \mathcal{L}, f)$ as follows:

$$\begin{aligned} \widetilde{\tau}_a(\widetilde{q}) &= \tau_a(q), \quad a = 1, \dots, K, \\ \widetilde{\tau}_{K+1}(\widetilde{q}) &= \log \mathsf{X}(q,x). \end{aligned} \tag{1.1}$$

See Proposition 6.14. As a consequence, we confirm that the diagram in Figure 1.1 is "commutative" in the sense that the vertical arrows are compatible with mirror symmetry. In particular, we obtain an alternative proof of the open mirror theorem [5, 56, 41, 42].

1.4 Extended Picard-Fuchs system and solutions

Components of the (non-equivariant) I-function of $\widetilde{\mathcal{X}}$ are known to give solutions to the (non-equivariant) *Picard-Fuchs system* $\widetilde{\mathcal{P}}$ of differential equations defined by the toric data of $\widetilde{\mathcal{X}}$ [50]. As originally proposed by [83, 70], $\widetilde{\mathcal{P}}$ can be viewed as an *extension* of the Picard-Fuchs system $\mathcal{P}$ associated to $\mathcal{X}$ and the difference between the two systems can be described in terms of the framed Aganagic-Vafa brane $(\mathcal{L}, f)$.

Continuing the study of the open/closed correspondence on the B-model side, we give an explicit comparison of the two Picard-Fuchs systems and confirm that solutions to $\mathcal{P}$ are indeed also solutions to $\widetilde{\mathcal{P}}$ (Proposition 7.1). Moreover, we characterize the extra solutions to $\widetilde{\mathcal{P}}$ in terms of the open string data. This includes the open mirror map $\log X(q,x)$ of $(\mathcal{X}, \mathcal{L}, f)$ (Proposition 7.4), which is identified with the additional closed mirror map of $\widetilde{\mathcal{X}}$ (1.1), and solutions whose power series part can be described in terms of the B-model disk functions $W_\lambda^{\mathcal{X},(\mathcal{L},f)}$ (Proposition 7.5).

1.5 Hori-Vafa mirrors, integral cycles, and periods

Our final developments of the B-model open/closed correspondence in this work are centered around *Hori-Vafa mirrors*. The Hori-Vafa mirrors of the toric Calabi-Yau orbifolds $\mathcal{X}$, $\widetilde{\mathcal{X}}$ are the families of hypersurfaces

$$\mathcal{X}_q^\vee := \{(u,v,X,Y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv = H(X,Y,q)\},$$
$$\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee := \{(u,v,X,Y,Z) \in \mathbb{C}^2 \times (\mathbb{C}^*)^3 : uv = \widetilde{H}(X,Y,Z,\widetilde{q})\}$$

parameterized by $q, \widetilde{q}$ respectively. Here, $H(X,Y,q)$ (resp. $\widetilde{H}(X,Y,Z,\widetilde{q})$) is a Laurent polynomial in X, Y (resp. X, Y, Z) whose Newton polyhedron is the 2- (resp. 3-) dimensional polyhedron Δ (resp. $\widetilde{\Delta}$) that is the convex hull of the generators of 1-cones of the fan of $\mathcal{X}$ (resp. $\widetilde{\mathcal{X}}$). It holds that the powers of Z in $\widetilde{H}(X,Y,Z,\widetilde{q})$ are non-negative, and under $\widetilde{q}_a = q_a$ for $a = 1, \dots, K$, we have

$$\widetilde{H}(X,Y,Z,\widetilde{q})\big|_{Z=0} = H(X,Y,q).$$

Then the projection $Z : \widetilde{\mathcal{X}}^\vee_{\widetilde{q}} \to \mathbb{C}^*$ defines a family of deformations of $\mathcal{X}^\vee_q$, which can be viewed as the central fiber over $Z = 0$.

It is well-known that period integrals of the middle-dimensional holomorphic form Ω_q (resp. $\widetilde{\Omega}_{\widetilde{q}}$) over integral cycles on the Hori-Vafa mirror $\mathcal{X}^\vee_q$ (resp. $\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}$) generate the solution space of the Picard-Fuchs system $\mathcal{P}$ (resp. $\widetilde{\mathcal{P}}$) [57, 67, 23]. As we view $\widetilde{\mathcal{P}}$ as the extended system of $\mathcal{P}$ by the open string data, we show that the additional solutions to $\widetilde{\mathcal{P}}$ can be recovered from periods of *relative* cycles on $\mathcal{X}^\vee_q$. To be more specific, we consider relative cycles with boundary in the family of hypersurfaces

$$\mathcal{Y}_{q,x} = \mathcal{X}^\vee_q \cap \{XY^f = -x\} = \{(u,v,Y) \in \mathbb{C}^2 \times \mathbb{C}^* : uv = H_0(Y,q,x) := H(X,Y,q)\big|_{XY^f=-x}\}$$

in $\mathcal{X}^\vee_q$ parameterized by the open-sector variable x. On its own, $\mathcal{Y}_{q,x}$ can be viewed as the Hori-Vafa mirror of a toric Calabi-Yau surface. The Newton polyhedron of the restricted Laurent polynomial $H_0(Y,q,x)$ is a closed interval Δ_0 that is the convex hull of the generators of 1-cones of the fan of the surface.

We establish the following correspondence of cycles and periods for the families $(\mathcal{X}^\vee_q, \mathcal{Y}_{q,x})$ and $\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}$ over a suitable domain $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$ where the Laurent polynomials involved the definitions above satisfy certain regularity properties.

Theorem 1.5 (See Theorem 8.6)**.** *For* $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$*, there is an isomorphism*

$$\iota : H_3(\mathcal{X}^\vee_q, \mathcal{Y}_{q,x}; \mathbb{Z}) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}; \mathbb{Z})$$

such that for any $\Gamma \in H_3(\mathcal{X}^\vee_q, \mathcal{Y}_{q,x}; \mathbb{Z})$*, we have*

$$\int_\Gamma \Omega_q = \frac{1}{2\pi\sqrt{-1}} \int_{\iota(\Gamma)} \widetilde{\Omega}_{\widetilde{q}}.$$

The construction of the isomorphism ι in Theorem 1.5 is based on the structure of $\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}$ as a family of deformations of $\mathcal{X}^\vee_q$ over $Z \in \mathbb{C}^*$, with $\mathcal{X}^\vee_q$ the central fiber over $Z = 0$. Given a relative

3-cycle $\Gamma \in H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{Z})$, included in the central fiber, we deform it over a small circle $\{|Z| = \epsilon'\}$ to obtain $\iota(\Gamma)$, which we show is an absolute 4-cycle without boundary.

We note that the original proposal [83, 70] considered relative cycles on $\mathcal{X}_q^\vee$ with boundary on certain complex 1-dimensional subspaces parameterized by x, known as Aganagic-Vafa B-branes and considered to be mirror to the A-brane $\mathcal{L}$ [5]. We show that this boundary condition is closely related to our boundary condition specified by the hypersurface $\mathcal{Y}_{q,x}$. See Section 8.6.

1.6 Mixed Hodge structures

As a dual statement of Theorem 1.5 on cycles, we also consider the deformations of forms as captured by mixed Hodge structures (MHS) [34, 35]. We consider the following formulation: The Hori-Vafa mirrors $\mathcal{X}_q^\vee$, $\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee$, $\mathcal{Y}_{q,x}$ are conic fibrations over the algebraic tori $(\mathbb{C}^*)^2$, $(\mathbb{C}^*)^3$, $\mathbb{C}^*$ respectively whose discriminant loci are the affine hypersurfaces

$$C_q \coloneqq \{H(X,Y,q) = 0\}, \qquad S_{\widetilde{q}} \coloneqq \{\widetilde{H}(X,Y,Z,\widetilde{q}) = 0\}, \qquad P_{q,x} \coloneqq \{H_0(Y,q,x) = 0\}$$

defined by the Laurent polynomials. Periods of the middle-dimensional holomorphic forms on the Hori-Vafa mirrors are known to correspond to periods of the standard holomorphic symplectic forms of the algebraic tori over relative cycles with boundary in the hypersurfaces [37, 23].

As a version of the open/closed correspondence of MHS, we show that the MHS of the middle-dimensional cohomology of the pair $((\mathbb{C}^*)^3, S_{\widetilde{q}})$ is an extension of that of $((\mathbb{C}^*)^2, C_q)$ by that of $(\mathbb{C}^*, P_{q,x})$.

Theorem 1.6 (See Theorem 9.2 and Corollary 9.3)**.** *For $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$, there is a short exact sequence of MHS*

$$0 \longrightarrow H^1(\mathbb{C}^*, P_{q,x}; \mathbb{C}) \longrightarrow H^3((\mathbb{C}^*)^3, S_{\widetilde{q}}; \mathbb{C}) \longrightarrow H^2((\mathbb{C}^*)^2, C_q; \mathbb{C}) \longrightarrow 0$$

under appropriate degree shifts of the Hodge and weight filtrations.

MHS of affine hypersurfaces in algebraic tori are studied by Batyrev [11], and subsequently by Stienstra [91] and Konishi-Minabe [67]. Their results imply that the MHS involed in Theorem 1.6 are equivalent[2] to $\mathbb{C}$-vector spaces

$$\mathcal{R}_H, \qquad \mathcal{R}_{\widetilde{H}}, \qquad \mathcal{R}_{H_0}$$

each equipped with two filtrations, all defined explicitly from the data of the polyhedra Δ, $\widetilde{\Delta}$, Δ_0 and the Laurent polynomials H, $\widetilde{H}$, H_0 in a way similar to the construction of *Jacobian rings*. We prove Theorem 1.6 by explicitly constructing the extension for these $\mathbb{C}$-vector spaces and verifying that the maps involved respect the Hodge and weight filtrations.

We note that Konishi-Minabe [67] showed that if Δ is *reflexive*, then $\mathcal{R}_H$ is also equivalent to the MHS on $H^3(\mathcal{X}_q^\vee; \mathbb{C})$. If this result could be generalized to arbitrary convex polyhedra of any dimensions, then we would be able to obtain an analog of Theorem 1.6 for the MHS of Hori-Vafa mirrors.

1.7 Generalizations and implications

The various levels of the open/closed correspondence that we develop in this work have further generalizations and implications. We briefly indicate some of those here and defer a rigorous treatment to future work.

1.7.1 Structures in open Gromov-Witten theory

On the A-model side, the open/closed correspondence can be used to study the structures of open Gromov-Witten invariants of toric Calabi-Yau 3-orbifolds. First, closed Gormov-Witten invariants are known to satisfy a set of recursive relations called the *Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations*, which give the associativity of quantum cohomology [68, 81, 87]. Solomon [89] proposed that the WDVV equations should have an analog in open Gromov-Witten

[2]This requires that the Laurent polynomials satisfy a regularity condition (Definition 8.2) with respect to their Newton polyhedra, which holds within our choice of the parameter domain $\widetilde{U}_\epsilon$.

theory, known as the *open WDVV equations*, which has inspired a series of subsequent developments; see e.g. [6, 10, 90, 26]. In particular, Solomon-Tukachinsky [90] established the open WDVV equations for a general class of pairs (X, L) of a symplectic manifold X and a Lagrangian submanifold L, and thereby endowing (X, L) with an associative relative quantum cohomology algebra and a flat relative quantum connection. From a more formal point of view, open WDVV equations induce *F-manifolds* with a compatible flat structure [82, 6], which generalize the notion of Frobenius manifolds. With the open/closed correspondence, one may attempt to establish the open WDVV equations or analogous relations, as well as the induced structures, for open Gromov-Witten invariants of toric Calabi-Yau 3-orbifolds.

1.7.2 Open/closed BPS correspondence

As a second example, closed Gromov-Witten invariants of Calabi-Yau 3-folds can be alternatively described by a set of invariants known as *BPS invariants*. While Gromov-Witten invariants are rational numbers in general, BPS invariants are conjectured by Gopakumar-Vafa [52, 53] to be *integers* and satisfy certain vainishing properties called *finiteness*, in accordance with physical interpretations; see [58, 36] for proofs via symplectic geometry. Klemm-Pandharipande [66] extended the conjecture to Calabi-Yau 4-folds and confirmed several cases. The conjecture can also been extended to open geometries on Calabi-Yau 3-folds, but just as for 4-folds, is not yet well-understood in general. For toric Calabi-Yau 3-folds, one may attempt to define and compute their open BPS invariants and identify them with the closed BPS invariants of toric Calabi-Yau 4-folds. Moreover, proving Gopakumar-Vafa integrality and finiteness on either side would have implications for the other side as well.

1.7.3 Inner branes

The open/closed correspondence can also be developed for an *inner* brane $\mathcal{L}$, that is, the unique torus invariant line l in X that the coarse moduli L intersects is compact. We expect the results and methods in this work to generalize to the case of inner branes with mild modifications.

1.7.4 Open-closed phase changes and wall-crossings

As emphasized by Lerche-Mayr [70], the open/closed correspondence should be compatible with *phase changes*. Given an open geometry $(\mathcal{X}_-, \mathcal{L}_-)$, we may change the closed phase by performing a crepant transformation, or a birational map that preserves the canonical sheaf, on the toric Calabi-Yau 3-orbifold $\mathcal{X}_-$; we may also change the open phase by moving the Aganagic-Vafa brane $\mathcal{L}_-$ to a different location. For a pair $(\mathcal{X}_+, \mathcal{L}_+)$ arising in this way, the corresponding 4-orbifold $\widetilde{\mathcal{X}}_+$ under the open/closed correspondence is exepcted to differ from the 4-orbifold $\widetilde{\mathcal{X}}_-$ arising from $(\mathcal{X}_-, \mathcal{L}_-)$ also by a crepant transformation. In other words, changes in open-closed phases on 3-orbifolds should be reflected by changes in closed phases on 4-orbifolds. See Section 2.6 for simple examples. Our results in this work and their generalizations to inner branes would be the main ingredients in rigorously establishing this framework.

The correspondence of phase changes above has implications on its own. For instance, it provides an alternative approach to the *Open Crepant Transformation Conjecture* for toric Calabi-Yau 3-orbifolds, which relates the disk invariants of open-closed phases $(\mathcal{X}_\pm, \mathcal{L}_\pm)$ related in the way above (see [94] and previous works summarized therein). Using the open/closed correspondence, one could give an alternative proof of this conjecture based on the well-established Crepant Transformation Conjecture that relates closed Gromov-Witten invariants of toric Calabi-Yau 4-orbifolds $\widetilde{\mathcal{X}}_\pm$ (see [32] and previous works summarized therein).

Chapter 2: Toric geometry and constructions

In this chapter, we define the toric Calabi-Yau 3-orbifold $\mathcal{X}$ and the framed Aganagic-Vafa brane $(\mathcal{L}, f)$. Then, we give the construction of the dual toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$ and describe its relations to the open geometry of $(\mathcal{X}, \mathcal{L}, f)$. In general, we use notations with with tilde ($\tilde{}$) while discussing $\widetilde{\mathcal{X}}$ and its closed Gromov-Witten invariants. We work over $\mathbb{C}$ throughout.

2.1 Preliminaries on toric orbifolds

We start by reviewing the basics of *toric orbifolds*, or *smooth toric Deligne-Mumford stacks* with trivial generic stablizer, and introducing some notations. We refer to [33, 47] for the general theory of toric varieties, and to [14, 45] for the general theory of smooth toric Deligne-Mumford stacks.

2.1.1 Extended stacky fan

Let $\mathcal{Z}$ be an r-dimensional toric orbifold specified by an *extended stacky fan* $\mathbf{\Xi} = (\mathbb{Z}^r, \Xi, \alpha)$ in the sense of Jiang [61], where Ξ is a finite simplicial fan in $\mathbb{R}^r = \mathbb{Z}^r \otimes \mathbb{R}$ and $\alpha : \mathbb{Z}^R \to \mathbb{Z}^r$ is a group homomorphism determined by a list of vectors $(b_1, \dots, b_R)$ in $\mathbb{Z}^r$. The coarse moduli space Z of $\mathcal{Z}$ is the simplicial toric variety defined by the fan Ξ. Since $\mathcal{Z}$ is an orbifold, the Deligne-Mumford torus acting on $\mathcal{Z}$ is also the dense algebraic torus acting on Z, which is isomorphic to $(\mathbb{C}^*)^r$.

For each $d = 0, \dots, r$, let $\Xi(d)$ denote the set of d-dimensional cones in Ξ. In particular, there exists $1 \le R' \le R$ such that

$$\Xi(1) = \{\mathbb{R}_{\ge 0} b_1, \dots, \mathbb{R}_{\ge 0} b_{R'}\}.$$

If $\alpha' : \mathbb{Z}^{R'} \to \mathbb{Z}^r$ is the group homomorphism determined by $(b_1, \dots, b_{R'})$, then the triple $(\mathbb{Z}^r, \Xi, \alpha')$ is the *stacky fan* of $\mathcal{Z}$ in the sense of Borisov-Chen-Smith [14].

For each $\sigma \in \Xi(d)$, let $\mathcal{V}(\sigma) \subseteq \mathcal{Z}$ denote the codimension-d $(\mathbb{C}^*)^r$-invariant closed substack of $\mathcal{Z}$ corresponding to σ. Let $V(\sigma) \subseteq Z$ denote the codimension-d $(\mathbb{C}^*)^r$-orbit closure in Z corresponding to σ, which is the coarse moduli space of $\mathcal{V}(\sigma)$. Let

$$\iota_\sigma : \mathcal{V}(\sigma) \to \mathcal{Z}, \quad V(\sigma) \to Z,$$

denote the inclusion maps.[1]

2.1.2 Stablizers

Let $\sigma \in \Xi(d)$. We set index sets

$$I'_\sigma := \{i \in \{1, \dots, R'\} : \rho_i \subseteq \sigma\}, \qquad I_\sigma := \{1, \dots, R\} \setminus I'_\sigma.$$

Note that $|I'_\sigma| = d$. The generic stablizer group of the substack $\mathcal{V}(\sigma)$, denoted G_σ, is a finite abelian group and can be identified as

$$G_\sigma \cong \left(\mathbb{Z}^r \cap \sum_{i \in I'_\sigma} \mathbb{R} b_i \right) \Big/ \sum_{i \in I'_\sigma} \mathbb{Z} b_i \ .$$

We define

$$\mathrm{Box}(\sigma) := \left\{ v \in \mathbb{Z}^r : v = \sum_{i \in I'_\sigma} c_i b_i \text{ for some } 0 \le c_i < 1 \right\},$$

which gives a set of representatives for G_σ. Given $v = \sum_{i \in I'_\sigma} c_i(v) b_i \in \mathrm{Box}(\sigma)$, we define

$$\mathrm{age}(v) := \sum_{i \in I'_\sigma} c_i(v).$$

Given cones $\tau \subseteq \sigma$ in Ξ, we have natural inclusions $G_\tau \subseteq G_\sigma$, $\mathrm{Box}(\tau) \subseteq \mathrm{Box}(\sigma)$.

[1] By an abuse of notation, in this work, the letter ι will be used to denote various natural inclusion maps of substacks/subvarieties, fixed loci of moduli spaces, or cones. The precise meaning and usage will be made clear in the context.

2.1.3 Fixed points, torus-invariant lines, fundamental groups, flags

For each $\sigma \in \Xi(r)$, let $\mathfrak{p}_\sigma := \mathcal{V}(\sigma)$ denote the corresponding $(\mathbb{C}^*)^r$-fixed point in $\mathcal{Z}$, and $p_\sigma := V(\sigma)$ denote the corresponding $(\mathbb{C}^*)^r$-fixed point in Z. For each $\tau \in \Xi(r-1)$, let $\mathfrak{o}_\tau \cong \mathbb{C}^* \times \mathcal{B}G_\tau$ denote the corresponding $(\mathbb{C}^*)^r$-orbit in $\mathcal{Z}$, and $o_\tau \cong \mathbb{C}^*$ denote the corresponding $(\mathbb{C}^*)^r$-orbit in Z. Let $\mathfrak{l}_\tau := \mathcal{V}(\tau)$ denote the corresponding closed $(\mathbb{C}^*)^r$-invariant line in $\mathcal{Z}$, which is the closure of $\mathfrak{o}_\tau$, and $l_\tau := V(\tau)$ denote the corresponding closed $(\mathbb{C}^*)^r$-invariant line in Z, which is the closure of o_τ. We set

$$\Xi(r-1)_c := \{\tau \in \Xi(r-1) : l_\tau \text{ is compact}\},$$

and define

$$\mathcal{Z}_c^1 := \bigcup_{\tau \in \Xi(r-1)_c} \mathfrak{l}_\tau, \quad Z_c^1 := \bigcup_{\tau \in \Xi(r-1)_c} l_\tau.$$

For each $\tau \in \Xi(r-1)$, let $H_\tau := \pi_1(\mathfrak{o}_\tau)$ be the fundamental group of $\mathfrak{o}_\tau$. The projection $\mathfrak{o}_\tau \to o_\tau$ to the coarse moduli space induces a map

$$\pi_\tau : H_\tau = \pi_1(\mathfrak{o}_\tau) \to \pi_1(o_\tau) \cong \mathbb{Z}$$

on fundamental groups, which fits into a split short exact sequence

$$1 \longrightarrow G_\tau \longrightarrow H_\tau \xrightarrow{\pi_\tau} \mathbb{Z} \longrightarrow 1. \tag{2.1}$$

Let

$$F(\Xi) := \{(\tau, \sigma) \in \Xi(r-1) \times \Xi(r) : \tau \text{ is a face of } \sigma\}$$

be the set of *flags* in Ξ. Given a flag $(\tau, \sigma) \in F(\Xi)$, we have $\mathfrak{p}_\sigma \subset \mathfrak{l}_\tau$ and $p_\sigma \in l_\tau$. Let

$$\chi_{(\tau,\sigma)} : G_\sigma \to \mathbb{C}^*$$

be the representation of G_σ on the tangent line $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\tau$. The image of $\chi_{(\tau,\sigma)}$ is $\mu_{\mathfrak{r}(\tau,\sigma)}$, where

$$\mathfrak{r}(\tau,\sigma) := \frac{|G_\sigma|}{|G_\tau|},$$

and for $r \in \mathbb{Z}_{>0}$, $\mu_r \subset \mathbb{C}^*$ is the cyclic group of r-th roots of unity. The map $\chi_{(\tau,\sigma)}$ and the inclusion $G_\tau \to G_\sigma$ fit into a short exact sequence

$$1 \longrightarrow G_\tau \longrightarrow G_\sigma \xrightarrow{\chi_{(\tau,\sigma)}} \mu_{\mathfrak{r}(\tau,\sigma)} \longrightarrow 1. \tag{2.2}$$

Let $\mathfrak{u}_{(\tau,\sigma)} := \mathfrak{o}_\tau \cup \mathfrak{p}_\sigma$, which is an open substack of $\mathfrak{l}_\tau$. The inclusion $\mathfrak{o}_\tau \to \mathfrak{u}_{(\tau,\sigma)}$ induces a surjective map

$$\pi_{(\tau,\sigma)} : H_\tau = \pi_1(\mathfrak{o}_\tau) \to \pi_1(\mathfrak{u}_{(\tau,\sigma)}) \cong G_\sigma \tag{2.3}$$

on fundamental groups which together with (2.1), (2.2) fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_\tau & \longrightarrow & H_\tau & \xrightarrow{\pi_\tau} & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi_{(\tau,\sigma)}} & & \downarrow{\scriptstyle d \mapsto e^{2\pi\sqrt{-1}d/\mathfrak{r}(\tau,\sigma)}} & & \\ 1 & \longrightarrow & G_\tau & \longrightarrow & G_\sigma & \xrightarrow[\chi_{(\tau,\sigma)}]{} & \mu_{\mathfrak{r}(\tau,\sigma)} & \longrightarrow & 1. \end{array}$$

2.1.4 Chen-Ruan orbifold cohomology

Let

$$\mathrm{Box}(\mathcal{Z}) := \bigcup_{\text{cone } \sigma \text{ in } \Xi} \mathrm{Box}(\sigma) = \bigcup_{\text{maximal cone } \sigma \text{ in } \Xi} \mathrm{Box}(\sigma),$$

which indexes the inertia components of $\mathbb{Z}$. The inertia stack of $\mathcal{Z}$ is

$$\mathcal{IZ} = \bigsqcup_{j \in \mathrm{Box}(\mathcal{Z})} \mathcal{Z}_j.$$

In particular, $\mathcal{Z}_{\vec{0}} = \mathcal{Z}$ is the untwisted sector. Let

$$\mathrm{inv}^* : \mathcal{IZ} \to \mathcal{IZ}$$

denote the involution on $\mathcal{IZ}$ which (z,g) with $z \in \mathcal{Z}, g \in \mathrm{Aut}(z)$ to (z,g^{-1})

As a graded vector space over $\mathbb{Q}$ (and as the state-space of relevant quantum theory in physics [95]), *Chen-Ruan cohomology group* [25] of $\mathcal{Z}$ is defined as

$$H^*_{\mathrm{CR}}(\mathcal{Z};\mathbb{Q}) := \bigoplus_{j \in \mathrm{Box}(\mathcal{Z})} H^*(\mathcal{Z}_j;\mathbb{Q})[2\mathrm{age}(j)],$$

where $[2\mathrm{age}(j)]$ denotes a degree shift by $2\mathrm{age}(j)$. We write $\mathbf{1}_j$ for the unit of $H^*(\mathcal{Z}_j;\mathbb{Q})$, viewed as an element of $H^{2\mathrm{age}(j)}_{\mathrm{CR}}(\mathcal{Z};\mathbb{Q})$. In addition, for any subtorus $Q \subseteq (\mathbb{C}^*)^r$, the *$Q$-equivariant* Chen-Ruan cohomology group of $\mathcal{Z}$ is

$$H^*_{\mathrm{CR},Q}(\mathcal{Z};\mathbb{Q}) := \bigoplus_{j \in \mathrm{Box}(\mathcal{Z})} H^*_Q(\mathcal{Z}_j;\mathbb{Q})[2\mathrm{age}(j)],$$

which is a module over $H^*_Q(\mathrm{pt};\mathbb{Q})$. The above definitions can be extended to $\mathbb{C}$-coefficients.

2.2 The toric Calabi-Yau 3-orbifold $\mathcal{X}$

Let $N \cong \mathbb{Z}^3$. Let $\mathcal{X}$ be a toric Calabi-Yau 3-orbifold defined by an extended stacky fan $\mathbf{\Sigma} = (N, \Sigma, \alpha)$ and X be the coarse moduli space of $\mathcal{X}$. Let $T = N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^3$ be the Deligne-Mumford torus acting on $\mathcal{X}$ (and X).

Assumption 2.1. We make the following assumptions:

- X is *Calabi-Yau*: The canonical bundle K_X of X is trivial.
- X is *semi-projective*: X is projective over its affinization $\mathrm{Spec}(H^0(X, \mathcal{O}_X))$.

Suppose the homomorphism $\alpha : \mathbb{Z}^R \to N$ is specified by $b_i = \alpha(e_i)$ for each i, where $\{e_1, \ldots, e_R\}$ is the standard basis for $\mathbb{Z}^R$. Let $M := \mathrm{Hom}(N, \mathbb{Z})$, which is canonically identified with the character lattice $\mathrm{Hom}(T, \mathbb{C}^*)$ of T. The Calabi-Yau condition implies the existence of a character $u_3 \in M$ such that, if $R' = |\Sigma(1)|$ is the number of rays, then $\langle u_3, b_i \rangle = 1$ for all $i = 1, \ldots, R'$. That is, $b_1, \ldots, b_{R'}$ belong to $N' \times \{1\}$ where $N' := \ker(u_3) \subset N$.

Let Δ be the cross section of the support $|\Sigma|$ of Σ in the hyperplane $N'_{\mathbb{R}} \times \{1\}$.[2] Then $|\Sigma|$ is the cone over Δ, and Σ induces a triangulation of Δ. Moreover, the semi-projectivity condition implies that Δ is convex. We assume that the additional lattice points $b_{R'+1}, \ldots, b_R$ are chosen in a way that $(b_1, \ldots, b_R)$ is a listing of the points in $\Delta \cap N$. In particular, the homomorphism α is surjective and fits into the following short exact sequence of lattices:

$$0 \longrightarrow \mathbb{L} \xrightarrow{\psi} \mathbb{Z}^R \xrightarrow{\alpha} N \longrightarrow 0, \tag{2.4}$$

where $\mathbb{L} := \ker(\alpha) \cong \mathbb{Z}^{R-3}$.

Let $G := \mathbb{L} \otimes (\mathbb{C}^*) \cong (\mathbb{C}^*)^{R-3}$. Let $\{\epsilon_1, \ldots, \epsilon_{R-3}\}$ be a basis for $\mathbb{L}$, and for each $a = 1, \ldots, R-3$, let

$$l^{(a)} = (l_1^{(a)}, \ldots, l_R^{(a)}) := \psi(\epsilon_a) \in \mathbb{Z}^R.$$

The vectors $l^{(a)}$ are known as *charge vectors*, which describe the linear action of G on $\mathbb{C}^R = \mathbb{Z}^R \otimes \mathbb{C} = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_R])$ induced by the inclusion ψ, as follows:

$$(s_1, \ldots, s_{R-3}) \cdot (x_1, \ldots, x_R) = \left(\prod_{a=1}^{R-3} s_a^{l_1^{(a)}} x_1, \ldots, \prod_{a=1}^{R-3} s_a^{l_R^{(a)}} x_R \right), \tag{2.5}$$

where $(s_1, \ldots, s_{R-3})$ are coordinates on G specified by the basis $\{\epsilon_1, \ldots, \epsilon_{R-3}\}$. Under this action, $\mathcal{X}$ can be described as the quotient stack

$$\mathcal{X} = \left[((\mathbb{C}^{R'} \setminus Z(\Sigma)) \times (\mathbb{C}^*)^{R-R'})/G \right], \tag{2.6}$$

where $Z(\Sigma)$ is a closed subvariety of $\mathbb{C}^{R'}$ defined by Σ. The semi-projectivity condition (Assumption 2.1) implies that the above is a GIT quotient.

Given a flag $(\tau, \sigma) \in F(\Sigma)$, G_τ is a cyclic subgroup of G_σ. We define

$$\mathfrak{m}(\tau, \sigma) := |G_\tau|.$$

[2] In this work, for a lattice L and a field $\mathbb{F} = \mathbb{R}, \mathbb{Q}$, or $\mathbb{C}$, we let $L_{\mathbb{F}}$ denote $L \otimes_{\mathbb{Z}} \mathbb{F}$.

2.3 The framed Aganagic-Vafa brane $(\mathcal{L}, f)$

In this section, we describe the symplectic structure on $\mathcal{X}$ and define the Aganagic-Vafa brane $\mathcal{L}$, following Fang-Liu-Tseng [42]. Let $G_{\mathbb{R}} \cong U(1)^{R-3}$ be the maximal compact subgroup of G, which carries a Hamiltonian action on $\mathbb{C}^R$ induced by (2.5). The moment map $\widetilde{\mu} : \mathbb{C}^R \to \mathfrak{g}_{\mathbb{R}}^*$ of the $G_{\mathbb{R}}$-action, where $\mathfrak{g}_{\mathbb{R}}^* \cong \mathbb{R}^{R-3}$ is the dual of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$, can be described as

$$\widetilde{\mu}(x_1, \ldots, x_R) = \left(\sum_{i=1}^{R} l_i^{(1)} |x_i|^2, \ldots, \sum_{i=1}^{R} l_i^{(R-3)} |x_i|^2 \right).$$

Applying $\mathrm{Hom}(-, \mathbb{Z})$ to (2.4), we obtain a short exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha^\vee} \mathbb{Z}^R \xrightarrow{\psi^\vee} \mathbb{L}^\vee \longrightarrow 0. \tag{2.7}$$

There is a canonical identification $\mathfrak{g}_{\mathbb{R}}^* \cong \mathbb{L}_{\mathbb{R}}^\vee$. Let $\{e_1^\vee, \ldots, e_R^\vee\}$ be the basis dual to $\{e_1, \ldots, e_R\}$. For each $i = 1, \ldots, R$, define

$$D_i := \psi^\vee(e_i^\vee) \in \mathbb{L}^\vee.$$

For each maximal cone $\sigma \in \Sigma(3)$, define the *extended σ-nef cone* as

$$\widetilde{\mathrm{Nef}}(\sigma) := \sum_{i \in I_\sigma} \mathbb{R}_{\geq 0} D_i.$$

The *extended nef cone* of $\mathcal{X}$ is defined to be

$$\widetilde{\mathrm{Nef}}(\mathcal{X}) := \bigcap_{\sigma \in \Sigma(3)} \widetilde{\mathrm{Nef}}(\sigma),$$

which is an $(R-3)$-dimensional simplicial cone in $\mathbb{L}_{\mathbb{R}}^\vee$.

Let $r = (r_1, \ldots, r_{R-3})$ be a point in the interior of $\widetilde{\mathrm{Nef}}(\mathcal{X})$, which can be viewed as an *extended Kähler class* of $\mathcal{X}$. Then $\mathcal{X}$ is the symplectic quotient

$$[\widetilde{\mu}^{-1}(r)/G_{\mathbb{R}}],$$

and the standard Kähler form

$$\frac{\sqrt{-1}}{2}\sum_{i=1}^{R} dx_i \wedge d\bar{x}_i$$

on $\mathbb{C}^R$ descends to a Kähler form ω_r on $\mathcal{X}$.

An *Aganagic-Vafa brane* [5, 42] $\mathcal{L}$ in $\mathcal{X}$ is a Lagrangian suborbifold of form

$$\mathcal{L} := \left[\left\{(x_1, \dots, x_R) \in \widetilde{\mu}^{-1}(r) : \sum_{i=1}^{R} l'_i |x_i|^2 = c', \sum_{i=1}^{R} l''_i |x_i|^2 = c'', \arg(\prod_{i=1}^{R} x_i) = c'''\right\} \Big/ G_{\mathbb{R}}\right],$$

where $c', c'', c''' \in \mathbb{R}$ are constants and vectors $l' = (l'_1, \dots, l'_R), l'' = (l''_1, \dots, l''_R) \in \mathbb{Z}^R$ satisfy

$$\sum_{i=1}^{R} l'_i = \sum_{i=1}^{R} l''_i = 0.$$

Let $L \subset X$ be the coarse moduli space of $\mathcal{L}$. The brane $\mathcal{L}$ intersects a unique T-invariant line $\mathfrak{l}_{\tau_0}$ in $\mathcal{X}$, where $\tau_0 \in \Sigma(2)$.

Assumption 2.2. We assume that $\mathcal{L}$ is an *outer* brane: $\tau_0 \notin \Sigma(2)_c$.

The inclusions $\mathcal{L} \cap \mathfrak{l}_{\tau_0} \to \mathcal{L}$, $\mathcal{L} \cap \mathfrak{l}_{\tau_0} \to \mathfrak{o}_{\tau_0}$ are homotopy equivalences. We have

$$\pi_1(\mathcal{L}) = \pi_1(\mathfrak{o}_{\tau_0}) = H_{\tau_0} \cong \mathbb{Z} \times G_{\tau_0},$$

which is abelian and thus also equal to $H_1(\mathcal{L}; \mathbb{Z})$.

Moreover, τ_0 is contained in a unique 3-cone $\sigma_0 \in \Sigma(3)$. After a possible permutation of indices, we assume that $I'_{\sigma_0} = \{1, 2, 3\}$ with b_1, b_2, b_3 appearing in $N' \times \{1\}$ in counterclockwise order, and that $I'_{\tau_0} = \{2, 3\}$. Let $\tau_2, \tau_3 \in \Sigma(2)$ be the other two facets of σ_0, with $I'_{\tau_2} = \{1, 3\}$ and $I'_{\tau_3} = \{1, 2\}$. See Figure 2.1.

Let

$$\mathfrak{m} := \mathfrak{m}(\tau_0, \sigma_0), \quad \mathfrak{r} := \mathfrak{r}(\tau_0, \sigma_0).$$

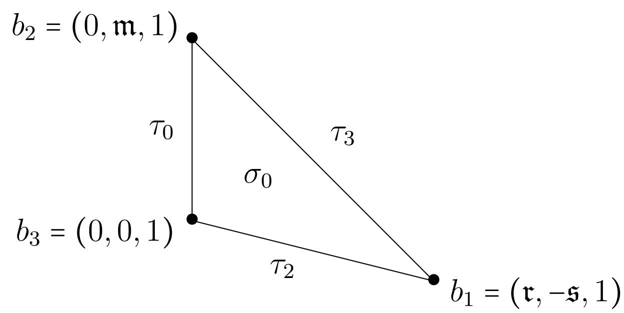

Figure 2.1: Distinguished rays and cones associated to the Aganagic-Vafa outer brane $\mathcal{L}$.

The flag (τ_0, σ_0) determines a basis $\{v_1, v_2, v_3\}$ for N under which

$$b_1 = (\mathfrak{r}, -\mathfrak{s}, 1), \quad b_2 = (0, \mathfrak{m}, 1), \quad b_3 = (0,0,1)$$

for some $\mathfrak{s} \in \{0, 1, \dots, \mathfrak{r} - 1\}$. For $i = 1, \dots, R$, let $(m_i, n_i, 1)$ be the coordiante of b_i under the basis $\{v_1, v_2, v_3\}$. Since $\mathcal{L}$ is outer, we have $m_i \geq 0$ for all i. Let $u_1, u_2 \in M$ such that $\{u_1, u_2, u_3\}$ is the basis dual to $\{v_1, v_2, v_3\}$. The corresponding characters $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3\}$ of T serve as equivariant parameters:

$$H_T^*(\mathrm{pt}; \mathbb{Z}) = \mathbb{Z}[\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3].$$

Let $T' := \ker(\mathsf{u}_3)$ be the 2-dimensional *Calabi-Yau subtorus* of T and $T'_{\mathbb{R}}$ be the maximal compact subgroup of T'. Then $\mathcal{L}$ is preserved under the $T'_{\mathbb{R}}$-action. We have

$$H_{T'}^*(\mathrm{pt}; \mathbb{Z}) = H_{T'_{\mathbb{R}}}^*(\mathrm{pt}; \mathbb{Z}) = \mathbb{Z}[\mathsf{u}_1, \mathsf{u}_2].$$

Let

$$\mathcal{Q}_{T'} := \mathbb{Q}(\mathsf{u}_1, \mathsf{u}_2)$$

be the fractional field of $H_{T'}^*(\mathrm{pt}; \mathbb{Q}) = H_{T'_{\mathbb{R}}}^*(\mathrm{pt}; \mathbb{Q})$.

2.3.1 Framing

Let $f \in \mathbb{Z}$ be an integer called the *framing* on $\mathcal{L}$. This determines a 1-dimensional *framing subtorus* $T_f := \ker(\mathsf{u}_2 - f\mathsf{u}_1)$ of T'. We have

$$H^*_{T_f}(\mathrm{pt};\mathbb{Z}) = \mathbb{Z}[\mathsf{u}_1].$$

Let

$$\mathcal{Q}_{T_f} := \mathbb{Q}(\mathsf{u}_1)$$

be the fractional field of $H^*_{T_f}(\mathrm{pt};\mathbb{Q})$.

2.4 The corresponding toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$

The corresponding toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$ is specified by the extended stacky fan $\widetilde{\mathbf{\Sigma}} = (\widetilde{\Sigma}, \widetilde{N}, \widetilde{\alpha})$ where:

- $\widetilde{N} := N \oplus \mathbb{Z}v_4 \cong \mathbb{Z}^4$ is a 4-dimensional lattice.
- Let $\mathbb{Z}^{R+2} \cong \mathbb{Z}^R \oplus \mathbb{Z}^2$ whose standard basis extends $\{e_1, \dots, e_R\}$ by e_{R+1}, e_{R+2}. The homomorphism $\widetilde{\alpha} : \mathbb{Z}^{R+2} \to \widetilde{N}$ maps $e_1, \dots, e_{R+2}$ to $\widetilde{b}_1, \dots, \widetilde{b}_{R+2} \in \widetilde{N}$, where

$$\widetilde{b}_i = (m_i, n_i, 1, 0) \quad \text{for } i = 1, \dots, R,$$

$$\widetilde{b}_{R+1} = (-1, -f, 1, 1), \quad \widetilde{b}_{R+2} = (0, 0, 1, 1).$$

- $\widetilde{\Sigma}$ has the following description: Let $\widetilde{M} := \mathrm{Hom}(\widetilde{N}, \mathbb{Z})$ with basis $\{u_1, u_2, u_3, u_4\}$ dual to $\{v_1, v_2, v_3, v_4\}$. We abuse notation here since $u_1, u_2, u_3 \in \widetilde{M}$ maps to $u_1, u_2, u_3 \in M$ respectively under the natural projection. Let $\widetilde{\Delta}$ be the convex hull of $\{\widetilde{b}_1, \dots, \widetilde{b}_{R+2}\}$ in $\widetilde{N}_{\mathbb{R}}$, which is a 3-dimensional convex polytope contained in the hyperplane $\widetilde{N}'_{\mathbb{R}} \times \{1\}$, where $\widetilde{N}' := \ker(u_3) \subset \widetilde{N}$. Note that $\widetilde{\Delta}$ contains Δ as a facet. Let $\widetilde{\Delta}_0$ be the convex hull of

$P \cup \{\widetilde{b}_{R+2}\}$. We triangulate $\widetilde{\Delta}_0$ by taking the cone over the triangulation of Δ induced by Σ, and extend this to a triangulation of $\widetilde{\Delta}$ that includes the simplex

$$\{2, 3, R+1, R+2\}.$$

Finally, let $\widetilde{\Sigma}$ be the fan which has support $|\widetilde{\Sigma}|$ equal to the cone over $\widetilde{\Delta}$ and is induced by the above triangulation of $\widetilde{\Delta}$.

Let $\widetilde{X}$ be the coarse moduli space of $\widetilde{\mathcal{X}}$, which is the simplicial toric 4-fold determined by the fan $\widetilde{\Sigma}$. By construction, $\widetilde{X}$ is both *Calabi-Yau* and *semi-projective*.

Note that

$$\widetilde{\Sigma}(1) = \{\widetilde{\rho}_1, \dots, \widetilde{\rho}_{R'}, \widetilde{\rho}_{R+1}, \widetilde{\rho}_{R+2}\} \quad \text{where } \widetilde{\rho}_i := \mathbb{R}_{\geq 0}\widetilde{b}_i.$$

Let $\widetilde{\Sigma}_0$ be the subfan of $\widetilde{\Sigma}$ whose support is the cone over $\widetilde{\Delta}_0$. We have

$$\widetilde{\Sigma}_0(1) = \{\widetilde{\rho}_1, \dots, \widetilde{\rho}_{R'}, \widetilde{\rho}_{R+2}\}.$$

The inclusions of fans

$$\Sigma \to \widetilde{\Sigma}_0 \to \widetilde{\Sigma}$$

induce inclusions of toric orbifolds

$$\mathcal{X} \to \mathcal{X} \times \mathbb{C} \to \widetilde{\mathcal{X}},$$

and we denote the composition by $\iota : \mathcal{X} \to \widetilde{\mathcal{X}}$.

The lattice map $\widetilde{\alpha} : \mathbb{Z}^{R+2} \to \widetilde{N}$ fits into the short exact sequence at the second row of the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{L} & \xrightarrow{\psi} & \mathbb{Z}^R & \xrightarrow{\alpha} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{\mathbb{L}} & \xrightarrow{\widetilde{\psi}} & \mathbb{Z}^{R+2} & \xrightarrow{\widetilde{\alpha}} & \widetilde{N} & \longrightarrow & 0,
\end{array} \tag{2.8}$$

where $\widetilde{\mathbb{L}} := \ker(\widetilde{\alpha}) \cong \mathbb{Z}^{R-2}$. We can identify $\mathbb{L}$ as a sublattice of $\widetilde{\mathbb{L}}$ and extend $\{\epsilon_1, \dots, \epsilon_{R-3}\}$ to a basis for $\widetilde{\mathbb{L}}$ by including an additional vector $\epsilon_{R-2} \in \widetilde{\mathbb{L}}$, in a way that

$$\widetilde{l}^{(a)} := \widetilde{\psi}(\epsilon_a) = (l_1^{(a)}, \dots, l_R^{(a)}, 0, 0) \quad \text{for } a = 1, \dots, R-3.$$

Denote $\widetilde{l}^{(R-2)} := \widetilde{\psi}(\epsilon_{R-2})$. Then similar to (2.6), $\widetilde{\mathcal{X}}$ can be described as a quotient stack $\widetilde{\mathcal{X}} = [((\mathbb{C}^{R'+2} \setminus Z(\widetilde{\Sigma})) \times (\mathbb{C}^*)^{R-R'})/\widetilde{G}]$ which is also a GIT quotient, where the linear action of $\widetilde{G} := \widetilde{\mathbb{L}} \otimes (\mathbb{C}^*) = (\mathbb{C}^*)^{R-2}$ on $\mathbb{C}^{R+2}$ is specified by $\widetilde{l}^{(1)}, \dots, \widetilde{l}^{(R-2)}$.

Let $\widetilde{T} = \widetilde{N} \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^4$ be the complex algebraic torus acting on $\widetilde{\mathcal{X}}$ (and $\widetilde{X}$), which contains T as a subtorus. The character lattice of $\widetilde{T}$ is canonically identified with $\widetilde{M}$. We have

$$H^*_{\widetilde{T}}(\mathrm{pt}; \mathbb{Z}) = \mathbb{Z}[\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4],$$

where $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4\}$ are characters corresponding to $\{u_1, u_2, u_3, u_4\}$. Let $\widetilde{T}' := \ker(\mathsf{u}_3)$ be the 3-dimensional *Calabi-Yau subtorus* of $\widetilde{T}$, which contains T' as a subtorus. We have

$$H^*_{\widetilde{T}'}(\mathrm{pt}; \mathbb{Z}) = \mathbb{Z}[\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4].$$

Let

$$\mathcal{Q}_{\widetilde{T}'} := \mathbb{Q}(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4)$$

be the fractional field of $H^*_{\widetilde{T}'}(\mathrm{pt}; \mathbb{Q})$.

2.5 Comparison of toric geometry and topology

We draw additional comparisons between the toric geometry and topology of $(\mathcal{X}, \mathcal{L})$ and $\widetilde{\mathcal{X}}$, mainly regarding the second homology groups, stablizer groups of torus-invariant substacks, and Chen-Ruan orbifold cohomology groups.

2.5.1 Cones and flags

Recall that we have an inclusion $\iota : X \to \widetilde{X}$. On the level of cones, we have an injective map

$$\iota : \Sigma(d) \to \widetilde{\Sigma}(d+1) \tag{2.9}$$

for each $d = 0, 1, 2, 3$, given by

$$I'_{\iota(\sigma)} = I'_\sigma \sqcup \{R+2\} \quad \text{for all } \sigma \in \Sigma(d).$$

In particular, $\iota(\Sigma(3))$ gives a set of 4-cones in $\widetilde{\Sigma}$. Given any other 4-cone $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$, we have that $R+1, R+2 \in I'_{\widetilde{\sigma}}$ and the other two indices specify a 2-cone $\delta_0(\widetilde{\sigma}) \in \Sigma(2) \smallsetminus \Sigma(2)_c$. This yields a map

$$\delta_0 : \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3)) \to \Sigma(2) \smallsetminus \Sigma(2)_c. \tag{2.10}$$

Then for any τ in the image of δ_0, we have $\iota(\tau) \in \widetilde{\Sigma}(3)_c$. In particular, let $\widetilde{\sigma}_0 \in \widetilde{\Sigma}(4)$ be the 4-cone with

$$I'_{\widetilde{\sigma}_0} = \{2, 3, R+1, R+2\}.$$

Then $\delta_0(\widetilde{\sigma}_0) = \tau_0$. For each $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$, we set

$$I'_{\delta_0(\widetilde{\sigma})} = \{i_2(\widetilde{\sigma}), i_3(\widetilde{\sigma})\}$$

such that $b_{i_2(\widetilde{\sigma})}, b_{i_3(\widetilde{\sigma})}$ appear on the boundary of Δ in $N'_{\mathbb{R}} \times \{1\}$ in counterclockwise order. In particular, $i_2(\widetilde{\sigma}_0) = 2$ and $i_3(\widetilde{\sigma}_0) = 3$. We have

$$I'_{\widetilde{\sigma}} = \{i_2(\widetilde{\sigma}), i_3(\widetilde{\sigma}), R+1, R+2\}.$$

There is an induced injective map of flags $\iota : F(\Sigma) \to F(\widetilde{\Sigma})$ given by $\iota(\tau, \sigma) = (\iota(\tau), \iota(\sigma))$. Moreover, for any 3-cone $\sigma \in \Sigma(3)$, we have $(\sigma, \iota(\sigma)) \in F(\widetilde{\Sigma})$. Any other flag in $\widetilde{\Sigma}$ consitutes of a

4-cone $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$ and one of its facets. In particular, we have $(\iota(\delta_0(\widetilde{\sigma})), \widetilde{\sigma}) \in F(\widetilde{\Sigma})$. The other facets $\delta_2(\widetilde{\sigma})$, $\delta_3(\widetilde{\sigma})$, $\delta_4(\widetilde{\sigma}) \in \widetilde{\Sigma}(3)$ are specified by

$$I'_{\delta_2(\widetilde{\sigma})} = \{i_3(\widetilde{\sigma}), R+1, R+2\}, \quad I'_{\delta_3(\widetilde{\sigma})} = \{i_2(\widetilde{\sigma}), R+1, R+2\}, \quad I'_{\delta_4(\widetilde{\sigma})} = \{i_2(\widetilde{\sigma}), i_3(\widetilde{\sigma}), R+1\}.$$

This yields maps

$$\delta_2, \delta_3, \delta_4 : \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3)) \to \widetilde{\Sigma}(3).$$

2.5.2 Stablizers

For any cone σ in Σ, we have

$$G_{\iota(\sigma)} = G_\sigma, \quad \mathrm{Box}(\iota(\sigma)) = \mathrm{Box}(\sigma).$$

Here and in the rest of this subsection, in view of the natural inclusion $N \to \widetilde{N}$ which also makes σ a cone in $\widetilde{\Sigma}$, it does not make a difference to define $\mathrm{Box}(\sigma)$ with respect to Σ or $\widetilde{\Sigma}$. In particular, if $\sigma \in \Sigma(3)$, we have

$$\mathfrak{r}(\sigma, \iota(\sigma)) = 1$$

for the flag $(\sigma, \iota(\sigma))$; for any flag $(\tau, \sigma) \in F(\Sigma)$, $G_{\iota(\tau)}$ is a cyclic subgroup of $G_{\iota(\sigma)}$ of order $\mathfrak{m}(\tau, \sigma)$, and

$$\mathfrak{r}(\iota(\tau, \sigma)) = \mathfrak{r}(\tau, \sigma).$$

Moreover, a direct computation gives the following charaterization of stablizers of cones in $\widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$.

Lemma 2.3. *For any $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$, $G_{\widetilde{\sigma}}$ is a cyclic group of order*

$$|G_{\widetilde{\sigma}}| = f(m_{i_3(\widetilde{\sigma})} - m_{i_2(\widetilde{\sigma})}) + (n_{i_2(\widetilde{\sigma})} - n_{i_3(\widetilde{\sigma})}).$$

If $|G_{\widetilde{\sigma}}| > 1$*, then a generator is given by the element*

$$\frac{|G_{\widetilde{\sigma}}|-1}{|G_{\widetilde{\sigma}}|}\widetilde{b}_{i_2(\widetilde{\sigma})} + \frac{1}{|G_{\widetilde{\sigma}}|}\widetilde{b}_{i_3(\widetilde{\sigma})} + \langle\frac{m_{i_3(\widetilde{\sigma})}-m_{i_2(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\rangle\widetilde{b}_{R+1} + \langle 1-\frac{m_{i_3(\widetilde{\sigma})}-m_{i_2(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\rangle\widetilde{b}_{R+2}$$

in $\mathrm{Box}(\widetilde{\sigma})$.

Elements of age at most 1 are precisely those contained in the cyclic subgroup

$$G_{\iota(\delta_0(\widetilde{\sigma}))} = G_{\delta_0(\widetilde{\sigma})} = G_{\delta_4(\widetilde{\sigma})} \cong \mu_{\gcd(|m_{i_2(\widetilde{\sigma})}-m_{i_3(\widetilde{\sigma})}|,|n_{i_2(\widetilde{\sigma})}-n_{i_3(\widetilde{\sigma})}|)}.$$

In the example of $\widetilde{\sigma}_0$, this subgroup is the entire $G_{\widetilde{\sigma}_0}$ and has order $\mathfrak{m}$; when $\mathfrak{m} > 1$, the generator presented in Lemma 2.3 is $(0,1,1,0)$. In addition, we have

$$G_{\delta_2(\widetilde{\sigma})} = G_{\delta_3(\widetilde{\sigma})} = \{1\}.$$

Elements in $G_{\widetilde{\sigma}} \setminus G_{\delta_0(\widetilde{\sigma})}$ all have age 2.

In summary, there is an inclusion

$$\mathrm{Box}(\mathcal{X}) \subseteq \mathrm{Box}(\widetilde{\mathcal{X}}),$$

where any $j \in \mathrm{Box}(\widetilde{\mathcal{X}}) \setminus \mathrm{Box}(\mathcal{X})$ is contained in $\mathrm{Box}(\widetilde{\sigma}) \setminus \mathrm{Box}(\delta_0(\widetilde{\sigma}))$ for some $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$ and has age 2.

2.5.3 Second homology

The inclusion $\iota : X \to \widetilde{X}$ induces an inclusion on second integral homology, described as follows. The inclusion $\iota : \Sigma(2) \to \widetilde{\Sigma}(3)$ restricts to an inclusion $\Sigma(2)_c \to \widetilde{\Sigma}(3)_c$. Then we have

$$\iota_* : H_2(X;\mathbb{Z}) \to H_2(\widetilde{X};\mathbb{Z}), \quad [l_\tau] \mapsto [l_{\iota(\tau)}] \quad \text{for all } \tau \in \Sigma(2)_c.$$

Moreover, the brane L bounds a holomorphic disk B in l_{τ_0}, which we orient by the holomorphic structure of X. Then

$$H_1(L;\mathbb{Z}) \cong \mathbb{Z}\partial[B], \quad H_2(X,L;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus \mathbb{Z}[B].$$

The map ι_* on second homology above can be extended to an inclusion

$$\iota_* : H_2(X,L;\mathbb{Z}) \to H_2(\widetilde{X};\mathbb{Z}) \tag{2.11}$$

which maps $[B]$ to $[l_{\iota(\tau_0)}]$. Note that $\iota(\tau_0) \in \widetilde{\Sigma}(3)_c$.

2.5.4 Chen-Ruan cohomology ring

The Chen-Ruan cohomology ring of a smooth toric Deligne-Mumford stack is equal to its orbifold Chow ring. Borisov-Chen-Smith [14] provided an explicit description of the orbifold Chow ring of smooth toric Deligne-Mumford stacks with projective coarse moduli spaces. For any toric Deligne-Mumford stack $\mathcal{Z}$, Borisov-Horja [15, Section 3] introduced the SR-cohomology ring which is determined by the stacky fan. When the coarse moduli space Z is projective, the SR-cohomology ring of $\mathcal{Z}$ coincides with the orbifold Chow ring of $\mathcal{Z}$ by [14, Theorem 1.1]; when Z is not projective, the SR-cohomology ring and the orbifold Chow ring can be different, even when $\mathcal{Z} = Z$ is a smooth toric variety. Jiang-Tseng [62] generalized [14, Theorem 1.1] to smooth toric Deligne-Mumford stacks with semi-projective coarse moduli spaces; the formula is in terms of the extended stacky fan introduced by Jiang [61]. Applying [62, Theorem 1.1] to $\mathcal{X}$ and $\widetilde{\mathcal{X}}$, we obtain the following statements.

- As graded $\mathbb{Q}$-algebras, $H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$ is generated by $\{\mathbf{1}_j : j \in \mathrm{Box}(\mathcal{X})\}$ and the divisor classes

$$\mathcal{D}_i := [\mathcal{V}(\rho_i)] \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Z}), \quad i = 1,\dots,R',$$

and $H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$ is generated by $\{\mathbf{1}_j : j \in \mathrm{Box}(\widetilde{\mathcal{X}})\}$ and the divisor classes

$$\widetilde{\mathcal{D}}_i := [\mathcal{V}(\widetilde{\rho}_i)] \in H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Z}), \quad i = 1, \dots, R', R+1, R+2.$$

- The inclusion $\iota : \mathcal{X} \to \widetilde{\mathcal{X}}$ induces a homogenous $\mathbb{Q}$-algebra homomorphism

$$\iota^* : H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q}) \to H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$$

specified by

$$\mathbf{1}_j \mapsto \begin{cases} \mathbf{1}_j & \text{if } j \in \mathrm{Box}(\mathcal{X}) \\ 0 & \text{if } j \in \mathrm{Box}(\widetilde{\mathcal{X}}) \setminus \mathrm{Box}(\mathcal{X}), \end{cases}$$

$$\widetilde{\mathcal{D}}_i \mapsto \mathcal{D}_i \text{ for } i = 1, \dots, R', \quad \widetilde{\mathcal{D}}_{R+1}, \widetilde{\mathcal{D}}_{R+2} \mapsto 0.$$

There is a canonical identification $H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q}) \cong \mathbb{L}^\vee_{\mathbb{Q}}$ which identifies $\mathcal{D}_i$ with D_i for each $i = 1, \dots, R'$ and $\mathbf{1}_{j(i)}$ with D_i if $b_i \in N$ is a representative of $j(i) \in \mathrm{Box}(\mathcal{X})$. On the other hand, applying $\mathrm{Hom}(-,\mathbb{Z})$ to (2.8), we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M} & \xrightarrow{\widetilde{\alpha}^\vee} & \mathbb{Z}^{R+2} & \xrightarrow{\widetilde{\psi}^\vee} & \widetilde{\mathbb{L}}^\vee & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\alpha^\vee} & \mathbb{Z}^R & \xrightarrow{\psi^\vee} & \mathbb{L}^\vee & \longrightarrow & 0 \end{array} \tag{2.12}$$

where the second row is (2.7). Let $\{e_1^\vee, \dots, e_{R+2}^\vee\}$ be the basis of $\mathbb{Z}^{R+2}$ dual to $\{e_1, \dots, e_{R+2}\}$, and for each $i = 1, \dots, R+2$ define

$$\widetilde{D}_i := \widetilde{\psi}^\vee(e_i^\vee) \in \widetilde{\mathbb{L}}^\vee. \tag{2.13}$$

For $i = 1, \dots, R$, $\widetilde{D}_i$ projects to $D_i \in \mathbb{L}^\vee$. Moreover,

$$\widetilde{D}_{R+1} = -\widetilde{D}_{R+2} \in \mathbb{Z}_{\neq 0}\epsilon^\vee_{R-2},$$

where $\{\epsilon_1^\vee, \dots, \epsilon_{R-2}^\vee\}$ be the basis of $\widetilde{\mathbb{L}}^\vee$ dual to $\{\epsilon_1, \dots, \epsilon_{R-2}\}$. There is a canonical identification

$H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q}) \cong \widetilde{\mathbb{L}}^\vee_{\mathbb{Q}}$ which identifies $\widetilde{\mathcal{D}}_i$ with $\widetilde{D}_i$ for each $i = 1, \dots, R', R+1, R+2$ and $\mathbf{1}_{j(i)}$ with $\widetilde{D}_i$ if $\widetilde{b}_i \in \widetilde{N}$ is a representative of $j(i) \in \mathrm{Box}(\widetilde{\mathcal{X}})$. The map $\iota^* : H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q}) \to H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$ is canonically identifed with the projection $\widetilde{\mathbb{L}}^\vee_{\mathbb{Q}} \to \mathbb{L}^\vee_{\mathbb{Q}}$.

Moreover, we define T'-equivariant divisor classes

$$\mathcal{D}_i^{T'} := [\mathcal{V}(\rho_i)] \in H^2_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q}), \quad i = 1, \dots, R'$$

of $\mathcal{X}$ and $\widetilde{T}'$-equivariant divisor classes

$$\widetilde{\mathcal{D}}_i^{\widetilde{T}'} := [\mathcal{V}(\widetilde{\rho}_i)] \in H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q}), \quad i = 1, \dots, R', R+1, R+2$$

of $\widetilde{\mathcal{X}}$. The non-equivariant limit of each $\mathcal{D}_i^{T'}$ (resp. $\widetilde{\mathcal{D}}_i^{\widetilde{T}'}$) is $\mathcal{D}_i$ (resp. $\widetilde{\mathcal{D}}_i$). Moreover, for $i = 1, \dots, R'$, $\widetilde{\mathcal{D}}_i^{\widetilde{T}'}$ restricts to $\mathcal{D}_i^{T'}$ under

$$H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q}) \to H^*_{\mathrm{CR},T'}(\widetilde{\mathcal{X}};\mathbb{Q}) \to H^*_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q}) \tag{2.14}$$

where the first map is induced by $T' \subset \widetilde{T}'$ and the second maps is induced by $\iota : \mathcal{X} \to \widetilde{\mathcal{X}}$.

2.5.5 Convention for equivariant lifts

The Gromov-Witten invariants considered in this work may take in suitable equivariant lifts of classes in $H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$ and $H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$ as insertions. Here, we specify our convention for choosing the lifts.

Convention 2.4. Given $\gamma \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$, we choose the unique lifts

$$\gamma^{T'} \in H^2_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q}), \quad \widetilde{\gamma} \in H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q}), \quad \widetilde{\gamma}^{\widetilde{T}'} \in H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$$

of γ that are consistent with the commutative diagram

$$
\begin{array}{ccccc}
H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q}) & \longrightarrow & H^2_{\mathrm{CR},T'}(\widetilde{\mathcal{X}};\mathbb{Q}) & \longrightarrow & H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q}) \\
 & \searrow & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\
 & & H^2_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q}) & \longrightarrow & H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\gamma}^{\widetilde{T}'} & \longmapsto & \widetilde{\gamma} \\
 & \searrow & \downarrow \\
 & \gamma^{T'} \longmapsto & \gamma
\end{array}
$$

and satisfy the following: If $\gamma \in H^2(\mathcal{X};\mathbb{Q})$, then

- $\iota^*_{\sigma_0}(\gamma^{T'}) = 0$;
- $\widetilde{\gamma}$ belongs to the span of $\widetilde{D}_1, \ldots, \widetilde{D}_{R'}$ in $H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$;
- $\iota^*_{\iota(\sigma_0)}(\widetilde{\gamma}^{\widetilde{T}'}) = \iota^*_{\widetilde{\sigma}_0}(\widetilde{\gamma}^{\widetilde{T}'}) = 0$.

