Derived Hecke Operators on Unitary Shimura Varieties

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Abstract

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We propose a coherent analogue of the non-archimedean case of Venkatesh's conjecture on the cohomology of locally symmetric spaces for Shimura varieties coming from unitary similitude groups. Let $\mathbf{G}$ be a unitary similitude group with an indefinite signature at at least one archimedean place. Let Π be an automorphic cuspidal representation of $\mathbf{G}$ whose archimedean component Π_∞ is a non-degenerate limit of discrete series and let $\mathcal{W}$ be an automorphic vector bundle such that Π contributes to the coherent cohomology of its canonical extension. We produce a natural action of the derived Hecke algebra of Venketesh with torsion coefficients via cup product coming from étale covers and show that under some standard assumptions this action coincides with the conjectured action of a certain motivic cohomology group associated to the adjoint representation $\mathrm{Ad}\,\rho_\Pi$ of the Galois representation attached to Π. We also prove that if the rank of $\mathbf{G}$ is greater than two, then the classes arising from the étale covers do not admit characteristic zero lifts, thereby showing that previous work of Harris-Venkatesh and Darmon-Harris-Rotger-Venkatesh is exceptional.

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Dedicated to my grandma Aleka,

for being an angel among humans.

Introduction

Motivation

Venkatesh's conjecture (see [PV16], [GV17], and [Ven19] for precise statements, and [Ven14] for exposition) on cohomology of symmetric spaces posits that for a given Hecke eigensystem χ coming from an automorphic cuspidal representation tempered at ∞, there is a natural action of the motivic cohomology of the coadjoint motive associated to χ that preserves the χ-isotypic component of the rational cohomology of locally symmetric spaces. This action thus explains the *degree spread* of the Hecke eigensystem and sheds light on phenomena in the Langlands program. The conjectures so far have been mostly restricted to the case where the reductive group G has no compact Cartan subgroup. In particular, this excludes all G that admit a Shimura datum. Outside of few limited cases, locally symmetric spaces that are not Shimura varieties have not been linked to algebraic geometry, and thus it is unclear how such motivic action could be realized.

Nevertheless, a similar conjecture for Shimura varieties is expected to exist for automorphic representations of irregular weights where a similar degree-spread phenomenon occurs. The first incarnations of these expectations are the papers [HV19] and [DHRV] concerning weight one modular forms, which appear in H^0 and H^1 of the modular curve for the same line bundle, and the role of the motivic group action is played by a certain group of Stark units. Up to date they are the only papers verifying numerical predictions coming from the non-archimedean motivic action of Venkatesh by relating the action to Stark's conjecture on special values of Artin L-functions. The aim of this thesis is to provide a coherent analogue of Venkatesh's conjecture and provide explication of the action.

Precise conjectures

Let $\mathbf{G}$ be a unitary (similitude) group associated to a vector space V of dimension n over a CM field F. We assume in this exposition that F is an imaginary quadratic number field. In particular, $\mathbf{G}$ has a unique archimedean component where $\mathbf{G}(\mathbf{R})$ has signature $(r, n-r)$ and we assume $1 < r < n$.

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum with a model over the reflex field F. Suppose $\Pi = \Pi_\infty \otimes \Pi_f$ is a cuspidal automorphic representation with archimedean component Π_∞ being a non-degenerate limit of discrete series, i.e. its Harish-Chandra parameter λ lies on some of the wall of the Weyl chambers but is not orthogonal to any of the compact roots. Let $\mathcal{W}$ be an automorphic vector bundle such that Π contributes to

$$H^*(\underbrace{\mathrm{Sh}_{K_\emptyset}(\mathbf{G}, \mathbf{X})_\Sigma}_{X_\emptyset}, \mathcal{W}^{can}) \tag{1}$$

for some neat compact open $K_\emptyset \subseteq \mathbf{G}(\mathbf{A}^\infty)$. Here $\mathcal{W}^{can}$ is the canonical extension of $\mathcal{W}$ introduced by Mumford (see [Mum77] but also [Har85]) to the toroidal compactification $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$ which is assumed to possess the usual desired properties for a judicious choice of combinatorial data Σ. By the work of Lan [Lan16a], it admits an integral model over $\mathcal{O}_F[\frac{1}{S}]$ for some finite set of primes of bad reduction S. The contribution to the cohomology occurs in an interval of consecutive degrees $i \in [q_0, q_0 + l_0]$ around the middle cohomology.

Fix a closure $\overline{\mathbf{Q}}_\ell$ of ℓ-adic numbers. The Langlands conjectures predict the existence of a Galois representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathbf{G}^\vee(\overline{\mathbf{Q}}_\ell)$$

characterized by the classical compatibility between Hecke eigenvalues and values of ρ at Frobenius elements. Composing with the adjoint action, we obtain

$$\mathrm{Ad}\,\rho : \mathrm{Gal}(\overline{F}/F) \to \mathbf{G}^\vee(\overline{\mathbf{Q}}_\ell) \xrightarrow{\mathrm{Ad}} \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell).$$

Furthermore, we assume there is a motive M_Π, pure of weight 0 and dimension n, such that

$$L(M_\Pi, s) = L(\mathrm{Ad}, \Pi, s).$$

Let $E = \mathbf{Q}(\Pi)$ be the field containing all Hecke eigenvalues of Π. Set $H^1_{mot}(M_\Pi, E(1))$ to be the motivic cohomology group defined by Voevodski (see [MVC06, Def 3.4]), and let

$$\mathrm{V} := H^1_{mot}((M_\Pi)_{\mathcal{O}}, E(1))$$

be the subspace of integral cohomology classes introduced by Scholl [Sch00, Thm 1.1.6]. Denote by V^* its E-linear dual. The main conjecture is the following.

Conjecture 0.0.0.1. *There is a natural graded action of*

$$\wedge^* \mathrm{V}^* \circlearrowleft H^*((\mathrm{Sh}_{K_\emptyset}(\mathbf{G}, \mathbf{X})_\Sigma)_{\mathcal{O}_F[\frac{1}{S}]}, \mathcal{W}^{can})_\Pi$$

preserving the rational structure on the Π*-isotypic component.*

This thesis contains collection of results providing evidence for the above conjecture, which is a natural coherent analogue of the conjectures of [PV16] and [Ven19]. As such it could be viewed as a generalization of the setup in [HV19] and [DHRV], except their results are unconditional in the cases studied. Our attention is focused on the non-archimedean case at primes $q \neq p$. For archimedean conjecture in the coherent setting see [Hor20] and the forthcoming work of Gyujin Oh.

Main results

Let p be a sufficiently large prime such that $\mathcal{O}_E$ at p is unramified over $\mathbf{Z}$. Pick a prime $\mathfrak{p}$ in E over p. There is a natural comparison map

$$\mathrm{V} \otimes \mathcal{O}_{\mathfrak{p}} \to H^1_f(\mathbf{Q}, (\mathrm{Ad}^*\rho \otimes \mathcal{O}_{\mathfrak{p}})(1)) \tag{2}$$

expected to be an isomorphism. The main contributions of this paper are

- construction of an action by a *derived* Hecke algebra and providing a geometric interpetation of this action via algebras of "diamond operators," at least at Taylor-Wiles datum;
- relating asymptotically the action of the derived Hecke algebra to that of the Selmer group $H^1_f(\mathbf{Q}, (\mathrm{Ad}^*\rho \otimes \mathcal{O}_{\mathfrak{p}})(1))$.

We briefly outline how these are achieved

Derived Hecke action at Taylor-Wiles datum

Let $\mathfrak{q}$ be a prime such that $N_{F/\mathbf{Q}}\mathfrak{q} \equiv 1\ (mod\ p^m)$ which also satisfies few additional Galois conditions, depending on $\bar{\rho}$. In the standard setup of the Taylor-Wiles method, there are maps of spaces

$$\overbrace{\mathbb{S}_{1,\Sigma}(\mathfrak{q}) \to \underbrace{\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q}) \to \mathbb{S}_{0,\Sigma}(\mathfrak{q})}_{\Delta_\mathfrak{q}}}^{\mathbf{T}(\mathbf{F}_\mathfrak{q})} \to \mathbb{S}_{\emptyset,\Sigma}, \tag{3}$$

with $\mathbf{T}$ being the torus inside a fixed borel $\mathbf{B}$ of $\mathbf{G}$, $\Delta_\mathfrak{q}$ being the largest p-power product inside $\mathbf{T}(\mathbf{F}_\mathfrak{q})$ and where all integral models are over $\mathrm{Spec}(R)$ for a p-adic ring R. The fact that we may arrange the combinatorial data Σ such that the map

$$\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q}) \to \mathbb{S}_{0,\Sigma}(\mathfrak{q})$$

remains étale even over the toroidal compactification is nontrivial and is the content of Appendix A. By comparison between étale and Zariski cohomology, we obtain for any $z \in \mathrm{Hom}(R, \Delta_\mathfrak{q})$ a *Shimura class*

$$z\mathfrak{S} \in H^1_{\text{Zar}}(\mathbb{S}_{0,\Sigma}(\mathfrak{q})_R, \mathcal{O}).$$

For an automorphic vector bundle $\mathcal{W}$ and a conjugacy class $\gamma \in G(\mathcal{O}_\mathfrak{q})\backslash G(\mathbf{Q}_\mathfrak{q})/G(\mathcal{O}_\mathfrak{q})$, consider the derived Hecke operator $T_{\mathfrak{q},\gamma,z}$ given by

$$\begin{aligned} H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}^{\text{can}}) \xrightarrow{\pi_1^*} H^*((\mathbb{S}_{K^\gamma,\Sigma})_R, \mathcal{W}^{\text{can}}) &\xrightarrow{\cup z\mathfrak{S}_\gamma} H^*((\mathbb{S}_{K^\gamma,\Sigma})_R, \mathcal{W}^{\text{can}}) \\ &\xrightarrow{[\cdot\gamma]\circ\pi_{2,*}} H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}^{\text{can}}), \end{aligned} \tag{4}$$

where π_1, π_2 are the projections from $K^\gamma := K \cap \gamma K \gamma^{-1}$ to K and $\gamma K \gamma^{-1}$-levels, respectively, and $\mathfrak{S}_\gamma$ is the pullback of $\mathfrak{S}$ to K^γ level. We show that these operators match, at Taylor-Wiles datum, the action of Venkatesh's derived Hecke algebra given as Ext-algebra. The algebra obtained over all such primes is commutative and these operators commute with the classical Hecke operators; in particular, they preserve the Hecke eigensystems.

Let us comment on a crucial subtlety in this construction – we necessarily work with torsion-coefficients R. In the Betti case of [Ven19], this is necessary since H^1 in characteristic zero is almost always empty. For coherent cohomology, the works [HV19] and [DHRV] leverage the fact that for $G = \mathrm{GL}_2$

$$H^1_{\mathrm{Zar}}(\text{structure sheaf})$$

is nonempty and admits a lift of $z\mathfrak{S}$ to characteristic zero realized by the weight two Eisenstein series. When we first started working on this project we expected to have similar characteristic zero lifts at least for $\mathbf{G} = GU(n-1,1)$ since in this case H^1 is known to be nonempty and furthermore has nice description – all classes are represented by theta lifts of characters from $GU(1)$. Unfortunately, as we show in §6.2,6.3 there are no characteristic zero lifts of $z\mathfrak{S}$ outside the GL_2 case, possibly with the exception of the case $\mathbf{G} = GU(n-1,1)$ with $n = 4k$. We do not expect this case to actually admit characteristic zero lifts, just our method does not apply then. This is unfortunate since pairings

$$[T_{\mathfrak{q},\gamma,z} f, \hat{f}]$$

via Serre duality of $f \in \Pi$ of irregular weights with its contragredient $\hat{f} \in \hat{\Pi}$ produce numerically falsifiable statements about Venkatesh's conjecture only if the Shimura covers lift to characteristic zero. Studying such pairings is currently the only attempted way to extract numerics for the conjectured non-archimedian motivic action, which is still only defined up to scalar.

Reciprocity law

Establishing that the natural derived Hecke action constructed in (4) recovers a "natural" action of the Selmer group of (2) follows in the coherent case very similarly to the general arguments of §6-8 of [Ven19] under a very similar (long) list of assumptions. We briefly point out a few of differences, the ideas for which were already present in [HV19].

- The derived Hecke algebra naturally acts on the cohomology of a certain complex of so-called *derived invariants*. We work with with Godement resolutions for the automorphic vector bundles and to establish the formal properties it is crucial to show that covers in (3) remain étale over the toroidal compactification.
- In [Ven19] the link between these covers and the derived Hecke algebra is achieved via pullbacks from a classifying space while we use the inclusion of the constants into the global sections of the structure sheaf.

Lastly, one expects that the two actions – one via derived Hecke correspondences and the other of the Selmer group – can only be shown to coincide asymptotically (see Cor 10.4.1.6) since they are obtained from torsion classes via a patching argument.

The invariant l_0

In general, if the same Hecke eigensystem appears in multiple (necessarily consecutive) degrees of the same automorphic vector bundle on a Shimura variety, then each instance is represented by the $(\mathfrak{P}, K)$-cohomology of a different automorphic representation. Each of these representations shares the same finite π_f part but differs in the archimedean component π_∞, which varies within the same archimedean L-packet. By a theorem of Mirkovic [Har90, Thm.3.5], the representations π_∞ are necessarily non-degenerate limits of discrete series, and by a theorem of Harris [Har90, Thm.3.4] the number l_0 of consecutive degrees supporting this eigensystem equals the number of Weyl chamber walls on which the associated Harish-Chandra parameter belongs to.

The invariant l_0 is also of interest from the standpoint of automorphy lifting results following the Calegari-Geraghty method, wherein one patches complexes rather than just modules as in the classical Taylor-Wiles method. With Π and $K_\emptyset$ as above, suppose $\mathrm{C}_\emptyset^\bullet$ computes the localized at

Π cohomology of (1). Then one produces a limit ring S_∞ of "diamond operators" at various compatible sets Q of auxiliary Taylor-Wiles primes as well as a complete local S_∞-algebra R_∞, and a complex $\mathrm{C}_\infty^\bullet$ of S_∞-modules such that

$$R_\infty \otimes_{S_\infty} \mathcal{O}_\mathfrak{p} = R, \quad \mathrm{C}_\infty^\bullet \otimes_{S_\infty} \mathcal{O}_\mathfrak{p} = \mathrm{C}_\emptyset^\bullet$$

and equipped with a map

$$R_\infty \to \mathrm{End}_{D(S_\infty)}(\mathrm{C}_\infty^\bullet),$$

where R is a universal Galois deformation ring adapted to Π. Then

$$l_0 = \dim S_\infty - \dim R_\infty.$$

In [Cal18], the invariant is decomposed as

$$l_0 = l_{0,p} + l_{0,\infty} + l_{0,G}.$$

In the case of Betti cohomology studied in [Ven19], one has $l_{0,p} = l_{0,\infty}$ and $l_{0,G} = \delta$ is the "defect." In contrast, in the case of coherent cohomology, $\delta = l_{0,G} = l_{0,\infty} = 0$ but $l_{0,p}$ is nonzero. Following the classical assumptions as in [Cal18, § 2.8] we show that this invariant recovers l_0 of Harris's theorem. It is worth pointing out that l_0 as defined in Coherent on [CG18, p. 300] is only correct when $\mathbf{G}$ has signature $(n-1,1)$ at the archimedean place.

Lastly, under the expected isomorphism (2), and assuming the Beilinson's conjectures Venkatesh shows that

$$\dim \mathrm{V} = l_0.$$

In this case, the invariant l_0 is computed as the order of vanishing of the L-function of the adjoint representation at $s = 0$. The coincidence of these three invariants is predicted by [GV17] and [Ven19] but to our knowledge the verification of the equality between the first and the second have not be written down before.

Organization of the thesis

In §1, we introduce Shimura varieties, their minimal and toroidal compactifications and describe the coherent cohomology theories for their automorphic vector bundles.

In §2, we briefly review the structure theory of representations of unitary (similitude) groups, their parameters and introduce limits of discrete series.

In §3, we gather a collection of results concerning the existence of integral models for Shimura varieties (resp. automorphic vector bundles) and their (resp. extensions over) toroidal compactifications.

In §4, we introduce the coadjoint motive and describe it in details in the simplest case when it is an Artin motive. We also introduce the Stark units group.

In §5, we produce Shimura covers over toroidal compactifications and use them to construct derived Hecke operators at Taylor-Wiles datum.

In §6, we prove that Shimura classes do not lift to characteristic zero outside of the case of GL_2, thereby showing that the results of [HV19] and [DHRV] are exceptional.

In §7, we introduce the derived Hecke algebra of Venkatesh, verify the necessary constructions so that it acts on coherent cohomology of automorphic vector bundles, and show that, at Taylor-Wiles datum, it coincides with the algebra of derived Hecke operators of §5.

In §9, we introduce the relevant complexes and their desired properties needed for the patching argument. We also show that, under assumptions, the global derived Hecke algebra obtained as a limit from the torsion ones is sufficient to explain the degree spread of Hecke eigenclasses.

In §10, we construct the action of the Selmer group and show that asymptotically it coincides with that of the derived Hecke algebra.

Chapter 1: Shimura varieties and automorphic vector bundles

1.1 Shimura varieties

Let F be a CM number field over a maximal totally real field F_0 of degree d with nontrivial F/F_0 automorphism $a \mapsto \bar{a} = a^c$. Suppose L is an imaginary quadratic field such that $F = F_0 L$, and for which the prime p splits: $p = uu^c$. Fix a place w of F above u. Let B be a division algebra with centre F such that

- the opposite algebra $B^{\mathrm{op}} \simeq B \otimes_{E,c} E$;
- B is split at w;
- at any place v of F which is not split over F_0, B_v is split; and
- at any place v of F which is split over F_0, B_v is either split or a division algebra.

Let $r = (\dim_F B)^{1/2}$ and $n = (\dim_F \mathrm{End}_B V)^{1/2}$. Let $*$ be a positive involution of the second kind on B, i.e. $*\big|_F = c$ and $\mathrm{Tr}_{B/\mathbf{Q}}(xx^*) > 0$, and let V be a nonzero finitely generated left B-module, equipped with a pairing $(\cdot, \cdot)$ such that $(bx, y) = (x, b^* y)$.

The signatures of V at the archimedean places determine a CM type Φ of F as follows. We set Φ to be a CM system (unique unless n is even and V has signature $(n/2, n/2)$ at one of the archimedean places) such that

$$\mathrm{sg}(V_\varphi) = (r_\varphi, r_{\bar{\varphi}}), \quad \text{s.t.} \quad r_\varphi \geq r_{\bar{\varphi}}, \quad \forall \varphi \in \Phi, \tag{1.1}$$

where $V_\varphi := V \otimes_{F,\varphi} \mathbf{C}$, $\varphi \in \operatorname{Hom}(F, \overline{\mathbf{Q}})$, and $\bar{\varphi}$ denotes the precomposition of φ with the nontrivial Galois element from F/F_0. Since $\dim V = n$, we may rewrite (1.1) as

$$\operatorname{sg}(V_\varphi) = (r_\varphi, n - r_\varphi), \quad \varphi \in \Phi.$$

Consider the linear algebraic $\mathbf{Q}$-group given by

$$\mathbf{G}(R) = \{g \in \operatorname{End}_B(V) \otimes_{\mathbf{Q}} R : gg^* = \nu(g) \in R^\times\}$$

for any $\mathbf{Q}$-algebra R. This is the unitary similitude group of signatures $(r_\varphi, n - r_\varphi),\ \varphi \in \Phi$ at the archimedean places. For each $\varphi \in \Phi$, choose a $\mathbf{C}$-basis of V_φ with respect to which the matrix of $(\cdot, \cdot)$ is given by

$$\operatorname{diag}(1_{r_\varphi}, -1_{n-r_\varphi}).$$

The inclusion $\mathbf{G} \hookrightarrow \mathrm{GL}$ gives raise to

$$\mathbf{G}_{\mathbf{R}} \to \mathrm{GL}_{F\otimes\mathbf{R}}(V \otimes \mathbf{R}) \underset{\sim}{\overset{\Phi}{\to}} \prod_{\varphi\in\Phi} \mathrm{GL}_{\mathbf{C}}(V_\varphi), \tag{1.2}$$

which in turn gives rise to a homomorphism from the Deligne torus

$$h = h_{K_\infty} : \mathbf{S} = \operatorname{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m \to \mathbf{G}_{\mathbf{R}},$$

given component-wise by

$$z \mapsto \operatorname{diag}(z \cdot 1_{r_\varphi}, \bar{z} \cdot 1_{n-r_\varphi}).$$

Let K_∞ be the stabilizer of this homomorphism under the action of $\mathbf{G}(\mathbf{R})$ by conjugation. Then K_∞ is Zariski connected as a centralizer of a torus and, furthermore, by the first two axioms on

Deligne's list [Del79, p. 2.1.1], it is also a maximal compact modulo center subgroup of $G :=$ $\mathbf{G}(\mathbf{R})$. The homomorphism $h = h_{K_\infty}$ lands inside K_∞, so it may be viewed as a homomorphism $\mathbf{S} \to \mathbf{G}_{\mathbf{R}}$, while a priori it is only defined as a morphism to $\prod_{\varphi\in\Phi} \mathrm{GL}_{\mathbf{C}}(V_\varphi)$. More explicitly, upon fixing an ordering of the d embeddings in Φ, we have

$$G = \mathbf{G}(\mathbf{R}) = \left\{ g = (g_1, \dots, g_d) \in \prod_{i=1}^{d} GU(r_i, n - r_i) \mid \nu_{r_1, n-r_1}(g_1) = \dots = \nu_{r_d, n-r_d}(g_d) \right\}$$

and

$$K_\infty = \left\{ g = (g_1, \dots, g_d) \in G \mid g_i = s \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} \text{ for } A_i \in U(r_i),\ D_i \in U(n - r_i) \text{ and } s \in \mathbf{R}^\times \right\}.$$

We also pick the maximal torus $A_\infty \subset K_\infty$ of diagonal matrices.

Let $\mathbf{X}$ be the $\mathbf{G}(\mathbf{R})$-conjugacy class of the homomorphism $h = h_{K_\infty}$, so that $(\mathbf{G}, \mathbf{X})$ is a Shimura pair, satisfying Deligne's list of axioms. The corresponding Shimura variety is given by

$$\mathrm{Sh}(\mathbf{G}, \mathbf{X}) := \varprojlim_K \mathrm{Sh}_K(\mathbf{G}, \mathbf{X}), \text{ where } \mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) := \mathbf{G}(\mathbf{Q}) \backslash \mathbf{X} \times \mathbf{G}(\mathbf{A}^\infty)/K,$$

where K runs over the compact open subgroups of $\mathbf{G}(\mathbf{A}^\infty)$. These have a model over the **reflex field** $E := E(\mathbf{G}, \mathbf{X})$, independent of K, which is the subfield of $\overline{\mathbf{Q}}$ given by

$$\mathrm{Gal}(\overline{\mathbf{Q}}/E) = \{\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) : \sigma^*(r_\varphi) = r_\varphi \text{ for all } \varphi \in \Phi\}.$$

Let $K \subseteq \mathbf{G}(\mathbf{A}^\infty)$ be a compact open subgroup. Let $\mathcal{F}_K$ be the category fibered in groupoids over the category LNSch/E of locally noetherian schemes over $\mathrm{Spec}(E(\mathbf{G}, \mathbf{X}))$, associating to each E-scheme S the quadruples $(A, \iota, \lambda, \bar{\eta})$, where

- A is an abelian scheme over S;
- $\iota : F \to \mathrm{End}^\circ(A)$ is an action of F on A defined up to isogeny;
- λ is a quasi-polarization on A; and
- $\bar{\eta}$ is a K-orbit of symplectic similitudes

$$\eta : \big(\prod_\ell T_\ell(A)\big) \otimes \mathbf{Q} \xrightarrow{\sim} V \otimes \mathbf{A}^\infty,$$

  between the rational adelic Tate module of A and $V(\mathbf{A}^\infty)$ (see [Kot92, §5]).

Let $\zeta \in F$ be a traceless element such that $\bar{\zeta} = -\zeta$, and let $\langle \cdot, \cdot \rangle$ be the $\mathbf{Q}$-valued form $\mathrm{Tr}_{F/\mathbf{Q}} \zeta(\cdot, \cdot)$ on V. We assume that

$$\mathrm{Ros}_\lambda\big(\iota(a)\big) = \iota\big(\bar{a}\big) \quad \text{for all} \quad a \in F,$$

where Ros_λ is the Rosati involution associated to the quasi-polarization λ, and that A satisfies the Kottwitz condition

$$\text{char}\big(\iota(a) |\, \mathrm{Lie}(A)\big) = \prod_{\varphi \in \mathrm{Hom}(F, \overline{\mathbf{Q}})} \big(T - \varphi(a)\big)^{r_\varphi} \quad \text{for all} \quad a \in F,$$

where the left-hand side denotes the characteristic polynomial of $\iota(a)$ acting on the locally free $\mathcal{O}_S$-module $\mathrm{Lie}(A)$, and the right-hand side is regarded as an element of $\mathcal{O}_S[T]$ via the structure morphism.

The morphisms $(A, \iota, \lambda, \bar{\eta}) \to (A', \iota', \lambda', \bar{\eta}')$ in this groupoid are the F-linear quasi-isogenies $A \to A'$ for which, Zariski-locally on S, the pullback of λ' is a $\mathbf{Q}^\times$-multiple of λ, and the pullback of $\bar{\eta}'$ is $\bar{\eta}$.

Theorem 1.1.0.1 (Kottwitz)**.** *The moduli problem $\mathcal{F}_K$ is representable by a Deligne-Mumford stack M_K over $\mathrm{Spec}(E)$. For even n,*

$$M_K(\mathbf{C}) = \mathrm{Sh}_K(\mathbf{G}, \mathbf{X}),$$

compatible with changing of K. For odd n, then $M_K(\mathbf{C})$ is a finite disjoint union of copies of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$, again compatible with changing K. The copies are enumerated by the elements of

$$\ker^1(\mathbf{Q}, \mathbf{G}) := \ker\left(H^1(\mathbf{Q}, \mathbf{G}(\overline{\mathbf{Q}})) \to H^1(\mathbf{Q}, \mathbf{G}(\overline{\mathbf{A}}))\right).$$

We usually assume that the compact open subgroups $K \subseteq \mathbf{G}(\mathbf{A}^\infty)$ are neat, i.e. $\mathbf{G}(\mathbf{Q}) \cap gKg^{-1}$ is torsion-free for any $g \in \mathbf{G}(\mathbf{A}^\infty)$. This makes $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ into a manifold instead of an orbifold and, more importantly, allows for the automorphic vector bundles defined in the latter sections to extend to the (smooth) toroidal compactifications of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$. Let $K_1, K_2 \subset \mathbf{G}(\mathbf{A}^\infty)$ be two neat compact open subgroups and let $g \in \mathbf{G}(\mathbf{A}^\infty)$. Suppose that $K_1 \subseteq gK_2g^{-1}$. Denote by $[\cdot g]_{K_1,K_2}$ the map:

$$\begin{gathered} \mathrm{Sh}_{K_1}(\mathbf{G}, \mathbf{X})(\mathbf{C}) = \mathbf{G}(\mathbf{Q})\backslash\mathbf{X} \times \mathbf{G}(\mathbf{A}^\infty)/K_1 \to \mathbf{G}(\mathbf{Q})\backslash\mathbf{X} \times \mathbf{G}(\mathbf{A}^\infty)/K_2 = \mathrm{Sh}_{K_2}(\mathbf{G}, \mathbf{X})(\mathbf{C}) \\ [(x, k)] \mapsto [(x, kg)] \end{gathered} \tag{1.3}$$

This map descends to a finite étale $\mathbf{Q}$-morphism $\mathrm{Sh}_{K_1}(\mathbf{G}, \mathbf{X}) \to \mathrm{Sh}_{K_2}(\mathbf{G}, \mathbf{X})$, which we also denote by $[\cdot g]_{K_1,K_2}$. When the context is clear we will simply write $[\cdot g]$, and call $[\cdot g]$ a **Hecke operator**.

1.2 Minimal and toroidal compactifications

For each $x \in \mathbf{X}$, the cocharacter μ_x given by $z \mapsto h_{x,\mathbf{C}}(z,1)$ yields a decreasing filtration Filt(μ_x) on $\mathrm{Rep}_{\mathbf{C}}(G)$ induced by the characters $t \mapsto t^n$. Let $\check{X}$ be the $\mathbf{G}(\mathbf{C})$-conjugacy class of filtrations of $\mathrm{Rep}_{\mathbf{C}}(G)$ containing Filt(μ_x). For any $x \in \mathbf{X}$, let P_x be the subgroup of $\mathbf{G}_{\mathbf{C}}$ fixing Filt(μ_x). Then P_x is parabolic subgroup and there is a bijection

$$\mathbf{G}(\mathbf{C})/P_x(\mathbf{C}) \to \check{\mathbf{X}},$$

endowing $\check{\mathbf{X}}$ with the structure of smooth projective variety over $\mathbf{C}$, which is called the **dual symmetric hermitian space of** $\mathbf{X}$. The natural map

$$\beta : \mathbf{X} \to \check{\mathbf{X}}, \quad x \mapsto \text{Filt}(\mu_x)$$

is an open embedding, called the **Borel embedding**.

1.3 Bailey-Borel compactification

Let D be a connected component of $\mathbf{X}$. The **minimal** or **Baily-Borel compactification** of D is its closure $\overline{D}$ in $\check{D}$ under the Borel embeddings $\beta : D \hookrightarrow \check{D}$ analogously constructed with $\mathbf{G}^+(\mathbf{R})$, the stabilizer of D, in place of $\mathbf{G}(\mathbf{R})$. Then the continuous action of $\mathbf{G}^+(\mathbf{R})$ extends to a continuous action on $\overline{D}$. The points of the space $\overline{D}$ decompose into equivalence classes, called **boundary components**, under the following equivalence relation: $x \sim y$ if there exists a holomorphic map $\lambda : \mathbb{D} \to \check{D}$ from the unit disk $\mathbb{D}$, whose image is fully contained in $\overline{D}$ and contains both points x and y. A boundary component F[1] is **rational** if its normalizer is defined over $\mathbf{Q}$, i.e. $N_F = \mathcal{N}(\mathbf{R})^+$

[1]This is standard notation and it should not be confused with the CM field, again denote by F. It will be clear from context which one is used.

for some algebraic subgroup $\mathcal{N}$ defined over $\mathbf{Q}$. If we decompose $\mathbf{G}$ into its Q-simple factors as $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_k$, and let $D = D_1 \times \cdots \times D_k$ be the corresponding decomposition, then a boundary component $F = F_1 \times \cdots \times F_k$ is rational if and only if each F_i is rational. For convenience, we assume G is simple in the remaining discussion. Each rational boundary component F corresponds to its stabilizer P_F, which is a maximal rational parabolic inside G.

For a neat arithmetic group $\Gamma \subseteq \mathbf{G}(\mathbf{Q})^+$, we embed the quasiprojective variety $S_\Gamma(D) := \Gamma\backslash D$ into its Bailey-Borel compactification

$$\overline{S}_\Gamma(D) = \Gamma\backslash\bar{D} = \Gamma\backslash D \cup \Gamma\backslash D^\infty,$$

endowed with Satake topology (see [Sat60]), where $D^\infty := \overline{D} - D$.

The compactification $\overline{S}_\Gamma(D)$ is canonical and does not depend on any choices, but is often singular. For instance, if $\dim(V) \geq 3$, the boundary $\overline{S}_\Gamma(D) - S_\Gamma(D)$ is of codimension at least two.

1.4 Toroidal compactification

For any level subgroup K, the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ has a family of toroidal compactifications, each one attached to a collection of **fans** $\Sigma := \sqcup\Sigma_F$ adapted to the neat level K, where the fans $\Sigma_F = \{\sigma\}_F$ are indexed by the rational boundary components. Each F corresponds to its stabilizer P_F, which is a maximal rational parabolic inside G. The **fan** Σ_F is a polyhedral cone decomposition of a certain open, convex, self-adjoint with respect to a $\mathbf{Q}$-rational positive definite quadratic form, cone C_F inside $U_F(\mathbf{R})$, where U_F is the center of the unipotent radical W_F of P_F. More precisely, each element of Σ_F is a strongly convex rational polyhedral cone and the fan Σ_F satisfies the following properties:

i) every face of a cone in Σ_F is also in Σ_F;

ii) if σ and σ' are both in Σ_F, then $\sigma \cap \sigma'$ is a face of both σ and σ';

iii) $C_F = \cup_{\sigma \in \Sigma_F} \sigma$.

The fan also needs to satisfy few conditions ensuring its compatibility with the neat subgroup K; most important is the transitivity of the action of $P_F \cap K$ on Σ_F (see [AMRT, §2, Def.4.10] and [Har89, (2.5.1)] for a complete list of properties in the classical and in the adelic setting, respectively).

Fix a rational boundary component $F \in D^\infty$ and let $P_F(\mathbf{Q})$ be the parabolic subgroup of $G(\mathbf{Q})$ fixing it. For a neat congruence $\Gamma \subseteq \mathbf{G}(\mathbf{Q})^+$, set $\Gamma_{P_F} := \Gamma \cap P_F(\mathbf{Q}) = \Gamma \cap N_F(\mathbf{Q})$ with center $\Gamma_{U_F} := \Gamma \cap U_F(\mathbf{Q})$, and consider their quotient Λ_F, making the sequence

$$1 \to \Gamma_{U_F} \to \Gamma_{P_F} \to \Lambda_F \to 1 \tag{1.4}$$

exact. The group F_{U_F} is an arithmetic subgroup of U_F and hence a lattice in the real vector space; $T_F := U_F(\mathbf{C})/\Gamma_{U_F}$ is then a complex torus with cocharacter group Γ_{U_F}.

Let $D(F) = U_F(\mathbf{C}) \cdot D$ be the Siegel domain of third kind containing D as explained in [ARMT, §4]. The arguments there produce a principal fiber bundle

$$D(F) \to D(F)' := D(F)/U_F(\mathbf{C}) \to F$$

with structure group $U_F(\mathbf{C})$. Passing to quotient, we obtain the principal torus bundle

$$T_F \to D(F)/\Gamma_{U_F} \to D(F)'.$$

Write $T = T_F = \operatorname{Spec}(\mathbf{Q}[X^*(T)])$, where $X^*(T) = \operatorname{Hom}(T, \mathbf{G}_m)$ is the character group of T_F.

