The arithmetic of del Pezzo surfaces and Hilbert schemes of points

Morena Porzio

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Abstract

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Motivated by the Cassels–Swinnerton-Dyer Conjecture for cubic surfaces, this thesis investigates the stable birational class of Hilb^n_X, the Hilbert scheme of length n closed subschemes on a given surface X. The primary focus is to determine for which pairs of positive integers (n, n') the varieties Hilb^n_X and $\mathrm{Hilb}^{n'}_X$ are stably birational, specifically when X is a surface with irregularity $q(X) = 0$.

After establishing general results for such surfaces, the study narrows its scope to geometrically rational surfaces. In this case, it is shown that, among the Hilb^n_X's, there exist only finitely many stable birational classes. A corollary of this finding is the rationality of the motivic zeta function $\zeta_{\mathrm{mot}}(X, t)$ in $K_0(\mathrm{Var}/k)/([\mathbb{A}^1_k])[[t]]$ over fields of characteristic zero.

Returning to cubic surfaces, the thesis further examines the stable birational types of Hilb^n_X both asymptotically and for small values of n.

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Introduction

The Cassels–Swinnerton-Dyer conjecture

Let F be an irreducible homogeneous polynomial of degree d in $k[x_0, \dots, x_r]$ over a field k. A key question in arithmetic geometry is: does $V(F) \subset \mathbb{P}_k^r$ have a k-rational point, that is $V(F)(k) \neq \varnothing$? This question has in general negative answer, but if we relax it and ask for the existence of closed points, then the answer is always positive. Indeed, there is always a finite extension k'/k such that the degree $[k' : k]$ divides d and $V(F)(k') \neq \varnothing$. Therefore a more refined question is the following.

Question. Let us assume that $V(F)(k') \neq \varnothing$ for some extension k'/k of degree $[k' : k]$ *coprime* with d. Is it true that then $V(F)(k) \neq \varnothing$?

More generally, consider the *index* of $V(F)$:

$$\mathrm{ind}(V(F)/k) \coloneqq \gcd\{[k' : k] \text{ s.t. } V(F)(k') \neq \varnothing\}.$$

If $V(F)(k) \neq \varnothing$ then the index $\mathrm{ind}(V(F)/k)$ is equal to 1. When is the converse true?

Known Answers.

1. If $d = 1$ then $V(F)$ is an hyperplane isomorphic to $\mathbb{P}_k^{r-1}$. So is has always a k-rational point.

2. If $d = 2$ then the converse is true as well: in characteristic different from 2, it is a reformulation of Springer's Theorem [Spr52] (see [Mah]), and a characteristic free proof is given by [Cor76, Prop. 2.1].

3. If $d \geq 4$ there are many counterexamples given by Kollár [Kol04, Example 6] and Shatz [Sha72, Section IV.5] of index 1 hypersurfaces $V(F)$ with no k-rational point.

4. The degree 3 case is of most interest. If $V(F)$ is a curve, then the answer is positive. This was known to Poincaré [Poi01] and Coray in [Cor76, Prop. 2.3] retraces his proof. Note that if $V(F)$ is smooth, then the result follows from Riemann-Roch Theorem.

5. For $d = 3$ and $r \geq 3$, we have the following cases in which the answer is positive:

   - if $V(F)(k') \neq \emptyset$ for a degree 2 extension k'/k [Cor76, Prop. 2.2],
   - if k is a C_1 field (this follows from the positive result for cubic curves),
   - if k is a global field and $V(F)$ satisfies the *Hasse principle* [Cor76, Prop. 4.8],
   - if $r = 3$ (so $V(F)$ is a surface) and k is a *quasi-local* field [Cor76, Thm. 4.7].

Moreover, for smooth cubic hypersurfaces, there are results with no assumptions on the field k.

Theorem (Coray [Cor76, Thm. 7.1] for k perfect & Ma [Ma19, Thm. 2.6] in general)**.** *Let k be an arbitrary field k. Let $V(F)$ be a smooth cubic surface of index 1. Then it has a closed point of degree n_0 equal to either* 1 *or* 4 *or* 10*.*

Theorem (Ma [Ma19, Thm. 3.6])**.** *Let k be an arbitrary field k. Let $V(F)$ be a smooth cubic 3-fold (resp. 4-fold) of index 1. Assume $V(F)$ has a closed point of degree 7 (resp. 8). Then it has a closed point of degree either* 1 *or* 2 *or* 4 *or* 5 *(resp.* $1, 2, 4, 5$ *or* 7*).*

Cassels–Swinnerton-Dyer (CS) Conjecture. Let $V(F) \subseteq \mathbb{P}^r_k$ be a cubic hypersurface. Then

$$\operatorname{ind}(V(F)/k) = 1 \text{ if and only if } V(F) \text{ has a } k\text{-rational point.}$$

The main motivation behind my PhD thesis was to study whether or not it can happen that a smooth cubic surface has a degree 4 (or degree 10) point but no k-rational point.