If $\gamma = \mathbf{1}_j$ for some $j \in \mathrm{Box}(\mathcal{X}) \subset \mathrm{Box}(\widetilde{\mathcal{X}})$, then all lifts are $\mathbf{1}_j$.

Given $\widetilde{\gamma} \in H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$, we choose the unique lift

$$\widetilde{\gamma}^{\widetilde{T}'} \in H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$$

of $\widetilde{\gamma}$ that satisfies the following: If $\widetilde{\gamma} \in H^2(\widetilde{\mathcal{X}};\mathbb{Q})$, then $\iota^*_{\widetilde{\sigma}_0}(\widetilde{\gamma}^{\widetilde{T}'}) = 0$. If $\widetilde{\gamma} = \mathbf{1}_j$ for some $j \in \mathrm{Box}(\mathcal{X}) \subset \mathrm{Box}(\widetilde{\mathcal{X}})$, then $\widetilde{\gamma}^{\widetilde{T}'} = \mathbf{1}_j$.

2.6 Examples of construction

In this section, we provide three examples for our construction of $\widetilde{\mathcal{X}}$ from $(\mathcal{X}, \mathcal{L}, f)$.

2.6.1 $\mathbb{C}^3$

Let $\mathcal{X} = \mathbb{C}^3$ and $\mathcal{L}$ be an outer brane. Then, we have

$$
\widetilde{\mathcal{X}} = \begin{cases}
\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(f) \oplus \mathcal{O}_{\mathbb{P}^1}(-f-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) & \text{if } f = 0, -1 \\
\mathrm{Tot}(\mathcal{O}_{\mathbb{P}(1,1,f)}(-f-1) \oplus \mathcal{O}_{\mathbb{P}(1,1,f)}(-1)) & \text{if } f > 0 \\
\mathrm{Tot}(\mathcal{O}_{\mathbb{P}(1,1,-f-1)}(f) \oplus \mathcal{O}_{\mathbb{P}(1,1,-f-1)}(-1)) & \text{if } f < -1.
\end{cases}
$$

We note that for $f \notin \{-2,-1,0,1\}$, $\widetilde{\mathcal{X}}$ is not a smooth manifold, even though $\mathcal{X}$ is. See Figure 2.2 for an illustration in the case $f = 1$, where the fan of $\mathcal{X}$ is the cone over the triangle on the left and the position of the brane $\mathcal{L}$ is indicated by the short boldfaced dash, and the fan of $\widetilde{\mathcal{X}}$ is the cone over the triangulated 3-dimensional polytope on the right.

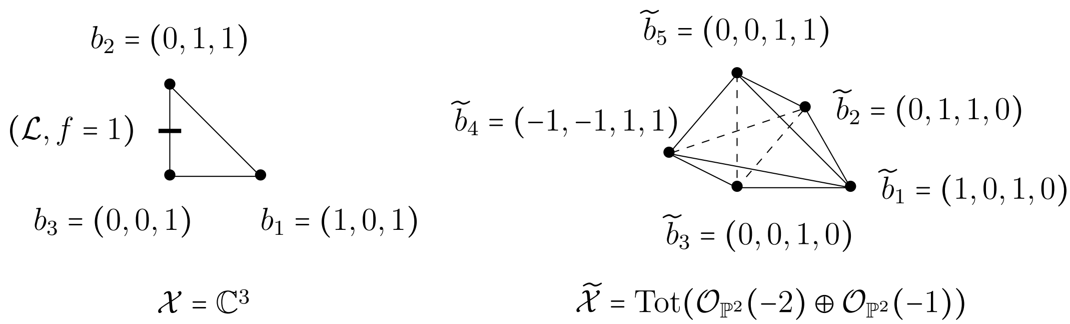

Figure 2.2: Construction of the 4-fold $\widetilde{\mathcal{X}}$ corresponding to $\mathcal{X} = \mathbb{C}^3$ and $f = 1$.

We will use this example as the running example throught this work (Examples 4.3, 6.6, 7.8, 8.16, and 9.7).

2.6.2 Local $\mathbb{P}^2$

Consider $\mathcal{X}_- = [\mathbb{C}^3/\mathbb{Z}_3]$ and its crepant resolution $\mathcal{X}_+ = \mathrm{Tot}(K_{\mathbb{P}^2})$. An outer brane $\mathcal{L}_-$ in $\mathcal{X}_-$ corresponds to an outer brane $\mathcal{L}_+$ in $\mathcal{X}_+$. We illustrate in Figure 2.3 the construction of corresponding 4-orbifolds $\widetilde{\mathcal{X}}_\pm$ when the branes $\mathcal{L}_\pm$ have framing 0, which agrees with the construction in [70]. We note that the triangulations on the "+"-side are refinements of those on the "−"-side. This holds for any framing and corresponds to that that $\mathcal{X}_+$ (resp. $\widetilde{\mathcal{X}}_+$) is a crepant (partial) resolution of $\mathcal{X}_-$ (resp. $\widetilde{\mathcal{X}}_-$).

As discussed in Section 1.7.4, in general, starting with a pair $\mathcal{X}_\pm$ differing by a toric crepant transformation and corresponding framed branes in them, the corresponding 4-orbifolds also differ by a toric crepant transformation. See also the subsequent example.

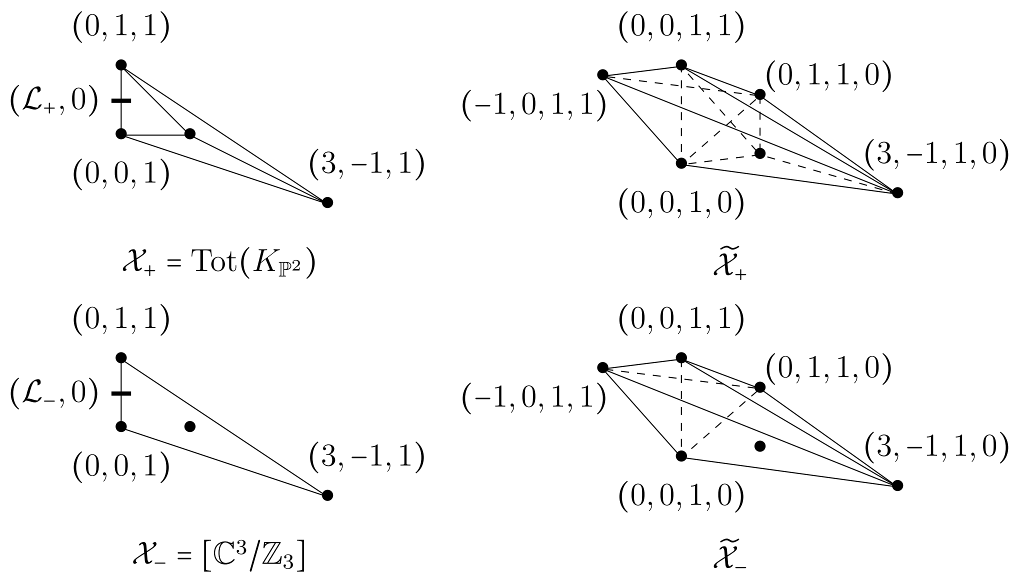

Figure 2.3: Construction of the 4-folds $\widetilde{\mathcal{X}}_\pm$ corresponding to $\mathcal{X}_+ = \mathrm{Tot}(K_{\mathbb{P}^2})$ and $\mathcal{X}_- = [\mathbb{C}^3/\mathbb{Z}_3]$ and corresponding framed branes.

2.6.3 A_1-singularities

Consider $\mathcal{X}_- = [\mathbb{C}^2/\mathbb{Z}_2] \times \mathbb{C}$ and its crepant resolution $\mathcal{X}_+$. Let $\mathcal{L}_-$ be the ineffective outer brane in $\mathcal{X}_-$, with generic stabilizer group μ_2, which correspond to two effective outer branes $\mathcal{L}_+^1, \mathcal{L}_+^2$ in $\mathcal{X}_+$. We illustrate in Figure 2.4 the construction of corresponding 4-orbifolds $\widetilde{\mathcal{X}}_\pm$ when $\mathcal{L}_-$, $\mathcal{L}_+^1, \mathcal{L}_+^2$ have framings $-1, -1, 0$ respectively, noting that the triples $(\mathcal{X}_+, \mathcal{L}_+^1, -1)$, $(\mathcal{X}_+, \mathcal{L}_+^2, 0)$ give rise to the same $\widetilde{\mathcal{X}}_+$. We note that for any framing, $\widetilde{\mathcal{X}}_+$ is a crepant (partial) resolution of $\widetilde{\mathcal{X}}_-$.

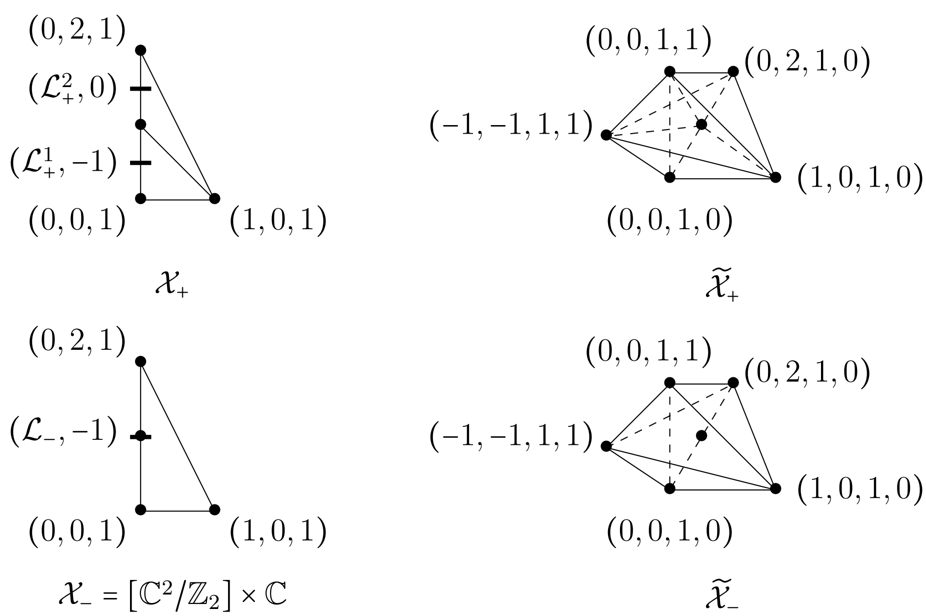

Figure 2.4: Construction of the 4-folds $\widetilde{\mathcal{X}}_\pm$ corresponding to $\mathcal{X}_- = [\mathbb{C}^2/\mathbb{Z}_2] \times \mathbb{C}$ and its crepant resolution $\mathcal{X}_+$.

Chapter 3: Gromov-Witten invariants and localization

In this chapter, we give the definitions of the disk invariants of $(\mathcal{X}, \mathcal{L}, f)$ and the closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$. Since $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are non-compact, these invariants are defined and computed using virtual localization [13, 54, 55]. We summarize the localization computations, following Fang-Liu-Tseng [42] on the open side and Liu [76] on the closed side.

3.1 Moduli of twisted stable maps to toric orbifolds

We start with some preliminaries of Gromov-Witten theory for toric orbifolds and localization computations. Orbifold Gromov-Witten theory is developed on the symplectic side by Chen-Ruan [24] and on the algebraic side by Abramovich-Graber-Vistoli [1, 2]. Here, we review the moduli spaces of twisted stable maps to toric orbifolds, induced torus actions on them, and the description of the torus-fixed loci in terms of decorated graphs. We provide additional details, specifically on Hurwitz-Hodge integrals and twisted covers of proper torus-invariant lines, in Appendix A. Our exposition follows [76]. We restrict our attention to genus zero.

In this section and the next, as in Section 2.1, let $\mathcal{Z}$ be an r-dimensional toric orbifold specified by an extended stacky fan $\boldsymbol{\Xi} = (\mathbb{Z}^r, \Xi, \alpha)$, Z be the coarse moduli space of $\mathcal{Z}$, and $(\mathbb{C}^*)^r$ be the r-dimensional Deligne-Mumford torus of $\mathcal{Z}$.

Let $n \in \mathbb{Z}_{\geq 0}$. A genus-zero, n-pointed *twisted curve* is a connected, proper, 1-dimensional Deligne-Mumford stack $\mathcal{C}$ together with n disjoint closed substacks $\mathfrak{x}_1, \dots, \mathfrak{x}_n$, such that:

- $\mathcal{C}$ is étale locally a nodal curve.
- Formally locally near a node, $\mathcal{C}$ is isomorphic to

$$[\mathrm{Spec}(\mathbb{C}[x,y]/(xy))/\mu_r]$$

for some $r \in \mathbb{Z}_{>0}$, where $\zeta \in \mu_r$ acts by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$.

- Each $\mathfrak{x}_i$ is contained in the smooth locus of $\mathcal{C}$.
- Each $\mathfrak{x}_i$ is an étale gerbe over $\mathrm{Spec}(\mathbb{C})$ with a section.
- $\mathcal{C}$ is a scheme outside $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ and the singular locus.
- The coarse moduli space C is a nodal curve of arithmetic genus zero.

If $\pi : \mathcal{C} \to C$ is the projection to the coarse moduli space and $x_i := \pi(\mathfrak{x}_i)$, then $x_1, \ldots, x_n$ are distinct smooth points of C and $(C, x_1, \ldots, x_n)$ is a genus-zero, n-pointed prestable curve.

Let $\beta \in H_2(Z; \mathbb{Z})$ be an effective class. A genus-zero, n-pointed, degree-β *twisted stable map* to $\mathcal{Z}$ is a representable morphism $u : (\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n) \to \mathcal{Z}$, where $(\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n)$ is a genus-zero, n-pointed twisted curve, such that the induced map $\bar{u} : (C, x_1, \ldots, x_n) \to Z$ between the coarse moduli spaces is a genus-zero, n-pointed, degree-β stable map to Z.

Let $\overline{\mathcal{M}}_{0,n}(\mathcal{Z}, \beta)$ be the moduli space of genus-zero, n-pointed, degree-β twisted stable maps to $\mathcal{Z}$, which is a Deligne-Mumford stack. For $i = 1, \ldots, n$, there is an evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(\mathcal{Z}, \beta) \to \mathcal{I}\mathcal{Z}$ associated to the i-th twisted point $\mathfrak{x}_i$. Given any $\vec{j} = (j_1, \ldots, j_n) \in \mathrm{Box}(\mathcal{Z})^n$, we define

$$\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta) := \bigcap_{i=1}^{n} \mathrm{ev}_i^{-1}(\mathcal{Z}_{j_i}).$$

Then $\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)$ is a union of connected components of $\overline{\mathcal{M}}_{0,n}(\mathcal{Z}, \beta)$, and admits a perfect obstruction theory of virtual dimension

$$\int_\beta c_1(T\mathcal{Z}) + r - 3 + n - \sum_{i=1}^{n} \mathrm{age}(j_i).$$

Moreover, we have

$$\overline{\mathcal{M}}_{0,n}(\mathcal{Z}, \beta) = \bigsqcup_{\vec{j} \in \mathrm{Box}(\mathcal{Z})^n} \overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta).$$

Let $\epsilon : \overline{\mathcal{M}}_{0,n}(\mathcal{Z}, \beta) \to \overline{\mathcal{M}}_{0,n}(Z, \beta)$ be the natural forgetful map. For $i = 1, \ldots, n$, define the *descendant class*

$$\bar{\psi}_i := \epsilon^* \psi_i \in A^1(\overline{\mathcal{M}}_{0,n}(\mathcal{Z}, \beta)).$$

These classes pull back to descendant classes on $\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)$ for each $\vec{j} \in \mathrm{Box}(\mathcal{Z})^n$.

The $(\mathbb{C}^*)^r$-action on $\mathcal{Z}$ induces a $(\mathbb{C}^*)^r$-action on $\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)$ for any $\vec{j}, \beta$. This makes the virtual tangent bundle of $\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)$ and the evaluation maps $\mathrm{ev}_1, \dots, \mathrm{ev}_n$ $(\mathbb{C}^*)^r$-equivariant. The fixed locus $\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)^{(\mathbb{C}^*)^r}$ is a proper, closed substack.

3.2 Torus-fixed locus and decorated graphs

Components of the $(\mathbb{C}^*)^r$-fixed loci of the moduli spaces of stable maps to $\mathcal{Z}$ can be described by *decorated graphs*, defined as follows:

Definition 3.1. Let $n \in \mathbb{Z}_{\geq 0}$, $\vec{j} = (j_1, \dots, j_n) \in \mathrm{Box}(\mathcal{Z})^n$, and $\beta \in H_2(Z; \mathbb{Z})$ be an effective curve class. A genus-zero, $\vec{j}$-twisted, degree-β *decorated graph* for $\mathcal{Z}$ is a tuple $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{s}, \vec{k})$, where:

- Γ is a compact, connected, 1-dimensional CW complex. Let $V(\Gamma)$ denote the vertex set of Γ, $E(\Gamma)$ denote the edge set of Γ, and

  $$F(\Gamma) := \{(e, v) \in E(\Gamma) \times V(\Gamma) : v \in e\}$$

  denote the set of flags.

- $\vec{f} : V(\Gamma) \sqcup E(\Gamma) \to \Xi(r) \sqcup \Xi(r-1)_c$ is the *label* map that sends each $v \in V(\Gamma)$ to an r-cone $\sigma_v \in \Xi(r)$ and each $e \in E(\Gamma)$ to an $(r-1)$-cone $\tau_e \in \Xi(r-1)_c$ such that for each flag $(e, v) \in F(\Gamma)$, (τ_e, σ_v) is a flag in $F(\Xi)$. We denote $G_v := G_{\sigma_v}$ for each $v \in V(\Gamma)$.

- $\vec{d}$ is the *degree map* that sends each edge $e \to E(\Gamma)$ to an element $\gamma_e \in H_{\tau_e}$, such that $d_e := \pi_{\tau_e}(\gamma_e)$ is a positive integer (see (2.1)).

- $\vec{s} : \{1, \dots, n\} \to V(\Gamma)$ is the *marking map*, defined if $n > 0$.

- $\vec{k}$ is the *twisting map* that sends each flag $(e,v) \in F(\Gamma)$ to some $k_{(e,v)} \in G_v$, and each marking $i \in \{1,\dots,n\}$ to some $k_i \in G_{\vec{s}(i)}$.

such that the following conditions are satisfied:

- The graph $\Gamma = (V(\Gamma), E(\Gamma))$ is a tree:

$$|E(\Gamma)| - |V(\Gamma)| + 1 = 0.$$

- $\sum_{e \in E(\Gamma)} d_e [l_{\tau_e}] = \beta.$
- (Compatibility along an edge) For any edge $e \in E(\Gamma)$, if $v, v' \in V(\Gamma)$ are the two incident vertices, then

$$\pi_{(\tau_e, \sigma_v)}(\gamma_e) = k_{(e,v)}, \quad \pi_{(\tau_e, \sigma_{v'})}(\gamma_e) = k_{(e,v')}$$

(see (2.3)).

- (Compatibility at a vertex) For any vertex $v \in V(\Gamma)$, the equation

$$\prod_{(e,v) \in F(\Gamma)} k_{(e,v)}^{-1} \prod_{i \in \vec{s}^{-1}(v)} k_i = 1$$

holds in G_v.

- (Compatibility with $\vec{j}$) For each $i = 1, \dots, n$, the pair $(p_{\sigma_{\vec{s}(i)}}, k_i)$ represents a point in the inertia component $\mathcal{X}_{j_i}$.

Let $\Gamma_{0,\vec{j}}(\mathcal{Z}, \beta)$ be the set of all genus-zero, $\vec{j}$-twisted, degree-β decorated graphs for $\mathcal{Z}$. We set up the following additional notations on a decorated graph $\vec{\Gamma} \in \Gamma_{0,\vec{j}}(\mathcal{Z}, \beta)$:

- For each $v \in V(\Gamma)$, let

$$E_v := \{e \in E(\Gamma) : (e,v) \in F(\Gamma)\}, \quad S_v := \vec{s}^{-1}(v),$$

and $\mathrm{val}(v) := |E_v|$, $n_v := |S_v|$. Let $\vec{k}_v := (k_{(e,v)}^{-1}, k_i) \in G_v^{E_v \cup S_v}$.

- Let

$$V^S(\vec{\Gamma}) := \{v \in V(\Gamma) : \mathrm{val}(v) + n_v - 2 > 0\}$$

  be the set of *stable* vertices of Γ, and

$$V^1(\vec{\Gamma}) := \{v \in V(\Gamma) : \mathrm{val}(v) = 1, n_v = 0\},$$
$$V^{1,1}(\vec{\Gamma}) := \{v \in V(\Gamma) : \mathrm{val}(v) = n_v = 1\},$$
$$V^2(\vec{\Gamma}) := \{v \in V(\Gamma) : \mathrm{val}(v) = 2, n_v = 0\}$$

  be a partition of the *unstable* vertices.

- Let $\mathrm{Aut}(\vec{\Gamma})$ be the *automorphism group* of $\vec{\Gamma}$, which consists of all automorphisms of Γ that make the maps $\vec{f}, \vec{d}, \vec{s}, \vec{k}$ invariant.

- For each $(e, v) \in F(\Gamma)$, let $r_{(e,v)}$ be the order of $k_{(e,v)}$ in G_v. For each $v \in V^2(\vec{\Gamma})$, if $E_v = \{e_1, e_2\}$, let $r_v := r_{(e_1,v)} = r_{(e_2,v)}$.

- Let

$$c_{\vec{\Gamma}} := \frac{1}{|\mathrm{Aut}(\vec{\Gamma})| \cdot \prod_{e \in E(\Gamma)} (d_e |G_e|)} \cdot \prod_{(e,v) \in F(\Gamma)} \frac{|G_v|}{r_{(e,v)}}. \tag{3.1}$$

Given a twisted stable map $u : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n) \to \mathcal{Z}$ that represents a point in the $(\mathbb{C}^*)^r$-fixed locus $\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)^{(\mathbb{C}^*)^r}$, we can assign a decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{s}, \vec{k}) \in \Gamma_{0,\vec{j}}(\mathcal{Z}, \beta)$ as follows. Let $\bar{u} : (C, x_1, \dots, x_n) \to Z$ be the induced stable map between coarse moduli spaces.

- The image of $\bar{u}$ lies in $Z_c^1 \subset Z$. The vertex set $V(\Gamma)$ is in one-to-one correspondence with the set of connected components in $\bar{u}^{-1}(Z^{(\mathbb{C}^*)^r})$. For $v \in V(\Gamma)$, let C_v denote the component associated to v, and $\mathcal{C}_v$ be the preimage of C_v under the projection $\mathcal{C} \to C$. Set $\vec{f}(v) = \sigma_v \in \Xi(r)$ such that the image of $\mathcal{C}_v$ under u is $\mathfrak{p}_{\sigma_v}$, or equivalently, the image of C_v under $\bar{u}$ is p_{σ_v}.

- The edge set $E(\Gamma)$ is in one-to-one correspondence with the set of irreducible components of C that do not map constantly to Z under $\bar{u}$. For $e \in E(\Gamma)$, let C_e denote the component associated to e, and $\mathcal{C}_e$ be the preimage of C_e under the projection $\mathcal{C} \to C$. Set $\vec{f}(e) = \tau_e \in \Xi(r-1)$ such that the image of $\mathcal{C}_e$ under u is $\mathfrak{l}_{\tau_e}$, or equivalently, the image of C_e under $\bar{u}$ is l_{τ_e}.

- The flag set $F(\Gamma)$ consists of all pairs (e, v) such that $\mathcal{C}_e \cap \mathcal{C}_v \neq \varnothing$. For $(e, v) \in F(\Gamma)$, let $\mathfrak{n}(e, v) := \mathcal{C}_e \cap \mathcal{C}_v$. Set $k_{(e,v)} \in G_v$ to be the image of the generator of the generic stablizer group of $\mathfrak{n}(e, v)$ in $\mathcal{C}_e$ under u.

- For an edge $e \in E(\Gamma)$ incident to vertices $v, v' \in V(\Gamma)$, we have $\mathcal{C}_e \cong \mathcal{C}_{r_{(e,v)}, r_{(e,v')}}$ (see Section A.2). Let $\gamma_e \in H_{\tau_e}$ be the element defined by $u|_{\mathcal{C}_e}$ and set $d_e = \pi_{\tau_e}(\gamma_e)$. The compatibility conditions $\pi_{(\tau_e, \sigma_v)}(\gamma_e) = k_{(e,v)}$ and $\pi_{(\tau_e, \sigma'_v)}(\gamma_e) = k_{(e,v')}$ are satisfied.

- For each marking $i = \{1, \dots, n\}$, set $\vec{s}(i) = v \in V(\Gamma)$ such that $\mathfrak{x}_i \subseteq \mathcal{C}_v$. Then $\mathfrak{x}_i$ is mapped by u to a point $(\mathfrak{p}_{\sigma_v}, k)$ in $\mathcal{Z}_{j_i}$, where $k \in G_v$ maps to j_i under $G_v \cong \mathrm{Box}(\sigma_v) \to \mathrm{Box}(\mathcal{Z})$. We set $k_i = k$. Then for each $v \in V^S(\vec{\Gamma})$, $u|_{\mathcal{C}_v}$ represents a point in $\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)$.

The above assignment gives a decomposition

$$\overline{\mathcal{M}}_{0,\vec{j}}(\mathcal{Z}, \beta)^{(\mathbb{C}^*)^r} = \bigsqcup_{\vec{\Gamma} \in \Gamma_{0,\vec{j}}(\mathcal{Z},\beta)} \mathcal{F}_{\vec{\Gamma}}$$

into connected components, where $\mathcal{F}_{\vec{\Gamma}}$ denotes the component corresponding to $\vec{\Gamma} \in \Gamma_{0,\vec{j}}(\mathcal{Z}, \beta)$. Upto a finite morphism, $\mathcal{F}_{\vec{\Gamma}}$ can be identified with

$$\mathcal{M}_{\vec{\Gamma}} := \prod_{v \in V^S(\vec{\Gamma})} \overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v),$$

and in $A_*(\mathcal{M}_{\vec{\Gamma}})$, we have[1]

$$[\mathcal{F}_{\vec{\Gamma}}] = \frac{1}{|\mathrm{Aut}(\vec{\Gamma})| \cdot \prod_{e \in E(\Gamma)} (d_e |G_e|)} \cdot \prod_{v \in V^S(\vec{\Gamma}), e \in E_v} \frac{|G_v|}{r_{(e,v)}} \cdot \prod_{v \in V^2(\vec{\Gamma})} \frac{|G_v|}{r_v} \cdot [\mathcal{M}_{\vec{\Gamma}}]. \tag{3.2}$$

[1]The coefficient below differs from $c_{\vec{\Gamma}}$ (see (3.1)) by factors associated to unstable vertices of Γ, and such difference is accounted for in [76, Section 9.3.3] by the integration conventions (A.2) at unstable vertices.

Finally, we set

$$\Gamma_{0,n}(\mathcal{Z},\beta) := \bigsqcup_{\vec{j}\in\mathrm{Box}(\mathcal{Z})^n} \Gamma_{0,\vec{j}}(\mathcal{Z},\beta)$$

to be the set of all genus-zero, n-pointed, degree-β decorated graphs for $\mathcal{Z}$. We have

$$\overline{\mathcal{M}}_{0,n}(\mathcal{Z},\beta)^{(\mathbb{C}^*)^r} = \bigsqcup_{\vec{\Gamma}\in\Gamma_{0,n}(\mathcal{Z},\beta)} \mathcal{F}_{\vec{\Gamma}}.$$

Remark 3.2. For our toric Calabi-Yau 3-orbifold $\mathcal{X}$, we will use T'-equivariant localization on the moduli spaces of stable maps. Note that the T'-fixed points and T'-invariant lines of $\mathcal{X}$ are the same as the T-fixed points and T-invariant lines. Therefore, the T'-fixed loci of the moduli spaces can be identified with the T-fixed loci and described by decorated graphs in the same way as above. Similarly, for our toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$, we will use $\widetilde{T}'$-equivariant localization on the moduli spaces of stable maps and describe the $\widetilde{T}'$-fixed loci by decorated graphs.

3.3 Disk invariants of $(\mathcal{X},\mathcal{L},f)$

In this section, we give the definition of the disk invariants of $(\mathcal{X},\mathcal{L},f)$, which are *open Gromov-Witten invariants* [28, 41, 42, 64] that encode twisted stable maps from genus-zero domains with a single boundary component. We follow Fang-Liu-Tseng [42] and refer the reader to there for additional details.

3.3.1 Twisted open stable maps and their moduli

Open stable maps to symplectic orbifolds with Lagrangian boundary conditions are defined by Cho-Poddar [28], generalizing the manifold case defined by Katz-Liu [64]; see also [46, 75]. The domain of such a map is a prestable bordered orbifold Riemann surface which allows stacky points at interior nodes and interior marked points. For our target $(\mathcal{X},\mathcal{L})$, we follow the definition of [42], which is closely related to the topological vertex [73, 41]. As observed by [42], the definition of [28] assumes that the Lagrangian suborbifold is a smooth manifold. In the more general setting where the Lagrangian contains stacky points, as is the case for our $\mathcal{L}$, in order to obtain compactness of the moduli when the target orbifold and the Lagrangian are both compact, one needs to allow orbifold structures at boundary nodes and boundary marked points of the domain. However, for $(\mathcal{X}, \mathcal{L})$, since the open Gromov-Witten invariants are defined by T'-equivariant localization and $\mathcal{L}$ does not contain any T'-fixed points, there is no need to allow orbifold structures on the boundary of the domain.

Let $n \in \mathbb{Z}_{\geq 0}$, $\beta' \in H_2(X, L; \mathbb{Z})$, and $(d, \lambda) \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times G_{\tau_0}$. Let

$$\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda)) = \overline{\mathcal{M}}_{(0,1),n}(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$$

be the moduli space of maps

$$u : ((\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n), \partial \mathcal{C}) \to (\mathcal{X}, \mathcal{L})$$

where

- $(\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n)$ is a prestable bordered orbifold Riemann surface of topological type $(0, 1)$ with n interior marked points $\mathfrak{x}_1, \dots, \mathfrak{x}_n$. Here, if $\pi : \mathcal{C} \to C$ is the projection to the coarse moduli space and $x_i := \pi(\mathfrak{x}_i)$, then $(C, x_1, \dots, x_n)$ is a prestable bordered Riemann surface of topological type $(0, 1)$ with n interior marked points. The topological type of $(0, 1)$ means that topologically, C is a nodal Riemann surface of arithmetic genus 0 with a single open disk removed. In particular, $\partial \mathcal{C}$ is connected and topologically a circle, and contains no orbifold points.

- Let $\nu : \hat{\mathcal{C}} \to \mathcal{C}$ be the normalization map, so that $\hat{\mathcal{C}}$ is a possibly disconnected bordered orbifold Riemann surface with no nodes. Then the map $\nu \circ u : \hat{\mathcal{C}} \to \mathcal{X}$ is holomorphic.

- The automorphism group of u is finite.

- Let $\bar{u} : (C, \partial C) \to (X, L)$ be the induced map between coarse moduli spaces. Then $\bar{u}_*[C] = \beta' \in H_2(X, L; \mathbb{Z})$.

- $u_*[\partial \mathcal{C}] = (d, \lambda) \in H_1(\mathcal{L}; \mathbb{Z})$, where $d \in \mathbb{Z}$ is the winding number and $\lambda \in G_{\tau_0}$ is the monodromy.

$\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ is a possibly singular stack with corners, equipped with a virtual tangent bundle which is a virtual real vector bundle. For $i = 1, \ldots, n$, there is an evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda)) \to \mathcal{I}\mathcal{X}$ associated to the i-th interior marked point $\mathfrak{x}_i$. Given any $\vec{j} = (j_1, \ldots, j_n) \in \mathrm{Box}(\mathcal{X})^n$, we define

$$\overline{\mathcal{M}}_{\vec{j}}(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda)) := \bigcap_{i=1}^{n} \mathrm{ev}_i^{-1}(\mathcal{X}_{j_i}).$$

Then $\overline{\mathcal{M}}_{\vec{j}}(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ is a union of connected components of $\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$, and the (real) rank of the virtual tangent bundle over $\overline{\mathcal{M}}_{\vec{j}}(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ is

$$2 \sum_{i=1}^{n} (1 - \mathrm{age}(j_i)).$$

3.3.2 Definition of disk invariants

The action of the compact Calabi-Yau 2-torus $T'_{\mathbb{R}}$ on $(\mathcal{X}, \mathcal{L})$ induces a $T'_{\mathbb{R}}$-action on $\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ and makes the virtual tangent bundle of $\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ and the evaluation maps $\mathrm{ev}_1, \ldots, \mathrm{ev}_n$ $T'_{\mathbb{R}}$-equivariant. Let $\mathcal{F} := \overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))^{T'_{\mathbb{R}}}$ be the $T'_{\mathbb{R}}$-fixed locus. Then each connected component of $\mathcal{F}$ is a compact orbifold, on which the virtual tangent bundle agrees with the tangent bundle. We have

$$[\mathcal{F}]^{\mathrm{vir}} = [\mathcal{F}].$$

Definition 3.3. Given $\gamma_1, \ldots, \gamma_n \in H^*_{\mathrm{CR}, T'_{\mathbb{R}}}(\mathcal{X}; \mathbb{Q}) = H^*_{\mathrm{CR}, T'}(\mathcal{X}; \mathbb{Q})$, we define

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathcal{X}, \mathcal{L}}_{\beta', (d, \lambda)} := \int_{[\mathcal{F}]^{\mathrm{vir}}} \frac{\iota^* \left(\prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \right)}{e_{T'_{\mathbb{R}}}(N^{\mathrm{vir}})} \in \mathcal{Q}_{T'},$$

where $\iota : \mathcal{F} \to \overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ is the inclusion and N^{vir} is the virtual normal bundle of $\mathcal{F}$ in $\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$.

Suppose that for each $i = 1, \dots, n$, we have $\gamma_i \in H^{2a_i}_{T'_{\mathbb{R}}}(\mathcal{X}_{j_i};\mathbb{Q})$ (viewed as a $\mathbb{Q}$-vector subspace of $H^{2(a_i+\mathrm{age}(j_i))}_{\mathrm{CR},T'_{\mathbb{R}}}(\mathcal{X};\mathbb{Q})$) for some $j_i \in \mathrm{Box}(\mathcal{X})$ and $a_i \in \mathbb{Z}_{\geq 0}$. Set $\vec{j} = (j_1, \dots, j_n)$. Then only the connected components of $\mathcal{F}$ contained in $\overline{\mathcal{M}}_{\vec{j}}(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$ contribute to $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},\mathcal{L}}_{\beta',(d,\lambda)}$. Therefore, $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},\mathcal{L}}_{\beta',(d,\lambda)}$ is a homogenous rational function in $\mathsf{u}_1, \mathsf{u}_2$ of degree

$$\sum_{i=1}^{n} (\mathrm{age}(j_i) - 1 + a_i).$$

Definition 3.4. Let $\gamma_1, \dots, \gamma_n \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$ and choose equivariant lifts in $H^2_{\mathrm{CR},T'_{\mathbb{R}}}(\mathcal{X};\mathbb{Q}) = H^2_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q})$ as in Convention 2.4. We define the *disk* invariant

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)} := \langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},\mathcal{L}}_{\beta',(d,\lambda)}\Big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} \in \mathcal{Q}_{T_f}.$$

Observe that $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)}$ is homogeneous of degree 0 and thus in $\mathbb{Q}$. We will confirm, as a consequence of the numerical open/closed correspondence, that $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},\mathcal{L}}_{\beta',(d,\lambda)}$ has no pole along $\mathsf{u}_2 - f\mathsf{u}_1 = 0$ and thus $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)}$ is defined. See Remark 4.9.

We note that $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},\mathcal{L}}_{\beta',(d,\lambda)}$ and $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)}$ are defined up to a sign depending on a choice of orientation on $\overline{\mathcal{M}}_n(\mathcal{X}, \mathcal{L} \mid \beta', (d, \lambda))$. Our choice will be specified by the computation result given in Proposition 3.7. See also Remark 3.5.

3.4 Localization computations of disk invariants

In this section, we summarize the localization computations of the disk invariants of $(\mathcal{X}, \mathcal{L}, f)$ by Fang-Liu-Tseng [42].

3.4.1 Tangent T'-weights

Given any flag $(\tau, \sigma) \in F(\Sigma)$, we define

$$\mathbf{w}(\tau, \sigma) := e_{T'}(T_{\mathfrak{p}_\sigma} \mathfrak{l}_\tau) \in H^2_{T'}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}\mathsf{u}_1 \oplus \mathbb{Q}\mathsf{u}_2,$$

and $w(\tau,\sigma) \in \mathbb{Q}$ such that

$$\mathbf{w}(\tau,\sigma)|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} = w(\tau,\sigma)\mathsf{u}_1.$$

In particular, for the three flags associated to the cone σ_0, we have

$$\begin{aligned}
\mathbf{w}_0 &:= \mathbf{w}(\tau_0,\sigma_0) = \frac{1}{\mathfrak{r}}\mathsf{u}_1, & w_0 &:= w(\tau_0,\sigma_0) = \frac{1}{\mathfrak{r}},\\
\mathbf{w}_2 &:= \mathbf{w}(\tau_2,\sigma_0) = \frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1 + \frac{1}{\mathfrak{m}}\mathsf{u}_2, & w_2 &:= w(\tau_2,\sigma_0) = \frac{\mathfrak{s}+\mathfrak{r}f}{\mathfrak{r}\mathfrak{m}},\\
\mathbf{w}_3 &:= \mathbf{w}(\tau_3,\sigma_0) = -\frac{\mathfrak{m}+\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2, & w_3 &:= w(\tau_3,\sigma_0) = -\frac{\mathfrak{m}+\mathfrak{s}+\mathfrak{r}f}{\mathfrak{r}\mathfrak{m}}.
\end{aligned} \tag{3.3}$$

3.4.2 The disk factor

Let $(d,\lambda) \in H_1(\mathcal{L};\mathbb{Z}) \cong \mathbb{Z} \times G_{\tau_0}$ such that $d > 0$. Define

$$h(d,\lambda) := \pi_{(\tau_0,\sigma_0)}(d,\lambda) \in G_{\sigma_0}$$

(see (2.3)). The *disk factor* [20, 86] is defined as

$$D_{d,\lambda} := \langle \mathbf{1}_{h(d,\lambda)} \rangle^{\mathcal{X},\mathcal{L}}_{d[B],(d,\lambda)} \in \mathcal{Q}_{T'},$$

which is homogeneous of degree $\mathrm{age}(h(d,\lambda)) - 1$. Here, the disk B is defined in Section 2.5.3. The disk factor is a key ingredient in the localization computation of the disk invariants. To state the formula for the disk factor given by [42], which is based on [86], we first set up some notations. First, we have

$$\chi_{(\tau_0,\sigma_0)}(\lambda) = \chi_{(\tau_2,\sigma_0)}(\lambda)\chi_{(\tau_3,\sigma_0)}(\lambda) = 1.$$

Let $\bar{\lambda} \in \{0,1,\dots,\mathfrak{m}-1\}$ such that

$$\chi_{(\tau_3,\sigma_0)}(\lambda) = e^{\frac{2\pi\sqrt{-1}\bar{\lambda}}{\mathfrak{m}}}. \tag{3.4}$$

Then

$$\chi_{(\tau_0,\sigma_0)}(h(d,\lambda)) = e^{2\pi\sqrt{-1}dw_0}, \quad \chi_{(\tau_2,\sigma_0)}(h(d,\lambda)) = e^{2\pi\sqrt{-1}(dw_2-\frac{\bar{\lambda}}{\mathfrak{m}})}, \quad \chi_{(\tau_3,\sigma_0)}(h(d,\lambda)) = e^{2\pi\sqrt{-1}(dw_3+\frac{\bar{\lambda}}{\mathfrak{m}})}.$$

We set

$$\epsilon_2 := \langle dw_2 - \tfrac{\bar{\lambda}}{\mathfrak{m}} \rangle, \quad \epsilon_3 := \langle dw_3 + \tfrac{\bar{\lambda}}{\mathfrak{m}} \rangle. \tag{3.5}$$

Then

$$\langle dw_0 \rangle + \epsilon_2 + \epsilon_3 = \mathrm{age}(h(d,\lambda)).$$

With the above notations, by choosing an orientation on the moduli space $\overline{\mathcal{M}}_1(\mathcal{X}, \mathcal{L} \mid d[B], (d,\lambda))$, [42, Section 3.11] gives the following formula:

$$\begin{aligned} D_{d,\lambda} &= (-1)^{\lfloor dw_3-\epsilon_3 \rfloor + d} \left(\frac{\mathfrak{r}\mathbf{w}_0}{d} \right)^{\mathrm{age}(h(d,\lambda))-1} \cdot \frac{1}{d\mathfrak{m} \cdot \lfloor dw_0 \rfloor !} \cdot \prod_{a=1}^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1} \left(\frac{d\mathbf{w}_2}{\mathfrak{r}\mathbf{w}_0} + a - \epsilon_2 \right) \\ &= (-1)^{\lfloor dw_3-\epsilon_3 \rfloor + d} \left(\frac{\mathsf{u}_1}{d} \right)^{\mathrm{age}(h(d,\lambda))-1} \cdot \frac{1}{d\mathfrak{m} \cdot \lfloor dw_0 \rfloor !} \cdot \prod_{a=1}^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1} \left(\frac{d\mathbf{w}_2}{\mathsf{u}_1} + a - \epsilon_2 \right). \end{aligned} \tag{3.6}$$

Remark 3.5. We note that the sign convention of formula (3.6) above differs from that in [42], yet agrees with that in [41] in the smooth case. Our choice of sign (and orientation on the moduli space of open stable maps) ensures that the numerical open/closed correpondence holds without a sign difference. See Theorem 4.1.

3.4.3 Localization computations

Given any $n \in \mathbb{Z}_{\geq 0}$ and effective class $\beta \in H_2(X;\mathbb{Z})$, the T'-action on $\mathcal{X}$ induces a T'-action on the moduli space $\overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)$ of stable maps to $\mathcal{X}$. This makes the virtual tangent bundle and the evaluation maps $\mathrm{ev}_1, \dots, \mathrm{ev}_{n+1}$ T'-equivariant. [42] directly relates the $T'_{\mathbb{R}}$-fixed loci of the moduli spaces of open stable maps to $(\mathcal{X},\mathcal{L})$ and the T'-fixed loci of the moduli spaces of stable maps to $\mathcal{X}$ and compares their tangent-obstruction theories. As a consequence, via the disk factor, [42, Proposition 3.3] relates the disk invariants of $(\mathcal{X},\mathcal{L},f)$ to the Gromov-Witten invariants of $\mathcal{X}$, as follows:

Theorem 3.6 ([42]). *Let $n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X;\mathbb{Z})$ be an effective class, $(d,\lambda) \in H_1(\mathcal{L};\mathbb{Z}) \cong \mathbb{Z} \times G_{\tau_0}$ such that $d > 0$, and $\gamma_1, \dots, \gamma_n \in H^2_{\mathrm{CR},T'_{\mathbb{R}}}(\mathcal{X};\mathbb{Q}) = H^2_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q})$. Set*

$$\beta' = \beta + d[B].$$

Then

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)} = \mathfrak{rm} D_{d,\lambda} \cdot \int_{[\overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)^{T'}]^{\mathrm{vir}}} \frac{\iota^*\left(\mathrm{ev}^*_{n+1}(\phi_{\sigma_0,h(d,\lambda)^{-1}}) \cdot \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\right)}{e_{T'}(N^{\mathrm{vir}}) \cdot (\frac{\mathsf{u}_1}{d} - \bar{\psi}_{n+1})}\bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0}, \tag{3.7}$$

where $\iota : \overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)^{T'} \to \overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)$ is the inclusion, N^{vir} is the virtual normal bundle of $\overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)^{T'}$, and

$$\phi_{\sigma_0,h(d,\lambda)^{-1}} = \iota_{\sigma_0,*}(\mathbf{1}_{h(d,\lambda)^{-1}}) \in H^*_{\mathrm{CR},T'}(\mathcal{X};\mathbb{Q})$$

is the T'-equivariant Poincaré dual of the point $(\mathfrak{p}_{\sigma_0}, h(d,\lambda)^{-1})$.

Recall from Section 3.2 that connected components of $\overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)^{T'}$ are indexed by decorated graphs in $\Gamma_{0,n+1}(\mathcal{X},\beta)$. Using the study of the tangent-obstruction theory by [76, Theorem 137] (which is also used in the proof of [42, Proposition 3.3]), we can rewrite (3.7) in terms of contributions from decorated graphs as

$$\begin{aligned}\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)} =& \mathfrak{rm} D_{d,\lambda} \cdot \sum_{\vec{\Gamma} \in \Gamma_{0,n+1}(\mathcal{X},\beta)} c_{\vec{\Gamma}} \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V(\Gamma)} \left(\prod_{i \in S_v} \iota^*_{\sigma_v}(\gamma_i) \right) \\ & \cdot \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\mathbf{h}(v)}{\left(\frac{\mathsf{u}_1}{d} - \bar{\psi}_{n+1}\right)^{\delta_{v,n+1}} \cdot \prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)}\bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0},\end{aligned} \tag{3.8}$$

where:

- For each $\vec{\Gamma} \in \Gamma_{0,n+1}(\mathcal{X},\beta)$, the coefficient $c_{\vec{\Gamma}}$ is defined in (3.1). To give the definitions of the other quantities in (3.8), we pick a stable map $u : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_{n+1}) \to \mathcal{X}$ whose associated decorated graph is $\vec{\Gamma}$ (see Section 3.2). The definitions do not depend on the choice of u.

- For each $e \in E(\Gamma)$, define

$$\mathbf{h}(e) := \frac{e_{T'}(H^1(\mathcal{C}_e, (u|_{\mathcal{C}_e})^* T\mathcal{X})^m)}{e_{T'}(H^0(\mathcal{C}_e, (u|_{\mathcal{C}_e})^* T\mathcal{X})^m)}, \tag{3.9}$$

  which is explicitly computed in [76, Lemma 130] in terms of tangent T'-weights. Here and throughout this work, the superscript "m" represents the *moving part*: Any complex representation V of a torus $(\mathbb{C}^*)^r$ decomposes into a direct sum of 1-dimensional representations and can thus be written as $V = V^f \oplus V^m$, where the *fixed part* V^f is the direct sum of all trivial 1-dimensional representations and the moving part V^m is the direct sum of all non-trivial ones.

- For each $(e, v) \in F(\Gamma)$, define

$$\mathbf{h}(e, v) := e_{T'}\left((T_{\mathfrak{p}_{\sigma_v}} \mathcal{X})^{k_{(e,v)}}\right) = \prod_{\substack{(\tau, \sigma_v) \in F(\Sigma) \\ k_{(e,v)} \in G_\tau}} \mathbf{w}(\tau, \sigma_v).$$

  Here, $(T_{\mathfrak{p}_{\sigma_v}} \mathcal{X})^{k_{(e,v)}}$ is the maximal subspace of $T_{\mathfrak{p}_{\sigma_v}} \mathcal{X}$ that is invariant under the action of $k_{(e,v)}$.

- We set

$$\gamma_{n+1} := \phi_{\sigma_0, h(d,\lambda)^{-1}}$$

  for convenience.

- For each $v \in V(\Gamma)$, the marked points and corresponding descendant classes of $\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)$ are indexed by $E_v \cup S_v$. The integral over $\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)$ is a Hurwitz-Hodge integral (see Section A.1), and we adopt the integration convention (A.2) for unstable vertices.

- For each stable vertex $v \in V^S(\vec{\Gamma})$, define

$$\mathbf{h}(v) := \frac{e_{T'}(H^1(\mathcal{C}_v, (u|_{\mathcal{C}_v})^* T\mathcal{X})^m)}{e_{T'}(H^0(\mathcal{C}_v, (u|_{\mathcal{C}_v})^* T\mathcal{X})^m)},$$

which is explicitly computed in [76, Lemma 126] in terms of Hurwitz-Hodge classes and tangent T'-weights. For each unstable vertex $v \notin V^S(\vec{\Gamma})$, define

$$\mathbf{h}(v) := \begin{cases} \mathbf{h}(e,v)^{-1} & \text{if } v \in V^1(\vec{\Gamma}) \cup V^{1,1}(\vec{\Gamma}), E_v = \{e\}, \\ \mathbf{h}(e_1,v)^{-1} = \mathbf{h}(e_2,v)^{-1} & \text{if } v \in V^2(\vec{\Gamma}), E_v = \{e_1, e_2\}. \end{cases}$$

- $\delta_{v,n+1}$ is the indicator function

$$\delta_{v,n+1} := \begin{cases} 1 & \text{if } n+1 \in S_v \\ 0 & \text{otherwise.} \end{cases}$$

- For each $(e,v) \in F(\Gamma)$, define

$$\mathbf{w}_{(e,v)} := e_{T'}(T_{\mathfrak{n}(e,v)}\mathcal{C}_e) = \frac{\mathfrak{r}(\tau_e,\sigma_v)\mathbf{w}(\tau_e,\sigma_v)}{r_{(e,v)}d_e},$$

where the T'-action on $T_{\mathfrak{n}(e,v)}\mathcal{C}_e$ is induced from that on $T_{\mathfrak{p}_{\sigma_v}}\mathfrak{l}_{\tau_e}$.

To simplify (3.8), we note that for any $\sigma \in \Sigma(3)$, $\sigma \neq \sigma_0$,

$$\iota_\sigma^*(\gamma_{n+1}) = \iota_\sigma^*(\phi_{\sigma_0, h(d,\lambda)^{-1}}) = 0.$$

Thus only the decorated graphs in the subset

$$\Gamma_{0,n+1}^{0,(d,\lambda)}(\mathcal{X},\beta) := \{\vec{\Gamma} \in \Gamma_{0,n+1}(\mathcal{X},\beta) : \vec{f} \circ \vec{s}(n+1) = \sigma_0, \vec{k}(n+1) = h(d,\lambda)^{-1}\} \tag{3.10}$$

can contribute. We simplify (3.8) as follows:

Proposition 3.7. *Let $n, \beta, (d,\lambda), \beta'$, and $\gamma_1, \dots, \gamma_n$ be as in Theorem 3.6. Then*

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)} = \mathfrak{rm} D_{d,\lambda} \cdot \sum_{\vec{\Gamma} \in \Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta)} c_{\vec{\Gamma}} \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V(\Gamma)} \left(\prod_{i \in S_v} \iota^*_{\sigma_v}(\gamma_i) \right)$$
$$\cdot \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\mathbf{h}(v)}{\left(\frac{\mathsf{u}_1}{d} - \bar{\psi}_{n+1}\right)^{\delta_{v,n+1}} \cdot \prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)} \Bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0}. \tag{3.11}$$

3.5 Closed invariants of $\widetilde{\mathcal{X}}$

In this section, we define closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$ and compute them by localization following [76].

3.5.1 Definition of closed Gromov-Witten invariants

Let $n \in \mathbb{Z}_{\geq 0}$ and $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z})$ be an effective class. The action of the Calabi-Yau 3-torus $\widetilde{T}'$ on $\widetilde{\mathcal{X}}$ induces a $\widetilde{T}'$-action on the moduli space $\overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}}, \widetilde{\beta})$. This makes the virtual tangent bundle and the evaluation maps $\mathrm{ev}_1, \dots, \mathrm{ev}_n$ $\widetilde{T}'$-equivariant. The $\widetilde{T}'$-fixed locus $\overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}}, \widetilde{\beta})^{\widetilde{T}'}$ can be identified with the $\widetilde{T}$-fixed locus.

Definition 3.8. Given $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \in H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$, we define

$$\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \rangle^{\widetilde{\mathcal{X}},\widetilde{T}'}_{\widetilde{\beta}} := \int_{[\overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}},\widetilde{\beta})^{\widetilde{T}'}]^{\mathrm{vir}}} \frac{\iota^*\left(\prod_{i=1}^n \mathrm{ev}_i^*(\widetilde{\gamma}_i)\right)}{e_{\widetilde{T}'}(N^{\mathrm{vir}})} \in \mathcal{Q}_{\widetilde{T}'},$$

where $\iota : \overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}},\widetilde{\beta})^{\widetilde{T}'} \to \overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}},\widetilde{\beta})$ is the inclusion and N^{vir} is the virtual normal bundle of $\overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}},\widetilde{\beta})^{\widetilde{T}'}$ in $\overline{\mathcal{M}}_{0,n}(\widetilde{\mathcal{X}},\widetilde{\beta})$.

Suppose that for each $i = 1, \dots, n$, we have $\widetilde{\gamma}_i \in H^{2a_i}_{\widetilde{T}'}(\widetilde{\mathcal{X}}_{j_i};\mathbb{Q})$ (viewed as a $\mathbb{Q}$-vector subspace of $H^{2(a_i+\mathrm{age}(j_i))}_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$) for some $j_i \in \mathrm{Box}(\widetilde{\mathcal{X}})$ and $a_i \in \mathbb{Z}_{\geq 0}$. Then $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \rangle^{\widetilde{\mathcal{X}}}_{\widetilde{\beta}}$ is a homogenous rational function in $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4$ of degree

$$-1 + \sum_{i=1}^n (\mathrm{age}(j_i) - 1 + a_i).$$

Definition 3.9. Let $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \in H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$ and choose equivariant lifts in $H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$ as in Convention 2.4. For $\lambda \in G_{\tau_0}$, let $\widetilde{\gamma}_\lambda$ be the class in $H^4_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$ defined by

$$\widetilde{\gamma}_\lambda := \begin{cases} \dfrac{\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_3^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}}{\frac{f}{\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2 - \mathsf{u}_4} & \text{if } \lambda = 1 \\ \widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}\mathbf{1}_{\lambda^{-1}} & \text{if } \lambda \neq 1, \end{cases} \tag{3.12}$$

where recall $\widetilde{\mathcal{D}}_i^{\widetilde{T}'} = [\mathcal{V}(\widetilde{\rho}_i)]$ is the $\widetilde{T}'$-equivariant Poincaré dual of the divisor $\mathcal{V}(\widetilde{\rho}_i)$. Then, we define the *closed* Gromov-Witten invariant

$$\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}},T_f} := \langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}},\widetilde{T}'}\Big|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0} \in \mathcal{Q}_{T_f}.$$

Observe that $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}},T_f}$ is homogeneous of degree 0 and thus in $\mathbb{Q}$.

Lemma 3.10. *Under the setup of Definition 3.9,* $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}},\widetilde{T}'}$ *has no pole along* $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$*. In particular,* $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}},T_f}$ *is defined.*

We defer the proof, which is based on mirror symmetry, to Section 6.4.

3.5.2 Tangent $\widetilde{T}'$-weights

Given any flag $(\widetilde{\tau}, \widetilde{\sigma}) \in F(\widetilde{\Sigma})$, we define

$$\widetilde{\mathbf{w}}(\widetilde{\tau}, \widetilde{\sigma}) := e_{\widetilde{T}'}(T_{\mathfrak{p}_{\widetilde{\sigma}}}\mathfrak{l}_{\widetilde{\tau}}) \in H^2_{\widetilde{T}'}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}\mathsf{u}_1 \oplus \mathbb{Q}\mathsf{u}_2 \oplus \mathbb{Q}\mathsf{u}_4.$$

For any $(\tau, \sigma) \in F(\Sigma)$, we have

$$\widetilde{\mathbf{w}}(\iota(\tau), \iota(\sigma))\big|_{\mathsf{u}_4=0} = \mathbf{w}(\tau, \sigma). \tag{3.13}$$

Moreover, for any $\sigma \in \Sigma(3)$, we have

$$\widetilde{\mathbf{w}}(\sigma, \iota(\sigma)) = \mathsf{u}_4.$$

Specifically for the cone $\iota(\sigma_0)$, we have

$$\widetilde{\mathbf{w}}(\iota(\tau_0),\iota(\sigma_0)) = \frac{1}{\mathfrak{r}}\mathsf{u}_1, \qquad \widetilde{\mathbf{w}}(\sigma_0,\iota(\sigma_0)) = \mathsf{u}_4,$$
$$\widetilde{\mathbf{w}}_2 := \widetilde{\mathbf{w}}(\iota(\tau_2),\iota(\sigma_0)) = \frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1 + \frac{1}{\mathfrak{m}}\mathsf{u}_2, \qquad \widetilde{\mathbf{w}}_3 := \widetilde{\mathbf{w}}(\iota(\tau_3),\iota(\sigma_0)) = -\frac{\mathfrak{m}+\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2 - \mathsf{u}_4.$$

Tangent $\widetilde{T}'$-weights at a fixed point $\mathfrak{p}_{\widetilde{\sigma}}$ for $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$ are given in Section B.2. Specifically for the cone $\widetilde{\sigma}_0$, we have

$$\widetilde{\mathbf{w}}(\iota(\tau_0),\widetilde{\sigma}_0) = -\mathsf{u}_1, \qquad \widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}_0),\widetilde{\sigma}_0) = \mathsf{u}_1 + \mathsf{u}_4,$$
$$\widetilde{\mathbf{w}}(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0) = -\frac{f}{\mathfrak{m}}\mathsf{u}_1 + \frac{1}{\mathfrak{m}}\mathsf{u}_2, \qquad \widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0) = \frac{f}{\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2 - \mathsf{u}_4,$$

where $\delta_2(\widetilde{\sigma}_0), \delta_3(\widetilde{\sigma}_0), \delta_4(\widetilde{\sigma}_0)$ are facets of $\widetilde{\sigma}_0$ described by

$$I'_{\delta_2(\widetilde{\sigma}_0)} = \{3, R+1, R+2\}, \quad I'_{\delta_3(\widetilde{\sigma}_0)} = \{2, R+1, R+2\}, \quad I'_{\delta_4(\widetilde{\sigma}_0)} = \{2, 3, R+1\}. \tag{3.14}$$

3.5.3 Localization computations

Given any $n \in \mathbb{Z}_{\geq 0}$ and effective class $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z})$, the $\widetilde{T}'$-action on $\widetilde{\mathcal{X}}$ induces a $\widetilde{T}'$-action on the moduli space $\overline{\mathcal{M}}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$ of stable maps to $\widetilde{\mathcal{X}}$. Recall from Section 3.2 that connected components of $\overline{\mathcal{M}}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})^{\widetilde{T}'}$ are indexed by decorated graphs in $\Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$. Using the study of the tangent-obstruction theory by [76, Theorem 137], we have the following computation:

Proposition 3.11. *Let $n \in \mathbb{Z}_{\geq 0}$, $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z})$ be an effective class, $\lambda \in G_{\tau_0}$, and $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \in H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$. Then*

$$\begin{aligned}\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\widetilde{\beta}} = &\sum_{\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})} c_{\vec{\Gamma}} \cdot \prod_{e \in E(\Gamma)} \widetilde{\mathbf{h}}(e) \cdot \prod_{(e,v) \in F(\Gamma)} \widetilde{\mathbf{h}}(e,v) \cdot \prod_{v \in V(\Gamma)} \left(\prod_{i \in S_v} \iota^*_{\sigma_v}(\widetilde{\gamma}_i) \right) \\ &\cdot \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\widetilde{\mathbf{h}}(v)}{\prod_{e \in E_v} \left(\widetilde{\mathbf{w}}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)} \Bigg|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0}.\end{aligned} \tag{3.15}$$

We explain the notations used in (3.15) above:

- For each $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}}, \widetilde{\beta})$, the coefficient $c_{\vec{\Gamma}}$ is defined in (3.1). To give the definitions of the other quantities in (3.8), we pick a stable map $u : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_{n+1}) \to \widetilde{\mathcal{X}}$ whose associated decorated graph is $\vec{\Gamma}$ (see Section 3.2). The definitions do not depend on the choice of u.

- For each $e \in E(\Gamma)$, define

$$\widetilde{\mathbf{h}}(e) := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}_e, (u|_{\mathcal{C}_e})^* T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}_e, (u|_{\mathcal{C}_e})^* T\widetilde{\mathcal{X}})^m)},$$

  which is explicitly computed in [76, Lemma 130] in terms of tangent $\widetilde{T}'$-weights.

- For each $(e, v) \in F(\Gamma)$, define

$$\widetilde{\mathbf{h}}(e, v) := e_{\widetilde{T}'}\left((T_{\mathfrak{p}_{\sigma_v}} \widetilde{\mathcal{X}})^{k_{(e,v)}}\right) = \prod_{\substack{(\widetilde{\tau}, \sigma_v) \in F(\widetilde{\Sigma}) \\ k_{(e,v)} \in G_{\widetilde{\tau}}}} \widetilde{\mathbf{w}}(\widetilde{\tau}, \sigma_v).$$

  Here, $(T_{\mathfrak{p}_{\sigma_v}} \widetilde{\mathcal{X}})^{k_{(e,v)}}$ is the maximal subspace of $T_{\mathfrak{p}_{\sigma_v}} \widetilde{\mathcal{X}}$ that is invariant under the action of $k_{(e,v)}$.

- We set

$$\widetilde{\gamma}_{n+1} := \widetilde{\gamma}_\lambda$$

  for convenience.