Each polyhedral cone $\sigma \in \Sigma_F$ induces an invariant embedding $T \hookrightarrow T_\sigma = \mathrm{Spec}(\mathbf{Q}[X^*(T) \cap \check{\sigma}])$, where $\check{\sigma}$ is the dual of σ. Recall that Γ_{U_F} is identified with the cocharacter gorup fo T_F. Let $\overline{\Gamma}_F \subseteq \mathrm{Aut}(\mathrm{Lie}(U_F))$ be the subgroup of automorphisms induced by Γ_{P_F}. A choice $\{\sigma_\alpha\}$ of $\overline{\Gamma}_F$-**admissible polyhedral decomposition** of $C(F) \subseteq U_F(\mathbf{C})$ (see [AMRT, §2, Def.4.10]) admits an equivariant embedding

$$\begin{array}{ccc}
T_F & \hookrightarrow & (T_F)_{\{\sigma_\alpha\}} \\
\downarrow & & \downarrow \\
D(F)/\Gamma_{U_F} & \hookrightarrow & D(F)/\Gamma_{U_F} \times^{T_F} (T_F)_{\{\sigma_\alpha\}} \\
\downarrow & & \downarrow \\
D(F)' & \overset{\sim}{\longrightarrow} & D(F)'
\end{array}$$

Let $(D/\Gamma_{U_F})_{\{\sigma_\alpha\}}$ be the interior of the closure of $D(F)/\Gamma_{U_F}$ in $D(F)/\Gamma_{U_F} \times^{T_F} (T_F)_{\{\sigma_\alpha\}}$. If Σ_F is $\overline{\Gamma}_F$-admissible for every F, then $\Sigma := \sqcup \Sigma_F$ is called a Γ**-admissible collection**, and the main theorem of [AMRT] asserts that $(D/\Gamma_{U_F})_{\{\sigma_\alpha\}_F}$ glue together to give charts for the compactifications of $\Gamma\backslash D$.

Proposition 1.4.0.1 ([AMRT, Thm III.5.2]). *There exists a unique compact Hausdorff complex analytic space $S_\Gamma(D)_\Sigma$ containing $S_\Gamma(D)$ as an open dense subset such that, for any rational boundary component F of D, there are open analytic morphisms $(\phi_\sigma)_{\sigma\in\Sigma_F}$ making the following diagram commutative*

$$\begin{array}{ccc}
D/\Gamma_{U_F} & \hookrightarrow & (D/\Gamma_{U_F})_{\{\sigma_\alpha\}} \\
\downarrow & & \downarrow \phi_\sigma \\
S_\Gamma(D) & \hookrightarrow & S_\Gamma(D)_\Sigma
\end{array}$$

and such that every point of $S_\Gamma(D)$ is in the image of one of the maps ϕ_σ.

When Γ is neat, φ_σ is a local homeomorphism, and therefore if every $\sigma \in \Sigma_F \subseteq \Sigma$ is generated by a subset of a basis for Γ_{U_F}, then $S_\Gamma(D)_\Sigma$ is smooth and $S_\Gamma(D)_\Sigma - S_\Gamma(D)$ is a normal crossings divisor, in which case Σ and the compactification is called SNC. By [AMRT, Cor. III.7.6] every Γ-admissible collection Σ admits an SNC refinement Σ' such that $S_\Gamma(D)_{\Sigma'} \to S_\Gamma(D)_\Sigma$ is projective, and by [AMRT, p. IV.2.1] there is an Γ-admissible Σ such that $S_\Gamma(D)_\Sigma$ is projective. Hence, $S_\Gamma(D)$ admits a projective SNC toroidal compactification. The construction also produces a map

$$\varphi_\Gamma : S_\Gamma(D)_\Sigma \to \overline{S}_\Gamma(D),$$

which is trivial on $S_\Gamma(D)$, and can be described as blow-downs along the boundary divisors. Intuitively, the toroidal compactification is obtained by blowing-up along the singular boundary divisors of $\overline{S}_\Gamma(D)$.

The above procedure can be adelized as is done in [Har89], and we obtain a corresponding toroidal compactification of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$, denoted $\mathrm{Sh}_K(G, X)_\Sigma$, where K is such that $K \cap \mathbf{G}(\mathbf{Q})^+ = \Gamma$. If Σ is $K \cap G(\mathbf{Q})$-admissible in the sense of [AMRT, Definition 5.1], the corresponding toroidal compactification $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$ is smooth, and if Σ is furthermore defined by cocores then $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$ is projective by Tai's theorem [AMRT, IV, §2]. Lastly, we assume that Σ is such that $Z_\Sigma := \mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma - \mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ is a divisor with normal crossing. This is equivalent to the hypothesis that for all $\sigma \in \Sigma$, the semigroup $\sigma \cap (U_F(\mathbf{Q}) \cap K)$ is generated by a subset of a basis for the free abelian group $U_F(\mathbf{Q}) \cap K$. Combinatorial data Σ satisfying all of the above conditions exists and is constructed for instance in [Har89].

When K is allowed to vary through a suitably chosen collection of open compact subgroups of $G(\mathbf{A}^\infty)$, we choose $\Sigma(K)$ adapted to K in such a way that, if $K' \subset K$ with both K' and K in the collection, the natural covering map $S_{K'}(G, X) \to S_K(G, X)$ extends to a map

$$S_{K'}(G, X)_{\Sigma(K')} \to S_K(G, X)_{\Sigma(K)}$$

of toroidal compactifications.

1.5 Automorphic vector bundles

To every finite dimensional representation (ρ, W) of K_∞, we may associate a vector bundle $\mathcal{W}$ on $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$. As a complex analytic vector bundle over $\mathrm{Sh}(\mathbf{G}, \mathbf{X})(\mathbf{C})$, the bundle $\mathcal{W}$ is given by

$$\mathcal{W} = \varprojlim_{K} \big(\mathbf{G}(\mathbf{Q})\backslash\mathbf{G}(\mathbf{A})\big) \times W/K_\infty K,$$

where K runs over the compact opens in $\mathbf{G}(\mathbf{A}^\infty)$ and K_∞ acts on the right on $\mathbf{G}(\mathbf{A})$ and via ρ on W. The bundles thus constructed are called **automorphic vector bundles**. If μ is the highest weight of (ρ, W), we write $W = W_\mu$ and denote by $\mathcal{W}_\mu$ the associated automorphic vector bundle. A full description of the admissible parameters μ is done in §2.1. As with Shimura variety, the bundle $\mathcal{W}_\mu$ is defined over a reflex field, denoted $E(\mathcal{W}_\mu)$, again independent of the level K. Hereafter all levels K are assumed to be neat.

By the works of Mumford [Mum77] and Harris [Har85], each automorphic vector bundle $\mathcal{W}_K$ at the finite neat level K has a canonical extension $\mathcal{W}_K^{\mathrm{can}} = \mathcal{W}_{K,\Sigma}^{\mathrm{can}}$ to a vector bundle on $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$. Let $\mathcal{J}_\Sigma$ be the the sheaf of the ideals defining Z_Σ. Set $\mathcal{W}_{K,\Sigma}^{\mathrm{sub}} := \mathcal{W}_{K,\Sigma}^{\mathrm{can}} \otimes_{\mathcal{O}} \mathcal{J}_\Sigma$, where $\mathcal{O}$ is the structure sheaf of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$. Denote the corresponding automorphic vector bundles on $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ by $\mathcal{W}^{\mathrm{sub}}$ and $\mathcal{W}^{\mathrm{can}}$, respectively. When the level is clear from context, we use the same notation for the finite level versions of these bundles.

Set

$$\tilde{H}^*(\mathcal{W}^{\text{can}}) := \varinjlim_{K,\Sigma} H^*(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_\Sigma, \mathcal{W}^{\text{can}}),$$
$$\tilde{H}^*(\mathcal{W}^{\text{sub}}) := \varinjlim_{K,\Sigma} H^*(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_\Sigma, \mathcal{W}^{\text{sub}}),$$
$$H_!^*(\mathcal{W}) := \mathbf{Im}\Big(\tilde{H}^*(\mathcal{W}^{\text{sub}}) \to \tilde{H}^*(\mathcal{W}^{\text{can}})\Big).$$

Unless the signature is different from $(n-1,1)$ (or we are in the anisotropic case) the toroidal compactifications are not canonical and depend on the choice of Σ_K, an admissible collection of fans that also depends on the level K. Therefore, for $K' \subseteq K$, in order to extend the map $\pi_{K,K'} : S_{K'}(\mathbf{G},\mathbf{X}) \to \mathrm{Sh}_K(\mathbf{G},\mathbf{X})$ to

$$\pi^{\text{tor}}_{K,K'} : \mathrm{Sh}_{K'}(\mathbf{G},\mathbf{X})_{\Sigma_{K'}} \to \mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma_K},$$

we have to pick Σ_K compatible with K' as well. Furthermore, the definition of $\tilde{H}^i(\mathcal{W}^{\text{can}})$ involves taking a limit over all refinements Σ'_K of Σ_K even at a fixed level K. These rather delicate issues are at least partly ameliorated when passing to the bounded derived category $D(E(\mathcal{W}))$ of $E(\mathcal{W})$-vector spaces, where $E(\mathcal{W})$ is the reflex field of the automorphic vector bundle $\mathcal{W}$.

Proposition 1.5.0.1 ([AH, Prop. 9])**.** *Let $K' \subseteq K \subseteq \mathbf{G}(\mathbf{A}^\infty)$ be neat subgroups, and Σ_K an admissible fan. Let $\mathcal{W}$ be an automorphic vector bundle.*

i) Let Σ'_K be a refinement of Σ_K. Then the morphism

$$\pi_{\Sigma,\Sigma'} : \mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma'_K} \to \mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma_K}$$

induces a canonical quasi-isomorphism

$$R\pi_{\Sigma,\Sigma',*} : R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma'_K}, \mathcal{W}^{can}) \cong R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma_K}, \mathcal{W}^{can})$$

in the category $D^b(E(\mathcal{W}))$ of bounded complexes of $E(\mathcal{W})$-vector spaces. Similarly, $\pi_{\Sigma,\Sigma'}$ also induces a canonical quasi-isomorphism

$$R\pi_{\Sigma,\Sigma',*} : R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma'_K}, \mathcal{W}^{sub}) \cong R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma_K}, \mathcal{W}^{sub})$$

Hence, the objects

$$R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma_K}, \mathcal{W}^{can}) \quad \textit{and} \quad R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_{\Sigma_K}, \mathcal{W}^{sub})$$

are well-defined in $D^b(E(\mathcal{W}))$ and are independent of the choice of Σ_K. Hereafter, we denote them by $R\Gamma^{can}(\mathcal{W}_K)$ and $R\Gamma^{sub}(\mathcal{W}_K)$, respectively.

ii) Let $t \in \mathbf{G}(\mathbf{A}^\infty)$. For any neat K, the natural isomorphism

$$*t^{-1} : \mathrm{Sh}_K(\mathbf{G},\mathbf{X}) \to \mathrm{Sh}_{tKt^{-1}}(\mathbf{G},\mathbf{X})$$

defines isomorphisms in $D^b(E(\mathcal{W}))$ by pullback

$$*t^{-1} : R\Gamma^{can}(\mathcal{W}_{tKt^{-1}}) \simeq R\Gamma^{can}(\mathcal{W}_K),$$
$$*t^{-1} : R\Gamma^{sub}(\mathcal{W}_{tKt^{-1}}) \simeq R\Gamma^{sub}(\mathcal{W}_K).$$

iii) Let $K' \subseteq K$ be compact open. Then

$$\pi^*_{K,K'} : R\Gamma^\star(\mathcal{W}_{K'}) \simeq R\Gamma^\star(\mathcal{W}_K),$$

$$\pi_{K,K',*} : R\Gamma^\star(\mathcal{W}_{K'}) \simeq R\Gamma^\star(\mathcal{W}_K).$$

with $\star \in \{\mathrm{can}, \mathrm{sub}\}$.

iv) Suppose K and K' are both unramified at p, and $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}), \mathrm{Sh}_{K'}(\mathbf{G}, \mathbf{X}), \mathcal{W}_K, \mathcal{W}_{K'}$ all admit compatible models over $\mathcal{O}$, where $\mathcal{O}$ is a localization of $E(\mathcal{W})$ at a prime above p. Then all objects defined have $\mathcal{O}$-models, i.e. there are (perfect) complexes in $D^b(\mathcal{O})$ recovering $R\Gamma(\mathbb{S}_K(\mathbf{G}, \mathbf{X}), \mathcal{W}^{can})$ and $R\Gamma(\mathbb{S}_K(\mathbf{G}, \mathbf{X}), \mathcal{W}^{sub})$, for which the conclusions of (1) and (3) hold true. Furthermore, if $t \in \mathbf{G}(\mathbf{A}^{\infty,p})$, the conclusions from (2) hold for the corresponding elements in $D^b(\mathcal{O})$.

Proof. Most of the above results follow from [Har90]. The fact that we may pick these complexes to be perfect is an application of Lemma 1 in Ch.II.5 of [Mum08]. □

Lemma 1.5.0.2 ([Har90]). *The spaces $\tilde{H}^*(\mathcal{W}^{can})$, $\tilde{H}^*(\mathcal{W}^{sub})$, and $H_!(\mathcal{W})$ are admissible $\mathbf{G}(\mathbf{A}^\infty)$-modules and the natural morphisms $\tilde{H}^*(\mathcal{W}^{sub}) \to \tilde{H}^*(\mathcal{W}^{can})$ are morphisms of $\mathbf{G}(\mathbf{A}^\infty)$-modules. Moreover, these spaces all possess canonical $E(\mathcal{W})$-rational structures, which are invariant under the action of $\mathbf{G}(\mathbf{A}^\infty)$.*

The cohomology classes of these bundles are related to automorphic forms with coefficients in the corresponding highest weight module. The connection is achieved through Dolbeault cohomology (see (2.3)). The following result was announced by Franke but its first proof appeared only recently in Jun Su's thesis [Su18].

Theorem 1.5.0.3. *There is a canonical isomorphism*

$$H^*_{\bar{\partial}}(\mathcal{A}(\mathbf{G}) \otimes W_\mu) = H^*(\mathfrak{P}, K_\infty; \mathcal{A}(\mathbf{G}) \otimes W_\mu) \xrightarrow{\sim} \tilde{H}^*(\mathcal{W}^{can}_\mu)$$

of $\mathbf{G}(\mathbf{A}^\infty)$*-modules, where* $\mathcal{A}(\mathbf{G})$ *are the automorphic forms on* $\mathbf{G}(\mathbf{Q})\backslash\mathbf{G}(\mathbf{A})$*, which are assumed to be* $Z_\infty \cdot K_\infty$*-finite with* Z_∞ *being the center of* $\mathbf{G}$*.*

Restriction to cusp forms $\mathcal{A}_0(\mathbf{G}) \subseteq \mathcal{A}(\mathbf{G})$ leads to contribution to the interior cohomology.

Theorem 1.5.0.4. *For any* i*, the following morphism induced by the above isomorphism*

$$H^i_{\bar{\partial}}(\mathcal{A}_0(\mathbf{G}) \otimes W_\mu) = H^i(\mathfrak{P}, K_\infty; \mathcal{A}_0(\mathbf{G}) \otimes W_\mu) \hookrightarrow H^i_!(\mathcal{W}^{can}_\mu)$$

is injective. Denote by $\underline{\text{Lift}}$ *its inverse map.*

Therefore, the study of the cohomology $\tilde{H}^*(\mathcal{W}^{\text{can}})$ reduces to that of the cohomologies $H^*_{\bar{\partial}}(M \otimes W_\mu)$ for unitary $(\mathfrak{g}, K_\infty)$-modules M. Particular interest among the unitary $(\mathfrak{g}, K_\infty)$-modules are the tempered representations. They are the representations whose matrix coefficients lie in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, and are the main constituents in the Langlands classification of irreducible $(\mathfrak{g}, K_\infty)$-modules. The following results reduce the computation of the tempered part of a coherent cohomology of an automorphic vector bundle to a bookkeeping involving Harish-Chandra parameters. See §2.3 for the relevant notation and definitions.

Theorem 1.5.0.5 (Mirković)**.** *Suppose* M *is a tempered representation such that there exists a* K_∞*-module* W_μ *with* $H^*_{\bar{\partial}}(M \otimes W_\mu) \neq 0$*. Then* M *is either discrete series or a non-degenerate limit of discrete series.*

If $M = \pi_\lambda$ or $\pi(\lambda, \psi)$ with $\psi \supset \Delta_c^+$, the cohomology group $H^*_{\bar{\partial}}(M \otimes W_\mu)$ is fully determined by λ and ψ.

Theorem 1.5.0.6 (Thm 3.4 of [Har90]). *Let $\lambda \in \mathbf{\Lambda} + \rho$ be such that $\langle \lambda, \alpha \rangle > 0,\ \forall \alpha \in \psi_c$ and λ is ψ-dominant with respect to some positive root system $\psi \supset \Delta_c^+$. Let $\mu \in \mathfrak{a}^*_{\infty,\mathbf{C}}$ be an Δ_c^+-dominant integral weight and (ρ_μ, W_μ) be the finite-dimensional irreducible representation of K_∞ of highest weight μ. Then*

(i) $H^q(\mathfrak{P}, K_\infty; \hat{\pi}(\lambda, \psi) \otimes V_\mu) = 0$ unless $q = q_{\lambda,\psi} = \#(\psi \cap \Delta_n^+)$ and $\Lambda = \lambda - \rho$.

(ii) The character $\mu \in \mathfrak{a}^*_{\infty,\mathbf{C}}$ is Δ_c^+-dominant and integral, and

$$\dim H^{q_{\lambda,\psi}}(\mathfrak{P}, K_\infty; \hat{\pi}(\lambda, \psi) \otimes V_\mu) = 1.$$

If λ is regular, the analogous statement holds for the discrete series π_λ in place of $\pi(\lambda, \psi)$.

Let $F_0 = \mathbf{Q}$ so that $d = 1$, and suppose $\mathbf{G}$ has a signature $(r, n-r)$ at the unique archimedean place. In this case, $q_{\lambda,\psi}$ varies among k consecutive integers, where k is the number of pairs $a_i = a_j$ with $1 \le i \le r,\ r+1 \le j \le n$.

Remark 1.5.0.7. *We say that $\hat{\pi}(\lambda, \psi)$* ***belongs to the*** *$(q_{\lambda,\psi} + 1)^{th}$* ***chamber****. Note for discrete series we have $q_\lambda = \Delta_n^+ \cap \Delta_n^+(\lambda)$, where $\Delta_n^+(\lambda)$ is the set of non-compact roots* positive *for $\lambda = (a_1, \ldots, a_r; a_{r+1}, \ldots, a_n; c)$. Suppose $\mathbf{G}$ has signature $(n-1, 1)$ at the unique archimedean place. Then for a non-degenerate limit of discrete series $\hat{\pi}(\lambda, \psi)$ in the notation of §2.3, we have*

$$q_{\lambda,\psi} = \begin{cases} i & \text{if } e_i - e_n \in \psi, \\ i-1 & \text{if } e_i - e_n \notin \psi. \end{cases}$$

In this situation, to distinguish between these cases set $\pi(\lambda, i)^+$ (resp. $\pi(\lambda, i)^-$) for the non-degenerate limit of discrete series $\hat{\pi}(\lambda, \psi)$ where $\lambda = (a_1, \ldots, a_i, \ldots, a_{n-1}; a_i; c)$ and $e_i - e_n \in \psi$ (resp. $e_i - e_n \notin \psi$).

We now define an invariant that computes the length of the amplitude of the complexes computing the relevant cohomology theories.

Definition 1.5.0.8 (the invariant l_0)**.** *Let u be a real place of F_0 and fix v a place in F above it. Assume $\mathbf{G}(F_v)$ is of signature $(r_v, n-r_v)$. Let $\lambda = (a_1, \ldots, a_{r_v}; a_{r_v+1}, \ldots, a_n; c) \in \mathbf{\Lambda} + \rho$ be such that $\langle \lambda, \alpha \rangle > 0,\ \forall \alpha \in \psi_c$ and λ is ψ-dominant with respect to some positive root system $\psi \supset \Delta_c^+$. Set $l_0(\lambda)$ to be the number*

$$l_{0,v}(\lambda) \coloneqq \#\{(i,j) \in \{1, \ldots, r_v\} \times \{r_v+1, \ldots, n\} :\ a_i = a_j\}.$$

Set

$$l_0 \coloneqq \sum_v l_{0,v}.$$

In light of Thm 1.5.0.6, the invariant $l_0(\lambda)$ gives the range of support of (localized) coherent cohomology for any NLDS $\hat{\pi}(\lambda, \psi)$ with Harish-Chandra parameter λ. Note that l_0 is not the same as the one in [BW80], which is zero since $\mathbf{G}$ admits Shimura datum. It is worth pointing out that, as stated, the setup of Coherent in [CG18, p.300] is true only for LDS of signature $(n-1, 1)$; the correct definition of l_0 for general coherent cohomology in the Shimura setting is ours. We also have the following.

Proposition 1.5.0.9. *The invariant l_0 defined above coincides with l_0 of deformation theory origin as defined in [Cal18, §2.8] with the assumptions therein (local and global deformation rings are complete intersections and the dimension of Kisin deformation ring depends only on the Hodge-Tate weights).*

Proof. Fix a rational prime $p > n$ which is totally split in F. The Galois representations of interests are

$$\bar{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathcal{G}(\mathbf{F}_{\mathfrak{p}})$$

and a lift ρ (see §8.2), which we assume to satisfy

- a "big image" assumption;
- be odd at infinity;
- crystalline at all primes above p.

Here $\mathfrak{p}$ is a prime above p in the rings of integers of a number field E containing F and all Hecke eigenvalues of ρ. In the notation of [Cal18, §2.8], we therefore have[2]

$$l_0 = l_{0,p} + l_{0,G} + l_{0,\infty}.$$

The invariant $l_{0,G}$ equals the "defect" δ from [BW80], and is given by

$$l_{0,G} = \sum_{v|\infty} \operatorname{rk}(G(F_v)) - \operatorname{rk}(K(F_v)) - \operatorname{rk}(A(F_v)) = 0,$$

where A is maximally split torus. This quantity is zero either by direct computation or by appealing to Harish-Chandra's theorem since $\mathbf{G}(\mathbf{R})$ admits discrete series. For each real place v of F, fix the complex conjugation $c_v \in \operatorname{Gal}(\overline{F_v}/F_v)$. Since ρ is odd, we have

$$l_{0,\infty} = \frac{1}{2} \sum_{v-\text{real}} \big(\operatorname{Tr}(c_v | \operatorname{Lie}(G)) - (\operatorname{rk}(G(R)) - 2 \cdot \operatorname{rk}(K(\mathbf{R}))\big) = \sum_{v-\text{real}} 0 = 0.$$

The contribution from $l_{0,p}$ now remains. Pick $v|p$. Consider its conjugation to a fixed Borel and then restricted to its torus:

$$\tilde{\rho} : I_v \to \mathcal{B}(\overline{\mathbf{Q}}_p) \to \mathcal{T}(\overline{\mathbf{Q}}_p),$$

[2] note that group in [Cal18] is semisimple, while ours is reductive. To deal with this, we may just fix determinants for all of our deformation rings.

which, since ρ is crystalline at v and thus semistable, $\tilde{\rho}$ is a sum of integral power of cyclotomic character. We may view it as a cocharacter $\tilde{\rho} \in \mathrm{Hom}(\mathbf{G}_m, \mathcal{T})$. Arguing as in [Cal18, p.42], we then obtain

$$l_{0,v} = \dim \mathcal{U}_v,$$

where $\mathcal{U}_v = \mathrm{Stab}_{W_{\mathbf{G}}}(\tilde{\rho})$. Ignoring the GL_1-factor of $\mathbf{G}$ we may think of $W_{\mathbf{G}}$ as S_n acting on the diagonal entries by permutation. The size of this stabilizer equals the number of diagonal entries which are equal, which is exactly $l_{0,v}$ of Definition 1.5.0.8. The claim follows by summing over all $v|p$. $\square$

Chapter 2: Structure theory and parameters for unitary (similitude) groups

2.1 Root systems and structure theory

Given a Shimura datum $(\mathbf{G}, \mathbf{X})$, fix a point $x \in \mathbf{X}$, identified with a homomorphism $h_x : \mathbf{S} \to \mathbf{G}_{\mathbf{R}}$. By the Deligne's axioms, the homomorphism $\mathrm{Ad} \circ h_x$ gives a Hodge structure on $\mathfrak{g}$ of weight 0 and type $\{(-1,1),(0,0),(1,-1)\}$. Consider the Hodge decomposition

$$\mathrm{Lie}(G)_{\mathbf{C}} = \mathfrak{g}_{\mathbf{C}} = H^{0,0}(h) \oplus H^{-1,1}(h) \oplus H^{1,-1}(h), \tag{2.1}$$

which specifies a filtration $\mathbf{F}^*(h)$ with $\mathbf{F}^0(h) = H^{0,0} \oplus H^{1,-1}$, $\mathbf{F}^1(h) = H^{1,-1}$, and $\mathbf{F}^2 = \{0\}$.

Recall that $G := \mathbf{G}(\mathbf{R})$ acts by conjugation on $\mathbf{X}$ and the stabilizer $K_\infty = K_x$ of the point $x \in \mathbf{X}$ is a maximal compact subgroup of G, which is Zariski connected, and compact modulo the centre. Let A_∞ be a maximal torus inside K_∞ containing $h(\mathbf{S}(\mathbf{R}))$; in particular, it is also maximal inside G. Denote by $\Delta = \Delta(G, A_\infty)$, $\Delta_c = \Delta(K_\infty, A_\infty)$, and $\Delta_n = \Delta - \Delta_c$ the set of roots, compact roots, and non-compact roots of A_∞ in G, respectively. For $\alpha \in \Delta$, denote its roots space inside $\mathfrak{g}_{\mathbf{C}}$ by $\mathfrak{g}_{\mathbf{C}}^{\alpha}$. Let $W = W(G, A_\infty)$ and $W_c = W(K_\infty, A_\infty)$ be the Weyl (resp. compact Weyl) group of G.

The Hodge decomposition (2.1) preserves the root space, i.e. for any $\alpha \in \Delta$ the root space $\mathfrak{g}_{\mathbf{C}}^{\alpha}$ is fully contained inside one of $H^{0,0}(h)$, $H^{1,-1}(h)$, or $H^{-1,1}(h)$. Moreover,

$$H^{0,0}(h) = \mathfrak{z}(\mathfrak{g}) \oplus \oplus_{\alpha \in \Delta_c} \mathfrak{g}_{\mathbf{C}}^{\alpha} = \mathfrak{k},$$

while $H^{1,-1}(h) \oplus H^{-1,1}(h)$ is the direct sum of all non-compact roots. We assume hereafter that non-compact $\alpha \in \Delta_n$ is positive, denoted $\alpha \in \Delta_n^+$, if $\mathfrak{g}_{\mathbf{C}}^{\alpha} \subset H^{-1,1}$. Since $H^{2,2}(h) = \{0\}$, the sum of two positive non-compact roots is not a root, while $[H^{0,0}(h) \oplus H^{1,-1}(h)] \subset H^{-1,1}$ shows that the sum of a positive non-compact and any compact root, if a root, it is necessarily positive. Therefore, any choice Δ_c^+ of positive compact roots is compatible with Δ_n^+ to complete a positive root system $\Delta^+ = \Delta_c^+ \cup \Delta_n^+$. Set $\rho_c = \frac{1}{2}\sum_{\alpha\in\Delta_c^+}\alpha$, $\rho_n = \frac{1}{2}\sum_{\alpha\in\Delta_n^+}\alpha$, and $\rho = \rho_c + \rho_n$. Set $\mathfrak{P} = \mathbf{F}^0(h)$ and let $\mathfrak{p}^+ := H^{-1,1}(h)$ (resp. $\mathfrak{p}^- := H^{1,-1}(h)$). Hence, $\mathfrak{P} = \mathfrak{k} \oplus \mathfrak{p}^-$, so that the cochain in Dolbealt cohomology (see (2.3)) are given by $C^q(\mathfrak{P}, K_\infty; W) = \mathrm{Hom}_{K_\infty}(\wedge^q \mathfrak{p}^-, W) \simeq \left(\wedge^q \mathfrak{p}^{-,*}, W\right)^{K_\infty}$ for a $(\mathfrak{g}, K_\infty)$-module W.

2.2 Roots and weights for unitary (similitude) groups

Following [Gol14], we embed $G = \mathbf{G}(\mathbf{R}) \hookrightarrow \mathrm{GL}(2n, \mathbf{R})^d$ via $\mathrm{GL}(n, \mathbf{C}) \to \mathrm{GL}(2n, \mathbf{R}) : X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & -X \end{pmatrix}$. Denote by $\mathrm{diag}(a_1, \ldots, a_n)$ the diagonal matrix with $a_1, \ldots, a_n$ along the diagonal. Under this embedding we may identify $\mathfrak{a}_{\infty,\mathbf{C}}$ with the set of matrices

$$\mathfrak{a}_{\infty,\mathbf{C}} \simeq \left\{ b = (b_i) \in \mathfrak{gl}(2n, \mathbf{R})^d : b_i = \begin{pmatrix} \mathrm{diag}(t, \ldots, t) & -\mathrm{diag}(a_{i1}, \ldots, a_{in}) \\ \mathrm{diag}(a_{i1}, \ldots, a_{in}) & \mathrm{diag}(t, \ldots, t) \end{pmatrix},\ a_{ij},\ t \in \mathbf{C} \right\}.$$

Furthermore, using the functional

$$\begin{aligned}(a_1, \ldots, a_{n-1}; a_n; c) : \mathfrak{a}_{\infty,\mathbf{C}} &\to \mathbf{C} \\ b &\mapsto ct + \sum_{k=1}^{n} \mathbf{i}\lambda_k a_k,\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n, c \in \mathbf{C}$, we identify $\mathfrak{a}^*_{\infty,\mathbf{C}}$ with $\mathbf{C}^n \oplus \mathbf{C}$. Denote by $\mathbf{X}^*(A_\infty)$ the algebraic characters of $\mathfrak{a}_\infty$. Under the above identification, the weight space $\mathbf{\Lambda}$ of differentials of algebraic characters of $\mathfrak{a}_\infty$ is given by

$$\mathbf{\Lambda} = \Big\{(a_{ij}), c) \in M_{d\times n}(\mathbf{Z}^n) \oplus \mathbf{Z} : \sum_{i=1}^{d}\sum_{j=1}^{n} a_{ij} = c \pmod 2\Big\}. \tag{2.2}$$

The parameter c plays little role in the theory, so we may often omit it, and think of $\mathbf{\Lambda}$ as really parameterized by d in number n-tuples of integers. In the remaining sections, we assume that F is an imaginary quadratic field, so that $d = 1$. We record the relevant parameters in this case, so that $\mathbf{G}(\mathbf{R})$ has signature $(r, n-r)$ at the unique archimedean place. Note that $\mathfrak{u}(r, n-r)$ has the same complexification as $\mathfrak{gl}(n)$, so we may work more conveniently with the latter to determine roots.

Let e_i be the vector with 1 in the i^{th} entry and all other entries equal to zero. The set Δ of roots is

$$\Delta = \{(e_i - e_j; 0) : 1 \le i, j \le n,\ i \ne j\}.$$

The choice of positive non-compact roots Δ_n^+ from §2.1 translates to

$$\Delta_n^+ = \{(e_i - e_j; 0) : 1 \le i \le r,\ r+1 \le j \le n\}.$$

The set of compact roots Δ_c is given by

$$\Delta_c = \{(e_i - e_j; 0) : 1 \le i, j \le r \text{ or } r+1 \le i, j \le n\}.$$

Since, as remarked in §2.1, any system of positive compact roots is compatible with Δ_n^+, so we pick

$$\Delta_c^+ = \{(e_i - e_j; 0) : 1 \le i < j \le r \text{ or } r+1 \le i < j \le n\}$$

Hence,

$$\rho = \Big(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-3}{2}, -\frac{n-1}{2}; 0\Big).$$

The inner product induced by the Killing form for $\mathfrak{gl}(n)$ is given by $(X, Y) \mapsto 4n\mathrm{Tr}(XY)$. Hence, given our identification and upon rescaling, the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*_{\infty,\mathbf{C}}$ restricted to Λ is simply the standard dot product on $\mathbf{Z}^n$. Hence, $(a_1, \ldots, a_r; a_{r+1}, \ldots, a_n)$ is Δ_c^+- (resp. Δ^+-) dominant if $a_1 \geq a_2 \geq \ldots \geq a_r; a_{r+1} \geq \ldots \geq a_n$ (resp. $a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq a_n$).

Recall that the Weyl group $W_{\mathbf{G}}$ acts simply transitively on the set of Weyl chambers, which are the closures of the connected components of $\big(\mathbf{X}^*(A_\infty) \otimes \mathbf{R}\big)^{\mathrm{reg}}$. In turn, the set of Weyl chambers is in one-to-one correspondence with systems of positive roots of $\mathbf{G}$ relative to A_∞. Now, since $W_{\mathbf{G}} \simeq S_n$, the symmetric group on n letters, a choice of positive roots Δ^+ is equivalent to choosing an ordering on the entries.

2.3 Harish-Chandra parameter, discrete series, and a limit of discrete series

Let (π, V_π) be an irreducible $(\mathfrak{g}, K_\infty)$-module. We extend the action to the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbf{C}}$. The center $\mathfrak{z}(\mathfrak{g}_{\mathbf{C}})$ then acts by a scalar on V_π, and the corresponding homomorphism $\chi_\pi : \mathfrak{z}(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{C}$ is called **infinitesimal character** of π. These characters correspond to linear forms on $\mathfrak{a}_{\infty,\mathbf{C}}$ modulo $W_{\mathbf{G}}$, and we identify them by n-tuples of complex numbers. More precisely, by the work of Harish-Chandra, every morphism $\mathfrak{z}(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{C}$ is of the form $Z \mapsto \lambda(\gamma(Z)) =: \chi_\lambda(Z)$, where $\gamma : \mathfrak{z}(\mathfrak{g}_{\mathbf{C}}) \to \mathrm{Sym}(\mathfrak{a}_{\infty,\mathbf{C}})$ is the Harish-Chandra morphism, and $\lambda \in \mathfrak{a}^*_{\infty,\mathbf{C}}$ is a linear form. The linear form λ is uniquelly determined modulo the compact Weyl group $(W_{\mathbf{G}})_c \simeq S_r \times S_{n-r}$, and is called the **Harish-Chandra parameter** of π.

The **highest weights** of a finite-dimensional irreducible representations of K_∞ are the weights with corresponding n-tuples of integers satisfying $a_1 \geq \ldots \geq a_r; a_{r+1} \geq \ldots \geq a_n$. An n-tuple $\lambda = (a_1, \ldots, a_n)$ is **regular** if all entries are distinct. This is equivalent to $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$.

The $\bar{\partial}$-**cohomology** (or **Dolbeault cohomology**) of a $\mathfrak{k}$-module π with coefficients in a finite dimensional K_∞-module W is

$$H^*_{\bar{\partial}}(\mathfrak{g}, K_\infty; \pi \otimes W) := H^*(\mathfrak{P}, K_\infty; \pi \otimes W). \tag{2.3}$$

The cochains are given by $C^q(\mathfrak{P}, K_\infty; W) = \mathrm{Hom}_{K_\infty}(\wedge^q \mathfrak{p}^-, W) \simeq \big(\wedge^q \mathfrak{p}^{-,*} \otimes W\big)^{K_\infty}$. Modifying the proof of [BW80, §II, Prop. 3.1] (see [Har90, Prop. 4.5]), we obtain

$$H^q(\mathfrak{P}, K_\infty; \pi \otimes W) = C^q(\mathfrak{P}, K_\infty; \pi \otimes W) \simeq \big(\wedge^q \mathfrak{p}^{-,*} \otimes W\big)^{K_\infty},$$

whenever the cohomology is nonzero.