Reformulation in terms of Hilbert schemes of points

In [Ma19] Ma reinterpreted the (CS) conjecture in terms of existence of rational maps between n-th symmetric powers $\mathrm{Sym}^n_{V(F)}$'s of $V(F)$: indeed, he proved the following.

Theorem (Ma [Ma19, Prop.s 4.1 & 4.3])**.** *Let k be a field. Let $V(F) \subseteq \mathbb{P}^3_k$ be a smooth cubic surface.*

1. *For any positive integer n such that $3 \nmid n$, there exists a rational map $\mathrm{Sym}^n_{V(F)} \dashrightarrow \mathrm{Sym}^{n_0}_{V(F)}$ for n_0 either equal to 1 or 4 or 10.*
2. *On the other hand, if there exists a rational map $f : \mathrm{Sym}^n_{V(F)} \dashrightarrow \mathrm{Sym}^{n'}_{V(F)}$ and $V(F)$ has a length n closed subscheme on it, then $\mathrm{Sym}^{n'}_{V(F)}(k) \neq \emptyset$.*

In the second part of the theorem, it is important to note that the assumption is not $\mathrm{Sym}^n_{V(F)}(k) \neq \emptyset$. Indeed, f is merely rational, and if the k-point lies in the singular locus of $\mathrm{Sym}^n_{V(F)}$ and outside the locus where f is defined, it cannot be pushed forward to $\mathrm{Sym}^{n'}_{V(F)}$. Instead, the existence of a length n subscheme on $V(F)$ is equivalent to the existence of a (smooth) k-point on $\mathrm{Hilb}^n_{V(F)}$, the *Hilbert scheme of points of degree n on X*, which precisely parametrizes the length n closed subschemes on $V(F)$.
When X is a geometrically connected, smooth, projective surface over a field k, Hilb^n_X is also a geometrically connected, smooth, projective k-scheme (Fogarty [Fog68, Thm. 2.4]).
Furthermore, there exists a map called the *Hilbert-Chow morphism* (Fogarty [Fog68]):

$$h_X : \mathrm{Hilb}^n_X \longrightarrow \mathrm{Sym}^n_X, \quad Z \mapsto \mathrm{Supp}(Z)$$

which serves as a resolution of singularities for Sym^n_X. The second part of the above Ma's Theorem is then derived from the Lang–Nishimura Theorem stated below:

Theorem (See [Poo17, p. 3.6.11])**.** *Let $X \dashrightarrow Y$ be a rational map between k-varieties (that is, separated schemes of finite type over k). Assume Y is proper. If X has a smooth k-point, then Y*

has a k-point.

Therefore a very related question to the (CS) Conjecture is the following.

New Question. Let $V(F)$ be a smooth cubic surface. Does there exist a rational map

$$\mathrm{Sym}^4_{V(F)} \dashrightarrow V(F) \text{ (resp. } \mathrm{Sym}^{10}_{V(F)} \dashrightarrow V(F))?$$

Following the results of Coray and Ma, for my thesis work I searched for geometric constructions that define such rational maps, which in turn brought me to study the (stable) birational type of Sym^n_X.

Indeed, by definition, Sym^n_X and $\mathrm{Sym}^{n'}_X$ are stably birational if and only if there exists two non-negative integers N, N' and opens $\mathcal{U} \subseteq \mathrm{Sym}^n_X \times_k \mathbb{P}^{N'}_k$ and $\mathcal{U}' \subseteq \mathrm{Sym}^{n'}_X \times_k \mathbb{P}^N_k$ which are isomorphic:

$$\begin{tikzcd}
 & \mathcal{U} \arrow[rr, leftrightarrow, "\simeq"] \arrow[d, hook] & & \mathcal{U}' \arrow[d, hook] & \\
\mathrm{Sym}^n_X \times_k (\mathbb{P}^{N'}_k(k)) \arrow[r, hook] & \mathrm{Sym}^n_X \times_k \mathbb{P}^{N'}_k \arrow[d, "pr_1"] & & \mathrm{Sym}^{n'}_X \times_k \mathbb{P}^N_X \arrow[d, "pr_1"] & \mathrm{Sym}^{n'}_X \times_k (\mathbb{P}^N_X(k)) \arrow[l, hook'] \\
 & \mathrm{Sym}^n_X & & \mathrm{Sym}^{n'}_X . &
\end{tikzcd}$$