- For each stable vertex $v \in V^S(\vec{\Gamma})$, define

$$\widetilde{\mathbf{h}}(v) := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}_v, (u|_{\mathcal{C}_v})^* T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}_v, (u|_{\mathcal{C}_v})^* T\widetilde{\mathcal{X}})^m)},$$

  which is explicitly computed in [76, Lemma 126] in terms of Hurwitz-Hodge classes and tangent $\widetilde{T}'$-weights. For each unstable vertex $v \notin V^S(\vec{\Gamma})$, define

$$\widetilde{\mathbf{h}}(v) := \begin{cases} \widetilde{\mathbf{h}}(e, v)^{-1} & \text{if } v \in V^1(\vec{\Gamma}) \cup V^{1,1}(\vec{\Gamma}), E_v = \{e\}, \\ \widetilde{\mathbf{h}}(e_1, v)^{-1} = \widetilde{\mathbf{h}}(e_2, v)^{-1} & \text{if } v \in V^2(\vec{\Gamma}), E_v = \{e_1, e_2\}. \end{cases}$$

- For each $(e,v) \in F(\Gamma)$, define

$$\widetilde{\mathbf{w}}_{(e,v)} := e_{\widetilde{T}'}(T_{\mathfrak{n}(e,v)}\mathcal{C}_e) = \frac{\mathfrak{r}(\tau_e,\sigma_v)\widetilde{\mathbf{w}}(\tau_e,\sigma_v)}{r_{(e,v)}d_e},$$

where the $\widetilde{T}'$-action on $T_{\mathfrak{n}(e,v)}\mathcal{C}_e$ is induced from that on $T_{\mathfrak{p}_{\sigma_v}}\mathfrak{l}_{\tau_e}$.

We note that for any $\widetilde{\sigma} \in \widetilde{\Sigma}(4)$, $\widetilde{\sigma} \neq \widetilde{\sigma}_0$, and any $\lambda \in G_{\tau_0}$,

$$\iota_{\widetilde{\sigma}}^*(\widetilde{\gamma}_{n+1}) = 0.$$

Thus only the decorated graphs satisfying $\vec{f} \circ \vec{s}(n+1) = \widetilde{\sigma}_0$ can contribute.

Chapter 4: Numerical correspondence

In this chapter, we establish the open/closed correspondence at the numerical level (Theorem 4.1). That is, we identify the disk invariants of $(\mathcal{X}, \mathcal{L}, f)$ and the closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$ in corresponding curve classes.

4.1 The statement

Recall from Section 2.5 the inclusion

$$\iota_* : H_2(X, L; \mathbb{Z}) \to H_2(\widetilde{X}; \mathbb{Z})$$

which maps $[B]$ to $[l_{\iota(\tau_0)}]$. Moreover, Convention 2.4 specifies equivariant lifts of classes in $H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$ taken as insertions in the Gromov-Witten invariants.

Theorem 4.1. *Let $n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X; \mathbb{Z})$ be an effective class, and $(d, \lambda) \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times G_{\tau_0}$ such that $d > 0$. Set*

$$\beta' = \beta + d[B], \quad \widetilde{\beta} = \iota_*(\beta').$$

Let $\gamma_1, \dots, \gamma_n \in H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$ and, by an abuse of notation, $\gamma_1, \dots, \gamma_n \in H^2_{\mathrm{CR},T'}(\mathcal{X}; \mathbb{Q})$, $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \in H^2_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{Q})$ be the equivariant lifts chosen as in Convention 2.4. Then

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)} = \langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\widetilde{\beta}}.$$

Remark 4.2. The open and closed invariants involved in the correspondence above can also be related to the genus-zero *relative* Gromov-Witten invariants [71, 72] of a log Calabi-Yau pair $(\mathcal{X} \sqcup \mathcal{D}, \mathcal{D})$, where $\mathcal{D}$ is an additional irreducible toric divisor. The correspondence between the open and relative invariants is already established by Fang-Liu [41] for all genera, while the correspondence between the relative and closed invariants can be interpreted as a version of the *log-local principle* of van Garrel-Graber-Ruddat [48] in the non-compact setting. We give a more detailed discussion of these correspondences in the case where $\mathcal{X} = X$ is smooth in Appendix C.

Example 4.3. Let $\mathcal{X} = \mathbb{C}^3$ and $\mathcal{L}$ be an outer brane, as in Section 2.6.1. Given an effective class $\beta' = d[B] \in H_2(X, L; \mathbb{Z})$ with $d \in \mathbb{Z}_{>0}$, the disk invariant (without insertions) is given by

$$\langle\rangle^{\mathcal{X},(\mathcal{L},f)}_{d[B],(d,1)} = \frac{(-1)^{df}}{d^2}\frac{\prod_{a=1}^{d-1}(df+a)}{(d-1)!},$$

which specializes to the famous formula $1/d^2$ at zero framing $f = 0$. On the other hand, when $f = 0$ we have

$$\widetilde{\mathcal{X}} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathbb{C} \times \mathcal{X}',$$

where $\mathcal{X}' = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ is the resolved conifold. The closed invariant $\langle \widetilde{\gamma}_{\lambda=1} \rangle^{\widetilde{\mathcal{X}},T_f}_{\iota_*(\beta')}$ is reduced to the well-known genus-zero degree-d Gromov-Witten invariant of the resolved conifold:

$$\langle H \rangle^{\mathcal{X}'}_{0,1,d[l_{\iota(\tau_0)}]} = \frac{1}{d^2},$$

which coincides with the disk invariant. Here, H is the hyperplane class and the invariant differs from the Aspinwall-Morrison formula [8] $1/d^3$ of genus-zero, degree-d covers of the zero section by the divisor equation.

For the rest of this section, we give an outline of the proof of Theorem 4.1 and set up some notations. Recall that the localization computations in Propositions 3.7 and 3.11 express $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)}$ as a sum of contributions from decorated graphs in $\Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X}, \beta)$ (see (3.10)) and $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\widetilde{\beta}}$ as a sum of contributions from decorated graphs in $\Gamma_{0,n+1}(\widetilde{\mathcal{X}}, \widetilde{\beta})$. There are two main steps in our proof of Theorem 4.1. First, we set up a one-to-one correspondence between $\Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X}, \beta)$ and a subset

$$\Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}}, \widetilde{\beta})$$

of decorated graphs in $\Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$, and directly relate the contributions from corresponding decorated graphs. Second, we show that there is no contribution from any decorated graph in $\Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$ outside the subset $\Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$.

To define $\Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$, we consider a map

$$\epsilon : \Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta) \to \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta}),$$

defined as follows: Given $\vec{\Gamma} \in \Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta)$, to obtain $\epsilon(\vec{\Gamma})$, we first post-compose the label map $\vec{f}$ with the map ι of cones (see (2.9)). Then, we replace the $(n+1)$-th marked point in $\vec{\Gamma}$, with marking $v_0 = v_0(\vec{\Gamma}) := \vec{s}(n+1)$, by

- a new vertex $\widetilde{v}_0 = \widetilde{v}_0(\epsilon(\vec{\Gamma}))$ with label $\vec{f}(\widetilde{v}_0) = \widetilde{\sigma}_0$;
- a new edge $e_0 = e_0(\epsilon(\vec{\Gamma}))$ between v_0 and $\widetilde{v}_0$, with label $\vec{f}(e_0) = \iota(\tau_0)$ and degree $\vec{d}(e_0) = (d,\lambda) \in H_{\iota(\tau_0)} = H_{\tau_0} \cong \mathbb{Z} \times G_{\tau_0}$;
- a new $(n+1)$-th marked point with marking $\vec{s}(n+1) = \widetilde{v}_0$ and twisting $\vec{k}(n+1) = \lambda \in G_{\widetilde{\sigma}_0} = G_{\tau_0}$.

See Figure 4.1. The twisting at the new flags are $k_{(e_0,v_0)} = h(d,\lambda)$, $k_{(e_0,\widetilde{v}_0)} = \lambda$ and the compatibility at vertices $v_0, \widetilde{v}_0$ in $\epsilon(\vec{\Gamma})$ are satisfied. It is straightforward to check that $\epsilon(\vec{\Gamma})$ indeed belongs to $\Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$ and that ϵ is injective. We define $\Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$ to be the image of ϵ.

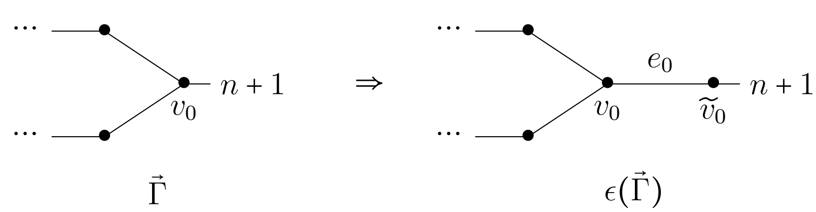

Figure 4.1: Newly added vertex $\widetilde{v}_0$ and edge e_0 in the graph $\epsilon(\vec{\Gamma})$.

We set up some additional notations. For each $\vec{\Gamma} \in \Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta)$, let

$$\begin{aligned} C_{\vec{\Gamma}} :=& \mathfrak{rm} D_{d,\lambda} \cdot c_{\vec{\Gamma}} \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V(\Gamma)} \left(\prod_{i \in S_v} \iota^*_{\sigma_v}(\gamma_i) \right) \\ & \cdot \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\mathbf{h}(v)}{\left(\frac{\mathsf{u}_1}{d} - \bar{\psi}_{n+1}\right)^{\delta_{v,n+1}} \cdot \prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)} \in \mathcal{Q}_{T'} \end{aligned} \tag{4.1}$$

denote the contribution of $\vec{\Gamma}$ to $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)}$ as in (3.11) in Proposition 3.7 before the weight restriction $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. Moreover, for an effective class $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z})$ and $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$, let

$$\begin{aligned} \widetilde{C}_{\vec{\Gamma}} :=& c_{\vec{\Gamma}} \cdot \prod_{e \in E(\Gamma)} \widetilde{\mathbf{h}}(e) \cdot \prod_{(e,v) \in F(\Gamma)} \widetilde{\mathbf{h}}(e,v) \cdot \prod_{v \in V(\Gamma)} \left(\prod_{i \in S_v} \iota^*_{\sigma_v}(\widetilde{\gamma}_i) \right) \\ & \cdot \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\widetilde{\mathbf{h}}(v)}{\prod_{e \in E_v} \left(\widetilde{\mathbf{w}}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)} \in \mathcal{Q}_{\widetilde{T}'} \end{aligned}$$

denote the contribution of $\vec{\Gamma}$ to $\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\widetilde{\beta}}$ as in (3.15) in Proposition 3.11 before the weight restriction $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$.

4.2 Matching contributions

As the first step in proving Theorem 4.1, we show the following lemma matching the contributions from decorated graphs in $\Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta)$ and $\Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta}) = \epsilon(\Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta))$.

Lemma 4.4. *For the quantities defined as in Theorem 4.1, we have that for each* $\vec{\Gamma} \in \Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta)$,

$$\widetilde{C}_{\epsilon(\vec{\Gamma})}\big|_{\mathsf{u}_4=0} = C_{\vec{\Gamma}}.$$

In particular, by Proposition 3.7, we have

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',(d,\lambda)} = \sum_{\vec{\Gamma} \in \Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})} \widetilde{C}_{\vec{\Gamma}}\big|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0}.$$

Proof. Let $\vec{\Gamma} \in \Gamma^{0,(d,\lambda)}_{0,n+1}(\mathcal{X},\beta)$ and $\vec{\Gamma}' = \epsilon(\vec{\Gamma}) \in \Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$. The underlying graph $\Gamma = (V(\Gamma), E(\Gamma))$

of $\vec{\Gamma}$ is a subgraph of the underlying graph $\Gamma' = (V(\Gamma'), E(\Gamma'))$ of $\vec{\Gamma}'$. Let $v_0 = v_0(\vec{\Gamma}) \in V(\Gamma)$, $\widetilde{v}_0 = \widetilde{v}_0(\vec{\Gamma}') \in V(\Gamma')$, $e_0 = e_0(\vec{\Gamma}') \in E(\Gamma')$. We carry out the comparison between $\widetilde{C}_{\vec{\Gamma}'}\big|_{\mathsf{u}_4=0}$ and $C_{\vec{\Gamma}}$ piece by piece, while supplying computational details in Appendix B.

- (Coefficients) Note that $\mathrm{Aut}(\vec{\Gamma}) \cong \mathrm{Aut}(\vec{\Gamma}')$. By definition (3.1), we have

$$c_{\vec{\Gamma}'} = c_{\vec{\Gamma}} \cdot \frac{\frac{G_{v_0}}{r_{(e_0,v_0)}} \cdot \frac{G_{\widetilde{v}_0}}{r_{(e_0,\widetilde{v}_0)}}}{d_{e_0}|G_{e_0}|} = c_{\vec{\Gamma}} \cdot \frac{\mathfrak{rm}}{dr_{(e_0,v_0)}r_{(e_0,\widetilde{v}_0)}}.$$

- (Edges) For each $e \in E(\Gamma)$, by Lemma B.1, we have

$$\left(\mathsf{u}_4\widetilde{\mathbf{h}}(e)\right)\big|_{\mathsf{u}_4=0} = \mathbf{h}(e).$$

  For the edge e_0, by the computation of $\widetilde{\mathbf{h}}(e_0)$ in Lemma B.10, we have

$$\widetilde{\mathbf{h}}(e_0)\big|_{\mathsf{u}_4=0} = \begin{cases} -\mathfrak{m}\left(\frac{\mathsf{u}_1}{d}\right)^{-1}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}}\right)^{-1} \cdot D_{d,1} & \text{if } \lambda = 1, \\ -\mathfrak{m}\left(\frac{\mathsf{u}_1}{d}\right)^{-1} \cdot D_{d,\lambda} & \text{if } \lambda \neq 1. \end{cases}$$

- (Flags) For each $(e,v) \in F(\Gamma)$, by Lemma B.2, we have

$$\frac{\widetilde{\mathbf{h}}(e,v)}{\mathsf{u}_4}\bigg|_{\mathsf{u}_4=0} = \mathbf{h}(e,v).$$

  For the flag (e_0, v_0), we have

$$\widetilde{\mathbf{h}}(e_0,v_0) = \prod_{\substack{\widetilde{\tau}\in\{\iota(\tau_0),\iota(\tau_2),\iota(\tau_3),\sigma_0\}\\ h(d,\lambda)\in G_{\widetilde{\tau}}}} \widetilde{\mathbf{w}}(\widetilde{\tau},\iota(\sigma_0)) = \mathsf{u}_4 \cdot \prod_{\substack{\widetilde{\tau}\in\{\iota(\tau_0),\iota(\tau_2),\iota(\tau_3)\}\\ h(d,\lambda)\in G_{\widetilde{\tau}}}} \widetilde{\mathbf{w}}(\widetilde{\tau},\iota(\sigma_0)).$$

  Therefore,

$$\frac{\widetilde{\mathbf{h}}(e_0,v_0)}{\mathsf{u}_4}\bigg|_{\mathsf{u}_4=0} = \prod_{\substack{\tau\in\{\tau_0,\tau_2,\tau_3\}\\ h(d,\lambda)\in G_\tau}} \mathbf{w}(\tau,\sigma_0) = \iota_{\sigma_0}^*(\phi_{\sigma_0,h(d,\lambda)^{-1}}). \tag{4.2}$$

  For the flag $(e_0, \widetilde{v}_0)$, note that $k_{(e_0,\widetilde{v}_0)} = \lambda$ belongs to G_{τ_0} and $G_{\delta_1(\widetilde{\sigma}_0)}$, but not $G_{\delta_2(\widetilde{\sigma}_0)}$ or

$G_{\delta_3(\widetilde{\sigma}_0)}$. Then

$$\widetilde{\mathbf{h}}(e_0, \widetilde{v}_0)\big|_{\mathsf{u}_4=0} = \begin{cases} \mathsf{u}_1^2 \left(\dfrac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}} \right)^2 & \text{if } \lambda = 1, \\ -\mathsf{u}_1^2 & \text{if } \lambda \neq 1. \end{cases}$$

- (Insertions) For any $i = 1, \dots, n$, with $i \in S_v$ for $v \in V(\Gamma)$, since $\iota^*(\widetilde{\gamma}_i) = \gamma_i$, we have

$$\iota^*_{\widetilde{\sigma}_v}(\widetilde{\gamma}_i)\big|_{\mathsf{u}_4} = \iota^*_{\sigma_v}(\gamma_i).$$

The insertion $\iota^*_{\sigma_0}(\gamma_{n+1}) = \iota^*_{\sigma_0}(\phi_{\sigma_0, h(d,\lambda)^{-1}})$ is related by (4.2) above to $\widetilde{\mathbf{h}}(e_0, v_0)$. Moreover, we have

$$\iota^*_{\widetilde{\sigma}_0}(\widetilde{\gamma}_{n+1}) = \begin{cases} -\mathsf{u}_1 \cdot \dfrac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}} & \text{if } \lambda = 1, \\ -\mathsf{u}_1 \mathbf{1}_{\lambda^{-1}} & \text{if } \lambda \neq 1. \end{cases} \tag{4.3}$$

- (Vertices) For any $v \in V(\Gamma) \smallsetminus \{v_0\}$, by Lemmas B.2 (when v is unstable), B.3, and B.4, we have

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\mathsf{u}_4 \widetilde{\mathbf{h}}(v)}{\prod_{e \in E_v} \left(\widetilde{\mathbf{w}}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)} \Bigg|_{\mathsf{u}_4=0} = \int_{\overline{\mathcal{M}}_{0,\vec{k}_v}(\mathcal{B}G_v)} \frac{\mathbf{h}(v)}{\prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)}.$$

For the vertex v_0, note that the $(n+1)$-th marked point in $\vec{\Gamma}$ is replaced by the flag (e_0, v_0) in $\vec{\Gamma}'$, and the twisting k_{n+1} in $\vec{\Gamma}$ is identified with $k^{-1}_{(e_0,v_0)} = h(d,\lambda)^{-1}$. Therefore, the integral

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}_{v_0}}(\mathcal{B}G_{v_0})} \frac{\mathsf{u}_4 \widetilde{\mathbf{h}}(v_0)}{\prod_{e \in E_{v_0}} \left(\widetilde{\mathbf{w}}_{(e,v_0)} - \frac{\bar{\psi}_{(e,v_0)}}{r_{(e,v_0)}} \right)} \Bigg|_{\mathsf{u}_4=0}$$

in $\widetilde{C}_{\vec{\Gamma}'}$ (after multiplied by u_4 and restricted to $\mathsf{u}_4 = 0$) is identified with

$$r_{(e_0,v_0)} \cdot \int_{\overline{\mathcal{M}}_{0,\vec{k}_{v_0}}(\mathcal{B}G_{v_0})} \frac{\mathbf{h}(v_0)}{\left(\frac{\mathsf{u}_1}{d} - \bar{\psi}_{n+1} \right) \cdot \prod_{e \in E_{v_0}} \left(\mathbf{w}_{(e,v_0)} - \frac{\bar{\psi}_{(e,v_0)}}{r_{(e,v_0)}} \right)}$$

where the integral is the one in $C_{\vec{\Gamma}}$.

Finally, for the unstable vertex $\widetilde{v}_0 \in V^{1,1}(\vec{\Gamma}'))$, we have by definition that $\widetilde{\mathbf{h}}(\widetilde{v}_0)\big|_{\mathsf{u}_4=0} = \left(\widetilde{\mathbf{h}}(e_0,\widetilde{v}_0)\big|_{\mathsf{u}_4=0}\right)^{-1}$. By (A.2), we have

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}_{\widetilde{v}_0}}(\mathcal{B}G_{\widetilde{v}_0})} \frac{\widetilde{\mathbf{h}}(\widetilde{v}_0)}{\widetilde{\mathbf{w}}_{(e_0,\widetilde{v}_0)} - \frac{\bar{\psi}_{(e_0,\widetilde{v}_0)}}{r_{(e_0,\widetilde{v}_0)}}}\Bigg|_{\mathsf{u}_4=0} = \frac{r_{(e_0,\widetilde{v}_0)}}{|G_{\widetilde{v}_0}|}\widetilde{\mathbf{h}}(\widetilde{v}_0)\big|_{\mathsf{u}_4=0} = \frac{r_{(e_0,\widetilde{v}_0)}}{\mathfrak{m}} \cdot \left(\widetilde{\mathbf{h}}(e_0,\widetilde{v}_0)\big|_{\mathsf{u}_4=0}\right)^{-1}.$$

We now piece together the above comparisons. First, we collect the powers of u_4 appearing in the pieces of $\widetilde{C}_{\vec{\Gamma}'}$. Each edge, flag, vertex in Γ contributes a power of $-1, 1, -1$ respectively. Since the graph Γ is a tree, we have

$$-|E(\Gamma)| + |F(\Gamma)| - |V(\Gamma)| = -1.$$

In addition, the extra flag (e_0, v_0) in Γ' contributes a power of 1, while there is no contribution from e_0, $(e_0,\widetilde{v}_0)$, or $\widetilde{v}_0$. Therefore, the total power of u_4 in $\widetilde{C}_{\vec{\Gamma}'}$ is 0. Summarizing the comparisons of the pieces above, we have that for $\lambda = 1$,

$$\widetilde{C}_{\vec{\Gamma}'}\big|_{\mathsf{u}_4=0} = C_{\vec{\Gamma}} \cdot \frac{1}{d r_{(e_0,v_0)} r_{(e_0,\widetilde{v}_0)}} \cdot (-1)\mathfrak{m}\left(\frac{\mathsf{u}_1}{d}\right)^{-1}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}}\right)^{-1} \cdot \left(-\mathsf{u}_1 \cdot \frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}}\right) \cdot r_{(e_0,v_0)} \cdot \frac{r_{(e_0,\widetilde{v}_0)}}{\mathfrak{m}} = C_{\vec{\Gamma}}.$$

For $\lambda \neq 1$, we have

$$\widetilde{C}_{\vec{\Gamma}'}\big|_{\mathsf{u}_4=0} = C_{\vec{\Gamma}} \cdot \frac{1}{d r_{(e_0,v_0)} r_{(e_0,\widetilde{v}_0)}} \cdot (-1)\mathfrak{m}\left(\frac{\mathsf{u}_1}{d}\right)^{-1} \cdot (-\mathsf{u}_1) \cdot r_{(e_0,v_0)} \cdot \frac{r_{(e_0,\widetilde{v}_0)}}{\mathfrak{m}} = C_{\vec{\Gamma}}.$$

□

4.3 Vanishing arguments

In this section, we study the contributions from decorated graphs in $\Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta}) \setminus \Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$. Recall first that we made the following observation at the end of Section 3.5.

Observation 4.5. *For any effective class $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z})$ and $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$, $\widetilde{C}_{\vec{\Gamma}} = 0$ unless $\vec{s}(n+1) = \widetilde{v}_0$ and $\vec{f}(\widetilde{v}_0) = \widetilde{\sigma}_0$. In particular, if $\widetilde{\beta} \in H_2(X;\mathbb{Z})$, then $\widetilde{C}_{\vec{\Gamma}} = 0$ for all $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$.*

Now we introduce some notations for an effective class $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z}) \smallsetminus H_2(X;\mathbb{Z})$ and $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$. We partition $V(\Gamma)$ into two subsets

$$V_0 := \{v \in V(\Gamma) : \vec{f}(v) \in \iota(\Sigma(3))\}, \quad V_1 := \{v \in V(\Gamma) : \vec{f}(v) \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))\}$$

and let $\Gamma_0 = (V_0, E_0)$, $\Gamma_1 = (V_1, E_1)$ be induced subgraphs of Γ on them, where

$$E_0 := \{e \in E(\Gamma) : \vec{f}(e) \in \iota(\Sigma(2)_c)\}, \quad E_1 := \{e \in E(\Gamma) : \vec{f}(e) \in \widetilde{\Sigma}(3)_c \smallsetminus \iota(\Sigma(2))\}.$$

The two subgraphs can be disconnected, but the components are connected by edges in

$$E_2 := E(\Gamma) \smallsetminus (E_0 \cup E_1) = \{e \in E(\Gamma) : \vec{f}(e) \in \iota(\Sigma(2) \smallsetminus \Sigma(2)_c)\}.$$

See Figure 4.2. Restricting the decorations of $\vec{\Gamma}$ to a component of Γ_0 yields a decorated graph whose degree is a curve class in $H_2(X;\mathbb{Z})$. Since $\widetilde{\beta} \notin H_2(X;\mathbb{Z})$, the graph Γ_1 must be non-empty. We further partition $F(\Gamma)$ into two subsets

$$F_0 := \{(e,v) \in F(\Gamma) : v \in V_0\}, \qquad F_1 := \{(e,v) \in F(\Gamma) : v \in V_1\}.$$

Moreover, we denote

$$V_1' := \{v \in V_1 \cap V^2(\vec{\Gamma}) : \vec{f}(E_v) = \{\delta_2(\vec{f}(v)), \delta_3(\vec{f}(v))\}\}.$$

We now prove the following preparatory lemma.

Lemma 4.6. *Let $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z}) \smallsetminus H_2(X;\mathbb{Z})$ be an effective class and $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$. Unless*

$$|E_1| - |V_1'| = |E_2| - c_0 = 0, \tag{4.4}$$

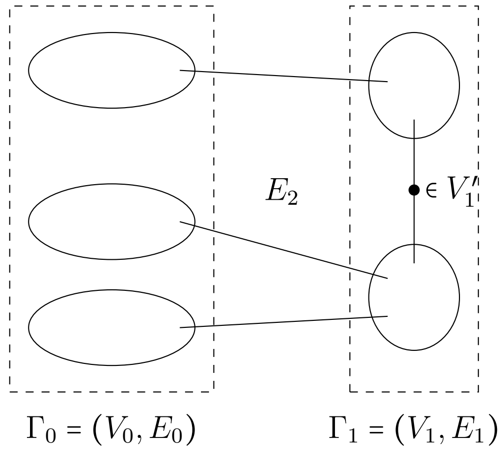

Figure 4.2: Subgraphs Γ_0, Γ_1 connected by E_2, and a vertex in V_1'.

we have

$$\widetilde{C}_{\vec{\Gamma}}\Big|_{\mathsf{u}_4=0} = 0.$$

Note that condition (4.4) is only possible if Γ_1 consists of a single vertex, denoted by $\widetilde{v}_0$, and each edge in E_2 connects a component of Γ_0 to $\widetilde{v}_0$. In this case, all edges in E_2 are labeled by $\iota(\tau_0)$.

Proof. We consider the power of u_4 in $\widetilde{C}_{\vec{\Gamma}}$. By Lemmas B.1 – B.7, B.9 – B.12, each edge in E_0 or vertex in $V_0 \cup V_1'$ contributes a power of -1, each flag in F_0 or edge in E_1 contributes a power of 1, and there are no other contributions. Since each connected component of Γ_0 is a tree, we see that the total power of u_4 is

$$|F_0| - |E_0| - |V_0| + |E_1| - |V_1'| = \big(|E_2| - c_0\big) + \big(|E_1| - |V_1'|\big), \tag{4.5}$$

where c_0 denotes the number of connected components of Γ_0. Since the subgraph Γ_1 is non-empty, we have $|E_2| \geq c_0$. Moreover, since each vertex in V_1' is incident to two distinct edges in E_1, we have $|E_1| \geq |V_1'|$. Therefore, the power (4.5) is always non-negative, and is zero only if $|E_1| - |V_1'| = |E_2| - c_0 = 0$. This implies the lemma. □

Lemma 4.6 implies the following result, which we will use in Chapter 5.

Lemma 4.7. *For any effective class* $\widetilde{\beta} \in H_2(\widetilde{X};\mathbb{Z}) \smallsetminus \iota_*(H_2(X,L;\mathbb{Z}))$ *and* $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$*, we have*

$$\widetilde{C}_{\vec{\Gamma}}\big|_{\mathsf{u}_4=0} = 0.$$

Proof. It suffices to observe that such $\vec{\Gamma}$ cannot satisfy condition (4.4), and thus Lemma 4.6 applies. □

Now we show the following lemma as the second main ingredient in proving Theorem 4.1.

Lemma 4.8. *Consider the setup of Theorem 4.1. Assume that* $f \in \mathbb{Z}$ *is* generic *(with respect to* $\widetilde{\beta}$*), i.e. avoiding a finite set of integers. Then for any* $\vec{\Gamma} \in \Gamma_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta}) \smallsetminus \Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$*, we have*

$$\widetilde{C}_{\vec{\Gamma}}\big|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0} = 0.$$

Proof. By Lemma 4.6, it suffices to consider the case where condition (4.4) holds. We again denote the only vertex in Γ_1 by $\widetilde{v}_0$. By Observation 4.5, it suffices to consider the case $\vec{s}(n+1) = \widetilde{v}_0$ and $\vec{f}(\widetilde{v}_0) = \widetilde{\sigma}_0$.

We assume without loss of generality that for each $i = 1,\dots,n$, $\widetilde{\gamma}_i$ is contained in $H^*_{\widetilde{T}'}(\widetilde{\mathcal{X}}_{j_i};\mathbb{Q})$ for some $j_i \in \mathrm{Box}(\widetilde{\mathcal{X}})$. If $j_i = \vec{0}$, then the way the lift $\widetilde{\gamma}_i$ is chosen (Convention 2.4) implies that we must have $\vec{s}(i) \neq \widetilde{v}_0$ in order for $\widetilde{C}_{\vec{\Gamma}}$ to be non-zero. Therefore, $S_{\widetilde{v}_0}$ consists of $n+1$ and a subset of $\{i \in \{1,\dots,n\} \mid j_i \neq \vec{0}\}$. Recall our notation that $\vec{k}_{\widetilde{v}_0} = (k^{-1}_{(e,\widetilde{v}_0)}, k_i) \in G^{E_2 \cup S_{\widetilde{v}_0}}_{\iota(\tau_0)}$. In particular, $k_{n+1} = \lambda$ and $k_i \neq 1$ for any other $i \in S_{\widetilde{v}_0}$.

We note that since $\vec{\Gamma} \notin \Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}},\widetilde{\beta})$, either $|E_2| > 1$, or $|E_2| = 1$ and $S_{\widetilde{v}_0}$ contains a marking other than $n+1$. In either case, $\widetilde{v}_0$ is a stable vertex.

Now we consider the power of $\mathsf{u}_2 - f\mathsf{u}_1$ in $\widetilde{C}_{\vec{\Gamma}}\big|_{\mathsf{u}_4=0}$ for a generic choice of $f \in \mathbb{Z}$. We consider several cases below and show that in each case, the total power is positive, which implies that $\widetilde{C}_{\vec{\Gamma}}\big|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0} = 0$.

- *Case I:* $\lambda = 1$. Note from (4.3) that $\iota^*_{\widetilde{\sigma}_0}(\widetilde{\gamma}_{n+1})$ contributes a power of 1 in the case. By Lemmas B.5 – B.7, B.9, B.10, the only additional powers come from the vertex $\widetilde{v}_0$, edges in E_2, and flags consisting of $\widetilde{v}_0$ and an incident edge. If $k_{(e,\widetilde{v}_0)} = 1$ for all $e \in E_2$ and $S_{\widetilde{v}_0} = \{n+1\}$, then $\widetilde{v}_0$ contributes a power of -2, each edge $e \in E_2$ contributes a power of -1, and the corresponding flag $(e, \widetilde{v}_0)$ contributes a power of 2. Thus the total power of $\mathsf{u}_2 - f\mathsf{u}_1$ is

$$1 - 2 + |E_2| = |E_2| - 1.$$

Since $\vec{\Gamma} \notin \Gamma^{0,\lambda}_{0,n+1}(\widetilde{\mathcal{X}}, \widetilde{\beta})$, we have $|E_2| > 1$, which makes the above power strictly positive.

If, on the other hand, that not all $k_{(e,\widetilde{v}_0)}$ are 1, or $S_{\widetilde{v}_0} \neq \{n+1\}$. Then $\widetilde{v}_0$ has no contribution, any edge $e \in E_2$ with $k_{(e,\widetilde{v}_0)} = 1$ contributes a power of -1, and the corresponding flag $(e, \widetilde{v}_0)$ contributes a power of 2. Thus the total power of $\mathsf{u}_2 - f\mathsf{u}_1$ is

$$1 + |\{e \in E_2 : k_{(e,\widetilde{v}_0)} = 1\}| > 0.$$

- *Case II:* $\lambda \neq 1$. In this case, there is no contribution from $\iota^*_{\widetilde{\sigma}_0}(\widetilde{\gamma}_{n+1})$. Moreover, $k_i \neq 1$ for all $i \in S_{\widetilde{v}_0}$. Similar to above, there is no contribution from the denominator of $\widetilde{\mathbf{h}}(\widetilde{v}_0)$, and the total contribution from edges in E_2 and the associated flags is

$$|\{e \in E_2 : k_{(e,\widetilde{v}_0)} = 1\}|.$$

This is positive if there exists some $e \in E_2$ such that $k_{(e,\widetilde{v}_0)} = 1$. In the remaining case where $k_{(e,\widetilde{v}_0)} \neq 1$ for all $e \in E_2$, Lemma B.8 implies that $\mathsf{u}_2 - f\mathsf{u}_1$ is a power of the numerator of $\widetilde{\mathbf{h}}(\widetilde{v}_0)$, which makes the total power positive.

$\square$

4.4 Completing the proof

Proof of Theorem 4.1. For generic $f \in \mathbb{Z}$ (with respect to β' and $\widetilde{\beta}$), Theorem 4.1 directly follows from Lemmas 4.4 and 4.8. We now prove the theorem for any arbitrary f. Let

$$C = C(\mathsf{u}_1, \mathsf{u}_2) := \langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X}, \mathcal{L}}_{\beta', (d, \lambda)} \in \mathbb{Q}(\mathsf{u}_1, \mathsf{u}_2),$$

which for generic $f \in \mathbb{Z}$ restricts to $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X}, (\mathcal{L}, f)}_{\beta', (d, \lambda)}$ under the weight restriction $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. On the other hand, we consider the dependence of the toric Calabi-Yau 4-orbifold $\widetilde{\mathcal{X}}$ and its closed invariants on $f \in \mathbb{Z}$. Fixing $\mathcal{X}$, $\mathcal{L}$, (d, λ), $\beta' = \beta + d[B]$, and $\gamma_1, \dots, \gamma_n$ as in Theorem 4.1, we let $\widetilde{\mathcal{X}}_f$ denote the 4-orbifold that we construct for $(\mathcal{X}, \mathcal{L}, f)$ and $\widetilde{\beta}_f \in H_2(\widetilde{\mathcal{X}}_f; \mathbb{Z})$ be the image of β'. Let

$$\widetilde{C} = \widetilde{C}(f, \mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4) := \langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}_f, \widetilde{T}'}_{\widetilde{\beta}_f} \in \mathbb{Q}(f, \mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4).$$

By Lemma 3.10, $\widetilde{C}$ has no pole along $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$, and thus restricts to

$$N = N(f) := \langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}_f, T_f}_{\widetilde{\beta}_f} \in \mathbb{Q}(f).$$

In fact, by Lemma 4.6, only decorated graphs satisfying condition (4.4) contribute to $\widetilde{C}$ and N. For such graphs, the dependence on f only comes from edges labeled by $\iota(\tau_0)$, which is *polynomial* by Lemma B.10. Thus $N \in \mathbb{Q}[f]$.

We now consider the difference $D := C(\mathsf{u}_1, \mathsf{u}_2) - N(f)$, as well as the smooth affine hypersurface $H = \{\mathsf{u}_2 - f\mathsf{u}_1\} \subset \mathbb{C}^3 = \mathrm{Spec}(\mathbb{C}[f, \mathsf{u}_1, \mathsf{u}_2])$. By Lemmas 4.4 and 4.8, D does not have a pole along H and thus restricts to a rational function on H. Moreover, $D|_H$ vanishes on an infinite collection of divisors $\{\{f = 0\} \subset H : f \in \mathbb{Z} \text{ generic}\}$ in H. Therefore, $D|_H$ is constantly zero, which implies that the weight restriction

$$C\Big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} = D\Big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} + N$$

is defined for all $f \in \mathbb{Z}$ and identified with $N = \langle \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n, \widetilde{\gamma}_\lambda \rangle_{\widetilde{\beta}_f}^{\widetilde{\mathcal{X}}_f, T_f}$. □

Remark 4.9. The above proof confirms that the disk invariant $\langle \gamma_1, \ldots, \gamma_n \rangle_{\beta', (d,\lambda)}^{\mathcal{X}, (\mathcal{L}, f)}$ is defined for any arbitrary framing $f \in \mathbb{Z}$.

Chapter 5: Generating functions

In this chapter, we use the numerical open/closed correspondence (Theorem 4.1) to obtain a correspondence at the level of the generating functions (Theorem 5.4). With this, we further show that the generating function of disk invariants of $(\mathcal{X}, \mathcal{L}, f)$ can be recovered from the $\widetilde{T}'$-equivariant J-function of $\widetilde{\mathcal{X}}$ (Theorem 5.8).

5.1 Definitions

We choose a basis $\{u_1, \dots, u_{R-3}\}$ of $H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q}) \cong \mathbb{Q}^{R-3}$ such that $u_1, \dots, u_{R'-3}$ is a basis of $H^2(\mathcal{X}; \mathbb{Q}) \cong \mathbb{Q}^{R'-3}$; we further assume that $u_1, \dots, u_{R-3}$ satisfy the conditions specified in Section 6.2. For $a = 1, \dots, R-3$, let $\widetilde{u}_a \in H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}}; \mathbb{Q})$ be the lift of u_a chosen as in Convention 2.4, and let $\widetilde{u}_{R-2} := \widetilde{D}_{R+1}$. Then $\{\widetilde{u}_1, \dots, \widetilde{u}_{R-2}\}$ is a basis of $H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}}; \mathbb{Q}) \cong \mathbb{Q}^{R-2}$, $\{\widetilde{u}_1, \dots, \widetilde{u}_{R'-3}, \widetilde{u}_{R-2}\}$ is a basis of $H^2(\widetilde{\mathcal{X}}; \mathbb{Q}) \cong \mathbb{Q}^{R'-2}$.

5.1.1 Generating functions of closed invariants of $\widetilde{\mathcal{X}}$

Let

$$\widetilde{\boldsymbol{\tau}}_2 = \sum_{a=1}^{R-2} \widetilde{\tau}_a \widetilde{u}_a = \widetilde{\boldsymbol{\tau}}'_2 + \widetilde{\boldsymbol{\tau}}''_2$$

where $\widetilde{\tau}_1, \dots, \widetilde{\tau}_{R-2}$ are complex variables,

$$\widetilde{\boldsymbol{\tau}}'_2 = \sum_{a=1}^{R'-3} \widetilde{\tau}_a \widetilde{u}_a + \widetilde{\tau}_{R-2} \widetilde{u}_{R-2} \in H^2(\widetilde{\mathcal{X}}; \mathbb{C}), \quad \widetilde{\boldsymbol{\tau}}''_2 = \sum_{a=R'-2}^{R-3} \widetilde{\tau}_a \widetilde{u}_a.$$

We choose $\widetilde{T}'$-equivariant lifts of $\widetilde{u}_1, \dots, \widetilde{u}_{R-2}$ as in Convention 2.4. Let

$$\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'} := H^*_{\widetilde{T}'}(\mathrm{pt}; \mathbb{C}) = \mathbb{C}[\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4],$$

and let $\mathcal{Q}^{\mathbb{C}}_{\widetilde{T}'} = \mathbb{C}(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4) = \mathcal{Q}_{\widetilde{T}'} \otimes_{\mathbb{Q}} \mathbb{C}$ be the fractional field of $\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}$. Define

$$\widetilde{Q}_a = e^{\widetilde{\tau}_a}, \quad a = 1, \ldots, R' - 3, R - 2.$$

Given any commutative ring $\mathbb{S}$, define

$$\mathbb{S}[\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!] := \mathbb{S}[\![\widetilde{Q}_1, \ldots, \widetilde{Q}_{R'-3}, \widetilde{Q}_{R-2}, \widetilde{\tau}_{R'-2}, \ldots, \widetilde{\tau}_{R-3}]\!].$$

Let $\mathrm{NE}(\widetilde{X}) \subset H_2(\widetilde{X}; \mathbb{R})$ be the Mori cone generated by effective curve classes in the coarse moduli space $\widetilde{X}$ of $\widetilde{\mathcal{X}}$. Let $E(\widetilde{X})$ denote the semigroup $\mathrm{NE}(\widetilde{X}) \cap H_2(\widetilde{X}; \mathbb{Z})$.

Definition 5.1. Given $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n \in H^*_{\mathrm{CR}, \widetilde{T}'}(\mathcal{X}; \mathbb{C}) \otimes_{\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}} \mathcal{Q}^{\mathbb{C}}_{\widetilde{T}'}$, we define the following generating function of genus zero $\widetilde{T}'$-equivariant closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$:

$$\langle\!\langle \gamma_1 \bar{\psi}^{a_1}, \cdots, \gamma_n \bar{\psi}^{a_n} \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2) := \sum_{\widetilde{\beta} \in E(\widetilde{X})} \sum_{l=0}^{\infty} \frac{1}{l!} \langle \widetilde{\gamma}_1 \bar{\psi}^{a_1}, \cdots, \widetilde{\gamma}_n \bar{\psi}^{a_n}, \widetilde{\boldsymbol{\tau}}_2^l \rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}_{\widetilde{\beta}}.$$

In particular,

$$\begin{aligned} \langle\!\langle \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2) \quad &:= \quad \sum_{\widetilde{\beta} \in E(\widetilde{X})} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle \widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n, \widetilde{\boldsymbol{\tau}}_2^l \rangle^{\mathcal{X}, \widetilde{T}'}_{\widetilde{\beta}} \\ &= \quad \sum_{\widetilde{\beta} \in E(\widetilde{X})} e^{\int_{\widetilde{\beta}} \widetilde{\boldsymbol{\tau}}'_2} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle \widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n, (\widetilde{\boldsymbol{\tau}}''_2)^l \rangle^{\mathcal{X}, \widetilde{T}'}_{\widetilde{\beta}} \in \mathcal{Q}^{\mathbb{C}}_{\widetilde{T}'}[\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!] \end{aligned}$$

where the second equality follows from the divisor equation.

Definition 5.2. For any $\lambda \in G_{\tau_0} \cong \mu_{\mathfrak{m}}$, we define

$$\langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}}, T_f}(\widetilde{\boldsymbol{\tau}}_2) := \sum_{\widetilde{\beta} \in E(\widetilde{X})} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle \widetilde{\boldsymbol{\tau}}_2^l, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}, T_f}_{\widetilde{\beta}} = \sum_{\widetilde{\beta} \in E(\widetilde{X})} e^{\int_{\widetilde{\beta}} \widetilde{\boldsymbol{\tau}}'_2} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle (\widetilde{\boldsymbol{\tau}}''_2)^l, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}, T_f}_{\widetilde{\beta}} \in \mathbb{C}[\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!].$$

By Observation 4.5 and Lemma 4.7, the closed invariant $\langle \widetilde{\boldsymbol{\tau}}_2^l, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}}, T_f}_{\widetilde{\beta}}$ vanishes for any $\widetilde{\beta} \in E(\widetilde{X})$ that is not of the form $\iota_*(\beta + d[B])$ for some $\beta \in E(X)$ and $d \in \mathbb{Z}_{>0}$, where $E(X)$ is the semigroup $\mathrm{NE}(X) \cap H_2(X;\mathbb{Z})$. Thus we have

$$\langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}},T_f}(\widetilde{\boldsymbol{\tau}}_2) = \sum_{\beta \in E(X)} \sum_{d \in \mathbb{Z}_{>0}} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle \widetilde{\boldsymbol{\tau}}_2^l, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\iota_*(\beta + d[B])}.$$

5.1.2 Generating functions of disk invariants of $(\mathcal{X}, \mathcal{L}, f)$

Now we give the definition of the generating function of disk invariants of $(\mathcal{X}, \mathcal{L}, f)$ following [42, Section 3.13]. Let

$$\boldsymbol{\tau}_2 = \sum_{a=1}^{R-3} \tau_a u_a$$

where $\tau_1, \dots, \tau_{R-3}$ are complex variables. We choose T'-equivariant lifts of $u_1, \dots, u_{R-3}$ and thus of $\boldsymbol{\tau}_2$ as in Convention 2.4. Let X be an additional variable. Set

$$\xi_0 := e^{-\frac{\pi\sqrt{-1}}{\mathfrak{m}}} \tag{5.1}$$

and for $\lambda \in G_{\tau_0}$, let $\bar{\lambda}$ be defined as in (3.4).

Definition 5.3. For any $\lambda \in G_{\tau_0}$, define

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) := \sum_{\beta \in E(X)} \sum_{d \in \mathbb{Z}_{>0}} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{\langle \boldsymbol{\tau}_2^l \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta + d[B],(d,\lambda)}}{l!} \mathsf{X}^d$$

which takes values in $\mathbb{C}$. Define

$$F^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) := \sum_{\lambda \in G_{\tau_0}} F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}$$

which takes values in $H^*_{\mathrm{CR}}(\mathcal{B}G_{\tau_0};\mathbb{C}) \cong \mathbb{C}^{\mathfrak{m}}$.

5.2 Correspondence of generating functions

We now identify the generating function of genus-zero closed Gromov-Witten invariants of $\widetilde{\mathcal{X}}$ with that of the disk invariants of $(\mathcal{X}, \mathcal{L}, f)$.

Theorem 5.4. *For any $\lambda \in G_{\tau_0}$, the correspondence*

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) = \langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}},T_f}(\widetilde{\boldsymbol{\tau}}_2)$$

holds under the relation $\widetilde{\tau}_a = \tau_a$ for $a = 1, \dots, R-3$ and $\widetilde{\tau}_{R-2} = \log \mathsf{X}$.

Proof. Fix an effective class $\beta \in E(X)$ and $(d, \lambda) \in H_1(\mathcal{L}; \mathbb{Z})$. We denote $\hat{\boldsymbol{\tau}}_2 := \sum_{a=1}^{R-3} \widetilde{\tau}_a \widetilde{u}_a = \widetilde{\boldsymbol{\tau}}_2 - \widetilde{\tau}_{R-2}\widetilde{u}_{R-2}$, which is the lift of $\boldsymbol{\tau}_2$ chosen as in Convention 2.4 under the identification $\widetilde{\tau}_a = \tau_a$ for $a = 1, \dots, R-3$. We have

$$\begin{aligned}
\sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle \widetilde{\boldsymbol{\tau}}_2^l, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\iota_*(\beta + d[B])} &= \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{l!} \langle (\hat{\boldsymbol{\tau}}_2 + \widetilde{\tau}_{R-2}\widetilde{u}_{R-2})^l, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\iota_*(\beta + d[B])} \\
&= \sum_{l \in \mathbb{Z}_{\geq 0}} \sum_{k=0}^{l} \frac{(d\widetilde{\tau}_{R-2})^k \langle (\hat{\boldsymbol{\tau}}_2)^{l-k}, \widetilde{\gamma}_\lambda \rangle^{\widetilde{\mathcal{X}},T_f}_{\iota_*(\beta + d[B])}}{k!(l-k)!} \\
&= \sum_{l \in \mathbb{Z}_{\geq 0}} \sum_{k=0}^{l} \frac{(d\widetilde{\tau}_{R-2})^k \langle \boldsymbol{\tau}_2^{l-k} \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta + d[B],(d,\lambda)}}{k!(l-k)!} \\
&= \sum_{k,m \in \mathbb{Z}_{\geq 0}} \frac{(d\widetilde{\tau}_{R-2})^k}{k!} \cdot \frac{\langle \boldsymbol{\tau}_2^m \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta + d[B],(d,\lambda)}}{m!} \\
&= \sum_{m \in \mathbb{Z}_{\geq 0}} \frac{1}{m!} \langle \boldsymbol{\tau}_2^m \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta + d[B],(d,\lambda)} \exp(d\widetilde{\tau}_{R-2})
\end{aligned}$$

where the second equality follows from the divisor equation applied with $\langle \widetilde{D}_{R+1}, \iota_*(\beta + d[B]) \rangle = d$, and the third equality follows from the numerical correspondence (Theorem 4.1). The final line is then identified with

$$\sum_{m \in \mathbb{Z}_{\geq 0}} \frac{1}{m!} \langle \boldsymbol{\tau}_2^m \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta + d[B],(d,\lambda)} \mathsf{X}^d$$

under $\widetilde{\tau}_{R-2} = \log \mathsf{X}$, or equivalently $\widetilde{Q}_{R-2} = \mathsf{X}$. □

5.3 Equivariant small quantum cohomology of $\widetilde{\mathcal{X}}$

As an application of Theorem 5.4, we show in Section 5.4 that $F_\lambda^{\mathcal{X},(\mathcal{L},f)}$ can be recovered from the $\widetilde{T}'$-equivariant small J-function of $\widetilde{\mathcal{X}}$. In preparation, we review the $\widetilde{T}'$-equivariant small quantum cohomology of $\widetilde{\mathcal{X}}$ in this section. We refer to [32, Section 2] for the equivariant quantum cohomology of a general smooth Deligne-Mumford stack that has a semi-projective coarse moduli space and admits a torus action, as well as [51, 85, 59] for additional details.

5.3.1 Quantum product

Let $(-,-)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}$ denote the $\widetilde{T}'$-equivariant orbifold Poincaré pairing of $\widetilde{\mathcal{X}}$, defined as

$$(\widetilde{u},\widetilde{v})^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} := \int_{\mathcal{I}\widetilde{\mathcal{X}}} \widetilde{u} \cup \mathrm{inv}^*(\widetilde{v}), \quad \widetilde{u},\widetilde{v} \in H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q}).$$

Here, the integral is defined by $\widetilde{T}'$-equivariant localization [9] on $\mathcal{I}\widetilde{\mathcal{X}}$ and takes value in $\mathcal{Q}_{\widetilde{T}'}$.

The $\widetilde{T}'$-equivariant small *quantum product* of $\widetilde{\mathcal{X}}$ at $\widetilde{\boldsymbol{\tau}}_2$ is an associative, commutative product $\star_{\widetilde{\boldsymbol{\tau}}_2}$ on

$$H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{C}) \otimes_{\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}} \mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\![\widetilde{Q},\widetilde{\boldsymbol{\tau}}''_2]\!]$$

defined by

$$a \star_{\widetilde{\boldsymbol{\tau}}_2} b := \sum_{\widetilde{\beta}\in E(\widetilde{X})} \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{1}{n!}\mathrm{inv}^*\mathrm{ev}_{3,*}\left(\mathrm{ev}_1^*(a)\mathrm{ev}_2^*(b)\prod_{i=4}^{n+3}\mathrm{ev}_i^*(\widetilde{\boldsymbol{\tau}}_2) \cap [\overline{\mathcal{M}}_{0,n+3}(\widetilde{\mathcal{X}},\widetilde{\beta})]^{\mathrm{vir}}\right).$$

Note that the semi-projectivity of $\widetilde{X}$ ensures that $\mathrm{ev}_3 : \overline{\mathcal{M}}_{0,n+3}(\widetilde{\mathcal{X}},\widetilde{\beta}) \to \mathcal{I}\widetilde{\mathcal{X}}$ is proper. Equivalently, $\star_{\widetilde{\boldsymbol{\tau}}_2}$ is defined by

$$(a \star_{\widetilde{\boldsymbol{\tau}}_2} b, c)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} = \langle\!\langle a,b,c\rangle\!\rangle^{\widetilde{\mathcal{X}},\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2)$$

where the right hand side is defined in Definition 5.1.

5.3.2 Quantum differential equations

For the remainder of this chapter, let A denote $\dim_{\mathbb{C}} H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{C})$. Then

$$H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{C}) \otimes_{\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}} \mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\![\widetilde{Q},\widetilde{\boldsymbol{\tau}}''_2]\!]$$

is a free module of rank A over the ring $\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!]$, so it defines a vector bundle E of rank A over the formal scheme $\hat{H} := \mathrm{Spec}\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!]$. The $\widetilde{T}'$-equivariant small *quantum connection* is a family of flat connections $\{\nabla^z : z \in \mathbb{C}^*\}$ on E defined by

$$\nabla^z_{\frac{\partial}{\partial \widetilde{\tau}_a}} s = \frac{\partial s}{\partial \widetilde{\tau}_a} - \frac{1}{z} \widetilde{u}_a \star_{\widetilde{\boldsymbol{\tau}}_2} s, \quad a = 1, \dots, R-2.$$

Flat sections of ∇^z are solutions of $\widetilde{T}'$-equivariant small *quantum differential equations*

$$\frac{\partial s}{\partial \widetilde{\tau}_a} = \frac{1}{z} \widetilde{u}_a \star_{\widetilde{\boldsymbol{\tau}}_2} s, \quad a = 1, \dots, R-2, \tag{5.2}$$

which can be rewritten as

$$\begin{aligned} \widetilde{Q}_a \frac{\partial s}{\partial \widetilde{Q}_a} &= \frac{1}{z} \widetilde{u}_a \star_{\widetilde{\boldsymbol{\tau}}_2} s, \quad a = 1, \dots, R'-3, R-2; \\ \frac{\partial s}{\partial \widetilde{\tau}_a} &= \frac{1}{z} \widetilde{u}_a \star_{\widetilde{\boldsymbol{\tau}}_2} s, \quad a = R'-2, \dots, R-3. \end{aligned} \tag{5.3}$$

5.3.3 The fundamental solution

The $\widetilde{T}'$-equivariant small quantum differential equations (5.3) are defined over the ring $\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!]$, and have regular singularities along $\widetilde{Q}_a = 0$. We now describe a fundamental solution to (5.3) which is defined over the larger ring

$$\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\widetilde{\boldsymbol{\tau}}'_2][\![z^{-1}]\!][\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!] := \mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\widetilde{\tau}_1, \dots, \widetilde{\tau}_{R'-3}, \widetilde{\tau}_{R-2}][\![z^{-1}]\!][\![\widetilde{Q}_1, \dots, \widetilde{Q}_{R'-3}, \widetilde{Q}_{R-2}, \widetilde{\tau}_{R'-2}, \dots, \widetilde{\tau}_{R-3}]\!]$$

(see [32, Section 3.2]). The S*-operator*

$$S(\widetilde{\boldsymbol{\tau}}_2, z) \in \mathrm{End}(H^*_{\mathrm{CR}, \widetilde{T}'}(\mathcal{X}; \mathbb{C})) \otimes_{\mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}} \mathcal{R}^{\mathbb{C}}_{\widetilde{T}'}[\widetilde{\boldsymbol{\tau}}'_2][\![z^{-1}]\!][\![\widetilde{Q}, \widetilde{\boldsymbol{\tau}}''_2]\!]$$

is defined as follows. For any $\widetilde{u}, \widetilde{v} \in H^*_{\mathrm{CR},\widetilde{T}'}(\mathcal{X};\mathbb{C})$,

$$(\widetilde{u}, S(\widetilde{\boldsymbol{\tau}}_2, z)\,\widetilde{v})^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} = (\widetilde{u}, \widetilde{v})^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} + \langle\!\langle \widetilde{u}, \frac{\widetilde{v}}{z - \bar{\psi}} \rangle\!\rangle^{\widetilde{\mathcal{X}},\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2)$$

where

$$\langle\!\langle \widetilde{u}, \frac{\widetilde{v}}{z - \bar{\psi}} \rangle\!\rangle^{\widetilde{\mathcal{X}},\widetilde{T}'} = \sum_{k=0}^{\infty} z^{-k-1} \langle\!\langle \widetilde{u}, \widetilde{v}\bar{\psi}^k \rangle\!\rangle^{\widetilde{\mathcal{X}},\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2)$$

More explicitly, we complete $\{\widetilde{u}_1, \ldots, \widetilde{u}_{R-2}\}$ into a homogeneous basis $\{\widetilde{u}_1, \ldots, \widetilde{u}_A\}$ of $H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$ over $\mathcal{Q}_{\widetilde{T}'}$, and let $\{\widetilde{u}^1, \ldots, \widetilde{u}^A\}$ be the basis dual to $\{\widetilde{u}_1, \ldots, \widetilde{u}_A\}$ under the $\widetilde{T}'$-equivariant Poincaré pairing $(-,-)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}$. Then

$$S(\widetilde{\boldsymbol{\tau}}_2, z)\widetilde{v} = \widetilde{v} + \sum_{a=1}^{A} \langle\!\langle \widetilde{u}_a, \frac{\widetilde{v}}{z - \bar{\psi}} \rangle\!\rangle^{\widetilde{\mathcal{X}},\widetilde{T}'}_{\widetilde{\beta}}(\widetilde{\boldsymbol{\tau}}_2)\widetilde{u}^a.$$

As observed by [32, Proposition 2.4], the following is a straightforward equivariant generalization of results in [51, 85, 59]:

Proposition 5.5 ([51, 85, 59, 32]). *For any $\widetilde{v} \in H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{C})$, $S(\widetilde{\boldsymbol{\tau}}_2, z)\widetilde{v}$ is a solution to the $\widetilde{T}'$-equivariant quantum differential equation* (5.2)*:*

$$\frac{\partial}{\partial \widetilde{\tau}_a} \mathcal{S}(\widetilde{\boldsymbol{\tau}}_2, z)\widetilde{v} = \frac{1}{z}\widetilde{u}_a \star_{\widetilde{\boldsymbol{\tau}}_2} S(\widetilde{\boldsymbol{\tau}}_2, z)\widetilde{v}, \qquad a = 1, \ldots, R-2.$$

Note that in the large radius limit $\widetilde{Q} \to 0$,

$$\lim_{\widetilde{Q} \to 0} S(\widetilde{\boldsymbol{\tau}}_2, z)\widetilde{v} = e^{\widetilde{\boldsymbol{\tau}}_2/z}\widetilde{v},$$

which is a solution to the following classical limit of the quantum differential equations.

$$\frac{\partial s}{\partial \widetilde{\tau}_a} = \frac{1}{z}\widetilde{u}_a s, \quad a = 1, \ldots, R-2.$$

5.4 Equivariant small J-function of $\widetilde{\mathcal{X}}$ and disk potential of $(\mathcal{X}, \mathcal{L}, f)$

The $\widetilde{T}'$-equivariant small J-function [92, 31, 50] of $\widetilde{\mathcal{X}}$

$$J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\boldsymbol{\tau}_2, z)$$

is characterized by

$$(J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\boldsymbol{\tau}_2, z), \widetilde{v})_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} = (\mathbf{1}, S(\boldsymbol{\tau}_2, z)\widetilde{v})_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}$$

for any $\widetilde{v} \in H^*_{\widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{C})$; in other words, it is defined by

$$J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z) := \mathbf{1} + \sum_{a=1}^{A} \langle\!\langle \mathbf{1}, \frac{\widetilde{u}_a}{z - \bar{\psi}} \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2) \widetilde{u}^a,$$

which takes value in $H^*_{\mathrm{CR}, \widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{C})$.

Notation 5.6. Let $\mathbb{S}$ be a commutative ring, and let

$$h(z) = \sum_{n=0}^{\infty} h_n z^{-n} \in \mathbb{S}[\![z^{-1}]\!]$$

where $h_n \in \mathbb{S}$. For any $n \in \mathbb{Z}_{\geq 0}$, define

$$[z^{-n}]h(z) = h_n.$$

The following lemma follows immediately from the definition and the string equation.

Lemma 5.7. *For any* $\widetilde{v} \in H^*_{\mathrm{CR}, \widetilde{T}'}(\widetilde{\mathcal{X}}; \mathbb{C})$,

$$(J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{v})_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} = (1, \widetilde{v})_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} + z^{-1}(\widetilde{\boldsymbol{\tau}}_2, \widetilde{v})_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} + \sum_{k=0}^{\infty} z^{-2-k} \langle\!\langle \widetilde{v} \bar{\psi}^k \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2).$$

In particular,

$$[z^{-2}](J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{v})_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} = \langle\!\langle \widetilde{v} \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2). \tag{5.4}$$

Under the correspondence of generating functions (Theorem 5.4), (5.4) immediately implies that the generating function $F^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X})$ of disk invariants of $(\mathcal{X}, \mathcal{L}, f)$ can be retrieved from the $\widetilde{T}'$-equivariant small J-function of $\widetilde{\mathcal{X}}$.

Theorem 5.8. *For any $\lambda \in G_{\tau_0}$,*

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) = [z^{-2}] \left(J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1=0}$$

under the relation $\widetilde{\tau}_a = \tau_a$ for $a = 1, \dots, R-3$ and $\widetilde{\tau}_{R-2} = \log \mathsf{X}$.

Chapter 6: B-model hypergeometric functions and mirror symmetry

In this chapter, we start to develop of the open/closed correspondence on the B-model side of mirror symmetry. We first establish the counterpart of Theorem 5.8 that the B-model disk function of $(\mathcal{X}, \mathcal{L}, f)$ [41, 42] can be recovered from the $\widetilde{T}'$-equivariant I-function of $\widetilde{\mathcal{X}}$ (Theorem 6.5). Furthermore, we show that the above results are compatible with the closed mirror symmetry for $\widetilde{\mathcal{X}}$ and the open mirror symmetry of $(\mathcal{X}, \mathcal{L}, f)$.