The infinitesimal character of a finite dimensional representation V_μ with a highest weight $\mu \in \mathbf{Z}^n$ is $\mu + \rho$, which in the cases of our interest will always be regular. By results of Harish-Chandra we know that the discrete series representations of G, or discrete series $(\mathfrak{g}, K_\infty)$-modules, are parameterized by the regular elements $(\mathbf{\Lambda} + \rho)^{\text{reg}}$ in $\mathbf{\Lambda} + \rho \subseteq (\mathfrak{a}_{\infty,\mathbf{C}})^*$, up to conjugation by $(W_{\mathbf{G}})_c$. Given $\lambda \in \mathbf{\Lambda} + \rho$, we denote the corresponding discrete series by π_λ with Harish-Chandra parameter λ. By the above discussion the infinitesimal character is $\lambda \ (mod\ W_{\mathbf{G}})$, so that there are precisely $|W_{\mathbf{G}}/(W_{\mathbf{G}})_c| = n$ discrete series $(\mathfrak{g}, K_\infty)$-modules sharing the same infinitesimal character $\Lambda + \rho$ with a given finite-dimensional representation V_Λ. We call the collection of these representation an L**-packet** associated to the dominant weight Λ. Herearefter, we identify a regular infinitesimal character with the unique Δ^+-dominant element in its $W_{\mathbf{G}}$-orbit and a Harish-Chandra parameteter with the unique Δ^+_c-dominant element in its $(W_{\mathbf{G}})_c$-orbit. Among the unitary representation, π_λ is parametrized by its infinitesimal character and its Blattner (lowest) K_∞-type τ_λ. The Blattner type occurs with multipicity one in π_λ and all other K_∞-types have longer highest weight; we identify it with the n-tuple of its highest weight.

The discrete series are parametrized by Harish-Chandra parameters λ belonging to the interior of the Weyl chambers. Allowing the Harish-Chandra parameters to belong to some of the walls of the Weyl chambers, Zuckermann constructs a representation $\pi(\lambda, \phi)$, called (non-degenerate)[1] **limit of discrete series representation** (NLDS), where $\phi \supset \Delta_n^+$ is a positive root system and $\lambda \in \mathbf{\Lambda} + \rho$ is dominant for ϕ[2], not orthogonal to any of the compact roots. Note that a regular λ determines unique ϕ_λ with these properties, and $\pi_\lambda = \pi(\lambda, \phi_\lambda)$. Similarly to discrete series, the representation $\pi(\lambda, \phi)$ is parametrized by its infinitesimal character and its Blattner K_∞-type τ_λ, which again occurs with multiplicity one and all other K_∞-types have longer highest weight. Given our choice of $\Delta^+ = \Delta_c^+ \cup \Delta_n^+$ in §2.2, we have the following description of NLDS: For $\lambda = (a_1, \ldots, a_r; a_{r+1}, \ldots, a_n; c) \in \mathbf{\Lambda} + \rho$ that is dominant with respect to a positive root system $\phi \supset \Delta_c^+$, the representation $\pi(\lambda, \phi)$ is a NLDS if and only if

1. $a_i \neq a_j$ if $1 \leq i < j \leq r$ or $r + 1 \leq i < j \leq n - 1$; and

2. $a_i = a_j$ for some $1 \leq i \leq r$ and $r + 1 \leq j \leq n - 1$.

Note that $a_i = a_j$ may occur for more than two entries as long as the inequalities on their indices are satisfied.

So far all construction of associated Galois representations to limits of discrete series on unitary (similitude) groups are achieved via congruences, see [Gol14], [PS16], and [GK19].

[1]there is also a degenerate limit of discrete series, wherein the Harish-Chandra parameter is allowed to be orthogonal to some of the compact roots (but necessarily not the ϕ-simple ones). We do not treat these here as they do not appear in the coherent cohomology of automorphic vector bundles.

[2]this condition is equivalent to $\pi(\lambda, \phi)$ being nonzero by the so-called *Hecht–Schmid identity* [HS76]. Whenever it is nonzero, it is known to be irreducible.

Chapter 3: Integral models

In this section we record relevant results concerning the integral models of the (open) Shimura variety $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ and of its toroidal compactification $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$ for suitably chosen Σ. The integral properties of PEL Shimura varieties are studied by Kottwitz [Kot92] who used their interpretation as a moduli space of abelian varieties with extra structure. The open Shimura variety $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ is of PEL-type and $\mathbf{G}$ is of type A, and thus fall in the scope of [Kot92].

The construction of appropriate arithmetic integral models for $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma$ is much more delicate and practically all known constructions employ the theory of degenerations of abelian varieties developed by Chai and Faltings in [FC90]. In his thesis [Lar92], Larsen constructs integral models for the compactifications of the Picard modular surfaces, which are the Shimura varieties of signature $(2, 1)$ at infinity. The case of general PEL-type Shimura varities is achieved by Lan in his thesis [Lan13]. These constructions are purely algebraic in nature with no analytic argument involved; for detailed comparison between the algebraic construction in [Lan13] and the analytic one in [AMRT], see [Lan12].

3.1 Kottwitz data and $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$

Consider the quintuple

$$\mathcal{U} = (B, V, *, \langle \cdot, \cdot \rangle, h),$$

satisfying

1. B is a finite-dimensional simple $\mathbf{Q}$-algebra;

2. V is finitely generated left B-module;

3. $*$ is a positive involution on B, i.e. $\mathrm{Tr}_{B(\mathbf{R})/\mathbf{R}}(x) > 0$ for all nonzero $x \in B(\mathbf{R})$;

4. $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{Q}$ is a non-degenerate symplectic form such that $\langle bx, y\rangle = \langle x, b^* y\rangle$ for all $b \in B$ and $x, y \in V$;

5. $h : \mathbf{C} \to \mathrm{End}_B V \otimes \mathbf{R}$ is a homomorphism of $\mathbf{R}$-algebras with involutions, where the involution on $\mathbf{C}$ is complex conjugation; and

6. the symmetric bilinear form

$$V(\mathbf{R}) \times V(\mathbf{R}) \to \mathbf{R}$$

$$(x, y) \mapsto \langle x, h(i)y\rangle$$

is positive definite.

Then, associated to $\mathcal{U}$ is a $\mathbf{Q}$-algebraic group $\mathbf{G} = \mathbf{G}(\mathcal{U})$, for which

$$\mathbf{G}(R) = \{g \in \mathrm{End}_B(V) \otimes_{\mathbf{Q}} R : gg^* = \nu(g) \in R^\times\}$$

for any $\mathbf{Q}$-algebra R.

If the center of B is a CM field F, then this construction recovers the unitary similitude group of §1. In this notation, the reflex field $E(\mathbf{G}, \mathbf{X})$ of the associated Shimura datum (with $\mathbf{X}$ constructed by the well-known recipe) is the following. Decompose $V(\mathbf{C}) = V_+ \oplus V_-$ as $B(\mathbf{C})$-module, wherein h acts on V_+ (resp. V_-) by z (resp. by $\bar{z}$). Then $E := E(\mathbf{G}, \mathbf{X})$ is the field of definition of the $B(\mathbf{C})$-module V_+; it is a number field.

Suppose now ℓ is a rational prime such that $B(\mathbf{Q}_\ell)$ is a product of matrix algebras over unramified extensions of $\mathbf{Q}_\ell$. Let $\mathcal{O}_B$ be a $\mathbf{Z}_{(\ell)}$-order of B with its ℓ-adic completion $\widehat{\mathcal{O}_B}$ being a maximal order of $B(\mathbf{Q}_\ell)$. Assume that there exists a lattice Λ inside $V(\mathbf{Q}_\ell)$ that is self-dual for $\langle\cdot,\cdot\rangle$ and is preserved by $\mathcal{O}_B$. We remark that under these assumptions ℓ is unramified in F and $\mathbf{G}$ is unramified at ℓ. Then for every $K = K^{(\ell)} \times K_\ell \subseteq \mathbf{G}(\mathbf{A}_f)$ with K_ℓ hyperspecial and $K^{(\ell)} \subseteq \mathbf{G}(\mathbf{A}^{\infty,\ell})$ neat, there is an analogous functor to $\mathcal{F}_K$ of §1 wherein each component of the quadruple $(A, \lambda, i, \bar{\eta})$ is well-behaved at the prime ℓ (see [Kot92, §5] for more details), that is representable by a smooth, quasi-projective scheme $\mathbb{S}_K := \mathbb{S}_K(\mathbf{G}, \mathbf{X})$ over $\mathcal{O}_{E,\ell} = \mathcal{O}_E \otimes Z_{(\ell)}$, whose basechange to E is $\mathbb{S}_K \otimes E = \mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$.

3.2 Integral models over the toroidal compactifications

Treating the construction of [Lan13] in any details is beyond the scope of this work. We just record the relevant statement.

Proposition 3.2.0.1 (Faltings-Chai, Lan). *Let ℓ be a prime at which G is unramified, and fix a hyperspecial maximal compact $K_\ell \subseteq G(\mathbf{Q}_\ell)$. Then for all neat $K \supset K_\ell$, there is a compatible choice of admissible smooth rational polyhedral data Σ (see [Lan13, Def. 6.3.3.2]) such that the $\mathbb{S}_K(\mathbf{G}, \mathbf{X})_\Sigma$ is a smooth proper scheme over a ℓ-adic integer ring $\mathcal{O}$ which contains $\mathbb{S}_K(\mathbf{G}, \mathbf{X})$ from above as an open dense subscheme. Furthermore, Σ may be chosen so that $\mathbb{S}_K(\mathbf{G}, \mathbf{X})_\Sigma -$ $\mathbb{S}_K(\mathbf{G}, \mathbf{X})$, viewed as a closed reduced subscheme, is a divisor with normal crossings. There is a canonical strata-preserving isomorphism between the basechange to $\mathrm{Spec}(\mathbf{C})$ of this integral model and the classical analytical construction as in [AMRT].*

3.3 Integral models of the automorphic bundles

Suppose ℓ is a rational prime satisfying all assumptions from §3.1. Let the quadruple $(A, \lambda, i, \bar{\eta})$ represent the universal class in the groupoid that is integral at ℓ. The abelian variety A comes equipped with an action by the $\mathbf{Z}_{(\ell)}$-order $\mathcal{O}_B$. Set Ω_A to be the isomorphism class of the relative sheaf of differentials of A – this is well-defined in the groupoid parameterizing the quadruples. The decomposition of Ω_A with respect to the $\mathcal{O}_B$-action is well-defined over a finite extension E_Ω of $E = E(\mathbf{G}, \mathbf{X})$. Possibly enlarging E_Ω we may assume that (i) E_Ω contains all embeddings of F into $\mathbf{C}$ and (ii) B is split over E_Ω. One can then show that for every $K = K^{(\ell)} \times K_\ell \subseteq \mathbf{G}(\mathbf{A}_f)$ with K_ℓ hyperspecial and $K^{(\ell)} \subseteq \mathbf{G}(\mathbf{A}^{\infty,\ell})$ neat and for any highest weight μ, there exists an automorphic vector bundle $\mathcal{W}_\mu$ on $\mathbb{S}_K$,whose base change recovers the classical $\mathcal{W}_\mu$ arising from an irreducible representation of K_∞ as in §1.5. For a more detailed construction of these using integral Schur functors, see [Gol14, §3.8 and Thm. 5.5.1].

Chapter 4: Motivic cohomology of the coadjoint motive

4.1 Artin motives

Let $\mathrm{Gal}_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the absolute Galois group. We work over the base field $\mathbf{Q}$ and with coefficients in some number field E. The category of **Artin motives over Q with coefficients in** E is the (Karoubian envelope) of the dual category with objects varieties over $\mathbf{Q}$ of dimension 0, and morphisms given by correspondences defined over $\mathbf{Q}$ with coefficients in E.The dual motive of M is denoted $M^{\vee}$. For example, given a number field F, the motive $h^0(F)$ is a self-dual Artin motive.

Let us be more explicit. A variety over $\mathbf{Q}$ of dimension zero $X = \mathrm{Spec}(\prod_i^n F_i)$ is the spectrum of a finite product of number fields. A correspondence between two such varieties X and Y is formal linear combinations of the connected components of $X \times Y$, i.e.

$$\mathrm{Cor}(X, Y) = \Big\{ \sum a_i Z_i \ : \ a_i \in E \Big\},$$

where $\{Z_i\}$ is the set of connected components of $X \times Y$. Mapping each Z_i to its characteristic function then identifies $\mathrm{Cor}(X, Y)$ with E-valued $\mathrm{Gal}_{\mathbf{Q}}$-invariant functions on $X \times Y(\overline{\mathbf{Q}})$. These can be naturally identified with matrices whose rows and columns are indexed by $X(\overline{\mathbf{Q}})$ and $Y(\overline{\mathbf{Q}})$, respectively.

Given a zero-dimensional X, we have an E-valued Galois representation given by the natural $\mathrm{Gal}_{\mathbf{Q}}$ action on $E^{X(\overline{\mathbf{Q}}}$. Furthermore, each correspondence $A \in \mathrm{Cor}(X, Y)$ given by the matrix A yields a $\mathrm{Gal}_{\mathbf{Q}}$-linear map $E^{Y(\overline{\mathbf{Q}})} \to E^{X(\overline{\mathbf{Q}})}$ given by the matrix A^t. In particular each projector

$p \in \mathrm{Cor}(X, X)$ yields $p^t : E^{X(\overline{\mathbf{Q}})} \to E^{X(\overline{\mathbf{Q}})}$. It is not hard to verify now that we have an equivalence of categories

$$\left\{ \begin{array}{l} \text{Artin motives over } \mathbf{Q} \\ \text{with } E \text{ coefficients} \end{array} \right\} \to \left\{ \begin{array}{l} E\text{-rational Galois rep-} \\ \text{resentations of } \mathrm{Gal}_{\mathbf{Q}} \end{array} \right\}$$

given by $(X, p) \mapsto \mathrm{im}(p^t)$ and $A \mapsto A^t$. Thus, we may identify these motives with their (Artin) Galois representations. It is often possible to find an invariant lattice $\Lambda \subseteq E^n$ so that for a given Artin motive X, the associated Galois representation is of the form

$$\rho_X : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_n(\Lambda).$$

We would also restrict to motives which are direct summand of the motive of a number field.

4.1.1 Artin motive attached to a cusp form of weight 1

Let $f \in \sum_n a_n q_n \in S_1(N, \epsilon)$ be a normalized newform of weight 1.[1] Let $E = \mathbf{Q}(a_n)$ be the number field generated by the Fourier coefficients. By a result of Deligne-Serre [DS74b, Thm. 4.1.], there is an associated Galois representation

$$\rho_f : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_E)$$

such that $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p)) = a_p$ and $\det(\rho_f(\mathrm{Frob}_p)) = \epsilon(p)$ for $p \nmid N$. We may indeed pick $\mathcal{O}_E$ coefficient by [DS74b, p. 521, footnote]. In light of the preceeding discussion, we may view ρ_f as the motive associated to f. Let us also denote $\pi = \pi_f$ the automorphic representation associated to π. For forms f of weight k, the Galois representation ρ_π associated to π differs from

[1] For the theory in the case of Hilbert modular forms, see Horawa's thesis [Hor20]

ρ_f by a $\det^{\frac{k-1}{2}}$ twist of the determinant. When $k = 1$ there is no difference, so we would use ρ_π and ρ_f interchangeably. In any case, the difference disappears when considering the adjoint representations.

We are particularly interested in the (co)adjoint motive $\mathrm{Ad}^*\rho_f$. Using ρ_f, we construct an action, denoted $\mathrm{Ad}\rho_f$, of $\mathrm{Gal}_{\mathbf{Q}}$ on $M_{2\times 2}(E)$ by conjugation. It decomposes as $\mathrm{Ad}\rho_f = \mathrm{Ad}^0\rho_f \oplus 1$, where $\mathrm{Ad}^0\rho_f$ is the restricted action on the traceless matrices. We thus have

$$\mathrm{Ad}^0\rho_f : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_3(E).$$

On the automorphic side, $\mathrm{Ad}^0\pi$ is realized as a strong lift from $H = \mathrm{GL}_2$ to $G = \mathrm{GL}_3$. Indeed, at each place, the L-homomorphism between ${}^L H^0 = \mathrm{GL}_2(\mathbf{C})$ and ${}^L G^0 = \mathrm{GL}_3(\mathbf{C})$ is

$$\mathrm{diag}(\alpha, \beta) \mapsto \mathrm{diag}(\alpha/\beta, 1, \beta/\alpha) \tag{4.1}$$

and the existence of the lift in this case is a theorem of Gelbart and Jacquet [GJ78, Thm. 9.3]

4.1.2 Stark unit group

We define the group of **Stark units** following [Sta75] associated to a general Artin motive M. Eventually we restrict only to the case of Artin motives arising from weight one representations but for clarity we work in full generality. By (4.1), it suffices to consider the associated Artin representations ρ_M, i.e.

$$\begin{array}{ccc} \mathrm{Gal}_{\mathbf{Q}} & \xrightarrow{\quad\rho_M\quad} & GL_n(\Lambda) \\ & \searrow \quad \nearrow_{\rho_M} & \\ & \mathrm{Gal}(H/\mathbf{Q}) & \end{array}$$

where $H/\mathbf{Q}$ is a finite extension through which ρ factors (by finite image). Set $G_{H/\mathbf{Q}} := \mathrm{Gal}(H/\mathbf{Q})$.

Definition 4.1.2.1. *The group of* ***Stark units*** *associated to* $\rho_M : G_{H/\mathbf{Q}} \to \mathrm{GL}_n(\Lambda)$ *is*

$$U_H[\rho] = \mathrm{Hom}_{\mathcal{O}_E[G_{H/\mathbf{Q}}]}(\Lambda, \mathcal{O}_H^\times \otimes \mathcal{O}_E).$$

Many of the relevant facts about $U_H[\rho]$ are verified in [Hor20, §1], which we just record below. Fix $\tau : L \hookrightarrow \mathbf{C}$ and let c_0 be the complex conjugation on L. Note that $\mathrm{rk}\, \mathcal{O}_L^\times = \#\big(G_{H/\mathbf{Q}}/\langle c_0\rangle\big) - 1$ by Dirichlet's unit theorem.

Proposition 4.1.2.2. *Suppose* $\rho : G_{H/\mathbf{Q}} \to \mathrm{GL}_n(\Lambda)$ *is an Artin representation. Then*

i) $U_H[\rho] \otimes E$ *is independent of the choice of splitting field* $H/\mathbf{Q}$.

ii) Suppose ρ *does not contain the trivial representation. Let* $d = \dim_E(\Lambda \otimes E)^{\langle c_0\rangle}$ *be the dimension of invariants under the complex conjugation. Then* $\mathrm{rk}\, U_H[\rho] = d$.

iii) If $\gcd(p, \mathcal{O}_H^{\times,\mathrm{tors}}) = 1$, *then* $U_H[\rho] \otimes \mathbf{Z}_p$ *is free* $\mathcal{O}_E \otimes \mathbf{Z}_p$*-module of rank* d. *In particular, for* $\mathfrak{p}|p$ *in* E*, we have that* $U_H[\rho] \otimes \mathcal{O}_E/\mathfrak{p}^n$ *is free* $\mathcal{O}_E/\mathfrak{p}^n$*-module of rank* d.

4.2 Stark units of a weight one form

Let f be as above. Considering $f \otimes \varepsilon^{-1}$ if necessary, we assume that f has trivial nebentypus. Let $p \nmid N$ and further that f is regular at p, i.e. has distinct Satake parameters. Let α_p and β_p be the eigenvalues (equivalently, the Satake parameters) of ρ_f at p. By regularity these are distinct, and since $\alpha_p\beta_p = \epsilon(p) = 1$ these are also p-units. Set

$$f_{\alpha_p}(z) = f(z) - \beta_p f(pz), \quad \text{and} \quad f_{\beta_p}(z) = f(z) - \alpha_p f(pz).$$

These are the *p-stabilized (ordinary) newforms of tame level N attached to f.* We follow [DLR15] in defining the Stark units associated to f as well as Stark elements u_{f^α} and u_{f^β}, whose p-adic logarithms are of interest. Let $\mathrm{Ad}^0\rho_f$ be the adjoint representation on the traceless matrices, and let H_f be its field of definition, which is potentially smaller than H. By (4.1), at p, the adjoint representation has Satake parameters $(\alpha_p/\beta_p)^{\pm 1}, 1$, so $\mathrm{Ad}^0\rho_f$ remains regular at p.

Definition 4.2.0.1. *The group(s) of **Stark units** associated to f are*

$$U_\pi = U_f = \mathrm{Hom}_{\mathcal{O}_E[G_{H/\mathbf{Q}}]}(\mathrm{Ad}\rho_f, \mathcal{O}_{H_f}^\times \otimes \mathcal{O}_E)$$
$$U_\pi^\circ = U_f^\circ = \mathrm{Hom}_{\mathcal{O}_E[G_{H/\mathbf{Q}}]}(\mathrm{Ad}^0\rho_f, \mathcal{O}_{H_f}^\times \otimes \mathcal{O}_E).$$

Similarly, we denote these groups by U_π and U_π°, where $\pi = \pi_f$ is an automorphic representation

By [Hor20, Cor 1.9], we have

$$U_f = U_L[\mathrm{Ad}^0\rho_f]$$

as f is cuspidal. In particular, U_f is one-dimensional. Let ϕ be a basis, and let

$$(\mathcal{O}_{H_f}^\times)_E^{\mathrm{Ad}^0\rho_f} := \phi(\mathrm{Ad}^0\rho_f) \subseteq \mathcal{O}_{H_f}^\times \otimes \mathcal{O}_E.$$

This module is of dimension $d \leq 3$ over E, where d is as in Prop 4.1.2.2 ii). By (4.1) Frob_p acts by one of the three potential eigenvalues. Fix an ordered basis (α_p, β_p) for Frob_p, and set

$$U_{f_\alpha} := \{u \in (\mathcal{O}_{H_f}^\times)_E^{\mathrm{Ad}^0\rho_f} : \mathrm{Frob}_p(u) = \frac{\alpha_p}{\beta_p}u\}.$$

In [DLR15, § 1.2], certain elements nonzero elements $u_{f_\alpha} \in U_{f_\alpha}$, again called Stark units attached to f_α, were produced on a case by case analysis on the type of f, provided that ρ_f is not induced from a character of a real quadratic field in which p splits. In that case, and only in it, U_{f_α} is trivial. Under our assumption of regularity at p, in the remaining cases $\dim_E U_{f_\alpha} = 1$. Under some assumptions, we may have a preferred choice of u_{f_α}. The p-adic logarithms of these Stark units are conjectured to give p-adic information about the first nonzero term of a Ranking-Selber L-function at $s = 1$. These are also related to the values of the trilinear form

$$\begin{aligned} I : S_2(N)_{\mathbf{C}} \times M_1(N, \epsilon)_{\mathbf{C}} \times M_1(N, \epsilon^{-1})_{\mathbf{C}} &\to \mathbf{C} \\ (\theta, f, f') &\mapsto \langle \theta, f f' \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Peterson scalar product.

4.3 Motivic group for Artin motive and the regulator map

Let $M = h(X)(n)$ with $w = i - 2n < 0$ be a motive over $\mathbf{Q}$. By definition,

$$H^{i+1}_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(n)) := [K_{2n-(i+1)}(X) \otimes \mathbf{Q}]^{(n)},$$

where the subscript n stands for the weight n subspace of the Adams operators. We are solely interested in the case of $i = 0$ and $n = 1$ and the integral subclasses

$$H^1_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(1))_{\mathbf{Z}} \subseteq H^1_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(1)),$$

which are defined as follows. Let $\mathcal{X}$ be a proper flat model over $\mathbf{Z}$ for X, and then

$$[K_1(X) \otimes \mathbf{Q}]^{(1)}_{\mathbf{Z}} = \mathrm{im}([K_1(\mathcal{X}) \otimes \mathbf{Q}] \to [K_1(X) \otimes \mathbf{Q}]^{(1)},$$

where we have taken the projection to the $n = 1$ subspace. For general i and n it is conjectured by Beilinson that this construction is independent on the choice of model $\mathcal{X}$. In fact, it is know that two proper regular models indeed yield the same result but such models are not known to exist in generally. For any Chow motive, Scholl defines $H^1_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(1))_{\mathbf{Z}}$ unconditionally. In general, we have a map

$$\mathcal{O}(X)^* \to K_1(X),$$

which is an isomorphism when X is the spectrum of the field (or a ring with Euclidean division algorithm). Thus, for an Artin motive $X = \mathrm{Spec}(F)$, we have

$$H^1_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(1)) \simeq F^{\times} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

This group comes with two types of regulator maps, coming from the de Rham and the étale realizations. In our case, these are given by

$$\begin{aligned} r_\infty : H^1_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(1)) &= [K_1(X) \otimes \mathbf{Q}]^{(1)} \to (F \otimes R)^{\times} \otimes \mathbf{Q}^{(1)} \xrightarrow{\mathrm{ch}_1} H_{\mathcal{D}}(X_{\mathbf{R}}, 1) \\ r_p : H^1_{\mathcal{M}_{\mathbf{Q}}}(X, \mathbf{Q}(1)) &= [K_1(X) \otimes \mathbf{Q}]^{(1)} \to (F \otimes \mathbf{Q}_p)^{\times} \otimes \mathbf{Q}^{(1)} \xrightarrow{\mathrm{ch}_1} H^1_{et}(X, \mathbf{Q}_p(1)), \end{aligned} \tag{4.2}$$

where $H_{\mathcal{D}}$ is the Deligne cohomology. The first Chern character maps are given by the usual and the p-adic logarithm, respectively. According to Beilinson's conjectures and the refinement by Bloch-Kato, the (determinant of the) archimedean regulator r_∞ governs the transcendental part of the special value of L-function, while r_p – the remaining rational multiple. In the case of interest the Stark unit group is essentially equivalent to the étale realization of the motivic group of the associated Artin motive (see Prop. 4.3.1.1 below). From this perspective, the fact that $\log_p(u_{f_\alpha})$ is related to the triple L-function is expected by the Bloch-Kato's conjecture.

4.3.1 Motivic cohomology of coadjoint motive

It is conjectured that there is a 3-dimensional, pure of weight 0, Chow motive M_{coad} over F, the *coadjoint motive of* f, with coefficients in a number field E[2], associated to $\mathrm{Ad}^*\rho_f$. The motivic group of interest is

$$H^1_{\mathcal{M}_{\mathbf{Q}}}(M_{coad}, E(1)),$$

and, as remarked earlier, for these School has associated

$$H^1_{\mathcal{M}_{\mathbf{Q}}}(M_{coad}, E(1))_{\mathbf{Z}} \subseteq H^1_{\mathcal{M}_{\mathbf{Q}}}(M_{coad}, E(1))$$

of integral classes. It is convenient to work with its étale realization. For $\mathfrak{p}$ in E, we have

$$H^*_{et}(M_{coad} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, E_{\mathfrak{p}}) \simeq \mathrm{Ad}^*\rho_f \otimes_E E_{\mathfrak{p}},$$

concentrated at degree 0. There is a natural map

$$H^1_{\mathcal{M}_{\mathbf{Q}}}(M_{coad}, E(1))_{\mathbf{Z}} \to H_{\mathcal{M}_H}(M_{coad}, E(1))_{\mathbf{Z}}^{G_{H_f/\mathbf{Q}}},$$

as well a $\mathfrak{p}$-adic realization map for every $\mathfrak{p}$ of E

$$H^1_{\mathcal{M}_{\mathbf{Q}}}(M_{coad}, E(1)) \otimes \mathcal{O}_{\mathfrak{p}} \to H^1_f(\mathbf{Q}, (\mathrm{Ad}^*\rho_f \otimes \mathcal{O}_{\mathfrak{p}})(1)), \tag{4.3}$$

both expected to be isomorphisms (see [HV19, (2.8)] and [BK07, 5.3(ii)]. The former is verified for the étale realization in the proof of the following.

[2] In [PV16, A.3] a procedure for general Chow motives coming from cohomological representations showing descend from $\overline{\mathbf{Q}}$ coefficients to a number field E is presented. We assume this to be the case for us too even though we study non-cohomological representations.

Proposition 4.3.1.1. *[Hor20, Prop. 1.17] There is an isomorphism*

$$H^1_f(\mathbf{Q}, (\mathrm{Ad}^*\rho_f \otimes \mathcal{O}_\mathfrak{p})(1)) \simeq U_f \otimes \mathbf{Q} \otimes \mathcal{O}_\mathfrak{p}$$

for all $\mathfrak{p}$ *such that* $N\mathfrak{p}$ *is coprime to* $[H : \mathbf{Q}]$.

Chapter 5: Shimura classes and derived Hecke operators

Let $F/\mathbf{Q}$ be an imaginary quadratic field, so that $F_0 = \mathbf{Q}$. Then the group $\mathbf{G} = GU(V)$ with $\dim V = n$ has a unique archimedean component. Let $\mathcal{O}$ be a large enough integer ring; usually we take $\mathcal{O} = \mathcal{O}_{E_\Pi}$, where $E_\Pi = \mathbf{Q}(\Pi)$ is the coefficient ring of an automorphic representation $\Pi = \Pi^\infty \otimes \Pi_\infty$, whose archimedean component Π_∞ is a nondegenerate limit of discrete series, and furthermore assume that E contains F. The same notation $\mathcal{O}$ is used for the structure sheaf of the Shimura variety but it should be clear from context which usage is meant. We suppose that to Π is associated a Galois representation

$$\rho_\Pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathcal{O}_{E_\Pi})$$

and let

$$\bar{\rho}_\Pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathcal{O}_{E_\Pi}/\mathfrak{p}) = \mathrm{GL}_n(\mathbf{F}_\mathfrak{p})$$

be the corresponding residual representation. A more refined description of the desired properties of the Galois representations is outlined in §8. We fix a neat level $K_\emptyset$, which we furthermore assume decomposes as $K_\emptyset = \prod K_v$. In applications, we would also assume that $\Pi_\emptyset^K \neq 0$. Let R be a $\mathcal{O}_F[\frac{1}{\mathrm{disc}(F)\cdot N}]$-algebra, where N is the product of the primes dividing the level $K_\emptyset$. Fix a prime $p \nmid K_\emptyset$, i.e. $K \cap \mathbf{G}(\mathbf{Q}_p)$ is hyperspecial, and also $p \nmid |W_\mathbf{G}|$. Let $\mathfrak{p}$ be an unramified prime in E above p. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ and let $\mathbf{T}$ be a torus inside it.

For a prime q in F with $N_{F/\mathbf{Q}}(\mathfrak{q}) \equiv 1 \ (mod\ p^m)$, set $\Delta_\mathfrak{q}$ to be the maximal product of p-power quotients of $\mathbf{T}(\mathbf{F}_{\mathfrak{q}}) = \big((\mathcal{O}_F/\mathfrak{q})^\times\big)^n$, and fix a map

$$\mathbf{T}(\mathbf{F}_{\mathfrak{q}}) \xrightarrow{\log_p} \Delta_{\mathfrak{q}} = (\mathbf{Z}/p^m)^n, \tag{5.1}$$

which exists by our assumption on $\mathfrak{q}$. Formally there are different $\log_p$ depending on $\mathfrak{q}$ but it would be clear from the context which $\mathfrak{q}$ is used, so we stick to the uniform notation $\log_p$. Set $R\langle 1\rangle := R\otimes\Delta_{\mathfrak{q}}$ and $R\langle -1\rangle := \mathrm{Hom}_R\big(\Delta_{\mathfrak{q}}, R\big)$. Similarly, if we have a collection $Q = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ of primes $N_{F/\mathbf{Q}}(\mathfrak{q}_i) \equiv 1\ (mod\ p^m)$ for $i = 1, \ldots, s$, we write $\Delta_Q = \prod_{i=1}^r \Delta_{\mathfrak{q}_i}$.

5.1 Classical Hecke algebra

Let $K \subseteq \mathbf{G}(\mathbf{A}_F^\infty)$ be a neat compact open, and let S be the set of rational primes q dividing the level K, i.e. these q for which $K \cap \mathbf{G}(\mathbf{Q}_q)$ is not hyperspecial. Let $\mathcal{W} = \mathcal{W}_\mu$ be an automorphic vector bundle. We focus particularly on μ corresponding to Π according to Thm 1.5.0.6. Consider the subset $G^S \subseteq \mathbf{G}(\mathbf{A}_F^\infty)$ of all elements g whose local components g_v are trivial for $v \in S$. The **unramified Hecke algebra** is

$$\mathbb{T}_K^j(\mathcal{W}) \subseteq \mathrm{End}(H^j(\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}), \mathcal{W}))$$

spanned by the operators $T_g := \pi_1^* \cdot [\cdot g] \circ \pi_{2,*}$ with $g \in G^S$. These are correspondences

$$\begin{array}{ccccc} & & \mathrm{Sh}_{K\cap gKg^{-1}}(\mathbf{G}, \mathbf{X}) & & \\ & \swarrow^{\pi_1} & & \searrow^{[\cdot g]\circ\pi_2} & \\ \mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) & & & & \mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \end{array} \tag{5.2}$$

where the maps π_1 and π_2 are the standard projections to $S_K(G, X)$ and $S_{gKg^{-1}}(G, X)$, respectively, and $[\cdot g] : S_{gKg^{-1}}(G, X) \xrightarrow{\sim} S_K(G, X)$ is Hecke operator given by multiplication by g.

It is sufficient to only consider one representative for each double coset $K_v \backslash \mathbf{G}(F_v)/K_v$, where $K_v \subset \mathbf{G}(F_v)$ is hyperspecial. These cosets are parameterized by the dominant chamber in the cocharacter lattice. As a $\mathbf{Z}$-module $\mathbb{T}^j_K(\mathcal{W})$ is thus generated by the cosets

$$\mathrm{GL}(n, (\mathcal{O}_F)_v)\, \mathrm{diag}(\varpi_v^{s_1}, \ldots, \varpi_v^{s_n})\mathrm{GL}(n, (\mathcal{O}_F)_v) \times (\mathcal{O}_F)_v^{\times},$$

with $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$ and $v \notin S$. As an algebra, it is generated by the operators $T_{i,v} \in \mathcal{O}[\mathbf{G}(F_v)//\mathbf{G}(\mathcal{O}_v)]$ given by the double cosets

$$g_{i,\mathfrak{q}} := \mathrm{GL}(n, \mathcal{O}_v)\, \mathrm{diag}(\varpi_v \cdot 1_i, 1_{n-i})\mathrm{GL}(n, \mathcal{O}_v) \times \mathcal{O}_v^{\times}. \tag{5.3}$$

For a double coset $[\gamma] \in K_v \backslash G(F_v)/K_v$ denote $K_v^{\gamma} := K_v \cap \gamma K_v \gamma^{-1}$. Lastly, let $K^{\gamma} := K^{(v)} \times K_v^{\gamma}$.