In particular, if $\left(\mathrm{Sym}^n_X \times_k (\mathbb{P}^{N'}_k(k))\right) \cap \mathcal{U}$ is non empty (for instance if k is infinite), then choosing an appropriate k-point x_0 in $\mathbb{P}^{N'}_k$ we get a rational map

$$\mathrm{Sym}^n_X \dashrightarrow \left(\mathrm{Sym}^n_X \times_k \{x_0\}\right) \cap \mathcal{U} \longrightarrow \mathcal{U}' \xrightarrow{pr_1} \mathrm{Sym}^{n'}_X .$$

Stable birationality of Hilbert schemes of points

Let X be a smooth, geometrically connected, projective surface over an arbitrary field k. Consider the Hilbert scheme Hilb^n_X of length n closed subschemes on X. As mentioned before, this is a smooth geometrically connected projective scheme of dimension $2n$ over k, birational to Sym^n_X. In particular (for surfaces), studying the (stable) birational type of Sym^n_X is equivalent to studying the one of Hilb_X^n. Furthermore, the question depends only on the stable birationality type of X itself thanks to the followin fact.

Fact (Kollár [Kol18, Cor. 8])**.** Let X and Y be two geometrically integral (positive dimensional) separated schemes of finite type over k. If X is stably birational to Y, then Sym_X^n is stably birational to Sym_Y^n.

The study of the (stable) birational type of the symmetric powers of a (quasi-projective) scheme has been studied by various people already. For instance:

Theorem (Kollár [Kol18])**.** *Let P be a Brauer Severi k-variety of dimension d (namely $P_{\overline{k}} \cong \mathbb{P}^d_{\overline{k}}$). Let* $\mathrm{ind}(P/k)$ *be the index of P. Then for any $n \geq 0$ the variety* Sym_P^n *is stably birational to* $\mathrm{Sym}_P^{\gcd(n,\mathrm{ind}(P/k))}$.

Theorem (Litt [Lit14, Thm. 19])**.** *Let k be a field of characteristic 0. Let X be a smooth, geometrically connected, projective surface with $H^0(X, \omega_X) \neq 0$. Then for any $n \geq 0$ there exists an integer $N'(n)$ such that, for all $n' \geq N'(n)$,* Sym_X^n *is* not *stably birational to* $\mathrm{Sym}_X^{n'}$.

Theorem (Wood in personal communication to [Lit14])**.** *Let k be a field of characteristic 0. Let X be a smooth, geometrically connected, projective surface with non-negative Kodaira dimension. Then for any n such that either $H^0(X, \omega_X^n)$ or $H^0(X, \omega_X^{2n})$ is non-zero, then* Sym_X^n *is stably birational to* $\mathrm{Sym}_X^{n'}$ *if and only if $n = n'$.*

Theorem (Ma [Ma19, Prop.s 6.1 & 6.2])**.** *Let k be a field of characteristic 0.*

1. *Let $X_{(2)}$ be a smooth cubic surface. Then* $\mathrm{Sym}_{X_{(2)}}^3$ *is stably birational to* $\mathrm{Sym}_{X_{(2)}}^6$.

2. *Let $X_{(3)}$ be a smooth cubic 3-fold. Then* $\mathrm{Sym}_{X_{(3)}}^5$ *is stably birational to* $\mathrm{Sym}_{X_{(3)}}^7$.

3. *Let $X_{(4)}$ be a smooth cubic 4-fold. Then* $\mathrm{Sym}_{X_{(4)}}^7$ *is stably birational to* $\mathrm{Sym}_{X_{(4)}}^8$.

Given the results, it makes sense to focus on the case when the Kodaira dimension of X is negative, or equivalently $\kappa(X_{\overline{k}}) = -\infty$. Since the Kodaira dimension is a birational invariant for

$X_{\overline{k}}$, by the classification of surfaces over algebraically closed fields of arbitrary characteristic (see Bădescu [Băd01]), we have two cases:

1. the surface X is geometrically rational, namely $X_{\overline{k}}$ is birational to $\mathbb{P}^2_{\overline{k}}$, or

2. the surface $X_{\overline{k}}$ is birational to $\mathbb{P}^1_{\overline{k}} \times_{\overline{k}} C$, where C is a smooth curve of positive genus.