6.1 Extended Nef cones and Mori cones

Recall that we defined the extended nef cone of $\mathcal{X}$ in Section 2.3. We now define the extended Mori cone of $\mathcal{X}$. For each maximal cone $\sigma \in \Sigma(3)$, define

$$\mathbb{K}_\sigma^\vee := \sum_{i \in I_\sigma} \mathbb{Z} D_i,$$

which is a sublattice of $\mathbb{L}^\vee$ of finite index. Let

$$\mathbb{K}_\sigma := \{\beta \in \mathbb{L}_\mathbb{Q} : \langle D, \beta \rangle \in \mathbb{Z} \text{ for all } D \in \mathbb{K}_\sigma^\vee\}$$

be the dual lattice of $\mathbb{K}_\sigma^\vee$ viewed as an overlattice of $\mathbb{L}$ in $\mathbb{L}_\mathbb{Q}$, where $\langle -, - \rangle$ denotes the pairing between $\mathbb{L}_\mathbb{Q}^\vee$ and $\mathbb{L}_\mathbb{Q}$. The map

$$v : \mathbb{K}_\sigma \to N, \quad \beta \mapsto \sum_{i=1}^{R} \lceil \langle D_i, \beta \rangle \rceil b_i$$

induces a bijection $\mathbb{K}_\sigma/\mathbb{L} \to \mathrm{Box}(\sigma)$, which we also refer to as v by abusive notation. Let

$$\mathbb{K}(\mathcal{X}) \coloneqq \bigcup_{\sigma\in\Sigma(3)} \mathbb{K}_\sigma.$$

For each $\sigma \in \Sigma(3)$, define the *extended σ-Mori cone* as

$$\widetilde{\mathrm{NE}}(\sigma) \coloneqq \{\beta \in \mathbb{L}_{\mathbb{R}} : \langle D, \beta\rangle \geq 0 \text{ for all } D \in \widetilde{\mathrm{Nef}}(\sigma)\},$$

which is the dual cone of $\widetilde{\mathrm{Nef}}(\sigma)$. Let

$$\mathbb{K}_{\mathrm{eff},\sigma} \coloneqq \mathbb{K}_\sigma \cap \widetilde{\mathrm{NE}}(\sigma).$$

The *extended Mori cone* of $\mathcal{X}$ is defined to be

$$\widetilde{\mathrm{NE}}(\mathcal{X}) \coloneqq \bigcup_{\widetilde{\sigma}\in\widetilde{\Sigma}(4)} \widetilde{\mathrm{NE}}(\widetilde{\sigma}).$$

Moreover, define

$$\mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \coloneqq \mathbb{K} \cap \widetilde{\mathrm{NE}}(\mathcal{X}) = \bigcup_{\sigma\in\Sigma(3)} \mathbb{K}_{\mathrm{eff},\sigma}.$$

We now give an analog of the above definitions for $\widetilde{\mathcal{X}}$. Let $\widetilde{D}_i \in \widetilde{\mathbb{L}}^\vee$, $i = 1, \ldots, R+2$ be defined as in (2.13). For each maximal cone $\widetilde{\sigma} \in \widetilde{\Sigma}(4)$, define the *extended $\widetilde{\sigma}$-nef cone* as

$$\widetilde{\mathrm{Nef}}(\widetilde{\sigma}) \coloneqq \sum_{i\in I_{\widetilde{\sigma}}} \mathbb{R}_{\geq 0}\widetilde{D}_i.$$

The *extended nef cone* of $\widetilde{\mathcal{X}}$ is defined to be

$$\widetilde{\mathrm{Nef}}(\widetilde{\mathcal{X}}) \coloneqq \bigcap_{\widetilde{\sigma}\in\widetilde{\Sigma}(4)} \widetilde{\mathrm{Nef}}(\widetilde{\sigma}),$$

which is an $(R-2)$-dimensional simplicial cone in $\widetilde{\mathbb{L}}^\vee_{\mathbb{R}}$. Under the projection $\widetilde{\mathbb{L}}^\vee \to \mathbb{L}^\vee$, the image of $\widetilde{\mathrm{Nef}}(\widetilde{\mathcal{X}})$ is $\widetilde{\mathrm{Nef}}(\mathcal{X})$.

For each $\widetilde{\sigma} \in \widetilde{\Sigma}(4)$, define

$$\mathbb{K}_{\widetilde{\sigma}}^{\vee} := \sum_{i \in I_{\widetilde{\sigma}}} \mathbb{Z}\widetilde{D}_i,$$

which is a sublattice of $\widetilde{\mathbb{L}}^{\vee}$ of finite index. Let

$$\mathbb{K}_{\widetilde{\sigma}} := \{\widetilde{\beta} \in \widetilde{\mathbb{L}}_{\mathbb{Q}} : \langle \widetilde{D}, \widetilde{\beta} \rangle \in \mathbb{Z} \text{ for all } \widetilde{D} \in \mathbb{K}_{\widetilde{\sigma}}^{\vee}\}$$

be the dual lattice of $\mathbb{K}_{\widetilde{\sigma}}^{\vee}$ viewed as an overlattice of $\widetilde{\mathbb{L}}$ in $\widetilde{\mathbb{L}}_{\mathbb{Q}}$, where $\langle -, - \rangle$ denotes the pairing between $\widetilde{\mathbb{L}}_{\mathbb{Q}}^{\vee}$ and $\widetilde{\mathbb{L}}_{\mathbb{Q}}$. The map

$$\widetilde{v} : \mathbb{K}_{\widetilde{\sigma}} \to \widetilde{N}, \quad \widetilde{\beta} \mapsto \sum_{i=1}^{R+2} \lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil \widetilde{b}_i$$

induces a bijection $\mathbb{K}_{\widetilde{\sigma}}/\widetilde{\mathbb{L}} \to \mathrm{Box}(\widetilde{\sigma})$, which we also refer to as $\widetilde{v}$ by abusive notation. Let

$$\mathbb{K}(\widetilde{\mathcal{X}}) := \bigcup_{\widetilde{\sigma} \in \widetilde{\Sigma}(4)} \mathbb{K}_{\widetilde{\sigma}}.$$

For each $\widetilde{\sigma} \in \widetilde{\Sigma}(4)$, define the *extended $\widetilde{\sigma}$-Mori cone* as

$$\widetilde{\mathrm{NE}}(\widetilde{\sigma}) := \{\widetilde{\beta} \in \widetilde{\mathbb{L}}_{\mathbb{R}} : \langle \widetilde{D}, \widetilde{\beta} \rangle \geq 0 \text{ for all } \widetilde{D} \in \widetilde{\mathrm{Nef}}(\widetilde{\sigma})\},$$

which is the dual cone of $\widetilde{\mathrm{Nef}}(\widetilde{\sigma})$. Let

$$\mathbb{K}_{\mathrm{eff},\widetilde{\sigma}} := \mathbb{K}_{\widetilde{\sigma}} \cap \widetilde{\mathrm{NE}}(\widetilde{\sigma}).$$

The *extended Mori cone* of $\widetilde{\mathcal{X}}$ is defined to be

$$\widetilde{\mathrm{NE}}(\widetilde{\mathcal{X}}) := \bigcup_{\widetilde{\sigma} \in \widetilde{\Sigma}(4)} \widetilde{\mathrm{NE}}(\widetilde{\sigma}).$$

Moreover, define

$$\mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) := \mathbb{K}(\widetilde{\mathcal{X}}) \cap \widetilde{\mathrm{NE}}(\widetilde{\mathcal{X}}) = \bigcup_{\widetilde{\sigma} \in \widetilde{\Sigma}(4)} \mathbb{K}_{\mathrm{eff},\widetilde{\sigma}}.$$

We now make a few observations to be used later.

Observation 6.1. *Given $\widetilde{\beta} \in \widetilde{\mathbb{L}}_{\mathbb{Q}}$ that is contained in $\mathbb{L}_{\mathbb{Q}}$, we have*

$$\langle \widetilde{D}, \widetilde{\beta} \rangle = \langle D, \widetilde{\beta} \rangle$$

for any $\widetilde{D} \in \widetilde{\mathbb{L}}^{\vee}$ projecting to $D \in \mathbb{L}^{\vee}$. Moreover, the following hold:

- $\langle \widetilde{D}_i, \widetilde{\beta} \rangle = \langle D_i, \widetilde{\beta} \rangle$, $i = 1, \ldots, R$.
- $\langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle = \langle \widetilde{D}_{R+2}, \widetilde{\beta} \rangle = 0$.
- *If $\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}})$, then $\widetilde{v}(\widetilde{\beta}) \in N \subset \widetilde{N}$ and agrees with $v(\widetilde{\beta})$.*

Lemma 6.2. *We have*

$$\widetilde{D}_{R+1} \in \widetilde{\mathrm{Nef}}(\widetilde{\mathcal{X}}).$$

Proof. For any $\widetilde{\sigma} \in \iota(\Sigma(3))$, we have $R + 1 \in I_{\widetilde{\sigma}}$ and thus $\widetilde{D}_{R+1} \in \widetilde{\mathrm{Nef}}(\widetilde{\sigma})$. Now we consider cones in $\widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$. Note that the exactness of the first row of (2.12) implies that

$$\widetilde{D}_{R+1} = \sum_{i=1}^{R} m_i \widetilde{D}_i = \sum_{\substack{i \in \{1,\ldots,R\} \\ m_i > 0}} m_i \widetilde{D}_i. \tag{6.1}$$

Moreover, for any $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$, we have

$$I_{\widetilde{\sigma}} \supseteq \{ i \in \{1, \ldots, R\} : m_i > 0 \}.$$

This implies that $\widetilde{D}_{R+1} \in \widetilde{\mathrm{Nef}}(\widetilde{\sigma})$. $\square$

6.2 Hypergeometric functions

6.2.1 Choices of bases

We choose elements

$$H_1,\ldots,H_{R-3}\in\mathbb{L}^\vee\cap\widetilde{\mathrm{Nef}}(\mathcal{X}),\quad \widetilde{H}_1,\ldots,\widetilde{H}_{R-2}\in\widetilde{\mathbb{L}}^\vee\cap\widetilde{\mathrm{Nef}}(\widetilde{\mathcal{X}})$$

that satisfy the following conditions:

- $\{H_1,\ldots,H_{R-3}\}$ is a $\mathbb{Q}$-basis for $\mathbb{L}^\vee_{\mathbb{Q}}$. $\{\widetilde{H}_1,\ldots,\widetilde{H}_{R-2}\}$ is a $\mathbb{Q}$-basis for $\widetilde{\mathbb{L}}^\vee_{\mathbb{Q}}$.
- The images of $H_1,\ldots,H_{R'-3}$ under the *Kirwan map*

  $$\kappa:\mathbb{L}^\vee\cong H^2_G(\mathbb{C}^R;\mathbb{Z})\to H^2(\mathcal{X};\mathbb{Z})$$

  form a $\mathbb{Q}$-basis for $H^2(\mathcal{X};\mathbb{Q})$. The images of $\{\widetilde{H}_1,\ldots,\widetilde{H}_{R'-3},\widetilde{H}_{R-2}\}$ under

  $$\widetilde{\kappa}:\widetilde{\mathbb{L}}^\vee\cong H^2_{\widetilde{G}}(\mathbb{C}^{R+2};\mathbb{Z})\to H^2(\mathcal{X};\mathbb{Z})$$

  is a $\mathbb{Q}$-basis for $H^2(\widetilde{\mathcal{X}};\mathbb{Q})$.
- For each $a=1,\ldots,R'-3$, $\widetilde{\kappa}(\widetilde{H}_a)$ is the lift of $\kappa(H_a)$ chosen as in Convention 2.4. In particular, $\widetilde{H}_a$ projects to H_a under $\widetilde{\mathbb{L}}^\vee\to\widetilde{\mathbb{L}}$.
- For each $a=R'-2,\ldots,R-3$, $H_a=D_{3+a}$ and $\widetilde{H}_a=\widetilde{D}_{3+a}$.
- $\widetilde{H}_{R-2}=\widetilde{D}_{R+1}$ (see Lemma 6.2).

Recall from Section 5.2 that we used bases $\{u_1,\ldots,u_{R-3}\}$ for $H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$, $\{\widetilde{u}_1,\ldots,\widetilde{u}_{R-2}\}$ for $H^2_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$ when defining generating functions of Gromov-Witten invariants. We now fix the choices

$$u_a = \kappa(H_a), \qquad a = 1, \dots, R' - 3,$$

$$\widetilde{u}_a = \widetilde{\kappa}(\widetilde{H}_a), \qquad a = 1, \dots, R' - 3, R - 2,$$

$$u_a = \widetilde{u}_a = \mathbf{1}_{j(3+a)}, \qquad a = R' - 2, \dots, R - 3$$

where $j(3+a) \in \mathrm{Box}(\mathcal{X}) \subseteq \mathrm{Box}(\widetilde{\mathcal{X}})$ is represented by b_{3+a} (or $\widetilde{b}_{3+a}$). As before, for $a = 1, \dots, R-2$, we choose T'-equivariant lifts of u_a and $\widetilde{T}'$-equivariant lifts of $\widetilde{u}_a$ as in Convention 2.4.

Let $q = (q_1, \dots, q_{R-3})$, $\widetilde{q} = (\widetilde{q}_1, \dots, \widetilde{q}_{R-2})$ be formal variables. For each $\beta \in \mathbb{K}(\mathcal{X})$, $\widetilde{\beta} \in \mathbb{K}(\widetilde{\mathcal{X}})$, we set

$$q^\beta := q_1^{\langle H_1, \beta\rangle} \cdots q_{R-3}^{\langle H_{R-3}, \beta\rangle}, \quad \widetilde{q}^{\widetilde{\beta}} := \widetilde{q}_1^{\langle \widetilde{H}_1, \widetilde{\beta}\rangle} \cdots \widetilde{q}_{R-2}^{\langle \widetilde{H}_{R-2}, \widetilde{\beta}\rangle}.$$

6.2.2 B-model disk function of $(\mathcal{X}, \mathcal{L}, f)$

Recall from (3.3) that

$$w_0 = \frac{1}{\mathfrak{r}}, \quad w_2 = \frac{\mathfrak{s} + \mathfrak{t} f}{\mathfrak{r}\mathfrak{m}}, \quad w_3 = -\frac{\mathfrak{m} + \mathfrak{s} + \mathfrak{t} f}{\mathfrak{r}\mathfrak{m}}.$$

Following [42], we set

$$\mathbb{K}_{\mathrm{eff}}(\mathcal{X}, \mathcal{L}) := \{(\beta, d) \in \mathbb{K}_{\mathrm{eff}, \sigma_0} \times \mathbb{Z} : \langle D_1, \beta\rangle + d w_0 \in \mathbb{Z}_{\geq 0}, d \neq 0\}.$$

Let x be a formal variable. The *B-model disk function* of $(\mathcal{X}, \mathcal{L}, f)$ [41, 42] is defined as follows.

Definition 6.3. Define

$$W^{\mathcal{X}, \mathcal{L}, f}(q, x) := \sum_{\lambda \in G_{\tau_0}} W_\lambda^{\mathcal{X}, \mathcal{L}, f}(q, x) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}},$$

where

$$W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x) = \sum_{\substack{(\beta,d)\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ v(\beta)=h(d,\lambda)}} q^\beta x^d \frac{(-1)^{\lfloor\langle D_3,\beta\rangle + dw_3\rfloor + d}}{\mathfrak{m}d(\langle D_1,\beta\rangle + dw_0)!\prod_{i=4}^R \langle D_i,\beta\rangle!}\cdot\frac{\prod_{m=1}^\infty(-\langle D_3,\beta\rangle - dw_3 - m)}{\prod_{m=0}^\infty(\langle D_2,\beta\rangle + dw_2 - m)}.$$

Here, ξ_0 is defined as in (5.1) and $\bar\lambda$ for $\lambda\in G_{\tau_0}$ is defined as in (3.4). Note that $W^{\mathcal{X},\mathcal{L},f}(q,x)$ takes value in $H^*_{\mathrm{CR}}(\mathcal{B}G_{\tau_0};\mathbb{C})$, while each $W_\lambda^{\mathcal{X},\mathcal{L},f}(q,x)$ takes values in $\mathbb{C}$.

Remark 6.4. We note that the sign convention of $W^{\mathcal{X},(\mathcal{L},f)}(q,x)$ above differs from that in [42], yet agrees with that in [41] in the smooth case. This is to be consistent with our sign convention of the disk invariants. See Remark 3.5.

6.2.3 Equivariant I-function of $\widetilde{\mathcal{X}}$

Let $\widetilde{q}_0$ be a formal variable. The $\widetilde{T}'$-equivariant *I-function* of $\widetilde{\mathcal{X}}$ is defined as

$$\begin{aligned}
I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}_0,\widetilde{q},z) &:= e^{\frac{1}{z}\left(\widetilde{q}_0 + \sum_{a\in\{1,\dots,R'-3,R-2\}}\widetilde{u}_a\log\widetilde{q}_a\right)} \\
&\cdot\sum_{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}})}\widetilde{q}^{\widetilde{\beta}}\prod_{i\in\{1,\dots,R',R+1,R+2\}}\frac{\prod_{m=\lceil\langle\widetilde{D}_i,\widetilde{\beta}\rangle\rceil}^\infty(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} + (\langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)z)}{\prod_{m=0}^\infty(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} + (\langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)z)}\cdot\prod_{i=R'+1}^R\frac{\prod_{m=\lceil\langle\widetilde{D}_i,\widetilde{\beta}\rangle\rceil}^\infty(\langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)z}{\prod_{m=0}^\infty(\langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)z}\mathbf{1}_{\widetilde{v}(\widetilde{\beta})} \\
&= e^{\frac{1}{z}\left(\widetilde{q}_0 + \sum_{a\in\{1,\dots,R'-3,R-2\}}\widetilde{u}_a\log\widetilde{q}_a\right)} \\
&\cdot\sum_{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}})}\widetilde{q}^{\widetilde{\beta}}\prod_{i\in\{1,\dots,R',R+1,R+2\}}\frac{\prod_{m=\lceil\langle\widetilde{D}_i,\widetilde{\beta}\rangle\rceil}^\infty(\frac{\widetilde{\mathcal{D}}_i^{\widetilde{T}'}}{z} + \langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)}{\prod_{m=0}^\infty(\frac{\widetilde{\mathcal{D}}_i^{\widetilde{T}'}}{z} + \langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)}\cdot\prod_{i=R'+1}^R\frac{\prod_{m=\lceil\langle\widetilde{D}_i,\widetilde{\beta}\rangle\rceil}^\infty(\langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)}{\prod_{m=0}^\infty(\langle\widetilde{D}_i,\widetilde{\beta}\rangle - m)}\frac{\mathbf{1}_{\widetilde{v}(\widetilde{\beta})}}{z^{\mathrm{age}(\widetilde{v}(\widetilde{\beta}))}},
\end{aligned}$$

which takes value in $H^*_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$. Here the second equality follows from that $\widetilde{\mathcal{X}}$ is Calabi-Yau, and thus $\sum_{i=1}^{R+2}\widetilde{D}_i = 0$. We set

$$I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q},z) := I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(0,\widetilde{q},z). \tag{6.2}$$

6.3 Correspondence of hypergeometric functions

6.3.1 The statement

We now establish the following correspondence that retrieves the disk function $W^{\mathcal{X},(\mathcal{L},f)}(q,x)$ from the $\widetilde{T}'$-equivariant I-function of $\widetilde{\mathcal{X}}$.

Theorem 6.5. *For each $\lambda \in G_{\tau_0}$,*

$$W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x) = [z^{-2}]\left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q},z), \widetilde{\gamma}_\lambda\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}\Bigg|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0}$$

under the relation $\widetilde{q}_a = q_a$ for $a = 1,\dots,R-3$ and $\widetilde{q}_{R-2} = x$.

Note that Theorem 6.5 is the B-model analog of Theorem 5.8. See Section 6.4 below for additional discussions in the context of mirror symmetry.

Example 6.6. Let $\mathcal{X} = \mathbb{C}^3$, $\mathcal{L}$ be an outer brane, and $f = 1$, as in Section 2.6.1. In this case,

$$\widetilde{\mathcal{X}} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)).$$

The B-model disk function of $(\mathcal{X},\mathcal{L},1)$ is

$$W_{\lambda=1}^{\mathcal{X},(\mathcal{L},1)}(x) = \sum_{d\in\mathbb{Z}_{>0}} x^d \frac{(-1)^d(2d-1)!}{d\cdot(d!)^2}.$$

For $\widetilde{\mathcal{X}}$, we have

$$\widetilde{H}_1 = \widetilde{D}_1 = \widetilde{D}_2 = \widetilde{D}_4 = -\widetilde{D}_5, \qquad \widetilde{D}_3 = -2\widetilde{H}_1,$$

$$\widetilde{u}_1 = \widetilde{\mathcal{D}}_1 = \widetilde{\mathcal{D}}_2 = \widetilde{\mathcal{D}}_4 = -\widetilde{\mathcal{D}}_5, \qquad \widetilde{\mathcal{D}}_3 = -2\widetilde{u}_1,$$

and the $\widetilde{T}'$-equivariant lift we choose for $\widetilde{u}_1$ is $\widetilde{\mathcal{D}}_1^{\widetilde{T}'}$. The $\widetilde{T}'$-equivariant I-function is

$$
\begin{aligned}
I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}_1, z) &= e^{\frac{\widetilde{\mathcal{D}}_1^{\widetilde{T}'} \log \widetilde{q}_1}{z}} \sum_{d \in \mathbb{Z}_{\geq 0}} \widetilde{q}_1^d \frac{\prod_{a=0}^{2d-1} (\frac{\widetilde{\mathcal{D}}_3^{\widetilde{T}'}}{z} - a) \cdot \prod_{a=0}^{d-1} (\frac{\widetilde{\mathcal{D}}_5^{\widetilde{T}'}}{z} - a)}{\prod_{a=1}^{d} (\frac{\widetilde{\mathcal{D}}_1^{\widetilde{T}'}}{z} + a)(\frac{\widetilde{\mathcal{D}}_2^{\widetilde{T}'}}{z} + a)(\frac{\widetilde{\mathcal{D}}_5^{\widetilde{T}'}}{z} + a)} \\
&= e^{\frac{\widetilde{\mathcal{D}}_1^{\widetilde{T}'} \log \widetilde{q}_1}{z}} \left(1 + \sum_{d \in \mathbb{Z}_{>0}} \widetilde{q}_1^d \frac{\widetilde{\mathcal{D}}_3^{\widetilde{T}'} \widetilde{\mathcal{D}}_5^{\widetilde{T}'}}{z^2} \frac{(-1)^d (2d-1)!}{d \cdot (d!)^2} \right).
\end{aligned}
$$

To see that $W_1^{\mathcal{X},(\mathcal{L},f)}(x) = [z^{-2}] \left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}_1, z), \widetilde{\gamma}_1 \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Big|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0}$ under $x = \widetilde{q}_1$, it suffices to note that

$$
\left(\widetilde{\mathcal{D}}_3^{\widetilde{T}'} \widetilde{\mathcal{D}}_5^{\widetilde{T}'}, \widetilde{\gamma}_1 \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} = \left(\widetilde{\mathcal{D}}_3^{\widetilde{T}'} \widetilde{\mathcal{D}}_5^{\widetilde{T}'}, \frac{\widetilde{\mathcal{D}}_2^{\widetilde{T}'} \widetilde{\mathcal{D}}_3^{\widetilde{T}'} \widetilde{\mathcal{D}}_4^{\widetilde{T}'}}{f\mathsf{u}_1 - \mathsf{u}_2 - \mathsf{u}_4} \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} = 1,
$$

and that $\widetilde{\gamma}_1$, which is supported on $\widetilde{\sigma}_0$, pairs to zero with any class involving $\widetilde{\mathcal{D}}_1^{\widetilde{T}'}$.

We will prove Theorem 6.5 in two steps. First, in Lemma 6.7, we identify $W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q, x)$ with the pairing $\widetilde{\gamma}_\lambda$ with the terms in $[z^{-2}] I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z)$ corresponding to classes in

$$
\mathbb{K}_{\text{eff}, \widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}}.
$$

Second, in Lemmas 6.8 and 6.9, we show that terms corresponding to classes outside $\mathbb{K}_{\text{eff}, \widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}}$ do not contribute after pairing with $\widetilde{\gamma}_\lambda$ and taking weight restrictions.

6.3.2 The proof

Recall that $\iota_{\widetilde{\sigma}}^*(\widetilde{\gamma}) = 0$ for any $\widetilde{\sigma} \in \widetilde{\Sigma}(4)$, $\widetilde{\sigma} \neq \widetilde{\sigma}_0$. For each $\widetilde{\beta}$, we set

$$
I_{\widetilde{\beta}}(\widetilde{q}, z) := \widetilde{q}^{\widetilde{\beta}} \prod_{i \in \{1, \ldots, R', R+1, R+2\}} \frac{\prod_{m = \lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil}^{\infty} (\frac{\widetilde{\mathcal{D}}_i^{\widetilde{T}'}}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m)}{\prod_{m=0}^{\infty} (\frac{\widetilde{\mathcal{D}}_i^{\widetilde{T}'}}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m)} \cdot \prod_{i=R'+1}^{R} \frac{\prod_{m = \lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil}^{\infty} (\langle \widetilde{D}_i, \widetilde{\beta} \rangle - m)}{\prod_{m=0}^{\infty} (\langle \widetilde{D}_i, \widetilde{\beta} \rangle - m)} \frac{\mathbf{1}_{\widetilde{v}(\widetilde{\beta})}}{z^{\operatorname{age}(\widetilde{v}(\widetilde{\beta}))}}. \tag{6.3}
$$

That is,

$$
I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z) := e^{\frac{1}{z} (\sum_{a \in \{1, \ldots, R'-3, R-2\}} \widetilde{u}_a \log \widetilde{q}_a)} \cdot \sum_{\widetilde{\beta} \in \mathbb{K}_{\text{eff}}(\widetilde{\mathcal{X}})} I_{\widetilde{\beta}}(\widetilde{q}, z).
$$

Lemma 6.7. *For each $\lambda \in G_{\tau_0}$,*

$$W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x) = [z^{-2}] \left(\sum_{\widetilde{\beta} \in \mathbb{K}_{\text{eff},\widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}}} I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Bigg|_{\mathsf{u}_4=0}$$

under the relation $\widetilde{q}_a = q_a$ for $a = 1, \dots, R-3$ and $\widetilde{q}_{R-2} = x$.

See Notation 5.6 for the notation $[z^{-2}]$.

Proof. We first set up a one-to-one correspondence between $\mathbb{K}_{\text{eff},\widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}}$ and $\mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L})$. Let

$$l^{(0)} := (w_0, w_2, w_3, 0, \dots, 0, 1, -1) \in \mathbb{Q}^{R+2}, \tag{6.4}$$

which satisfies $\widetilde{\alpha}(l^{(0)}) = 0$ (see (2.8)) and does not belong to the span of $\{\widetilde{l}^{(1)}, \dots, \widetilde{l}^{(R-3)}\}$. We write

$$\widetilde{l}^{(R-2)} = \widetilde{l}_{R+1}^{(R-2)} l^{(0)} + c_1 \widetilde{l}^{(1)} + \cdots + c_{R-3} \widetilde{l}^{(R-3)},$$

where $\widetilde{l}_{R+1}^{(R-2)}$ is the $(R+1)$-th component of $\widetilde{l}^{(R-2)}$ and $c_1, \dots, c_{R-3} \in \mathbb{Q}$. Then, we define a map

$$\widetilde{\mathbb{L}}_{\mathbb{Q}} \to \mathbb{L}_{\mathbb{Q}} \times \mathbb{Q}, \tag{6.5}$$

$$\widetilde{\beta} = (\beta_1, \dots, \beta_{R-2}) \mapsto (\beta, d) = ((\beta_1 + \beta_{R-2} c_1, \dots, \beta_{R-3} + \beta_{R-2} c_{R-3}), \beta_{R-2} \widetilde{l}_{R+1}^{(R-2)}).$$

Note that

$$d = \langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle. \tag{6.6}$$

If $\widetilde{\beta} \in \mathbb{K}_{\text{eff},\widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}}$, we have $d \neq 0$, and $\langle \widetilde{D}_i, \widetilde{\beta} \rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in I_{\widetilde{\sigma}_0} = \{1, 4, \dots, R\}$. Then (6.1) gives

$$d = \langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle = \langle m_1 \widetilde{D}_1 + \sum_{i=4}^{R} m_i \widetilde{D}_i, \widetilde{\beta} \rangle \in \mathbb{Z}_{\geq 0}.$$

This verifies that the image of any class in $\mathbb{K}_{\text{eff},\widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}}$ satisfies $d \in \mathbb{Z}_{\neq 0}$. Now denote $l^{(0)} =$

$(l_1^{(0)}, \ldots, l_{R+2}^{(0)})$. It is straightforward to check that for any $i = 1, \ldots, R$,

$$\langle \widetilde{D}_i, \widetilde{\beta} \rangle = \langle D_i, \beta \rangle + d l_i^{(0)}. \tag{6.7}$$

Therefore, (6.5) induces a bijection $\mathbb{K}_{\mathrm{eff}, \widetilde{\sigma}_0} \setminus \mathbb{L}_{\mathbb{Q}} \to \mathbb{K}_{\mathrm{eff}}(\mathcal{X}, \mathcal{L})$.

From now on, we fix corresponding classes $\widetilde{\beta}$ and (β, d) under the above bijection. Moreover, let $\lambda \in G_{\tau_0}$ be the element that corresponds to $-\widetilde{v}(\widetilde{\beta})$. From the definitions, we can verify that

$$v(\beta) = h(d, \lambda).$$

Note that for each $a = 1, \ldots, R-3$, H_a can be written as a linear combination of $D_4, \ldots, D_R$ and $\widetilde{H}_a$ can be written as a linear combination of $\widetilde{D}_4, \ldots, \widetilde{D}_R$ with the same coefficients (Convention 2.4). Then (6.7) and $l_4^{(0)} = \cdots = l_R^{(0)} = 0$ imply that

$$\langle \widetilde{H}_a, \widetilde{\beta} \rangle = \langle H_a, \beta \rangle.$$

This combined with (6.6) gives that

$$\widetilde{q}^{\widetilde{\beta}} = q^{\beta} x^d$$

under $\widetilde{q}_a = q_a$ for $a = 1, \ldots, R-3$ and $\widetilde{q}_{R-2} = x$.

Now, we compute that

$$\begin{aligned}
\iota_{\widetilde{\sigma}_0}^*(I_{\widetilde{\beta}}(\widetilde{q}, z)) &= \frac{\widetilde{q}^{\widetilde{\beta}}}{\prod_{i \in I_{\widetilde{\sigma}_0}} \langle \widetilde{D}_i, \widetilde{\beta} \rangle!} \cdot \prod_{i \in \{2,3,R+1,R+2\}} \frac{\prod_{m=\lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil}^{\infty} \left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_i^{\widetilde{T}'})}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m \right)}{\prod_{m=0}^{\infty} \left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_i^{\widetilde{T}'})}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m \right)} \frac{\mathbf{1}_{\widetilde{v}(\widetilde{\beta})}}{z^{\mathrm{age}(\widetilde{v}(\widetilde{\beta}))}} \\
&= \frac{\widetilde{q}^{\widetilde{\beta}}}{(\langle D_1, \beta \rangle + d w_0)! \prod_{i=4}^{R} \langle D_i, \beta \rangle!} \cdot \prod_{i \in \{2,3,R+1,R+2\}} \frac{\prod_{m=\lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil}^{\infty} \left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_i^{\widetilde{T}'})}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m \right)}{\prod_{m=0}^{\infty} \left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_i^{\widetilde{T}'})}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m \right)} \frac{\mathbf{1}_{\widetilde{v}(\widetilde{\beta})}}{z^{\mathrm{age}(\widetilde{v}(\widetilde{\beta}))}}
\end{aligned}$$

where the second equality follows from (6.7). Recall that

$$\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_2^{\widetilde{T}'}) = -\frac{f}{\mathfrak{m}}\mathsf{u}_1 + \frac{1}{\mathfrak{m}}\mathsf{u}_2, \quad \iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_3^{\widetilde{T}'}) = \frac{f}{\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2 - \mathsf{u}_4, \quad \iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}) = -\mathsf{u}_1, \quad \iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}) = \mathsf{u}_1 + \mathsf{u}_4.$$

For $i = R+1, R+2$, since $\langle \widetilde{D}_{R+1}, \widetilde{\beta}\rangle = -\langle \widetilde{D}_{R+2}, \widetilde{\beta}\rangle = d > 0$, we have

$$\begin{aligned}
&\frac{\prod_{m=\lceil \langle \widetilde{D}_{R+1},\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'})}{z} + \langle \widetilde{D}_{R+1}, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'})}{z} + \langle \widetilde{D}_{R+1}, \widetilde{\beta}\rangle - m\right)} \cdot \frac{\prod_{m=\lceil \langle \widetilde{D}_{R+2},\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'})}{z} + \langle \widetilde{D}_{R+2}, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'})}{z} + \langle \widetilde{D}_{R+2}, \widetilde{\beta}\rangle - m\right)}\Bigg|_{\mathsf{u}_4=0} \\
&= \frac{\frac{\mathsf{u}_1}{z}\prod_{m=1}^{d-1}\left(\frac{\mathsf{u}_1}{z} - m\right)}{\prod_{m=1}^{d}\left(\frac{-\mathsf{u}_1}{z} + m\right)} \\
&= (-1)^{d-1}\frac{\mathsf{u}_1}{z}\frac{1}{-\frac{\mathsf{u}_1}{z} + d}.
\end{aligned}$$

For $i = 2, 3$, if $\widetilde{v}(\widetilde{\beta}) = \vec{0}$, or equivalently $\lambda = 1$, we have

$$\begin{aligned}
&\frac{\prod_{m=\lceil \langle \widetilde{D}_2,\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_2^{\widetilde{T}'})}{z} + \langle \widetilde{D}_2, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_2^{\widetilde{T}'})}{z} + \langle \widetilde{D}_2, \widetilde{\beta}\rangle - m\right)} \cdot \frac{\prod_{m=\lceil \langle \widetilde{D}_3,\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_3^{\widetilde{T}'})}{z} + \langle \widetilde{D}_3, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_3^{\widetilde{T}'})}{z} + \langle \widetilde{D}_3, \widetilde{\beta}\rangle - m\right)}\Bigg|_{\mathsf{u}_4=0} \\
&= (-1)^{\langle \widetilde{D}_3,\widetilde{\beta}\rangle - 1}\frac{f\mathsf{u}_2 - \mathsf{u}_1}{\mathfrak{m}z} \cdot \frac{\prod_{m=1}^{\infty}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}z} - \langle \widetilde{D}_3, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}z} + \langle \widetilde{D}_2, \widetilde{\beta}\rangle - m\right)} \\
&= (-1)^{\langle D_3,\beta\rangle + dw_3 - 1}\frac{f\mathsf{u}_2 - \mathsf{u}_1}{\mathfrak{m}z} \cdot \frac{\prod_{m=1}^{\infty}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}z} - \langle D_3, \beta\rangle - dw_3 - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}z} + \langle D_2, \beta\rangle + dw_2 - m\right)},
\end{aligned}$$

where the last equality follows from (6.7). If otherwise $\widetilde{v}(\widetilde{\beta}) \neq \vec{0}$, or equivalently $\lambda \neq 1$, we have

$$\begin{aligned}
&\frac{\prod_{m=\lceil \langle \widetilde{D}_2,\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_2^{\widetilde{T}'})}{z} + \langle \widetilde{D}_2, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_2^{\widetilde{T}'})}{z} + \langle \widetilde{D}_2, \widetilde{\beta}\rangle - m\right)} \cdot \frac{\prod_{m=\lceil \langle \widetilde{D}_3,\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_3^{\widetilde{T}'})}{z} + \langle \widetilde{D}_3, \widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\iota_{\widetilde{\sigma}_0}^*(\widetilde{\mathcal{D}}_3^{\widetilde{T}'})}{z} + \langle \widetilde{D}_3, \widetilde{\beta}\rangle - m\right)}\Bigg|_{\mathsf{u}_4=0} \\
&= (-1)^{\lfloor \langle D_3,\beta\rangle + dw_3\rfloor - 1}\frac{\prod_{m=1}^{\infty}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}z} - \langle D_3, \beta\rangle - dw_3 - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}z} + \langle D_2, \beta\rangle + dw_2 - m\right)}.
\end{aligned}$$

Summarizing the above computations, we see that under $\widetilde{q}_a = q_a$ for $a = 1, \ldots, R-3$ and

$\widetilde{q}_{R-2} = x$, the z^{-2}-part of $\iota^*_{\widetilde{\sigma}_0}(I_{\widetilde{\beta}}(\widetilde{q}, z))\big|_{\mathsf{u}_4=0}$ is

$$q^\beta x^d \frac{\mathsf{u}_1(f\mathsf{u}_1 - \mathsf{u}_2)}{\mathfrak{m} z^{-2}} \cdot \frac{(-1)^{\langle D_3,\beta\rangle + dw_3 + d}}{d(\langle D_1, \beta\rangle + dw_0)! \prod_{i=4}^R \langle D_i, \beta\rangle!} \cdot \frac{\prod_{m=1}^\infty(-\langle D_3, \beta\rangle - dw_3 - m)}{\prod_{m=0}^\infty(\langle D_2, \beta\rangle + dw_2 - m)}$$

when $\widetilde{v}(\widetilde{\beta}) = \vec{0}$ or equivalently $\lambda = 1$, and

$$q^\beta x^d \frac{\mathsf{u}_1 \mathbf{1}_{\lambda^{-1}}}{z^{-2}} \cdot \frac{(-1)^{\lfloor\langle D_3,\beta\rangle + dw_3\rfloor + d}}{d(\langle D_1, \beta\rangle + dw_0)! \prod_{i=4}^R \langle D_i, \beta\rangle!} \cdot \frac{\prod_{m=1}^\infty(-\langle D_3, \beta\rangle - dw_3 - m)}{\prod_{m=0}^\infty(\langle D_2, \beta\rangle + dw_2 - m)}$$

when $\widetilde{v}(\widetilde{\beta}) \neq \vec{0}$ or equivalently $\lambda \neq 1$. Therefore,

$$[z^{-2}]\left(I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\gamma}_\lambda\right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}\Big|_{\mathsf{u}_4=0}$$

yields the term in $W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x)$ that corresponds to (β, d). The lemma thus follows as we sum over all classes $\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff},\widetilde{\sigma}_0} \smallsetminus \mathbb{L}_{\mathbb{Q}}$. $\square$

Lemma 6.8. *For any $\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \smallsetminus \mathbb{K}_{\mathrm{eff},\widetilde{\sigma}_0}$ and any $\lambda \in G_{\tau_0}$,*

$$\left(I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\gamma}_\lambda\right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} = 0. \tag{6.8}$$

Proof. By definition, there exists $\widetilde{\sigma} \in \widetilde{\Sigma}(4)$, $\widetilde{\sigma} \neq \widetilde{\sigma}_0$ such that $\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff},\widetilde{\sigma}}$. We first consider the case $\widetilde{\sigma} = \iota(\sigma_0)$. In this case, $\langle \widetilde{D}_i, \widetilde{\beta}\rangle \in \mathbb{Z}_{\geq 0}$ for any $i \in I_{\iota(\sigma_0)} = \{4, \dots, R, R+1\}$. By (6.1), we have

$$\langle \widetilde{D}_1, \widetilde{\beta}\rangle = \langle \widetilde{D}_{R+1} - \sum_{i=4}^R m_i \widetilde{D}_i, \widetilde{\beta}\rangle \in \mathbb{Z}.$$

We cannot have $\langle \widetilde{D}_1, \widetilde{\beta}\rangle \in \mathbb{Z}_{\geq 0}$ as that would imply $\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff},\widetilde{\sigma}_0}$. Thus $\langle \widetilde{D}_1, \widetilde{\beta}\rangle \in \mathbb{Z}_{<0}$. This implies that $I_{\widetilde{\beta}}(\widetilde{q}, z)$ contains $\widetilde{\mathcal{D}}_1^{\widetilde{T}'}$ as a factor. Then (6.8) holds since $\iota^*_{\widetilde{\sigma}_0}(\widetilde{\mathcal{D}}_1^{\widetilde{T}'}) = 0$.

Now we consider the case $\widetilde{\sigma} \neq \iota(\sigma_0)$. In this case, if $\widetilde{v}(\widetilde{\beta}) \neq \vec{0}$, then $\widetilde{v}(\widetilde{\beta})$ is a non-trivial element in $\mathrm{Box}(\widetilde{\sigma})$ and thus cannot represent any element of $\mathrm{Box}(\tau_0)$. However, $\widetilde{\gamma}_\lambda$ belongs to a sector corresponding to an element in $\mathrm{Box}(\tau_0)$. Thus (6.8) holds. Now suppose on the other hand that $\widetilde{v}(\widetilde{\beta}) = \vec{0}$, which means that $\widetilde{\beta} \in \widetilde{\mathbb{L}}$ and $\langle \widetilde{D}_i, \widetilde{\beta}\rangle \in \mathbb{Z}$ for all $i = 1, \dots, R+2$. Since $\widetilde{\beta} \notin \mathbb{K}_{\mathrm{eff},\widetilde{\sigma}_0}$, we have in particular that $\widetilde{\beta} \notin \widetilde{\mathrm{NE}}(\widetilde{\sigma}_0)$, i.e. there exists $i \in I_{\widetilde{\sigma}_0} = \{1, 4, \ldots, R\}$ such that $\langle \widetilde{D}_i, \widetilde{\beta} \rangle < 0$. This implies that $I_{\widetilde{\beta}}(\widetilde{q}, z)$ contains $\widetilde{\mathcal{D}}_i^{\widetilde{T}'}$ as a factor. Then (6.8) holds since $\iota^*_{\widetilde{\sigma}_0}(\widetilde{\mathcal{D}}_i^{\widetilde{T}'}) = 0$. ☐

Lemma 6.9. *For any $\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \cap \mathbb{L}_{\mathbb{Q}}$ and any $\lambda \in G_{\tau_0}$, we have*

$$[z^{-2}] \left(I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\gamma}_\lambda \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \bigg|_{\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} = 0. \tag{6.9}$$

Proof. Note from Observation 6.1 that $\langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle = \langle \widetilde{D}_{R+2}, \widetilde{\beta} \rangle = 0$, which implies that $[z^{-2}] I_{\widetilde{\beta}}(\widetilde{q}, z)$ is either 0 or has form

$$c \widetilde{q}^{\widetilde{\beta}} \cdot \begin{cases} \widetilde{\mathcal{D}}_i^{\widetilde{T}'} \widetilde{\mathcal{D}}_{i'}^{\widetilde{T}'} & \text{if } \widetilde{v}(\widetilde{\beta}) = \vec{0}, \\ \widetilde{\mathcal{D}}_i^{\widetilde{T}'} \mathbf{1}_{\widetilde{v}(\widetilde{\beta})} & \text{if } \widetilde{v}(\widetilde{\beta}) \neq \vec{0} \end{cases}$$

for some $i, i' \in \{1, \ldots, R'\}$ and $c \in \mathbb{Q}$, $c \neq 0$. The pullback of the above term to $\widetilde{\sigma}_0$ is zero unless $i, i' \in \{2, 3\}$ and $\widetilde{v}(\widetilde{\beta}) \in \mathrm{Box}(\tau_0)$. Moreover, (6.9) is zero unless $\lambda \in G_{\tau_0}$ corresponds to $-\widetilde{v}(\beta) \in \mathrm{Box}(\tau_0)$. In the remainder of this proof we assume $\widetilde{v}(\beta) \in \mathrm{Box}(\tau_0)$ and $\lambda \in G_{\tau_0}$ is the element corresponding to $-\widetilde{v}(\beta)$.

If $\widetilde{v}(\widetilde{\beta}) = \vec{0}$, then

$$\left(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} \widetilde{\mathcal{D}}_{i'}^{\widetilde{T}'}, \widetilde{\gamma}_\lambda \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \bigg|_{\mathsf{u}_4 = 0} = \left(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} \widetilde{\mathcal{D}}_{i'}^{\widetilde{T}'}, \frac{\widetilde{\mathcal{D}}_2^{\widetilde{T}'} \widetilde{\mathcal{D}}_3^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}}{\frac{f}{\mathfrak{m}} \mathsf{u}_1 - \frac{1}{\mathfrak{m}} \mathsf{u}_2 - \mathsf{u}_4} \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \bigg|_{\mathsf{u}_4 = 0} = \frac{\iota^*_{\widetilde{\sigma}_0}(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} \widetilde{\mathcal{D}}_{i'}^{\widetilde{T}'})}{\mathfrak{m} \left(\frac{f}{\mathfrak{m}} \mathsf{u}_1 - \frac{1}{\mathfrak{m}} \mathsf{u}_2 - \mathsf{u}_4 \right) \iota^*_{\widetilde{\sigma}_0}(\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'})} \bigg|_{\mathsf{u}_4 = 0} = \pm \frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}^2 \mathsf{u}_1},$$

which further restricts to 0 under $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. If on the other hand $\widetilde{v}(\widetilde{\beta}) \neq \vec{0}$ then

$$\left(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} \mathbf{1}_{\widetilde{v}(\widetilde{\beta})}, \widetilde{\gamma}_\lambda \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \bigg|_{\mathsf{u}_4 = 0} = \left(\widetilde{\mathcal{D}}_i^{\widetilde{T}'} \mathbf{1}_{\widetilde{v}(\widetilde{\beta})}, \widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'} \mathbf{1}_{\lambda^{-1}} \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \bigg|_{\mathsf{u}_4 = 0} = \frac{\iota^*_{\widetilde{\sigma}_0}(\widetilde{\mathcal{D}}_i^{\widetilde{T}'})}{\mathfrak{m} \iota^*_{\widetilde{\sigma}_0}(\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'})} \bigg|_{\mathsf{u}_4 = 0} = \pm \frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}^2 \mathsf{u}_1},$$

which again restricts to 0 under $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. Therefore (6.9) holds. ☐

Proof of Theorem 6.5. By our choice of $\widetilde{T}'$-equivariant lifts, we have $\iota^*_{\widetilde{\sigma}_0}(\widetilde{u}_a) = 0$ for all $a = 1, \ldots, R-2$, and thus

$$\iota^*_{\widetilde{\sigma}_0} \left(e^{\frac{1}{z} (\sum_{a \in \{1, \ldots, R'-3, R-2\}} \widetilde{u}_a \log \widetilde{q}_a)} \right) = 1.$$

Therefore,

$$\left(I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q},z),\widetilde{\gamma}_\lambda\right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} = \left(\sum_{\widetilde{\beta}\in\mathbb{K}_{\text{eff}}(\widetilde{\mathcal{X}})} I_{\widetilde{\beta}}(\widetilde{q},z),\widetilde{\gamma}_\lambda\right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}.$$

We can then conclude by Lemmas 6.7, 6.8, and 6.9. □

6.4 Toric mirror symmetry

In this section and the next, we situate our correspondences in the context of mirror symmetry. Recall the web of relations in Figure 1.1, where the vertical arrows in Figure 1.1 are our correspondences.

For the bottom arrow, the *mirror theorem* of [50, 27, 30] relates the $\widetilde{T}'$-equivariant J- and I-functions of $\widetilde{\mathcal{X}}$ in the following way.

Theorem 6.10 ([50, 27, 30])**.** *We have*

$$e^{\frac{1}{z}\widetilde{\tau}_0(\widetilde{q}_0,\widetilde{q})} J^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{\boldsymbol{\tau}}_2(\widetilde{q}),z) = I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q}_0,\widetilde{q},z),$$

where the $\widetilde{T}'$-equivariant closed mirror map $\widetilde{\tau}_0 = \widetilde{\tau}_0(\widetilde{q}_0,\widetilde{q})$*,* $\widetilde{\boldsymbol{\tau}}_2 = \widetilde{\boldsymbol{\tau}}_2(\widetilde{q})$ *is determined by the first-order term in the expansion of the I-function in powers of z^{-1}:*

$$I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q}_0,\widetilde{q},z) = 1 + z^{-1}\left(\widetilde{\tau}_0(\widetilde{q}_0,\widetilde{q}) + \widetilde{\boldsymbol{\tau}}_2(\widetilde{q})\right) + o(z^{-1})$$

where terms in $o(z^{-1})$ involves z^{-k} for some $k \geq 2$.

We now give an explicit description of the closed mirror map for $\widetilde{\mathcal{X}}$, starting with the following definitions.

- For $i = 1,\dots,R', R+1, R+2$, let

$$\widetilde{\Omega}_i := \{\widetilde{\beta}\in\mathbb{K}_{\text{eff}}(\widetilde{\mathcal{X}}) : \widetilde{v}(\widetilde{\beta}) = \vec{0}, \langle\widetilde{D}_i,\widetilde{\beta}\rangle < 0, \langle\widetilde{D}_{i'},\widetilde{\beta}\rangle \geq 0 \text{ for } i' \in \{1,\dots,R+2\}\smallsetminus\{i\}\},$$

and

$$\widetilde{A}_i(\widetilde{q}) := \sum_{\widetilde{\beta}\in\widetilde{\Omega}_i} \widetilde{q}^{\widetilde{\beta}} \frac{(-1)^{-\langle \widetilde{D}_i, \widetilde{\beta}\rangle -1}(-\langle \widetilde{D}_i, \widetilde{\beta}\rangle - 1)!}{\prod_{i'\in\{1,\dots,R+2\}\setminus\{i\}} \langle \widetilde{D}_{i'}, \widetilde{\beta}\rangle!}.$$

- For $i = R'+1, \dots, R$, let

$$\widetilde{\Omega}_i := \{\widetilde{\beta} \in \mathbb{K}_{\text{eff}}(\widetilde{\mathcal{X}}) : \widetilde{v}(\widetilde{\beta}) = \widetilde{b}_i, \langle \widetilde{D}_{i'}, \widetilde{\beta}\rangle \notin \mathbb{Z}_{<0} \text{ for } i' = 1, \dots, R+2\},$$

and

$$\widetilde{A}_i(\widetilde{q}) := \sum_{\widetilde{\beta}\in\widetilde{\Omega}_i} \widetilde{q}^{\widetilde{\beta}} \prod_{i'=R'+1}^{R} \frac{\prod_{m=\lceil \langle \widetilde{D}_{i'}, \widetilde{\beta}\rangle\rceil}^{\infty} (\langle \widetilde{D}_{i'}, \widetilde{\beta}\rangle - m)}{\prod_{m=0}^{\infty} (\langle \widetilde{D}_{i'}, \widetilde{\beta}\rangle - m)}.$$

- For $i = 1, \dots, R+2$, we write

$$\widetilde{D}_i = \sum_{a=1}^{R-2} \widetilde{m}_i^{(a)} \widetilde{H}_a$$

for $\widetilde{m}_i^{(a)} \in \mathbb{Q}$. Moreover, set

$$\widetilde{\lambda}_i := \iota_{\widetilde{\sigma}_0}^* (\widetilde{\mathcal{D}}_i^{\widetilde{T}'}),$$

which is zero unless $i \in \{2, 3, R+1, R+2\}$. Since $\iota_{\widetilde{\sigma}_0}^*(\widetilde{u}_a) = 0$ for all a, we have

$$\widetilde{\mathcal{D}}_i^{\widetilde{T}'} = \sum_{a=1}^{R-2} \widetilde{m}_i^{(a)} \widetilde{u}_a + \widetilde{\lambda}_i.$$

- For $a = 1, \dots, R'-3, R-2$, let

$$\widetilde{S}_a(\widetilde{q}) := \sum_{i\in\{1,\dots,R',R+1,R+2\}} \widetilde{m}_i^{(a)} \widetilde{A}_i(\widetilde{q}).$$

With the above definitions, the I-function of $\widetilde{\mathcal{X}}$ can be written as

$$I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}_0, \widetilde{q}, z) = 1 + z^{-1}\left(\widetilde{q}_0 + \sum_{i=1}^{R+2} \widetilde{\lambda}_i \widetilde{A}_i(\widetilde{q}) + \sum_{a\in\{1,\dots,R'-3,R-2\}} \left(\log \widetilde{q}_a + \widetilde{S}_a(\widetilde{q})\right) \widetilde{u}_a + \sum_{i=R'+1}^{R} \widetilde{A}_i(\widetilde{q}) \mathbf{1}_{\widetilde{b}_i}\right) + o(z^{-1}).$$

As stated in Theorem 6.10, the closed mirror map for $\widetilde{\mathcal{X}}$ is thus given by

$$\widetilde{\tau}_0(\widetilde{q}_0,\widetilde{q}) = \widetilde{q}_0 + \sum_{i=1}^{R+2} \widetilde{\lambda}_i \widetilde{A}_i(\widetilde{q}),$$

$$\widetilde{\tau}_a(\widetilde{q}) = \begin{cases} \log \widetilde{q}_a + \widetilde{S}_a(\widetilde{q}), & a \in \{1,\dots,R'-3, R-2\}, \\ \widetilde{A}_{a+3}(\widetilde{q}), & a = R'-2,\dots,R-3. \end{cases}$$

We now make some observations that will simplify the closed mirror map above.

Lemma 6.11. *We have*

(i) $(\widetilde{m}^{(1)}_{R+1},\dots,\widetilde{m}^{(R-2)}_{R+1}) = -(\widetilde{m}^{(1)}_{R+2},\dots,\widetilde{m}^{(R-2)}_{R+2}) = (0,\dots,0,1)$.

(ii) $(\widetilde{m}^{(R-2)}_{1},\dots,\widetilde{m}^{(R-2)}_{R+2}) = l^{(0)} = (w_0, w_2, w_3, 0,\dots,0,1,-1)$ *(see* (6.4)*).*

(iii) $\widetilde{m}^{(a)}_i = m^{(a)}_i$ *for each* $i = 1,\dots,R$*,* $a = 1,\dots,R-3$*.*

Proof. (1) follows from our choice that $\widetilde{H}_{R-2} = \widetilde{D}_{R+1} = -\widetilde{D}_{R+2}$. For (2), as we observed in the proof of Lemma 6.7, $\{\widetilde{H}_1,\dots,\widetilde{H}_{R-3}\}$ gives a $\mathbb{Q}$-basis for the span of $\{\widetilde{D}_4,\dots,\widetilde{D}_R\}$, which implies that $\widetilde{m}^{(R-2)}_4 = \cdots = \widetilde{m}^{(R-2)}_R = 0$. Then (2) follows from that $\widetilde{\alpha}(\widetilde{m}^{(R-2)}_1,\dots,\widetilde{m}^{(R-2)}_{R+2}) = 0$ (see (2.8)) and $\widetilde{m}^{(R-1)}_{R+1} = 1$. Finally, (3) follows from (2) and our construction that under the projection $\widetilde{\mathbb{L}}^\vee \to \mathbb{L}^\vee$, $\widetilde{D}_i$ projects to D_i for $i = 1,\dots,R$, $\widetilde{D}_{R+1}$ and $\widetilde{D}_{R+2}$ projects to 0, and $\widetilde{H}_a$ projects to H_a for $a = 1,\dots,R-3$. □

Lemma 6.12. *We have*

$$\widetilde{\Omega}_{R+1} = \widetilde{\Omega}_{R+2} = \varnothing, \quad \widetilde{A}_{R+1}(\widetilde{q}) = \widetilde{A}_{R+2}(\widetilde{q}) = 0.$$

Proof. Note that $\widetilde{D}_1 + \cdots + \widetilde{D}_R = 0$ and $\{\widetilde{D}_1,\dots,\widetilde{D}_R\}$ contains a $\mathbb{Q}$-basis for $\widetilde{\mathbb{L}}^\vee_{\mathbb{Q}}$. Then, any $\widetilde{\beta} \in \widetilde{\mathbb{L}}_{\mathbb{Q}}$ that satisfies $\langle \widetilde{D}_i, \widetilde{\beta}\rangle \geq 0$ for all $i = 1,\dots,R$ must be zero. This implies that $\widetilde{\Omega}_{R+1} = \widetilde{\Omega}_{R+2} = \varnothing$. □

With Lemmas 6.11 and 6.12, we simplify the mirror map as follows:

$$
\begin{aligned}
\widetilde{\tau}_0(\widetilde{q}_0, \widetilde{q}) &= \widetilde{q}_0 + \widetilde{\lambda}_2 \widetilde{A}_2(\widetilde{q}) + \widetilde{\lambda}_3 \widetilde{A}_3(\widetilde{q}) \\
&= \widetilde{q}_0 + \left(-\frac{f}{\mathfrak{m}}\mathsf{u}_1 + \frac{1}{\mathfrak{m}}\mathsf{u}_2\right)\widetilde{A}_2(\widetilde{q}) + \left(\frac{f}{\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2 - \mathsf{u}_4\right)\widetilde{A}_3(\widetilde{q}), \\
\widetilde{\tau}_a(\widetilde{q}) &= \log \widetilde{q}_a + \widetilde{S}_a(\widetilde{q}) = \log \widetilde{q}_a + \sum_{i=1}^{R'} \widetilde{m}_i^{(a)} \widetilde{A}_i(\widetilde{q}), \quad a = 1, \dots, R' - 3, \\
\widetilde{\tau}_a(\widetilde{q}) &= \widetilde{A}_{a+3}(\widetilde{q}), \quad a = R' - 2, \dots, R - 3, \\
\widetilde{\tau}_{R-2}(\widetilde{q}) &= \log \widetilde{q}_{R-2} + w_0 \widetilde{A}_1(\widetilde{q}) + w_2 \widetilde{A}_2(\widetilde{q}) + w_3 \widetilde{A}_3(\widetilde{q}).
\end{aligned}
\tag{6.10}
$$

Now we prove Lemma 3.10 that the closed Gromov-Witten invariants under consideration are defined.

Proof of Lemma 3.10. By Lemma 5.7 and Theorem 6.10, we have

$$
\langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2) = [z^{-2}](J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{\gamma}_\lambda)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} = [z^{-2}]\left(e^{-\frac{1}{z}\widetilde{\tau}_0(0,\widetilde{q})} I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}
$$

under the mirror map (6.10). Note that

$$
\widetilde{\tau}_0(0, \widetilde{q})\Big|_{\mathsf{u}_4=0} = \frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}}(-\widetilde{A}_2(\widetilde{q}) + \widetilde{A}_3(\widetilde{q})).
$$

By a similar vanishing argument as in the proof of Lemma 6.9, we have

$$
[z^{-2}]\left(e^{-\frac{1}{z}\widetilde{\tau}_0(0,\widetilde{q})} I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}\Bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} = [z^{-2}]\left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}\Bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0}, \tag{6.11}
$$

which is identified by Theorem 6.5 with $W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q, x)$ under the relation $q_a = \widetilde{q}_a$ for $a = 1, \dots, R-3$ and $x = \widetilde{q}_{R-2}$. By definition (Definition 6.3), $W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q, x)$ is a power series in q, x with $\mathbb{Q}$-coefficients. Therefore, considering $[z^{-2}]\left(e^{-\frac{1}{z}\widetilde{\tau}_0(0,\widetilde{q})} I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}$ as a power series in $\widetilde{q}$ with coefficients in $\mathcal{Q}_{\widetilde{T}'}$, we see that none of such coefficients have a pole along $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$. Since the mirror map $\widetilde{\boldsymbol{\tau}}_2 = \widetilde{\boldsymbol{\tau}}_2(\widetilde{q})$ (6.10) is non-equivariant, if we consider the generating function $\langle\!\langle \widetilde{\gamma}_\lambda \rangle\!\rangle^{\widetilde{\mathcal{X}}, \widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2)$ as a power series in $\widetilde{Q}, \widetilde{\boldsymbol{\tau}}_2''$ with coefficients in $\mathcal{Q}_{\widetilde{T}'}^{\mathbb{C}}$ (Definition 5.1), then none of such coefficients have a pole along $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$ either. The lemma thus follows. □

6.5 Open mirror symmetry and compatibility

The mirror theorem of [50, 27, 30] also applies to the Calabi-Yau 3-orbifold $\mathcal{X}$, which relates the (T'-equivariant) J- and I-functions of $\mathcal{X}$ via the closed mirror map $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_2(q)$. As conjectured by [5] and verified by [56, 41, 42], such mirror symmetry can be extended to the open sector in the sense that the A- and B-model disk functions can be identified via the closed mirror map and an additional *open* mirror map $\mathsf{X} = \mathsf{X}(q,x)$. This gives the top arrow in Figure 1.1.

Theorem 6.13 ([5, 56, 41, 42])**.** *For all $\lambda \in G_{\tau_0}$,*

$$F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2(q), \mathsf{X}(q,x)) = W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q,x)$$

under the open-closed mirror map $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_2(q)$, $\mathsf{X} = \mathsf{X}(q,x)$, given in (6.12) *below.*

We now describe the open-closed mirror map of $(\mathcal{X}, \mathcal{L}, f)$, again starting with some definitions.

- For $i = 1, \dots, R'$, let

  $$\Omega_i := \{\beta \in \mathbb{K}_{\mathrm{eff}}(\mathcal{X}) : v(\beta) = \vec{0}, \langle D_i, \beta\rangle < 0, \langle D_{i'}, \beta\rangle \geq 0 \text{ for } i' \in \{1, \dots, R\} \setminus \{i\}\},$$

  and

  $$A_i(q) := \sum_{\beta \in \Omega_i} q^\beta \frac{(-1)^{-\langle D_i, \beta\rangle - 1}(-\langle D_i, \beta\rangle - 1)!}{\prod_{i' \in \{1,\dots,R\} \setminus \{i\}} \langle D_{i'}, \beta\rangle!}.$$

- For $i = R'+1, \dots, R$, let

  $$\Omega_i := \{\beta \in \mathbb{K}_{\mathrm{eff}}(\mathcal{X}) : v(\beta) = b_i, \langle D_{i'}, \beta\rangle \notin \mathbb{Z}_{<0} \text{ for } i' = 1, \dots, R\},$$

  and

  $$A_i(q) := \sum_{\beta \in \Omega_i} q^\beta \prod_{i'=R'+1}^{R} \frac{\prod_{m=\lceil\langle D_{i'}, \beta\rangle\rceil}^{\infty}(\langle D_{i'}, \beta\rangle - m)}{\prod_{m=0}^{\infty}(\langle D_{i'}, \beta\rangle - m)}.$$

- For $i = 1, \dots, R$, we write

$$D_i = \sum_{a=1}^{R-3} m_i^{(a)} H_a$$

  for $m_i^{(a)} \in \mathbb{Q}$.

- For $a = 1, \dots, R' - 3$, let

$$S_a(q) := \sum_{i=1}^{R'} m_i^{(a)} A_i(q).$$

Then, the open-closed mirror map of $\mathcal{X}$ is given by

$$\tau_a(q) = \begin{cases} \log q_a + S_a(q), & a \in \{1, \dots, R'-3\}, \\ A_{a+3}(q), & a = R'-2, \dots, R-3, \end{cases} \tag{6.12}$$

$$\log \mathsf{X} = \log x + w_0 A_1(q) + w_2 A_2(q) + w_3 A_3(q).$$

For the remainder of this section, we show that our correspondences can be used to provide an alternative proof of Theorem 6.13. In other words, we show that the diagram in Figure 1.1 is "commutative" by showing that the top arrow can be recovered from the other three arrows. We first identify the open-closed mirror map of $(\mathcal{X}, \mathcal{L}, f)$ with the closed mirror map of $\widetilde{\mathcal{X}}$.