5.2 Shimura classes and cyclic covers

We now construct another class of operators which will commute with the classical Hecke operators. These have geometric origin and arise from étale covers of added level Q, where Q is a product of primes $\mathfrak{q}$, which arise in the Taylor-Wiles method. We reserve the gothic letter $\mathfrak{q}$ for primes in F above the rational prime q.

Definition 5.2.0.1. [1] *A set of primes $Q = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ is a **Taylor-Wiles datum of level** m relative to the pair $(\Pi, \mathfrak{p})$, denoted $Q \in \mathrm{TW}_m(\Pi, \mathfrak{p})$ or simply $Q \in \mathrm{TW}_m$ when the pair is clear, is a set of primes*

- *each $\mathfrak{q}_i \in Q$ does not divide lie above any primes in S;*

[1] In the classical Taylor-Wiles method setup (see [Har13] for axiomatic treatment), one usually works with primes in F_0 the totally real subfield of F (in this section $F_0 = \mathbf{Q}$) which furthermore split in F. Since we are interested in operators at primes in F, we implicitly work with a choice of a prime in F above these primes in F_0, and assume that no two primes in our datum lie above the same prime in F_0.

- *for each* $i = 1, \dots, s$*, the rational prime* $N_{F/\mathbf{Q}}(q_i) \equiv 1 \ (mod\ \mathfrak{p}^m)$*;*

- *for each* $i = 1, \dots, s$*,* $\bar{\rho}_\Pi(\mathrm{Frob}_\mathfrak{q})$ *is conjugate to a "strongly regular" element* $\mathrm{Frob}^T_{\mathfrak{q}_i} \in T^\vee(\mathbf{F}_\mathfrak{p})$*, i.e.*

$$\mathbf{Cent}_{G^\vee}(\mathrm{Frob}^T_{\mathfrak{q}_i}) = T^\vee.$$

  Explicitly, there exist $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{F}^n_\mathfrak{p}$ *of pairwise distinct* α_i*'s such that*

$$\bar{\rho}_\Pi(\mathrm{Frob}_\mathfrak{q}) \textit{ is conjugate to } \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n).$$

  Note there are $|W_\mathbf{G}|$ *choices of* $\mathrm{Frob}^T_{\mathfrak{q}_i}$*.*

- *we have the following cohomological condition. In the exact sequence*

$$H^1_f(\mathcal{O}_F[\frac{1}{S \sqcup Q_m}], \mathrm{Ad}\,\bar{\rho}_\Pi) \to H^1(\mathcal{O}_F[\frac{1}{S \sqcup Q_m}], \mathrm{Ad}\,\bar{\rho}_\Pi) \xrightarrow{A} \frac{H^1(\Gamma_{F_S}, (\mathrm{Ad}\,\bar{\rho}_\Pi)_\mathfrak{p})}{H^1_f(\Gamma_{F_S}, (\mathrm{Ad}\,\bar{\rho}_\Pi)_\mathfrak{p})} \to$$
$$H^2_f(\mathcal{O}_F[\frac{1}{S \sqcup Q_m}], \mathrm{Ad}\,\bar{\rho}_\Pi) \to H^2(\mathcal{O}_F[\frac{1}{S \sqcup Q_m}], \mathrm{Ad}\,\bar{\rho}_\Pi) \xrightarrow{B} \otimes_{\mathfrak{q} \in Q_m} H^2(\Gamma_{F_\mathfrak{q}}, (\mathrm{Ad}\,\bar{\rho}_\Pi)_\mathfrak{q}).$$

  This sequence is defined in Appendix B of [GV17]. The big image assumption assures that B *is an isomorphism.*

Remark 5.2.0.2. *These exist by the Chebotarev density theorem, and a big image assumption on* $\bar{\rho}_\Pi$*.*

We reserve the notation $Q_m \in \mathrm{TW}_m$ for collection $Q_m = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ of Taylor-Wiles primes of level m. Quite often, we only talk about the primes $\mathfrak{q}_i$ but we should remember that each one of them comes with a distinguished choice of a conjugacy class of the image of the $\mathrm{Frob}_\mathfrak{q}$ under the residual representation, and of course this depends on the automorphic representation Π.

Consider one such collection $Q = Q_m = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ of level m. Let $K_\emptyset \subseteq \mathbf{G}(\mathbf{A}^\infty_F)$ be a neat compact open subgroup which we assume to admit a factorization $K_\emptyset = K^Q \times K_Q$, where $K_Q = \prod_{\mathfrak{q}\in Q} K_\mathfrak{q}$ is a neat open compact subgroup such that $K \cap \mathbf{G}(F_\mathfrak{q})$ is hyperspecial. Let $\mathbf{U} \subseteq \mathbf{G}$ be the corresponding unitary group, and set $K^U = K \cap \mathbf{U}(\mathbf{A}_F^\infty)$ and similarly $K_\mathfrak{q}^U = K_\mathfrak{q} \cap \mathbf{U}(F_\mathfrak{q})$. Assume further that $K_\mathfrak{q}^U$ is hyperspecial in $\mathbf{U}(F_\mathfrak{q})$ for each $\mathfrak{q} \in Q$.

For each $q \in Q$, we may pick two subgroups $K_1(\mathfrak{q}) \subset K_0(\mathfrak{q}) \subset K_\mathfrak{q}^U$ with $K_1(\mathfrak{q})$ normal in $K_0(\mathfrak{q})$, and

$$K_0(\mathfrak{q})/K_1(\mathfrak{q}) \xrightarrow{\sim} \mathbf{T}(\mathbf{F}_\mathfrak{q}).$$

Indeed, we could pick

$$K_0(\mathfrak{q}) = \{m \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{q}) \mid m \equiv \begin{pmatrix} a(m) & b \\ 0 & d \end{pmatrix} \ (mod\, \mathfrak{q})\}$$

with $a(m) \in (\mathcal{O}_F/\mathfrak{q})^\times$, $b \in \mathrm{Mat}_{1\times(n-1)}(\mathcal{O}_F/\mathfrak{q})$ and $d \in \mathrm{GL}_{n-1}(\mathcal{O}_F/\mathfrak{q})$, and $K_1(\mathfrak{q})$ corresponding to the matrices with $a(m) = 1$.

Define $K_0(Q) = K^Q \times \prod_{q\in Q} K_0(\mathfrak{q})$ and similarly for $K_1(Q)$. Set $\Delta_\mathfrak{q}$ to be the product of the maximal p-power quotients of the coordinates of $\mathbf{T}(\mathbf{F}_\mathfrak{q})$, and let $\Delta_Q \coloneqq \prod_{q\in Q} \Delta_q$.

There is a map[2] $p_\Delta : K_0(Q) \to \Delta_Q$, and let $K_\Delta(Q) = \ker(p_\Delta)$. In this case $K_0(Q) \subset K_\Delta(Q)$, and let

$$S_0(Q) = \mathrm{Sh}_{K_0(Q)}(\mathbf{G}, \mathbf{X}), \quad S_\Delta(Q) = \mathrm{Sh}_{K_\Delta(Q)}(\mathbf{G}, \mathbf{X})$$

with corresponding smooth integral models $\mathbb{S}_0(Q)$ and $\mathbb{S}_\Delta(Q)$, respectively. Write $\mathbb{S}_\emptyset$ for the integral model at level $K_\emptyset$. The inclusion $K_0(Q) \subset K_\Delta(Q)$ yields an étale map $S_\Delta(Q) \to S_0(Q)$, which descends to an étale map $\oint_Q : \mathbb{S}_\Delta(Q) \to \mathbb{S}_0(Q)$ on the integral models. Although étale maps do not always remain étale on the toroidal compactification, our choices of compact open

[2]given by projection to the Levi (toral) factor and reducing modulo q!

subgroups allow us to extend it to an étale $\oint_Q : \mathbb{S}_{\Delta,\Sigma}(Q) \to \mathbb{S}_{0,\Sigma}(Q)$. These covers give raise to diamond operators, which are one of the ingredients necessary for [AH].

Proposition 5.2.0.3. *Let R be $\mathcal{O}_{\mathfrak{p}}$-algebra. In the setup above, we have*

i) For any normal subgroup U_1 of U_2, the map

$$\oint_{U_1,U_2} : (\mathbb{S}_{K^{(\mathfrak{q})}\times U_2,\Sigma})_R \to (\mathbb{S}_{K^{(\mathfrak{q})}\times U_1,\Sigma})_R$$

extending the canonical projection is finite, and the quotient of $(\mathbb{S}_{K^{(\mathfrak{q})}\times U_2,\Sigma})_R$ by U_2/U_1 can be identified with $(\mathbb{S}_{K^{(\mathfrak{q})}\times U_1,\Sigma})_R$.

ii) for any automorphic vector bundle $\mathcal{W}$, we have a natural isomorphism

$$\oint_{U_1,U_2}^{*} \mathcal{W}_{U_1}^{can} \simeq \mathcal{W}_{U_2}^{can}$$

over R.

iii) If, furthermore, U_1/U_2 is a p-group, then $\oint_{U_1,U_2}$ is étale.

Proof. Part i) follows from [Har89, Lem. 2.6]. For ii) see [Har85]. The proof of iii) is in Appendix A. □

5.3 Hecke algebra in the derived category

It is convenient to now cast the Hecke operators as endomorphisms in the derived category. Let T be the set of places v that satisfy at least one of the following:

i) $K_{\emptyset,v}$ is not hyperspecial,

ii) $\mathbf{G}$ is ramified at v,

iii) $v|\operatorname{disc}(F)$,

iv) Π_v is ramified at v,

Set

$$S = T \cup \{v : v|p\}.$$

For any $t \in \mathbf{G}(\mathbf{A}_F^{\infty,S})$, and (good) $K \subseteq K_\emptyset$ that differs only at K_v for $v \in S$, consider $T(t) \in \operatorname{End}(\mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_K))$ given by

$$\mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_K) \xrightarrow{\pi^*_{K^t,K}} \mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_{K^t}) \xrightarrow{\pi_{tkt^{-1},K(t),*}} \mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_{tkt^{-1}}) \xrightarrow{*t^{-1}} \mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_K). \tag{5.4}$$

Then $T_{i,v} = T(g_{i,v})$ recoves the classical Hecke operators.

Suppose $K' \subseteq K$ be neat open subgroups such that K/K' is a pro-p group Δ. By Prop 5.2.0.3iii) we have an étale morphism $f : X = \mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma \to \mathrm{Sh}_{K'}(\mathbf{G}, \mathbf{X})_\Sigma = Y$. Let $\mathcal{W}_Y^{\text{can}}$ be an automorphic vector bundle on Y. Recall that by construction $f^*\mathcal{W}_Y^{\text{can}} = \mathcal{W}_X^{\text{can}}$, so we may suppress the level from the notation. By [Nak84, Thm.1], we have a (finite) perfect complex $C^\bullet(K)$ consisting of free $F_v[\Delta]$-modules, such that

$$H^i(C^\bullet(K)) = \tilde{H}^i(\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma, \mathcal{W}^{\text{can}})$$

for all i. Thus we obtain an action $F_v[\Delta] \circlearrowleft \mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_K)$ (by Prop. 1.5.0.1 we may supress Σ_K). For an automorphic vector bundle $\mathcal{W}$, set

$$\mathbb{T}^S(\mathcal{W}, K) \subseteq \operatorname{End}_{D(\mathcal{O})}(\mathrm{R}\Gamma^{\text{can}}(\mathcal{W}_K)) \tag{5.5}$$

to be the algebra generated by $T_{i,v},\ v \notin S$. We descend to $\mathcal{O}$-coefficients in light of Prop 1.5.0.1 iv). It is a commutative algebra, because the Hecke operators at places q at which K_q is hyperspecial commute with one another. Note that there is a natural map

$$\mathbb{T}^S(\mathcal{W}, K) \to \big(\bigoplus H^i(\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})_\Sigma, \mathcal{W}^{\mathrm{can}})\big),$$

which is not necessarily isomorphism onto its image and thus $\mathbb{T}^S(\mathcal{W}, K)$ may be non-reduced. For example, by [KT14, Lemma 2.5], the map to the usual (homological) Hecke algebra has nilpotent kernel.

5.4 The Shimura classes $\mathfrak{S}$

For this and next section only, we set $R = \mathcal{O}/\mathfrak{p}^m$ so that we work with torsion-coefficients. We construct a class in $\mathfrak{S} \in H^1_{\mathrm{Zar}}(\mathbb{S}_{0,\Sigma}, \mathcal{O}_{\mathbb{S}_{0,\Sigma}})$ coming from the étale cover $\oint_{\mathfrak{q}}$ from the previous section. We call this a **Shimura class** in analogy to the Shimura coverings of [Maz77, §II].

Consider $\mathrm{Iw}(\mathfrak{q}) \subseteq \mathbf{G}(\mathbf{A}_F^\infty)$ obtained by adding Iwahori structure at q, i.e. changing the $K_{\mathfrak{q}}$ component of $K_\emptyset$ to the Borel $\mathbf{B}(\mathcal{O}_{\mathfrak{q}}) \subseteq \mathrm{GL}(n, \mathcal{O}_{\mathfrak{q}})$. Similarly, let $\mathrm{Iw}_1(\mathfrak{q})$ be the subgroup coming from the unipotent radical in the Borel. Thus,

$$\mathrm{Iw}(\mathfrak{q})/\mathrm{Iw}_1(\mathfrak{q}) \xrightarrow{\sim} \mathbf{T}(\mathbf{F}_{\mathfrak{q}}).$$

Letting $U_1 = \mathrm{Iw}_1(\mathfrak{q})$ and $U_0 = \mathrm{Iw}(\mathfrak{q})$ in Prop 5.2.0.3, we obtain an étale $\oint_{\mathfrak{q}} : \mathbb{S}_{\Delta,\Sigma}(\mathfrak{q}) \to \mathbb{S}_{0,\Sigma}(\mathfrak{q})$, where $\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q})$ and $\mathbb{S}_{0,\Sigma}(\mathfrak{q})$ are the smooth integral models coming from the choice of $\mathrm{Iw}_1(\mathfrak{q}) \subset \mathrm{Iw}(\mathfrak{q})$. After base-change to R, this map furnishes a Shimura class $\mathfrak{S}$ in $H^1_{\text{ét}}(\mathbb{S}_{0,\Sigma}(\mathfrak{q})_R, R\langle 1\rangle)$.

The natural map $R \to \mathbb{G}_a$ then yields a class in

$$H^1_{\text{ét}}(\mathbb{S}_{0,\Sigma}(\mathfrak{q})_R, \mathbb{G}_a\langle 1\rangle) \simeq H^1_{\text{Zar}}(\mathbb{S}_{0,\Sigma}(\mathfrak{q})_R, \mathcal{O})\langle 1\rangle,$$

which we still denote by $\mathfrak{S}$. Note that $\mathfrak{S}$ depends on the Taylor-Wiles prime; we suppress this dependency from the notation. Clearly, the same construction could be extended to a Taylor-Wiles datum Q.

5.5 Geometric version of "derived" Hecke operators

For an automorphic vector bundle $\mathcal{W}$ and a conjugacy class $\gamma \in K\backslash G/K$, and $z \in R\langle -1\rangle$, consider the **derived Hecke operator** $T_{\mathfrak{q},\gamma,z}$ given by

$$\begin{aligned} H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}^{\text{can}}) \xrightarrow{\pi_1^*} H^*((\mathbb{S}_{K^\gamma,\Sigma})_R, \mathcal{W}^{\text{can}}) &\xrightarrow{\cup z\mathfrak{S}_\gamma} H^*((\mathbb{S}_{K^\gamma,\Sigma})_R, \mathcal{W}^{\text{can}}) \\ &\xrightarrow{[\cdot\gamma]\circ\pi_{2,*}} H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}^{\text{can}}), \end{aligned} \tag{5.6}$$

where π_1, π_2 are as (5.2) and $\mathfrak{S}_\gamma$ is the pullback of $\mathfrak{S}$ to K^γ level. Without the inserted cup product with $z\mathfrak{S}_{g_{i,\mathfrak{q}}}$ this construction recovers the classical Hecke operator $T_{i,\mathfrak{q}}$. As we see shortly (see Corollary 7.3.0.3 below) the class $z\mathfrak{S}_\gamma$ is *Hecke trivial* in the sense of [Ven19, Def.2.9]; this is precisely the reason why the derived operators $T_{\mathfrak{q},\gamma,z}$ commute with the classical Hecke operators. This ad hoc definition may seem a bit unnatural and not derived in the sense of algebraic geometry but in §7, we see that, under local to global compatibility, it recovers the action of a more natural object, called **derived Hecke algebra**, at least at Taylor-Wiles primes.

Chapter 6: Cohomology classes in $H^1(\mathrm{Sh}(\mathbf{G}, \mathbf{X}), \mathcal{O})$

In this section, we study the classes appearing in the first cohomology of the Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$. In later chapters these will be compared with certain geometric classes, the so-called *Shimura classes*, arising from étale covers at suitably chosen levels. For convenience, we assume again that $F/\mathbf{Q}$ is an imaginary quadratic field, so that $d = 1$ and $\mathbf{G}$ has a unique archimedean component at which $\mathbf{G}$ has signature $(n-r, r)$. We study separately the case when $r = 1$ wherein these classes are constructed as theta lifts of characters, before proceeding to study the case of general r.

6.1 Ladder representations of $U(n-1,1)$ and the small automorphic representation

In this section we focus on the ladder representations of $U(n-1,1)$, which are the unique $(\mathfrak{g}, K_\infty)$-modules admitting nontrivial first cohomology. They are obtained as theta lifts of characters, so we first briefly review the classical theory of theta lifts. In this chapter and in this chapter alone V is not the vector space defined in §1.

6.1.1 Theta correspondence

Let L/K be a quadratic extension, possibly split, of local fields of characteristic zero. Let V and V' be vector spaces over L of dimensions n and m, respectively, equipped with a skew-Hermitian form $\langle\ ,\ \rangle : V \times V \to L$ and a Hermitian $(\ ,\ ) : V' \times V' \to L$ form. Then the isometry groups $U(V)$ and $U(V')$ form a dual reductive pair in $\mathrm{Sp}(\mathbb{W})$, where the space $\mathbb{W} :=$

$\mathrm{Res}_{L/K} V \otimes_L V'$ is equipped with the symplectic form

$$\underline{\langle} v_1 \otimes v_1', v_2 \otimes v_2' \underline{\rangle} = \frac{1}{2}\mathrm{Tr}_{L/K}\big((v_1, v_1')\langle v_2, v_2'\rangle^{\iota}\big),$$

with ι denoting the nontrivial automorphism in $\mathrm{Gal}(L/K)$.

Fix a nontrivial additive character ψ of E. The metaplectic cover $\mathrm{Mp}(\mathbb{W})$ is a central extension by $\mathbf{C}$ of $\mathrm{Sp}(\mathbb{W})$, i.e.

$$1 \to \mathbf{C} \to \mathrm{Mp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W}) \to 1.$$

Let $U'(V)$ and $U'(V')$ be the preimages of $U(V)$ and $U(V')$ in $\mathrm{Mp}(\mathbb{W})$. They still commute with each other, so one may restrict the Weil representation ω_ψ of $\mathrm{Mp}(\mathbb{W})$ to $U'(V) \times U'(V')$ on the Schwartz space $\mathcal{S}(\mathbb{X})$, where $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ is a complete polarization. Then, for any irreducible smooth representation π of $U'(V)$, we denote by $\Theta(\pi)$ the maximal quotient of $\pi \otimes \mathcal{S}$, where $U'(V)$ acts trivially. We similarly define $\Theta(\pi')$ for any irreducible admissible representation π' of $U'(V')$.

Denote by $\varepsilon_{L/K}$ the quadratic character associated to L/K by class field theory. For any pair of characters (χ, χ') of $L^\times$ satisfying

$$\chi\big|_{K^\times} = \varepsilon_{L/K}^{\dim V'} \quad \text{and} \quad \chi'\big|_{K^\times} = \varepsilon_{L/K}^{\dim V} \tag{6.1}$$

there exist a lift $\tilde{i}_{\chi,\chi'} : U(V) \times U(V') \to \mathrm{Mp}(\mathbb{W})$ such that

$$\begin{array}{ccc} & & \mathrm{Mp}(\mathbb{W}) \\ & \overset{\tilde{i}_{\chi,\chi'}}{\nearrow} & \downarrow \\ U(V) \times U(V') & \xrightarrow{i_{\chi,\chi'}} & \mathrm{Sp}(\mathbb{W}) \end{array}$$

The explicit construction is described in [HKS96, §1]. Hence, pulling back the smooth Weil representation ω_ψ of $\mathrm{Mp}(\mathbb{W})$ via $\tilde{i}_{\chi,\chi'}$, we obtain a representation $\omega_{\chi,\chi'}$ of $U(V) \times U(V')$ on the Schwartz space $\mathcal{S}(\mathbb{X})$. Analogously define $\Theta_{\chi,\chi'}(\pi)$ (resp. $\Theta_{\chi,\chi'}(\pi')$) to be the corresponding $U(V')$-(resp. $U(V)$-) representation obtained from a smooth representation π on $U(V)$ (resp. π' on $U(V')$) by pulling back along $\tilde{i}_{\chi,\chi'}$. In later section, we may fix $\mu : F^\times \to \mathbf{C}^\times$ such that $\mu\big|_{\mathbf{Q}^\times} = \eta_{F/\mathbf{Q}}$, and take $\chi = \mu^{\dim V'}, \chi' = \mu^{\dim V}$.

6.1.2 Ladder representations

We follow the conventions of [Har99]. Assume π is a $(\mathfrak{g}, K_\infty)$-module and V_μ is a finite-dimensional representation of G such that $H^1(\mathfrak{g}, K_\infty; \pi \otimes V_\mu) \neq 0$. Then μ is one of the following:

$$\mu(\alpha,\beta) \coloneqq (\alpha+\beta, \beta, \dots, \beta; \beta),$$
$$\mu(-\alpha,\beta) \coloneqq (\beta, \beta, \dots, \beta; \beta-\alpha).$$

for integers $\alpha \geq 0$ and β. Denote by $W^+(-\alpha,\beta)$ and $W^-(\alpha,\beta)$ the K_∞-modules corresponding to $\mu^+(-\alpha,\beta)$ and $\mu^-(\alpha,\beta)$, respectively. There are situations:

i) If $\alpha > 0$ and $\mu = \mu(\alpha,\beta)$ or $\mu = \mu(-\alpha,\beta)$, there is a unique $(\mathfrak{g}, K_\infty)$-module π such that $H^1(\mathfrak{g}, K_\infty; \hat{\pi} \otimes V_\mu) \neq 0$.

ii) When $\alpha = 0$ there are two such π and we denote them by $\mathbb{L}^-(\alpha,\beta)$ and $\mathbb{L}^+(-\alpha,\beta)$, respectively. They are distinguished by their $\bar{\partial}$-cohomology by

$$\dim H^0_{\bar{\partial}}\big(\widehat{\mathbb{L}}^+(-\alpha,\beta) \otimes W^+(-\alpha,\beta)\big) = \dim H^1_{\bar{\partial}}\big(\widehat{\mathbb{L}}^-(\alpha,\beta) \otimes W^-(\alpha,\beta)\big) = 1, \tag{6.2}$$

where $W^-(\alpha,\beta)$ and $W^+(-\alpha,\beta)$ are the K_∞-modules corresponding to $\mu^-(\alpha,\beta) = (\alpha + \beta, \beta, \dots, \beta; \beta)$ and $\mu^+(-\alpha,\beta) = (\beta, \dots, \beta, \beta-\alpha-1; \beta+1)$, respectively.

The Blattner types of $\mathbb{L}^-(\alpha, \beta)$ and $\mathbb{L}^+(-\alpha, \beta)$, identified with their highest weights, are given by

$$\tau^-(\alpha, \beta) = (\alpha + \beta + 1, \beta, \dots, \beta, \beta; \beta - 1),$$
$$\tau^+(-\alpha, \beta) = \Lambda^+(-\alpha, \beta) = (\beta, \dots, \beta, \beta - \alpha - 1; \beta + 1).$$

We also distinguish a certain irreducible holomorphic highest weight module $\widehat{\mathbb{M}}^+(a, \beta)$, which exists for a positive integer a, and has the property that

$$\dim H^0_{\bar\partial}(\widehat{\mathbb{M}}^+(a, \beta) \otimes T^+(a, \beta)) \neq 0,$$

where $T^+(a, \beta)$ is the 1-dimensional K_∞-type $(\beta, \dots, \beta; a + \beta)$.

6.1.3 Small automorphic representations

Suppose that $\mathbf{G}$ has signature $(n-1, 1)$ at the unique archimedean place. In this subsection we record facts about the automorphic representations π of $\mathbf{G}$ that contribute to the first cohomology of the associated Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$. Their archimedean components π_∞ are the ladder representations of §6.1.2.

Definition 6.1.3.1. *A* ***small automorphic representation*** *of* $\mathbf{G}$ *is any representation of the form*

$$\Theta_{\psi,\chi,\chi'}(\eta) := \Theta_{\psi,\chi,\chi'}(GU(W) \to GU(V), \eta),$$

where W is a one-dimensional hermitian space over F, and $\eta : GU(\mathbf{Q})\backslash GU(\mathbf{A}) \to \mathbf{C}^\times$ is a character.

Under appropriate conditions on η_∞ (e.g, see (6.3) below), the local component $\Theta_{\psi,\chi,\chi'}(\eta)_\infty$ is cohomological. Let $j : \mathrm{Sh}(\mathbf{G}, \mathbf{X}) \to \mathrm{Sh}(\mathbf{G}, \mathbf{X})^*$ be the standard (Bailey-Borel) embedding. Then, for any local system $\mathcal{F}$ on $\mathrm{Sh}(\mathbf{G},\mathbf{X})$, we have a natural isomorphism

$$H^1(\mathrm{Sh}(\mathbf{G},\mathbf{X})^*, j_{*!}\mathcal{F}) \to H^1(\mathrm{Sh}(\mathbf{G},\mathbf{X}),\mathcal{F}),$$

where $j_{*!}\mathcal{F}$ is the middle perverse extension of $\mathcal{F}$. By Zucker's conjecture, the left hand side may be identified with the L^2-cohomology of $\mathrm{Sh}(\mathbf{G},\mathbf{X})$. As a consequence, we have

Proposition 6.1.3.2. *Suppose the local component $\Theta_{\psi,\chi,\chi'}(\eta)_\infty$ is of cohomological type. Then the automorphic representation $\Theta_{\psi,\chi,\chi'}(\eta)$ is contained un $L^2(\mathbf{G}(\mathbf{Q})\backslash\mathbf{G}(\mathbf{A}))$.*

The component $\Theta_{\psi,\chi,\chi'}(\eta)_\infty$ is cohomological, for example, when $k \in \mathbf{N}$ and

$$\alpha(\chi') = 1,$$
$$\eta^{-1} \text{ is of type } \begin{cases} \pm(k+(n-2)/2) & n-\text{even}, \\ k+(n-1)/2 \text{ or } -k-(n-3)/2, & n-\text{odd}. \end{cases} \tag{6.3}$$

Despite Prop. 6.1.3.2, the small automorphic representations are the most degenerate among all infinitely-dimensional representations. For $n \geq 2$, they are nontempered at each finite place. The case $n = 3$ has been studied by Rogawski (see [Rog90, Rog92]) – these representations are then either cuspidal or residual, while for $n \geq 4$ they are generally residual, rather than cuspidal.

6.2 Lifts of $\mathfrak{S}$ to characteristic zero in signature $(n-1,1)$

Often, the first cohomology $H^1(\mathbb{S}_K, \mathcal{O}_{\mathbb{S}_K})$ is empty (see §6.3 below). Notable exceptions are the cases of signature $(n-1,1)$ and $(n,0)$. For $n = 2$, i.e. the case of $U(1,1)$, in [HV19] the Eisenstein class E_2 is shown to be a lift of the Shimura class. For $n \geq 3$ and signature $(n-1,1)$ all classes in $H^1(\mathrm{Sh}_K(\mathbf{G},\mathbf{X}), \mathcal{O}_{\mathrm{Sh}_K(\mathbf{G},\mathbf{X})})$ are the ladder representations of §6.1 (see [Har99]; see also [GeRo]). By a result of Li (for a discussion, see [Har07, E1]), it follows that in fact π with π_∞ being a ladder representation is globally a theta lift of character, and in particular π_v is local theta lift for each v. These are potential candidates for characteristic zero lifts of $\mathfrak{S}$. Nevertheless, we show that, assuming Adams conjectures [Ad89], this is never the case for any $n \geq 3$. Therefore, the case of weight one forms studied in [HV19] and [DHRV] is exceptional.

6.2.1 Small automorphic representations under Hecke

We now specify the Hecke action on $\Theta_{\chi,\chi',\psi}(\eta)$ at primes ℓ where χ, χ', ψ, and η are unramified. The formulas differ based on the parity of n. They are most conveniently expressed in terms of the standard L-function of the base-change $\mathrm{BC}(\pi)$ of π to $\mathrm{GL}(n, F_\ell)$, whose Euler factor at ℓ is $L_\ell(s,\pi,\mathrm{St}) := L_\ell(s,\mathrm{BC}(\pi))$, which is interpreted as a product of two factors when ℓ splits in F. Recall that if $\pi_q = \pi(\eta_1,\ldots,\eta_n)$ for unramified character η_i of $\mathbf{Q}_q$, then

$$\mathrm{BC}(\pi)_v = \pi(\eta_1 \circ \|\cdot\|, \ldots, \eta_n \circ \|\cdot\|), \tag{6.4}$$

where $\|\cdot\| := N_{F_v/\mathbf{Q}_q} : F_v^\times \to \mathbf{Q}_q^\times$ is the norm map and v is a prime above q. Set $\tilde{\eta} := \eta \circ \|\cdot\|$.

Proposition 6.2.1.1. *[Har99, IV, Prop.1.3] Let p be a prime at which χ, ω and ψ are unramified.*

(i) For n even and $\chi' \equiv 1$, we have

$$L_p(s,\Theta_{\chi,\chi',\psi}(\omega^{-1}),St) = L_{F,p}(s,\mathrm{BC}(\omega)\chi)L_{F,p}(s,\chi)\prod_{j=1}^{\frac{n}{2}-1}\left[L_{F,p}(s,\chi\|\cdot\|^j)L_{F,p}(s,\chi\|\cdot\|^{-j})\right]$$

(ii) For n odd and $\chi' = \chi^{\pm 1}$, we have

$$L_p(s,\Theta_{\chi,\chi',\psi}(\omega^{-1}),St) = L_{F,p}(s,\mathrm{BC}(\omega)\cdot\chi/\chi')\prod_{j=1}^{\frac{n-1}{2}}\left[L_{F,p}(s,\chi\|\cdot\|^j)L_{F,p}(s,\chi\|\cdot\|^{-j})\right]$$

Remark 6.2.1.2. *For a more general formula see §I,(3.15.2), ibid. As remarked after it, the formulas remain true regardless if ℓ is split or inert, even though they are written in the split notation. For a conjectured behavior at ramified places, see Adams Conjecture [HKS96, Conj.7.2]. Note the different normalization of HKS and H99!*

Recall that for a character ω of $\mathbf{G}(\mathbf{Q}_\ell)$ we have

$$\mathrm{BC}(\omega)(x) = \omega\Big(\frac{x}{x^\iota}\Big),$$

where x^ι is the Galois conjugate of x in the quadratic extension over $\mathbf{Q}_\ell$.

6.2.2 Shimura cover under Hecke

We record here a relevant computation due to Tamagawa.

Lemma 6.2.2.1. *[Tam63, Lem.12] We have*

$$\deg(T_{i,q}) = q^{-(i/2)(i+1)} \sum_{1 \le r_1 < \cdots < r_i \le n} q^{r_1 + \cdots + r_i}.$$

Given an eigenform $f \in \Theta_{\chi,\chi',\psi}(\omega^{-1})$, we have at unramified places that

$$T_{i,q} \cdot f = \big(q^{\frac{(n-i)i}{2}} \sigma_i[\xi_1(q), \ldots, \xi_n(q)]\big) \cdot f, \tag{6.5}$$

where $\sigma_i(X_1, \ldots, X_N)$ is the i-th symmetric polynomial on the variables $X_1, \ldots, X_n$, and $\{\xi_i(q)\}_i$ are the Satake parameters at q. We will contract the i-th symmetric polynomial of the Satake parameters to just σ_i. Combining Tamagawa's computation and Cor. 7.3.0.3, this implies

$$\sigma_i = q^{\frac{-(n+1)i}{2}} \sum_{1 \le r_1 < \cdots < r_i \le n} q^{r_1 + \cdots + r_i}.$$

Applying this formula for $\pi_{z\mathfrak{S}}$ then yields that the Satake parameters at q are given by

$$\{q^{1-\frac{n+1}{2}}, q^{2-\frac{n+1}{2}}, \cdots, q^{n-\frac{n+1}{2}}\},$$

i.e.

$$\{\xi_1, \ldots, \xi_n\} = \{\|\cdot\|^{\frac{1-n}{2}}, \|\cdot\|^{\frac{3-n}{2}}, \ldots, \|\cdot\|^{\frac{n-1}{2}}\}.$$

Alternatively, this could be seen by noting that the Langlands normalization is so that the trivial character corresponds to ρ. Recall that we have

$$L(s, \mathrm{BC}_{F/\mathbf{Q}}(\pi)) = L(s, \pi)L(s, \pi \cdot \varepsilon_{F/\mathbf{Q}})$$

and recall that the Frobenius on F_v restricts to the square of the Frobenius of $\mathbf{Q}_v$.

We are now finally in position to prove the assertion made in the beginning of the section.