In this thesis ultimately, we want to focus on the first family: geometrically rational surfaces over arbitrary fields (with a special care for cubic surfaces). Since the cohomology group $H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a birational invariant for $X_{\overline{k}}$ (see [Har77, Prop. V.3.4]), this is equivalent to say that we focus on surfaces of negative Kodaira dimension whose $H^1(X, \mathcal{O}_X)$ is trivial.

Therefore, in the first part of this thesis, the vanishing of $H^1(X, \mathcal{O}_X)$ will be the only cohomological assumption that we put on our surface X: only later we specialize to geometrically rational ones.

Results for surfaces of irregularity zero

The input for our general study is a triple $(X, \mathcal{L}, Q)$ where

- X is a smooth, geometrically connected, projective surface over a field k, whose $H^1(X, \mathcal{O}_X)$ is trivial; namely a surface of irregularity $q(X) = \dim_k H^1(X, \mathcal{O}_X) = 0$,

- $\mathcal{L}$ is an ample line bundle on X, and

- Q is a geometrically reduced, effective 0-cycle on X of degree d.

We call these $(X, \mathcal{L}, Q)$'s *suitable triples*: by extension, if X, $\mathcal{L}$ or Q have the above properties we call them *suitable*.

Remark. As said above, we do not assume X to have negative Kodaira dimension from the start. We will see that this becomes a *necessary* condition when we look at results for Hilb^n_X with $n >> 0$, in accordance with Litt's and Wood's Theorems (cfr. Theorem 4.1.1 with Proposition 4.1.7).

Remark. Notice that smooth, geometrically connected, projective surfaces X have many suitable $\mathcal{L}$ and Q on them. Indeed, if the field k is infinite, we can embed X into a projective space and use smooth hyperplane sections to define smooth effective 0-cycles on X: this is classical Bertini's theorem [Har77, Thm. II.8.18]. If the field k is finite, we can use Poonen's version of Bertini's theorem [Poo04, Thm. 1.1] and intersect X with hypersurfaces of sufficiently big degrees. Therefore, there are many suitable triples $(X, \mathcal{L}, Q)$ - the ample line bundles come from X being projective.

It turns out that if a suitable triple $(X, \mathcal{L}, Q)$ satisfies certain assumptions with respect to an integer n then we have the following theorem.

Theorem A. *Let X be a suitable surface. Assume there exist $(\mathcal{L}, Q)$ suitable such that the following assumptions hold true (with respect to n):*

1. *the restriction map $H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}|_Q)$ is surjective,*
2. $\dim_k H^0(X, \mathcal{L}) \geq n + 1 + 2\deg(Q)$*,*
3. *the sheaf $\mathcal{L} \otimes \mathcal{I}_Q$ is generated by global sections,*
4. *for any $\overline{k}$-point $x \in X_{\overline{k}} \setminus Q_{\overline{k}}$, the map $H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)_{\overline{k}} \longrightarrow \mathcal{L}_x/m_x^2\mathcal{L}_x$ is surjective,*
5. *for any curve C in the linear system $|\mathcal{L}|$, the arithmetic genus $p_a(C) \leq n$.*

Then

$$\mathrm{Hilb}_X^n \sim_{\text{stably birational}} \mathrm{Hilb}_X^{n+\deg(Q)}.$$

Inspired by Coray's paper [Cor76], the idea is to find appropriate curves C passing through Q. We can then use Riemann-Roch, the description of the Abel-Jacobi map $\mathrm{Hilb}_C^n \to \mathrm{Pic}_C^n$ and Brill-Noether theory on C to infer the existence of certain effective 0-cycles on X. The first four assumptions in Theorem A ensure the existence of those curves: in particular, the first two ensure that we have enough freedom in choosing sections of $\mathcal{L}$ passing by Q, and the next two guarantee a Bertini's theorem like for sections of $\mathcal{L} \otimes \mathcal{I}_Q$. The fifth assumption works "against" the others, but it is there to make sure that the Abel-Jacobi map is surjective, and we have a useful Brill-Nother theory.

After establishing this general theorem for surfaces of irregularity zero, we focus on finding sufficient and necessary conditions on n so that we have triples satisfying the five assumptions with respect to n.