Proposition 6.14. *Under the relation* $\widetilde{q}_a = q_a$ *for* $a = 1, \dots, R-3$ *and* $\widetilde{q}_{R-2} = x$*, we have*

$$\widetilde{\tau}_a(\widetilde{q}) = \tau_a(q), \quad a = 1, \dots, R-3,$$

$$\widetilde{\tau}_{R-2}(\widetilde{q}) = \log \mathsf{X}(q, x).$$

Proof. Observe that if $\widetilde{\beta} \in \widetilde{\mathbb{L}}$ satisfies that both $\langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle$ and $\langle \widetilde{D}_{R+2}, \widetilde{\beta} \rangle$ are non-negative (or both are non-positive), then both are in fact zero and $\widetilde{\beta} \in \mathbb{L}$. Thus, by Observation 6.1, it is straightforward to verify that $\widetilde{q}^{\widetilde{\beta}}$ does not involve $\widetilde{q}_{R-2}$ and equals q^{β}, and that for any $i = 1, \dots, R$,

$$\widetilde{\Omega}_i = \Omega_i, \quad \widetilde{A}_i(\widetilde{q}) = A_i(q), \tag{6.13}$$

under the relation $\widetilde{q}_a = q_a$, $a = 1, \dots, R-3$. This combined with Lemma 6.11 (3) gives

$$\widetilde{S}_a(\widetilde{q}) = S_a(q)$$

for $a = 1, \dots, R-3$. We then conclude by the descriptions (6.10), (6.12) of the mirror maps. $\square$

Now we give our alternative proof of Theorem 6.13.

Proof of Theorem 6.13 via Theorems 5.8 and 6.5. We have the following:

$$\begin{aligned}
&F_\lambda^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, \mathsf{X}) \\
&= [z^{-2}] \left(J_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Big|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} && \text{(by Theorem 5.8 under } \tau_a = \widetilde{\tau}_a,\ \log \mathsf{X} = \widetilde{\tau}_{R-2}) \\
&= [z^{-2}] \left(e^{-\frac{1}{z}\widetilde{\tau}_0(0,\widetilde{q})} I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Big|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} && \text{(by Theorem 6.10 under closed mirror map (6.10))} \\
&= [z^{-2}] \left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\gamma}_\lambda \right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'} \Big|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} && \text{(by (6.11))} \\
&= W_\lambda^{\mathcal{X},(\mathcal{L},f)}(q, x) && \text{(by Theorem 6.5 under } q_a = \widetilde{q}_a,\ x = \widetilde{q}_{R-2}).
\end{aligned}$$

Moreover, Proposition 6.14 implies that the dependence of $\boldsymbol{\tau}_2$ and X on q and x through the above chain of correspondences agrees with the open-closed mirror map (6.12) of $(\mathcal{X}, \mathcal{L}, f)$. $\square$

Remark 6.15. As a consequence of the "commutativity" of the diagram in Figure 1.1, we may deduce any one of Theorems 5.8, 6.5, and 6.13 (i.e. the left, right, and top arrows) from the other two. This yields alternative proofs of both Theorems 5.8 and 6.5.

Chapter 7: Extended Picard-Fuchs system and solutions

In this chapter, we compare the Picard-Fuchs systems associated to $\mathcal{X}$, $\widetilde{\mathcal{X}}$ and show that the latter is an extension of the former (Proposition 7.1). We further describe the additional solutions to the extended system in terms of the open string data $(\mathcal{L}, f)$ on $\mathcal{X}$ (Propositions 7.4 and 7.5).

7.1 Extended Picard-Fuchs system

We start by defining the Picard-Fuchs systems. Recall from Sections 6.4 and 6.5 that in describing the mirror maps for $\mathcal{X}$ and $\widetilde{\mathcal{X}}$, we have the following expression of $D_1, \ldots, D_R \in \mathbb{L}^\vee$ (resp. $\widetilde{D}_1, \ldots, \widetilde{D}_{R+2} \in \widetilde{\mathbb{L}}^\vee$) in terms of the preferred basis $\{H_1, \ldots, H_{R-3}\}$ (resp. $\{\widetilde{H}_1, \ldots, \widetilde{H}_{R-2}\}$):

$$\begin{aligned} D_i &= \sum_{a=1}^{R-3} m_i^{(a)} H_a, && i = 1, \ldots, R; \\ \widetilde{D}_i &= \sum_{a=1}^{R-2} \widetilde{m}_i^{(a)} \widetilde{H}_a, && i = 1, \ldots, R+2. \end{aligned}$$

(See Section 6.2 for the definition of these bases.) For each $\beta \in \mathbb{L}$, we define the differential operator

$$\mathcal{P}_\beta := q^\beta \prod_{\substack{i \in \{1,\ldots,R\} \\ \langle D_i, \beta \rangle < 0}} \prod_{m=0}^{-\langle D_i, \beta \rangle - 1} (\mathcal{P}_i - m) - \prod_{\substack{i \in \{1,\ldots,R\} \\ \langle D_i, \beta \rangle > 0}} \prod_{m=0}^{-\langle D_i, \beta \rangle - 1} (\mathcal{P}_i - m)$$

where for each $i = 1, \ldots, R$,

$$\mathcal{P}_i := \sum_{a=1}^{R-3} m_i^{(a)} q_a \frac{\partial}{\partial q_a}.$$

Similarly, for each $\widetilde{\beta} \in \widetilde{\mathbb{L}}$, we define the differential operator

$$\widetilde{\mathcal{P}}_{\widetilde{\beta}} := \widetilde{q}^{\widetilde{\beta}} \prod_{\substack{i \in \{1,\ldots,R+2\} \\ \langle \widetilde{D}_i, \widetilde{\beta} \rangle < 0}} \prod_{m=0}^{-\langle \widetilde{D}_i, \widetilde{\beta} \rangle - 1} (\widetilde{\mathcal{P}}_i - m) - \prod_{\substack{i \in \{1,\ldots,R+2\} \\ \langle \widetilde{D}_i, \widetilde{\beta} \rangle > 0}} \prod_{m=0}^{-\langle \widetilde{D}_i, \widetilde{\beta} \rangle - 1} (\widetilde{\mathcal{P}}_i - m)$$

where for each $i = 1, \dots, R+2$,

$$\widetilde{\mathcal{P}}_i := \sum_{a=1}^{R-2} \widetilde{m}_i^{(a)} \widetilde{q}_a \frac{\partial}{\partial \widetilde{q}_a}.$$

The *Picard-Fuchs* systems associated to $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are given by

$$\mathcal{P} := \{\mathcal{P}_\beta\}_{\beta \in \mathbb{L}}, \qquad \widetilde{\mathcal{P}} := \{\widetilde{\mathcal{P}}_{\widetilde{\beta}}\}_{\widetilde{\beta} \in \widetilde{\mathbb{L}}}$$

respectively, and we refer to $\widetilde{\mathcal{P}}$ as the *extended* Picard-Fuchs system. We now show that every solution to $\mathcal{P}$ is also a solution to $\widetilde{\mathcal{P}}$.

Proposition 7.1. *Suppose $F(q_1, \dots, q_{R-3})$ is such that*

$$\mathcal{P}_\beta F(q_1, \dots, q_{R-3}) = 0 \qquad \textit{for all } \beta \in \mathbb{L}.$$

Then

$$\widetilde{\mathcal{P}}_{\widetilde{\beta}} F(\widetilde{q}_1, \dots, \widetilde{q}_{R-3}) = 0 \qquad \textit{for all } \widetilde{\beta} \in \widetilde{\mathbb{L}}.$$

Proof. Under the identification $\widetilde{q}_a = q_a$ for $a = 1, \dots, R-3$, we have by Lemma 6.11 that

$$\widetilde{\mathcal{P}}_1 = \mathcal{P}_1 + w_0 \widetilde{q}_{R-2} \frac{\partial}{\partial \widetilde{q}_{R-2}}, \quad \widetilde{\mathcal{P}}_2 = \mathcal{P}_2 + w_2 \widetilde{q}_{R-2} \frac{\partial}{\partial \widetilde{q}_{R-2}}, \quad \widetilde{\mathcal{P}}_3 = \mathcal{P}_3 + w_3 \widetilde{q}_{R-2} \frac{\partial}{\partial \widetilde{q}_{R-2}},$$

$$\widetilde{\mathcal{P}}_i = \mathcal{P}_i \qquad \text{for } i = 4, \dots, R,$$

$$\widetilde{\mathcal{P}}_{R+1} = \widetilde{q}_{R-2} \frac{\partial}{\partial \widetilde{q}_{R-2}}, \quad \widetilde{\mathcal{P}}_{R+2} = -\widetilde{q}_{R-2} \frac{\partial}{\partial \widetilde{q}_{R-2}}.$$

Therefore, for any function $G = G(q_1, \dots, q_{R-3})$ that only depends on the first $R-3$ variables, we have

$$\widetilde{\mathcal{P}}_i G(\widetilde{q}_1, \dots, \widetilde{q}_{R-3}) = \mathcal{P}_i G(q_1, \dots, q_{R-3}) \qquad \text{for } i = 1, \dots, R, \tag{7.1}$$

$$\widetilde{\mathcal{P}}_{R+1} G = \widetilde{\mathcal{P}}_{R+2} G = 0. \tag{7.2}$$

Now we consider the operator $\widetilde{\mathcal{P}}_{\widetilde{\beta}}$ corresponding to some $\widetilde{\beta} \in \widetilde{\mathbb{L}}$. If $\beta \in \mathbb{L}$, then Observation 6.1

and (7.1) together imply that

$$\widetilde{\mathcal{P}}_{\widetilde{\beta}} F(\widetilde{q}_1, \ldots, \widetilde{q}_{R-3}) = \mathcal{P}_{\widetilde{\beta}} F(q_1, \ldots, q_{R-3}),$$

which is 0 by assumption. On the other hand, if $\widetilde{\beta} \notin \mathbb{L}$, then $\langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle = -\langle D_{R+2}, \widetilde{\beta} \rangle \neq 0$. It follows that one of the two products in $\widetilde{\mathcal{P}}_{\widetilde{\beta}}$ contains $\widetilde{\mathcal{P}}_{R+1}$ as a multiplicative factor, and the other contains $\widetilde{\mathcal{P}}_{R+2}$ as a multiplicative factor. Then (7.2) implies that $\widetilde{\mathcal{P}}_{\widetilde{\beta}} F = 0$. □

Given Proposition 7.1, our goal is then to understand the extra solutions to the extended system $\widetilde{\mathcal{P}}$ that do not come from solutions to $\mathcal{P}$. By Iritani [59], we have the following characterization of the dimensions of solution spaces (over $\mathbb{C}$).

Theorem 7.2 ([59]). *The dimension of the solution space to the system $\mathcal{P}$ is*

$$\dim_{\mathbb{C}} H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{C}) = \mathrm{Vol}(\Delta),$$

and the dimension of the solution space to the extended system $\widetilde{\mathcal{P}}$ is

$$\dim_{\mathbb{C}} H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}}; \mathbb{C}) = \mathrm{Vol}(\widetilde{\Delta}),$$

where $\mathrm{Vol}(\Delta)$, $\mathrm{Vol}(\widetilde{\Delta})$ are the normalized volumes[1] *of the polyhedra Δ, $\widetilde{\Delta}$ respectively.*

Note that

$$\mathrm{Vol}(\widetilde{\Delta}) = \sum_{\widetilde{\sigma} \in \widetilde{\Sigma}(4)} |G_{\widetilde{\sigma}}| = \sum_{\sigma \in \Sigma(3)} |G_\sigma| + \sum_{\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))} |G_{\widetilde{\sigma}}| = \mathrm{Vol}(\Delta) + \sum_{\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))} |G_{\widetilde{\sigma}}|.$$

Thus the number of linearly independent solutions to $\widetilde{\mathcal{P}}$ that are not solutions to $\mathcal{P}$ is

$$\sum_{\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))} |G_{\widetilde{\sigma}}| = \sum_{\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))} (n_{i_2(\widetilde{\sigma})} - f m_{i_2(\widetilde{\sigma})}) - (n_{i_3(\widetilde{\sigma})} - f m_{i_3(\widetilde{\sigma})}),$$

[1] This means that the volume of a standard simplex of any dimension is normalized to 1.

by Lemma 2.3. Here, recall from Section 2.5 that the indices $i_2(\widetilde{\sigma}), i_3(\widetilde{\sigma}) \in I'_{\delta_0(\widetilde{\sigma})} \subset I'_{\widetilde{\sigma}}$ are defined so that $b_{i_2(\widetilde{\sigma})}, b_{i_3(\widetilde{\sigma})}$ appear on the boundary of Δ in counterclockwise order. We fix a total ordering of the cones in $\widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$ by

$$\widetilde{\sigma}^1, \ldots, \widetilde{\sigma}^S, \qquad S := |\widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))|, \tag{7.3}$$

such that $b_{i_2(\widetilde{\sigma}^1)}, \ldots, b_{i_2(\widetilde{\sigma}^S)}$ appear on the boundary of Δ in counterclockwise order. In other words,

$$i_3(\widetilde{\sigma}^s) = i_2(\widetilde{\sigma}^{s+1}) \qquad \text{for all } s = 1, \ldots, S-1.$$

We denote

$$\tau^s := \delta_0(\widetilde{\sigma}^s), \qquad s = 1, \ldots, S.$$

See Figure 7.1 for an illustration (which also illustrates the interval Δ_0 to be defined in Section 8.1).

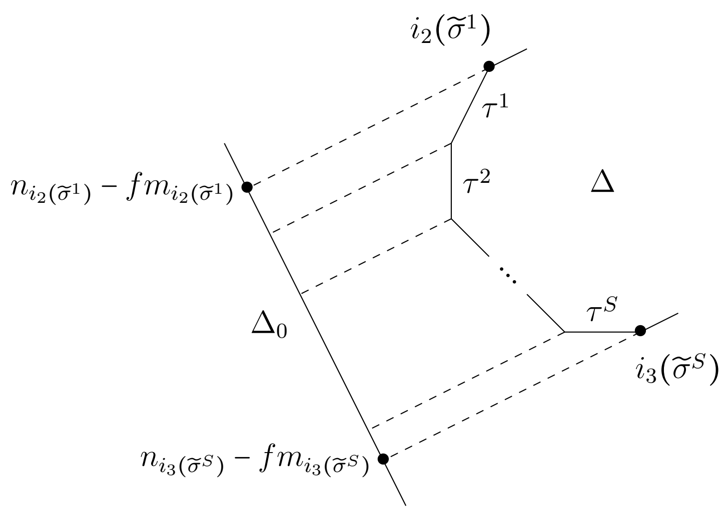

Figure 7.1: Ordering of cones in $\widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$ and relation between Δ and Δ_0.

Then, the quantity

$$n_{i_2(\widetilde{\sigma}^s)} - fm_{i_2(\widetilde{\sigma}^s)}$$

is strictly monotone decreasing in s. The difference in the dimension of solutions spaces to $\mathcal{P}$ and

$\widetilde{\mathcal{P}}$ can be rewritten as

$$(n_{i_2(\widetilde{\sigma}^1)} - fm_{i_2(\widetilde{\sigma}^1)}) - (n_{i_3(\widetilde{\sigma}^S)} - fm_{i_3(\widetilde{\sigma}^S)}). \tag{7.4}$$

7.2 Solutions as coefficients of non-equivariant I-functions

To characterize the extra solutions to the extended system $\widetilde{\mathcal{P}}$, we use the description due to Givental [50] of the solutions to Picard-Fuchs systems as coefficients of the *non-equivariant I-functions* of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$:

$$I_{\mathcal{X}}(q,z) := e^{\frac{1}{z}\left(\sum_{a\in\{1,\dots,R'-3\}} u_a \log q_a\right)}$$
$$\cdot \sum_{\beta\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X})} q^{\beta} \prod_{i\in\{1,\dots,R'\}} \frac{\prod_{m=\lceil\langle D_i,\beta\rangle\rceil}^{\infty}\left(\frac{\mathcal{D}_i}{z} + \langle D_i,\beta\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\mathcal{D}_i}{z} + \langle D_i,\beta\rangle - m\right)} \cdot \prod_{i=R'+1}^{R} \frac{\prod_{m=\lceil\langle D_i,\beta\rangle\rceil}^{\infty}\left(\langle D_i,\beta\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\langle D_i,\beta\rangle - m\right)} \frac{\mathbf{1}_{v(\beta)}}{z^{\mathrm{age}(v(\beta))}},$$

$$I_{\widetilde{\mathcal{X}}}(\widetilde{q},z) := e^{\frac{1}{z}\left(\sum_{a\in\{1,\dots,R'-3,R-2\}} \widetilde{u}_a \log \widetilde{q}_a\right)}$$
$$\cdot \sum_{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}})} \widetilde{q}^{\widetilde{\beta}} \prod_{i\in\{1,\dots,R',R+1,R+2\}} \frac{\prod_{m=\lceil\langle \widetilde{D}_i,\widetilde{\beta}\rangle\rceil}^{\infty}\left(\frac{\widetilde{\mathcal{D}}_i}{z} + \langle \widetilde{D}_i,\widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\frac{\widetilde{\mathcal{D}}_i}{z} + \langle \widetilde{D}_i,\widetilde{\beta}\rangle - m\right)} \cdot \prod_{i=R'+1}^{R} \frac{\prod_{m=\lceil\langle \widetilde{D}_i,\widetilde{\beta}\rangle\rceil}^{\infty}\left(\langle \widetilde{D}_i,\widetilde{\beta}\rangle - m\right)}{\prod_{m=0}^{\infty}\left(\langle \widetilde{D}_i,\widetilde{\beta}\rangle - m\right)} \frac{\mathbf{1}_{\widetilde{v}(\widetilde{\beta})}}{z^{\mathrm{age}(\widetilde{v}(\widetilde{\beta}))}}.$$

(See also [59, Lemma 4.6]; see [43, Section 3.9 – 3.11] for a discussion for toric Calabi-Yau 3-orbifolds.) Note that $I_{\widetilde{\mathcal{X}}}(\widetilde{q},z)$ is the non-equivariant limit of the $\widetilde{T}'$-equivariant I-function of $\widetilde{\mathcal{X}}$ (6.2) considered in Section 6.2. Note furthermore that

$$\iota^*\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q},z)\right) = I_{\mathcal{X}}(q,z) \tag{7.5}$$

under $\widetilde{q}_a = q_a$, $a = 1,\dots,R-3$. (Here, all terms involving $\widetilde{q}_{R-2}$ pull back to 0.)

The functions $I_{\mathcal{X}}(q,z)$, $I_{\widetilde{\mathcal{X}}}(\widetilde{q},z)$ take value in the non-equivariant cohomology $H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$, $H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{Q})$ respectively. To extract their coefficients, we consider the perfect pairings

$$\begin{aligned}(-,-)_{\mathcal{X}} &: H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) \times H^{6-*}_{\mathrm{CR},c}(\mathcal{X};\mathbb{C}) \to \mathbb{C},\\ (-,-)_{\widetilde{\mathcal{X}}} &: H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{C}) \times H^{8-*}_{\mathrm{CR},c}(\widetilde{\mathcal{X}};\mathbb{C}) \to \mathbb{C},\end{aligned} \tag{7.6}$$

where the subscript "c" stands for cohomology *with compact support*. There is a pushforward map

$$\iota_* : H^*_{\mathrm{CR},c}(\mathcal{X};\mathbb{C}) \to H^{*+2}_{\mathrm{CR},c}(\widetilde{\mathcal{X}};\mathbb{C}), \qquad \gamma_c \mapsto \gamma_c \widetilde{\mathcal{D}}_{R+2}$$

such that for any $\widetilde{\gamma} \in H^*_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{C})$ and $\gamma_c \in H^*_{\mathrm{CR},c}(\mathcal{X};\mathbb{C})$, we have

$$(\iota^*(\gamma), \gamma_c)_{\mathcal{X}} = (\widetilde{\gamma}, \iota_*(\gamma_c))_{\widetilde{\mathcal{X}}} = \left(\widetilde{\gamma}, \gamma_c \widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}. \tag{7.7}$$

Theorem 7.3 ([50])**.** *A basis for the solution space to the Picard-Fuchs system $\mathcal{P}$ consists of*

$$[z^{-\deg(\gamma_c)/2}]\,(I_{\mathcal{X}}(q,z), \gamma_c)_{\mathcal{X}}$$

where γ_c ranges through any homogeneous basis for $H^*_{\mathrm{CR},c}(\mathcal{X};\mathbb{C})$. A basis for the solution space to the Picard-Fuchs system $\widetilde{\mathcal{P}}$ consists of

$$[z^{-\deg(\widetilde{\gamma}_c)/2}]\,(I_{\widetilde{\mathcal{X}}}(\widetilde{q},z), \widetilde{\gamma}_c)_{\widetilde{\mathcal{X}}}$$

where $\widetilde{\gamma}_c$ ranges through any homogeneous basis for $H^*_{\mathrm{CR},c}(\widetilde{\mathcal{X}};\mathbb{C})$.

See Notation 5.6 for the notation $[z^{-k}]$. We now consider the expansion of the I-functions in powers of z^{-1}. The zeroth-order terms are

$$[z^0]I_{\mathcal{X}}(q,z) = 1 = (I_{\mathcal{X}}(q,z), [\mathrm{pt}])_{\mathcal{X}}, \qquad [z^0]I_{\widetilde{\mathcal{X}}}(\widetilde{q},z) = 1 = (I_{\widetilde{\mathcal{X}}}(\widetilde{q},z), [\mathrm{pt}])_{\widetilde{\mathcal{X}}}.$$

Recall from Sections 6.4 and 6.5 that the first-order terms in the expansion give the closed mirror maps:

$$[z^{-1}]I_{\mathcal{X}}(q,z) = \sum_{a=1}^{R-3} \tau_a(q) u_a, \qquad [z^{-1}]I_{\widetilde{\mathcal{X}}}(\widetilde{q},z) = \sum_{a=1}^{R-2} \widetilde{\tau}_a(\widetilde{q}) \widetilde{u}_a.$$

In particular, $\tau_1(q), \ldots, \tau_{R-3}(q)$ are solutions to $\mathcal{P}$, and $\widetilde{\tau}_1(\widetilde{q}), \ldots, \widetilde{\tau}_{R-2}(\widetilde{q})$ are solutions to $\widetilde{\mathcal{P}}$. (To obtain these solutions from the pairings (7.6) as in Theorem 7.3, we may pair the I-function with the basis for $H^4_{\mathrm{CR},c}(\mathcal{X};\mathbb{C})$ (resp. $H^6_{\mathrm{CR},c}(\widetilde{\mathcal{X}};\mathbb{C})$) dual to $\{u_1,\dots,u_{R-3}\}$ (resp. $\{\widetilde{u}_1,\dots,\widetilde{u}_{R-2}\}$).) Proposition 6.14 gives us the identification

$$\tau_a(q) = \widetilde{\tau}_a(\widetilde{q}), \qquad a = 1,\dots,R-3$$

of solutions, while $\widetilde{\tau}_{R-2}(\widetilde{q})$ is an extra solution to $\widetilde{\mathcal{P}}$.

Now we turn to the second-order terms in the expansion. Note that there are no higher-order terms since $H^{\geq 6}_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) = H^{\geq 6}_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{C}) = 0$. For the remainder of this section, let A denote $\dim_{\mathbb{C}} H^4_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$. Let $\{u_{1,c},\dots,u_{A,c}\}$ be a basis for $H^2_{\mathrm{CR},c}(\mathcal{X};\mathbb{C})$. Then the coefficients of the dual basis for $H^4_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ in $I_{\mathcal{X}}(q,z)$ are

$$(I_{\mathcal{X}}(q,z), u_{a,c})_{\mathcal{X}}, \qquad a = 1,\dots,A,$$

which are the remaining solutions to $\mathcal{P}$. From the above basis, we can construct the following basis for $H^4_{\mathrm{CR},c}(\widetilde{\mathcal{X}};\mathbb{C}) \cong H^4_{\mathrm{CR}}(\widetilde{\mathcal{X}};\mathbb{C})$:

$$\begin{aligned} &\{\iota_*(u_{1,c}),\dots,\iota_*(u_{A,c})\} \sqcup \{\widetilde{\mathcal{D}}_{i_2(\widetilde{\sigma}^s)}\widetilde{\mathcal{D}}_{R+2} : s = 2,\dots,S\} \\ &\sqcup \{\mathbf{1}_j\widetilde{\mathcal{D}}_{R+2} : j \in \mathrm{Box}(\tau^s), s = 1,\dots,S, j \neq \vec{0}\} \\ &\sqcup \{\mathbf{1}_j : j \in \mathrm{Box}(\widetilde{\mathcal{X}}) \setminus \mathrm{Box}(\mathcal{X})\}. \end{aligned} \tag{7.8}$$

Here, observe first that

$$\{i_2(\widetilde{\sigma}^2),\dots,i_2(\widetilde{\sigma}^S)\}$$

is the set of indices i such that the divisor $\mathcal{V}(\rho_i)$ of $\mathcal{X}$ is non-compact but codimension-2 $\widetilde{T}$-invariant closed substack of $\widetilde{\mathcal{X}}$ corresponding to the cone spanned by $\widetilde{\rho}_i$, $\widetilde{\rho}_{R+2}$ is compact. In terms of fans, ρ_i lies on the boundary of the support of Σ while $\widetilde{\rho}_i$ lies in the interior of the support of $\widetilde{\Sigma}$. Similarly,

$$\bigcup_{s=1}^{S} \mathrm{Box}(\tau^s) \setminus \{\vec{0}\}$$

is the set of age-1 elements $j \in \mathrm{Box}(\mathcal{X}) \subseteq \mathrm{Box}(\widetilde{\mathcal{X}})$ such that if $j \in \mathrm{Box}(\tau^s)$, then the line $\mathfrak{l}_{\tau^s}$ in the twisted sector $\mathcal{X}_j$ is non-compact but the line in $\widetilde{\mathcal{X}}_j$ corresponding to the cone spanned by τ^s and $\widetilde{\rho}_{R+2}$ is compact. Moreover, recall from Section 2.5 that any element $j \in \mathrm{Box}(\widetilde{\mathcal{X}}) \setminus \mathrm{Box}(\mathcal{X})$ has age 2 and thus $\mathbf{1}_j$ is a degree-4 class. By (7.5) and (7.7),

$$(I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \iota_*(u_{a,c}))_{\widetilde{\mathcal{X}}} = (I_{\mathcal{X}}(q, z), u_{a,c})_{\mathcal{X}}$$

for $a = 1, \dots, A$ are the coefficients of $I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z)$ corresponding to those of $I_{\mathcal{X}}(q, z)$. The additional solutions result from pairing $I_{\widetilde{\mathcal{X}}}$ with the additional basis elements. We check that the number of such solutions is equal to

$$\left| \bigcup_{s=1}^{S} \mathrm{Box}(\tau^s) \setminus \{\vec{0}\} \right| + \left| \mathrm{Box}(\widetilde{\mathcal{X}}) \setminus \mathrm{Box}(\mathcal{X}) \right| = \sum_{s=1}^{S} |G_{\widetilde{\sigma}^s}| - 1,$$

which is consistent with the discussion in Section 7.1 above.

7.3 Extra solutions from open strings

In this section, we characterize certain solutions to the extended Picard-Fuchs system $\widetilde{\mathcal{P}}$ by the open geometry $(\mathcal{X}, \mathcal{L}, f)$. First, Proposition 6.14 directly translates into the following result.

Proposition 7.4. *The extra solution $\widetilde{\tau}_{R-2}(\widetilde{q})$ is given by the open mirror map*

$$\log \mathsf{X}(\widetilde{q}_1, \dots, \widetilde{q}_{R-3}, \widetilde{q}_{R-2})$$

of $(\mathcal{X}, \mathcal{L}, f)$ defined in 6.12. This solution has form

$$\log \widetilde{q}_{R-2} + \textit{power series in } \widetilde{q}_1, \dots, \widetilde{q}_{R-3}.$$

Now, we study the extra solutions resulting from pairing the I-function of $\widetilde{\mathcal{X}}$ with degree-4 classes that are related to the distinguished cone $\widetilde{\sigma}_0$, using our open/closed correspondence for hypergeometric functions (Theorem 6.5). We show that when the ray $\widetilde{\rho}_2$ or $\widetilde{\rho}_3$ is shared with another cone in $\widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$, which happens for certain framings f, the power series part of the corresponding solution has a description involving the B-model disk function $W^{\mathcal{X},(\mathcal{L},f)}$ in the untwisted sector. Moreover, for any non-trivial $j \in \mathrm{Box}(\tau_0)$, we show that the corresponding solution can be described in terms of $W^{\mathcal{X},(\mathcal{L},f)}$ in the corresponding twisted sector.

We introduce some notations. Let $s \in \{1, \dots, S\}$ be the order of $\widetilde{\sigma}_0$ in the list (7.3), i.e. $\widetilde{\sigma}_0 = \widetilde{\sigma}^s$. When $s > 1$, $\widetilde{\sigma}_0$ shares the ray $\widetilde{\rho}_2$ with $\widetilde{\sigma}^{s-1}$ and $\widetilde{\mathcal{D}}_2\widetilde{\mathcal{D}}_{R+2}$ is an element of the basis (7.8) for $H^4_{\mathrm{CR},c}(\widetilde{\mathcal{X}};\mathbb{C})$. In this case we define

$$\widetilde{\gamma}_1^- := \frac{\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}^{\widetilde{T}'}_{i_2(\widetilde{\sigma}^{s-1})}\widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+1}}{\widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}^{s-1}), \widetilde{\sigma}^{s-1})} \in H^4_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q}),$$

which is the analog of the class $\widetilde{\gamma}_1$ (3.12) in the untwisted sector but defined for the cone $\widetilde{\sigma}^{s-1}$. When $s < S$, $\widetilde{\sigma}_0$ shares the ray $\widetilde{\rho}_3$ with $\widetilde{\sigma}^{s+1}$ and $\widetilde{\mathcal{D}}_3\widetilde{\mathcal{D}}_{R+2}$ is an element of the basis (7.8). In this case we define

$$\widetilde{\gamma}_1^+ := \frac{\widetilde{\mathcal{D}}_3^{\widetilde{T}'}\widetilde{\mathcal{D}}^{\widetilde{T}'}_{i_3(\widetilde{\sigma}^{s+1})}\widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+1}}{\widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}^{s+1}), \widetilde{\sigma}^{s+1})} \in H^4_{\mathrm{CR},\widetilde{T}'}(\widetilde{\mathcal{X}};\mathbb{Q})$$

which is the analog of the class $\widetilde{\gamma}_1$ (3.12) in the untwisted sector but defined for the cone $\widetilde{\sigma}^{s+1}$.

Proposition 7.5. *(1) When $s > 1$, the solution $[z^{-2}]\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q},z), \widetilde{\mathcal{D}}_2\widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}$ to $\widetilde{\mathcal{P}}$ has form*

$$\begin{aligned} &c^-(\log\widetilde{q}_{R-2})^2 + (\log\widetilde{q}_{R-2})G_1^-(\widetilde{q}_1,\dots,\widetilde{q}_{R-3}) + G_2^-(\widetilde{q}_1,\dots,\widetilde{q}_{R-3}) \\ &\qquad - W_1^{\mathcal{X},(\mathcal{L},f)}(\widetilde{q}_1,\dots,\widetilde{q}_{R-3},\widetilde{q}_{R-2}) + [z^{-2}]\left(I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q},z), \widetilde{\gamma}_1^-\right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}\Bigg|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0} \end{aligned}$$

for some constant $c^- \in \mathbb{Q}$ and functions G_1^-, G_2^- in $\widetilde{q}_1,\dots,\widetilde{q}_{R-3}$.

(2) When $s < S$, the solution $[z^{-2}]\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q},z), \widetilde{\mathcal{D}}_3\widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}$ to $\widetilde{\mathcal{P}}$ has form

$$\begin{aligned} &c^+(\log\widetilde{q}_{R-2})^2 + (\log\widetilde{q}_{R-2})G_1^+(\widetilde{q}_1,\dots,\widetilde{q}_{R-3}) + G_2^+(\widetilde{q}_1,\dots,\widetilde{q}_{R-3}) \\ &\qquad + W_1^{\mathcal{X},(\mathcal{L},f)}(\widetilde{q}_1,\dots,\widetilde{q}_{R-3},\widetilde{q}_{R-2}) - [z^{-2}]\left(I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q},z), \widetilde{\gamma}_1^+\right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}\Bigg|_{\mathsf{u}_4=0,\mathsf{u}_2-f\mathsf{u}_1=0} \end{aligned}$$

for some constant $c^+ \in \mathbb{Q}$ *and functions* G_1^+, G_2^+ *in* $\widetilde{q}_1, \ldots, \widetilde{q}_{R-3}$.

(3) *For any* $j \in \mathrm{Box}(\tau_0)$, $j \neq \vec{0}$, *we have*

$$[z^{-2}]\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \mathbf{1}_j \widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}} = G_j(\widetilde{q}_1, \ldots, \widetilde{q}_{R-3}) - W_\lambda^{\mathcal{X},(\mathcal{L},f)}(\widetilde{q}_1, \ldots, \widetilde{q}_{R-3}, \widetilde{q}_{R-2})$$

for some function G_j *in* $\widetilde{q}_1, \ldots, \widetilde{q}_{R-3}$, *where* $\lambda^{-1} \in G_{\tau_0}$ *corresponds to* j.

Proof. We first prove (1), and (2) will follow from a very similar proof. We consider $\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \widetilde{\mathcal{D}}_2 \widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}$ as the non-equivariant limit of the $\widetilde{T}'$-equivariant Poincaré pairing

$$\left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \widetilde{\mathcal{D}}_2^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'},$$

and we may consider the weight restriction $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$. As our focus is the part of the solution that does not involve any logarithms, it suffices to consider the pairing of $\widetilde{\mathcal{D}}_2^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}$ with

$$\sum_{\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}})} \widetilde{q}^{\widetilde{\beta}} \prod_{i \in \{1, \ldots, R', R+1, R+2\}} \frac{\prod_{m=\lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil}^{\infty} \left(\frac{\widetilde{\mathcal{D}}_i^{\widetilde{T}'}}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m\right)}{\prod_{m=0}^{\infty} \left(\frac{\widetilde{\mathcal{D}}_i^{\widetilde{T}'}}{z} + \langle \widetilde{D}_i, \widetilde{\beta} \rangle - m\right)} \cdot \prod_{i=R'+1}^{R} \frac{\prod_{m=\lceil \langle \widetilde{D}_i, \widetilde{\beta} \rangle \rceil}^{\infty} \left(\langle \widetilde{D}_i, \widetilde{\beta} \rangle - m\right)}{\prod_{m=0}^{\infty} \left(\langle \widetilde{D}_i, \widetilde{\beta} \rangle - m\right)} \frac{\mathbf{1}_{\widetilde{v}(\widetilde{\beta})}}{z^{\mathrm{age}(\widetilde{v}(\widetilde{\beta}))}}$$
$$= \sum_{\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}})} I_{\widetilde{\beta}}(\widetilde{q}, z)$$

using the notation (6.3). Moreover, our focus is on the untwisted sector and on terms that depend on $\widetilde{q}_{R-2}$. Since $\widetilde{q}^{\widetilde{\beta}}$ is independent of $\widetilde{q}_{R-2}$ for any $\widetilde{\beta} \in \mathbb{L}$, it suffices to consider the pairing of $\widetilde{\mathcal{D}}_2^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}$ with each $I_{\widetilde{\beta}}(\widetilde{q}, z)$ for $\widetilde{\beta}$ in

$$\{\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \setminus \mathbb{L} : \widetilde{v}(\widetilde{\beta}) = \vec{0}\} = \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \cap (\widetilde{\mathbb{L}} \setminus \mathbb{L}).$$

Fix such a $\widetilde{\beta}$. We may assume that the degree of $I_{\widetilde{\beta}}(\widetilde{q}, z)$ is 4, as the pairing is otherwise 0. Since $\langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle = -\langle \widetilde{D}_{R+2}, \widetilde{\beta} \rangle \neq 0$, $[z^{-2}] I_{\widetilde{\beta}}(\widetilde{q}, z)$ has form

$$c \widetilde{q}^{\widetilde{\beta}} \widetilde{\mathcal{D}}_i^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'} \qquad \text{or} \qquad c \widetilde{q}^{\widetilde{\beta}} \widetilde{\mathcal{D}}_i^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}$$

for some $i \in \{1, \dots, R'\}$ and $c \in \mathbb{Q}$, $c \neq 0$. Note that for any $\widetilde{\sigma} \in \iota(\Sigma(3))$, $\iota_{\widetilde{\sigma}}^*(\widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}) = 0$ and $\iota_{\widetilde{\sigma}}^*(\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}) = \mathsf{u}_4$. Moreover, the only cones in $\widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$ that contains the ray $\widetilde{\rho}_2$ are $\widetilde{\sigma}_0$ and $\widetilde{\sigma}^{s-1}$. It then follows that

$$\begin{aligned}
\left(I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}\Bigg|_{\mathsf{u}_4=0} &= \sum_{\widetilde{\sigma}\in\widetilde{\Sigma}(4)} \frac{\iota_{\widetilde{\sigma}}^*\left(\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'} I_{\widetilde{\beta}}(\widetilde{q}, z)\right)}{|G_{\widetilde{\sigma}}| e_{\widetilde{T}'}(T_{\mathfrak{p}_{\widetilde{\sigma}}}\widetilde{\mathcal{X}})}\Bigg|_{\mathsf{u}_4=0} \\
&= \frac{\iota_{\widetilde{\sigma}_0}^*\left(\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'} I_{\widetilde{\beta}}(\widetilde{q}, z)\right)}{|G_{\widetilde{\sigma}_0}| e_{\widetilde{T}'}(T_{\mathfrak{p}_{\widetilde{\sigma}_0}}\widetilde{\mathcal{X}})}\Bigg|_{\mathsf{u}_4=0} + \frac{\iota_{\widetilde{\sigma}^{s-1}}^*\left(\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'} I_{\widetilde{\beta}}(\widetilde{q}, z)\right)}{|G_{\widetilde{\sigma}^{s-1}}| e_{\widetilde{T}'}(T_{\mathfrak{p}_{\widetilde{\sigma}^{s-1}}}\widetilde{\mathcal{X}})}\Bigg|_{\mathsf{u}_4=0} \\
&= \left(I_{\widetilde{\beta}}(\widetilde{q}, z), -\widetilde{\gamma}_1\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}\Bigg|_{\mathsf{u}_4=0} + \left(I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\gamma}_1^-\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}\Bigg|_{\mathsf{u}_4=0}.
\end{aligned}$$

Here, the last equality follows from that

$$\iota_{\widetilde{\sigma}_0}^*\left(\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}\right)\Big|_{\mathsf{u}_4=0} = -\iota_{\widetilde{\sigma}_0}^*\left(\widetilde{\gamma}_1\right)\Big|_{\mathsf{u}_4=0}, \qquad \iota_{\widetilde{\sigma}^{s-1}}^*\left(\widetilde{\mathcal{D}}_2^{\widetilde{T}'}\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}\right)\Big|_{\mathsf{u}_4=0} = \iota_{\widetilde{\sigma}^{s-1}}^*\left(\widetilde{\gamma}_1^-\right)\Big|_{\mathsf{u}_4=0}$$

and that $\iota_{\widetilde{\sigma}}^*\left(\widetilde{\gamma}_0\right) = 0$ for any $\widetilde{\sigma} \neq \widetilde{\sigma}_0$, $\iota_{\widetilde{\sigma}}^*\left(\widetilde{\gamma}_1^-\right) = 0$ for any $\widetilde{\sigma} \neq \widetilde{\sigma}^{s-1}$. Statement (1) of the proposition then follows from Theorem 6.5.

Now we prove (3). Again we consider $\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \mathbf{1}_j\widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}$ as the non-equivariant limit of the $\widetilde{T}'$-equivariant Poincaré pairing

$$\left(I_{\widetilde{\mathcal{X}}}^{\widetilde{T}'}(\widetilde{q}, z), \mathbf{1}_j\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}\right)_{\widetilde{\mathcal{X}}}^{\widetilde{T}'},$$

and we may consider the weight restriction $\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0$. Since we are considering the twisted sector specified by j, the above pairing is equal to the pairing of $\mathbf{1}_j\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}$ with

$$\sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \\ \widetilde{v}(\widetilde{\beta})=-j}} I_{\widetilde{\beta}}(\widetilde{q}, z).$$

Again, since we focus on terms that depend on $\widetilde{q}_{R-2}$, we consider the pairing of $\mathbf{1}_j\widetilde{\mathcal{D}}_{R+2}^{\widetilde{T}'}$ with each $I_{\widetilde{\beta}}(\widetilde{q}, z)$ for $\widetilde{\beta}$ in

$$\{\widetilde{\beta} \in \mathbb{K}_{\mathrm{eff}}(\widetilde{\mathcal{X}}) \setminus \mathbb{L}_{\mathbb{Q}} : \widetilde{v}(\widetilde{\beta}) = -j\}.$$

Fix such a $\widetilde{\beta}$. We may assume that the degree of $I_{\widetilde{\beta}}(\widetilde{q}, z)$ is 4, as the pairing is otherwise 0. Since $\langle \widetilde{D}_{R+1}, \widetilde{\beta} \rangle = -\langle \widetilde{D}_{R+2}, \widetilde{\beta} \rangle \neq 0$, $[z^{-2}]I_{\widetilde{\beta}}(\widetilde{q}, z)$ has form

$$c\widetilde{q}^{\widetilde{\beta}} \widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+1} \mathbf{1}_j \qquad \text{or} \qquad c\widetilde{q}^{\widetilde{\beta}} \widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+2} \mathbf{1}_j$$

for some $c \in \mathbb{Q}$, $c \neq 0$. Note that j only belongs to the Box of the 4-cones $\iota(\sigma_0)$ and $\widetilde{\sigma}_0$, and $\iota^*_{\iota(\sigma_0)}(\widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+1}) = 0$, $\iota^*_{\iota(\sigma_0)}(\widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+2}) = \mathsf{u}_4$. It then follows that

$$\left(I_{\widetilde{\beta}}(\widetilde{q}, z), \mathbf{1}_j \widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+2} \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \Bigg|_{\mathsf{u}_4=0} = \frac{\iota^*_{\widetilde{\sigma}_0} \left(\mathbf{1}_j \widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+2} I_{\widetilde{\beta}}(\widetilde{q}, z) \right)}{|G_{\widetilde{\sigma}_0}| e_{\widetilde{T}'}(T_{\mathfrak{p}_{\widetilde{\sigma}_0}} \widetilde{\mathcal{X}}_j)} \Bigg|_{\mathsf{u}_4=0} = \left(I_{\widetilde{\beta}}(\widetilde{q}, z), -\widetilde{\gamma}_\lambda \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \Bigg|_{\mathsf{u}_4=0} + \left(I_{\widetilde{\beta}}(\widetilde{q}, z), \widetilde{\gamma}_1^- \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \Bigg|_{\mathsf{u}_4=0}.$$

Here, the last equality follows from that

$$\iota^*_{\widetilde{\sigma}_0} \left(\mathbf{1}_j \widetilde{\mathcal{D}}^{\widetilde{T}'}_{R+2} \right) \big|_{\mathsf{u}_4=0} = -\iota^*_{\widetilde{\sigma}_0} \left(\widetilde{\gamma}_\lambda \right) \big|_{\mathsf{u}_4=0}$$

and that $\iota^*_{\iota(\sigma_0)}(\widetilde{\gamma}_\lambda) = 0$. Statement (3) of the proposition then follows from Theorem 6.5. □

Remark 7.6. In (1) and (2) of Proposition 7.5, the terms

$$[z^{-2}] \left(I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \widetilde{\gamma}_1^- \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \Bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1=0}, \qquad [z^{-2}] \left(I^{\widetilde{T}'}_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \widetilde{\gamma}_1^+ \right)^{\widetilde{T}'}_{\widetilde{\mathcal{X}}} \Bigg|_{\mathsf{u}_4=0, \mathsf{u}_2 - f\mathsf{u}_1=0} \tag{7.9}$$

that describe the $\widetilde{q}_{R-2}$-dependence of the power series part aside from $W_1^{\mathcal{X},(\mathcal{L},f)}$ can also be interpreted as the B-model disk function (in the untwisted sector) of $\mathcal{X}$ relative to an Aganagic-Vafa outer brane $\mathcal{L}'$ intersecting $\mathfrak{l}_{\tau^{s\pm1}}$, with framining f' depending on f. The framing f' is in general not integral. For instance, in the example $\mathcal{X} = \mathbb{C}^3$ and $f > 0$, we have

$$f' = \frac{-f - 1}{f}.$$

In some cases, including $\mathcal{X} = \mathbb{C}^3$ (see Example 7.8 below) and $\mathcal{X} = \mathrm{Tot}(K_{\mathbb{P}^2})$, it turns out that (7.9) is a scalar multiple of $W_1^{\mathcal{X},(\mathcal{L},f)}$, which then completely describes the $\widetilde{q}_{R-2}$-dependence of the power series part of the solution.

Remark 7.7. Similarly, for $s = 2, \dots, S$, the $\widetilde{q}_{R-2}$-dependence of the power series part of the solution $[z^{-2}]\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \widetilde{\mathcal{D}}_{i_2(\widetilde{\sigma}^s)}\widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}$ to $\widetilde{\mathcal{P}}$ can be described in terms of the B-model disk functions (in the untwisted sector) of $\mathcal{X}$ relative to framed Aganagic-Vafa branes intersecting $\mathfrak{l}_{\tau^{s-1}}$, $\mathfrak{l}_{\tau^s}$ respectively. For any non-trivial $j \in \mathrm{Box}(\tau^s)$, $s = 1, \dots, S$, the $\widetilde{q}_{R-2}$-dependence of the solution $[z^{-2}]\left(I_{\widetilde{\mathcal{X}}}(\widetilde{q}, z), \mathbf{1}_j\widetilde{\mathcal{D}}_{R+2}\right)_{\widetilde{\mathcal{X}}}$ to $\widetilde{\mathcal{P}}$ can be described in terms of the B-model disk function of $\mathcal{X}$ relative to a framed Aganagic-Vafa brane intersecting $\mathfrak{l}_{\tau^s}$, in the corresponding twisted sector.

Example 7.8. Let $\mathcal{X} = \mathbb{C}^3$, $\mathcal{L}$ be an outer brane, and $f = 1$, as in Section 2.6.1 and Example 6.6. The non-equivariant I-function of $\widetilde{\mathcal{X}} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1))$ is

$$\begin{aligned} I_{\widetilde{\mathcal{X}}}(\widetilde{q}_1, z) &= e^{\frac{\widetilde{H}_1 \log \widetilde{q}_1}{z}} \left(1 + \sum_{d \in \mathbb{Z}_{>0}} \widetilde{q}_1^d \frac{2\widetilde{H}_1^2}{z^2} \frac{(-1)^d (2d-1)!}{d \cdot (d!)^2}\right) \\ &= 1 + \frac{\widetilde{H}_1}{z} \log \widetilde{q}_1 + \frac{\widetilde{H}_1^2}{z^2}\left(\frac{1}{2}(\log \widetilde{q}_1)^2 + 2W_1^{\mathcal{X},(\mathcal{L},1)}(\widetilde{q}_1)\right). \end{aligned}$$

The three coefficients give three linearly independent solutions to $\widetilde{\mathcal{P}}$. Among them, the constant 1 is the only solution to $\mathcal{P}$. The solution $\log \widetilde{q}_1$ in the $[z^{-1}]$ part is the open mirror map (Proposition 7.4). The power series part of the solution in the $[z^{-2}]$ part is a scalar multiple of the B-model disk function $W_1^{\mathcal{X},(\mathcal{L},1)}$ (Proposition 7.5).

Chapter 8: Integral cycles and periods

In this chapter, we study the open/closed correspondence from the perspective of B-model mirror families. We establish a correspondence between integral relative 3-cycles on the Hori-Vafa mirror of $\mathcal{X}$ and integral 4-cycles on that of $\widetilde{\mathcal{X}}$ under which the periods of holomorphic volume forms are preserved (Theorem 8.6).

8.1 Laurent polynomials

We start by introducing the Laurent polynomials that will be used to define the mirror families. Consider the bases $\{H_1, \dots, H_{R-3}\}$, $\{\widetilde{H}_1, \dots, \widetilde{H}_{R-2}\}$ defined in Section 6.2. For $a = 1, \dots, R-3$, we write H_a in terms of the basis $\{D_4, \dots, D_R\}$ (corresponding to the cone $\sigma_0 \in \Sigma(3)$) as

$$H_a = \sum_{i=4}^{R} s_{ai} D_i$$

for $s_{ai} \in \mathbb{Q}_{\geq 0}$. By rescaling if necessary, we may assume that $\{H_a\}$ is chosen such that $s_{ai} \in \mathbb{Z}_{\geq 0}$ for any $a = 1, \dots, R-3$, $i = 4, \dots, R$. By construction, we can write $\{\widetilde{H}_a\}$ in terms fo the basis $\{\widetilde{D}_4, \dots, \widetilde{D}_R, \widetilde{D}_{R+1}\}$ (corresponding to the cone $\iota(\sigma_0) \in \widetilde{\Sigma}(4)$) as

$$\widetilde{H}_a = \sum_{i=4}^{R} s_{ai} \widetilde{D}_i, \qquad a = 1, \dots, R-3,$$

$$\widetilde{H}_{R-2} = \widetilde{D}_{R+1}.$$

For $i = 4, \dots, R$, define

$$s_i(q) := \prod_{a=1}^{R-3} q_a^{s_{ai}}$$

which is a monomial in $q_1, \dots, q_{R-3}$. For convenience we set

$$s_1(q) = s_2(q) = s_3(q) := 1.$$

We will write $s_i(\widetilde{q})$ as the same monomial but in $\widetilde{q}_1, \dots, \widetilde{q}_{R-3}$. We define the following Laurent polynomials parameterized by q and $\widetilde{q}$:

$$H(X,Y,q) := X^{\mathfrak{r}}Y^{-\mathfrak{s}} + Y^{\mathfrak{m}} + 1 + \sum_{i=4}^{R} s_i(q) X^{m_i} Y^{n_i} \qquad \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}],$$
$$\widetilde{H}(X,Y,Z,\widetilde{q}) := X^{\mathfrak{r}}Y^{-\mathfrak{s}} + Y^{\mathfrak{m}} + 1 + \sum_{i=4}^{R} s_i(\widetilde{q}) X^{m_i} Y^{n_i} + \widetilde{q}_{R-2} X^{-1} Y^{-f} Z + Z \quad \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}].$$

Note that under $\widetilde{q}_a = q_a$ for $a = 1, \dots, R-3$, we have

$$\widetilde{H}(X,Y,Z,\widetilde{q}) = H(X,Y,q) + \widetilde{q}_{R-2} X^{-1} Y^{-f} Z + Z, \tag{8.1}$$

where the two additional terms correspond to the two additional vectors $\widetilde{b}_{R+1} = (-1, -f, 1, 1)$, $\widetilde{b}_{R+2} = (0, 0, 1, 1)$. Moreover, we use the open parameter x to define the following restriction of $H(X,Y,q)$:

$$\begin{aligned} H_0(Y,q,x) &:= H(X,Y,q)\big|_{XY^f = -x} = H(-xY^{-f}, Y, q) \\ &= (-x)^{\mathfrak{r}} Y^{-\mathfrak{s} - f\mathfrak{r}} + Y^{\mathfrak{m}} + 1 + \sum_{i=4}^{R} s_i(q)(-x)^{m_i} Y^{n_i - f m_i} \qquad \in \mathbb{C}[Y^{\pm 1}]. \end{aligned}$$

We consider the following domains for the parameters $q, x, \widetilde{q}$:

Definition 8.1. Let $\epsilon > 0$. Let $U_\epsilon \subset (\mathbb{C}^*)^{R'-3} \times \mathbb{C}^{R-R'}$ be the set of $(c_1, \dots, c_{R-3})$ such that

- $c_1, \dots, c_{R'-3} \in \mathbb{C}^*$, $c_{R'-2}, \dots, c_{R-3} \in \mathbb{C}$;
- $|c_a| < \epsilon$ for all $a = 1, \dots, R-3$.

Moreover, Let $\widetilde{U}_\epsilon \subset (\mathbb{C}^*)^{R'-2} \times \mathbb{C}^{R-R'}$ be the set of $(c_1, \dots, c_{R-2})$ such that

- $(c_1, \ldots, c_{R-3}) \in U_\epsilon$;
- $c_{R-2} \in \mathbb{C}^*$, $|c_{R-2}| < \epsilon$.

Given $q \in U_\epsilon$, $\widetilde{q} \in \widetilde{U}_\epsilon$, the Newton polyhedra[1] of $H(X,Y,q)$ and $\widetilde{H}(X,Y,Z,\widetilde{q})$ are $\Delta \subset \mathbb{R}^3$ and $\widetilde{\Delta} \subset \mathbb{R}^4$ respectively. Moreover, for ϵ sufficiently small and $(q,x) \in \widetilde{U}$, the Newton polyhedron of $H_0(Y,q,x)$ is the integral polyhedron $\Delta_0 \in \mathbb{R}$ defined as the convex hull of the integers in

$$\{n_i - fm_i : i = 1, \ldots, R\}.$$

Note that the quantity $n_i - fm_i$ achives maximum at $i = i_2(\widetilde{\sigma}^1)$ and minimum at $i = i_3(\widetilde{\sigma}^S)$ (see the discussion at the end of Section 7.1). This gives the following description of Δ_0:

$$\Delta_0 = [n_{i_3(\widetilde{\sigma}^S)} - fm_{i_3(\widetilde{\sigma}^S)}, n_{i_2(\widetilde{\sigma}^1)} - fm_{i_2(\widetilde{\sigma}^1)}]. \tag{8.2}$$

See Figure 7.1 for an illustration. We have from (7.4) that

$$\mathrm{Vol}(\widetilde{\Delta}) = \mathrm{Vol}(\Delta) + \mathrm{Vol}(\Delta_0). \tag{8.3}$$

In this work, we will choose a sufficiently small radius ϵ for the parameter domains such that the Laurent polynomials satisfy the following regularity condition of Batyrev [11] (see also [38, 39, 49]).

Definition 8.2. Let

$$F(X_1, \ldots, X_r) = \sum_{a=(a_1,\ldots,a_r)\in\mathbb{Z}^r} c_a X_1^{a_1} \cdots X_r^{a_r} \qquad \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$$

be a Laurent polynomial whose Newton polyhedron $\Delta_F \subset \mathbb{R}^r$ is r-dimensional. F is said to be

[1] The *Newton polyhedron* of a Laurent polynomial $F(X_1, \ldots, X_r) \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ is the convex hull in $\mathbb{R}^r$ of lattice points $(a_1, \ldots, a_r) \in \mathbb{Z}^r$ such that the coefficient of $X_1^{a_1} \cdots X_r^{a_r}$ in F is non-zero.

Δ_F-*regular* if for every d-dimensional face $\Delta' \subseteq \Delta$, $0 < d \leq r$, the functions

$$F^{\Delta'} := \sum_{a \in \Delta' \cap \mathbb{Z}^r} c_a X_1^{a_1} \cdots X_r^{a_r}, \qquad X_1 \frac{\partial F^{\Delta'}}{\partial X_1}, \quad \cdots, \quad X_r \frac{\partial F^{\Delta'}}{\partial X_r}$$

have no common zeroes in $(\mathbb{C}^*)^r$.

Assumption 8.3. We fix a sufficiently small $\epsilon > 0$ such that for any $(q, x), \widetilde{q} \in \widetilde{U}_\epsilon = U_\epsilon \times \{c \in \mathbb{C}^* : |c| < \epsilon\}$, the following hold:

- $H(X, Y, q)$ is Δ-regular;
- $\widetilde{H}(X, Y, Z, \widetilde{q})$ is $\widetilde{\Delta}$-regular;
- $H_0(Y, q, x)$ has Newton polyhedron Δ_0 and is Δ_0-regular.

The existence of such an ϵ is implied by [59, Lemma 3.8], which studies a closely related non-degeneracy condition of Kouchnirenko [69]. Assumption 8.3 will only be used in this chapter to ensure the smoothness of the mirror families, and will be used in full in Chapter 9 when we study mixed Hodge structures.

8.2 Hori-Vafa mirrors and correspondence of cycles and periods

The Hori-Vafa mirror family of $\mathcal{X}$ is $(\mathcal{X}_q^\vee, \Omega_q)$ defined over $q \in U_\epsilon$, where

$$\mathcal{X}_q^\vee := \{(u, v, X, Y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv = H(X, Y, q)\}$$

is a non-compact Calabi-Yau 3-fold and

$$\Omega_q := \mathrm{Res}_{\mathcal{X}_q^\vee} \frac{du \wedge dv \wedge \frac{dX}{X} \wedge \frac{dY}{Y}}{H(X, Y, q) - uv}$$

is a holomorphic 3-form on $\mathcal{X}_q^\vee$. Similarly, the Hori-Vafa mirror family of $\widetilde{\mathcal{X}}$ is $(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee, \widetilde{\Omega}_{\widetilde{q}})$ defined over $\widetilde{q} \in \widetilde{U}_\epsilon$, where

$$\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee := \{(u, v, X, Y, Z) \in \mathbb{C}^2 \times (\mathbb{C}^*)^3 : uv = \widetilde{H}(X, Y, Z, \widetilde{q})\}$$

is a non-compact Calabi-Yau 4-fold and

$$\widetilde{\Omega}_{\widetilde{q}} := \mathrm{Res}_{\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee} \frac{du \wedge dv \wedge \frac{dX}{X} \wedge \frac{dY}{Y} \wedge \frac{dZ}{Z}}{\widetilde{H}(X, Y, Z, \widetilde{q}) - uv}$$

is a holomorphic 4-form on $\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee$. Note that the projection $Z : \widetilde{\mathcal{X}}_{\widetilde{q}}^\vee \to \mathbb{C}^*$ given by the Z-coordinate can be extended to $Z = 0$, and by relation (8.1), the fiber over $Z = 0$ is isomorphic to $\mathcal{X}_{(\widetilde{q}_1, \dots, \widetilde{q}_{R-3})}^\vee$.

Moreover, we consider a family $(\mathcal{Y}_{q,x}, \Omega_{q,x}^0)$ defined over $(q, x) \in \widetilde{U}_\epsilon$, where

$$\mathcal{Y}_{q,x} := \{(u, v, Y) \in \mathbb{C}^2 \times \mathbb{C}^* : uv = H_0(Y, q, x)\}$$

is a non-compact Calabi-Yau surface and

$$\Omega_{q,x}^0 := \mathrm{Res}_{\mathcal{Y}_{q,x}} \frac{du \wedge dv \wedge \frac{dY}{Y}}{H_0(Y, q, x) - uv}$$

is a holomorphic 2-form on $\mathcal{Y}_{q,x}$. $(\mathcal{Y}_{q,x}, \Omega_{q,x}^0)$ can be interpreted as the Hori-Vafa mirror family of a toric Calabi-Yau surface whose fan has support equal to the cone over Δ_0. Note that for any q, $\mathcal{Y}_{q,x}$ can be viewed as a family of hypersurfaces in $\mathcal{X}_q^\vee$ parameterized by x:

$$\mathcal{Y}_{q,x} = \mathcal{X}_q^\vee \cap \{XY^f = -x\}.$$

There is also an identification of $\mathcal{Y}_{q,x} \times \mathbb{C}^*$ with the hypersurface

$$\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee \cap \{XY^f = -\widetilde{q}_{R-2}\}$$

given by $((u,v,Y),Z) \mapsto (u,v,xY^{-f},Y,Z)$ under $\widetilde{q}_a = q_a$, $a = 1,\dots,R-3$ and $\widetilde{q}_{R-2} = x$. Under the extension of the fibration $Z : \widetilde{\mathcal{X}}^\vee_{\widetilde{q}} \to \mathbb{C}^*$ to $Z = 0$, this recovers the embedding of $\mathcal{Y}_{q,x}$ in $\mathcal{X}^\vee_q$.

By Assumption 8.3, in particular the regularity condition applied to the top-dimensional faces of the polyhedra themselves, the three families $\mathcal{X}^\vee_q$, $\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}$, and $\mathcal{Y}_{q,x}$ are smooth over $(q,x), \widetilde{q} \in \widetilde{U}_\epsilon$.

It is a well-known fact that period integrals of the holomorphic form over middle-dimensional integral cycles on the Hori-Vafa mirror generate the solution space to the Picard-Fuchs system.

Theorem 8.4 (Cf. [57, 67, 23]). *The period integrals*

$$\int_\Gamma \Omega_q, \qquad \Gamma \in H_3(\mathcal{X}^\vee_q;\mathbb{Z})$$

give a $\mathbb{C}$-basis for the solution space to the Picard-Fuchs system $\mathcal{P}$ as Γ ranges through a basis. The period integrals

$$\int_{\widetilde{\Gamma}} \widetilde{\Omega}_{\widetilde{q}}, \qquad \widetilde{\Gamma} \in H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}};\mathbb{Z})$$

give a $\mathbb{C}$-basis for the solution space to the Picard-Fuchs system $\widetilde{\mathcal{P}}$ as $\widetilde{\Gamma}$ ranges through a basis.

Remark 8.5. Similarly, as Γ_0 ranges through a basis for $H_2(\mathcal{Y}_{q,x};\mathbb{Z})$, the period integrals

$$\int_{\Gamma_0} \Omega^0_{q,x}$$

give a $\mathbb{C}$-basis for the solution space of a Picard-Fuchs system defined by the polyhedron Δ_0. The dimension of the solution space is $\mathrm{Vol}(\Delta_0)$, which is the difference between that of $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ (see Theorem 7.2 and (7.4), (8.3)). In particular, (8.3) translates into

$$\mathrm{rank}(H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}};\mathbb{Z})) = \mathrm{rank}(H_3(\mathcal{X}^\vee_q;\mathbb{Z})) + \mathrm{rank}(H_2(\mathcal{Y}_{q,x};\mathbb{Z})). \tag{8.4}$$

Our main result of this chapter is that periods of $\widetilde{\Omega}_{\widetilde{q}}$ over 4-cycles on $\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}$, and thus solutions to the extended system $\widetilde{\mathcal{P}}$, can be completely recovered from period of Ω_q over *relative* 3-cycles on $\mathcal{X}^\vee_q$ with boundary on the hypersurface $\mathcal{Y}_{q,x}$. This gives the open/closed correspondence on the level of cycles and periods.

Theorem 8.6. *For $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$, there is an isomorphism*

$$\iota : H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{Z}) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee; \mathbb{Z})$$

such that for any $\Gamma \in H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{Z})$, we have

$$\int_\Gamma \Omega_q = \frac{1}{2\pi\sqrt{-1}} \int_{\iota(\Gamma)} \widetilde{\Omega}_{\widetilde{q}}.$$

We note that Li-Lian-Yau [74] considered compact Calabi-Yau hypersurfaces in projective toric 4-folds and studied periods of relative 3-cycles with boundary in a family of divisors. They showed that the relative periods satisfy a GKZ system associated to an enhanced polytope [7], which is very similar to our construction of $\widetilde{\Delta}$, and our Theorem 8.6 can be viewed as an extension to the non-compact setting. The method of [74] is based on the variations of the holomorphic 3-form of the Calabi-Yau hypersurface in relative cohomology. We will return to this perspective in Chapter 9.