Proposition 6.2.2.2. *Let $n \geq 3$ and suppose $4 \nmid n$. Suppose G has signature $(n-1, 1)$ at one archimedean place. Then $\mathfrak{S}$ does not have a char zero lift in $H^1(\mathbb{S}_K, \mathcal{O}_{\mathbb{S}_K})$.*

Proof. By the general discussion around § 6.2, we may assume that G has unique archimedean place of signature $(n-1, 1)$. Suppose $n \geq 3$. Assume the Shimura class $z\mathfrak{S}$ admits a characteristic zero lift. It is of the form $\Theta_{\chi_0, \chi_0^n, \psi}(\eta)\|\cdot\|^a$ for some unitary character η. Since $z\mathfrak{S}$ has trivial central character, then η, by virtue of being $z\mathfrak{S}$ central character, is trivial. The crux of the argument is then the following observation: Adams conjectures assert that theta lifts are endoscopic lifts from triv_{n-1} and $\mathrm{BC}(\eta) = \mathrm{triv}$ (with suitably chosen splitting characters used in the construction of the map of L-groups), while the potential characteristic zero lift of the Shimura class should be triv_n at unramified places. The Satake parameters of the Shimura class (see Prop. 6.2.1.1 and 7.3.0.2) are either integral or half-integral based on the parity of n. Since triv_n and triv_{n-1} are always of different parity, we are always off by $|\cdot|^{1/2}$, which cannot be compensated by $\|\cdot\|^a$. We explicate the above observation in two cases based on the parity of n. To avoid repetition, all primes below are picked so that all relevant data is unramified.

i) Let $n \geq 3$ be odd. Since the Shimura class is Hecke-trivial, we have

$$L_\ell(s, z\mathfrak{S}, St) \equiv \zeta_K(s) \prod_{j=1}^{\frac{n-1}{2}} \zeta_K(s-j)\zeta_K(s+j) \ (mod\, p)$$

while by Prop 6.2.1.1, we have

$$\begin{aligned} L_\ell(s, z\mathfrak{S}, \mathrm{St}) &= L_\ell(s, \Theta_{\chi,\chi',\psi}(\eta), St) \\ &= L_{K,\ell}(s, \mathrm{BC}(\eta)\cdot\chi/\chi') \prod_{j=1}^{\frac{n-1}{2}} \left[L_{K,\ell}(s, \chi\|\cdot\|^{j/2}) L_{K,\ell}(s, \chi\|\cdot\|^{-j/2})\right] \ (mod\, p). \end{aligned}$$

All of these equalities are at places $\ell \neq p$, where all data is unramified. Suppose further that $\varepsilon_{K/Q}(\ell) = 1$, so that the characters χ and χ' by (6.1) give 1. Since η is trivial character, then we have equality of Euler factor of standard L-functions (which in this case is product of (translations of) Dedekind L-functions):

$$\begin{aligned} &\zeta_{K,\ell}(s) \prod_{j=1}^{\frac{n-1}{2}} \zeta_{K,\ell}(s-j)\zeta_{K,\ell}(s+j) \\ &\qquad \equiv \zeta_{K,\ell}(s-a) \prod_{j=1}^{\frac{n-1}{2}-1} \zeta_{K,\ell}(s-a+\frac{2j+1}{2})\zeta_{K,\ell}(s-a-\frac{2j+1}{2}) \ (mod\, p). \end{aligned}$$

Since $n \geq 2$, at least one of the two products on the right hand side contains half-integral powers of ℓ. If a is integral, then it's the $\zeta_{K,\ell}(s-a)$ - factor, while if a is half-integral, it is the remaining product. Thus, if we picked ℓ which is quadratic non-residue modulo p, then the above congruence is absurd.

ii) Let $n \geq 3$ be even. In this case it is more convenient to argue via Hecke eigenvalues. Pick a normalized eigenform $f \in \Theta_{\chi_0,\chi_0^n,\psi}(\text{triv})\|\cdot\|^{a+\frac{1}{2}}$ for integer a.[1] Then, by Satake isomorphism, we have

$$T_{1,\ell}f = \ell^{\frac{n-1}{2}}\sigma_1(\text{Satake parameters of } \Theta_{\chi_0,\chi_0^n,\psi}(\text{triv})\|\cdot\|^{a+\frac{1}{2}})f,$$

where $T_{1,\ell}$ is the Hecke operator coming from the double coset represented by $\text{diag}(\ell, 1, \ldots, 1)$, and σ_1 is the first elementary symmetric polynomial. Using Prop. 6.2.1.1, for ℓ such that $\varepsilon_{K/Q}(\ell) = 1$, we have

$$T_{1,\ell}f = \ell^a(\ell^{n-1} + \ell^{n-2} + \cdots + \ell^{n/2+1} + 2\ell^{n/2} + \ell^{n/2-1} + \ldots + \ell)f$$

By Tamagawa's computation we also have

$$T_{1,\ell}f = (\ell^{n-1} + \ldots + \ell + 1)f,$$

so that

$$\ell^a(\ell^{n-1} + \ell^{n-2} + \cdots + \ell^{n/2+1} + 2\ell^{n/2} + \ell^{n/2-1} + \ldots + \ell) \equiv \ell^{n-1} + \ldots + \ell + 1 \ (mod\ p) \quad (6.6)$$

Suppose $n = 4k + 2$ and pick $\ell \equiv -1(mod\ p)$. Multiplying both sides by $\ell - 1 \neq 0(mod\ p)$, may simplify, after dividing by $\ell^{n/2} - 1 \neq 0\ (mod\ p)$, to

$$\ell^{n/2} + 1 \equiv (\ell^{n/2} + \ell)\ell^a\ (mod\ p),$$

[1] we need to twist by $\|\cdot\|^{\frac{1}{2}}$ to ensure the class is motivic.

which in turn is equivalent to

$$-1+1 \equiv (-1)^a(-1-1)\ (mod\, p),$$

which is absurd since p is odd.

$\square$

Remark 6.2.2.3. *The case $n = 4k$ appears is trickier. Simulations with all primes p up to $25,000$ and $n = \{8, 12, 16\}$ always produce many residue classes modulo p violating (6.6), regardless of the value of a.*

Remark 6.2.2.4. *It is believe that all classes in $H^1(\mathbb{S}_K, \mathcal{O})$ when $G = GU(n, 0)$ are also theta lifts but we were unable to find a source justifying it.*

6.3 Lifts for general signature $(n-r, r)$

We now proceed to show that there are no characteristic zero lifts of the Shimura class $\mathfrak{S}$ in all remaining non-compact signatures. The argument in these cases is more direct – we simply show that $H^1(\mathbb{S}_K, \mathcal{O}_{\mathbb{S}_K})$ is empty. For convenience, assume that G has unique archimedean place of signature $(n-r, r)$ with $2 \leq r \leq n/2$. Using our choices of root systems in §2.2, by Thm 3.1 of [GH67], we have that all classes in $H^1(\mathrm{Sh}_K, \mathcal{O})$ are square-integrable (see also the discussion towards the end of p. 303 and Thm. 3 on next page in [R68]). Since all classes in $H^1(\mathrm{Sh}_K, \mathcal{O})$ are square-integrable, they are in particular unitary. By the vanishing result of Kumaresan (see Thm. 8.1 and especially Table 8.2. in [VZ84]), we then see that

$$H^1(\mathrm{Sh}_K, \mathcal{O}) = 0.$$

The discussion so far concerns the first cohomology group H^1 of the open Shimura variety, not that of its toroidal compactification. Note that if $2 \leq r \leq n-r$, we have for every $1 \leq i < r$

$$r(n-r) - (r-i)(n-r-i) > 2,$$

so by Higher Koecher Principle (see Thm 4.7 of [Lan16b, Thm.4.7]) for $U = \overline{\mathrm{Sh}}$, the full Bailey-Borel compactification, we have that

$$H^1(\mathrm{Sh}_{K,\Sigma}, \mathcal{W}^{\mathrm{can}}) \xrightarrow{\sim} H^1(\mathrm{Sh}_K, \mathcal{W})$$

for any automorphic vector bundle. In particular, setting $\mathcal{W} = \mathcal{O}$, we obtain that

$$H^1(\mathrm{Sh}_{K,\Sigma}, \mathcal{O}^{\mathrm{can}}) = 0.$$

Chapter 7: Derived Hecke algebra

7.1 Derived Hecke algebra

Fix a prime p, where $\mathbf{G}(\mathbf{Q}_p)$ is hyperspecial, and $p > n$ so that $p \nmid |W_{\mathbf{G}}| = n!$ does not divide the order of the Weyl group. Let $\mathfrak{p}$ be a rational prime in $E(\Pi)$ above p. Let R be a p-adic ring. Fix q a prime at which G is unramified. In §5.5, we gave an ad hoc definition of a "derived" Hecke algebra. In this section, we outline Venkatesh's proposed derived Hecke algebra, which will be a graded algebra whose first degree recovers the constructed action via cup product with Shimura classes. Let v be a prime in F where $\mathbf{G}$ is not ramified and set $G_v := \mathbf{G}(F_v)$. Let $U_v \subseteq G_v$ be a compact open subgroup. To motivate the definition, we recast the classical Hecke algebra as an endomorphism algebra of a certain group algebra. The classical Hecke algebra $H_R(G_v, U_v)$ for the pair (G_v, U_v) with coefficients in R, given by double-cosets, is isomorphic to the following endomorphism algebra:

$$H_R(G_v, U_v) := R[G_v//U_v] = R[G_v]^{U_v \times U_v} \simeq \mathrm{Hom}_{R[G_v]}(R[G_v/U_v], R[G_v/U_v])$$

$$\varphi([eU_v]) \leftarrow\!\shortmid \varphi$$

Hereafter, the non-curly version H_R will be reserved for classical Hecke algebras, whereas the curly $\mathcal{H}_R$ version will denote the derived Hecke algebra, defined below. As it will become apparent from the definition

$$H_R = \mathcal{H}_R^{(0)},$$

i.e. the classical Hecke algebra is the zeroth degree component of the graded derived Hecke algebra.

Definition 7.1.0.1. *For a compact open $U_v \subseteq G_v$, the* ***derived Hecke algebra for*** (G_v, U_v) ***over a finite ring*** R *is defined to be*

$$\mathcal{H}_R(G_v, U_v) \coloneqq \mathrm{Ext}^*_{R[G_v]}(R[G_v/U_v], R[G_v/U_v]). \tag{7.1}$$

It is convenient to have an explicit model. If $\mathbf{P}^\bullet \to R[G_v/U_v]$ is a projective resolution, obtained as an induction of a projective resolution $\mathbf{Q}^\bullet \to R$ of R as a trivial G_v-module, then

$$\mathcal{H}_R(G_v, U_v) = H^*\big(\underline{\mathrm{Hom}}_{R[G_v]}(\mathbf{P}^\bullet, \mathbf{P}^\bullet)\big),$$

where $\underline{\mathrm{Hom}}$ is computed in the derived category. For instance, one may pick $\mathbf{Q}^\bullet$ to be the standard "bar" resolution of R given by free $R[G_v/U_v]$-modules, viewed as a complex of smooth G_v-representations.

The classical Hecke algebra acts on cohomology as outlined in §5.1 and §5.3. The derived Hecke algebra serves a similar role but we need to be more careful with introducing the correct objects.

Definition 7.1.0.2. *Let $\boldsymbol{M}^\bullet$ be a complex of G_v-modules. The* ***derived*** U_v***-invariants of*** $\boldsymbol{M}^\bullet$ *is the* $\underline{\mathrm{Hom}}_{G_v}(\boldsymbol{P}^\bullet, \boldsymbol{M}^\bullet)$.

Clearly, we have an action $\underline{\mathrm{End}}_{R[G_v]}(\mathbf{P}^\bullet) \circlearrowleft \underline{\mathrm{Hom}}_{R[G_v]}(\mathbf{P}^\bullet, \mathbf{M}^\bullet)$ by precomposition and, passing to cohomology, an action

$$\mathcal{H}_R(G_v, U_v) \circlearrowleft H^*(\text{derived } U_v\text{-invariants of } \mathbf{M}^\bullet) = \mathbb{H}^*(U_v, \mathbf{M}^\bullet),$$

where $\mathbb{H}^*$ stands for the hypercohomology.

7.2 Double coset description of $\mathcal{H}_R$

Recall that the classical Hecke algebra H_R has a nice description via double-coset classes. Following [Ven19, Appendix A] we now show that so does $\mathcal{H}_R$ although the multiplication rules are rather opaque. In particular, $\mathcal{H}_R$ is seldom commutative.

Let L be a smooth representation of U_v. Then

$$\mathrm{res}^{G_v}_{U_v}\left(\mathrm{ind}^{G_v}_{U_v}\right) \simeq \bigoplus_{[\alpha]\in G_v/U_v} R[U_v\alpha U_v]\otimes_{R[U_v]} L \simeq R[U_v]\otimes_{R[U_q^\alpha]} L_\alpha \simeq \bigoplus_{[\alpha]} \mathrm{ind}^{U_v}_{U_q^\alpha}, \tag{7.2}$$

where $U_v^\alpha = U_v \cap \alpha U_v \alpha^{-1}$, which is the stabilizer of α in U_v, and L_α is the U_v^α-module with underlying space L but with U_v^α-action given by

$$u \star l = \left(\alpha^{-1}u\alpha\right) l.$$

The second isomorphism in (7.2) is given by

$$u_1\alpha u_2 \otimes l = u_1\alpha \otimes u_2 l \mapsto u_1 \otimes u_2 l$$

with an inverse

$$u \otimes l \mapsto u\alpha \otimes l$$

For a set of representatives $[U_v\backslash G_v/U_v] \subseteq G_v/U_v$ of the left U_v-orbits on G_v/U_v, we have the following isomorphism

$$\bigoplus_{[\gamma]\in[U_v\backslash G_v/U_v]} H^*(U^\gamma, R) \xrightarrow{\sim} \mathcal{H}_{q,R}. \tag{7.3}$$

We use the map (7.2) to explicate this isomorphism. Let $\mathbf{P}^\bullet$ and $\mathbf{Q}^\bullet$ be as in the previous section. We have

$$\underline{\mathrm{Hom}}_{R[G_v]}\left(\mathbf{P}^\bullet, \mathbf{P}^\bullet\right) = \underline{\mathrm{Hom}}_{R[U_v]}\left(\mathbf{Q}^\bullet, \mathrm{ind}_{U_v}^{G_v}\mathbf{Q}^\bullet\right) \xleftarrow{(7.2)} \bigoplus_{[\alpha]\in U_v\backslash G_v/U_v} \underline{\mathrm{Hom}}_{U_v}\left(\mathbf{Q}^\bullet, \mathrm{ind}_{U_v^\alpha}^{U_v}\mathbf{Q}_\alpha^\bullet\right) \quad (7.4)$$

It suffices to show that, upon passing to cohomology, the above map is an isomorphism. Note that $\mathbf{Q}_\alpha^\bullet$ is cohomologically concentrated in degree zero and therefore so is $\oplus_\alpha \mathrm{ind}_{U_v^\alpha}^{U_v}\mathbf{Q}_\alpha^\bullet$. Since the map $\oplus_i \underline{\mathrm{Hom}}_{U_v}\left(\mathbf{Q}^\bullet, L_i\right) \to \underline{\mathrm{Hom}}_{U_v}\left(\mathbf{Q}^\bullet, \oplus_i L_i\right)$ is a quasi-isomorphism, we see that $H^*(\underline{\mathrm{Hom}}_{U_v}\left(\mathbf{Q}^\bullet, -\right))$ commutes with the infinite direct sum $\oplus_\alpha \mathrm{ind}_{U_v^\alpha}^{U_v}\mathbf{Q}_\alpha^\bullet$. Observe further that $\mathbf{Q}^\bullet$ and $\mathbf{Q}_\alpha^\bullet$ are both resolutions of R and $\mathbf{Q}_\alpha^\bullet$ is a U_v^α-module. Therefore,

$$H^*\Big(\bigoplus_{[\alpha]\in U_v\backslash G_v/U_v} \underline{\mathrm{Hom}}_{U_v}\left(\mathbf{Q}^\bullet, \mathrm{ind}_{U_v^\alpha}^{U_v}\mathbf{Q}_\alpha^\bullet\right)\Big) \simeq \otimes_\alpha H^*\left(U_v^\alpha, R\right)$$

and thus (7.4) shows that the map (7.2) explicates the isomorphism (7.3).

Working with this description is rather difficult so we briefly mention another description. Justifying the equivalence of these two definition is the content of [Ven19, Appendix A]. The derived Hecke algebra $\mathcal{H}_R(G_v, U_v)$ may also be viewed as "G_v-equivariant cohomology classes in $G_v/U_v \times G_v/U_v$ with finite support modulo G_v." In this model $h \in \mathcal{H}_R(G_v, U_v)$ assigns to $(x, y) \in G_v/U_v \times G_v/U_v$ an element $h(x, y) \in H^*(G_v^{xy}, R)$, where G_v^{xy} denotes the pointwise stabilizer of (x, y). The assignment h is assumed to be G_v-invariant under pullback by $\mathrm{Ad}(g)$ and of finite support modulo G, i.e. h is supported only on a finite subset of $G_v/U_v \times G_v/U_v$. This rather roundabout model is convenient for explicating the multiplicative structure within $\mathcal{H}_R$, which is given by the following rule

$$h_1 \star h_2(x,z) = \sum_{[y] \in G_v/U_v} h_1(x,y) \cup h_2(y,z),$$

where the cup product is computed with both elements restricted to $H^*(G_v^{xyz}, R)$. If $G_v/U_v = \sqcup O_i$ as a disjoint union of orbits under the G_v^{xz}, then for each i, we interpret

$$\sum_{[y] \in O_i} h_1(x,y) \cup h_2(y,z) \coloneqq \mathrm{Cores}_{G_v^{xz}}^{G_v^{xyz}}(h_1(x,y) \cup h_2(y,z)).$$

7.3 Derived Hecke on coherent cohomology

Let $\mathcal{W}$ be an automorphic vector bundle. We now use the above abstract nonsense construction to produce an action of the derived Hecke algebra $\mathcal{H}_{q,R} \coloneqq \mathcal{H}_R(G_v, K_v)$ on the coherent cohomology $H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}^{\mathrm{can}})$ of the (canonical extension of an) automorphic vector bundle on the toroidal compactification. We succeed in doing so explicitly in the cases of primes $\mathfrak{q}$ arising from the Taylor-Wiles process following [HV19, §7], at least for the degree one component.

Let $N_{F/\mathbf{Q}}(\mathfrak{q}) \equiv 1 \ (mod\, p)$ be a prime and recall that R is a p-adic algebra. Fix a level structure away from q, i.e. a compact open $K^{(\mathfrak{q})} \subseteq \mathbf{G}(\mathbf{A}_F^{(\infty,\mathfrak{q})})$, which we furthermore assume can be written as $K^{(\mathfrak{q})} = \prod_{v \neq \mathfrak{q}} K_v$ with K_v hyperspecial for all but finitely many places v. We will look at a pro-system of varieties

$$U \mapsto \mathrm{Sh}_\Sigma(U) \coloneqq \mathrm{Sh}_{K^{(\mathfrak{q})} \times U}(\mathbf{G}, \mathbf{X})_\Sigma,$$

indexed by the compact opens $U \subseteq \mathbf{G}(F_\mathfrak{q})$ (hereafter, we drop the subscript $\mathfrak{q}$ on the compact opens U). By [FC90] and, more generally [Lan13], these admit integral models over $\mathrm{Spec}(R)$, which we will denote by $\mathbb{S}_\Sigma(U)$. The isomorphisms

$$\mathrm{Sh}_\Sigma(U) \xrightarrow{\sim} \mathrm{Sh}_\Sigma(g^{-1}Ug), \qquad \mathbb{S}_\Sigma(U) \xrightarrow{\sim} \mathbb{S}_\Sigma(g^{-1}Ug) \tag{7.5}$$

induce a $G_\mathfrak{q} := \mathbf{G}(F_\mathfrak{q})$-action. Even though the action is canonical on the open Shimura variety, it extends to the compactifications only if we vary the Σ along with U. Later on we are solely interested on the corresponding induced action of $G_\mathfrak{q}$ on $R\Gamma(\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_\Sigma, \mathcal{W}^{\mathrm{can}})$; in light of Prop 1.5.0.1 ii), the dependence on Σ causes no concern.

Denote by $\mathcal{O}_{\mathbb{S}_\Sigma}$ the structure sheaf of the Shimura variety, where the level is to be read off from the context. Recall that the Godement functor is an assignment

$$\mathcal{F} \mapsto \left(U \mapsto \prod_{x \in U} \mathcal{F}_x \right)$$

of a sheaf $\mathcal{F}$ to its sheaf of discontinuous sections, viewed as $\mathcal{O}$-modules. The Godement resolution $\mathcal{G}^\bullet(U)$ of $\mathcal{F}(U)$ is then a complex of $\mathcal{O}_{\mathrm{Sh}_{K^{(\mathfrak{q})}\times U}(G,X)}$-modules on $\mathrm{Sh}_{K^{(\mathfrak{q})}\times U}(G,X)$. Let $\mathbf{M}^\bullet(U)$ and $\mathbf{N}^\bullet(U)$ be the global sections of the Godement resolutions of $\mathcal{W}^{\mathrm{can}}$ and $\mathcal{O}_{\mathbb{S}}^{\mathrm{can}}$, respectively, at level $K^{(\mathfrak{q})} \times U$. These are complexes of R-modules.

Set

$$\mathbf{M}^\bullet_\infty := \varinjlim \mathbf{M}^\bullet(U),$$

which is a complex of R-modules and comes with a $G_\mathfrak{q}$-action, lifting the action $G_\mathfrak{q}$ action (7.5).

Note that Prop. 5.2.0.3 recovers the desired properties (i)-(iii) of §7.1 in [HV19], and thus Lemma 7.1-7.3 hold in our setting of Shimura varieties with the automorphic vector bundle $\mathcal{W}$ in place of the Hodge bundle ω. We record them as they are necessary for explicating some of the maps in the following chapter.

Lemma 7.3.0.1. *Let $V \lhd U \subseteq G_v$ be a normal pro-p subgroup of finite index, and let*

$$f : \mathbb{S}_\Sigma(V) \to \mathbb{S}_\Sigma(U)$$

be the covering map, which is étale by Prop. 5.2.0.3. Then

1. *there is a quasi-isomorphism $(\boldsymbol{M}^\bullet_\infty)^V \simeq \boldsymbol{M}^\bullet(V)$.*

2. *there is a quasi-isomorphism*

$$\boldsymbol{M}^\bullet(U) \to \boldsymbol{M}^\bullet(V)^{U/V}$$

induced by the pushforward f_* of global section.

3. *let $\boldsymbol{Q}^\bullet \to R$ be the augmentation from the previous subsection, then there is a quasi-isomorphism*

$$\boldsymbol{M}^\bullet(U) \to \underline{\mathrm{Hom}}_{R[U]}(\boldsymbol{Q}^\bullet, \boldsymbol{M}^\bullet_\infty) = \underline{\mathrm{Hom}}_{R[G]}(\boldsymbol{P}^\bullet, \boldsymbol{M}^\bullet_\infty).$$

In the case of signature $(n-1,1)$ (or in the anisotropic case) the compactification is canonical but in the remaining signatures one needs to furthemore vary Σ used to define the compactification with the level K. We appeal to Prop 1.5.0.1 i) to deal with this ambigiuity.

By (3) above, the identification of R with global sections of $\mathcal{O}$ on all levels U via the constant section yields a map $R \hookrightarrow \varinjlim \mathbf{N}^\bullet(U) =: \mathbf{N}^\bullet_\infty$, which in turn induces a natural homomorphism

$$H^*_{\mathrm{grp}}(U, R) \to H^*_{\mathrm{Zar}}((\mathbb{S}_{K^q \times U, \Sigma})_R, \mathcal{O}), \tag{7.6}$$

from group cohomology to Zariski (coherent) cohomology, which we denote $\alpha \mapsto \langle \alpha \rangle$.

From now on, set $R = \mathcal{O}_{E_\Pi}/\mathfrak{p}^m = \mathcal{O}/\mathfrak{p}^m$, where $\mathcal{O}_{E_\Pi} = \mathcal{O}$ as in §5 is the field of definition for the automorphic representation Π. Recall the notation from §5 that $\mathcal{O}/\mathfrak{p}^m\langle -1 \rangle :=$

$\mathrm{Hom}_{\mathcal{O}/\mathfrak{p}^m}(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m)$, where $\Delta_{\mathfrak{q}}$ is the maximal product of p-power quotients of $\mathbf{T}(\mathbf{F}_{\mathfrak{q}}) = ((\mathcal{O}_F/\mathfrak{q})^\times)^n$.

Proposition 7.3.0.2. *Let $\langle\alpha\rangle$ be the image of the class $\alpha \in H^i(U, R)$. Then for any classical Hecke operator T supported at a prime w of F which does not divide K, p and such that $\mathbf{G}(F_w)$ is split, we have*

$$T\langle\alpha\rangle = \deg(T)\langle\alpha\rangle.$$

In other words, $\langle\alpha\rangle$ transforms like the constant class.

Proof. This is [Ven19, Lem 2.8] with the map (7.6) in place of (29) therein. □

A delicate analysis of the construction in [Ven19] (especially Appendix A.10 and B.4) and [HV19], shows that the Shimura class is in the image of the map (7.6) for a class in H^1. More precisely, consider

$$p_\Delta : K_0(Q) \to \Delta_Q$$

given by projection to the Levi (toral) factor as in §5.2. For $z \in R\langle -1\rangle = \mathrm{Hom}_R(\Delta_Q, R)$, the homomorphism

$$z \circ p_\Delta : K_0(Q) \xrightarrow{p_\Delta} \Delta_Q \xrightarrow{z} R$$

satisfies $\langle z \circ p_\Delta\rangle = z\mathfrak{S}$. Therefore, we have the following.

Corollary 7.3.0.3. *Let $[\gamma] \in K\backslash G/K$. Then for any classical unramified Hecke operator T and $z \in \mathcal{O}/\mathfrak{p}^m\langle -1\rangle$, it holds that*

$$T(z\mathfrak{S}_\gamma) = \deg(T)(z\mathfrak{S}_\gamma).$$

Following §7.5, 7.6 in [HV19], we describe the action of $T_{\gamma,q,z} \in \mathrm{End}(H^*(\mathbb{S}_{K,\Sigma}, \mathcal{O}_{\mathbb{S}}))$ through an action on the hypercohomology of the Godement resolution.

Lemma 7.3.0.4. *Under the identification of $H^*(\mathbb{S}_{U,\Sigma}, \mathcal{W}^{can})$ with the hypercohomology $\mathbb{H}^*(U, \boldsymbol{M}_\infty^\bullet)$*

from Lemma 7.3.0.1, the cup product with $\langle \alpha \rangle$ in coherent cohomology corresponds to a cup product with α in hypercohomology.

Proof. The proof of [HV19, Lemma 7.5] works for $\mathcal{W}$ since all automorphic bundles are constructed from the Hodge bundle (see [LS13, §4.2]). $\square$

In light of the above result, a restatement of Lemma A.10 in [Ven19] gives the following.

Lemma 7.3.0.5. *The action of $h_\gamma \in H^*(K^\gamma, R)$ on $\mathbb{H}^*(K_v, M^\bullet_\infty)$ is given by*

$$\mathbb{H}^*(K, M^\bullet_\infty) \xrightarrow{\mathrm{Ad}(\gamma^{-1})^*} \mathbb{H}^*(K^\gamma, M^\bullet_\infty) \xrightarrow{m \mapsto \gamma m} \mathbb{H}^*(K^\gamma, M^\bullet_\infty) \xrightarrow{-\cup h_\gamma} \mathbb{H}^*(K^\gamma, M^\bullet_\infty) \xrightarrow{\text{Cores}} \mathbb{H}^*(K, M^\bullet_\infty)$$

We recover the action of $T_{\mathfrak{q},\gamma,z}$ by taking $h_\gamma = \langle z \circ p^\gamma_\Delta \rangle \in H^1(K^\gamma, R)$ where $p^\gamma_\Delta : K^\gamma \to \Delta_q$ is defined analogously to p_Δ.

7.4 Iwahori-Hecke algebra

Suppose that $q_v \equiv 1 (mod\ p^m)$ and recall that $p \nmid W_{\mathbf{G}}$. In this subsection, we study the classical Hecke algebra $H_R(G_v, U_v) = \mathcal{H}^{(0)}_R(G_v, U_v)$ and focus on the cases where U_v equal to $K_v = \mathrm{GL}_n(\mathcal{O}_v)$ or $\mathrm{Iw}(v)$, where the latter is an Iwahori group. For specific choice, we may choose

$$\mathrm{Iw}(v) = \text{preimage of } \mathcal{B}(F_v) \text{ inside } \mathcal{G}(\mathcal{O}_v),$$

where $\mathcal{B}$ and $\mathcal{G}$ are integral models of $\mathbf{B}$ and $\mathbf{G}$. The choices for $\mathrm{Iw}(v)$ are all $W_{\mathbf{G}}$-conjugates.

These two arise naturally in the Taylor-Wiles method and this subsection is dedicated to explicating the relationship between H_{K_v} and $H_{\mathrm{Iw}(v)}$. By our assumption, $p \nmid |W_{\mathbf{G}}|$. Since $[\mathrm{GL}_n(\mathcal{O}_v) : \mathrm{Iw}(v)] \equiv |W_{\mathbf{G}}| (mod\ p)$, this index is invertible in R.

Let $H_K := H_R(G_v, K_v)$ and $H_I := H_R(G_v, \mathrm{Iw}(v))$. We summarize the relevant results from [Ven19]. Introduce the idempotent inside H_I given by

$$e_K = \mathbf{1}_{K_v}/\mu(K_v),$$

where the measure μ is normalized, so that $\mu(\mathrm{Iw}(v)) = 1$. Set

$$H_{IK} = e_K \star H_I, \quad H_{KI} = H_I \star e_K.$$

These two bimodules will later facilitate transfers between hyperspecial and Iwahori levels. Another more convenient description is

$$H_{IK} = \mathrm{Hom}_{R[G_v]}(R[G_v/G_v], R[G_v/\mathrm{Iw}(v)])$$
$$H_{KI} = \mathrm{Hom}_{R[G_v]}(R[G_v/\mathrm{Iw}(v)], R[G_v/K_v])$$

Note that $e_K H_{KI} = H_{IK} e_K = e_K H_I e_K = H_K$. Let $X_*(\mathbf{T}) := \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ be the cocharacter group of $\mathbf{T}$.

Lemma 7.4.0.1. *For $q_v \equiv 1$ $(mod\ p)$, we have an isomorphism*

$$H_I \simeq R[X_*(\mathbf{T}) \rtimes W_{\mathbf{G}}]$$
$$\mathbf{1}_{\mathrm{Iw}\beta\mathrm{Iw}} \mapsto \beta.$$

Proof. This follows from the Bernstein's representation of the Iwahori-Hecke algebra as argued in [CKM, Thm 4.1]. Crucially, as $q_v \equiv 1$ $(mod\ p)$, the identity $(T_s - 1)(T_s - q) = 1$ reduces to $T_s^2 = 1$. For more details, see [KT14, Lem 5.1] □

There is an identification

$$X_*(\mathbf{T}) \to \mathbf{T}(\mathbf{F}_v)/\mathbf{T}(\mathbf{F}_v) \cap K_v$$
$$\chi \mapsto \chi(\varpi), \tag{7.7}$$

where ϖ is an uniformizer. We clearly have

$$Z := R[X_*]^{W_{\mathbf{G}}} \to Z(H_I)$$

to the center of the Iwahori-Hecke algebra. The **discriminant** is the following element

$$d_I := \prod_\alpha (1 - \alpha^*) \in Z, \tag{7.8}$$

where the product is taken over roots α and $\alpha^* = (\alpha^\vee)^{m_\alpha}$ with m_α being the largest integers so that $\alpha/m_\alpha \in X^*$. Let Z_{d_I} be its stabilizer. Understanding this opaque definition is not so important for us. It is mostly useful in order to state the following.

Lemma 7.4.0.2. *[Ven19, Lem 4.5] Let d_I be as above. Set $H'_K := H_K \otimes_Z Z_{d_I}$ and same for H'_{IK}, H'_{KI}, H'_I. Then there is an equivalence between the categories of H'_K and H'_I-modules induced by the bimodules H'_{KI} and H'_{IK}.*

7.5 Derived Satake isomorphism

The derived Hecke algebra is difficult to analyze for all q. Nevertheless, for unramified Taylor-Wiles primes, it is related to the derived Hecke algebra of a torus reminiscent of the classic Satake isomorphism. This result will be very fruitful in giving a geometric interpretation of the derived Hecke operators in degree one via cup product with classes in covering spaces (see §10.10).

Denote by G_v, K_v and T_v the groups $\mathbf{G}(\mathbf{F}_v)$, a choice of a maximal compact, and a torus subgroup of G_v, respectively.

Theorem 7.5.0.1 (Thm.3.3 in [Ven19])**.** *In the notation from above, there is an isomorphism*

$$\mathcal{H}_R(G_v, K_v) \xrightarrow{\sim} \mathcal{H}_R(T_v, T_v \cap K_v)^{W_{G_v}}$$

given by restriction. In the second model from §7.2, the restriction map associates to the assignment

$$h : (x, y) \in (G_v/K_v)^2 \mapsto h(x, y) \in H^*(G_v^{xy}, R)$$

the assignment

$$h' : (x, y) \in (T_v/T_v \cap K_v)^2 \hookrightarrow (G_v/K_v)^2 \xrightarrow{h} h(x, y) \in H^*(G_v^{xy}, R) \xrightarrow{\iota^*} h'(x, y) \in H^*(T_v^{xy}, R),$$

where the last map is pullback along $\iota : T_v^{xy} \to G_v^{xy}$

Corollary 7.5.0.2. *Suppose $k > m$ and $p^k | q_v - 1$. Then the induced map*

$$\mathcal{H}_{\mathcal{O}/\mathfrak{p}^k}(G_v, K_v) \to \mathcal{H}_{\mathcal{O}/\mathfrak{p}^m}(G_v, K_v)$$

is surjective.

Proof. By the derived Satake isomorphism, it suffices to show that $H^*(\Lambda, \mathcal{O}/\mathfrak{p}^k) \to H^*(\Lambda, \mathcal{O}/\mathfrak{p}^m)$ is surjective for a cyclic group Λ whose order is divisible by p^k, which can be shown directly. □

7.6 Perfect complexes and coherent homology

In this section we just collect the necessary construction from [CG18] concerning the existence of perfect complexes computing coherent homology. All results are shown in §7.2 in ibid.

Suppose $Y \to X$ is a finite étale map with Galois group Δ, where Y and X will be proper and smooth over $\mathrm{Spec}(\mathcal{O})$ toroidal compactifications of Shimura varieties. Set $R := \mathcal{O}[\Delta]$. Then there exist perfect complexes $\mathrm{C}_m^\bullet$ and $\tilde{\mathrm{C}}_m^\bullet$ of $R/\mathfrak{p}^m$-modules computing

$$H_i(Y, \mathcal{W} \otimes \mathcal{O}/\mathfrak{p}^m) \quad \text{and} \quad H_i(Y, \mathcal{W} \otimes \mathcal{O}/\mathfrak{p}^m)_{\mathfrak{m}},$$

where $\mathcal{W}$ is an automorphic vector bundle and $\mathfrak{m}$ is a maximal ideal of the (classical) Hecke algebra of operators at unramified places and diamond operators at TW primes. The term coherent 'homology' is perhaps confusing but it is natural to use homology for patching purposes, and the homology in this case is defined via Poincaré duality:

$$H_i(Y, \mathcal{W} \otimes \mathcal{O}/\mathfrak{p}^m) := H^i(Y, \mathcal{W}^* \otimes_{\mathcal{O}_Y} \omega_Y \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p}^m)^\vee, \tag{7.9}$$

where ω_Y is the determinant of $\Omega^1_{Y/\mathcal{O}}$. Therefore, we have

$$H^*\Big(\mathrm{Hom}_{\mathcal{O}/p^m[\Delta]}(\mathrm{C}_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\Big) \simeq H^*(X, \mathcal{W} \otimes \mathcal{O}/\mathfrak{p}^m),$$
$$H^*\Big(\mathrm{Hom}_{\mathcal{O}/p^m[\Delta]}(\tilde{\mathrm{C}}_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\Big) \simeq H^*(X, \mathcal{W} \otimes \mathcal{O}/\mathfrak{p}^m)_{\mathfrak{m}}.$$

Note that we compute the cohomology of X, not Y. This is because of the action Δ by deck transformations and the fact that $Y \to X$ is étale with covering group Δ. Similarly, we may take $\mathrm{C}_\infty^\bullet$ computing the $\mathcal{O}$-coefficient (co)homology.