8.3 Conic and circle fibrations

Before proving Theorem 8.6, we first review the structure of Hori-Vafa mirrors as conic fibrations over algebraic tori in this section, following [23, Section 4.2]; see also [37, Section 5.1]. We detail the construction for $\mathcal{X}_q^\vee$ (see also [43, Section 4.4] for this case) and the constructions for $\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee, \mathcal{Y}_{q,x}$ are similar.

Consider the following families of affine hypersurfaces in algebraic tori of different dimensions, defined over $(q,x), \widetilde{q} \in \widetilde{U}_\epsilon$:

$$\begin{aligned} C_q &:= \{(X,Y) \in (\mathbb{C}^*)^2 : H(X,Y,q) = 0\}, \\ S_{\widetilde{q}} &:= \{(X,Y,Z) \in (\mathbb{C}^*)^3 : \widetilde{H}(X,Y,Z,\widetilde{q}) = 0\}, \qquad (8.5) \\ P_{q,x} &:= \{Y \in \mathbb{C}^* : H_0(Y,q,x) = 0\}. \end{aligned}$$

By Assumption 8.3, C_q (resp. $S_{\widetilde{q}}$) is a family of smooth algebraic curves (resp. surfaces) and $P_{q,x}$ is a family of $\mathrm{Vol}(\Delta_0)$ distinct points. The projection $Z : S_{\widetilde{q}} \to \mathbb{C}^*$ can be extended to $Z = 0$ and the fiber over Z is isomorphic to $C_{(q_1,\dots,q_{R-3})}$ by (8.1). For any q, $P_{q,x}$ can be viewed as a set of points on C_q parameterized by x:

$$P_{q,x} = C_q \cap \{XY^f = -x\}.$$

There is also an identification of $P_{q,x} \times \mathbb{C}^*$ with the hypersurface

$$S_{\widetilde{q}} \cap \{XY^f = -\widetilde{q}_{R-2}\}$$

given by $(Y, Z) \mapsto (xY^{-f}, Y, Z)$ under $\widetilde{q}_a = q_a$, $a = 1, \dots, R-3$ and $\widetilde{q}_{R-2} = x$. Under the extension of the fibration $Z : S_{\widetilde{q}} \to \mathbb{C}^*$ to $Z = 0$, this recovers the embedding of $P_{q,x}$ in C_q.

Consider the following Hamiltonian $U(1)$-action on $\mathcal{X}_q^\vee$:

$$e^{\theta\sqrt{-1}} \cdot (u, v, X, Y) := (e^{\theta\sqrt{-1}}u, e^{-\theta\sqrt{-1}}v, X, Y),$$

whose moment map is given by

$$\mu : \mathcal{X}_q^\vee \to \mathbb{R}, \qquad (u, v, X, Y) \mapsto \frac{1}{2}\left(|u|^2 - |v|^2\right).$$

Now consider the conic fibration

$$\mathcal{X}_q^\vee \to (\mathbb{C}^*)^2, \qquad (u, v, X, Y) \mapsto (X, Y),$$

whose discriminant locus is the curve C_q (8.5). Restricting the above to the level set

$$\mu^{-1}(0) = \{(u, v, X, Y) \in \mathcal{X}_q^\vee : |u| = |v|\},$$

we obtain a circle fibration

$$\pi : \mu^{-1}(0) \to (\mathbb{C}^*)^2$$

where the fiber degenerates to a point precisely over $(X, Y) \in C_q$. The map $\gamma \mapsto \pi^{-1}(\gamma)$ induces an isomorphism

$$\alpha : H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z}) \xrightarrow{\sim} H_3(\mathcal{X}_q^\vee; \mathbb{Z}).$$

By identical constructions, the conic fibrations

$$\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee \to (\mathbb{C}^*)^3, \qquad \mathcal{Y}_{q,x} \to (\mathbb{C}^*)$$

restrict to circle fibrations

$$\widetilde{\pi} : \mu^{-1}(0) \subset \widetilde{\mathcal{X}}_{\widetilde{q}}^\vee \to (\mathbb{C}^*)^3, \qquad \pi_0 : \mu^{-1}(0) \subset \mathcal{Y}_{q,x} \to (\mathbb{C}^*)$$

whose discriminant loci are the surface $S_{\widetilde{q}}$ and the point set $P_{q,x}$ respectively. There are induced isomorphisms

$$\widetilde{\alpha} : H_3((\mathbb{C}^*)^3, S_{\widetilde{q}}; \mathbb{Z}) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee; \mathbb{Z}), \qquad \alpha_0 : H_1(\mathbb{C}^*, P_{q,x}; \mathbb{Z}) \xrightarrow{\sim} H_2(\mathcal{Y}_{q,x}; \mathbb{Z}).$$

The standard holomorphic forms

$$\omega := \frac{dX}{X} \wedge \frac{dY}{Y}, \qquad \widetilde{\omega} := \frac{dX}{X} \wedge \frac{dY}{Y} \wedge \frac{dZ}{Z}, \qquad \omega^0 := \frac{dY}{Y} \tag{8.6}$$

on $(\mathbb{C}^*)^2$, $(\mathbb{C}^*)^3$, $\mathbb{C}^*$ respectively vanish on the hypersurfaces C_q, $S_{\widetilde{q}}$, $P_{q,x}$ and thus represent relative cohomology classes in

$$H^2((\mathbb{C}^*)^2, C_q; \mathbb{C}), \qquad H^3((\mathbb{C}^*)^3, S_{\widetilde{q}}; \mathbb{C}), \qquad H^1(\mathbb{C}^*, P_{q,x}; \mathbb{C}).$$

The following result states that the relative periods of these relative forms are matched with periods of the holomorphic forms on the Hori-Vafa mirrors via the isomorphisms α, $\widetilde{\alpha}$, α_0 on homology.

Theorem 8.7 ([37, 23])**.** *We have the following correspondences of period integrals:*

$$\int_\gamma \omega = \frac{1}{2\pi\sqrt{-1}} \int_{\alpha(\gamma)} \Omega_q \qquad \text{for all } \gamma \in H_2((\mathbb{C}^*)^2, C_q; \mathbb{Z});$$
$$\int_{\widetilde{\gamma}} \widetilde{\omega} = \frac{1}{2\pi\sqrt{-1}} \int_{\widetilde{\alpha}(\widetilde{\gamma})} \widetilde{\Omega}_{\widetilde{q}} \qquad \text{for all } \widetilde{\gamma} \in H_3((\mathbb{C}^*)^3, S_{\widetilde{q}}; \mathbb{Z});$$
$$\int_{\gamma_0} \omega^0 = \frac{1}{2\pi\sqrt{-1}} \int_{\alpha_0(\gamma_0)} \Omega^0_{q,x} \qquad \text{for all } \gamma_0 \in H_1(\mathbb{C}^*, P_{q,x}; \mathbb{Z}).$$

In fact, the relative periods of $\omega, \widetilde{\omega}$ also generate the solution spaces to the Picard-Fuchs systems $\mathcal{P}$, $\widetilde{\mathcal{P}}$ respectively (see Theorem 8.4).

8.4 Decomposing relative 3-cycles

Now we start to prove Theorem 8.6. Fix $(q, x) = \widetilde{q} \in \widetilde{U}_\epsilon$. Consider the long exact sequence of integral homology associated to the pair $(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x})$:

$$\cdots \to H_3(\mathcal{Y}_{q,x}; \mathbb{Z}) \to H_3(\mathcal{X}_q^\vee; \mathbb{Z}) \to H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{Z}) \to H_2(\mathcal{Y}_{q,x}; \mathbb{Z}) \to H_2(\mathcal{X}_q^\vee; \mathbb{Z}) \to \cdots. \tag{8.7}$$

Our goal of this section is to show the following.

Proposition 8.8. *The following sequence induced by* (8.7)

$$0 \to H_3(\mathcal{X}_q^\vee; \mathbb{Z}) \to H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{Z}) \to H_2(\mathcal{Y}_{q,x}; \mathbb{Z}) \to 0 \tag{8.8}$$

is a split short exact sequence.

We prove Proposition 8.8 in a series of lemmas below.

Lemma 8.9. *We have*

$$H_3(\mathcal{Y}_{q,x}; \mathbb{Z}) = 0.$$

Proof. Consider the conic fibration $\pi : \mathcal{Y}_{q,x} \to \mathbb{C}^*$. We write

$$\mathbb{C}^* = A_0 \cup B_0,$$

where B_0 is the union of disjoint open disks, one around each point in $P_{q,x}$, and A_0 is the complement in $\mathbb{C}^*$ of disjoint closed disks, one around each point in $P_{q,x}$. Note that B_0 deformation retracts to $P_{q,x}$ while A_0 is homotopy equivalent to $\mathbb{C}^* \setminus P_{q,x}$. Their intersection $A_0 \cap B_0$ deformation retracts to a disjoint union of circles, one around each point in $P_{q,x}$.

Then $\mathcal{Y}_{q,x}$ is covered by the sets $A = \pi^{-1}(A_0)$ and $B = \pi^{-1}(B_0)$, and Mayer-Vietoris gives

$$\cdots \to H_3(A;\mathbb{Z}) \oplus H_3(B;\mathbb{Z}) \to H_3(\mathcal{Y}_{q,x};\mathbb{Z}) \to H_2(A \cap B;\mathbb{Z}) \to H_2(A;\mathbb{Z}) \oplus H_2(B;\mathbb{Z}) \to \cdots.$$

The fiber of π over each point in $P_{q,x}$ is $\{uv = 0\} = \mathbb{C} \cup \mathbb{C} \subset \mathbb{C}^2$. Thus B deformation retracts to $P_{q,x}$, which implies that $H_3(B;\mathbb{Z}) = H_2(B;\mathbb{Z}) = 0$. Moreover, A is a trivial $\mathbb{C}^*$-bundle over A_0 with a trivialization given by

$$A \cong \mathbb{C}^* \times A_0, \qquad \left(u, v = \frac{H_0(Y,q,x)}{u}, Y\right) \leftrightarrow (u, Y),$$

and $A \cap B$ is a subbundle over $A_0 \cap B_0$. Thus $H_3(A;\mathbb{Z}) = 0$ and moreover, the map $H_2(A \cap B) \to H_2(A)$ induced by inclusion is injective by the injectivity of the map $H_1(A_0 \cap B_0) \to H_1(A_0)$ induced by inclusion. □

Lemma 8.10. *The map*

$$H_1(C_q;\mathbb{Z}) \to H_1((\mathbb{C}^*)^2;\mathbb{Z})$$

induced by inclusion is surjective.

We note that [43, Section 4.4] studies the above map and shows that it has finite cokernel. We confirm that the cokernel is indeed trivial.

Proof. Denote the above map by ι_*. We use the analysis of the family of curves C_q in [43, Sections

4.4 and 5.8]. First we consider the case $n_1 = \mathfrak{s} = 0$. Given a 1-cycle $\gamma \in H_1(C_q;\mathbb{Z})$, we will consider the pairing of $\iota_*(\gamma)$ with the basis of 1-forms $\frac{dX}{2\pi\sqrt{-1}X}, \frac{dY}{2\pi\sqrt{-1}Y} \in H^1((\mathbb{C}^*)^2;\mathbb{Z})$. For any flag $(\tau,\sigma) \in F(\Sigma)$, [43] constructs an element $\gamma^{(\tau,\sigma)} \in H_1(C_q;\mathbb{Z})$ by considering the degeneration of C_q in the limit $q \to 0$ and taking the vanishing cycles on the irreducible component corresponding to σ around punctures corresponding to τ. [43] further computes the pairing of $\iota_*(\gamma^{(\tau,\sigma)})$ with $\frac{dX}{2\pi\sqrt{-1}X}, \frac{dY}{2\pi\sqrt{-1}Y}$. In our present case $\mathfrak{s} = 0$, a direct computation shows that for the cycles $\gamma^{(\tau_0,\sigma_0)}$, $\gamma^{(\tau_2,\sigma_0)}$ corresponding to flags (τ_0,σ_0), (τ_2,σ_0) respectively, we have

$$\begin{aligned} \left\langle \iota_*(\gamma^{(\tau_0,\sigma_0)}), \frac{dX}{2\pi\sqrt{-1}X} \right\rangle &= 1, \qquad \left\langle \iota_*(\gamma^{(\tau_0,\sigma_0)}), \frac{dY}{2\pi\sqrt{-1}Y} \right\rangle = 0, \\ \left\langle \iota_*(\gamma^{(\tau_2,\sigma_0)}), \frac{dX}{2\pi\sqrt{-1}X} \right\rangle &= 0, \qquad \left\langle \iota_*(\gamma^{(\tau_2,\sigma_0)}), \frac{dY}{2\pi\sqrt{-1}Y} \right\rangle = 1. \end{aligned}$$

This implies that $\iota_*(\gamma^{(\tau_0,\sigma_0)})$, $\iota_*(\gamma^{(\tau_2,\sigma_0)})$ form a basis for $H_1((\mathbb{C}^*)^2;\mathbb{Z})$ and the surjectivity of ι_* follows.

Now we consider the case $n_1 = -\mathfrak{s} < 0$. Note that the vector

$$(1,0,1) = \frac{1}{\mathfrak{r}} b_1 + \frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}} b_2 + \left(1 - \frac{1}{\mathfrak{r}} - \frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\right) b_3$$

always lies inside the cone σ_0 (recall that $\mathfrak{s} \le \mathfrak{r} - 1$). Let $\mathcal{X}'$ be the crepant partial resolution of $\mathcal{X}$ whose fan is obtained from Σ by taking the star subdivision along a new ray generated by $(1,0,1)$. By [94, Section 5.1], the family of curves $\{C_q\}_{q \in U_\epsilon}$ and the analogous family defined by the toric Calabi-Yau 3-orbifold $\mathcal{X}'$ fit into a global family of affine curves in $(\mathbb{C}^*)^2$ over a base $\mathcal{M}$. Moreover, $H_1(C_q;\mathbb{Z})$ forms a local system of lattices over $\mathcal{M}$ and maps to the trivial system $H_2((\mathbb{C}^*)^2;\mathbb{Z})$ via inclusion. Now, note that $\mathcal{X}'$ satisfies the condition $\mathfrak{s} = 0$ of the previous case and ι_* is surjective for the family of curves defined by $\mathcal{X}'$. It follows that ι_* is also surjective for $\{C_q\}_{q \in U_\epsilon}$ since surjectivity is preserved under parallel transport over $\mathcal{M}$. □

Lemma 8.11. *The map*

$$H_1(C_q, P_{q,x};\mathbb{Z}) \to H_1((\mathbb{C}^*)^2, P_{q,x};\mathbb{Z})$$

induced by inclusion is surjective.

Proof. The lemma follows from Lemma 8.10 and the Five Lemma applied to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(C_q;\mathbb{Z}) & \longrightarrow & H_1(C_q,P_{q,x};\mathbb{Z}) & \longrightarrow & H_0(P_{q,x};\mathbb{Z}) & \longrightarrow & H_0(C_q;\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_1((\mathbb{C}^*)^2;\mathbb{Z}) & \longrightarrow & H_1((\mathbb{C}^*)^2,P_{q,x};\mathbb{Z}) & \longrightarrow & H_0(P_{q,x};\mathbb{Z}) & \longrightarrow & H_0(\mathbb{C}^*;\mathbb{Z}). \end{array}$$

□

Lemma 8.12. *There is a section*

$$s : H_2(\mathcal{Y}_{q,x};\mathbb{Z}) \to H_3(\mathcal{X}_q^\vee,\mathcal{Y}_{q,x};\mathbb{Z})$$

of the boundary map $\partial : H_3(\mathcal{X}_q^\vee,\mathcal{Y}_{q,x};\mathbb{Z}) \to H_2(\mathcal{Y}_{q,x};\mathbb{Z})$*, i.e. the composition* $\partial \circ s$ *is the identity on* $H_2(\mathcal{Y}_{q,x};\mathbb{Z})$*. In particular,* ∂ *is surjective.*

Proof. Recall from Section 8.3 that there is an isomorphism $\alpha_0 : H_1(\mathbb{C}^*,P_{q,x};\mathbb{Z}) \xrightarrow{\sim} H_2(\mathcal{Y}_{q,x};\mathbb{Z})$ induced by the circle fibration π_0. Consider the inclusion

$$\mathbb{C}^* \to (\mathbb{C}^*)^2, \qquad Y \mapsto \left(-xY^{-f}, Y\right),$$

over which the inclusion $\mathcal{Y}_{q,x} \to \mathcal{X}_q^\vee$ is the induced inclusion of conic bundles. Note that the induced map $H_1(\mathbb{C}^*,P_{q,x};\mathbb{Z}) \to H_1((\mathbb{C}^*)^2,P_{q,x};\mathbb{Z})$ is injective.

Now let $\gamma_0 \in H_1(\mathbb{C}^*,P_{q,x};\mathbb{Z}) \subseteq H_1((\mathbb{C}^*)^2,P_{q,x};\mathbb{Z})$. By Lemma 8.11, γ_0 is homologous in $(\mathbb{C}^*)^2$ to a relative 1-cycle γ_0' contained in C_q relative to $P_{q,x}$. Now let γ be a 2-chain in $(\mathbb{C}^*)^2$ such that $\partial\gamma = \gamma_0 - \gamma_0'$. Then the circle fibration map

$$\pi : \{|u| = |v| : (u,v,X,Y) \in \mathcal{X}_q^\vee\} \to (\mathbb{C}^*)^2$$

lifts γ to a relative 3-chain $\pi^{-1}(\gamma)$ in $\mathcal{X}_q^\vee$ whose boundary is

$$\partial(\pi^{-1}(\gamma)) = \pi_0^{-1}(\gamma_0).$$

(Note that γ_0' is contained in the discriminant locus C_q of π.) We may then define $s(\pi_0^{-1}(\gamma_0)) := \pi^{-1}(\gamma)$. See Figure 8.1 for an illustration of the construction. $\square$

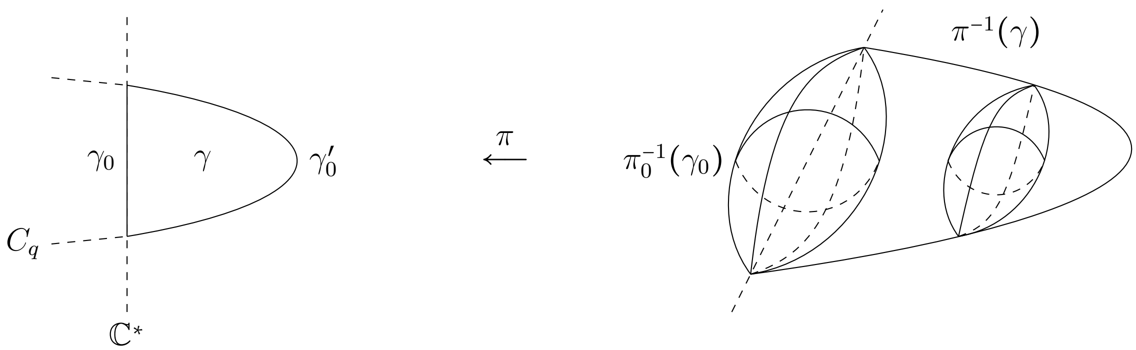

Figure 8.1: Construction of the section s in Lemma 8.12.

Proof of Proposition 8.8. Lemma 8.9 implies that (8.8) is exact at $H_3(\mathcal{X}_q^\vee;\mathbb{Z})$. Lemma 8.12 implies that (8.8) is exact at $H_2(\mathcal{Y}_{q,x};\mathbb{Z})$ and is split. $\square$

8.5 Constructing 4-cycles and matching periods

In this section, we prove Theorem 8.6. In view of Proposition 8.8, we will construct the desired isomorphism $\iota : H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x};\mathbb{Z}) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee;\mathbb{Z})$ on the direct summands $H_3(\mathcal{X}_q^\vee;\mathbb{Z})$ and $s(H_2(\mathcal{Y}_{q,x};\mathbb{Z}))$. Again we fix $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$.

Lemma 8.13. *There is an injective map*

$$\iota_1 : H_3(\mathcal{X}_q^\vee;\mathbb{Z}) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee;\mathbb{Z})$$

such that for any $\Gamma \in H_3(\mathcal{X}_q^\vee;\mathbb{Z})$*, we have*

$$\int_\Gamma \Omega_q = \frac{1}{2\pi\sqrt{-1}} \int_{\iota_1(\Gamma)} \widetilde{\Omega}_{\widetilde{q}}. \tag{8.9}$$

Proof. Recall from Section 8.2 that the projection $Z : \widetilde{\mathcal{X}}_{\widetilde{q}}^\vee \to \mathbb{C}^*$ given by the Z-coordinate can be extended to $Z = 0$, and the fiber over $Z = 0$ is equal to $\mathcal{X}_q^\vee$. Take a sufficiently small $\epsilon' > 0$ such that over $\{|Z| \leq \epsilon'\}$, the projection is a trivial $\mathcal{X}_q^\vee$-bundle. Then given $\Gamma \in H_3(\mathcal{X}_q^\vee;\mathbb{Z})$, we may extend Γ to a locally constant section, and define

$$\iota_1(\Gamma) := \Gamma \times \{|Z| = \epsilon'\} \in H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee;\mathbb{Z}).$$

Moreover, we confirm (8.9) by integrating along the Z-direction:

$$\int_{\Gamma\times\{|Z|=\epsilon'\}} \widetilde{\Omega}_{\widetilde{q}} = 2\pi\sqrt{-1}\int_\Gamma \mathrm{Res}_{Z=0}\widetilde{\Omega}_{\widetilde{q}} = 2\pi\sqrt{-1}\int_\Gamma \Omega_q.$$

The injectivity of ι_1 follows from that the corresponding periods (8.9) are linearly independent as Γ ranges through a basis for $H_3(\mathcal{X}_q^\vee;\mathbb{Z})$ (Theorem 8.4). □

Lemma 8.14. *The section* $s : H_2(\mathcal{Y}_{q,x};\mathbb{Z}) \to H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x};\mathbb{Z})$ *constructed in (the proof of) Lemma 8.12 satisfies that for any* $\Gamma_0 \in H_2(\mathcal{Y}_{q,x};\mathbb{Z})$*, we have*

$$x\frac{\partial}{\partial x}\int_{s(\Gamma_0)} \Omega_q = \int_{\Gamma_0} \Omega_{q,x}^0.$$

Proof. We reparametrize $(\mathbb{C}^*)^2$ by Y and a new coordinate

$$\bar{X} := -XY^f.$$

Note that $\frac{d\bar{X}}{\bar{X}} \wedge \frac{dY}{Y} = \frac{dX}{X} \wedge \frac{dY}{Y}$. The hypersurface $\mathcal{Y}_{q,x}$ in $\mathcal{X}_q^\vee$ is now defined by the constant equation

$$\bar{X} = x.$$

Now we fix a branch of $\log \bar{X}$ defined on the complement of a ray avoiding x. Then for any $\Gamma_0 \in H_2(\mathcal{Y}_{q,x};\mathbb{Z})$, the construction of $s(\Gamma_0)$ in the proof of Lemma 8.12 may be performed within this branch. (Note that Lemmas 8.10 and 8.11 used in the construction are valid under the pullback $\mathbb{C} \to \mathbb{C}^*$ to the universal cover in the $\bar{X}$-coordinate.) Since the boundary $\partial(s(\Gamma_0)) = \Gamma_0$ is contained in $\{\bar{X} = x\}$, or $\{\log \bar{X} = \log x\}$, we have

$$\frac{\partial}{\partial \log x}\int_{s(\Gamma_0)} \Omega_q = \int_{\Gamma_0} \mathrm{Res}_{\mathcal{X}_q^\vee \cap \{\bar{X}=x\}} \frac{du \wedge dv \wedge \frac{dY}{Y}}{H(X,Y,q)\big|_{\bar{X}=x} - uv} = \int_{\Gamma_0} \Omega^0_{q,x}.$$

□

Lemma 8.15. *There is an injective map*

$$\iota_2 : s(H_2(\mathcal{Y}_{q,x};\mathbb{Z})) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}};\mathbb{Z}),$$

where $s : H_2(\mathcal{Y}_{q,x};\mathbb{Z}) \to H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x};\mathbb{Z})$ *is the section in Lemma 8.12, such that for any* $\Gamma_0 \in H_2(\mathcal{Y}_{q,x};\mathbb{Z})$*, we have*

$$\int_{s(\Gamma_0)} \Omega_q = \frac{1}{2\pi\sqrt{-1}} \int_{\iota_2(s(\Gamma_0))} \widetilde{\Omega}_{\widetilde{q}}. \qquad (8.10)$$

Proof. Let $\Gamma_0 \in H_2(\mathcal{Y}_{q,x};\mathbb{Z})$. As in the construction of $s(\Gamma_0)$ in the proof of Lemma 8.12, we take $\gamma_0 \in H_1(\mathbb{C}^*, P_{q,x};\mathbb{Z})$ inducing Γ_0 under the circle fibration π_0, $\gamma_0' \in H_1(C_q, P_{q,x};\mathbb{Z})$ homologous to γ_0 in $(\mathbb{C}^*)^2$, and 2-chain γ in $(\mathbb{C}^*)^2$ with $\partial\gamma = \gamma_0 - \gamma_0'$. Without loss of generality, we assume that γ_0 is a smooth path with endpoints in $P_{q,x}$. Since $s(\Gamma_0) = \pi^{-1}(\gamma)$, we have

$$\int_\gamma \omega = \frac{1}{2\pi\sqrt{-1}} \int_{s(\Gamma_0)} \Omega_q.$$

See Theorem 8.7 and its proof in [23, Lemma 13].

Now, similar to the proof of Lemma 8.13, we view the pair $((\mathbb{C}^*)^2, C_q)$ in the fibration $Z : ((\mathbb{C}^*)^3, S_{\widetilde{q}}) \to \mathbb{C}^*$ as the extended fiber over $Z = 0$, and deform γ to a 3-chain $\widetilde{\gamma}_1 \cong \gamma \times \{|Z| = \epsilon'\}$ over $\{|Z| = \epsilon'\}$ for some sufficiently small $\epsilon' > 0$. This can be done such that the deformation of $\gamma_0' \subset \partial\gamma$ is contained in $S_{\widetilde{q}}$ and the deformation of $\gamma_0 \subset \partial\gamma$ has the following description:

Let $(X(t), Y(t))$, $t \in [0,1]$ be a smooth parametrization of γ_0 and choose a smooth function $r : [0,1] \to [0,1]$ such that $r(t) = 0$ only at $t = 0, 1$. Then the deformation of γ_0 is parameterized by an angle $\theta \in [0, 2\pi]$ as

$$X(t,\theta) = (1 - r(t)\epsilon' e^{\theta\sqrt{-1}})X(t), \qquad Y(t,\theta) = Y(t), \qquad Z(t,\theta) = Z_0(t,\theta) = \epsilon' e^{-\theta\sqrt{-1}}.$$

See Figure 8.2 for an illustration of the 3-chain $\widetilde{\gamma}_1$. Note that $Y(t,\theta)$ is independent of θ, and at $t = 0, 1$ the point $(X(t,\theta), Y(t,\theta))$ is an original endpoint of γ_0 contained in $P_{q,x}$. Moreover, we may choose $r(t)$ such that $H(X(t,\theta), Y(t,\theta), q) \neq 0$ whenever $t \neq 0, 1$.

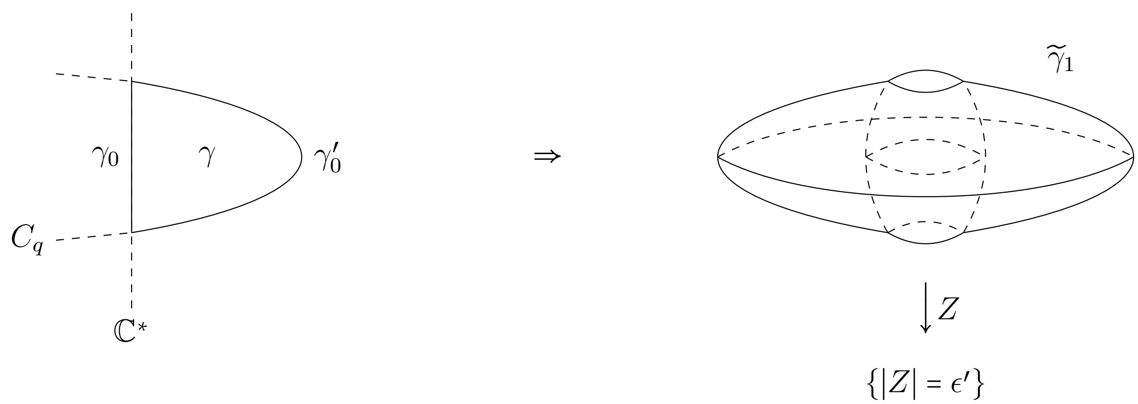

Figure 8.2: Construction of the 3-chain $\widetilde{\gamma}_1$ in Lemma 8.15.

Then, we add to $\widetilde{\gamma}_1$ a second 3-chain $\widetilde{\gamma}_2$ defined as follows: For any $t \in [0,1]$, $\widetilde{\gamma}_2$ is swept out by a homotopy between the loops $\ell_0 = (X(t,\theta), Y(t), Z_0(t,\theta))$ and $\ell_1 = (X(t,\theta), Y(t,\theta), Z_1(t,\theta))$ where

$$Z_1(t,\theta) = -\frac{H(X(t,\theta), Y(t), q)}{1 + \widetilde{q}_{R-2} X(t,\theta)^{-1} Y(t)^{-f}}.$$

Note that when $t \in (0,1)$, $(X(t,\theta), Y(t))$ is purturbed off the hypersurface $\{XY^f = -\widetilde{q}_{R-2}\}$ and the denominator above is non-zero. To further validate this definition, we first note that the limit

$$\lim_{t \to 0,1} Z_1(t,\theta)$$

is non-zero (for any θ) since the derivative $\frac{\partial H_0(Y,q,x)}{\partial Y}$ does not vanish at the endpoints $Y(0), Y(1) \in$

$P_{q,x}$ due to the regularity of H_0 (Assumption 8.3). Thus the loop ℓ_1 is defined in the limit $t \to 0, 1$. Moreover, for any $t \in (0,1)$, $X(t)Y(t)^f = -\widetilde{q}_{R-2}$ implies that

$$-\frac{H(X(t,\theta),Y(t),q)}{1+\widetilde{q}_{R-2}X(t,\theta)^{-1}Y(t)^{-f}} = -\frac{H(X(t,\theta),Y(t),q)}{1-(1-r(t)\epsilon' e^{\theta\sqrt{-1}})^{-1}} = ((r(t)\epsilon')^{-1}e^{-\theta\sqrt{-1}}-1)H(X(t,\theta),Y(t),q).$$

Since ϵ' is sufficiently small and all powers of X in $H(X,Y,q)$ (the m_i's) are non-negative, it follows that ℓ_1 has winding number -1 around $Z=0$ and is thus indeed homotopic to ℓ_0. By construction, ℓ_1 is contained in $S_{\widetilde{q}}$ for any $t \in [0,1]$. We may take the homotopy to be one in the Z-coordinate only, i.e. it fixes $(X(t,\theta),Y(t))$ for every t,θ.

We take $\widetilde{\gamma} := \widetilde{\gamma}_1 + \widetilde{\gamma}_2$. Its boundary $\partial\widetilde{\gamma}$ consists of three 2-chains: The first is the deformation of γ_0' over $\{|Z| = \epsilon'\}$ and is taken to be contained in $S_{\widetilde{q}}$. The second is the union of all loops ℓ_1 for $t \in [0,1]$, which is also contained in $S_{\widetilde{q}}$. The third is the defining homotopy of $\widetilde{\gamma}_2$ restricted to $t = 0,1$, which is contained in $P_{q,x} \times \mathbb{C}^* \subset S_{\widetilde{q}}$. Therefore, $\widetilde{\gamma}$ is a relative 3-cycle in $H_3((\mathbb{C}^*)^3, S_{\widetilde{q}}; \mathbb{Z})$, and we may define $\iota_2(s(\Gamma_0)) := \widetilde{\alpha}(\widetilde{\gamma}) = \widetilde{\pi}^{-1}(\widetilde{\gamma}) \in H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}}; \mathbb{Z})$.

We have by Theorem 8.7 that

$$\int_{\iota_2(s(\Gamma_0))} \widetilde{\Omega}_{\widetilde{q}} = 2\pi\sqrt{-1}\int_{\widetilde{\gamma}} \widetilde{\omega} = 2\pi\sqrt{-1}\int_{\widetilde{\gamma}_1} \widetilde{\omega} + 2\pi\sqrt{-1}\int_{\widetilde{\gamma}_2} \widetilde{\omega}.$$

The construction of $\widetilde{\gamma}_1$ implies that

$$2\pi\sqrt{-1}\int_{\widetilde{\gamma}_1} \widetilde{\omega} = (2\pi\sqrt{-1})^2\int_{\gamma} \omega = 2\pi\sqrt{-1}\int_{s(\Gamma_0)} \Omega_q$$

(again by Theorem 8.7). Moreover, on $\widetilde{\gamma}_2$, locally (X,Y) are the coordinates of the parameterized curve γ_0 and the defining homotopy is in the Z-coordinate only. Thus we have

$$\int_{\widetilde{\gamma}_2} \widetilde{\omega} = 0.$$

This confirms (8.10).

Finally, the injectivity of ι_2 is equivalent to that of $\iota_2 \circ s$. By Lemma 8.14, applying $x\frac{\partial}{\partial x} = \widetilde{q}_{R-2}\frac{\partial}{\partial \widetilde{q}_{R-2}}$ to the periods (8.10) gives

$$\int_{\Gamma_0} \Omega^0_{q,x},$$

which are linearly independent as Γ_0 ranges through a basis for $H_2(\mathcal{Y}_{q,x};\mathbb{Z})$ by an analog of Theorem 8.4 (see Remark 8.5). The injectivity of $\iota_2 \circ s$ thus follows. □

Proof of Theorem 8.6. By Proposition 8.8, we have

$$H_3(\mathcal{X}^\vee_q, \mathcal{Y}_{q,x};\mathbb{Z}) = H_3(\mathcal{X}^\vee_q;\mathbb{Z}) \oplus s(H_2(\mathcal{Y}_{q,x};\mathbb{Z})).$$

We define the desired map $\iota : H_3(\mathcal{X}^\vee_q, \mathcal{Y}_{q,x};\mathbb{Z}) \xrightarrow{\sim} H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}};\mathbb{Z})$ by the injective maps ι_1 and ι_2 on the direct summands defined in Lemmas 8.13 and 8.15 respectively. Note that ι is injective, since periods of cycles in the image of ι_1 are independent of $\widetilde{q}_{R-2} = x$ while periods of cycles in the image of ι_2 have non-trivial dependence on $\widetilde{q}_{R-2} = x$. Moreover, ι is surjective, since (8.4) implies that $\mathrm{rank}(H_3(\mathcal{X}^\vee_q, \mathcal{Y}_{q,x};\mathbb{Z})) = \mathrm{rank}(H_4(\widetilde{\mathcal{X}}^\vee_{\widetilde{q}};\mathbb{Z}))$. □

Example 8.16. We illustrate the discussion so far with the basic example $\mathcal{X} = \mathbb{C}^3$, $\mathcal{L}$ an outer brane, and $f = 1$, as in Section 2.6.1 and Example 7.8. Among the three linearly independent solutions to $\widetilde{\mathcal{P}}$ found in Example 7.8, the constant 1 is the period of $\widetilde{\omega}$ over the 3-cycle represented by the real 3-torus $(U(1))^3$ in $H_3((\mathbb{C}^*)^3;\mathbb{Z}) \subset H_3((\mathbb{C}^*)^3, S_{\widetilde{q}};\mathbb{Z})$, and thus the period of $\widetilde{\Omega}_{\widetilde{q}_1}$ over the corresponding 4-cycle on $\widetilde{\mathcal{X}}^\vee_{\widetilde{q}_1}$.

As for the other two solutions, we first note that $\Delta_0 = [-1,1]$ and the family $(\mathcal{Y}_x, \Omega^0_x)$ is the Hori-Vafa mirror of the toric Calabi-Yau surface $\mathrm{Tot}(K_{\mathbb{P}^1})$. At the base of the conic fibration, the set P_x is the set of two points in $\mathbb{C}^*$ defined by

$$0 = H_0(Y,x) = -xY^{-1} + Y + 1.$$

In view of Lemmas 8.14 and 8.15, we apply $\widetilde{q}_1 \frac{\partial}{\partial \widetilde{q}_1}$ to the two additional solutions

$$\log \widetilde{q}_1, \qquad \frac{1}{2}(\log \widetilde{q}_1)^2 + 2W_1^{\mathcal{X},(\mathcal{L},1)}(\widetilde{q}_1)$$

to $\widetilde{\mathcal{P}}$ to obtain

$$1, \qquad \log \widetilde{q}_1 + 2\widetilde{q}_1 \frac{\partial}{\partial \widetilde{q}_1} W_1^{\mathcal{X},(\mathcal{L},1)}(\widetilde{q}_1).$$

These are the periods of ω^0 over the following 1-cycles in $H_1(\mathbb{C}^*, P_x; \mathbb{Z})$ respectively: a loop around $Y = 0$ (based at a point in P_x); a path between the two points in P_x, and thus the periods of Ω_x^0 over the corresponding 2-cycles on $\mathcal{Y}_x$. We can also see that these are the two linearly independent solutions to the Picard-Fuchs system of $\mathrm{Tot}(K_{\mathbb{P}^1})$: the constant and the closed mirror map (see Remark 8.5).

8.6 Relation to Aganagic-Vafa B-branes

In this section, we discuss how our constructions and results in this chapter relate to those based on Aganagic-Vafa B-branes in the literature. As originally proposed by Aganagic-Vafa [5], these are complex 1-dimensional subspaces of the Hori-Vafa mirror $\mathcal{X}_q^\vee$ of $\mathcal{X}$ parameterized by the open moduli parameter x. Moreover, the superpotential of the B-brane should recover the B-model disk function of the corresponding A-brane $\mathcal{L}$ in $\mathcal{X}$, which is a first formulation of the open mirror theorem (Theorem 6.13) and is studied in [56, 41, 42]. This is also the perspective taken by Mayr [83] in the original proposal of the open/closed correspondence.

We start with the definition of the B-branes. Let $(q, x) \in \widetilde{U}$. Fix a referencing point $p_* = (X_*, Y_*) \in C_q$. Let $\mathcal{C}_* \subset \mathcal{X}_q^\vee$ be the holomorphic curve defined by the equations

$$u = H(X, Y, q) = 0, \qquad X = X_*, \qquad Y = Y_*.$$

Note that $\mathcal{C}_* \cong \mathbb{C}$ and v gives a coordinate on $\mathcal{C}_*$. Now take a point $p = (X_p, Y_p) \in P_{q,x} \subset C_q$ and a path $\gamma_p \subset C_q$ from p to the referencing point p_*. This data defines a curve $\mathcal{C}_p$ in

$$\{u = H(X, Y, q) = 0\} \subset \mathcal{X}_q^\vee$$

consisting of (v, X, Y) parameterized by $v \in \mathbb{C}$ as follows: The coordinates (X, Y) only depends on the radius $|v|$. As $|v|$ varies from 0 to a fixed, sufficiently large threshold $\Lambda >> 0$, $(X(|v|), Y(|v|))$ traces out the path γ_p. When $|v| \geq \Lambda$, $(X(|v|), Y(|v|))$ stays constantly at p_*. Note that $\mathcal{C}_p$ is homeomorphic to $\mathbb{C}$ but is in general not holomorphic. See Figure 8.3 for an illustration of the B-branes.

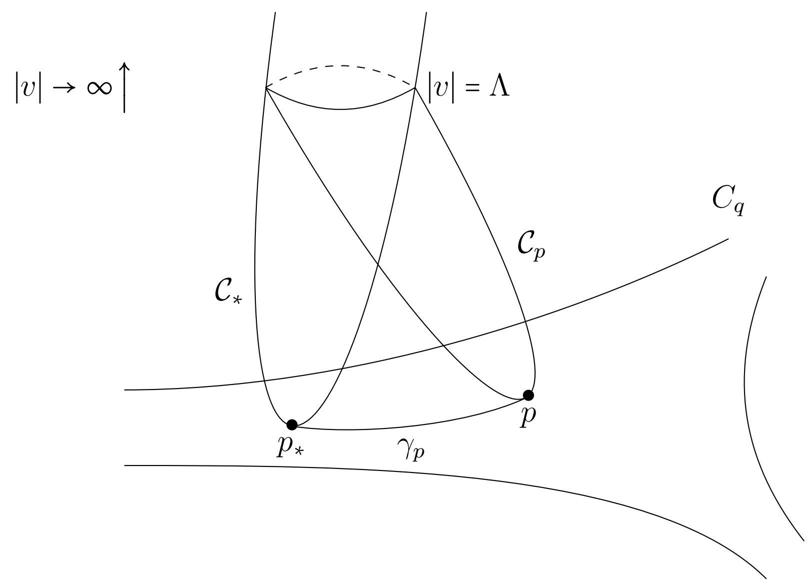

Figure 8.3: Aganagic-Vafa B-branes $\mathcal{C}_*$ and $\mathcal{C}_p$.

Fix a branch of $\log Y$ that contains p_* and $P_{q,x}$ and assume that the path γ_p is chosen within this branch. The *superpotential* of the brane $\mathcal{C}_p$ is an integral of a 2-form over all of $\mathcal{C}_p$ and is reduced, in the radial direction of v, to the line integral

$$W(\mathcal{C}_p) = 2\pi\sqrt{-1}\int_{p_*}^{p} \log Y \frac{d\bar{X}}{\bar{X}}$$

along γ_p in C_q. Here we use the coordinate $\bar{X} = -XY^f$ as in the proof of Lemma 8.14. Similarly, we have $W(\mathcal{C}_*) = 0$ for the referencing brane $\mathcal{C}_*$. Then the open mirror theorem, in the case where the A-brane $\mathcal{L}$ is effective ($\mathfrak{m} = 1$), states that for an appropriate choice of p and γ_p,

$$\left(x\frac{\partial}{\partial x}\right)^2 W^{\mathcal{X},\mathcal{L},f}(q,x) = \frac{1}{2\pi\sqrt{-1}} x\frac{\partial}{\partial x}\left(W(\mathcal{C}_p) - W(\mathcal{C}_*)\right) = x\frac{\partial}{\partial x}\int_{p_*}^{p} \log Y \frac{d\bar{X}}{\bar{X}} = \log Y_p(x). \tag{8.11}$$

See [42] for precise statements that also cover ineffective A-branes in general.

The difference

$$W(\mathcal{C}_p) - W(\mathcal{C}_*)$$

can be viewed as an integral on the 2-cycle Γ_0' in $\mathcal{C}_p \cup \mathcal{C}_*$ defined by $|v| \le \Lambda$ (which generates $H_2(\mathcal{C}_p \cup \mathcal{C}_*; \mathbb{Z})$). Moreover, Γ_0' bounds a relative 3-cycle Γ' in $\mathcal{X}_q^\vee$ consisting of all the points (v', X, Y) such that $(X,Y) \in \gamma_p$ and $|v'|$ is upper-bounded by the radius of v on $\mathcal{C}_p$. Then we may write

$$W(\mathcal{C}_p) - W(\mathcal{C}_*) = \int_{\Gamma'} \Omega_q.$$

Indeed, Mayr [83] proposed a map

$$H_3(\mathcal{X}_q^\vee, \mathcal{C}_p \cup \mathcal{C}_*; \mathbb{Z}) \to H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee; \mathbb{Z})$$

under which the periods are related in a way similar to our Theorem 8.6.

We now relate the above constructions to ours based on the complex hypersurface $\mathcal{Y}_{q,x}$ in $\mathcal{X}_q^\vee$. Let $p_1, p_2 \in P_{q,x}$ be distinct points and choose a path γ_0 in $\mathbb{C}^*$ from p_1 to p_2 passing through the referencing Y-coordinate Y_*. The relative 1-cycle $\gamma_0 \in H_1(\mathbb{C}^*, P_{q,x}; \mathbb{Z})$ induces a 2-cycle $\Gamma_0 = \alpha_0(\gamma_0) \in H_2(\mathcal{Y}_{q,x}; \mathbb{Z})$, which bounds a relative 3-cycle $\Gamma = s(\Gamma_0) \in H_3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{Z})$ (Lemma 8.12). Our Theorem 8.6 (and particularly Lemma 8.15) finds a 4-cycle $\widetilde{\Gamma} \in H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee; \mathbb{Z})$ such that

$$\frac{1}{2\pi\sqrt{-1}} \widetilde{q}_{R-2} \frac{\partial}{\partial \widetilde{q}_{R-2}} \int_{\widetilde{\Gamma}} \widetilde{\Omega}_{\widetilde{q}} = x\frac{\partial}{\partial x}\int_{\Gamma} \Omega_q = \int_{\Gamma_0} \Omega_{q,x}^0 = 2\pi\sqrt{-1}\int_{p_1}^{p_2} \frac{dY}{Y}.$$

We may assume that the above construction may be performed within our choice of branch for

$\log Y$. (Note that Lemmas 8.10 and 8.11 used in the construction are valid under the pullback $\mathbb{C} \to \mathbb{C}^*$ to the universal cover in the Y-coordinate.) This gives

$$\frac{1}{(2\pi\sqrt{-1})^2}\left(\widetilde{q}_{R-2}\frac{\partial}{\partial\widetilde{q}_{R-2}}\right)^2\int_{\widetilde{\Gamma}}\widetilde{\Omega}_{\widetilde{q}} = \frac{1}{2\pi\sqrt{-1}}\left(x\frac{\partial}{\partial x}\right)^2\int_{\Gamma}\Omega_q = \log Y_{p_2}(x) - \log Y_{p_1}(x).$$

On the other hand, by Lemma 8.11, we lift the path γ_0 to a path γ_0' in C_q from p_1 to p_2 passing through p_*, and use the two segments of γ_0' to define B-branes $\mathcal{C}_{p_1}$ and $\mathcal{C}_{p_2}$. Similar to above, let Γ_0' be the 2-cycle in $\mathcal{C}_{p_1} \cup \mathcal{C}_{p_2}$ defined by $|v| \le \Lambda$, which generates $H_2(\mathcal{C}_{p_1} \cup \mathcal{C}_{p_2}; \mathbb{Z})$ and bounds a relative 3-cycle $\Gamma' \in H_3(\mathcal{X}_q^\vee, \mathcal{C}_{p_1} \cup \mathcal{C}_{p_2}; \mathbb{Z})$. Our earlier discussion gives

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}x\frac{\partial}{\partial x}\int_{\Gamma'}\Omega_q = \frac{1}{2\pi\sqrt{-1}}x\frac{\partial}{\partial x}\left(W(\mathcal{C}_{p_2}) - W(\mathcal{C}_{p_1})\right) &= \log Y_{p_2}(x) - \log Y_{p_1}(x)\\ &= \frac{1}{(2\pi\sqrt{-1})^2}\left(\widetilde{q}_{R-2}\frac{\partial}{\partial\widetilde{q}_{R-2}}\right)^2\int_{\widetilde{\Gamma}}\widetilde{\Omega}_{\widetilde{q}}.\end{aligned} \tag{8.12}$$

In particular, we obtain a description of the $\widetilde{q}_{R-2}$-dependence of the power series part of the period $\int_{\widetilde{\Gamma}}\widetilde{\Omega}_{\widetilde{q}}$ in terms of superpotentials of the B-branes. In certain cases, for instance when p_1 and p_2 correspond to neighboring effective A-branes on $\mathcal{X}$, version (8.11) of the open mirror theorem relates the difference of B-brane superpotentials $W(\mathcal{C}_{p_2}) - W(\mathcal{C}_{p_1})$ to the difference of B-model disk functions of the corresponding A-branes. Then the description (8.12) is consistent with Proposition 7.5 which describes the $\widetilde{q}_{R-2}$-dependence of the power series part of certain solutions to $\widetilde{\mathcal{P}}$ in terms of disk functions of the A-branes.

Chapter 9: Mixed Hodge structures

In this chapter, we study the mixed Hodge structures (MHS) associated to the mirror families in Chapter 8 and extablish a version of the open/closed correspondence at the level of MHS (Theorem 9.2 and Corollary 9.3).

9.1 Vector spaces from Laurent polynomials

Recall that in Section 8.1 we defined families of Laurent polynomials

$$H(X,Y,q), \qquad \widetilde{H}(X,Y,Z,\widetilde{q}), \qquad H_0(Y,q,x)$$

whose Newton polyhedra are $\Delta, \widetilde{\Delta}, \Delta_0$ respectively. Over the parameter domain $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$, these polynomials are regular with respect to their Newton polyhedra (Assumption 8.3). We now recall the construction of Batyrev [11] of graded $\mathbb{C}$-vector spaces associated to these polynomials.

Let

$$\mathcal{S}_\Delta, \qquad \mathcal{S}_{\widetilde{\Delta}}, \qquad \mathcal{S}_{\Delta_0}$$

be the subring of

$$\mathbb{C}[X_0, X^{\pm 1}, Y^{\pm 1}], \qquad \mathbb{C}[X_0, X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}], \qquad \mathbb{C}[X_0, Y^{\pm 1}]$$

generated as a $\mathbb{C}$-vector space by monomials

$$\left\{X_0^k X^a Y^b : (a,b) \in k\Delta\right\}, \qquad \left\{X_0^k X^a Y^b Z^c : (a,b,c) \in k\widetilde{\Delta}\right\}, \qquad \left\{X_0^k Y^b : b \in k\Delta_0\right\}$$

respectively. Here we take the convention that the unit 1 is included in the sets of monomials above by $k = 0$. The subrings are graded by the degree of the variable X_0, i.e.

$$\deg X_0^k X^a Y^b = \deg X_0^k X^a Y^b Z^c = \deg X_0^k Y^b = k.$$

On $\mathcal{S}_\Delta$, consider the following differential operators defined by $H(X, Y, q)$:

$$\mathcal{D}_0 := X_0 \frac{\partial}{\partial X_0} + X_0 H(X, Y, q), \quad \mathcal{D}_X := X \frac{\partial}{\partial X} + X_0 X \frac{\partial H(X, Y, q)}{\partial X}, \quad \mathcal{D}_Y := Y \frac{\partial}{\partial Y} + X_0 Y \frac{\partial H(X, Y, q)}{\partial Y}.$$

Similarly, on $\mathcal{S}_{\widetilde{\Delta}}$, consider the following differential operators defined by $\widetilde{H}(X, Y, Z, \widetilde{q})$:

$$\widetilde{\mathcal{D}}_0 := X_0 \frac{\partial}{\partial X_0} + X_0 \widetilde{H}(X, Y, Z, \widetilde{q}), \qquad \widetilde{\mathcal{D}}_X := X \frac{\partial}{\partial X} + X_0 X \frac{\partial \widetilde{H}(X, Y, Z, \widetilde{q})}{\partial X},$$
$$\widetilde{\mathcal{D}}_Y := Y \frac{\partial}{\partial Y} + X_0 Y \frac{\partial \widetilde{H}(X, Y, Z, \widetilde{q})}{\partial Y}, \qquad \widetilde{\mathcal{D}}_Z := Z \frac{\partial}{\partial Z} + X_0 Z \frac{\partial \widetilde{H}(X, Y, Z, \widetilde{q})}{\partial Z}.$$

Finally, on $\mathcal{S}_{\Delta_0}$, consider the following differential operators defined by $H_0(Y, q, x)$:

$$\mathcal{D}_0^0 := X_0 \frac{\partial}{\partial X_0} + X_0 H_0(Y, q, x), \qquad \mathcal{D}_Y^0 := Y \frac{\partial}{\partial Y} + X_0 Y \frac{\partial H_0(Y, q, x)}{\partial Y}.$$

Then, define

$$\mathcal{R}_H := \mathcal{S}_\Delta \big/ (\mathcal{D}_0 \mathcal{S}_\Delta + \mathcal{D}_X \mathcal{S}_\Delta + \mathcal{D}_Y \mathcal{S}_\Delta)),$$
$$\mathcal{R}_{\widetilde{H}} := \mathcal{S}_{\widetilde{\Delta}} \big/ (\widetilde{\mathcal{D}}_0 \mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_X \mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_Y \mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_Z \mathcal{S}_{\widetilde{\Delta}})),$$
$$\mathcal{R}_{H_0} := \mathcal{S}_{\Delta_0} \big/ (\mathcal{D}_0^0 \mathcal{S}_{\Delta_0} + \mathcal{D}_Y^0 \mathcal{S}_{\Delta_0})).$$

It holds that

$$\dim_{\mathbb{C}} \mathcal{R}_H = \mathrm{Vol}(\Delta), \qquad \dim_{\mathbb{C}} \mathcal{R}_{\widetilde{H}} = \mathrm{Vol}(\widetilde{\Delta}), \qquad \dim_{\mathbb{C}} \mathcal{R}_{H_0} = \mathrm{Vol}(\Delta_0). \tag{9.1}$$

9.2 MHS of affine hypersurfaces in algebraic tori

Batyrev [11] showed that the $\mathbb{C}$-vector spaces $\mathcal{R}_H$, $\mathcal{R}_{\widetilde{H}}$, $\mathcal{R}_{H_0}$ are endowed with two filtrations such that the induced MHS can be used to describe the MHS (in the sense of Deligne [34, 35]) arising from the affine hypersurfaces C_q, $S_{\widetilde{q}}$, $P_{q,x}$ in algebraic tori defined by the Laurent polynomials. We review the filtrations and relevant results on MHS in this section, following [67].

The two filtrations on $\mathcal{R}_H$, $\mathcal{R}_{\widetilde{H}}$, $\mathcal{R}_{H_0}$ will descend from two filtrations on $\mathcal{S}_\Delta$, $\mathcal{S}_{\widetilde{\Delta}}$, $\mathcal{S}_{\Delta_0}$ respectively. First, the *$\mathcal{E}$-filtration* is a decreasing filtration

$$\cdots \supseteq \mathcal{E}^{-k} \supseteq \cdots \supseteq \mathcal{E}^{-1} \supseteq \mathcal{E}^0 = \mathbb{C}1 \supset \mathcal{E}^{-1} = 0$$

where $\mathcal{E}^{-k}$ is the $\mathbb{C}$-vector subspace spanned by monomials of degree $\leq k$. It holds that

$$\mathcal{E}^{-2}\mathcal{R}_H = \mathcal{R}_H, \qquad \mathcal{E}^{-3}\mathcal{R}_{\widetilde{H}} = \mathcal{R}_{\widetilde{H}}, \qquad \mathcal{E}^{-1}\mathcal{R}_{H_0} = \mathcal{R}_{H_0}.$$

Second, the *$\mathcal{I}$-filtration* is an increasing filtration defined as follows: In $\mathcal{S}_\Delta$, for $0 \leq l \leq 3$, let $\mathcal{I}_l\mathcal{S}_\Delta$ be the homogenous ideal generated as a $\mathbb{C}$-vector subspace by monomials $X_0^k X^a Y^b Z^c$ such that (a, b, c) does not belong to any codimension-l face of $k\Delta$. Note that $\mathcal{I}_3\mathcal{S}_\Delta$ is generated by all monomials with positive degrees. We set $\mathcal{I}_4\mathcal{S}_\Delta = \mathcal{S}_\Delta$. Descending to $\mathcal{R}_H$, and applying the same construction to $\mathcal{R}_{\widetilde{H}}$ and $\mathcal{R}_{H_0}$, we have

$$0 = \mathcal{I}_0\mathcal{R}_H \subseteq \cdots \subseteq \mathcal{I}_3\mathcal{R}_H \subset \mathcal{I}_4\mathcal{R}_H = \mathcal{R}_H;$$

$$0 = \mathcal{I}_0\mathcal{R}_{\widetilde{H}} \subseteq \cdots \subseteq \mathcal{I}_4\mathcal{R}_{\widetilde{H}} \subset \mathcal{I}_5\mathcal{R}_{\widetilde{H}} = \mathcal{R}_{\widetilde{H}};$$

$$0 = \mathcal{I}_0\mathcal{R}_{H_0} \subseteq \mathcal{I}_1\mathcal{R}_{H_0} \subseteq \mathcal{I}_2\mathcal{R}_{H_0} \subset \mathcal{I}_3\mathcal{R}_{H_0} = \mathcal{R}_{H_0}.$$

As observed by Stienstra [91, Theorem 7] and Konishi-Minabe [67, Theorem 4.2], the results of Batyrev [11] give isomorphisms between $\mathcal{R}_H, \mathcal{R}_{\widetilde{H}}, \mathcal{R}_{H_0}$ and

$$\mathcal{H} := H^2((\mathbb{C}^*)^2, C_q; \mathbb{C}), \qquad \widetilde{\mathcal{H}} := H^3((\mathbb{C}^*)^3, S_{\widetilde{q}}; \mathbb{C}), \qquad \mathcal{H}_0 := H^1(\mathbb{C}^*, P_{q,x}; \mathbb{C})$$

respectively under which the $\mathcal{E}$- and $\mathcal{I}$-filtrations correspond to the Hodge filtrations $\mathcal{F}^\bullet$ and weight filtrations $\mathcal{W}_\bullet$ respectively.

Theorem 9.1 ([11, 91, 67])**.** *There are isomorphisms*

$$\rho : \mathcal{R}_H \xrightarrow{\sim} \mathcal{H}, \qquad \widetilde{\rho} : \mathcal{R}_{\widetilde{H}} \xrightarrow{\sim} \widetilde{\mathcal{H}}, \qquad \rho_0 : \mathcal{R}_{H_0} \xrightarrow{\sim} \mathcal{H}_0$$

under which the following hold:

- *We have*

  $$\rho(1) = [\omega], \qquad \widetilde{\rho}(1) = [\widetilde{\omega}], \qquad \rho_0(1) = [\omega^0]$$

  where the forms $\omega, \widetilde{\omega}, \omega^0$ are defined in (8.6)*.*

- *For any k, we have*

  $$\rho(\mathcal{E}^{-k}\mathcal{R}_H) = \mathcal{F}^{-k+2}\mathcal{H}, \qquad \widetilde{\rho}(\mathcal{E}^{-k}\mathcal{R}_{\widetilde{H}}) = \mathcal{F}^{-k+3}\widetilde{\mathcal{H}}, \qquad \rho_0(\mathcal{E}^{-k}\mathcal{R}_{H_0}) = \mathcal{F}^{-k+1}\mathcal{H}_0.$$

- *For any l, we have*

  $$\rho(\mathcal{I}_l\mathcal{R}_H) = \mathcal{W}_l\mathcal{H}, \qquad \widetilde{\rho}(\mathcal{I}_l\mathcal{R}_{\widetilde{H}}) = \mathcal{W}_{l+1}\widetilde{\mathcal{H}}, \qquad \rho_0(\mathcal{I}_l\mathcal{R}_{H_0}) = \mathcal{W}_{l-1}\mathcal{H}_0.$$

9.3 Correspondence of MHS

Our main result of the chapter is the that the MHS $\mathcal{R}_{\widetilde{H}}$ is an extension of $\mathcal{R}_H$ by $\mathcal{R}_{H_0}$. This gives the open/closed correspondence on the level of MHS.

Theorem 9.2. *For* $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$*, there is a short exact sequence of MHS*

$$0 \longrightarrow \mathcal{R}_{H_0} \xrightarrow{\iota} \mathcal{R}_{\widetilde{H}} \xrightarrow{\pi} \mathcal{R}_H \longrightarrow 0 \tag{9.2}$$

in the sense that for any k*,* l*,*

$$\iota(\mathcal{E}^{-k}\mathcal{R}_{H_0}) \subseteq \mathcal{E}^{-k-1}\mathcal{R}_{\widetilde{H}}, \qquad \pi(\mathcal{E}^{-k}\mathcal{R}_{\widetilde{H}}) \subseteq \mathcal{E}^{-k}\mathcal{R}_H,$$
$$\iota(\mathcal{I}_l\mathcal{R}_{H_0}) \subseteq \mathcal{I}_{l+1}\mathcal{R}_{\widetilde{H}}, \qquad \pi(\mathcal{I}_l\mathcal{R}_{\widetilde{H}}) \subseteq \mathcal{I}_{l-1}\mathcal{R}_H.$$

In view of Theorem 9.1, Theorem 9.2 directly translates into the following result.

Corollary 9.3. *For* $(q,x) = \widetilde{q} \in \widetilde{U}_\epsilon$*, there is a short exact sequence of MHS*

$$0 \longrightarrow \mathcal{H}_0 \xrightarrow{\iota} \widetilde{\mathcal{H}} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0 \tag{9.3}$$

in the sense that for any k*,* l*,*

$$\iota(\mathcal{F}^k\mathcal{H}_0) \subseteq \mathcal{F}^{k+1}\widetilde{\mathcal{H}}, \qquad \pi(\mathcal{F}^k\widetilde{\mathcal{H}}) \subseteq \mathcal{F}^{k-1}\mathcal{H},$$
$$\iota(\mathcal{W}_l\mathcal{H}_0) \subseteq \mathcal{W}_{l+3}\widetilde{\mathcal{H}}, \qquad \pi(\mathcal{W}_l\widetilde{\mathcal{H}}) \subseteq \mathcal{W}_{l-2}\mathcal{H}.$$

Remark 9.4. Equation (9.3) can be viewed as a dual statement to the short exact sequence of homology

$$0 \to H_3(\mathcal{X}_q^\vee;\mathbb{Z}) \to H_4(\widetilde{\mathcal{X}}_{\widetilde{q}}^\vee;\mathbb{Z}) \to H_2(\mathcal{Y}_{q,x};\mathbb{Z}) \to 0$$

established in Theorem 8.6 and Proposition 8.8. Indeed, Konishi-Minabe [67] showed that if Δ is reflexive,[1] there is an isomorphism $H^3(\mathcal{X}_q^\vee;\mathbb{C}) \cong \mathcal{R}_H \cong \mathcal{H} = H^2((\mathbb{C}^*)^2, C_q;\mathbb{C})$ that respects the filtrations with appropriate degree shifts. One may attempt to extend this result to polyhedra of arbitrary dimensions and that are not necessarily reflexive, thereby obtaining an analog of Corollary

[1] For the 2-dimensional polyhedron Δ, this means that $\vec{0}$ is in the interior of Δ and the distance between $\vec{0}$ and the line generated by each codimension-1 face of Δ is 1. See [12, Section 4] for a more general and precise definition.