7.7 Global derived Hecke algebra

The action of the derived Hecke algebra produces an endomorphism algebra

$$\tilde{\mathbb{T}}_k \subseteq \mathrm{End}\left(H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^k)\right)$$

obtained form the derived Hecke algebras $\mathcal{H}_{v,\mathcal{O}/\mathfrak{p}^k}$ for all good primes v. Then we may define the global derived Hecke algebra

$$\tilde{\mathbb{T}} \subseteq \mathrm{End}\left(H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W})\right)$$

of all endomorphisms of the form $\varprojlim t_k$ for a **compatible system** $t_k \in \tilde{\mathbb{T}}_k$. A compatible system is one for which the following diagram commutes for $k > m$

$$\begin{array}{ccc} H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^k) & \xrightarrow{t_k} & H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^k) \\ \downarrow & & \downarrow \\ H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^m) & \xrightarrow{t_m} & H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^m) \end{array}$$

We briefly sketch the details of how to extract the limit via a compactness argument following [Ven19, § 2.13]. Let $\tilde{\mathbb{T}}_k^*$ be the collection of all compatible systems of elements $(t_k, t_{k-1}, \dots, t_1)$ with $t_i \in \tilde{\mathbb{T}}_i$. From the commutativity of the above diagram this means that $\tilde{\mathbb{T}}_k^* \circlearrowleft H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^k)$. Set

$$\tilde{\mathbb{T}} := \varprojlim \tilde{\mathbb{T}}_k^*.$$

Note that each $\tilde{\mathbb{T}}_m^*$ is a finite set, and thus the map

$$\tilde{\mathbb{T}}_k^* \to \tilde{\mathbb{T}}_m^* \tag{7.10}$$

for increasing $k > m$ stabilizes. Therefore, there exists a positive integer N_m and a subring $\tilde{\mathbb{T}}^*_{\infty,m}$ of $\tilde{\mathbb{T}}^*_m$ such that the image of (7.10) for all $k \geq N_m$ is $\tilde{\mathbb{T}}^*_{\infty,m}$, and thus

$$\tilde{\mathbb{T}} \to \tilde{\mathbb{T}}^*_{\infty,m}$$

is onto. We conclude by applying Corollary 7.5.0.2 as follows: if $p^{N_m} | q_v - 1$, the image of $\tilde{\mathbb{T}}^*_{N_m}$ inside $\mathrm{End}\,(H^*((\mathbb{S}_{K,\Sigma})_R, \mathcal{W}/\mathfrak{p}^m))$ contains the image of $\mathcal{H}_{v,\mathcal{O}/\mathfrak{p}^m}$, and thus the image of $\tilde{\mathbb{T}}^*_{\infty,m}$ also contains the image of $\mathcal{H}_{v,\mathcal{O}/\mathfrak{p}^m}$.

Chapter 8: Setup

Let $\mathrm{disc}(F)$ be the discriminant of $F/\mathbf{Q}$. Then $\mathrm{Sh}_K(\mathbf{G},\mathbf{X})$ admits a smooth integral model over $\mathrm{Spec}(\mathcal{O}_F[\frac{1}{\mathrm{disc}(F)\cdot N}])$, where N is the product of the primes dividing the level K. Below we have attempted to be consistent with the notations from [GV17] and [Ven19]. Most of the statements are contained in the above two sources with modifications necessary for coherent cohomology.

8.1 Assumption on Hecke algebra

1. Fix $K_\emptyset$ as base level, and write $\mathbb{S}_\emptyset$ for the integral model over $\mathrm{Spec}(\mathcal{O}_F[\frac{1}{\mathrm{disc}(F)\cdot N}])$ at level $K_\emptyset$, which is neat. Let $K \subseteq K_\emptyset$ be a deeper level structure.

2. Suppose $\Pi = \Pi_\infty \otimes \Pi_f$ be a cuspidal automorphic representation with archimedean component Π_∞ non-degenerate limit of discrete series, and such that $\Pi^{K_\emptyset} \neq 0$.

3. Let $\mathcal{W}$ be an automorphic vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})$ such that Π contributes to its interior cohomology.

4. Let $\mathbb{T}(\mathcal{W}, K_\emptyset)$, or simply $\mathbb{T}$, be the usual Hecke algebra at level $K_\emptyset$ generated at places relatively prime to $K_\emptyset$ and p as in (5.2), i.e.

$$\mathbb{T}(\mathcal{W}, K_\emptyset) \subseteq \mathrm{End}_{D(\mathcal{O})}(\mathrm{R}\Gamma^{\mathrm{can}}(\mathcal{W}_{K_\emptyset})).$$

5. Let $E_\Pi = \mathbf{Q}(\Pi)$ be the coefficient field of Π, i.e. the number field containing all Hecke eigenvalues of Π at unramified places. If necessary, we extend E_Π to contain F as well. Let

$\mathcal{O}_E$ be its ring of integers, so we have a ring homomorphism

$$\chi_{\Pi,\mathcal{W}} : \mathbb{T}_{K_\emptyset} \to \mathcal{O}_{E_\Pi}.$$

6. Let T be the set of places in F where either Π is ramified, $\mathcal{W}$ does not have integral model, or $K_\emptyset$ is not hyperspecial. Set $S = T \cup \{v : v|p\}$.

7. Let $p > |W|$ be a prime with $p \notin S$. Let $\mathfrak{p} \lhd \mathcal{O}_{E_\Pi}$ be a prime above the rational prime p. We assume

   i) $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W})$ is p-torsion-free, where $\mathbb{S}_{\emptyset,\Sigma}$ is an integral model of $\mathrm{Sh}_{K_\emptyset}(\mathbf{G},\mathbf{X})_\Sigma$;

   ii) "No congruence with other forms":

   If $\mathfrak{m} = \ker(\mathbb{T}_{K_0} \to \mathcal{O}_{E_\Pi}/\mathfrak{p} =: \mathbf{F}_\mathfrak{p})$, then $(\mathbb{T}_{K_0})_\mathfrak{m} \simeq \mathcal{O}_\mathfrak{p}$;

   iii) $\mathcal{O}_E$ over $\mathbf{Z}$ is unramified at $\mathfrak{p}$, so that $W(\mathbf{F}_p) = \mathcal{O}_\mathfrak{p}$;

   iv) $H_j(\mathbb{S}_\emptyset, \mathcal{W} \otimes \mathcal{O}_\mathfrak{p})_\mathfrak{m}$ vanishes outside $j \in [q_0, q_0 + l_0]$, where

   $$q_0 = \min_\psi (q_{\lambda,\psi}) \tag{8.1}$$

   in the notation of Thm. 1.5.0.6 and l_0 is as in Def 1.5.0.8 (or as in Prop. 1.5.0.9) $(\star)$.

In light of Thm. 1.5.0.6 condition $(\star)$ is expected to hold for sufficiently large p assuming (7.ii).

8.2 Assumption on the Galois representations and deformation rings

For $K \subseteq K_\emptyset$, let $\mathfrak{m} = \ker(\mathbb{T}_K \to \mathcal{O}_{E_\Pi} \to \mathbf{F}_\mathfrak{p})$. Even though $\mathfrak{m}$ depends on the level K, we shall use the same notation for all levels to avoid extra notational burden. Suppose there is a Galois representation

$$\rho = \rho_\Pi : \underbrace{\mathrm{Gal}(\overline{F}/F)}_{\Gamma_F} \to \mathcal{G}_n(\mathbb{T}_{K,m}) = (\mathrm{GL}_n \rtimes \mathrm{GL}_1)(\mathbb{T}_{K,\mathfrak{m}}),$$

where $\mathcal{G}_n$ is the c-group (see [BG14]) constructed in [CHT08], lifting the residual Galois representation

$$\bar{\rho}_\Pi : \Gamma_F \to \mathcal{G}_n(\mathbf{F}_\mathfrak{p}).$$

Note that Π only defines a representation of Γ_F, not $\Gamma_\mathbf{Q}$. Nevertheless, Venkatesh's conjectures concern $\mathrm{Ad}^*\rho$ which descends to $\Gamma_\mathbf{Q}$. Suppose further that ρ satisfies

1. For all primes $q \neq p$ not dividing the level $K_\emptyset$, the representation ρ is unramified at all primes $\mathfrak{q}$ above q. Let $\Gamma_\mathfrak{q} \subseteq \Gamma_F$ be a decomposition group at $\mathfrak{q}$; then the semisimplification of the restriction $\rho_\Pi\big|_{\Gamma_\mathfrak{q}}$ corresponds to the restriction to $\mathfrak{q}$ to the image of the Hecke operators at $\mathfrak{q}$ by the unramified Langlands correspondence as follows: let $\tau : \mathbb{T}_K \to \mathbf{F}_\mathfrak{q}$ there is an equality of polynomials in $\mathbf{F}_\mathfrak{q}[X]$

$$\det\left(1 - \rho_\Pi(\mathrm{Frob}_\mathfrak{q})X\right) = 1 + \sum (-1)^i \tau\left(N(\mathfrak{q})^{\frac{(n+1)i}{2}} T_{i,\mathfrak{q}}\right) X^i$$

   where the $T_{i,\mathfrak{q}}$ are the standard Hecke operators at $\mathfrak{q}$, normalized by the Satake isomorphism.

2. The representation $\bar{\rho}_\Pi$ has big image: the image $\bar{\rho}_\Pi\big|_{F(\zeta_{p^\infty})}$ contains the images of the $\mathbf{F}_\mathfrak{p}$-points of the simply connected cover. This implies that $\bar{\rho}_\Pi$ is Schur, i.e. $\mathrm{End}_{\mathbf{F}_\mathfrak{p}}(\bar{\rho}_\Pi) = \mathbf{F}_\mathfrak{p}^\times$.

3. The representation ρ is crystalline at all primes above p. The precise formal deformation theoretic sense is as in [GV17, Conj.6.1].

4. The representation $\bar{\rho}$ is odd at all real places $v \mid \infty$.

5. The local-global compatibility assumption (see Assumption 3) holds for all places $\mathfrak{q}$ belonging to a Taylor-Wiles datum, both at level $K_\emptyset$ and at the Iwahori level $K_0(\mathfrak{q})$.

6. Formal smoothness at S:

$$H^0(F_v, \mathrm{Ad}\bar{\rho}) = H^2(F_v, \mathrm{Ad}\bar{\rho}) \text{ for all } v \in S.$$

This implies that the local deformation ring $R_{\bar{\rho}_\Pi}$ is $\mathcal{O}_\mathfrak{p}$ if $v \in S$ and is formally smooth if $v|p$.

We also study the adjoint Galois representation

$$\mathrm{Ad}\ \rho_\Pi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathfrak{gl}_{n,E},$$

which extends to the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Note that even though ρ_Π is a representation for $\mathrm{Gal}(\overline{F}/F)$, the adjoint $\mathrm{Ad}\ \rho_\Pi$ descends to a representatin of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The two notions of c and L group of [BG14] coincide after conjugation, so we may simply use L-group.

Denote by ρ_m (resp. $\mathrm{Ad}\ \rho_m$) the reduction of ρ (resp. $\mathrm{Ad}\ \rho$) modulo $\mathfrak{p}^m$.

8.3 Level structure and diamond operators

Let $\mathbb{S}_{0,\Sigma}, \mathbb{S}_{\Delta,\Sigma}, \mathbb{S}_{1,\Sigma}$ be integral models as in §3 over $\mathcal{O}_E$, which may be extended if necessary to contain $\mathcal{O}_F$. Hereafter, we write $\mathcal{O}$ in place of this p-adic ring $\mathcal{O}_E$. We would hereafter entirely focus in the cases where Taylor-Wiles method applies and suppose that $p^m | N_{F/\mathbf{Q}}(\mathfrak{q}) - 1$. Then we have

$$\overbrace{\mathbb{S}_{1,\Sigma}(\mathfrak{q}) \to \underbrace{\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q}) \to \mathbb{S}_{0,\Sigma}(\mathfrak{q})}_{\Delta_\mathfrak{q}}}^{\mathbf{T}(\mathbf{F}_\mathfrak{q})} \to \mathbb{S}_{\emptyset,\Sigma}, \tag{8.2}$$

where $\mathbf{T}$ is a torus inside a Borel $\mathbf{B}$ of $\mathbf{G}$ as in §5. Note that $\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q})$ exists because $p^m | N_{F/\mathbf{Q}}(\mathfrak{q})-1$ and furthermore the map

$$\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q}) \to \mathbb{S}_{0,\Sigma}(\mathfrak{q})$$

may be arranged to be étale with Galois group $\Delta_{\mathfrak{q}}$ in light of Prop 5.2.0.3 iii). We always assume that Σ is chosen so that this is the case and the boundary components are normal crossings divisors.

Similarly, for a Taylor-Wiles datum $Q_m = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ of level m, we write $\mathbb{S}_{\Delta,\Sigma}(Q_m)$ for the fiber-product of all coverings $\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q})$ for $q \in Q_m$. Set Δ_{Q_m} to be the Galois group of the covering

$$\underbrace{\mathbb{S}_{\Delta,\Sigma}(Q_m) \to \mathbb{S}_{0,\Sigma}(Q_m)}_{\prod_{q\in Q_m}(\mathbf{T}(\mathbf{F}_{\mathfrak{q}})/p^m)=\prod_{q\in Q_m}\Delta_{\mathfrak{q}}},$$

so that there is a non-canonical identification

$$\Delta_{Q_m} \simeq (\mathbf{Z}/p^m)^{n|Q_m|} = (\mathbf{Z}/p^m)^R,$$

where $R := n|Q_m| = ns$, which will be the dimension of a limit ring S_∞ coming from the patching process, introduced in next section.

Note that for $\mathbf{G}$, a Shimura variety of unitary type, or general Shimura variety, we have that $\mathrm{rk}(\mathbf{T}) = \mathrm{rk}(\mathbf{G}) = n$. This is contrast with the cases of interest for [Ven19], where $\mathrm{rk}(\mathbf{G}) - \mathrm{rk}(\mathbf{T}) =: \delta_0 > 0$.

8.4 Diamond operators

We now introduce the **diamond operators** which are the rings generated by the deck transformations of the various étale covers $\mathbb{S}_{\Delta,\Sigma}(Q_m) \to \mathbb{S}_{0,\Sigma}(Q_m)$. Let

$$S_{Q_m} \coloneqq \mathcal{O}/\mathfrak{p}^m[\Delta_{Q_m}], \quad S'_{Q_m} \coloneqq \mathcal{O}[\Delta_{Q_m}],$$

which would act on the complexes $\tilde{\mathrm{C}}_m^\bullet$ and $\mathrm{C}_\infty^\bullet$, respectively; the details are explained in §7.6. Note that

$$S_{Q_m} = \mathcal{O}/\mathfrak{p}^m[x_1, \ldots, x_R]/((1+x_i)^{p^m} - 1).$$

Introduce also a limit ring

$$\mathrm{S}_\infty \coloneqq \mathcal{O}_\mathfrak{p}[\![x_1, \ldots, x_R]\!].$$

Viewing $\tilde{\mathrm{C}}_m^\bullet$ as a S_{Q_m}-module, we have the canonical isomorphisms (see the discussion in 7.6)

$$\begin{aligned} H_*(\tilde{\mathrm{C}}_m^\bullet) &\simeq H_*(\mathbb{S}_{\Delta,\Sigma}(Q_m), \mathcal{W}/\mathfrak{p}^m)_\mathfrak{m} \\ H^*(\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}_m^\bullet, \mathcal{W}/\mathfrak{p}^m)) &\simeq H^*(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W}/\mathfrak{p}^m)_\mathfrak{m} \\ H_*(\tilde{\mathrm{C}}_m^\bullet \otimes_{S_{Q_m}} \mathcal{O}/\mathfrak{p}^m) &\simeq H_*(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W}/\mathfrak{p}^m)_\mathfrak{m} \end{aligned} \tag{8.3}$$

where we adopted the shorthand $\mathcal{W}/p^m \coloneqq \mathcal{W} \otimes \mathcal{O}/\mathfrak{p}^m$. Since $\tilde{\mathrm{C}}_m^\bullet$ is perfect, $\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}_m^\bullet, \mathcal{W}/\mathfrak{p}^m)$ also computes the derived Hom from $\tilde{\mathrm{C}}_m^\bullet$ to $\mathcal{O}/\mathfrak{p}^m$. In the derived category, we can compose all the homomorphisms and obtain

$$H^*\Big(\mathrm{Hom}_{\mathcal{O}/p^m[\Delta]}(C_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\Big) \times \mathrm{Ext}^*_{S_m}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) \to H^*\Big(\mathrm{Hom}_{\mathcal{O}/p^m[\Delta]}(C_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\Big) \tag{8.4}$$

This opaque action of $\mathrm{Ext}^*_{S_m}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) = H^*(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m)$ should coincide with the perhaps more natural one arising from the Δ_{Q_m}-covering $\mathbb{S}_{\Delta,\Sigma}(Q_m) \to \mathbb{S}_{0,\Sigma}(Q_m)$. The identification between the two actions is far from obvious and is sketched in [Ven19, App.B4] in the topological case. It is worth remarking that the action thus produced is strictly a *right* action, and thus attention has to be paid to place the multiplication in the correct order. Nevertheless, we would be entirely focused on the cases arising from Taylor-Wiles datum, where by Thm. 7.5.0.1 the algebra is commutative; hence, we shall write the action as a left action as is customary with the classical Hecke algebra.

Assumption 1. *Suppose Q_m is a system of Taylor-Wiles primes. The action*

$$H^*(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m) \circlearrowleft H^*(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W}/\mathfrak{p}^m)_{\mathfrak{m}}$$

given by (8.4) coincides with the action of Δ_{Q_m} coming from the deck transformations.

From this perspective, the inserted $\cup z\mathfrak{S}$ in (5.6) corresponds to the action $H^1(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m)$.

For completeness we record also

Lemma 8.4.0.1. *The natural map*

$$\mathrm{Ext}^*_{S_{Q_m}}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) \to \mathrm{Ext}^*_{\mathbb{S}_\infty/\mathfrak{p}^m}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m)$$

is surjective, while the map

$$\mathrm{Ext}^*_{\mathbb{S}_\infty/\mathfrak{p}^m}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) \to \mathrm{Ext}^*_{\mathbb{S}_\infty}(\mathcal{O}_\mathfrak{p}, \mathcal{O}/\mathfrak{p}^m)$$

is an isomorphism.

Proof. This is the Lemma in [Ven19, §6.4]. □

8.5 Transfer between level 1 and level $\mathfrak{q}$

We fix a prime q not dividing the level $K_\emptyset$ which is also part of a Taylor-Wiles datum for p. The first part of the construction is to pass from hypespecial to Iwahori level at $\mathfrak{q}$. Fix $K^{(\mathfrak{q})} \subseteq \mathbf{G}(\mathbf{A}_F^{(\mathfrak{q},\infty)})$, and let

$$K_\emptyset = K^{(\mathfrak{q})} \times \mathrm{GL}_n(\mathcal{O}_\mathfrak{q}), \qquad K_0(\mathfrak{q}) = K^{(\mathfrak{q})} \times \mathrm{Iw}(\mathfrak{q})$$

and denote by $\mathbb{S}_{\emptyset,\Sigma}$ and $\mathbb{S}_{0,\Sigma}$ the integral models at level $K_\emptyset$ and $K_0(\mathfrak{q})$, respectively. There are natural maps

$$\begin{aligned} H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W} \otimes R) \otimes_{H_K} H_{KI} &\longrightarrow H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes R) \\ H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes R) \otimes_{H_I} H_{IK} &\longrightarrow H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W} \otimes R) \end{aligned} \tag{8.5}$$

These maps are given as follows: Identify $H_K = \mathrm{Hom}_{R[G_\mathfrak{q}]}(R[G_\mathfrak{q}/K_\mathfrak{q}], R[G_\mathfrak{q}/K_\mathfrak{q}])$ and $H_{KI} = \mathrm{Hom}_{R[G_\mathfrak{q}]}(R[G_\mathfrak{q}/K_\mathfrak{q}], R[G_\mathfrak{q}/\mathrm{Iw}(\mathfrak{q})])$. Then by Lem 7.3.0.1, we have that $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W} \otimes R)$ is computed by

$$\underline{\mathrm{Hom}}_{R[G]}(\mathbf{P}^\bullet, \mathbf{M}_\infty^\bullet),$$

where $\mathbf{P}^\bullet$ is a projective resolution of $R[G_\mathfrak{q}/K_\mathfrak{q}]$. For any $\psi \in \mathrm{Hom}_{R[G_\mathfrak{q}]}(R[G_\mathfrak{q}/K_\mathfrak{q}], R[G_\mathfrak{q}/\mathrm{Iw}(\mathfrak{q})])$, we have

$$\mathbf{P}^\bullet \to R[G_\mathfrak{q}/K_\mathfrak{q}] \xrightarrow{\psi} R[G_\mathfrak{q}/\mathrm{Iw}(\mathfrak{q})], \tag{8.6}$$

which can be viewed as a projective resolution of $R[G_\mathfrak{q}/\mathrm{Iw}(\mathfrak{q})]$ as $R[G_\mathfrak{q}]-$module. Applying again Lem 7.3.0.1, this resolution now computes $H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes R)$. The second map in (8.5) is analogous.

As long as we control for the action of the center of H_I, it turns out that the above maps are isomorphisms when properly localized. Note that $R[X_*]^{W_{\mathbf{G}}}$ is contained in the center of $R[X_*(\mathbf{T}) \rtimes W_{\mathbf{G}}]$, and by Prop 8.5.0.1 the ring $Z := R[X_*]^{W_{\mathbf{G}}}$ maps naturally to $Z(R[X_*(\mathbf{T}) \rtimes W_{\mathbf{G}}])$. Furthermore, the map

$$Z \to H_K$$
$$z \mapsto e_K z e_K$$

is a ring isomorphism, which is clear by looking at the generators for H_K.

Assumption 2 (Local-global compatibility)**.** *View H_K as a subring inside the center of H_I. Then*

$$H_K \circlearrowleft H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes \mathbf{F}_{\mathfrak{p}})_{\mathfrak{m}}$$

via the character $\overline{\chi}_{\Pi,\mathcal{W}}$, the reduction of the character $\chi_{\Pi,\mathcal{W}}$ modulo $\mathfrak{p}$ (see §8.1 (5)).

Proposition 8.5.0.1. *In the above notation, the maps (8.5) are isomorphisms when localized away from the discriminant d_I from (7.8). In particular, for a TW prime $\mathfrak{q}$, assuming the local-global compatibility above, we have*

$$H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W} \otimes R)_{\mathfrak{m}} \otimes_{H_K} H_{KI} \xrightarrow{\sim} H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes R)_{\mathfrak{m}}$$

Proof. The proof is identical to that of [Ven19, Lem. 6.6]. All we need to see is that the maps in (8.6) satisfy the general functorial setup in the proof of the Lemma, which follows by their explication above and Lemma 7.4.0.2. □

Corollary 8.5.0.2. *Let $Q_m \in \mathrm{TW}_m$ be a Taylor-Wiles datum of level m. Then*

$$H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W} \otimes R)_{\mathfrak{m}} \bigotimes_{\otimes_{\mathfrak{q}\in Q_m} H_{K_\mathfrak{q}}} \otimes_{\mathfrak{q}\in Q_m} H_{K_\mathfrak{q}, I_\mathfrak{q}} \xrightarrow{\sim} H^*(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W} \otimes R)_{\mathfrak{m}}$$
$$H^*(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W} \otimes R)_{\mathfrak{m}} \bigotimes_{\otimes_{\mathfrak{q}\in Q_m} H_{K_\mathfrak{q}}} \otimes_{\mathfrak{q}\in Q_m} H_{I_\mathfrak{q}, K_\mathfrak{q}} \xrightarrow{\sim} H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W} \otimes R)_{\mathfrak{m}}$$

This result translates the action from $\mathbb{S}_{\emptyset,\Sigma}$ to $\mathbb{S}_{0,\Sigma}$. Recall by Lemma 7.4.0.1 that H_K and H_I differ by the extra generators coming from $X_*^+ := X_*^+(\mathbf{T})$, where $X_*^+(\mathbf{T})$ is the positive cone corresponding to the Borel group. Explicitly, these correspond to the strictly dominant diagonal matrices with strictly descending entries along the diagonal. We produce the action of $\mathbf{F}_\mathfrak{p}[X_*^+]$ on $H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes R)_{\mathfrak{m}}$.

Note that we have

$$\mathbf{F}_\mathfrak{p}[X_*^+] \hookrightarrow H_I$$
$$\chi \mapsto \mathrm{Iw}(\mathfrak{q})\chi\mathrm{Iw}(\mathfrak{q})$$

An element $t \in T^\vee(\mathbf{F}_\mathfrak{p})$ produces a character $\chi_t : \mathbf{F}_\mathfrak{p}[X_*^+] \to \mathbf{F}_\mathfrak{p}$ via

$$t \in T^\vee(\mathbf{F}_\mathfrak{p}) \xrightarrow{\text{Local Langlands}} \text{unramified } \mathbf{T}(\mathbf{Q}_q) \to \mathbf{F}_\mathfrak{p}^\times \xrightarrow{\text{linear extension}} \mathbf{F}_\mathfrak{p}[X_*^+] \to \mathbf{F}_\mathfrak{p}$$

Then the action of this monoid algebra is easy to compute in light of the previous lemma. These operators play the role of the classical $U_\mathfrak{q}$-operators.

Corollary 8.5.0.3. *Let $\mathfrak{q} \in \mathrm{TW}_m$. Then the generalized eigenvalues of $\mathbf{F}_\mathfrak{p}[X_*^+] \circlearrowleft H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes R)_{\mathfrak{m}}$ are of the form χ_{Frob^T}, where $\mathrm{Frob}^T \in T^\vee(\mathbf{F}_\mathfrak{p})$ is a conjugate to $\mathrm{Frob}_\mathfrak{q}$ in the notation from above.*

Most importantly, we have

$$H^*(\mathbb{S}_{0,\Sigma}(\mathfrak{q}), \mathcal{W}/\mathfrak{p}^m)_{\mathfrak{m},\chi_{\mathrm{Frob}^T}} \simeq H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/\mathfrak{p}^m)_{\mathfrak{m}}$$

with the obvious maps – from left to right via pushforward, and from right to left pullback followed by projection to the Frob^T eigenspace (see Remark after [Ven19, Cor. 6.7]).

Proposition 8.5.0.4. *Let $\mathfrak{q}$ be a part of Taylor-Wiles datum. Assume also local-global compatibility (see Assumption 2) and §8.1 (6) that $H^*(\mathbb{S}_{\emptyset,\Sigma})$ is p-torsion-free. Then $H_j(\mathbb{S}_{\Delta,\Sigma}(\mathfrak{q}), \mathcal{W} \otimes \mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}}$ vanishes for $j \notin [q_0, q_0 + l_0]$ with q_0 as in (8.1) and l_0 as in Thm. 1.5.0.6 and Def 1.5.0.8.*

Proof. We may replace $\mathcal{O}_{\mathfrak{p}}$ with $\mathbf{F}_{\mathfrak{p}}$ by Assumption 7.i) from §8.1. Arguing as in [Ven19, §6.8], it's enough to prove the assertion at the $\mathbb{S}_{0,\Sigma}$-level. By virtue of (8.5), we can reduce to the $\mathbb{S}_{\emptyset,\Sigma}$ level. Then the claim follows by Assumption 7.iv. □

Chapter 9: Patching

In this section we go in details to construct inverse limits $\mathrm{S}_\infty, \mathrm{R}_\infty$ and $\mathrm{C}_\infty^\bullet$ of $S_{Q_m}, \mathrm{R}_{\bar{\rho},Q_m}$ and $\tilde{\mathrm{C}}_m^\bullet$, respectively. To do so, one employs a compactness argument and uses a specifically chosen Taylor-Wiles datums. The discussion below essentially follows §13 in [GV17].

9.1 Deformation rings at level Q_m

Consider the universal deformation ring $\mathrm{R}_{\bar{\rho},Q_m}$ parametrizing deformations ρ of $\bar{\rho} = \bar{\rho}_\Pi$ that are

- unramified outside $Q_m \cup T \cup \{p\}$
- crystalline at all primes above p

We also write $\mathrm{R}_{\bar{\rho}}$ if $Q_m = \emptyset$.

Let $\mathrm{R}_{\bar{\rho},Q_m}^{\leq m}$ be its quotient parametrizing deformations that are furthermore

- of inertial level $\leq m$ for all $\mathfrak{q} \in Q_m$, i.e the action of the tame inertia factors through $I_\mathfrak{q}/p^m$. This means that the representation $\rho\big|_{G_{F_\mathfrak{q}}}$, restricted to $\mathbf{F}_\mathfrak{q}^\times$ as before Eqn.(9.1) factors through $\mathbf{F}_\mathfrak{q}^\times/\mathfrak{p}^m$.

We now relate the Galois deformation rings at the level $\mathbb{S}_{0,\Sigma}$ and the Iwahori-Hecke algebra. Consider the universal deformation $\sigma : G_F \to G^\vee(\mathrm{R}_{\bar{\rho}})$ of $\bar{\rho}$. We have

Lemma 9.1.0.1. *[Ven19, Lem 6.12] If $Q_m \in \mathrm{TW}_m$ is a Taylor-Wiles datum and $\mathfrak{q} \in Q_m$, then $\sigma\big|_{G_{F_\mathfrak{q}}}$ can be uniquely conjugated to a representation*

$$G_{F_\mathfrak{q}} \to T^\vee(\mathrm{R}_{\bar{\rho},Q_m})$$

landing in the torus where the image of a fixed uniformizer is $\mathrm{Frob}_\mathfrak{q}^T$.

For convenience, in the following discussion assume that $Q_m = \{\mathfrak{q}\}$ is a singleton. By the above lemma, the restriction of σ to $G_{F_\mathfrak{q}}$ factors as

$$\widehat{F}_\mathfrak{q}^{ab} \to T^\vee(\mathrm{R}_{\bar{\rho},Q_m})$$

mapping the uniformizer to Frob^T. Restrict to $\mathbf{F}_\mathfrak{q} \subseteq F_\mathfrak{q}^\times$, we have

$$\mathbf{F}_\mathfrak{q}^\times \to \mathbf{F}_\mathfrak{q}^\times/\mathfrak{p}^m \to T^\vee(\mathrm{R}_{\bar{\rho},Q_m})$$

Pairing with character $X^*(T^\vee)$, we get a pairing $\mathbf{F}_\mathfrak{q}^\times \times X^*(T^\vee) \to (\mathrm{R}_{\bar{\rho},Q_m})^\times$. Lastly, by duality of $T^\vee$ and T, we obtain

$$\mathbf{T}(F_q) \to (\mathrm{R}_{\bar{\rho},Q_m})^\times \tag{9.1}$$

By our assumption about the existence of Galois representation we have a map

$$\mathrm{R}_{\bar{\rho},Q_m} \to \mathbb{T}_{K_\Delta(q),\mathfrak{m}}. \tag{9.2}$$

This produces an action of $\mathrm{R}_{\bar{\rho},Q_m}$ on $H^*(\mathbb{S}_{0,\Sigma}(q),\mathcal{W})_\mathfrak{m}$ and on its summand $H^*(\mathbb{S}_{0,\Sigma}(q),\mathcal{W})_{\mathfrak{m},\mathrm{Frob}^T}$.

In light of (9.1) this produces a $\mathbf{T}(\mathbf{F}_{\mathfrak{q}})$-action on the cohomology groups:

$$\mathbf{T}(\mathbf{F}_{\mathfrak{q}}) \to \mathrm{R}_{\bar{\rho},Q_m}^{\times} \to \mathrm{End}\left(H^*(\mathbb{S}_{0,\Sigma}(q), \mathcal{W})_{\mathfrak{m},\mathrm{Frob}^T}\right) \tag{9.3}$$

Later we will see that this action actually factors through $\mathrm{R}_{\bar{\rho},Q_m}^{\leq m}$.