9.3 for the MHS of Hori-Vafa mirrors. In particular, one may obtain a description of the MHS on the relative cohomology $H^3(\mathcal{X}_q^\vee, \mathcal{Y}_{q,x}; \mathbb{C})$ and compare with the results of [74].

To prove Theorem 9.2, we define and study the maps ι, π in (9.2) separately as follows.

Lemma 9.5. *There is a surjective $\mathbb{C}$-linear map*

$$\pi : \mathcal{R}_{\widetilde{H}} \to \mathcal{R}_H$$

such that for any k, l,

$$\pi(\mathcal{E}^{-k}\mathcal{R}_{\widetilde{H}}) \subseteq \mathcal{E}^{-k}\mathcal{R}_H, \qquad \pi(\mathcal{I}_l\mathcal{R}_{\widetilde{H}}) \subseteq \mathcal{I}_{l-1}\mathcal{R}_H.$$

Proof. By construction, the last coordinate of any point in $\widetilde{\Delta}$ is non-negative. Therefore, $\mathcal{S}_{\widetilde{\Delta}}$ is a subring of $\mathbb{C}[X_0, X^{\pm 1}, Y^{\pm 1}, Z]$. We then define

$$\pi : \mathcal{S}_{\widetilde{\Delta}} \to \mathcal{S}_\Delta, \qquad Z \mapsto 0.$$

Note that this definition is valid because Δ is the facet of $\widetilde{\Delta}$ where the last coordinate is 0. This also implies that for any k, l,

$$\pi(\mathcal{E}^{-k}\mathcal{S}_{\widetilde{\Delta}}) \subseteq \mathcal{E}^{-k}\mathcal{S}_\Delta, \qquad \pi(\mathcal{I}_l\mathcal{S}_{\widetilde{\Delta}}) \subseteq \mathcal{I}_{l-1}\mathcal{S}_\Delta.$$

Moreover, under the restriction $Z = 0$, the operator $\widetilde{\mathcal{D}}_Z$ is trivial, while (8.1) implies that $\widetilde{\mathcal{D}}_0$, $\widetilde{\mathcal{D}}_X$, $\widetilde{\mathcal{D}}_Y$ restrict to $\mathcal{D}_0$, $\mathcal{D}_X$, $\mathcal{D}_Y$ respectively. Thus π descends to the desired map from $\mathcal{R}_{\widetilde{H}}$ to $\mathcal{R}_H$. □

Lemma 9.6. *There is a $\mathbb{C}$-linear map*

$$\iota : \mathcal{R}_{H_0} \to \mathcal{R}_{\widetilde{H}}$$

such that:

- *For any k, l, we have*

$$\iota(\mathcal{E}^{-k}\mathcal{R}_{H_0}) \subseteq \mathcal{E}^{-k-1}\mathcal{R}_{\widetilde{H}}, \qquad \iota(\mathcal{I}_l\mathcal{R}_{H_0}) \subseteq \mathcal{I}_{l+1}\mathcal{R}_{\widetilde{H}},$$

- *The image of ι consists of all elements in the span of the monomials $X_0^kX^aY^bZ^c$ with $c \geq 1$.*

Proof. We start by defining a $\mathbb{C}$-linear map $\iota : \mathcal{S}_{\Delta_0} \to \mathcal{S}_{\widetilde{\Delta}}$. First we set

$$\iota(1) = X_0Z$$

which corresponds to the element $(0,0,1)$ (or $\widetilde{b}_{R+2}$) in $\widetilde{\Delta}$. Now consider $X_0Y^{b_0}$ where $b_0 \in \Delta_0 \cap \mathbb{Z}$. Recall from Sections 7.1 and 8.1 that Δ_0 is the union of intervals

$$[n_{i_3}(\widetilde{\sigma}^s) - fm_{i_3}(\widetilde{\sigma}^s), n_{i_2}(\widetilde{\sigma}^s) - fm_{i_2}(\widetilde{\sigma}^s)], \qquad s = 1, \ldots, S,$$

whose length is $|G_{\widetilde{\sigma}^s}|$ (see Lemma 2.3). Take s such that the above inverval contains b_0, and we may write

$$b_0 = \frac{c_0}{|G_{\widetilde{\sigma}^s}|}\left(n_{i_3}(\widetilde{\sigma}^s) - fm_{i_3}(\widetilde{\sigma}^s)\right) + \frac{|G_{\widetilde{\sigma}^s}| - c_0}{|G_{\widetilde{\sigma}^s}|}\left(n_{i_2}(\widetilde{\sigma}^s) - fm_{i_2}(\widetilde{\sigma}^s)\right)$$

for some $c_0 \in \{0, \ldots, |G_{\widetilde{\sigma}^s}|\}$. Then we set

$$\iota(X_0Y^{b_0}) = (-x)^{-a}X_0^2X^aY^bZ$$

where

$$a = \frac{|G_{\widetilde{\sigma}^s}| - c_0}{|G_{\widetilde{\sigma}^s}|}m_{i_2}(\widetilde{\sigma}^s) + \frac{c_0}{|G_{\widetilde{\sigma}^s}|}m_{i_3}(\widetilde{\sigma}^s), \qquad b = \frac{|G_{\widetilde{\sigma}^s}| - c_0}{|G_{\widetilde{\sigma}^s}|}n_{i_2}(\widetilde{\sigma}^s) + \frac{c_0}{|G_{\widetilde{\sigma}^s}|}n_{i_3}(\widetilde{\sigma}^s).$$

Note that $(a, b, 1)$ corresponds to c_0 times the generator presented in Lemma 2.3. When $c_0 = 0$ (resp. $c_0 = |G_{\widetilde{\sigma}^s}|$), $\left(\frac{a}{2}, \frac{b}{2}, \frac{1}{2}\right)$ is the the middle point of the edge between $\widetilde{b}_{i_2(\widetilde{\sigma}^s)}$ (resp. $\widetilde{b}_{i_3(\widetilde{\sigma}^s)}$) and $\widetilde{b}_{R+2}$; otherwise $\left(\frac{a}{2}, \frac{b}{2}, \frac{1}{2}\right)$ lies in the interior of the cone $\widetilde{\sigma}^s$. In particular, $\left(\frac{a}{2}, \frac{b}{2}, \frac{1}{2}\right)$ lies in the interior of $\widetilde{\Delta}$ unless b_0 is one of the two boundary points $n_{i_3}(\widetilde{\sigma}^S) - fm_{i_3}(\widetilde{\sigma}^S)$ or $n_{i_2}(\widetilde{\sigma}^1) - fm_{i_2}(\widetilde{\sigma}^1)$ of Δ_0, in which case $\left(\frac{a}{2}, \frac{b}{2}, \frac{1}{2}\right)$ is the middle point of the edge between $\widetilde{b}_{R+2}$ and $\widetilde{b}_{i_2(\widetilde{\sigma}^1)}$ or $\widetilde{b}_{i_3(\widetilde{\sigma}^S)}$.

In general, consider $X_0^k Y^{b_0}$ where $\frac{b_0}{k} \in [n_{i_3}(\widetilde{\sigma}^s) - fm_{i_3}(\widetilde{\sigma}^s), n_{i_2}(\widetilde{\sigma}^s) - fm_{i_2}(\widetilde{\sigma}^s)]$ for some $s \in \{1, \ldots, S\}$. We may write

$$b_0 = c_3\left(n_{i_3}(\widetilde{\sigma}^s) - fm_{i_3}(\widetilde{\sigma}^s)\right) + c_2\left(n_{i_2}(\widetilde{\sigma}^s) - fm_{i_2}(\widetilde{\sigma}^s)\right)$$

for $c_2, c_3 \in \mathbb{Q}_{\geq 0}$ with $c_2 + c_3 = k$. Then we set

$$\iota(X_0^k Y^{b_0}) = (-x)^{-a} X_0^{k+1} X^a Y^b Z$$

where

$$a = c_2 m_{i_2}(\widetilde{\sigma}^s) + c_3 m_{i_3}(\widetilde{\sigma}^s), \qquad b = c_2 n_{i_2}(\widetilde{\sigma}^s) + c_3 n_{i_3}(\widetilde{\sigma}^s).$$

Note that this recovers the definitions for $k = 0, 1$ above. Moreover, we have

$$b_0 = b - fa. \tag{9.4}$$

It follows from the construction that for any k, l,

$$\iota(\mathcal{E}^{-k}\mathcal{S}_{\Delta_0}) \subseteq \mathcal{E}^{-k-1}\mathcal{S}_{\widetilde{\Delta}}, \qquad \iota(\mathcal{I}_l \mathcal{S}_{\Delta_0}) \subseteq \mathcal{I}_{l+1}\mathcal{S}_{\widetilde{\Delta}}.$$

Now we verify that ι descends to a map $\mathcal{R}_{H_0} \to \mathcal{R}_{\widetilde{H}}$. Let $X_0^k Y^{b_0} \in \mathcal{S}_{\Delta_0}$ mapping to $X_0^{k+1} X^a Y^b Z$ as above. We compute that

$$\mathcal{D}_0^0(X_0^k Y^{b_0}) = k X_0^k Y^{b_0} + H_0(Y, q, x) X_0^{k+1} Y^{b_0} = k X_0^k Y^{b_0} + \sum_{i=1}^R s_i(q)(-x)^{m_i} X_0^{k+1} Y^{n_i - fm_i + b_0},$$

and

$$\iota(\mathcal{D}_0^0(X_0^k Y^{b_0})) = k(-x)^{-a} X_0^{k+1} X^a Y^b Z + \sum_{i=1}^R s_i(q)(-x)^{m_i} \iota(X_0^{k+1} Y^{n_i - fm_i + b_0}).$$

We now compare this to

$$
\begin{aligned}
(-x)^{-a}\widetilde{\mathcal{D}}_0(X_0^{k+1}X^aY^bZ) =&(k+1)(-x)^{-a}X_0^{k+1}X^aY^bZ + (-x)^{-a}\widetilde{H}(X,Y,Z,\widetilde{q})X_0^{k+2}X^aY^bZ \\
=&(k+1)(-x)^{-a}X_0^{k+1}X^aY^bZ + (-x)^{-a}(xX^{-1}Y^{-f}+1)X_0^{k+2}X^aY^bZ^2 \\
&+\sum_{i=1}^{R} s_i(q)(-x)^{-a}X_0^{k+2}X^{m_i+a}Y^{n_i+b}Z.
\end{aligned}
$$

We show below that the difference

$$
\begin{aligned}
&(-x)^{-a}\widetilde{\mathcal{D}}_0(X_0^{k+1}X^aY^bZ) - \iota(\mathcal{D}_0^0(X_0^kY^{b_0})) \\
&=(-x)^{-a}\left(X_0^{k+1}X^aY^bZ + (xX^{-1}Y^{-f}+1)X_0^{k+2}X^aY^bZ^2\right) \\
&+\sum_{i=1}^{R} s_i(q)\left((-x)^{-a}X_0^{k+2}X^{m_i+a}Y^{n_i+b}Z - (-x)^{m_i}\iota(X_0^{k+1}Y^{n_i-fm_i+b_0})\right)
\end{aligned}
$$

is contained in $\widetilde{\mathcal{D}}_Z\mathcal{S}_{\widetilde{\Delta}}$, where recall that

$$
\widetilde{\mathcal{D}}_Z = Z\frac{\partial}{\partial Z} + X_0Z\frac{\partial \widetilde{H}(X,Y,Z,\widetilde{q})}{\partial Z} = Z\frac{\partial}{\partial Z} + (xX^{-1}Y^{-f}+1)X_0Z.
$$

First, we have

$$
\widetilde{\mathcal{D}}_Z(X_0^{k+1}X^aY^bZ) = X_0^{k+1}X^aY^bZ + (xX^{-1}Y^{-f}+1)X_0^{k+2}X^aY^bZ^2.
$$

Now for each $i = 1,\dots,R$, we have

$$
\iota(X_0^{k+1}Y^{n_i-fm_i+b_0}) = (-x)^{a_i'}X_0^{k+2}X^{a_i'}Y^{b_i'}Z
$$

for some a_i', b_i' satisfying that

$$
b_i' - fa_i' = n_i - fm_i + b_0 = (n_i+b) - f(m_i+a)
$$

(see (9.4)). Now note that for any k, a', b',

$$\widetilde{\mathcal{D}}_Z(X_0^{k+1}X^{a'}Y^{b'}) = (xX^{-1}Y^{-f}+1)X_0^{k+2}X^{a'}Y^{b'}Z = X_0^{k+2}X^{a'}Y^{b'}Z - (-x)X_0^{k+2}X^{a'-1}Y^{b'-f}Z.$$

Applying this repeatedly as (a', b') ranges through the integral points on the segment between (m_i+a, n_i+b) and (a'_i, b'_i), we see that

$$(-x)^{-a}X_0^{k+2}X^{m_i+a}Y^{n_i+b}Z - (-x)^{m_i-a'_i}X_0^{k+2}X^{a'_i}Y^{b'_i}Z \in \widetilde{\mathcal{D}}_Z\mathcal{S}_{\widetilde{\Delta}}.$$

Therefore, we have verified that

$$(-x)^{-a}\widetilde{\mathcal{D}}_0(X_0^{k+1}X^aY^bZ) - \iota(\mathcal{D}_0^0(X_0^kY^{b_0})) \in \widetilde{\mathcal{D}}_Z\mathcal{S}_{\widetilde{\Delta}}.$$

A similar computation shows that

$$(-x)^{-a}(\widetilde{\mathcal{D}}_Y - f\widetilde{\mathcal{D}}_X)(X_0^{k+1}X^aY^bZ) - \iota(\mathcal{D}_Y^0(X_0^kY^{b_0})) \in \widetilde{\mathcal{D}}_Z\mathcal{S}_{\widetilde{\Delta}}.$$

In other words, we have

$$\iota(\mathcal{D}_0^0(X_0^kY^{b_0})), \iota(\mathcal{D}_Y^0(X_0^kY^{b_0})) \in \widetilde{\mathcal{D}}_0\mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_X\mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_Y\mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_Z\mathcal{S}_{\widetilde{\Delta}}$$

for any k, b_0. This implies that ι descends to a map $\mathcal{R}_{H_0} \to \mathcal{R}_{\widetilde{H}}$.

Finally, we characterize the image of ι by showing that for any k, a, b, c with $c \geq 1$,

$$X_0^kX^aY^bZ^c \in \iota(\mathcal{S}_{\Delta_0}) + \widetilde{\mathcal{D}}_0\mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_X\mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_Y\mathcal{S}_{\widetilde{\Delta}} + \widetilde{\mathcal{D}}_Z\mathcal{S}_{\widetilde{\Delta}}.$$

We induct on c. The base case $c = 1$ follows from the argument above, which shows that any $X_0^kX^aY^bZ$ can be modified via elements of the form $\widetilde{\mathcal{D}}_Z(X_0^{k-1}X^{a'}Y^{b'})$ into a multiple of $\iota(X_0^{k-1}Y^{b-fa})$.

Now consider a general $c > 1$. We have

$$\widetilde{\mathcal{D}}_0(X_0^{k-1}X^aY^bZ^{c-1}) = (k-1)X_0^{k-1}X^aY^bZ^{c-1} + H(X,Y,q)X_0^kX^aY^bZ^{c-1} + (\widetilde{q}_{R-2}X^{-1}Y^{-f}+1)X_0^kX^aY^bZ^c,$$
$$\widetilde{\mathcal{D}}_X(X_0^{k-1}X^aY^bZ^{c-1}) = aX_0^{k-1}X^aY^bZ^{c-1} + X\frac{\partial H(X,Y,q)}{\partial X}X_0^kX^aY^bZ^{c-1} - \widetilde{q}_{R-2}X^{-1}Y^{-f}\cdot X_0^kX^aY^bZ^c.$$

Note that $X_0^kX^aY^bZ^c$ differs from $\widetilde{\mathcal{D}}_0(X_0^{k-1}X^aY^bZ^{c-1}) + \widetilde{\mathcal{D}}_X(X_0^{k-1}X^aY^bZ^{c-1})$ by monomials where Z has power $c-1$. We may then conclude by the inductive hypothesis. □

Proof of Theorem 9.2. We use the maps π and ι constructed in Lemmas 9.5 and 9.6. Lemma 9.6 verifies that the image of ι is equal to the kernel of π, i.e. the sequence (9.2) is exact in the middle. Moreover, (8.3) and (9.1) imply that

$$\dim_{\mathbb{C}}\mathcal{R}_{\widetilde{H}} = \dim_{\mathbb{C}}\mathcal{R}_H + \dim_{\mathbb{C}}\mathcal{R}_{H_0}.$$

Thus (9.2) is exat on the left as well. □

Example 9.7. Let $\mathcal{X} = \mathbb{C}^3$, $\mathcal{L}$ be an outer brane, and $f = 1$, as in Section 2.6.1 and Example 8.16. In this case, the Laurent polynomials are

$$H(X,Y) = X + Y + 1,$$
$$\widetilde{H}(X,Y,Z,\widetilde{q}_1) = X + Y + 1 + (\widetilde{q}_1X^{-1}Y^{-1}+1)Z,$$
$$H_0(Y,x) = -xY^{-1} + Y + 1.$$

A direct computation from the definitions shows that

$$\mathcal{R}_H \cong \mathbb{C}1, \qquad \mathcal{R}_{\widetilde{H}} \cong \mathbb{C}1 + \mathbb{C}X_0Z + \mathbb{C}X_0^2Z, \qquad \mathcal{R}_{H_0} \cong \mathbb{C}1 + \mathbb{C}X_0.$$

The $\mathcal{E}$-filtrations are specified by powers of X_0, while the $\mathcal{I}$-filtrations are given by

$$\mathcal{I}_0\mathcal{R}_H = \cdots = \mathcal{I}_3\mathcal{R}_H = 0, \qquad \mathcal{I}_4\mathcal{R}_H = \mathcal{R}_H,$$

$$\mathcal{I}_0\mathcal{R}_{\widetilde{H}} = 0, \qquad \mathcal{I}_1\mathcal{R}_{\widetilde{H}} = \mathbb{C}X_0^2Z, \qquad \mathcal{I}_2\mathcal{R}_{\widetilde{H}} = \mathcal{I}_3\mathcal{R}_{\widetilde{H}} = \mathcal{I}_4\mathcal{R}_{\widetilde{H}} = \mathbb{C}X_0Z + \mathbb{C}X_0^2Z, \qquad \mathcal{I}_5\mathcal{R}_{\widetilde{H}} = \mathcal{R}_{\widetilde{H}},$$

$$\mathcal{I}_0\mathcal{R}_{H_0} = 0, \qquad \mathcal{I}_1\mathcal{R}_{H_0} = \mathcal{I}_2\mathcal{R}_{H_0} = \mathbb{C}X_0, \qquad \mathcal{I}_3\mathcal{R}_{H_0} = \mathcal{R}_{H_0}.$$

The map $\pi : \mathcal{R}_{\widetilde{H}} \to \mathcal{R}_H$ constructed in Lemma 9.5 maps X_0Z and X_0^2Z to 0, preserves the $\mathcal{E}$-filtration, and lowers degrees in the $\mathcal{I}$-filtration by 1. The map $\iota : \mathcal{R}_{H_0} \to \mathcal{R}_{\widetilde{H}}$ constructed in Lemma 9.6 maps 1 to X_0Z and X_0 to X_0^2Z, lowers degrees in the $\mathcal{E}$-filtration by 1, and raises degrees in the $\mathcal{I}$-filtration by 1.

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Appendix A: Preliminaries of orbifold Gromov-Witten theory

In this chapter, we collect additional premliminaries of orbifold Gromov-Witten theory, mainly to supplement Chapter 3.

A.1 Hurwitz-Hodge integrals

In this section, we briefly review *Hurwitz-Hodge integrals*, which are intersection numbers on moduli spaces of twisted stable maps to the classifying stack of a finite group. We restrict our attention to the genus zero case and the case where the finite group is abelian, which we will need in our localization computations. We fix a finite abelian group G in this section. The classifying stack $\mathcal{B}G = [\mathrm{pt}/G]$ is a smooth Deligne-Mumford stack, and

$$\mathcal{IB}G = \bigsqcup_{k \in G} (\mathcal{B}G)_k.$$

Let $n \in \mathbb{Z}_{\geq 0}$ and $\vec{k} = (k_1, \dots, k_n) \in G^n$. The definitions in Section 3.1 can be applied to define the moduli space $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G) := \overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G, 0)$ of genus-zero, $\vec{k}$-twisted stable maps to $\mathcal{B}G$. We assume that $n \geq 3$ and

$$k_1 \cdots k_n = 1 \in G.$$

Otherwise, $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)$ is empty.

Let $\pi : \mathcal{U} \to \overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)$ be the universal curve, $u : \mathcal{U} \to \mathcal{B}G$ be the universal map, and $\epsilon : \overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G) \to \overline{\mathcal{M}}_{0,n}$ be the natural forgetful map. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G, where V is a 1-dimensional vector space over $\mathbb{C}$. Then $\mathcal{E}_\rho := [V/G]$ is a line bundle over $\mathcal{B}G$, and

$$\pi_* u^* \mathcal{E}_\rho = \begin{cases} \mathcal{O}_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)} & \text{if } \rho \text{ is the trivial representation} \\ 0 & \text{otherwise.} \end{cases}$$

The ρ-twisted *Hurwitz-Hodge bundle* $\mathbb{E}_\rho$ is the dual of the vector bundle $R^1\pi_* f^* \mathcal{E}_\rho \to \overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)$. If $\rho = 1$ is the trivial representation, then $\mathbb{E}_1$ is the pullback of the Hodge bundle over $\overline{\mathcal{M}}_{0,n}$ under the map ϵ, and thus $\mathbb{E}_1 = 0$. If ρ is non-trivial, then we have

$$\operatorname{rank}(\mathbb{E}_\rho) = -1 + \sum_{i=1}^{n} c_i, \tag{A.1}$$

where for each $i = 1, \dots, n$, $c_i \in [0,1) \cap \mathbb{Q}$ is such that the eigenvalue of $\rho(k_i) \in \mathrm{GL}(V)$ is $\exp(2\pi\sqrt{-1}c_i)$.

We define the following classes on $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)$:

- *Hodge classes:* Given an irreducible representation ρ of G, and $j = 0, \dots, \operatorname{rank}(\mathbb{E}_\rho)$, let

$$\lambda_j^\rho := c_j(\mathbb{E}_\rho) \in A^j(\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)).$$

- *Descendant classes:* For $i = 1, \dots, n$, let

$$\bar{\psi}_i := \epsilon^* \psi_i \in A^1(\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)).$$

Moreover, we define

$$\Lambda_\rho^\vee(\mathbf{w}) = \sum_{j=0}^{\operatorname{rank}(\mathbb{E}_\rho)} (-1)^j \lambda_j^\rho \mathbf{w}^{\operatorname{rank}(\mathbb{E}_\rho) - j},$$

where $\mathbf{w}$ is a formal variable. We will use the following version of Mumford's relation [22, Proposition]:

Lemma A.1 ([22]). *Let G be a finite abelian group, $n \in \mathbb{Z}_{\geq 3}$, and $\vec{k} = (k_1, \dots, k_n) \in G^n$ such that*

$k_1 \cdots k_n = 1 \in G$. *Let* ρ *be an irreducible representation of* G*,* $\rho^\vee$ *denote its dual representation, and* $\mathbf{w}$ *be a formal variable. Then*

$$\Lambda_\rho^\vee(\mathbf{w})\Lambda_{\rho^\vee}^\vee(-\mathbf{w}) = (-1)^{\mathrm{rank}(\mathbb{E}_{\rho^\vee})}\mathbf{w}^{\mathrm{rank}(\mathbb{E}_\rho)+\mathrm{rank}(\mathbb{E}_{\rho^\vee})},$$

where $\mathbb{E}_\rho, \mathbb{E}_{\rho^\vee}$ *are Hurwitz-Hodge bundles over* $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)$ *defined above.*

Hurwitz-Hodge integrals are integrals of form

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)} \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n} \lambda_{j_1}^{\rho_1} \cdots \lambda_{j_m}^{\rho_m},$$

where $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, $\rho_1, \ldots, \rho_m$ are (not necessarily distinct) irreducible representations of G, and each $j_i \in \{0, \ldots, \mathrm{rank}(\mathbb{E}_{\rho_i})\}$. Zhou [96] gave an algorithm for computing these integrals, as follows: By Tseng's orbifold Riemann-Roch theorem [92], the Hurwitz-Hodge integrals can be reconstructed from descendant integrals, i.e. integrals of form

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)} \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n},$$

where $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$. By Jarvis-Kimura [60, Proposition 3.4], we have

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)} \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n} = \frac{1}{|G|} \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \begin{cases} \frac{(n-3)!}{a_1! \cdots a_n!} & \text{if } a_1 + \cdots + a_n = n-3 \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, it is straightforward to derive the following identity (see e.g. [76, Lemmas 61 and 123] for some special cases):

Lemma A.2. *Let* G *be a finite abelian group,* $n \in \mathbb{Z}_{\geq 3}$*, and* $\vec{k} = (k_1, \ldots, k_n) \in G^n$ *such that* $k_1 \cdots k_n = 1 \in G$*. Let* $S \subseteq \{1, \ldots, n\}$*, and for each* $i \in S$*, let* $\mathbf{w}_i$ *be a formal variable. Then*

$$\int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G)} \prod_{i \in S} \frac{1}{\mathbf{w}_i - \bar{\psi}_i} = \frac{1}{|G| \cdot \prod_{i \in S} \mathbf{w}_i} \left(\sum_{i \in S} \frac{1}{\mathbf{w}_i} \right)^{n-3}.$$

In our localization computations, we adopt the following integration conventions for

- $\overline{\mathcal{M}}_{0,(1)}(\mathcal{B}G)$, viewed as a (-2)-dimensional space, and
- $\overline{\mathcal{M}}_{0,(k,k^{-1})}(\mathcal{B}G)$, viewed as a (-1)-dimensional space, where $k \in G$.

Let $\mathbf{w}_1, \mathbf{w}_2$ be formal variables. We set

$$\begin{aligned}&\int_{\overline{\mathcal{M}}_{0,(1)}(\mathcal{B}G)} \frac{1}{\mathbf{w}_1 - \bar{\psi}_1} = \frac{\mathbf{w}_1}{|G|}, \quad \int_{\overline{\mathcal{M}}_{0,(k,k^{-1})}(\mathcal{B}G)} \frac{1}{\mathbf{w}_1 - \bar{\psi}_1} = \frac{1}{|G|}, \\ &\int_{\overline{\mathcal{M}}_{0,(k,k^{-1})}(\mathcal{B}G)} \frac{1}{(\mathbf{w}_1 - \bar{\psi}_1)(\mathbf{w}_2 - \bar{\psi}_2)} = \frac{1}{|G| \cdot (\mathbf{w}_1 + \mathbf{w}_2)}.\end{aligned} \tag{A.2}$$

Note that this is consistent with Lemma A.2.

A.2 Twisted covers of proper torus-invariant lines

In this section, we characterize non-constant representable morphisms from irreducible twisted curves to proper torus-invariant lines in toric orbifolds. Such maps are studied in detail by Johnson [63].

The domain of such a map has form $\mathcal{C}_{r_1,r_2}$ for some $r_1, r_2 \in \mathbb{Z}_{>0}$, which is the 1-dimensional toric orbifold defined by the stacky fan

$$\mathbf{\Sigma}_{r_1,r_2} = (\mathbb{Z}, \Sigma, \alpha_{r_1,r_2}).$$

Here, Σ is the complete fan in $\mathbb{R}$ and $\alpha_{r_1,r_2} : \mathbb{Z}^2 \to \mathbb{Z}$ is determined by $(r_1, -r_2)$. The coarse moduli space of $\mathcal{C}_{r_1,r_2}$ is isomorphic to $\mathbb{P}^1$. There are two $(\mathbb{C}^*)$-fixed points $\mathfrak{p}_1, \mathfrak{p}_2$ of $\mathcal{C}_{r_1,r_2}$, with generic stablizer groups μ_{r_1}, μ_{r_2} respectively. Let $p_1, p_2 \in \mathbb{P}^1$ be their images in the coarse moduli space, which are the two $(\mathbb{C}^*)$-fixed points.

Now let $\mathcal{Z}$ be an r-dimensional toric orbifold specified by an extended stacky fan $\mathbf{\Xi} = (\mathbb{Z}^r, \Xi, \alpha)$ as in Section 2.1, with Deligne-Mumford torus $(\mathbb{C}^*)^r$. Let $\tau \in \Xi(r-1)_c$ and $\sigma_1, \sigma_2 \in \Xi(r)$ be the two r-dimensional cones that contain τ. Let

$$u : \mathcal{C}_{r_1,r_2} \to \mathfrak{l}_\tau \tag{A.3}$$

be a representable morphism such that if $\bar{u} : C \to l_\tau$ is the induced map between coarse moduli spaces, then $\bar{u}(p_1) = p_{\sigma_1}$ and $\bar{u}(p_2) = p_{\sigma_2}$. Consider the restrictions to the open orbit

$$u|_{\mathbb{C}^*} : \mathbb{C}^* \to \mathfrak{o}_\tau, \quad \bar{u}|_{\mathbb{C}^*} : \mathbb{C}^* \to o_\tau \cong \mathbb{C}^*.$$

Let γ be the image of the generator of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ under the map $(u|_{\mathbb{C}^*}) : \pi_1(\mathbb{C}^*) \to \pi_1(\mathfrak{o}_\tau) = H_\tau$. Then $d = \pi_\tau(\gamma) > 0$ is the image of the generator of $\pi_1(\mathbb{C}^*)$ under the map $(\bar{u}|_{\mathbb{C}^*})_* : \pi_1(\mathbb{C}^*) \to \pi_1(o_\tau) \cong \mathbb{Z}$, and is the degree of the map $\bar{u}|_{\mathbb{C}^*}$. The map u is in fact uniquely determined by the element $\gamma \in H_\tau$ up to automorphisms, and we say that γ is the *degree* of u.

Let $k_1 \in G_{\sigma_1}$ (resp. $k_2 \in G_{\sigma_2}$) be the image of the generator of the stablizer group μ_{r_1} of $\mathfrak{p}_1$ (resp. μ_{r_2} of $\mathfrak{p}_2$) under u. The representability of u implies that k_1 (resp. k_2) has order r_1 (resp. r_2). Moreover, k_1 and k_2 are determined by γ as

$$\pi_{(\tau,\sigma_1)}(\gamma) = k_1, \quad \pi_{(\tau,\sigma_2)}(\gamma) = k_2$$

(see (2.3)).

Appendix B: Details of localization computations and comparisons

In this chapter, we supply the details of localization computations and comparisons needed in proving the numerical open/closed correspondence (Theorem 4.1) in Chapter 4.

B.1 Contributions from maps to $\mathcal{X}$

In this section, we consider maps from an irreducible twisted curve to a T'-fixed point (resp. proper T'-invariant line) in $\mathcal{X}$, which can also be viewed as maps to the corresponding $\widetilde{T}'$-fixed point (resp. proper $\widetilde{T}'$-invariant line) in $\widetilde{\mathcal{X}}$ via the inclusion $\mathcal{X} \to \widetilde{\mathcal{X}}$. We explicitly compare their contributions to the localization computations of the disk invariants (Proposition 3.7) and closed invariants (Proposition 3.11).

Lemma B.1 (Edges). *Let $\tau \in \Sigma(2)_c$, and*

$$u : \mathcal{C} = \mathcal{C}_{r_1,r_2} \to \mathfrak{l}_\tau \subset \mathcal{X} \subset \widetilde{\mathcal{X}}$$

be a morphism as in (A.3) *in Section A.2. Let*

$$\mathbf{h} := \frac{e_{T'}(H^1(\mathcal{C}, u^*T\mathcal{X})^m)}{e_{T'}(H^0(\mathcal{C}, u^*T\mathcal{X})^m)}, \quad \widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}.$$

Then

$$\left(\mathsf{u}_4\widetilde{\mathbf{h}}\right)\Big|_{\mathsf{u}_4=0} = \mathbf{h}.$$

Proof. Over $\mathfrak{l}_\tau = \mathfrak{l}_{\iota(\tau)}$, we have the following relation:

$$T\widetilde{\mathcal{X}}\Big|_{\mathfrak{l}_{\iota(\tau)}} \cong \mathcal{O} \oplus T\mathcal{X}\Big|_{\mathfrak{l}_\tau},$$

which implies that

$$\widetilde{\mathbf{h}} = \frac{1}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*\mathcal{O}))} \cdot \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*(T\mathcal{X}\big|_{\mathfrak{l}_\tau}))^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*(T\mathcal{X}\big|_{\mathfrak{l}_\tau}))^m)} = \frac{1}{\mathsf{u}_4} \cdot \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*(T\mathcal{X}\big|_{\mathfrak{l}_\tau}))^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*(T\mathcal{X}\big|_{\mathfrak{l}_\tau}))^m)}.$$

By (3.13), we may obtain T'-weights on $T\mathcal{X}\big|_{\mathfrak{l}_\tau}$ by applying the restriction $\mathsf{u}_4 = 0$ to $\widetilde{T}'$-weights. The lemma thus directly follows from the explicit computations of $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ in [76, Lemma 130].

□

Lemma B.2 (Flags)**.** *Let $\sigma \in \Sigma(3)$ and $k \in G_\sigma = G_{\iota(\sigma)}$. Let*

$$\mathbf{h} := e_{T'}\left((T_{\mathfrak{p}_\sigma}\mathcal{X})^k\right), \quad \widetilde{\mathbf{h}} := e_{\widetilde{T}'}\left((T_{\mathfrak{p}_{\iota(\sigma)}}\widetilde{\mathcal{X}})^k\right).$$

Then

$$\frac{\widetilde{\mathbf{h}}}{\mathsf{u}_4}\bigg|_{\mathsf{u}_4=0} = \mathbf{h}.$$

Proof. We have

$$\mathbf{h} = \prod_{(\tau,\sigma)\in F(\Sigma), k\in G_\tau} \mathbf{w}(\tau,\sigma), \quad \widetilde{\mathbf{h}} = \prod_{(\widetilde{\tau},\iota(\sigma))\in F(\widetilde{\Sigma}), k\in G_{\widetilde{\tau}}} \widetilde{\mathbf{w}}(\widetilde{\tau},\iota(\sigma)).$$

First, $\widetilde{\mathbf{h}}$ contains $\widetilde{\mathbf{w}}(\sigma,\iota(\sigma)) = \mathsf{u}_4$ as a factor. For the other facets of $\iota(\sigma)$, since $G_{\iota(\tau)} = G_\tau$ for any $\tau \in \Sigma(2)$, we have $k \in G_{\iota(\tau)}$ if and only if $k \in G_\tau$. Thus $\mathbf{w}(\iota(\tau),\iota(\sigma))$ is a factor of $\widetilde{\mathbf{h}}$ if and only if $\mathbf{w}(\tau,\sigma)$ is a factor of $\mathbf{h}$. The lemma then follows from (3.13). □

Lemma B.3 (Stable vertices)**.** *Let $\sigma \in \Sigma(3)$, $n \in \mathbb{Z}_{\geq 3}$, $\vec{k} = (k_1,\dots,k_n) \in G_\sigma^n = G_{\iota(\sigma)}^n$ such that $k_1 \cdots k_n = 1$, and*

$$u : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n) \to \mathfrak{p}_\sigma \subset \mathcal{X} \subset \widetilde{\mathcal{X}}$$

be a morphism that represents a point in $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G_\sigma)$. Let

$$\mathbf{h} := \frac{e_{T'}(H^1(\mathcal{C}, u^*T\mathcal{X})^m)}{e_{T'}(H^0(\mathcal{C}, u^*T\mathcal{X})^m)}, \quad \widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}.$$

Then

$$\left(\mathsf{u}_4\widetilde{\mathbf{h}}\right)\Big|_{\mathsf{u}_4=0} = \mathbf{h}.$$

Proof. We first recall the explicit computations of $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ in [76, Lemma 126]. Consider the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xrightarrow{\widetilde{u}} & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{u} & \mathcal{B}G_\sigma. \end{array}$$

Let $\hat{G} \subseteq G_\sigma$ denote the subgroup generated by the monodromies of the G_σ-cover $\widetilde{\mathcal{C}} \to \mathcal{C}$. For each flag $(\tau,\sigma) \in F(\Sigma)$ (resp. $(\widetilde{\tau},\iota(\sigma)) \in F(\widetilde{\Sigma})$), recall that $\chi_{(\tau,\sigma)}$ (resp. $\chi_{(\widetilde{\tau},\iota(\sigma))}$) denotes the G_σ-representation $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\tau$ (resp. $T_{\mathfrak{p}_{\iota(\sigma)}}\mathfrak{l}_{\widetilde{\tau}}$). Then, [76, Lemma 126] states that

$$\mathbf{h} = \frac{\prod_{(\tau,\sigma)\in F(\Sigma)} \Lambda^\vee_{\chi_{(\tau,\sigma)}}(\mathbf{w}(\tau,\sigma))}{\prod_{(\tau,\sigma)\in F(\Sigma),\hat{G}\subseteq G_\tau} \mathbf{w}(\tau,\sigma)}, \quad \widetilde{\mathbf{h}} = \frac{\prod_{(\widetilde{\tau},\iota(\sigma))\in F(\widetilde{\Sigma})} \Lambda^\vee_{\chi_{(\widetilde{\tau},\iota(\sigma))}}(\widetilde{\mathbf{w}}(\widetilde{\tau},\iota(\sigma)))}{\prod_{(\widetilde{\tau},\iota(\sigma))\in F(\widetilde{\Sigma}),\hat{G}\subseteq G_{\widetilde{\tau}}} \widetilde{\mathbf{w}}(\widetilde{\tau},\iota(\sigma))}.$$

For each $(\tau,\sigma) \in F(\Sigma)$, we have $\chi_{(\tau,\sigma)} = \chi_{(\iota(\tau),\iota(\sigma))}$. Thus by (3.13), we have

$$\Lambda^\vee_{\chi_{(\iota(\tau),\iota(\sigma))}}(\widetilde{\mathbf{w}}(\iota(\tau),\iota(\sigma)))\Big|_{\mathsf{u}_4=0} = \Lambda^\vee_{\chi_{(\tau,\sigma)}}(\mathbf{w}(\tau,\sigma)).$$

Moreover, $\chi_{(\sigma,\iota(\sigma))}$ is trivial, and thus

$$\Lambda^\vee_{\chi_{(\sigma,\iota(\sigma))}}(\widetilde{\mathbf{w}}(\sigma,\iota(\sigma))) = 1.$$

Finally, since $\hat{G} \subseteq G_\sigma$, $\widetilde{\mathbf{w}}(\sigma,\iota(\sigma)) = \mathsf{u}_4$ is a factor of the denominator of $\widetilde{\mathbf{h}}$. The lemma then follows from (3.13). □

Lemma B.4 (Vertex integrals)**.** *Let $\sigma \in \Sigma(3)$, $n \in \mathbb{Z}_{\geq 1}$, $\vec{k} = (k_1,\ldots,k_n) \in G_\sigma^n = G_{\iota(\sigma)}^n$ such that $k_1\cdots k_n = 1$. Let $0 \leq n' \leq n$, and $\tau_1,\ldots,\tau_{n'} \in \Sigma(2)_c$ such that each τ_i is a facet of σ and $k_i \in G_{\tau_i} = G_{\iota(\tau_i)}$. In addition, for each $i = 1,\ldots,n'$, let r_i be the order of k_i in G_σ, and take*

$d_i \in \mathbb{Z}_{>0}$. *Let*

$$\mathbf{h} := \int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G_\sigma)} \frac{1}{\prod_{i=1}^{n'} \left(\frac{\mathfrak{r}(\tau_i,\sigma)\mathbf{w}(\tau_i,\sigma)}{r_i d_i} - \frac{\bar{\psi}_i}{r_i} \right)}, \quad \widetilde{\mathbf{h}} := \int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G_{\iota(\sigma)})} \frac{1}{\prod_{i=1}^{n'} \left(\frac{\mathfrak{r}(\iota(\tau_i),\iota(\sigma))\widetilde{\mathbf{w}}(\iota(\tau_i),\iota(\sigma))}{r_i d_i} - \frac{\bar{\psi}_i}{r_i} \right)}.$$

Then

$$\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0} = \mathbf{h}.$$

Proof. We first show that u_4 is not a factor of the denominator of $\widetilde{\mathbf{h}}$, i.e. $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$ is defined. To see this, note by (3.13) that $\widetilde{\mathbf{w}}(\iota(\tau_i),\iota(\sigma)) \notin \mathbb{Q}\mathsf{u}_4$. In view of Lemma A.2, the only case where u_4 appears as a factor in the denominator is when $n = n' = 2$ and

$$r_1\widetilde{\mathbf{w}}(\iota(\tau_1),\iota(\sigma)) + r_2\widetilde{\mathbf{w}}(\iota(\tau_2),\iota(\sigma)) \in \mathbb{Q}\mathsf{u}_4,$$

but this equation cannot hold. Then, the lemma is a direct consequence of (3.13). □

Lemma B.5. *Let the quantity* $\mathbf{h}$ *be as defined in Lemma B.1, B.2, B.3, or B.4. Then for generic* $f \in \mathbb{Z}$, $\mathsf{u}_2 - f\mathsf{u}_1$ *is not a factor of* $\mathbf{h}$.

Proof. The lemma follows from that the data defining $\mathbf{h}$ is independent of f. □

B.2 Fixed points corresponding to cones in $\widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$

In this section, we study the tangent $\widetilde{T}'$-weights at a fixed point $\mathfrak{p}_{\widetilde{\sigma}}$ in $\widetilde{\mathcal{X}}$ corresponding to a cone $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$. In addition, we study the contributions of vertices and flags associated to such fixed points to the localization computations of the closed invariants (Proposition 3.11). See Section 2.5 for a description of such cones, their associated flags, and stablizers.

First, the tangent $\widetilde{T}'$-weights at $\mathfrak{p}_{\widetilde{\sigma}}$ are given by

$$\begin{aligned}
\widetilde{\mathbf{w}}(\iota(\delta_0(\widetilde{\sigma})),\widetilde{\sigma}) &= \frac{n_{i_3(\widetilde{\sigma})}-n_{i_2(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_1 + \frac{m_{i_2(\widetilde{\sigma})}-m_{i_3(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_2 + \frac{m_{i_2(\widetilde{\sigma})}n_{i_3(\widetilde{\sigma})}-m_{i_3(\widetilde{\sigma})}n_{i_2(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_4,\\
\widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}),\widetilde{\sigma}) &= \frac{n_{i_2(\widetilde{\sigma})}-n_{i_3(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_1 + \frac{m_{i_3(\widetilde{\sigma})}-m_{i_2(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_2 + \frac{|G_{\widetilde{\sigma}}|+m_{i_3(\widetilde{\sigma})}n_{i_2(\widetilde{\sigma})}-m_{i_2(\widetilde{\sigma})}n_{i_3(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_4,\\
\widetilde{\mathbf{w}}(\delta_2(\widetilde{\sigma}),\widetilde{\sigma}) &= -\frac{f}{|G_{\widetilde{\sigma}}|}\mathsf{u}_1 + \frac{1}{|G_{\widetilde{\sigma}}|}\mathsf{u}_2 + \frac{-fm_{i_3(\widetilde{\sigma})}+n_{i_3(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_4,\\
\widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}),\widetilde{\sigma}) &= \frac{f}{|G_{\widetilde{\sigma}}|}\mathsf{u}_1 - \frac{1}{|G_{\widetilde{\sigma}}|}\mathsf{u}_2 + \frac{fm_{i_2(\widetilde{\sigma})}-n_{i_2(\widetilde{\sigma})}}{|G_{\widetilde{\sigma}}|}\mathsf{u}_4.
\end{aligned}$$

Note that $\widetilde{\mathbf{w}}(\iota(\delta_0(\widetilde{\sigma})),\widetilde{\sigma})\big|_{\mathsf{u}_4=0}$ and $\widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}),\widetilde{\sigma})\big|_{\mathsf{u}_4=0}$ are nonzero and independent of f, and

$$\widetilde{\mathbf{w}}(\delta_2(\widetilde{\sigma}),\widetilde{\sigma})\big|_{\mathsf{u}_4=0} = \frac{1}{|G_{\widetilde{\sigma}}|}(\mathsf{u}_2 - f\mathsf{u}_1), \quad \widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}),\widetilde{\sigma})\big|_{\mathsf{u}_4=0} = -\frac{1}{|G_{\widetilde{\sigma}}|}(\mathsf{u}_2 - f\mathsf{u}_1).$$

We now consider maps from an irreducible twisted curve to a $\widetilde{T}'$-fixed point in $\widetilde{\mathcal{X}}$ corresponding to a cone in $\widetilde{\Sigma}(4)\setminus\iota(\Sigma(3))$. In particular, we study the powers of u_4 and $\mathsf{u}_2-f\mathsf{u}_1$ in the contributions of such maps to the localization computations of the closed invariants (Proposition 3.11).

Lemma B.6 (Flags)**.** *Let $\widetilde{\sigma}\in\widetilde{\Sigma}(4)\setminus\iota(\Sigma(3))$ and $k\in G_{\widetilde{\sigma}}$. Let*

$$\widetilde{\mathbf{h}} := e_{\widetilde{T}'}\left((T_{\mathfrak{p}_{\widetilde{\sigma}}}\widetilde{\mathcal{X}})^k\right).$$

Then u_4 is not a factor of $\widetilde{\mathbf{h}}$. Moreover, for generic $f\in\mathbb{Z}$, if $k=1$, then the power of $\mathsf{u}_2-f\mathsf{u}_1$ in $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$ is 2; otherwise, $\mathsf{u}_2-f\mathsf{u}_1$ is not a factor of $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$.

Proof. We have

$$\widetilde{\mathbf{h}} = \prod_{(\widetilde{\tau},\widetilde{\sigma})\in F(\widetilde{\Sigma}),k\in G_{\widetilde{\tau}}} \widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma}).$$

Since none of $\widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma})$ is a rational multiple of u_4, u_4 is not a factor of $\widetilde{\mathbf{h}}$. Now for generic $f\in\mathbb{Z}$, neither of $\widetilde{\mathbf{w}}(\iota(\delta_0(\widetilde{\sigma})),\widetilde{\sigma})\big|_{\mathsf{u}_4=0}$, $\widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}),\widetilde{\sigma})\big|_{\mathsf{u}_4=0}$ is a rational multiple of $\mathsf{u}_2-f\mathsf{u}_1$. If $k=1$, then $(\widetilde{\mathbf{w}}(\delta_2(\widetilde{\sigma}),\widetilde{\sigma})\widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}),\widetilde{\sigma}))\big|_{\mathsf{u}_4=0}$, which is a rational multiple of $(\mathsf{u}_2-f\mathsf{u}_1)^2$, is a factor of $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$. Otherwise, $k\neq 1$ is not contained in the trivial group $G_{\delta_2(\widetilde{\sigma})}=G_{\delta_3(\widetilde{\sigma})}=\{1\}$, which implies that

$\mathsf{u}_2 - f\mathsf{u}_1$ is not a factor of $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$. □

Lemma B.7 (Stable vertices)**.** *Let $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$, $n \in \mathbb{Z}_{\geq 3}$, $\vec{k} = (k_1, \ldots, k_n) \in G_{\widetilde{\sigma}}^n$ such that $k_1 \cdots k_n = 1$, and*

$$u : (\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n) \to \mathfrak{p}_{\widetilde{\sigma}} \subset \widetilde{\mathcal{X}}$$

be a morphism that represents a point in $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G_{\widetilde{\sigma}})$. Let

$$\widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}.$$

Then u_4 is not a factor of the denominator of $\widetilde{\mathbf{h}}$, i.e. $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$ is defined. Moreover, for generic $f \in \mathbb{Z}$, if $k_1 = \cdots = k_n = 1$, then the power of $\mathsf{u}_2 - f\mathsf{u}_1$ in $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$ is at least -2; otherwise, the power of $\mathsf{u}_2 - f\mathsf{u}_1$ in $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$ is at least 0, i.e. $\mathsf{u}_2 - f\mathsf{u}_1$ is not a factor of the denominator of $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$.

Proof. Similar to the proof of Lemma B.3, we use [76, Lemma 126] to compute $\widetilde{\mathbf{h}}$ as

$$\widetilde{\mathbf{h}} = \frac{\prod_{(\widetilde{\tau},\widetilde{\sigma}) \in F(\widetilde{\Sigma})} \Lambda^{\vee}_{\chi(\widetilde{\tau},\widetilde{\sigma})}(\widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma}))}{\prod_{(\widetilde{\tau},\widetilde{\sigma}) \in F(\widetilde{\Sigma}), \hat{G} \subseteq G_{\widetilde{\tau}}} \widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma})}$$

where $\hat{G} \subseteq G_{\widetilde{\sigma}}$ denotes the subgroup generated by the monodromies of the $G_{\widetilde{\sigma}}$-cover $\widetilde{\mathcal{C}} \to \mathcal{C}$ pulled back from $\mathrm{pt} \to \mathcal{B}G_{\widetilde{\sigma}}$ under u. We focus on the denominator of $\widetilde{\mathbf{h}}$. Observe first that none of $\widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma})$ is a rational multiple of u_4, which implies that u_4 is not a factor of the denominator of $\widetilde{\mathbf{h}}$. Now for generic $f \in \mathbb{Z}$, neither of $\widetilde{\mathbf{w}}(\iota(\delta_0(\widetilde{\sigma})),\widetilde{\sigma})\big|_{\mathsf{u}_4=0}$, $\widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}),\widetilde{\sigma})\big|_{\mathsf{u}_4=0}$ is a rational multiple of $\mathsf{u}_2 - f\mathsf{u}_1$. If $k_1 = \cdots = k_n = 1$, the group $\hat{G}$ is trivial, and thus $\left(\widetilde{\mathbf{w}}(\delta_2(\widetilde{\sigma}),\widetilde{\sigma})\widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}),\widetilde{\sigma})\right)\big|_{\mathsf{u}_4=0}$, which is a rational multiple of $(\mathsf{u}_2 - f\mathsf{u}_1)^2$, is a factor of the denominator of $\widetilde{\mathbf{h}}$ after the restriction $\mathsf{u}_4 = 0$. Otherwise, $\hat{G}$ is non-trivial and is not contained in the trivial group $G_{\delta_2(\widetilde{\sigma})} = G_{\delta_3(\widetilde{\sigma})} = \{1\}$, which implies that $\mathsf{u}_2 - f\mathsf{u}_1$ is not a factor of the denominator of $\widetilde{\mathbf{h}}$ after the restriction $\mathsf{u}_4 = 0$. □

Lemma B.8 (Vertices labeled by $\widetilde{\sigma}_0$ with all non-trivial twistings)**.** *Let $n \in \mathbb{Z}_{\geq 3}$, $\vec{k} = (k_1, \ldots, k_n) \in G_{\widetilde{\sigma}_0}^n$ such that $k_1 \cdots k_n = 1$ and $k_i \neq 1$ for all i, and*

$$u : (\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n) \to \mathfrak{p}_{\widetilde{\sigma}_0} \subset \widetilde{\mathcal{X}}$$

be a morphism that represents a point in $\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G_{\widetilde{\sigma}_0})$. Let

$$\widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}.$$

Then for generic $f \in \mathbb{Z}$, the power of $\mathsf{u}_2 - f\mathsf{u}_1$ in $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$ is $n-2 \geq 1$.

Proof. Similar to the proof of Lemma B.7, we use [76, Lemma 126] and that $k_i \neq 1$ for all i to compute $\widetilde{\mathbf{h}}$ as

$$\widetilde{\mathbf{h}} = \frac{\prod_{(\widetilde{\tau},\widetilde{\sigma}_0)\in F(\widetilde{\Sigma})} \Lambda^\vee_{\chi_{(\widetilde{\tau},\widetilde{\sigma}_0)}}(\widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma}_0))}{\widetilde{\mathbf{w}}(\iota(\tau_0),\widetilde{\sigma}_0)\widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}_0),\widetilde{\sigma}_0)} = \frac{\prod_{(\widetilde{\tau},\widetilde{\sigma}_0)\in F(\widetilde{\Sigma})} \Lambda^\vee_{\chi_{(\widetilde{\tau},\widetilde{\sigma}_0)}}(\widetilde{\mathbf{w}}(\widetilde{\tau},\widetilde{\sigma}_0))}{(-\mathsf{u}_1)(\mathsf{u}_1+\mathsf{u}_4)}.$$

We focus on the numerator of $\widetilde{\mathbf{h}}$. Observe first that $\mathfrak{r}(\iota(\tau_0),\widetilde{\sigma}_0) = \mathfrak{r}(\delta_4(\widetilde{\sigma}_0),\widetilde{\sigma}_0) = 1$, which implies that $\chi_{(\iota(\tau_0),\widetilde{\sigma}_0)}$ and $\chi_{(\delta_4(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}$ are trivial. Then,

$$\Lambda^\vee_{\chi_{(\iota(\tau_0),\widetilde{\sigma}_0)}}(\widetilde{\mathbf{w}}(\iota(\tau_0),\widetilde{\sigma}_0)) = \Lambda^\vee_{\chi_{(\delta_4(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}(\widetilde{\mathbf{w}}(\delta_4(\widetilde{\sigma}_0),\widetilde{\sigma}_0)) = 1.$$

Moreover, the product of the isomorphisms

$$\chi_{(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}, \chi_{(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0)} : G_{\widetilde{\sigma}_0} \to \mu_{\mathfrak{m}} \subset \mathbb{C}^*$$

is trivial. Lemma A.1 then implies that

$$\begin{aligned}
&\left(\Lambda^\vee_{\chi_{(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}(\widetilde{\mathbf{w}}(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0))\Lambda^\vee_{\chi_{(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}(\widetilde{\mathbf{w}}(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0))\right)\Big|_{\mathsf{u}_4=0} \\
&= \Lambda^\vee_{\chi_{(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}\left(\tfrac{\mathsf{u}_2-f\mathsf{u}_1}{\mathfrak{m}}\right)\Lambda^\vee_{\chi_{(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}\left(-\tfrac{\mathsf{u}_2-f\mathsf{u}_1}{\mathfrak{m}}\right) \\
&= \pm\left(\frac{\mathsf{u}_2-f\mathsf{u}_1}{\mathfrak{m}}\right)^{\mathrm{rank}(\mathbb{E}_{\chi_{(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}})+\mathrm{rank}(\mathbb{E}_{\chi_{(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}})}.
\end{aligned}$$

Finally, since $k_i \neq 1$ for each $i = 1, \ldots, n$, we have by (A.1) that

$$\mathrm{rank}(\mathbb{E}_{\chi_{(\delta_2(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}) + \mathrm{rank}(\mathbb{E}_{\chi_{(\delta_3(\widetilde{\sigma}_0),\widetilde{\sigma}_0)}}) = n-2.$$

□

Lemma B.9 (Vertex integrals). *Let* $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$, $n \in \mathbb{Z}_{\geq 1}$, $\vec{k} = (k_1, \dots, k_n) \in G_{\widetilde{\sigma}}^n$ *such that* $k_1 \cdots k_n = 1$. *Let* $0 \leq n' \leq n$, *and* $\widetilde{\tau}_1, \dots, \widetilde{\tau}_{n'} \in \widetilde{\Sigma}(3)_c$ *such that each* $\widetilde{\tau}_i$ *is a facet of* $\widetilde{\sigma}$ *and* $k_i \in G_{\widetilde{\tau}_i}$. *In addition, for each* $i = 1, \dots, n'$, *let* r_i *be the order of* k_i *in* $G_{\widetilde{\sigma}}$, *and take* $d_i \in \mathbb{Z}_{>0}$. *Let*

$$\widetilde{\mathbf{h}} \coloneqq \int_{\overline{\mathcal{M}}_{0,\vec{k}}(\mathcal{B}G_{\widetilde{\sigma}})} \frac{1}{\prod_{i=1}^{n'} \left(\frac{\mathfrak{r}(\widetilde{\tau}_i, \widetilde{\sigma}) \widetilde{\mathbf{w}}(\widetilde{\tau}_i, \widetilde{\sigma})}{r_i d_i} - \frac{\bar{\psi}_i}{r_i} \right)}.$$

If $n = n' = 2$ *and* $\{\widetilde{\tau}_1, \widetilde{\tau}_2\} = \{\delta_2(\widetilde{\sigma}), \delta_3(\widetilde{\sigma})\}$, *then the power of* u_4 *in* $\widetilde{\mathbf{h}}$ *is* -1; *otherwise,* u_4 *is not a factor of* $\widetilde{\mathbf{h}}$. *Moreover, if* $\widetilde{\tau}_i \in \iota(\Sigma(2))$ *for all* i *and* $f \in \mathbb{Z}$ *is generic, then* $\mathsf{u}_2 - f\mathsf{u}_1$ *is not a factor of* $\widetilde{\mathbf{h}}\big|_{\mathsf{u}_4=0}$.

Proof. Note by (3.13) that $\widetilde{\mathbf{w}}(\iota(\tau_i), \iota(\sigma)) \notin \mathbb{Q}\mathsf{u}_4$. In view of Lemma A.2, the only case where u_4 appears as a factor in the denominator of $\widetilde{\mathbf{h}}$ is when $n = n' = 2$ and

$$r_1 \widetilde{\mathbf{w}}(\iota(\tau_1), \iota(\sigma)) + r_2 \widetilde{\mathbf{w}}(\iota(\tau_2), \iota(\sigma)) \in \mathbb{Q}\mathsf{u}_4,$$

which holds only if $\{\widetilde{\tau}_1, \widetilde{\tau}_2\} = \{\delta_2(\widetilde{\sigma}), \delta_3(\widetilde{\sigma})\}$.

Now suppose $\widetilde{\tau}_i \in \iota(\Sigma(2))$ for all i. Then the data defining $\widetilde{\mathbf{h}}$ is independent of f, which implies the final claim of the lemma. □

B.3 Proper $\widetilde{T}'$-invariant lines corresponding to cones in $\widetilde{\Sigma}(3)_c \setminus \iota(\Sigma(2)_c)$

In this section, we consider maps from an irreducible twisted curve to a proper $\widetilde{T}'$-invariant line in $\widetilde{\mathcal{X}}$ corresponding to a cone $\widetilde{\tau} \in \widetilde{\Sigma}(3)_c \setminus \iota(\Sigma(2)_c)$ and their contributions to the localization computations of the closed invariants (Proposition 3.11). For the distinguished cone $\widetilde{\tau} = \iota(\tau_0)$, we explicitly compute the contribution and compare it to the disk factor (Section 3.4.2). For a general $\widetilde{\tau}$, we study the power of u_4 in the contribution.