Assumption 3 (Local-global compatibility)**.** *The above $\mathbf{T}(\mathbf{F}_{\mathfrak{q}})$-action coincides with the action coming from the Shimura covers, i.e. the deck transformations.*

9.2 Limit via patching

Pick a sequence of Taylor-Wiles data $\{Q_m\}$ with $m \to \infty$. After possibly passing to a subsequence, we assume that we have the following data (see [GV17, §13] for more details):

a) A sequence of Taylor-Wiles data Q_m of level m. This follows from Cheboratev density theorem and the big image assumption.

b) Let $\mathrm{S}_\infty = \mathcal{O}_{\mathfrak{p}}[\![x_1, \dots, x_R]\!]$ and denote $I_\infty = (x_1, \dots, x_R)$. Assume there is a perfect complex $\mathrm{C}_\infty^\bullet$ of finite free S_∞-modules concentrated in $[-(q_0 + l_0), -q_0]$ with

$$q_0 = \min_{\psi} q_{\lambda,\psi},$$

where $q_{\lambda,\psi}$ is as in Thm 1.5.0.6, l_0 as in Def 1.5.0.8, and λ is the Harish-Chandra parameter of Π. The complex $\mathrm{C}_\infty^\bullet$ furthermore satisfies

$$\mathrm{C}_\infty^\bullet \otimes_{\mathrm{S}_\infty}^{\mathbf{L}} S_{Q_m} \simeq \tilde{\mathrm{C}}_m^\bullet,$$

where $\tilde{\mathrm{C}}_m^\bullet$ is furthermore refined to compute the the localization both at $\mathfrak{m}$ and at the choice of

Frob^T in the notation of §7.6. Lastly, $\mathrm{C}_\infty^\bullet$ is quasi-isomorphic to $H_q(\mathrm{C}_\infty^\bullet)$ with the latter free over R_∞.

c) We have deformation rings

$$\mathrm{R}_{\bar\rho} \to \mathrm{End}_{D(\mathcal{O}_\mathfrak{p})}(\mathrm{C}_\infty^\bullet)$$

and a quotient

$$\overline{\mathrm{R}_{\bar\rho,Q_m}^{\leq m}} = \mathrm{R}_{\bar\rho,Q_m}^{\leq m}/(\mathfrak{p}^m, \mathfrak{m}^{k(m)})$$

There is a map $S_{Q_m} \to \mathrm{R}_{\bar\rho,Q_m}$ and $k(m) \geq 2m$ chosen so that $\mathrm{R}_{\bar\rho,Q_m} \circlearrowleft H_*(C_m)$ factors through $\overline{\mathrm{R}_{\bar\rho,Q_m}^{\leq m}}$.

d) A *limit deformation ring* $\mathrm{R}_\infty \simeq \mathcal{O}[\![x_1, \ldots, x_{R-l_0}]\!]$ equipped with maps $\mathrm{S}_\infty \to \mathrm{R}_\infty$, $\mathrm{R}_\infty \to \mathrm{R}_{\bar\rho,Q_m}$, and $\mathrm{R}_\infty \twoheadrightarrow R_\rho$ which are compatible, so that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{S}_\infty & \longrightarrow & \mathrm{R}_\infty & \longrightarrow & R_{\overline{\rho}} \\ \downarrow & & \downarrow & & \downarrow \\ S_{Q_m} & \longrightarrow & \overline{\mathrm{R}_{\bar\rho,Q_m}^{\leq m}} & \longrightarrow & R_{\overline{\rho}}/(\mathfrak{p}^m, \mathfrak{m}^{k(m)}) \end{array} \tag{9.4}$$

with $\mathrm{S}_\infty \to \mathrm{R}_\infty \to R_{\overline{\rho}}$ factoring through augmentation $\mathrm{S}_\infty \to \mathcal{O}$ and left square inducing $\mathrm{R}_\infty \otimes^{\mathbf{L}}_{\mathrm{S}_\infty} S_{Q_m} \simeq \mathrm{R}_{\bar\rho,Q_m}$. By the no congruence assumption, the map $S_\infty \twoheadrightarrow \mathrm{R}_\infty$ is surjective. By formal smoothness, we have $\mathrm{R}_\infty \twoheadrightarrow \mathrm{R}_{\bar\rho,Q_m}$,

e) An action of R_∞ on $H_*(\mathrm{C}_\infty^\bullet)$ compatible with the S_∞-action, which is also compatible via $\mathrm{R}_\infty \twoheadrightarrow \mathrm{R}_{\bar\rho,Q_m}$ with the maps $H_*(\mathrm{C}_\infty^\bullet) \to H_*(\tilde{\mathrm{C}}_m^\bullet)$, i.e a diagram

$$\begin{array}{ccc} \mathrm{R}_{\bar\rho,Q_m} & \longrightarrow & \mathrm{End}_{D(S_{Q_m})}(\tilde{\mathrm{C}}_m^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{R}_{\bar\rho,Q_m}/p^m & \longrightarrow & \mathrm{End}_{D(\mathcal{O}/p^m)}(\mathrm{C}_\infty^\bullet/\mathfrak{p}^m) \end{array}$$

9.3 Derived Hecke operators as limits

Under all of the above assumptions, we show that the derived Hecke algebra is rich enough to account for the degree spread.

Theorem 9.3.0.1. *Assume all the assumptions of §8.1 and §8.2. The cohomology* $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W})_{\mathfrak{m}}$ *is generated over the global Hecke algebra* $\tilde{\mathbb{T}}$ *of §7.7 by the lowest degree component* $H^q(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W})_{\mathfrak{m}}$.

Proof. The argument is identical to that in [Ven19, Thm 7.6], which we reproduce below due to its importance. By the definition of the global derived Hecke algebra $\tilde{\mathbb{T}}$, it is sufficient to prove the statement for the reductions $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}}$ with $m \geq 1$.

The proof consists of three parts:

i) establishing that the map

$$H^q(\mathrm{Hom}_{\mathrm{S}_\infty}(\mathrm{C}^\bullet_\infty, \mathcal{O}_\mathfrak{p})) \otimes^{\mathbf{L}} \mathrm{Ext}^j_{\mathrm{S}_\infty}(\mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}) \twoheadrightarrow H^{q+j}(\mathrm{Hom}_{\mathrm{S}_\infty}(\mathrm{C}^\bullet_\infty, \mathcal{O}_\mathfrak{p})) \tag{9.5}$$

is a surjection (compare with eqn (8.4)).

ii) Establish the surjection

$$H^q(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W})_{\mathfrak{m},\chi'} \times H^*(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m) \twoheadrightarrow H^*(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W})_{\mathfrak{m},\chi'}, \tag{9.6}$$

where χ' is a character coming from the U_q-eigen-decomposition.

iii) use the derived Iwahori-Hecke algebras to pass between $K_\emptyset$ and the $K_0(Q_n)$ levels, and finally passing to a limit.

Note that

$$H^q(\mathrm{Hom}_{S_\infty}(\mathrm{C}^\bullet_\infty, \mathcal{O}_\mathfrak{p})) = \mathrm{Hom}_{D(S_\infty)}(\mathrm{Hom}_{S_\infty}(\mathrm{C}^\bullet_\infty, \mathcal{O}_\mathfrak{p}[q]))$$

and

$$\mathrm{Ext}^j(\mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}) = \mathrm{Hom}_{D(S_\infty)}(\mathcal{O}_\mathfrak{p}[k], \mathcal{O}_\mathfrak{p}[q+j])$$

and thus the map (9.5) is the natural one. By the formal smoothness assumption (4) of §8.2, we may write $\mathrm{R}_\infty = \mathcal{O}_\mathfrak{p}[\![x_1, \ldots, x_{R-l_0}]\!]$. We also have the following

Lemma 9.3.0.2. *[Ven19, Lem. 7.5] There are generators x_i and y_j of S_∞ and R_∞ s.t*

$$\mathrm{S}_\infty = \mathcal{O}_\mathfrak{p}[\![x_1, \ldots, x_R]\!], \quad \mathrm{R}_\infty = \mathcal{O}_\mathfrak{p}[\![x_1, \ldots, x_{R-l_0}]\!]$$

such that $x_i \mapsto y_i$ for $i = 1, \ldots, R - l_0$, and $x_i \mapsto 0$ for $i > R - l_0$.

Lastly, $\mathrm{C}_\infty^\bullet$ is quasi-isomorphic to $\mathrm{S}_\infty/(x_R, \ldots, x_{R-l_0+1})$ concentrated in a single degree by b) of §9.2. The proof of i) now follows by the Koszul resolution calculation in [Ven19, Appendix B].

At level m, we have the diagram

$$\begin{array}{ccccc}
H^q\left(\mathrm{Hom}_{\mathrm{S}_\infty}(\mathrm{C}_\infty^\bullet, \mathcal{O}_\mathfrak{p})\right) & \times & \mathrm{Ext}^j_{\mathrm{S}_\infty}(\mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}) & \longrightarrow & H^{q+j}\left(\mathrm{Hom}_{\mathrm{S}_\infty}(\mathrm{C}_\infty^\bullet, \mathcal{O}_\mathfrak{p})\right) \\
\| & & \downarrow & & \twoheaddownarrow{\scriptstyle (\star\star\star)} \\
H^q\left(\mathrm{Hom}_{\mathrm{S}_\infty}(\mathrm{C}_\infty^\bullet, \mathcal{O}_\mathfrak{p})\right) & \times & \mathrm{Ext}^j_{\mathrm{S}_\infty}(\mathcal{O}_\mathfrak{p}, \mathcal{O}/\mathfrak{p}^m) & \longrightarrow & H^{q+j}\left(\mathrm{Hom}_{\mathrm{S}_\infty}(\mathrm{C}_\infty^\bullet, \mathcal{O}_\mathfrak{p}/\mathfrak{p}^m)\right) \\
{\scriptstyle (\star)}\downarrow & & \twoheaduparrow{\scriptstyle (\star\star)} & & \uparrow\simeq \\
H^q\left(\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\right) & \times & \mathrm{Ext}^j_{S_{Q_m}}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) & \longrightarrow & H^{q+j}\left(\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\right)
\end{array}$$

where the maps between the second and third row come under the identification $\mathrm{C}_\infty^\bullet \otimes^{\mathbf{L}} S_{Q_m} \simeq \tilde{\mathrm{C}}_m^\bullet$. This diagram is commutative in the sense that $(\star)$ and $(\star\star)$ are adjoint. The map $(\star\star)$ is surjective by Lemma 8.4.0.1, and $(\star\star\star)$ is surjective by the no-torsion assumption (7) of §8.2. The bottom row is now surjective by tracing the diagram

Note the complexes of §7.6 may be augmented to incorporate the U_q-eigen-decomposition (as is shown in §7.2 of [CG18]). The surjectivity of the bottom row implies $H^*\left(\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}_m^\bullet, \mathcal{O}/\mathfrak{p}^m)\right)$

is generated by $H^q\left(\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}^{\bullet}_m, \mathcal{O}/\mathfrak{p}^m)\right)$ over $\mathrm{Ext}^*_{S_{Q_m}}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m)$. Since

$$\mathrm{Ext}^*_{S_{Q_m}}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) = H^*(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m),$$

part ii) follows by Assumption 1.

Now, we have

- $H^{q+j}(\mathrm{Hom}_{S_{Q_m}}(\tilde{\mathrm{C}}^{\bullet}_m, \mathcal{O}/\mathfrak{p}^m)) \simeq H^{q+j}(\mathbb{S}_{0,\Sigma}(Q_m), \mathcal{W}/p^m)_{\mathfrak{m},\chi_{\mathrm{Frob}T}}$
- $\mathrm{Ext}^j_{S_{Q_m}}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) = H^j(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m)$ and the action factors through

$$\bigotimes_{\mathfrak{q}\in Q_m} \mathcal{H}_{I_{\mathfrak{q}}}$$

by local-global compatibility.

We also have three surjective maps

$$H^q(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}} \otimes_{\otimes_{\mathfrak{q}\in Q_m} H_{K_{\mathfrak{q}}}} \bigotimes_{\mathfrak{q}\in Q_m} H_{K_{\mathfrak{q}},I_{\mathfrak{q}}} \twoheadrightarrow H^q(\mathbb{S}_{0,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}}$$
$$H^q(\mathbb{S}_{0,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}} \otimes_{\otimes_{\mathfrak{q}\in Q_m} H_{K_{\mathfrak{q}}}} \bigotimes_{\mathfrak{q}\in Q_m} \mathcal{H}^{(j)}_{I_{\mathfrak{q}}} \twoheadrightarrow H^{q+j}(\mathbb{S}_{0,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}},$$
$$H^{q+j}(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}} \otimes_{\otimes_{\mathfrak{q}\in Q_m} H_{K_{\mathfrak{q}}}} \bigotimes_{\mathfrak{q}\in Q_m} H_{I_{\mathfrak{q}},K_{\mathfrak{q}}} \twoheadrightarrow H^{q+j}(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/p^m)_{\mathfrak{m}}$$

where we note that the second map involves the derived Iwahori-Hecke algebra. The second one is what we just proved whereas the first and the third follow by local-global compatibility and Prop 8.5.0.1. The morphisms

$$H_{K_{\mathfrak{q}},I_{\mathfrak{q}}} \otimes \mathcal{H}^{(j)}_{I_{\mathfrak{q}}} \otimes H_{I_{\mathfrak{q}},K_{\mathfrak{q}}} \to \mathcal{H}_{K_{\mathfrak{q}}}$$

arising from composition of the Ext^* are compatible with the action of the Hecke algebras in light of the description of the map (9.5) above. Hence, $\otimes_{q\in Q_m}\mathcal{H}_{K_\mathfrak{q}}$ acts surjectively on $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/p^m)_\mathfrak{m}$. The claim now follows by passing to the limit. □

Chapter 10: Galois action and Reciprocity

Recall that by Lemma 9.3.0.2 we have

$$\begin{array}{ccccc} \mathrm{S}_\infty & \longrightarrow & \mathrm{R}_\infty & \longrightarrow & \mathcal{O}_\mathfrak{p} \\ \cup & & \cup & & \\ I_\infty & & J_\infty & & \end{array}$$

where I_∞ and J_∞ are augmentation ideals. Set

$$\mathfrak{t}_{\mathrm{S}_\infty} \simeq \mathrm{Hom}_*(\mathrm{S}_\infty, \mathcal{O}_\mathfrak{p}[\varepsilon]/\varepsilon^2) \quad \text{and} \quad \mathfrak{t}_{\mathrm{R}_\infty} \simeq \mathrm{Hom}_*(\mathrm{R}_\infty, \mathcal{O}_\mathfrak{p}[\varepsilon]/\varepsilon^2),$$

where the Hom_* indicates that each homomorphism lifts the natural augmentations $\mathrm{S}_\infty \to \mathcal{O}_\mathfrak{p}$ and $\mathrm{R}_\infty \to \mathcal{O}_\mathfrak{p}$, respectively. The surjection $\mathrm{S}_\infty \twoheadrightarrow \mathrm{R}_\infty$ induces a surjection $I_\infty/I_\infty^2 \twoheadrightarrow J_\infty/J_\infty^2$. Since tangent spaces may be viewed as $\mathcal{O}_\mathfrak{p}$ linear duals of A/A^2, where A is the augmentation ideal, this yields an injection

$$\mathfrak{t}_{\mathrm{R}_\infty} \hookrightarrow \mathfrak{t}_{\mathrm{S}_\infty} \twoheadrightarrow W.$$

with W a free $\mathcal{O}_\mathfrak{p}$-module of dimension $R - (R - l_0) = l_0$.

Recall that S_∞ and R_∞ are obtained via limiting process from smaller rings S_{Q_m} and $\mathrm{R}_{\bar{\rho},Q_m}$, equipped with maps $S_{Q_m} \to \mathrm{R}_{\bar{\rho},Q_m} \to \mathcal{O}/\mathfrak{p}^m$ with compatible augmentations. We now proceed to study the tangent spaces of both sides at finite level.

For convenience, here is a list of all Galois deformation rings:

- $\mathrm{R}_{\bar{\rho}}$ - the crystalline deformation ring

- $\mathrm{R}_{\bar{\rho},Q_m}$ - the deformation ring parameterizing deformations unramified outside $Q_m \cup T \cup \{v : v|p\}$ and crystalline at all primes above p

- $\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}$ - the quotient of $\mathrm{R}_{\bar{\rho},Q_m}$ of Galois deformations of inertial level $\leq m$ for all $\mathfrak{q} \in Q_m$

- $\overline{\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}}$ - the quotient of $\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}/(\mathfrak{p}^m, \mathfrak{m}^{k(m)})$ with $k(m)$ as in §9.2 c).

10.1 Tangent spaces

Let $Q_m \in \mathrm{TW}_m$ be a Taylor-Wiles datum of level m.

10.1.1 The tangent space to S_{Q_m}

Fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$. It this section all Lie groups are over E_Π with integral models over localizations of $\mathcal{O}_{E_\Pi}$. Set $S_{Q_m} = \mathcal{O}/\mathfrak{p}^m[\Delta_{Q_m}]$ to be the group algebra of $\Delta_{Q_m} := \prod_{\mathfrak{q}\in Q_m}(\mathbf{Z}/p^m)^n = \prod_{\mathfrak{q}\in Q_m}\mathbf{T}(\mathbf{F}_\mathfrak{q})/p^m$. By [Ven19, Eqn. (88)], we have a canonical identification

$$\mathfrak{t}_{S_{Q_m}} \simeq \mathrm{Ext}^1_{S_{Q_m}}(\mathcal{O}/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) \simeq H^1(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m), \tag{10.1}$$

where the second identification is by definition. The tangent space $\mathfrak{t}_{S_{Q_m}}$ has also an interpretation in terms of Galois cohomology. Identify $H^1(F_\mathfrak{q}, \mathrm{Ad}\rho_m)$ with the set of lifts of ρ_m to $\mathbf{G}^\vee(\mathcal{O}/\mathfrak{p}^m[\epsilon])$ up to conjugation. By Lemma 9.1.0.1, because of our assumption on $\rho\big|_{F_\mathfrak{q}}$, any deformation can be conjugated to lie in $T^\vee$, and so it factors through $F_\mathfrak{q}^\times$.

Following [Ven19, §8.15], we have (depending on the choice of a fixed conjugate elements of $\bar{\rho}(\mathrm{Frob}_\mathfrak{q})$) an isomorphism

$$\frac{H^1(F_\mathfrak{q}, \mathrm{Ad}\rho_m)}{H^1_{\mathrm{ur}}(F_\mathfrak{q}, \mathrm{Ad}\rho_m)} \simeq \mathrm{Hom}(\mathbf{F}_\mathfrak{q}^\times, \mathrm{Lie}(T^\vee) \otimes \mathcal{O}/\mathfrak{p}^m).$$

Identifying $\mathrm{Lie}(T^\vee)$ with $X_*(T^\vee)$, and using (10.1) and interpreting the tangent space with classes in Galois cohomology up to unramified classes, we have

$$\begin{aligned}\mathfrak{t}_{S_{Q_m}} \simeq H^1(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m) \simeq \mathrm{Hom}(\mathbf{F}_\mathfrak{q}^\times, \mathrm{Lie}(T^\vee) \otimes \mathcal{O}/\mathfrak{p}^m) &\simeq \bigoplus_{\mathfrak{q}\in Q_m} \mathrm{Hom}(\mathbf{F}_\mathfrak{q}^\times/\mathfrak{p}^m, \mathcal{O}/\mathfrak{p}^m) \otimes X_*(T^\vee) \\ &\simeq \bigoplus_{\mathfrak{q}\in Q_m} \frac{H^1(F_\mathfrak{q}, \mathrm{Ad}\rho_m)}{H^1_{\mathrm{ur}}(F_\mathfrak{q}, \mathrm{Ad}\rho_m)},\end{aligned} \tag{10.2}$$

where the last one depends on the TW data (conjugacy classes of the Frobenii elements).

Consider the following diagram

$$\begin{array}{ccccc} \mathfrak{t}_{\mathrm{R}_{\bar\rho,Q_m}} & \longrightarrow & \mathfrak{t}_{S_{Q_m}} & \longrightarrow & \mathfrak{t}_{S_{Q_m}}/\mathfrak{t}_{\mathrm{R}_{\bar\rho,Q_m}} \\ \downarrow \simeq & & \downarrow \simeq & & \| \\ H^1_f(\mathcal{O}_F[\frac{1}{S\cup Q_m}], \mathrm{Ad}\ \rho_m) & \xrightarrow{\psi} & \bigotimes_{\mathfrak{q}\in Q_m} \frac{H^1(F_\mathfrak{q}, \mathrm{Ad}\ \rho_m)}{H^1_{ur}(F_\mathfrak{q}, \mathrm{Ad}\ \rho_m)} & \longrightarrow & \mathfrak{t}_{S_{Q_m}}/\mathfrak{t}_{\mathrm{R}_{\bar\rho,Q_m}} \end{array}$$

with ψ being the restriction map in Galois cohomology. The middle vertical isomorphism is (10.2), and the first is the computation of the tangent space to a deformation ring. This gives raise to a pairing

$$H^1_f(\mathcal{O}_F[\frac{1}{S\cup Q_m}], \mathrm{Ad}^*\rho_m(1)) \times \mathfrak{t}_{S_{Q_m}}/\mathfrak{t}_{\mathrm{R}_{\bar\rho,Q_m}} \to \mathcal{O}/\mathfrak{p}^m \tag{10.3}$$

given by

$$(\alpha, (\beta_\mathfrak{q})_{\mathfrak{q}\in Q_m}) \mapsto \sum_\mathfrak{q} (\mathrm{res}_\mathfrak{q}(\alpha), \beta_\mathfrak{q})_{\mathrm{Tate}},$$

where $\mathrm{res}_\mathfrak{q} : H^1_f(\mathcal{O}_F[\frac{1}{S}], \mathrm{Ad}^*\rho(1)) \to H^1(F_\mathfrak{q}, \mathrm{Ad}^*\rho(1))$ is the restriction map. The local pairing is just the cup product pairing

$$H^1(F_{\mathfrak{q}}, \mathrm{Ad}^* \rho_m(1)) \times H^1(F_{\mathfrak{q}}, \mathrm{Ad}\, \rho_m) \to H^2(\mu_{p^m}) = \mathcal{O}/\mathfrak{p}^m$$

Since the classes $\mathrm{res}_{\mathfrak{q}}(\alpha)$ are unramified, they pair trivially with

$$H^1_{ur}(F_{\mathfrak{q}}, \mathrm{Ad}\, \rho_m),$$

and thus the pairing is indeed well-defined. One of the major results of [Ven19, §8] is that under all the assumption of §8 and §9 the pairing (10.3) is perfect for a Taylor-Wiles datum Q_m, and thus we get identification

$$\mathfrak{t}_{S_{Q_m}}/\mathfrak{t}_{\mathrm{R}_{\bar{\rho},Q_m}} \simeq H^1_f(\mathcal{O}_F[\frac{1}{S \cup Q_m}], \mathrm{Ad}^*\rho_m(1))^\vee. \tag{10.4}$$

10.1.2 The tangent space to $\overline{\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}}$

Fix $Q_m \in \mathrm{TW}_m$. Recall from §9.1, that $\overline{\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}}$ is a quotient of the crystalline deformation ring $\mathrm{R}_{\bar{\rho},Q_m}$. Passing to a quotient does not lose salient information from the perspective of tangent spaces. Later on, we will see that this implies that the action (9.3) factors through $\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}$. By [Ven19, Lemma 8.14], we have an isomorphism of tangent spaces

$$\mathfrak{t}_{\mathrm{R}_{Q_m}} \simeq \mathfrak{t}_{\overline{\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}}}.$$

induced from the map $\mathrm{R}_{\bar{\rho},Q_m} \to \mathrm{R}_{\bar{\rho},Q_m}$. The key argument is to note that $T^\vee(\mathcal{O}/\mathfrak{p}^m)$ has exponent p^m, so any deformation has conductor on inertia $\leq m$. In light of this identification, we will write $\mathfrak{t}_{R_m}$ in place of either of these tangent spaces.

Of interest for us are the tangent spaces at the limits. Set

$$W_m = \mathrm{coker}(\mathfrak{t}_{R_m} \to \mathfrak{t}_{S_{Q_m}}),$$

which by (10.4), may be identified with the dual to the Selmer group. We show the following isomorphism of short exact sequences as [Ven19, Eqn(134)]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{t}_{R_m} & \longrightarrow & \mathfrak{t}_{S_{Q_m}} & \longrightarrow & W_m & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & \mathfrak{t}_{\mathrm{R}_{\bar{\rho}}}/p^m & \longrightarrow & \mathfrak{t}_{\mathrm{S}_\infty}/p^m & \longrightarrow & W/p^m & \longrightarrow & 0 \end{array}$$

The projections $\mathrm{S}_\infty \to S_{Q_m}$ and $\mathrm{R}_\infty \to \overline{\mathrm{R}^{\leq m}_{\bar{\rho},Q_m}}$ induce maps

$$\alpha : \mathfrak{t}_{S_{Q_m}} \to \mathfrak{t}_{\mathrm{S}_\infty}/\mathfrak{p}^m, \quad \beta : \mathfrak{t}_{R_m} \to \mathfrak{t}_{\mathrm{R}_\infty}/\mathfrak{p}^m$$

in light of the isomorphism

$$\mathfrak{t}_{\mathrm{S}_\infty}/\mathfrak{p}^m \simeq \mathrm{Hom}_*(\mathrm{S}_\infty, \mathcal{O}/\mathfrak{p}^m[\varepsilon]/\varepsilon^2), \quad \mathfrak{t}_{\mathrm{R}_\infty}/\mathfrak{p}^m \simeq \mathrm{Hom}_*(\mathrm{R}_\infty, \mathcal{O}/\mathfrak{p}^m[\varepsilon]/\varepsilon^2).$$

The first two vertical maps can then be seen as isomorphisms by direct computation using the coordinate representation of the coordinate rings R_m and S_{Q_m}, and the third one is induced on cokernels.

10.2 Reduction maps on Galois cohomology

10.2.1 Unramified classes

Let F_S/F be the largest unramified away from S extension of F. Denote by Γ_{F_S} its Galois group. Any Γ_{F_S}-module M can be viewed as an étale sheaf on $\mathrm{Spec}(\mathcal{O}_F[\frac{1}{S}])$, and write $H^1(\mathcal{O}_F[\frac{1}{S}], M)$ for the first group cohomology of M as Γ_{F_S}-module. We restrict further to the subspace

$$H^1(\mathcal{O}_F[\frac{1}{S}], M) \supset H^1_f(\mathcal{O}_F[\frac{1}{S}], M)$$

of classes that are crystalline at p, i.e. crystalline at all primes above p. These are the only local conditions we impose. Even though in general we have $H^1_f(F, M) \subseteq H^1_f(\mathcal{O}_F[\frac{1}{S}], M)$, for our Galois modules M, we would have

$$H^1(F_v, M) = 0 \text{ for } v \in T,$$

and thus $H^1_f(F, M) = H^1_f(\mathcal{O}_F[\frac{1}{S}], M)$.

10.2.2 Dual Selmer group

Let $E = E_\Pi$ be the field of definition of Π and let $\mathcal{O} = \mathcal{O}_{E_\Pi}$ the corresponding ring of integers. In the notation of §8.1, let $\rho : \Gamma_F \to \mathbf{G}^\vee(\mathcal{O}_\mathfrak{p})$ be the Galois representation coming from $\mathfrak{m}$ associated to Π at level $K_\emptyset$. Set

$$\mathrm{V} := H^1_f(\mathcal{O}_F[\frac{1}{S}], \mathrm{Ad}^* \rho_\Pi(1))^\vee, \tag{10.5}$$

where $-^\vee$ stands for the $\mathcal{O}_{E_\Pi}$-dual. Note again that $\mathrm{Ad}\rho$ descends to a $\Gamma_{\mathbf{Q}}$-representation even though ρ is a Γ_F-representation.

10.2.3 Explication of Galois action

Given the datum

- $\mathfrak{q}$ - Taylor-Wiles prime of level m with a choice of a strongly regular $\bar{\rho}(\mathrm{Frob}_{\mathfrak{q}})$
- a conjugacy class $[\gamma] \in K_{\mathfrak{q}} \backslash \mathbf{G}(F_{\mathfrak{q}})/K_{\mathfrak{q}}$ with K_q - hyperspecial;
- a class $\alpha \in H^1(\mathbf{T}(\mathbf{F}_{\mathfrak{q}}), \mathcal{O}/\mathfrak{p}^m)$.

we produce a map

$$[\mathfrak{q}, \gamma, \alpha] : H^1_f(\mathcal{O}_F[\frac{1}{S}], \mathrm{Ad}^*\rho(1)) \to \mathcal{O}/\mathfrak{p}^m.$$

Its significance lies in the fact that it is expected to recover the action of the derived Hecke operator $T_{\mathfrak{q},\gamma,\alpha}$ on the (localized) cohomology of $\mathcal{W}$.

For the prime $\mathfrak{q}$ of F with associated Frobenius element $\mathrm{Frob}_{\mathfrak{q}}$, let

$$\mathrm{Lie}(T^\vee) \text{ with trivial } \mathrm{Gal}(\overline{F_{\mathfrak{q}}}/F_{\mathfrak{q}}) \text{ action} \xhookrightarrow{\mathfrak{e}_{\nu_\gamma,\rho(\mathrm{Frob}_{\mathfrak{q}})}} \mathrm{Ad}\rho\big|_{\mathrm{Gal}(\overline{F_{\mathfrak{q}}}/F_{\mathfrak{q}})} \subseteq \mathbf{G}^\vee, \tag{10.6}$$

where $\mathfrak{e}_{\nu_\gamma,\rho(\mathrm{Frob}_{\mathfrak{q}})}$ is the averaging map inside $\mathrm{Lie}(T^\vee)$ given by

$$\sum_{\phi: T^\vee \xrightarrow{\sim} Z_{\rho(\mathrm{Frob}_{\mathfrak{q}})} = T^\vee} \langle \nu_\gamma, \phi^{-1}(\rho(\mathrm{Frob}_{\mathfrak{q}})) \rangle \cdot d\phi, \tag{10.7}$$

where the sum is over all conjugations from $T^\vee$ to the centralizer $Z_{\rho(\mathrm{Frob}_{\mathfrak{q}})} = T^\vee$ of the Frobenius $\rho(\mathrm{Frob}_{\mathfrak{q}})$, and ν_γ is the cocharacter corresponding to the conjugacy class of γ. Note that by design the obtained element is indeed $\rho(\mathrm{Frob}_{\mathfrak{q}})$-invariant. Similarly, we may obtain

$$X_*(T^\vee)/\mathfrak{p}^m \hookrightarrow \mathrm{Ad}\rho_m\big|_{\mathrm{Gal}(\overline{F_{\mathfrak{q}}}/F_{\mathfrak{q}})}.$$

Identify

$$H^1(\mathbf{T}(\mathbf{F}_{\mathfrak{q}}), \mathcal{O}/\mathfrak{p}^m) = \mathrm{Hom}(\mathbf{F}_{\mathfrak{q}}^{\times}, X_*(T^\vee)/\mathfrak{p}^m).$$

Again, identifying $X_*(T^\vee)$ and $\mathrm{Lie}(T^\vee)$, we get

$$\tilde{\alpha} \in \frac{H^1(F_{\mathfrak{q}}, \mathrm{Lie}(T^\vee)/p^m)}{H^1_{\mathsf{ur}}(F_{\mathfrak{q}}, \mathrm{Lie}(T^\vee)/p^m)} \xrightarrow{(10.6)} \bar{\alpha}^\gamma \in \frac{H^1(F_{\mathfrak{q}}, \mathrm{Ad}\rho_m)}{H^1_{\mathsf{ur}}(F_{\mathfrak{q}}, \mathrm{Ad}\rho_m)}$$

Lastly, by Tate duality we produce a map

$$\begin{aligned} [\mathfrak{q}, \gamma, \alpha] : H^1_f(\mathcal{O}_F[\frac{1}{S}], \mathrm{Ad}^*\rho(1)) &\to \mathcal{O}/\mathfrak{p}^m \\ \beta &\mapsto \langle \bar{\alpha}^\gamma, \mathrm{res}_{\mathfrak{q}}(\beta) \rangle_{\text{Tate}}. \end{aligned} \tag{10.8}$$

10.3 Reciprocity law

Let $z \in \mathcal{O}/\mathfrak{p}^m\langle -1 \rangle \coloneqq \mathrm{Hom}(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m)$. Recall that Δ_{Q_m} is the maximal p-power quotient of $\prod_{\mathfrak{q} \in Q_m} \mathbf{T}(\mathbf{F}_{\mathfrak{q}})$. The composition

$$\prod_{\mathfrak{q} \in Q_m} \mathbf{T}(\mathbf{F}_{\mathfrak{q}}) \xrightarrow{\log_p} \Delta_{Q_m} \xrightarrow{z} \mathcal{O}/\mathfrak{p}^m$$

with $\log_p$ as in (5.1) produces a class $\alpha_z \in \mathrm{Hom}(\mathbf{T}(\mathbf{F}_{\mathfrak{q}}), \mathcal{O}/\mathfrak{p}^m) = H^1(\mathbf{T}(\mathbf{F}_{\mathfrak{q}}), \mathcal{O}/\mathfrak{p}^m)$. For a fixed Taylor-Wiles prime q of level m (with choice of toral conjugate to the Frobenius), and $[\gamma] \in K\backslash G/K$, we combine the above map with (10.8) to produce

$$\begin{aligned} \varsigma_{\mathfrak{q}}^{\gamma} : \mathcal{O}/\mathfrak{p}^m\langle -1 \rangle &\to \mathrm{V}/\mathfrak{p}^m \\ z &\mapsto [\mathfrak{q}, \gamma, \alpha_z] \end{aligned} \tag{10.9}$$

10.4 The $\mathrm{V}/\mathfrak{p}^m$ action on coherent

Let $\mathfrak{q}$ be a Taylor-Wiles prime of level m, so that it comes with associated $\mathrm{Frob}_{\mathfrak{q}} \in \mathbf{T}^\vee(\mathbf{F}_\mathfrak{p})$. There is an action of $\mathrm{V}/\mathfrak{p}^m$ coming from $\theta : H^1(\Delta_\mathfrak{q}, \mathcal{O}/\mathfrak{p}^m) \to \mathcal{H}^{(1)}_{\mathfrak{q},\mathcal{O}/\mathfrak{p}^m}$, such that for $\alpha \in H^1(\mathbf{T}(\mathbf{F}_\mathfrak{q}), \mathcal{O}/\mathfrak{p}^m)$, it is characterized by

$$\text{Pullback to } \mathbb{S}_{K_0(\mathfrak{q}),\Sigma}\text{, project to } \mathrm{Frob}_\mathfrak{q}\text{-eigenspace, cup with } \alpha\text{, pushdown to } \mathbb{S}_{0,\Sigma} \tag{10.10}$$

Let us now explicate θ. By the derived Satake isomorphism (see Thm 7.5.0.1), we have

$$\mathcal{H}^{(1)}_{\mathfrak{q},\mathcal{O}/\mathfrak{p}^m} \xrightarrow{\sim} \left(\mathcal{O}/\mathfrak{p}^m[X_*] \otimes H^1(\Delta_\mathfrak{q}, \mathcal{O}/\mathfrak{p}^m)\right)^{W_\mathbf{G}},$$

where $W_\mathbf{G}$ is the Weyl group. Then $\mathrm{Frob}_\mathfrak{q} \in \mathbf{T}^\vee(\mathbf{F}_\mathfrak{p})$ yields a character $\chi_{\mathrm{Frob}_\mathfrak{q}} : \mathcal{O}/\mathfrak{p}^m[X_*] \to \mathbf{F}_\mathfrak{p}$ such that the kernel of

$$\mathcal{O}/\mathfrak{p}^m[X_*]^{W_\mathbf{G}} \to \mathcal{O}/\mathfrak{p}^m[X_*] \to \mathbf{F}_\mathfrak{p}$$

is $\mathfrak{m}$. Let $\tilde{\mathfrak{m}}$ be the extension of $\mathfrak{m}$ to $\mathcal{O}/\mathfrak{p}^m[X_*]$ (no longer necessarily maximal - it cuts out $\mathrm{Frob}_\mathfrak{q}$ and its Weyl conjugates). We have the identification

$$\mathcal{O}/\mathfrak{p}^m[X_*]_{\tilde{\mathfrak{m}}} \simeq \bigoplus_{w\in W} \mathcal{O}/\mathfrak{p}^m[X_*]_{w\chi}$$

of completions, where the completions of the right hand side are taken with respect to the maximal ideals corresponding to the kernels of the characters. The following composition is an isomorphism

$$\begin{aligned}\left(\mathcal{O}/\mathfrak{p}^m[X_*] \otimes H^1(\Delta_\mathfrak{q}, \mathcal{O}/\mathfrak{p}^m)\right)^{W_\mathbf{G}}_\mathfrak{m} &\to \mathcal{O}/\mathfrak{p}^m[X_*]_{\tilde{\mathfrak{m}}} \otimes H^1(\Delta_\mathfrak{q}, \mathcal{O}/\mathfrak{p}^m)\\ &\to \mathcal{O}/\mathfrak{p}^m[X_*]_\chi \otimes H^1(\Delta_\mathfrak{q}, \mathcal{O}/\mathfrak{p}^m)\end{aligned}$$

and the first map is the inclusion of Weyl-invariants. Composing with the Satake isomorphism, we obtain an isomorphism

$$\left(\mathcal{H}^{(1)}_{\mathfrak{q},\mathcal{O}/\mathfrak{p}^m}\right)_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}/\mathfrak{p}^m[X_*]_\chi \otimes H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m) \tag{10.11}$$

and then θ is given by

$$H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m) \ni h \mapsto 1 \otimes h \in \mathcal{O}/\mathfrak{p}^m[X_*]_\chi \otimes H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m) \to \left(\mathcal{H}^{(1)}_{\mathfrak{q},\mathcal{O}/\mathfrak{p}^m}\right)_{\mathfrak{m}},$$

where the last map is the inverse of (10.11). Note that the construction depends on the choice of $\mathrm{Frob}_{\mathfrak{q}}$; changing to $w\mathrm{Frob}_{\mathfrak{q}}$ with by a Weyl element w affects the construction by the natural action of $W_{\mathbf{G}}$ on $H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m)$. For justification why θ recovers (10.10), see [Ven19, pp. 91-92].