Lemma B.10 (Edges labeled by $\iota(\tau_0)$)**.** *Let*

$$u : \mathcal{C} = \mathcal{C}_{r_1,r_2} \to \mathfrak{l}_{\iota(\tau_0)} \subset \widetilde{\mathcal{X}}$$

be a morphism as in (A.3) *in Section A.2 with degree* $(d,\lambda) \in H_{\iota(\tau_0)} \cong \mathbb{Z} \times G_{\tau_0}$*. Let*

$$\widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}.$$

Then u_4 *is not a factor of* $\widetilde{\mathbf{h}}$*. Moreover, if* $\lambda = 1$*, we have*

$$\widetilde{\mathbf{h}}\Big|_{\mathsf{u}_4=0} = \frac{(-1)^{\lfloor dw_3-\epsilon_3 \rfloor + d+1}}{d \cdot \lfloor dw_0 \rfloor!} \left(\frac{\mathsf{u}_1}{d}\right)^{\mathrm{age}(h(d,\lambda))-2} \left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}}\right)^{-1} \cdot \prod_{a=1}^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1} \left(\frac{d\mathbf{w}_2}{\mathsf{u}_1} + a - \epsilon_2\right).$$

If $\lambda \neq 1$*, we have*

$$\widetilde{\mathbf{h}}\Big|_{\mathsf{u}_4=0} = \frac{(-1)^{\lfloor dw_3-\epsilon_3 \rfloor + d+1}}{d \cdot \lfloor dw_0 \rfloor!} \left(\frac{\mathsf{u}_1}{d}\right)^{\mathrm{age}(h(d,\lambda))-2} \cdot \prod_{a=1}^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1} \left(\frac{d\mathbf{w}_2}{\mathsf{u}_1} + a - \epsilon_2\right).$$

Here, $h(d,\lambda) = \pi_{(\tau_0,\sigma_0)}(d,\lambda) \in G_{\sigma_0}$*, the quantities* $\mathsf{w}_2, w_0, w_2, w_3$ *are defined in* (3.3)*, and* ϵ_2, ϵ_3 *are defined in* (3.5)*.*

Proof. We use the explicit computation of $\widetilde{\mathbf{h}}$ in [76, Lemma 130]. We set

$$\mathbf{b}_1 = \frac{1}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\mathfrak{l}_{\iota(\tau_0)})^m)}.$$

Moreover, the normal bundle $N_{\mathfrak{l}_{\iota(\tau_0)}/\widetilde{\mathcal{X}}}$ splits as a direct sum of $\widetilde{T}'$-equivariant line bundles L_2, L_3, L_4 given by the normal bundles of $\mathfrak{l}_{\iota(\tau_0)}$ in the 2-dimensional $\widetilde{T}$-invariant closed substacks of $\widetilde{\mathcal{X}}$ corresponding to the cones spanned by

$$\{\widetilde{\rho}_3, \widetilde{\rho}_{R+2}\}, \quad \{\widetilde{\rho}_2, \widetilde{\rho}_{R+2}\}, \quad \{\widetilde{\rho}_2, \widetilde{\rho}_3\}$$

respectively. We set

$$\mathbf{b}_i := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*L_i)^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*L_i)^m)}, \quad i = 2, 3, 4.$$

Then, we have

$$\widetilde{\mathbf{h}} = \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4,$$

where the $\mathbf{b}_i$'s are computed in [76, Lemma 130] as

$$\mathbf{b}_1 = \frac{(-1)^d}{\lfloor dw_0 \rfloor! d!}\left(\frac{\mathsf{u}_1}{d}\right)^{-\lfloor dw_0 \rfloor - d}, \quad \mathbf{b}_4 = \prod_{a=1}^{d-1}\left(\mathsf{u}_4 + \frac{a\mathsf{u}_1}{d}\right),$$

$$\mathbf{b}_2 = \begin{cases} \prod_{a=0}^{\lfloor dw_2-\epsilon_2 \rfloor}\left(\widetilde{\mathbf{w}}_2 - \frac{a+\epsilon_2}{d}\mathsf{u}_1\right)^{-1} & \text{if } w_2 \geq 0 \\ \prod_{a=1}^{\lfloor \epsilon_2 - dw_2 - 1 \rfloor}\left(\widetilde{\mathbf{w}}_2 + \frac{a-\epsilon_2}{d}\mathsf{u}_1\right) & \text{if } w_2 < 0, \end{cases} \quad \mathbf{b}_3 = \begin{cases} \prod_{a=0}^{\lfloor dw_3-\epsilon_3 \rfloor}\left(\widetilde{\mathbf{w}}_3 - \frac{a+\epsilon_3}{d}\mathsf{u}_1\right)^{-1} & \text{if } w_3 \geq 0 \\ \prod_{a=1}^{\lfloor \epsilon_3 - dw_3 - 1 \rfloor}\left(\widetilde{\mathbf{w}}_3 + \frac{a-\epsilon_3}{d}\mathsf{u}_1\right) & \text{if } w_3 < 0. \end{cases}$$

It follows that u_4 is not a factor of $\widetilde{\mathbf{h}}$. We can compute that

$$\mathbf{b}_4\Big|_{\mathsf{u}_4=0} = (d-1)!\left(\frac{\mathsf{u}_1}{d}\right)^{d-1}.$$

Moreover, if $\lambda = 1$, then

$$(\mathbf{b}_2\mathbf{b}_3)\Bigg|_{\mathsf{u}_4=0} = (-1)^{\lfloor dw_3-\epsilon_3 \rfloor+1}\left(\frac{\mathsf{u}_1}{d}\right)^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1}\left(\frac{\mathsf{u}_2 - f\mathsf{u}_1}{\mathfrak{m}}\right)^{-1} \cdot \prod_{a=1}^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1}\left(\frac{d\mathbf{w}_2}{\mathsf{u}_1} + a - \epsilon_2\right).$$

On the other hand, if $\lambda \neq 1$, then

$$(\mathbf{b}_2\mathbf{b}_3)\Big|_{\mathsf{u}_4=0} = (-1)^{\lfloor dw_3-\epsilon_3 \rfloor+1}\left(\frac{\mathsf{u}_1}{d}\right)^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1} \cdot \prod_{a=1}^{\lfloor dw_0 \rfloor + \mathrm{age}(h(d,\lambda))-1}\left(\frac{d\mathbf{w}_2}{\mathsf{u}_1} + a - \epsilon_2\right).$$

The lemma thus follows. $\square$

We now consider a more general cone $\widetilde{\tau} \in \iota(\Sigma(2) \smallsetminus \Sigma(2)_c)$. Here, $\widetilde{\tau} = \iota(\delta_0(\widetilde{\sigma}))$ for some $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \smallsetminus \iota(\Sigma(3))$ and $I'_{\widetilde{\tau}} = \{i_2(\widetilde{\sigma}), i_3(\widetilde{\sigma}), R+2\}$. We have

$$G_{\widetilde{\tau}} \cong \mu_{\gcd(|m_{i_2(\widetilde{\sigma})} - m_{i_3(\widetilde{\sigma})}|, |n_{i_2(\widetilde{\sigma})} - n_{i_3(\widetilde{\sigma})}|)} \subseteq G_{\widetilde{\sigma}} \cong \mu_{|G_{\widetilde{\sigma}}|}.$$

Given $\gamma \in G_{\widetilde{\tau}} \subseteq G_{\widetilde{\sigma}}$, we have $\chi_{(\widetilde{\tau},\widetilde{\sigma})}(\gamma) = \chi_{(\delta_4(\widetilde{\sigma}),\widetilde{\sigma})}(\gamma) = 1$, and since $\chi_{(\delta_2(\widetilde{\sigma}),\widetilde{\sigma})}, \chi_{(\delta_3(\widetilde{\sigma}),\widetilde{\sigma})} : G_{\widetilde{\sigma}} \to \mu_{|G_{\widetilde{\sigma}}|}$ are isomorphisms, $\chi_{(\delta_2(\widetilde{\sigma}),\widetilde{\sigma})}(\gamma) = \chi_{(\delta_3(\widetilde{\sigma}),\widetilde{\sigma})}(\gamma) = 1$ if and only if $\gamma = 1$.

Lemma B.11 (Edges with label in $\iota(\Sigma(2) \setminus \Sigma(2)_c)$)**.** *Let* $\widetilde{\tau} \in \widetilde{\Sigma}(3)_c \cap \iota(\Sigma(2) \setminus \Sigma(2)_c)$*, and*

$$u : \mathcal{C} = \mathcal{C}_{r_1,r_2} \to \mathfrak{l}_{\widetilde{\tau}} \subset \widetilde{\mathcal{X}}$$

be a morphism as in (A.3) *in Section A.2 with degree* $(d, \lambda) \in H_{\widetilde{\tau}} \cong \mathbb{Z} \times G_{\widetilde{\tau}}$*. Let*

$$\widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\widetilde{\mathcal{X}})^m)}.$$

Then u_4 *is not a factor of* $\widetilde{\mathbf{h}}$*.*

Proof. Similar to the proof of Lemma B.10, we use the explicit computation of $\widetilde{\mathbf{h}}$ in [76, Lemma 130]. We set

$$\mathbf{b}_1 = \frac{1}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*T\mathfrak{l}_{\widetilde{\tau}})^m)}.$$

Moreover, let $\widetilde{\sigma} \in \widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$ such that $\widetilde{\tau} = \iota(\delta_0(\widetilde{\sigma}))$. The normal bundle $N_{\mathfrak{l}_{\widetilde{\tau}}/\widetilde{\mathcal{X}}}$ splits as a direct sum of $\widetilde{T}'$-equivariant line bundles L_2, L_3, L_4, which are the normal bundles of $\mathfrak{l}_{\widetilde{\tau}}$ in the 2-dimensional $\widetilde{T}$-invariant closed substacks of $\widetilde{\mathcal{X}}$ corresponding to the cones spanned by

$$\{\widetilde{\rho}_{i_3(\widetilde{\sigma})}, \widetilde{\rho}_{R+2}\}, \quad \{\widetilde{\rho}_{i_2(\widetilde{\sigma})}, \widetilde{\rho}_{R+2}\}, \quad \{\widetilde{\rho}_{i_2(\widetilde{\sigma})}, \widetilde{\rho}_{i_3(\widetilde{\sigma})}\}$$

respectively. We set

$$\mathbf{b}_i := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^*L_i)^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^*L_i)^m)}, \quad i = 2, 3, 4.$$

Then, we have

$$\widetilde{\mathbf{h}} = \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4.$$

By the computations of the $\mathbf{b}_i$'s in [76, Lemma 130], u_4 is not a factor of any of them, which implies the lemma. □

Lastly, we consider a cone $\widetilde{\tau} \in \widetilde{\Sigma}(3)_c \setminus \iota(\Sigma(2))$. Here, $\widetilde{\tau}$ is a common facet of two distinct 4-cones in $\widetilde{\Sigma}(4) \setminus \iota(\Sigma(3))$, and $G_{\widetilde{\tau}} = \{1\}$, which implies that $H_{\widetilde{\tau}} \cong \mathbb{Z}$.

Lemma B.12 (Edges with label in $\widetilde{\Sigma}(3)_c \setminus \iota(\Sigma(2))$)**.** *Let* $\widetilde{\tau} \in \widetilde{\Sigma}(3)_c \setminus \iota(\Sigma(2))$*, and*

$$u : \mathcal{C} = \mathcal{C}_{r_1,r_2} \to \mathfrak{l}_{\widetilde{\tau}} \subset \widetilde{\mathcal{X}}$$

be a morphism as in (A.3) *in Section A.2 with degree* $d \in H_{\widetilde{\tau}} \cong \mathbb{Z}$*. Let*

$$\widetilde{\mathbf{h}} := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^* T\widetilde{\mathcal{X}})^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^* T\widetilde{\mathcal{X}})^m)}.$$

Then the power of u_4 *in* $\widetilde{\mathbf{h}}$ *is* 1.

Proof. We again use the explicit computation of $\widetilde{\mathbf{h}}$ in [76, Lemma 130]. We set

$$\mathbf{b}_2 = \frac{1}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^* T\mathfrak{l}_{\widetilde{\tau}})^m)}.$$

Moreover, let $I'_{\widetilde{\tau}} = \{i, R+1, R+2\}$. The normal bundle $N_{\mathfrak{l}_{\widetilde{\tau}}/\widetilde{\mathcal{X}}}$ splits as a direct sum of $\widetilde{T}'$-equivariant line bundles L_1, L_3, L_4, which are the normal bundles of $\mathfrak{l}_{\widetilde{\tau}}$ in the 2-dimensional $\widetilde{T}$-invariant closed substacks of $\widetilde{\mathcal{X}}$ corresponding to the cones spanned by

$$\{\widetilde{\rho}_i, \widetilde{\rho}_{R+2}\}, \quad \{\widetilde{\rho}_{R+1}, \widetilde{\rho}_{R+2}\}, \quad \{\widetilde{\rho}_i, \widetilde{\rho}_{R+1}\}$$

respectively. We set

$$\mathbf{b}_i := \frac{e_{\widetilde{T}'}(H^1(\mathcal{C}, u^* L_i)^m)}{e_{\widetilde{T}'}(H^0(\mathcal{C}, u^* L_i)^m)}, \quad i = 1, 3, 4.$$

Then, we have

$$\widetilde{\mathbf{h}} = \mathbf{b}_2 \mathbf{b}_1 \mathbf{b}_3 \mathbf{b}_4.$$

By the computations of the $\mathbf{b}_i$'s in [76, Lemma 130], u_4 is not a factor of any of $\mathbf{b}_2, \mathbf{b}_1, \mathbf{b}_4$, but is a factor of $\mathbf{b}_3$ with power 1. The lemma thus follows. □

Appendix C: A numerical open/relative/local correspondence

In this chapter, as mentioned in Remark 4.2, we extend the numerical open/closed correspondence (Theorem 4.1) to include the relative Gromov-Witten invariants of a log Calabi-Yau pair that is closely related to both the open and closed geometries. For simplicity, we restrict to the case where $\mathcal{X} = X$ is smooth and the open invariants have no insertions ($n = 0$). Similar results may be obtained in the orbifold setting as well.

We use notations with hat ($\hat{\ }$) while discussing the relative geometry and invariants.

C.1 Relative geometry

We start by defining the relative geometry. Let Y be the smooth toric 3-fold defined by a fan $\hat{\Sigma}$ in $N_{\mathbb{R}}$ specified as follows:

- $\hat{\Sigma}(1) = \Sigma(1) \sqcup \{\rho_{R+1}\}$, where $\rho_{R+1} = \mathbb{R}_{\geq 0} b_{R+1}$ and under the basis $\{v_1, v_2, v_3\}$ of N,

$$b_{R+1} = (-1, -f, 0) \in N.$$

- $\hat{\Sigma}(2) = \Sigma(2) \cup \{\hat{\tau}_2, \hat{\tau}_3\}$, where $I'_{\hat{\tau}_2} = \{3, R+1\}$, $I'_{\hat{\tau}_3} = \{2, R+1\}$.
- $\hat{\Sigma}(3) = \Sigma(3) \sqcup \{\hat{\sigma}_0\}$, where $I'_{\hat{\sigma}_0} = \{2, 3, R+1\}$.

Let $D := V(\rho_{R+1})$ be the toric divisor corresponding to the additional ray ρ_{R+1}. We have the following observations on the relation between the pair (Y, D) and X:

- We have $Y \setminus D = X$. The pair (Y, D) is log Calabi-Yau: $\bigwedge^3 \Omega_Y(\log D) \cong \mathcal{O}_Y$.
- We have $\hat{\Sigma}(2)_c = \Sigma(2)_c \sqcup \{\tau_0\}$ and $Y^1_c = X^1_c \cup l_{\tau_0}$.
- $F(\hat{\Sigma}) = F(\Sigma) \sqcup \{(\tau_0, \hat{\sigma}_0), (\hat{\tau}_2, \hat{\sigma}_0), (\hat{\tau}_3, \hat{\sigma}_0)\}$.

- Similar to the construction in Section 2.5.3, there is an isomorphism

$$H_2(X,L;\mathbb{Z}) \xrightarrow{\cong} H_2(Y;\mathbb{Z})$$

which maps $[B]$ to $[l_{\tau_0}]$, with τ_0 viewed as a cone in $\hat{\Sigma}(2)_c$.

The relative geometry (Y,D) is related to the closed geometry $\widetilde{\mathcal{X}}$ as follows: Let $\widetilde{X}' \subset \widetilde{X}$ be the smooth toric Calabi-Yau 4-fold defined by the subfan $\widetilde{\Sigma}'$ of $\widetilde{\Sigma}$ consisting of $\widetilde{\Sigma}_0$ and the 4-cone $\widetilde{\sigma}_0$ (together with its faces). See Section 2.4. Then, $\widetilde{X}'$ is isomorphic to the total space of the line bundle $\mathcal{O}_Y(-D)$ and describes the local geometry of Y in $\widetilde{X}$. In general, $\widetilde{X}'$ is not equal to $\widetilde{X}$ and not semi-projective. We have the following additional observations on the relation between (Y,D) and $\widetilde{X}'$:

- The inclusion maps (2.9) can be naturally extended to maps $\hat{\Sigma}(2) \to \widetilde{\Sigma}'(3) \subseteq \widetilde{\Sigma}(3)$ and $\hat{\Sigma}(3) \to \widetilde{\Sigma}'(4) \subseteq \widetilde{\Sigma}(4)$ by

$$\hat{\sigma}_0 \mapsto \widetilde{\sigma}_0, \quad \hat{\tau}_2 \mapsto \delta_2(\widetilde{\sigma}_0), \quad \hat{\tau}_3 \mapsto \delta_3(\widetilde{\sigma}_0).$$

(See (3.14).) We denote the extended maps also by ι.

- The map $\iota : \hat{\Sigma}(2) \to \widetilde{\Sigma}'(3)$ restricts to a bijection between $\hat{\Sigma}(2)_c$ and $\widetilde{\Sigma}'(3)_c$, which in particular maps τ_0 to $\iota(\tau_0)$. This induces the isomorphism $Y^1_c \cong (\widetilde{X}')^1_c$. The induced isomorphism on second homology fits into the following factorization of (2.11):

$$H_2(X,L;\mathbb{Z}) \xrightarrow{\cong} H_2(Y;\mathbb{Z}) \xrightarrow{\cong} H_2(\widetilde{X}';\mathbb{Z}) \longrightarrow H_2(\widetilde{X};\mathbb{Z}). \qquad \text{(C.1)}$$

C.2 A basic example

Before defining the Gromov-Witten invariants of (Y,D) and $\widetilde{X}'$ and stating the extended correspondence, we first consider the example $X = \mathbb{C}^3$ discussed in Section 2.6.1 and Example 4.3.

In this case,

$$Y = \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(f) \oplus \mathcal{O}_{\mathbb{P}^1}(-f-1)\right), \quad \widetilde{X}' = \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(f) \oplus \mathcal{O}_{\mathbb{P}^1}(-f-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\right),$$

and $D \cong \mathbb{C}^2$ is the fiber of the projection $Y \to l_{\tau_0} = \mathbb{P}^1$ over the fixed point $p_{\hat{\sigma}_0}$.

When $f = 0$, we have

$$Y = \mathbb{C} \times S, \quad \widetilde{X}' = \widetilde{X} = \mathbb{C} \times X',$$

where $S = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1))$ is the blowup of $\mathbb{C}^2$ at the origin, and $X' = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ is the resolved conifold. We observe that

$$\overline{\mathcal{M}}_{0,n}(\widetilde{X}', d[l_{\iota(\tau_0)}]) = \mathbb{C} \times \overline{\mathcal{M}}_{0,n}(X', d[l_{\iota(\tau_0)}]), \quad \overline{\mathcal{M}}(Y/D, d[l_{\tau_0}]) = \mathbb{C} \times \overline{\mathcal{M}}(S/F, d[l_{\tau_0}]),$$

where the second relation is for moduli spaces of relative stable maps to be defined later, and $F \cong \mathbb{C}$ is the fiber of the projection $S \to l_{\tau_0}$ over $p_{\hat{\sigma}_0}$.

Now let $d \in \mathbb{Z}_{>0}$. As in Example 4.3, the degree-d disk invariant of $(X, L, 0)$ is $1/d^2$. Moreover, the degree-d local Gromov-Witten invariant of $\widetilde{X}'$ that we will consider coincides with the closed invariant of $\widetilde{X}$, which is also $1/d^2$. On the other hand, the degree-d relative Gromov-Witten invariant of (Y, D) that we will consider is

$$\frac{(-1)^{d+1}}{d^2},$$

which coincides with the relative/log Gromov-Witten invariant of the pair (S, F) computed in the proof of Proposition 2.4 in [48]. Our extended numerical correspondence states that the relative invariant differs from either the disk or the local invariant by a sign $(-1)^{d+1}$.

C.3 Maximally tangent relative Gromov-Witten invariants of (Y, D)

Given the log Calabi-Yau 3-fold (Y, D), we consider its genus-zero *maximally-tangent relative Gromov-Witten invariants*, which are a special class of relative invariants. As our computations will show, these invariants can be recovered from the *formal* relative Gromov-Witten invariants of $(\hat{Y}, \hat{D})$ introduced by Li-Liu-Liu-Zhou [73], where $(\hat{Y}, \hat{D})$ is the formal completion of (Y, D) along the 1-skeleton Y_c^1. We use the moduli spaces of relative stable maps defined by Li [71, 72]. Our computations are similar to those in [73, 77].

C.3.1 Expanded targets and torus action

Let $\Delta(D)$ be the total space of the projective line bundle

$$\mathbb{P}(\mathcal{O}_D \oplus N_{D/Y}) \to D.$$

The action of the Calabi-Yau torus T' on D extends to an action on $\Delta(D)$, under which the fixed locus is the fiber over $p_{\hat{\sigma}_0}$.

For each $m \in \mathbb{Z}_{\geq 0}$, define

$$Y[m] = Y \cup \Delta_{(1)} \cup \cdots \cup \Delta_{(m)},$$

where each $\Delta_{(i)}$ is a copy of $\Delta(D)$ with distinguished sections $D_{(i-1)} = \mathbb{P}(\mathcal{O}_D \oplus 0)$ and $D_{(i)} = \mathbb{P}(0 \oplus N_{D/Y})$, and

$$Y \cap \Delta_{(1)} = D_{(0)}, \qquad \Delta_{(i)} \cap \Delta_{(i+1)} = D_{(i)} \quad \text{for } i = 1, \ldots, m-1.$$

In particular, for $m = 0$, we have $(Y[0], D_{(0)}) = (Y, D)$. We denote

$$Y(m) = \Delta_{(1)} \cup \cdots \cup \Delta_{(m)},$$

which admits a map to $D_{(0)} = D$ by projection. This induces a projection map

$$\pi_m : Y[m] \to Y.$$

We denote $p_{(0)} = p_{\hat{\sigma}_0}$. For each $i = 1, \ldots, m$, let $l_{(i)} \coloneqq \pi_m^{-1}(p_{(0)}) \cap \Delta_{(i)} \cong \mathbb{P}^1$ and $p_{(i)}$ be the unique point where $l_{(i)}$ and $D_{(i)}$ intersects. Let

$$Y_c^1(m) \coloneqq l_{(1)} \cup \cdots \cup l_{(m)},$$

which is a chain of m copies of $\mathbb{P}^1$'s, and $Y_c^1[m] \coloneqq Y_c^1 \cup Y_c^1(m)$. Then the torus T' acts on $Y(m)$ with fixed locus $Y_c^1(m)$ and the projection $\pi_m : Y[m] \to Y$ is T'-equivariant. In addition, there is a $(\mathbb{C}^*)^m$-action on $Y(m)$ that scales the fiber direction of each $\Delta_{(i)}$ and makes the projection $Y(m) \to D$ invariant. This action extends to an action on $Y[m]$ that pointwise fixes Y and makes π_m invariant, which restricts to an action on $Y_c^1[m]$ that pointwise fixes Y_c^1.

C.3.2 Definition

Let $\hat{\beta} \in H_2(Y; \mathbb{Z})$ be an effective curve class of Y such that $d \coloneqq \hat{\beta} \cdot D > 0$. Let

$$\overline{\mathcal{M}}(Y/D, \hat{\beta})$$

be the moduli space of morphisms

$$u : (C, x) \to (Y[m], D_{(m)})$$

where:

- (C, x) is a connected prestable Riemann surface of arithmetic genus 0 with a single marked point x.
- $m \in \mathbb{Z}_{\geq 0}$ and $(Y[m], D_{(m)})$ is as defined in Section C.3.1.

- $(\pi_m \circ u)_*[C] = \hat{\beta}$.

- $u(x) \in D_{(m)}$ and $u^{-1}(D_{(m)}) = dx$ as Cartier divisors.

- For each $i = 1, \dots, m-1$, the preimage $u^{-1}(D_{(i)})$ consists of nodes in C. If $q \in u^{-1}(D_{(i)})$ and C_1, C_2 are the two irreducible components of C that intersect at q, then $u|_{C_1}$ and $u|_{C_2}$ have the same contact order to $D_{(i)}$ at q.

- The automorphism group of u is finite. Here, an automorphism of u is a pair (α_1, α_2), where α_1 is an automorphism of (C, x) and α_2 is an automorphism of $(Y[m], D_{(m)})$ that makes π_m invariant, such that $u \circ \alpha_1 = \alpha_2 \circ u$.

$\overline{\mathcal{M}}(Y/D, \hat{\beta})$ is a Deligne-Mumford stack with a perfect obstruction theory of virtual dimension 1. The action of T' on the expanded targets $(Y[m], D_{(m)})$ induces an action on $\overline{\mathcal{M}}(Y/D, \hat{\beta})$ under which the fixed locus is proper.

Definition C.1. The genus-zero, degree-$\hat{\beta}$ *maximally-tangent relative Gromov-Witten invariant* of (Y, D) is defined by

$$\langle \mathsf{u}_2 - f\mathsf{u}_1 \rangle_{\hat{\beta}}^{(Y,D),T_f} := \int_{[\overline{\mathcal{M}}(Y/D,\hat{\beta})^{T'}]^{\mathrm{vir}}} \frac{\mathsf{u}_2 - f\mathsf{u}_1}{e_{T'}(N^{\mathrm{vir}})}\Bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} \in \mathbb{Q}$$

where N^{vir} is the virtual normal bundle of $\overline{\mathcal{M}}(Y/D, \hat{\beta})^{T'}$.

Remark C.2. Our choice of the class $\mathsf{u}_2 - f\mathsf{u}_1$ in the integration follows the choice by Li-Liu-Liu-Zhou [73]: The moduli space $\overline{\mathcal{M}}(Y/D, \hat{\beta})$ admits an evaluation map ev to the divisor D associated to the marked point x. Then our choice of class is $\mathrm{ev}^*(c_1^{T'}(\mathcal{O}_D(l_{\hat{\tau}_2}))) = \mathsf{u}_2 - f\mathsf{u}_1$.

Remark C.3. We will confirm, as a consequence of our extended correspondence between the open and relative invariants, that $\int_{[\overline{\mathcal{M}}(Y/D,\hat{\beta})^{T'}]^{\mathrm{vir}}} \frac{\mathsf{u}_2 - f\mathsf{u}_1}{e_{T'}(N^{\mathrm{vir}})}$ has no pole along $\mathsf{u}_2 - f\mathsf{u}_1$ and thus $\langle \mathsf{u}_2 - f\mathsf{u}_1 \rangle_{\hat{\beta}}^{(Y,D),T_f}$ is defined. In fact, the weight restriction $\mathsf{u}_2 - f\mathsf{u}_1 = 0$ will turn out to be unnecessary. By our localization computations in the subsequent section, the expression $\int_{[\overline{\mathcal{M}}(Y/D,\hat{\beta})^{T'}]^{\mathrm{vir}}} \frac{\mathsf{u}_2 - f\mathsf{u}_1}{e_{T'}(N^{\mathrm{vir}})}$

can be identified with a formal relative Gromov-Witten invariant of $(\hat{Y}, \hat{D})$ introduced by Li-Liu-Liu-Zhou [73], which is shown to be a rational number independent of $\mathsf{u}_1, \mathsf{u}_2$ (see [73, Theorem 4.8]).

C.4 Localization computations of relative invariants

In this section, we compute the relative invariant $\langle \mathsf{u}_2 - f\mathsf{u}_1 \rangle_{\hat{\beta}}^{(Y,D),T_f}$ (Definition C.1) by localization. The end result is given in Proposition C.4. We first describe connected components of the T'-fixed locus of $\overline{\mathcal{M}}(Y/D, \hat{\beta})$ in terms of decorated graphs, and then compute the contribution of each component by computing the virtual normal bundle from the moving part of the perfect obstruction theory.

C.4.1 Fixed locus of moduli

Let $u : (C, x) \to (Y[m], D_{(m)})$ be a relative stable map that represents a point $[u] \in \overline{\mathcal{M}}(Y/D, \hat{\beta})^{T'}$. Let $\widetilde{u} := \pi_m \circ u : C \to Y$ denote the composition. The image of u lies in $Y_c^1[m] \subset Y[m]$, and the image of $\widetilde{u}$ lies in $Y_c^1 \subset Y$. Then we associate to u the decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{s}) \in \Gamma_{0,1}(Y, \hat{\beta})$ associated to $\widetilde{u}$ as in Section 3.2. (Since Y is smooth, the twisting map $\vec{k}$ is trivial and thus omitted.) Note that $\vec{f} \circ \vec{s}(1) = \hat{\sigma}_0$. Let

$$\Gamma'_{0,1}(Y, \hat{\beta})$$

denote the set of all decorated graphs that arise this way, which indexes the connected components of $\overline{\mathcal{M}}(Y/D, \hat{\beta})^{T'}$.

Given $\vec{\Gamma} \in \Gamma'_{0,1}(Y, \hat{\beta})$, let $\mathcal{F}_{\vec{\Gamma}}$ denote the connected component of $\overline{\mathcal{M}}(Y/D, \hat{\beta})^{T'}$ indexed by $\vec{\Gamma}$. We partition $V(\Gamma)$ into two subsets

$$V_0(\Gamma) := \{v \in V(\Gamma) : \vec{f}(v) \neq \hat{\sigma}_0\}, \qquad V_1(\Gamma) := \vec{f}^{-1}(\hat{\sigma}_0).$$

Note that for any $v \in V_0(\Gamma)$, $n_v = 0$ and $S_v = \varnothing$. Moreover, $V_1(\Gamma) = \{\hat{v}_0(\vec{\Gamma})\}$ is a singleton set. Let $\mu(\vec{\Gamma}) = (\mu(\vec{\Gamma})_1, \dots, \mu(\vec{\Gamma})_{\ell(\mu(\vec{\Gamma}))})$ be the partition of d determined by the degrees of $\widetilde{u}$ restricted to the components in $\vec{f}^{-1}(\tau_0) \subset E(\Gamma)$. Note that $\ell(\mu(\vec{\Gamma})) = 1$ if and only if $m = 0$. We consider two cases separately:

- *Case I:* $\ell(\mu(\vec{\Gamma})) = 1$, i.e. $\mu(\vec{\Gamma}) = (d)$. In this case, $\vec{\Gamma}$ satisfies the following:

  - $C_{\hat{v}_0(\vec{\Gamma})} = \{x\}$.
  - There is a unique edge $e_0(\vec{\Gamma}) \in E(\Gamma)$ such that $\vec{f}(e_0(\vec{\Gamma})) = \tau_0$. We have $(e_0(\vec{\Gamma}), \hat{v}_0(\vec{\Gamma})) \in F(\Gamma)$ and $\vec{d}(e_0(\vec{\Gamma})) = d$.
  - $\vec{s}(1) = \hat{v}_0(\vec{\Gamma})$, $n_{\hat{v}_0(\vec{\Gamma})} = 1$, and $S_{\hat{v}_0(\vec{\Gamma})} = \{1\}$.

  There is a map $i_{\vec{\Gamma}} : \mathcal{M}_{\vec{\Gamma}} \to \overline{\mathcal{M}}(Y/D, \hat{\beta})$ with image $\mathcal{F}_{\vec{\Gamma}}$, under which $\mathcal{F}_{\vec{\Gamma}}$ can be identified as a quotient of $\mathcal{M}_{\vec{\Gamma}}$ by a finite group, as in Section 3.2.

- *Case II:* $\ell(\mu(\vec{\Gamma})) > 1$. In this case, for each relative stable map $u : C \to Y[m]$ whose associated graph is $\vec{\Gamma}$, the restriction to $C_{\hat{v}_0(\vec{\Gamma})}$ represents a point in

  $$\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}} := \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, \mu(\vec{\Gamma}), (d)) /\!\!/ \mathbb{C}^*,$$

  the moduli space of relative stable maps to the non-rigid $(\mathbb{P}^1, 0, \infty)$ with relative condition $\mu(\vec{\Gamma})$ at 0 and (d) at ∞. Such a map has form

  $$u' : (C', y_1, \dots, y_{\ell(\mu(\vec{\Gamma}))}, x) \to \mathbb{P}^1(m)$$

  where:

  - $(C', y_1, \dots, y_{\ell(\mu(\vec{\Gamma}))}, x)$ is a connected prestable Riemann surface of arithmetic genus 0 with $\ell(\mu(\vec{\Gamma})) + 1$ marked points.
  - $u'(x) = p_{(m)}$, and $(u')^{-1}(p_{(m)}) = dx$ as Cartier divisors; $u'(y_j) = p_{(0)}$ for each j, and $(u')^{-1}(p_{(0)}) = \mu(\vec{\Gamma})_1 y_1 + \cdots + \mu(\vec{\Gamma})_{\ell(\mu(\vec{\Gamma}))} y_{\ell(\mu(\vec{\Gamma}))}$ as Cartier divisors.

- For $i = 1, \ldots, m-1$, the preimage $(u')^{-1}(p_{(i)})$ consists of nodes in C'. If $q \in (u')^{-1}(p_{(i)})$ and C_1, C_2 are the two irreducible components of C' that intersect at q, then $u'|_{C_1}$ and $u'|_{C_2}$ have the same contact order to $p_{(i)}$ at q.
- The automorphism group of u' is finite. Here, an automorphism of u' is a pair (α_1, α_2), where α_1 is an automorphism of C' fixing x and each y_j and $\alpha_2 \in (\mathbb{C}^*)^m$ is an automorphism of $(\mathbb{P}^1(m), p_{(0)}, p_{(m)})$, such that $u' \circ \alpha_1 = \alpha_2 \circ u'$.

There is a map $i_{\vec{\Gamma}}$ from

$$\hat{\mathcal{M}}_{\vec{\Gamma}} := \left(\prod_{v \in V^S(\Gamma) \cap V_0(\Gamma)} \overline{\mathcal{M}}_{0,E_v} \right) \times \overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}}$$

to $\overline{\mathcal{M}}(Y/D, \hat{\beta})$ with image $\mathcal{F}_{\vec{\Gamma}}$, under which $\mathcal{F}_{\vec{\Gamma}}$ can be identified as a quotient of $\hat{\mathcal{M}}_{\vec{\Gamma}}$ by a finite group.

We denote

$$\Gamma^0_{0,1}(Y, \hat{\beta}) := \{\vec{\Gamma} \in \Gamma'_{0,1}(Y, \hat{\beta}) : \ell(\mu(\vec{\Gamma})) = 1\}.$$

C.4.2 Virtual normal bundle

At a point $[u : (C, x) \to (Y[m], D_{(m)})] \in \overline{\mathcal{M}}(Y/D, \hat{\beta})$, the tangent space T^1 and the obstruction space T^2 are determined by the following two exact sequences of complex vector spaces:

$$0 \to \mathrm{Ext}^0(\Omega_C(x), \mathcal{O}_C) \to H^0(\mathbf{D}^\bullet) \to T^1 \to \mathrm{Ext}^1(\Omega_C(x), \mathcal{O}_C) \to H^1(\mathbf{D}^\bullet) \to T^2 \to 0,$$

$$\begin{aligned} 0 \to &H^0(C, u^*\Omega_{Y[m]}(\log D_{(m)}))^\vee \to H^0(\mathbf{D}^\bullet) \to \bigoplus_{i=0}^{m-1} H^0_{\text{ét}}(\mathbf{R}^\bullet_i) \to \\ \to &H^1(C, u^*\Omega_{Y[m]}(\log D_{(m)}))^\vee \to H^1(\mathbf{D}^\bullet) \to \bigoplus_{i=0}^{m-1} H^1_{\text{ét}}(\mathbf{R}^\bullet_i) \to 0. \end{aligned}$$

Here, if $\{q_1, \dots, q_{n_i}\}$ is a list of nodes in $u^{-1}(D_{(i)})$, then

$$H^0_{\text{ét}}(\mathbf{R}_i^\bullet) \cong \bigoplus_{j=1}^{n_i} T_{q_j}(u^{-1}(\Delta_{(i)})) \otimes T^*_{q_j}(u^{-1}(\Delta_{(i)}));$$

$$H^1_{\text{ét}}(\mathbf{R}_i^\bullet) \cong \left(\bigoplus_{j=1}^{n_i} (u|_{\{q_j\}})^{-1} N_{D_{(i)}/\Delta_{(i-1)}} \otimes N_{D_{(i)}/\Delta_{(i)}} \right) / \mathbb{C}$$

where as all the 1-dimensional vector spaces $(u|_{\{q_j\}})^{-1} N_{D_{(i)}/\Delta_{(i-1)}} \otimes N_{D_{(i)}/\Delta_{(i)}}$ are isomorphic, we mod out the direct sum by the diagonal embedding of this vector space.

The obstruction theory of $\overline{\mathcal{M}}(Y/D, \hat{\beta})$ is T'-equivariant, and the virtual normal bundle N^{vir} of $\overline{\mathcal{M}}(Y/D, \hat{\beta})^{T'}$ is given by the moving part. From a calculation similar to that in [77, Appendix A], we obtain the following: For a decorated graph $\vec{\Gamma} \in \Gamma^0_{0,1}(Y, \hat{\beta})$, we have

$$i^*_{\vec{\Gamma}} \left(\frac{1}{e_{T'}(N^{\text{vir}})} \right) = \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma), v \in V_0(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V_0(\Gamma)} \frac{\mathbf{h}(v)}{\prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \psi_{(e,v)} \right)}.$$

Here, the quantities $\mathbf{h}(e,v)$, $\mathbf{h}(v)$, and $\mathbf{w}_{(e,v)}$ are defined in the same way as in Section 3.4.3. If $\widetilde{u} : (C, x) \to (Y, D)$ is the induced map of a relative stable map whose associated decorated graph is $\vec{\Gamma}$, then for each $e \in E(\Gamma)$, we define

$$\mathbf{h}(e) := \frac{e_{T'}(H^1(C_e, (\widetilde{u}|_{C_e})^* \Omega_Y(\log D)^\vee)^m)}{e_{T'}(H^0(C_e, (\widetilde{u}|_{C_e})^* \Omega_Y(\log D)^\vee)^m)}.$$

In the case $\vec{f}(e) \in \Sigma(2)_c$, this is consistent with (3.9) in since $(\Omega_Y(\log D)^\vee)|_X = TX$.

Moreover, for $\vec{\Gamma} \in \Gamma'_{0,1}(Y, \hat{\beta}) \setminus \Gamma^0_{0,1}(Y, \hat{\beta})$,

$$\begin{aligned} i^*_{\vec{\Gamma}} \left(\frac{1}{e_{T'}(N^{\text{vir}})} \right) =& (-\mathsf{u}_1 - \psi^t)^{\ell(\mu(\vec{\Gamma}))-1} \cdot \frac{\left((-f\mathsf{u}_1 + \mathsf{u}_2)(f\mathsf{u}_1 - \mathsf{u}_2)\right)^{\ell(\mu(\vec{\Gamma}))-1}}{\prod_{j=1}^{\ell(\mu(\vec{\Gamma}))} \left(-\frac{\mathsf{u}_1}{\mu(\vec{\Gamma})_j} - \psi_{(e, \hat{v}_0(\vec{\Gamma}))}\right)} \\ & \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma), v \in V_0(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V_0(\Gamma)} \frac{\mathbf{h}(v)}{\prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \psi_{(e,v)} \right)}, \end{aligned} \tag{C.2}$$

where ψ^t is the *target psi class* of $\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}} = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, \mu(\vec{\Gamma}), (d)) /\!\!/ \mathbb{C}^*$ at $0 \in \mathbb{P}^1$ (see e.g. [78, Section 5] for the definition). We note in particular that the term $(-\mathsf{u}_1 - \psi^t)^{\ell(\mu(\vec{\Gamma}))-1}$ comes from the moving part of $\bigoplus_{i=0}^{m-1} H^1_{\text{ét}}(\mathbf{R}_i^\bullet)$ in the perfect obstruction theory. For each $e \in E_{\hat{v}_0(\vec{\Gamma})}$, we have

$$\frac{-\mathsf{u}_1 - \psi^t}{-\frac{\mathsf{u}_1}{d_e} - \psi_{(e,\hat{v}_0(\vec{\Gamma}))}} = d_e.$$

Thus (C.2) can be simplified as follows:

$$\begin{aligned} i^*_{\vec{\Gamma}}\left(\frac{1}{e_{T'}(N^{\text{vir}})}\right) &= \prod_{j=1}^{\ell(\mu(\vec{\Gamma}))} \mu(\vec{\Gamma})_j \cdot (-1)^{\ell(\mu(\vec{\Gamma}))-1} (\mathsf{u}_2 - f\mathsf{u}_1)^{2\ell(\mu(\vec{\Gamma}))-2} \cdot \frac{1}{-\mathsf{u}_1 - \psi^t} \\ &\cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v)\in F(\Gamma), v \in V_0(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V_0(\Gamma)} \frac{\mathbf{h}(v)}{\prod_{e\in E_v}\left(\mathbf{w}_{(e,v)} - \psi_{(e,v)}\right)}. \end{aligned}$$

C.4.3 Summary of computation

We summarize our computation of the maximally-tangent relative Gromov-Witten invariants of (Y, D) as follows:

Proposition C.4. *Let $\hat{\beta} \in H_2(Y;\mathbb{Z})$ be an effective curve class of Y such that $d := \hat{\beta}\cdot D > 0$. Then*

$$\begin{aligned} \langle \mathsf{u}_2 - f\mathsf{u}_1\rangle_{\hat{\beta}}^{(Y,D),T_f} &= \sum_{\vec{\Gamma}\in\Gamma^0_{0,1}(Y,\hat{\beta})} c_{\vec{\Gamma}}(\mathsf{u}_2 - f\mathsf{u}_1) \cdot \prod_{e\in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v)\in F(\Gamma), v\in V_0(\Gamma)} \mathbf{h}(e,v) \\ &\quad \cdot \prod_{v\in V_0(\Gamma)} \int_{\overline{\mathcal{M}}_{0,|E_v|}} \frac{\mathbf{h}(v)}{\prod_{e\in E_v}\left(\mathbf{w}_{(e,v)} - \psi_{(e,v)}\right)}\Bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} \\ &+ \sum_{\vec{\Gamma}\in\Gamma'_{0,1}(Y,\hat{\beta})\setminus\Gamma^0_{0,1}(Y,\hat{\beta})} c_{\vec{\Gamma}}(\mathsf{u}_2 - f\mathsf{u}_1) \cdot \prod_{e\in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v)\in F(\Gamma), v\in V_0(\Gamma)} \mathbf{h}(e,v) \\ &\quad \cdot \prod_{v\in V_0(\Gamma)} \int_{\overline{\mathcal{M}}_{0,|E_v|}} \frac{\mathbf{h}(v)}{\prod_{e\in E_v}\left(\mathbf{w}_{(e,v)} - \psi_{(e,v)}\right)} \\ &\quad \cdot \prod_{j=1}^{\ell(\mu(\vec{\Gamma}))} \mu(\vec{\Gamma})_j \cdot (-1)^{\ell(\mu(\vec{\Gamma}))-1} (\mathsf{u}_2 - f\mathsf{u}_1)^{2\ell(\mu(\vec{\Gamma}))-2} \int_{\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}}} \frac{1}{-\mathsf{u}_1 - \psi^t}\Bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0}. \end{aligned}$$

Here, we adopt the integration convention (A.2) for the unstable vertices. We note that the dimension of $\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}}$ is $\ell(\mu(\vec{\Gamma})) - 2$, and

$$\int_{\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}}} \frac{1}{-\mathsf{u}_1 - \psi^t} = (-\mathsf{u}_1)^{1-\ell(\mu(\vec{\Gamma}))} \int_{\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}}} (\psi^t)^{\ell(\mu(\vec{\Gamma}))-2}.$$

C.5 Local Gromov-Witten invariants of $\widetilde{X}'$

In addition to the relative invariants of (Y, D), we also consider genus-zero Gromov-Witten invariants of the local geometry $\widetilde{X}' = \mathrm{Tot}(\mathcal{O}_Y(-D))$, which we show are identified with the closed invariants of $\widetilde{\mathcal{X}}$ defined and computed in Section 3.5.

Let $\widetilde{\beta} \in H_2(\widetilde{X}'; \mathbb{Z})$ be a non-zero effective curve class of $\widetilde{X}'$. Consider the moduli space $\overline{\mathcal{M}}_{0,1}(\widetilde{X}', \widetilde{\beta})$ of genus-zero, 1-pointed, degree-$\widetilde{\beta}$ stable maps to $\widetilde{X}'$, which is a Deligne-Mumford stack with a perfect obstruction theory of virtual dimension 2.

Definition C.5. Let $\widetilde{\gamma}'$ be the class in $H^4_{\widetilde{T}'}(\widetilde{X}'; \mathbb{Q})$ defined by

$$\widetilde{\gamma}' := \widetilde{\mathcal{D}}_2^{\widetilde{T}'} \widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}.$$

where $\widetilde{\mathcal{D}}_i^{\widetilde{T}'} = [V(\widetilde{\rho}_i)]$ is the $\widetilde{T}'$-equivariant Poincaré dual of the divisor $V(\widetilde{\rho}_i)$. Then, we define the *local* Gromov-Witten invariant

$$\langle \widetilde{\gamma}' \rangle_{\widetilde{\beta}}^{\widetilde{X}', T_f} := \int_{[\overline{\mathcal{M}}_{0,1}(\widetilde{X}', \widetilde{\beta})^{\widetilde{T}'}]^{\mathrm{vir}}} \frac{\iota^* \mathrm{ev}_1^*(\widetilde{\gamma}')}{e_{\widetilde{T}'}(N^{\mathrm{vir}})} \bigg|_{\mathsf{u}_4 = 0, \mathsf{u}_2 - f\mathsf{u}_1 = 0} \in \mathbb{Q},$$

where $\iota : \overline{\mathcal{M}}_{0,1}(\widetilde{X}', \widetilde{\beta})^{\widetilde{T}'} \to \overline{\mathcal{M}}_{0,1}(\widetilde{X}', \widetilde{\beta})$ is the inclusion and N^{vir} is the virtual normal bundle.

Note that the choice of class $\widetilde{\gamma}'$ is consistent with $\widetilde{\gamma}_1$ in (3.12) (for $\lambda = 1$) since in $\widetilde{X}'$, the only $\widetilde{T}'$-fixed point contained in the divisor $V(\widetilde{\rho}_3)$ is $p_{\widetilde{\sigma}_0}$. Applying the same reasoning for the divisor $V(\widetilde{\rho}_2)$ gives the following identity in $H^4_{\widetilde{T}'}(\widetilde{X}'; \mathbb{Q})$:

$$\widetilde{\gamma}' = (\mathsf{u}_2 - f\mathsf{u}_1) \widetilde{\mathcal{D}}_{R+1}^{\widetilde{T}'}.$$

The local invariant $\langle \widetilde{\gamma}' \rangle_{\widetilde{\beta}}^{\widetilde{X}', T_f}$ of $\widetilde{X}'$ can be computed using $\widetilde{T}'$-equivariant localization in terms of contributions from decorated graphs in $\Gamma_{0,1}(\widetilde{X}', \widetilde{\beta})$, in the same way as in Section 3.5.3. We now view $\widetilde{\beta}$ as a class in $H_2(\widetilde{X}; \mathbb{Z})$ under the inclusion $H_2(\widetilde{X}'; \mathbb{Z}) \to H_2(\widetilde{X}; \mathbb{Z})$ and compare $\langle \widetilde{\gamma}' \rangle_{\widetilde{\beta}}^{\widetilde{X}', T_f}$ to the closed invariant $\langle \widetilde{\gamma}_1 \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}}, T_f}$. Note that the inclusion $(\widetilde{X}')^1_c \subseteq \widetilde{X}^1_c$ induces the following inclusion on the sets of decorated graphs:

$$\Gamma_{0,1}(\widetilde{X}', \widetilde{\beta}) \to \Gamma_{0,1}(\widetilde{X}, \widetilde{\beta}).$$

If $\widetilde{\beta} \in H_2(X;\mathbb{Z})$, then by the reasoning of Observation 4.5, both invariants are zero. Otherwise, observe that any additional graph in $\Gamma_{0,1}(\widetilde{X}, \widetilde{\beta}) \smallsetminus \Gamma_{0,1}(\widetilde{X}', \widetilde{\beta})$ fails to satisfy condition (4.4) and thus does not contribute to $\langle \widetilde{\gamma}_1 \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}}, T_f}$ by Lemma 4.6. Therefore, the two invariants are always equal.

Lemma C.6. *For any effective class $\widetilde{\beta} \in H_2(\widetilde{X}';\mathbb{Z})$, we have*

$$\langle \widetilde{\gamma}' \rangle_{\widetilde{\beta}}^{\widetilde{X}', T_f} = \langle \widetilde{\gamma}_1 \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}}, T_f}.$$

C.6 Extended numerical correspondence

Our main result of the chapter is the following extension of the numerical open/closed correspondence (Theorem 4.1) that includes the relative and local invariants.

Theorem C.7. *Let $\beta \in H_2(X;\mathbb{Z})$ be an effective class and $d \in \mathbb{Z}_{>0}$. Set*

$$\beta' = \beta + d[B]$$

and $\hat{\beta} \in H_2(Y;\mathbb{Z})$, $\widetilde{\beta} \in H_2(\widetilde{X}';\mathbb{Z}) \subseteq H_2(\widetilde{X};\mathbb{Z})$ be the images of β' under the inclusions (C.1). *Then*

$$\langle \rangle_{\beta',(d,1)}^{X,(L,f)} = (-1)^{d+1} \langle \mathsf{u}_2 - f \mathsf{u}_1 \rangle_{\hat{\beta}}^{(Y,D),T_f} = \langle \widetilde{\gamma}' \rangle_{\widetilde{\beta}}^{\widetilde{X}', T_f} = \langle \widetilde{\gamma}_1 \rangle_{\widetilde{\beta}}^{\widetilde{\mathcal{X}}, T_f}. \tag{C.3}$$

Here, the last equality in (C.3) is Lemma C.6.

Remark C.8. The first equality $\langle \rangle_{\beta',(d,1)}^{X,(L,f)} = (-1)^{d+1} \langle \mathsf{u}_2 - f \mathsf{u}_1 \rangle_{\hat{\beta}}^{(Y,D),T_f}$ in (C.3) is a special case of a general correspondence between open Gromov-Witten invariants of (X, L, f) and relative Gromov-Witten invariants of (Y, D), which involves invariants of higher genus and general winding/tangency profiles. The general open/relative correspondence is already established by Fang-Liu

[41], whose proof builds upon a relation between the open Gromov-Witten invariants of (X, L, f) and the *formal* relative Gromov-Witten invariants of the $(\hat{Y}, \hat{D})$, the formal completion of (Y, D) along the toric 1-skeleton of Y. Formal relative Gromov-Witten invariants for a general formal toric Calabi-Yau 3-fold relative to a collection of boundary divisors are introduced by Li-Liu-Liu-Zhou [73] as a fundamental building block for a mathematical theory of the topological vertex. In our case, since the relative condition is specified by an irreducible divisor D, we can directly define and compute the relative invariants without resorting to formal geometry.

Remark C.9. The second equality $(-1)^{d+1}\langle \mathsf{u}_2 - f\mathsf{u}_1\rangle_{\hat{\beta}}^{(Y,D),T_f} = \langle \widetilde{\gamma}' \rangle_{\widetilde{\beta}}^{\widetilde{X}',T_f}$ in (C.3) can be viewed as an instantiation of the *log-local principle* of van Garrel-Graber-Ruddat [48], which at the numerical level conjectures a correspondence between the maximally-tangent *log* Gromov-Witten invariants of a projective variety Y relative to a normal crossing divisor D whose irreducible components $D_1, \dots, D_k$ are smooth and nef, and the local Gromov-Witten invariants of the total space of the vector bundle $\mathcal{O}_Y(-D_1) \oplus \cdots \oplus \mathcal{O}_Y(-D_k)$. Since its proposal, this principle has been verified in various cases and studied from different perspectives; see e.g. [18, 16, 17, 19, 21, 29, 84, 93]. It is also known not to hold in general when D is reducible [84]. Our Theorem C.7 provides a general class of examples for the log-local principle in the extended setting where the base Y is non-compact, in view of the identification of the log and relative Gromov-Witten invariants [3]. In particular, we note that Conjecture 1.1 of [16] proposes a variant of the log-local principle for more general log Calabi-Yau pairs, and our Theorem C.7 verifies this conjecture for the toric pairs (Y, D) arising from our constructions. Moreover, similar to [48], our numerical correspondence in can be derived from a cycle-level correspondence proven by degeneration; for this, we may use the degeneration formula of Li [72] in relative Gromov-Witten theory, or its variant in log Gromov-Witten theory by Kim-Lho-Ruddat [65], and equivariant intersection theory [40].

We now prove Theorem C.7. Based on Theorem 4.1 and Lemma C.6, it suffices to prove the first equality $\langle\rangle_{\beta',(d,1)}^{X,(L,f)} = (-1)^{d+1}\langle \mathsf{u}_2 - f\mathsf{u}_1\rangle_{\hat{\beta}}^{(Y,D),T_f}$ in (C.3). As mentioned in Remark C.8, this is already established by [41]. For completeness, we include a direct proof that does not involve formal geometry and invariants. Based on Propositions 3.7 and C.4, we prove this equality in two steps: First, in Lemma C.10, we identify $\langle\rangle^{X,(L,f)}_{\beta',(d,1)}$ with the contribution to $(-1)^{d+1}\langle \mathsf{u}_2 - f\mathsf{u}_1\rangle^{(Y,D),T_f}_{\hat{\beta}}$ from stable maps with an unexpanded target (Y, D). Second, in Lemma C.11, we show that for generic $f \in \mathbb{Z}$ (with respect to the curve class), there is no contribution from stable maps with an expanded target $(Y[m], D[m]), m > 0$ after the weight restriction $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. We may then deduce the result for any arbitrary f using the reasoning of Section 4.4.

To start, we observe that there is a natural bijection between the sets of decorated graphs

$$\hat{\epsilon} : \Gamma^{0,(d,1)}_{0,1}(X, \beta) \to \Gamma^{0}_{0,1}(Y, \hat{\beta}),$$

where given $\vec{\Gamma} \in \Gamma^{0,(d,1)}_{0,1}(X, \beta)$, we obtain $\hat{\epsilon}(\vec{\Gamma}) \in \Gamma^{0}_{0,1}(Y, \hat{\beta})$ by replacing the marked point of $\vec{\Gamma}$ by

- a new vertex $\hat{v}_0 = \hat{v}_0(\hat{\epsilon}(\vec{\Gamma}))$ with label $\vec{f}(\hat{v}_0) = \hat{\sigma}_0$;
- a new edge $e_0 = e_0(\hat{\epsilon}(\vec{\Gamma}))$ connecting $\hat{v}_0$ to $\vec{s}(1)$ with degree $\vec{d}(e_0) = d$ and label $\vec{f}(e_0) = \tau_0$;
- a new marked point 1 with marking $\vec{s}(1) = \hat{v}_0$.

The inverse map removes $\hat{v}_0$ and e_0 and moves the marked point to the place of the unique flag in $\vec{f}^{-1}(\tau_0, \sigma_0)$.

Recall from (4.1) that

$$C_{\vec{\Gamma}} = D_{d,1} \cdot c_{\vec{\Gamma}} \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{0,|E_v|}} \frac{\mathbf{h}(v)}{\left(\frac{\mathsf{u}_1}{d} - \psi_1\right)^{\delta_{v,1}} \cdot \prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \psi_{(e,v)}\right)}$$

denotes the contribution of $\vec{\Gamma}$ to $\langle\rangle^{X,(L,f)}_{\beta',(d,1)}$ as in (3.11) in Proposition 3.7 before the weight restriction $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. Here, the disk factor (3.6) simplifies to

$$D_{d,1} = \frac{(-1)^{df}}{d!\mathsf{u}_1} \cdot \prod_{a=1}^{d-1} \left(\frac{d\mathsf{u}_2}{\mathsf{u}_1} + a \right).$$

We set up some additional notations. For each $\vec{\Gamma} \in \Gamma^{0}_{0,1}(Y, \hat{\beta})$, we set

$$\hat{C}_{\vec{\Gamma}} := c_{\vec{\Gamma}}(\mathsf{u}_2 - f\mathsf{u}_1) \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v) \in F(\Gamma), v \in V_0(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V_0(\Gamma)} \int_{\overline{\mathcal{M}}_{0,|E_v|}} \frac{\mathbf{h}(v)}{\prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \psi_{(e,v)}\right)}.$$

to be the contribution of $\vec{\Gamma}$ to $\langle \mathsf{u}_2 - f\mathsf{u}_1 \rangle_{\hat{\beta}}^{(Y,D),T_f}$ before the weight restriction $\mathsf{u}_2 - f\mathsf{u}_1 = 0$. Similarly, for each $\vec{\Gamma} \in \Gamma'_{0,1}(Y,\hat{\beta}) \setminus \Gamma^0_{0,1}(Y,\hat{\beta})$, we set

$$\begin{aligned}\hat{C}_{\vec{\Gamma}} :=& c_{\vec{\Gamma}}(\mathsf{u}_2 - f\mathsf{u}_1) \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e) \cdot \prod_{(e,v)\in F(\Gamma), v \in V_0(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v \in V_0(\Gamma)} \int_{\overline{\mathcal{M}}_{0,|E_v|}} \frac{\mathbf{h}(v)}{\prod_{e \in E_v} \left(\mathbf{w}_{(e,v)} - \psi_{(e,v)}\right)} \\ &\cdot \prod_{j=1}^{\ell(\mu(\vec{\Gamma}))} \mu(\vec{\Gamma})_j \cdot (-1)^{\ell(\mu(\vec{\Gamma}))-1} (\mathsf{u}_2 - f\mathsf{u}_1)^{2\ell(\mu(\vec{\Gamma}))-2} \int_{\overline{\mathcal{M}}^{(1)}_{\vec{\Gamma}}} \frac{1}{-\mathsf{u}_1 - \psi^t}\end{aligned}$$

to be the contribution of $\vec{\Gamma}$ before the weight restriction.

Lemma C.10. *For the quantities defined as in Theorem C.7, we have that for each* $\vec{\Gamma} \in \Gamma^{0,(d,1)}_{0,1}(X,\beta)$,

$$C_{\vec{\Gamma}}\big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} = (-1)^{d+1} \hat{C}_{\hat{\epsilon}(\vec{\Gamma})}\big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0}.$$

Proof. Note that $\mathrm{Aut}(\vec{\Gamma}) \cong \mathrm{Aut}(\hat{\epsilon}(\vec{\Gamma}))$ and

$$c_{\hat{\epsilon}(\vec{\Gamma})} = \frac{c_{\vec{\Gamma}}}{d_{e_0}} = \frac{c_{\vec{\Gamma}}}{d}.$$

Comparing the contributions $C_{\vec{\Gamma}}$ and $\hat{C}_{\hat{\epsilon}(\vec{\Gamma})}$, we see that it amounts to showing

$$(-1)^{d+1}(\mathsf{u}_2 - f\mathsf{u}_1)\frac{\mathbf{h}(e_0)}{d}\bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} = D_{d,1}\big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} = \frac{(-1)^{df}}{d!\mathsf{u}_1} \cdot \prod_{a=1}^{d-1} (df + a). \tag{C.4}$$

We compute that if $f \geq 0$,

$$\begin{aligned}(\mathsf{u}_2 - f\mathsf{u}_1)\mathbf{h}(e_0)\bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} &= \frac{d^d}{d!\mathsf{u}_1^d} \cdot \frac{(-\frac{d-1}{d}\mathsf{u}_1 - \mathsf{u}_2)(-\frac{d-2}{d}\mathsf{u}_1 - \mathsf{u}_2)\cdots(\frac{df-1}{d}\mathsf{u}_1 - \mathsf{u}_2)}{\mathsf{u}_2(-\frac{\mathsf{u}_1}{d} + \mathsf{u}_2)(-\frac{2\mathsf{u}_1}{d} + \mathsf{u}_2)\cdots(-\frac{df-1}{d}\mathsf{u}_1 + \mathsf{u}_2)}\bigg|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} \\ &= \frac{d^d}{d!\mathsf{u}_1} \cdot \frac{(-\frac{d-1}{d} - f)(-\frac{d-2}{d} - f)\cdots(\frac{df-1}{d} - f)}{f(-\frac{1}{d} + f)(-\frac{2}{d} + f)\cdots(-\frac{df-1}{d} + f)} \\ &= \frac{(-1)^{d(f+1)+1}d}{d!\mathsf{u}_1} \cdot \prod_{a=1}^{d-1}(df + a);\end{aligned} \tag{C.5}$$

on the other hand, if $f < 0$,

$$
\begin{aligned}
(\mathsf{u}_2 - f\mathsf{u}_1)\mathbf{h}(e_0)\Big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} &= -\frac{d^d}{d!\mathsf{u}_1^d} \cdot \frac{(\frac{\mathsf{u}_1}{d} + \mathsf{u}_2)(\frac{2\mathsf{u}_1}{d} + \mathsf{u}_2)\cdots(-\frac{df+1}{d}\mathsf{u}_1 + \mathsf{u}_2)}{(-\mathsf{u}_1 - \mathsf{u}_2)(-\frac{d+1}{d}\mathsf{u}_1 - \mathsf{u}_2)(-\frac{d+2}{d}\mathsf{u}_1 - \mathsf{u}_2)\cdots(\frac{df+1}{d}\mathsf{u}_1 - \mathsf{u}_2)}\Big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} \\
&= -\frac{d^d}{d!\mathsf{u}_1} \cdot \frac{(\frac{1}{d} + f)(\frac{2}{d} + f)\cdots(-\frac{df+1}{d} + f)}{(-1 - f)(-\frac{d+1}{d} - f)(-\frac{d+2}{d} - f)\cdots(\frac{df+1}{d} - f)} \\
&= \frac{(-1)^{d(f+1)+1}d}{d!\mathsf{u}_1} \cdot \prod_{a=1}^{d-1}(df + a),
\end{aligned}
\tag{C.6}
$$

which is the same as the $f \geq 0$ case. Therefore, (C.4) directly follows. □

Lemma C.11. *For the quantities defined as in Theorem C.7 and generic $f \in \mathbb{Z}$ (with respect to $\hat{\beta}$), we have that for each $\vec{\Gamma} \in \Gamma'_{0,1}(Y, \hat{\beta}) \smallsetminus \Gamma^0_{0,1}(Y, \hat{\beta})$,*

$$\hat{C}_{\vec{\Gamma}}\Big|_{\mathsf{u}_2 - f\mathsf{u}_1 = 0} = 0.$$

Proof. We determine the power of $\mathsf{u}_2 - f\mathsf{u}_1$ in $\hat{C}_{\vec{\Gamma}}$ and show that it is positive. This would imply the lemma. For a generic choice of $f \in \mathbb{Z}$, $\mathbf{w}(\tau_e, \sigma_v) \neq \pm(\mathsf{u}_2 - f\mathsf{u}_1)$ for any $(\tau_e, \sigma_v) \in F(\Gamma)$. Then, Lemma A.2 implies that $\mathsf{u}_2 - f\mathsf{u}_1$ is not a factor of the denominator of

$$\prod_{v \in V_0(\Gamma)} \int_{\overline{\mathcal{M}}_{0,E_v}} \frac{1}{\prod_{e \in E_v}(\frac{\mathbf{w}(\tau_e, \sigma_v)}{d_e} - \psi_{(e,v)})}.$$

It suffices to focus on the term

$$(\mathsf{u}_2 - f\mathsf{u}_1)^{2\ell(\mu(\vec{\Gamma}))-1} \prod_{e \in \vec{f}^{-1}(\tau_0)} \mathbf{h}(e).$$

By a computation similar to (C.5) and (C.6), the power of $\mathsf{u}_2 - f\mathsf{u}_1$ in $\mathbf{h}(e)$ is -1 for any $e \in \vec{f}^{-1}(\tau_0)$. Therefore, the total power of $\mathsf{u}_2 - f\mathsf{u}_1$ is $\ell(\mu(\vec{\Gamma})) - 1$ which is strictly positive since $\vec{\Gamma} \notin \Gamma^0_{0,1}(Y, \hat{\beta})$. □

Proof of Theorem C.7. By Theorem 4.1 and Lemma C.6, it suffices to prove the first equality $\langle\rangle^{X,(L,f)}_{\beta',(d,1)} = (-1)^{d+1}\langle \mathsf{u}_2 - f\mathsf{u}_1 \rangle^{(Y,D),T_f}_{\hat{\beta}}$ in (C.3), which for generic $f \in \mathbb{Z}$ follows from Lemma C.10

and C.11. We may conclude for any arbitrary f using the argument of Section 4.4. $\square$