10.4.1 Explicating the action of $\mathrm{V}/\mathfrak{p}^m$ in degree 1

To produce an action of $\mathrm{V}/\mathfrak{p}^m$ we produce an action of $\mathfrak{t}_{S_{Q_m}}$, and show that this action factors through a quotient isomorphic to $\mathrm{V}/\mathfrak{p}^m$. In order for these actions to "patch" nicely to a characteristic zero action, we need assume a few more properties of the Taylor-Wiles data.

Definition 10.4.1.1. *A sequence of Taylor-Wiles data Q_m of level m is* ***convergent*** *if*

- *The map $H^1(\mathbf{Z}[\frac{1}{S\cup Q_m}], \mathrm{Ad}\,\bar\rho) \to \frac{H^1(F_p, \mathrm{Ad}\,\bar\rho)}{H^1_f(F_p, \mathrm{Ad}\,\bar\rho)}$ is surjective;*
- *The map $H^2(\mathbf{Z}[\frac{1}{S\cup Q_m}], \mathrm{Ad}\,\bar\rho) \to \prod_{q \text{ above } \mathfrak{q}\in Q_m} H^2(\mathbf{Q}_p, \mathrm{Ad}\,\bar\rho)$ is injective;*
- *one can pass to limit, i.e. all the assumption in §9 are met.*

Definition 10.4.1.2. *A Taylor-Wiles datum Q_m of level m is* ***strict*** *if the map*

$$\mathfrak{t}_{S_{Q_m}} \xrightarrow{\iota_{Q_m}} \mathrm{End} H^\bullet(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m)$$

factors through f_{Q_m}. Hence, a strict Taylor-Wiles datum of level m gives rise to an action of $\mathrm{V}/\mathfrak{p}^m$ on $H^\bullet(\Delta_\mathfrak{q}, \mathcal{O}/\mathfrak{p}^m)$.

Let $\{Q_m\}$ be a convergent sequence Taylor-Wiles data. We have an embedding

$$\iota_{Q_m} : \mathfrak{t}_{S_{Q_m}} \simeq H^1(\Delta_{Q_m}, \mathcal{O}/\mathfrak{p}^m) \hookrightarrow \big(\otimes_{\mathfrak{q}\in Q_m} (\mathcal{H}_{\mathfrak{q},\mathcal{O}/\mathfrak{p}^m})_\mathfrak{m}\big)^{(1)}, \tag{10.12}$$

by the identification of $\mathfrak{t}_{S_{Q_m}}$ with Ext^1. This yields a degree +1 action on the cohomology $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/\mathfrak{p}^m)_\mathfrak{m}$. It is worth iterating again that the embedding depends on the choice of $\mathrm{Frob}_\mathfrak{q}$ for each $\mathfrak{q} \in Q_m$, and thus so does the action of $\mathfrak{t}_{S_{Q_m}}$. Changing $\mathrm{Frob}_\mathfrak{q} \to w_\mathfrak{q}\mathrm{Frob}_\mathfrak{q}$ amends it by the natural action of $w_\mathfrak{q}$. For a thorough investigation of all dependencies of all choices, see [Ven19, §8.24].

The following key lemma allows to relate, at least for convergent Taylor-Wiles primes, the action of $\mathfrak{t}_{S_{Q_m}}$ to the one by $\mathrm{V}/\mathfrak{p}^m$.

Lemma 10.4.1.3. *[Ven19, Lem 8.20] Let $\{Q_m\}$ be a convergent sequence of Taylor-Wiles data. Then for each m the action of $\mathfrak{t}_{S_{Q_m}}$ on $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/\mathfrak{p}^m)_\mathfrak{m}$ is trivial on the image of $\mathfrak{t}_{R_m}$, and thus factors through $\mathfrak{t}_{S_{Q_m}} \to W_m \simeq \mathrm{V}/\mathfrak{p}^m$*

Note now that the actions furnished by ι_{Q_m} and f_{Q_m} depend solely on Q_m, and not on the limiting process. Thus we have

Lemma 10.4.1.4. *Let $\{Q_m\}$ be a convergent sequence of Taylor-Wiles data. Then, for each m, the datum Q_m is strict.*

Theorem 10.4.1.5. *Let us be in the setup of §8 with all assumptions and notation therein. Let V be as given in (10.5). Then there exists a sequence $\{a_k\}_{k\geq 1}$ of positive integers and an action of V on $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W})_\mathfrak{m}$ such that*

- *For any $k \geq 1$ and any $\mathfrak{q}$ part of a Taylor-Wiles datum of level a_k, the two actions of $H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m)$ on obtained by*

  $$\iota_{\mathfrak{q}} : H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m) \to \mathrm{V}/\mathfrak{p}^m$$
  $$f_{\mathfrak{q}} : H^1(\Delta_{\mathfrak{q}}, \mathcal{O}/\mathfrak{p}^m) \to \mathcal{H}^{(1)}_{q,\mathcal{O}/\mathfrak{p}^m}$$

  coincide.

Furthermore, this uniquely determines the action of $\mathrm{V}/\mathfrak{p}^m$ at Taylor-Wiles primes.

Proof. The uniqueness of the action follows from Cheboratev density theorem. The rest of the argument follows that preceding [Ven19, §8.26, (†)].

The action of V is obtained by a limiting process from the actions of $\mathrm{V}/\mathfrak{p}^m$ for a choice of convergent Taylor-Wiles sequence $\{Q_m\}$. One problem is to deal with choices. These come in two flavors:

- dependency on the choice of $\mathrm{Frob}_{\mathfrak{q}}$. These depend on a choice of Weyl element and we have the following diagram

$$\begin{array}{ccccc} \mathfrak{t}_{S_{Q_m}} & \times & H^1_f(\mathcal{O}_F[\frac{1}{S}], \mathrm{Ad}^*\rho(1)) & \xrightarrow{\mathrm{Frob}_{\mathfrak{q}}} & \mathcal{O}/\mathfrak{p}^m \\ \downarrow w & & \downarrow = & & \\ \mathfrak{t}_{S_{Q_m}} & \times & H^1_f(\mathcal{O}_F[\frac{1}{S}], \mathrm{Ad}^*\rho(1)) & \xrightarrow{w\mathrm{Frob}_{\mathfrak{q}}} & \mathcal{O}/\mathfrak{p}^m \end{array}$$

- dependency on the choice of Taylor-Wiles data. This is dealt with in [Ven19, Lem.8.24] and is worth pointing out that it hinges on the assumption that $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W})$ is free over the p-adic ring $\mathcal{O}$. It's unclear how to deal in the presence of torsion.

Now, we are left to show the coincidence of the two actions. We start with a sequence $\{Q_m\}$ of Taylor-Wiles data, and extract a convergent subsequence $\{Q_{m_k}\}$. Possibly passing to a subsequence, we may assume that the action of V on $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/\mathfrak{p}^k)_{\mathfrak{m}}$ is eventually constant for a fixed k. The coincidence of the actions now follows from Lem. 10.4.1.4. □

Corollary 10.4.1.6. *Let* $[\mathfrak{q}, \gamma, \alpha]$ *be as defined in §10.2.3,* $T_{\mathfrak{q},\gamma,z}$ *as in §5.5, and the map* $\varsigma_{\mathfrak{q}}^{\gamma}$ *is as in §10.3. Then* $\varsigma_{\mathfrak{q}}^{\gamma}(z)$ *and* $T_{\mathfrak{q},\gamma,z}$ *coincide asymptotically as follows:*

There is a $N_0(m)$ *such that for* $\mathfrak{q}$*,* γ*, and* α *as above with* $N_{F/\mathbf{Q}}(\mathfrak{q}) \equiv 1 \ (mod\ p^{N_0(m)})$*, the actions of* $T_{\mathfrak{q},\gamma,z}$ *and* $\varsigma_{\mathfrak{q}}^{\gamma}(z)$ *on* $H^*(\mathbb{S}_{\emptyset,\Sigma}, \mathcal{W}/\mathfrak{p}^m)_{\mathfrak{m}}$ *coincide.*

10.5 Conjecture

Denote by V_E be the classes in $\mathrm{V}_{E_{\mathfrak{p}}}$, whose pairing with the images of motivic classes along the regulator r_p of (4.2) lie in E. We are ready to state the main conjecture

Conjecture 10.5.0.1 (Analogue of Conj.8.8 in [Ven19])**.** *Suppose* Π *is an automorphic cuspidal representation appearing in the (coherent) cohomology of the automorphic vector bundle* $\mathcal{W}$*. Then the action of* V_E *described above preserves* $H^*((\mathbb{S}_{\emptyset,\Sigma})_{\mathcal{O}_E[\frac{1}{N}]}, \mathcal{W}^{can})_{\mathfrak{m}}$.

In light of Thm.1.5.0.6, if $\mathbf{G}$ has a unique non-compact palce which is of signature $(n-1,1)$, then $\dim H^i((\mathbb{S}_{\emptyset,\Sigma})_{\mathcal{O}_F[\frac{1}{N}]}, \mathcal{W}^{\mathrm{can}})_{\mathfrak{m}} \le 1$ for all i, so the Conjecture predicts the existence of a 'scaling factor' for the motivic action. For instance, for Artin motives coming from weight one forms, the conjecture predicts that $\log_p$(Stark unit) acts by rescaling, where $\log_p$ is as in (10.13) below. Furthermore, the work of Darmon, Lauder, and Rotger [DLR15, Conj. ES] suggest a relationship between this scalar and a value of triple product p-adic L-functions outside the range of interpolation.

10.5.1 Conjecture for Artin motives of weight one form

Let $f \in S_1(N, \epsilon)$ be a cuspidal weight one form. Then $\rho_f : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_E)$ constructed by Deligne and Serre can be viewed as its Artin motive. The main conjecture above speculates an action at Taylor-Wiles primes. Suppose $p \nmid 6N$ and that $p \nmid [H_f : \mathbf{Q}]$, where H_f is the field cut out by $\mathrm{Ad}\rho_f$. Let q be a Taylor-Wiles prime of level $m \geq 1$ relative to ρ_f and $\mathfrak{p}$. Fix a prime $\mathfrak{q}$ of H_f such that $\mathfrak{q} \cap H_f = (q)$. We may construct a *reduction modulo p* map by

$$\mathrm{red}_p : U_f^\circ = (H_f^\times \otimes \mathrm{Ad}^*\rho_f^\circ)^{G_{H_f/\mathbf{Q}}} \to ((H_f/\mathfrak{q})^\times \otimes \mathcal{O}_E)^{\mathrm{Frob}_\mathfrak{q}=1} = (\mathbf{Z}/q\mathbf{Z})^\times \otimes \mathcal{O}_E,$$

where the middle map is given by the inclusion of units into product of local units and the averaging operation from (10.7). This map is independent of the choice of $\mathfrak{q}$ above q as verified in [HV19, §2.8]. Thus we may construct a *discrete logarithm*, which could also be viewed as a part of the Taylor-Wiles data, given by

$$\log : (\mathbf{Z}/q\mathbf{Z})^\times \to \mathbf{Z}/p^m\mathbf{Z}. \tag{10.13}$$

Note that such map was used in the construction of the diamond operators (see p_Δ in §5.2). Note that both sides of the main conjecture are thus ambivalent in precisely the same way on the choice of $\log$. Upon fixing it for both sides in the same way, this ambivalence is resolved. Combining both maps, at Taylor-Wiles primes, we have

$$\theta_q^{(p)} : U_f^\circ \xrightarrow{\mathrm{red}_p} (\mathbf{Z}/q\mathbf{Z})^\times \otimes \mathcal{O}_E \xrightarrow{\log} \mathbf{Z}/p^m\mathbf{Z} \otimes \mathcal{O}_E = \mathcal{O}_E/p^m =: k. \tag{10.14}$$

We often contract this map to θ_q since p is part of the TW data for q. In light of (4.3) and Prop. 4.3.1.1, the main conjecture predicts an action of θ_q(Stark unit) preserving the rational structure. To get a meaningful prediction, similarly to [DLR15], we want to pair f with a form that has conjugate nebentypus. Naturally we pick f^*, the dual form, corresponding to the contragredient representation to π_f. Let

$$G(z) \coloneqq \mathrm{Tr}_{X_0(N)}^{X_0(Nq)}(f(z)f^*(qz)) \in H^0(X_0(N)_\mathcal{O}, \Omega^1)$$

be a weight two form. Then pairing with the Shimura class $\mathfrak{S}$ via Serre duality produces

$$\langle G, \mathfrak{S} \rangle \in \mathcal{O}_E/\mathfrak{p}^m.$$

Following through all the definitions, the Main Conjecture predicts

Conjecture 10.5.1.1 (Conjecture for weight one forms)**.** *Let $p \geq 5$ such that $p \nmid [H_f : \mathbf{Q}]$ and there are no p-th roots of unity inside H_f. Then there exists a Stark unit $u_f \in U_f^\circ$ such that, for all TW primes q,*

$$\langle G, \mathfrak{S} \rangle = \theta_q(u_f).$$

Remark 10.5.1.2. *This rephrasing shifts the problem from identifying the constant of proportionality as in [HV19] to finding appropriate Stark unit.*

This conjecture was established in [DHRV] for a dihedral f, i.e. one for which $[H_f : \mathbf{Q}] = 2$.

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Appendix A: Toroidal compactifications

Recall that if we decompose $\mathbf{G}$ into its $\mathbf{Q}$-simple factors as $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_d$, and let $D = D_1 \times \cdots \times D_k$ be the corresponding decomposition, then a boundary component $F = F_1 \times \cdots \times F_k$ is rational if and only if the F_i are rational. Thus it suffices to construct the relevant toroidal compactification in the case when $\mathbf{G}$ is simple, so that it is defined over an imaginary quadratic field L and $\mathbf{G}$ has a unique archimedean place.

A.1 Setup

Let $L/\mathbf{Q}$ be an imaginary quadratic field of discriminant D, ring of integers $\mathcal{O}_L$, and a non-trivial Galois automorphism $\alpha \mapsto \alpha^c$. Let V be a n-dimensional L-vector space admitting a non-degenerate Hermitian form $J : V \times V \to L$ of signature $(r, n-r)$ with $2r \leq n$ on $V(\mathbf{R})$. Suppose further that V admits an integral structure coming from a lattice Λ such that the restriction of J to Λ is $\mathcal{O}_L$-valued. Consider the linear algebraic $\mathbf{Q}$-group given by

$$\mathbf{G}(R) = \{g \in \mathrm{GL}_n(R \otimes_{\mathbf{Q}} V) : \exists r(g) \in R \text{ s.t. } J(gu, gv) = r(g)J(u,v) \text{ for all } u, v \in R \otimes_{\mathbf{Q}} V\}$$

for any $\mathbf{Q}$-algebra R. This is the unitary similitude group of signature $(r, n-r)$ at the unique archimedean place. Following Deligne's axiomatism, we may construct a locally symmetric space X such that $(\mathbf{G}, \mathbf{X})$ is a Shimura datum. Since any compactification of (the quotients of) the space X is done independently on all connected components, we work instead with the connected Shimura datum (G, D), where G is the stabilizer of the connected component D of X. In particular,

$G(\mathbf{R}) \simeq U(r, n-r)$ and $D = G(\mathbf{R})/K_G$ for some maximal compact $K_G \subset G(\mathbf{R})$.

For the purposes of toroidal compactifications we may work explicitly with the nonstandard unbounded realization of G as in [Lan17], namely

$$G(\mathbf{R}) = \{g \in \mathrm{GL}_n(\mathbf{C}) : g^t J_{r,n-r} g = J_{r,n-r}\},$$

where

$$J_{r,n-r} := \begin{pmatrix} & & I_{n-r} \\ & T & \\ -I_r & & \end{pmatrix}$$

is skew-Hermitian with $-iT$ is positive definite for a choice of skew-Hermitian matrix T. Then the stabilizer G of a connected component D is of $\mathbf{X}$ is of the form $U'_{r,n-r}$ and $D = \mathcal{H}_{r,n-r}$ in the notation of [Lan17]. This realization of D is part of the general idea that one toroidally compactifies $\Gamma\backslash D$ for an arithmetic group $\Gamma \subseteq G(\mathbf{Q})$ by first embedding it inside a Siegel domain of third kind by a "tube-like requirement."

As [Lan17, p.39], the boundary components F in the language of [AMRT] are all $G(Q)$-translates of $\mathcal{H}_{r-m,n-r-m}$, i.e. similar Hermitian domains of lower dimensions. More precisely, the Baily-Borel compactification is a quotient of

$$\mathcal{H}^*_{r,n-r} := \mathcal{H}_{r,n-r} \cup \underbrace{\big(\cup_{1\leq m \leq n-r} G(\mathbf{Q}) \cdot \mathcal{H}_{r-m,n-r-m} \big)}_{D^\infty}$$

A.2 Smooth compactifications over C

Fix an arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$, i.e. a group commensurable with $G(\mathbf{Z}) = \{g \in G(\mathbf{Q}) : gL = L\}$. We furthermore assume that Γ is congruence subgroup so that it contains an isomorphic copy of $\ker\big(\mathbf{G}(\mathbf{Z}) \to \mathbf{G}(\mathbf{Z}/n\mathbf{Z})\big)$ for some $n \in \mathbf{N}$. In this case Γ is discrete and acts properly discontinuous on D, so that the quotient

$$S_\Gamma(D) \coloneqq \Gamma \backslash D$$

is a complex orbifold. To avoid issues with the finite numbers of points stabilized by noncentral elements of Γ, we hereafter assume that Γ is neat, so that $S_\Gamma(D)$ is actually smooth.

A.3 Baily-Borel compactification

We may express the rational boundary components as

$$\mathcal{H}_{r-m,n-r-m} \simeq \begin{pmatrix} \infty_m & \\ & \mathcal{H}_{r-m,n-r-m} \end{pmatrix} \simeq G(\mathbf{Q})/P^{(m)}(\mathbf{Q}),$$

where $P^{(m)} \subset G$ is the *maximal* $\mathbf{Q}$-parabolic subgroup stabilizing a fixed rational isotropic subspace. In general, there exists a basis in which

$$P^{(m)}(\mathbf{Q}) = \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & & & * & \\ & * & * & * & * \end{bmatrix} \tag{A.1}$$

where the blocks are of sizes $m, n-r-m, a-b, m$, and $n-r-m$, both horizontally and vertically. In the Levi decomposition $P^{(m)}(\mathbf{Q}) = L^{(m)}(\mathbf{Q}) \ltimes N^{(m)}(\mathbf{Q})$, the unipotent radical is of the form

$$N^{(m)}(\mathbf{Q}) = \begin{bmatrix} 1 & * & * & * & * \\ & 1 & & * & \\ & & 1 & * & \\ & & & 1 & \\ & & & * & 1 \end{bmatrix}. \tag{A.2}$$

The center $U^{(m)}(\mathbf{Q}) \subset N^{(m)}(\mathbf{Q})$ is of the form

$$U^{(m)}(\mathbf{Q}) = \begin{bmatrix} 1 & & & * & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \simeq \mathrm{Herm}_m(E), \tag{A.3}$$

where $\mathrm{Herm}_m(E)$ is viewed as an algebraic group over $\mathbf{Q}$. Letting

$$\overline{D} = \mathcal{H}^*_{r,n-r} := \mathcal{H}_{r,n-r} \cup \big(\cup_{1 \le m \le n-r} G(\mathbf{Q}) \cdot \mathcal{H}_{r-m,n-r-m} \big) = D \cup D^\infty$$

with its Satake topology, we obtain the Baily- Borel compactification

$$\overline{S}_\Gamma(D) = \Gamma \backslash \bar{D} = \Gamma \backslash D \cup \Gamma \backslash D^\infty$$

which is a normal complex projective variety with boundary

$$\overline{S}_{\Gamma}^{\infty}(D) \coloneqq \Gamma\backslash D^{\infty} \simeq \cup_{1\leq m\leq n-r}\big(\Gamma\backslash G(\mathbf{Q})/P^{(m)}(\mathbf{Q})\big).$$

A.4 Smooth compactification

We now proceed by describing the smooth compactification $\widetilde{S}_{\Gamma}(D)$ resolving the singularities along $\Gamma\backslash G(\mathbf{Q})/P^{(m)}(\mathbf{Q})$, which would depend on a combinatorial data Σ.

Fix a boundary component $F \in D^{\infty}$ and let $P^{(m)}(\mathbf{Q})$ be the parabolic subgroup of $G(\mathbf{Q})$ fixing it. Up to conjugation, we may assume that it is of the form $(A.1)$ for some m. For neat Γ, set $\Gamma_{P^{(m)}} \coloneqq \Gamma \cap P^{(m)}(\mathbf{Q}) = \Gamma \cap N^{(m)}(\mathbf{Q})$ with center $\Gamma_{U^{(m)}} \coloneqq \Gamma \cap U^{(m)}(\mathbf{Q})$, and consider their quotient Λ_m, making the sequence

$$1 \to \Gamma_{U^{(m)}} \to \Gamma_{P^{(m)}} \to \Lambda_m \to 1 \tag{A.4}$$

exact.

Let $D(F) = U^{(m)}(\mathbf{C}) \cdot D$ be the "Siegel domain of third kind" containing D as explained in [ARMT, §4]. The arguments there produce a principal fiber bundle

$$D(F) \to D(F)' \coloneqq D(F)/U^{(m)}(\mathbf{C})$$

with structure group $U^{(m)}(\mathbf{C})$. Passing to quotient, we obtain the principal bundle

$$T(F) \coloneqq U^{(m)}(\mathbf{C})/\Gamma_{U^{(m)}} \to D(F)/\Gamma_{U^{(m)}} \to D(F)'.$$

A choice $\{\sigma_\alpha\}$ of $\bar{\Gamma}_F$-admissible polyhedral decomposition of $C(F) \subseteq U^{(m)}(\mathbf{C})$ (see [AMRT,

§2, Def.4.10]) admits an equivariant embedding

$$\begin{array}{ccc} T(F) & \hookrightarrow & T(F)_{\{\sigma_\alpha\}} \\ \downarrow & & \downarrow \\ D(F)/\Gamma_{U(m)} & \hookrightarrow & D(F)/\Gamma_{U(m)} \times^{T(F)} T(F)_{\{\sigma_\alpha\}} \\ \downarrow & & \downarrow \\ D(F)' & \overset{\sim}{\text{———}} & D(F)' \end{array}$$

Finally, set

$$\big(D(F)/\Gamma_{U(m)}\big)_{\{\sigma_\alpha\}} = \text{interior of closure of } D(F)/\Gamma_{U(m)} \text{ in } D(F)/\Gamma_{U(m)} \times^{T(F)} T(F)_{\{\sigma_\alpha\}}. \quad \text{(A.5)}$$

These $\big(D(F)/\Gamma_{U(m)}\big)_{\{\sigma_\alpha\}}$ will provide local charts of the partial compactifications along the F-direction. We may perform the same procedure for all rational boundary components. The smooth toroidal compactification $S_\Gamma(D)_\Sigma$ is then obtained as a quotient of

$$\sqcup_F \big(D(F)/\Gamma_{U(m)}\big)_{\{\sigma_\alpha\}}$$

by the action of Γ. This procedure yields a map

$$\varphi_\Gamma : S_\Gamma(D)_\Sigma \to \overline{S}_\Gamma(D),$$

which is trivial on $S_\Gamma(D)$, and can be described as blow-downs along the boundary divisors. Here Σ is a Γ-admissible collection of partial polyhedral decompositions, $\Sigma_F = \{\sigma_\alpha\}_F$, one for each boundary component F. The existence of such simplicial data Σ is asserted in [AMRT].

A.5 Adelic version

Given the Shimura datum $(\mathbf{G}, \mathbf{X})$ for any neat compact open $K \subseteq \mathbf{G}(\mathbf{Q})$, we consider the Shimura variety

$$\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) = \mathbf{G}(\mathbf{Q})\backslash \mathbf{X} \times \mathbf{G}(\mathbf{A}_f)/K.$$

Let $\mathbf{X}^\infty$ be the set of rational boundary components, which are identified with

$$\mathbf{X}^\infty = \cup_{1 \le m \le n-r} \mathbf{G}(\mathbf{Q})/P^{(m)}(\mathbf{Q})$$

for choices of maximal $\mathbf{Q}$-parabolic subgroups $P^{(m)}$. Set $\overline{\mathbf{X}} := \mathbf{X} \cup \mathbf{X}^\infty$, so that the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ is

$$\overline{\mathrm{Sh}}_K(\mathbf{G}, \mathbf{X}) = \mathbf{G}(\mathbf{Q})\backslash \overline{\mathbf{X}} \times \mathbf{G}(\mathbf{A}_f)/K.$$

The boundary is described by the adelic quotient

$$\begin{aligned} \overline{\mathrm{Sh}}_K^\infty(\mathbf{G}, \mathbf{X}) = \mathbf{G}(\mathbf{Q})\backslash \mathbf{X}^\infty \times \mathbf{G}(\mathbf{A}_f)/K &= \cup_{1 \le m \le n-r} \mathbf{G}(\mathbf{Q})\backslash \big(\mathbf{G}(\mathbf{Q})/P^{(m)}(\mathbf{Q})\big) \times \mathbf{G}(\mathbf{A}_f)/K \\ &= \cup_{1 \le m \le n-r} P^{(m)}(\mathbf{Q})\backslash \mathbf{G}(\mathbf{A}_f)/K. \end{aligned} \tag{A.6}$$

To construct the smooth compactification, we proceed as in the previous section. Let F be a rational boundary component, corresponding to $1 \le m \le n-r$, at level K represented by $(t, h) \in \mathbf{G}(\mathbf{Q}) \times \mathbf{G}(\mathbf{A}_f)$ as in (A.6). Then its stabilizer is $tP^{(m)}(\mathbf{Q})t^{-1}$ and

$$\Gamma_{P^{(m)}} = tP^{(m)}(\mathbf{Q})t^{-1} \cap hKh^{-1},$$

$$\Gamma_{U^{(m)}} = tU^{(m)}(\mathbf{Q})t^{-1} \cap hKh^{-1},$$

where $U^{(m)}(\mathbf{Q})$ is the center of the unipotent radical $N^{(m)}(\mathbf{Q})$ of $P^{(m)}(\mathbf{Q})$. The smooth compactification $\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_\Sigma$ is now obtained as §A.4 by resolving each boundary divisor. The existence of the required simplicial data Σ is obtained in [Har89, Remark 2.5.7].

A.6 Étaleness of maps between smooth compactifications

The general theory of Shimura varieties asserts that for neat $K \subseteq \mathbf{G}(\mathbf{A}_f)$, the locally symmetric space $\mathrm{Sh}_K(\mathbf{G},\mathbf{X})$ is a quasiprojective variety admitting a model over the reflex field $E(\mathbf{G},\mathbf{X}) = E$. One can show that $\overline{\mathrm{Sh}}_K^\infty(\mathbf{G},\mathbf{X})$ is also defined over E by studying automorphic forms of sufficiently high weight, and relating them to a linear system of divisors defined over E. Lastly, one might interpret the $\mathrm{Sh}_K(\mathbf{G},\mathbf{X})_\Sigma \to \overline{\mathrm{Sh}}_K(\mathbf{G},\mathbf{X})$ as a blow-up, which is a rational operation, and thus preserves the field of definition. It is worth remarking, however, that the individual components of $\overline{\mathrm{Sh}}_K^\infty(\mathbf{G},\mathbf{X})$ and $\mathrm{Sh}_K^\infty(\mathbf{G},\mathbf{X})_\Sigma$ are only defined over E^{ab}. Furthermore, a results of Chai-Faltings extended further by Kai-Wan Lan show that $\mathrm{Sh}_K(\mathbf{G},\mathbf{X})$ and its smooth toroidal compactification $\mathrm{Sh}_K(\mathbf{G},\mathbf{X})$ admits a smooth model over $\mathcal{O}[\frac{1}{D}]_{(S)}$, where S is a finite set of prime ideals depending on the level K. If K has a full level at p, then this descends to a model over $\mathcal{O}_\mathfrak{p}$ for $\mathfrak{p}|p$.

Fix a rational prime q. We would be interested in studying infinite towers by adding an infinite level at q. In general, these towers would not be étale integrally so our best hope is to constrain their ramification to be prime to a fixed prime p. This is sufficient for our purposes.

Fix a level $K^{(q)} = \prod_{v\neq q} K_v$ away from q such that K_v is hyperspecial for almost all v. Assume further that K_v is hyperspecial for $v = p$. Fix $\mathfrak{p} \lhd \mathcal{O}$ above p. For a subgroup $K \subseteq \mathbf{G}(\mathbf{Q}_q)$, let $Y(K) \coloneqq \mathrm{Sh}_{K^{(q)}\times K}(\mathbf{G},\mathbf{X})$ be the open Shimura variety and $X(K) \coloneqq \mathrm{Sh}_{K^{(q)}\times K}(\mathbf{G},\mathbf{X})_\Sigma$ its toroidal compactification along suitably chosen data Σ. For any $\mathcal{O}_\mathfrak{p}$-algebra R, these spaces admit models over $\mathrm{Spec}(R)$.

Consider two compact open subgroups $K' \lhd K \subseteq \mathbf{G}(\mathbf{Q}_q)$ such that $|K/K'| = p^\ell$. Then there is an étale map

$$\varphi : Y(K')_R \to Y(K)_R.$$

We will show that the natural extension

$$\varphi^\Sigma : X(K')_R \to X(K)_R$$

is étale. The morphism φ^Σ is of finite presentation and by results of Chai-Faltings it is also smooth.

Proposition A.6.0.1. *In the notation above, the map φ^Σ is étale for any $\mathcal{O}_\mathfrak{p}$-algebra R.*

Proof. Since étaleness is stable under base change, we may assume $R = \mathcal{O}_\mathfrak{p}$. Furthermore, since φ^Σ is étale on the open Shimura variety, by purity of the branch locus, it suffices to prove the assertion on the generic fibre of each irreducible boundary component, i.e. we may work over E. As the morphism is finitely presented, we can furthermore prove étaleness in the flat topology, so we do so after a faithfully flat base-change to $\mathbf{C}$. By abuse of notation, we write below K' and K in place of $K^{(q)} \times K$ and $K^{(q)} \times K'$.

Let F be a rational boundary component, corresponding to $1 \leq m \leq n - r$, at level K represented by $(t, h) \in \mathbf{G}(\mathbf{Q}) \times \mathbf{G}(\mathbf{A}_f)$ as in (A.6). Its stabilizer is given by $tP^{(m)}(\mathbf{Q})t^{-1}$ and, varying the level,

$$\Gamma_{P^{(m)}}(K') = tP^{(m)}(\mathbf{Q})t^{-1} \cap hK'h^{-1},$$

$$\Gamma_{U^{(m)}}(K') = tU^{(m)}(\mathbf{Q})t^{-1} \cap hK'h^{-1},$$

$$\Lambda_m(K') = \Gamma_{P^{(m)}}(K')/\Gamma_{U^{(m)}}(K')$$

with analogous formulas for $\Gamma_{P^{(m)}}(K'), \Gamma_{U^{(m)}}(K')$ and $\Lambda_r(K')$. Locally along the boundary F, the variety $X(K')$ is given by quotients of $\big(D(F)/\Gamma_{U^{(m)}(K')}\big)_{\{\sigma_\alpha(K')\}}$ by properly discontinuous action.

The étaleness of φ^{Σ} at the boundary component over F then would follow readily if we were to show that

$$\big(D(F)/\Gamma_{U^{(m)}(K')}\big)_{\{\sigma_\alpha(K')\}} = \big(D(F)/\Gamma_{U^{(m)}(K)}\big)_{\{\sigma_\alpha(K)\}}, \tag{A.7}$$

The fact that we may arrange the data Σ adapted both levels K and K' is shown in [Har89, Remark 2.5.7 (c)]. In other words, we may pick Σ adapted for both levels such that

$$\{\sigma_\alpha(K')\} = \{\sigma_\alpha(K)\}$$

Thus it is enough to establish the equality $\Gamma_{U^{(m)}}(K') = \Gamma_{U^{(m)}}(K)$.

Lemma A.6.0.2. *In the setup above,*

$$\Gamma_{U^{(m)}}(K') = \Gamma_{U^{(m)}}(K)$$

Proof. The assertion is equivalent to

$$tU^{(m)}(\mathbf{Q})t^{-1} \cap hK'h^{-1} = tU^{(m)}(\mathbf{Q})t^{-1} \cap hKh^{-1}.$$

By (A.3) we have $U(\mathbf{Q}_q) \subseteq U_n(\mathbf{Q}_q)$, where $U_n(\mathbf{Q}_q)$ is the standard unipotent of upper triangular matrices in $G(\mathbf{Q}_q) = \mathrm{GL}_n(\mathbf{Q}_q)$. Since $tU_n(\mathbf{Q}_q)t^{-1}$ is pro-q, while $hKh^{-1}/hK'h^{-1}$ is pro-p, we have

$$tU_n(\mathbf{Q}_q)t^{-1} \cap hK'h^{-1} = tU_n(\mathbf{Q}_q)t^{-1} \cap hKh^{-1}.$$

The claim now follows as

$$tU(\mathbf{Q})t^{-1} \cap hK'h^{-1} = tU(\mathbf{Q})t^{-1} \cap tU(\mathbf{Q}_q)t^{-1} \cap tU_n(\mathbf{Q}_q)t^{-1}hK'h^{-1}$$
$$= tU(\mathbf{Q})t^{-1} \cap tU(\mathbf{Q}_q)t^{-1} \cap tU_n(\mathbf{Q}_q)t^{-1}hKh^{-1} = tU(\mathbf{Q})t^{-1} \cap hK'h^{-1},$$

where we used $U(\mathbf{Q}) \subseteq U(\mathbf{Q}_q) \subseteq U_n(\mathbf{Q}_q)$. □

This lemma implies the equality (A.7), and the claim follows. □

Remark A.6.0.3. *The whole argument hinges on the containment* $U(\mathbf{Q}_q) \subseteq U_n(\mathbf{Q}_q)$*. This hold for* $G = GSp_n$*, so the above argument works in that setting and we may construct Shimura classes in the exact same way.*