Theorem B. *Let $(X, \mathcal{L}, Q)$ be a suitable triple. There exists an integer $e_0 = e_0(X, \mathcal{L}, Q)$ such that for any $e \geq e_0$ the triple $(X, \mathcal{L}^{\otimes e}, Q)$ satisfies the assumptions of Theorem A with respect to n if and only if n is in the interval*

$$I_e := \left[\frac{e^2 c_1(\mathcal{L})^2 + e\, c_1(\mathcal{L}) \cdot K_X}{2} + 1, \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + \dim_k H^2(X, \mathcal{O}_X) - 2\deg(Q) \right],$$

in which case Hilb^n_X *is then stably birational to* $\mathrm{Hilb}^{n+\deg(Q)}_X$.

Given the nature of Theorem B, it is natural to wonder about the non-emptiness and the growth of I_e (as e grows).

We already expect that for a surface X of non-negative Kodaira dimension the intervals I_e must be empty for $e >> 0$ (Litt's and Wood's Theorems above). This is confirmed by Proposition 4.1.7 where we will see that a necessary conditions for having non-empty intervals I_e for $e >> 0$ is for X to have negative Kodaira dimension (and hence to be geometrically rational).

Furthermore, being geometrically rational turns out to be a *sufficient* condition as well. Actually, it is a sufficient condition for having enough suitable triples $(X, \mathcal{L}, Q)$ so that their intervals I_e cover all the integers in a neighborhood of $+\infty$.

Theorem C. *Let X be a geometrically rational suitable surface. Let* $\mathrm{ind}(X/k)$ *be the index of X. Then there exists an integer n_0 such that*

$$\mathrm{Hilb}^n_X \sim_{\text{stably birational}} \mathrm{Hilb}^{n+\mathrm{ind}(X/k)}_X \quad \forall n \geq n_0.$$

Application to the rationality of the motivic zeta function

Theorem C has an interesting application to the rationality question of the *motivic zeta function*. First introduced by Kapranov [Kap] as a generalization of the Hasse-Weil zeta function $Z(X,t)$, the motivic zeta function is defined as the power series

$$\zeta_{\mathrm{mot}}(X,t) = 1 + \sum_{n=1}^{\infty} [\mathrm{Sym}^n_X] t^n \in K_0(Var/k)[[t]]$$

where $K_0(Var/k)$ is the Grothendieck ring of k-varieties (namely, reduced separated schemes of finite type over k).

The question about the rationality of the motivic zeta function rose naturally from the Theorem of the rationality of the *Hasse-Weil* zeta function (conjectured by Weil in [Wei49] and proved in general by Dwork in [Dwo60]).

In general the answer is negative: indeed, Larsen and Lunts in [LL04, Thm. 1.1] proved that for a complex surface X, $\zeta_{\mathrm{mot}}(X,t)$ is rational if and only if $\kappa(X) = -\infty$. On the other hand, Kapranov proved that the answer is positive for curves (with index 1, see [Kap, Thm. 1.1.9]), and Larsen and Lunts proved that the same is true for rational surfaces over algebraically closed fields $k = \overline{k}$ of characteristic zero ([LL04, Thm. 3.9]).

Modding out by the ideal generated by $[\mathbb{A}^1_k]$, we extend the latter result to geometrically rational surfaces over any field of characteristic zero.

Theorem D. *Let k be a field of characteristic zero. Let X be a geometrically rational surface. Then the motivic zeta function $\zeta_{\mathrm{mot}}(X,t)$ of X is rational in the quotient $K_0(Var/k)/([\mathbb{A}^1_k])[[t]]$.*

Alternative ways to relate Hilbert schemes of points

The idea of considering special curves C lying on the surface X can be also exploited without fixing a specific cycle Q. Indeed, we can also use line bundles $\mathcal{M}$'s on X and restrict them to C. Following this idea, we have the following analogue of Theorem A.

Theorem E. *Let $(X, \mathcal{L})$ be a suitable pair. Let $\mathcal{M}$ be a line bundle on X. Let n be a positive integer such that $n' = c_1(\mathcal{L})c_1(\mathcal{M}) - n$ is positive. Assume that $\mathcal{L}$ satisfies the following assumptions with respect to $(n, \mathcal{M})$:*

1. $h^0(X, \mathcal{L}) \geq \max\{n, n'\} + 1$,
2. *the bundle $\mathcal{L}$ is globally generated,*
3. *for any $\overline{k}$-point $x \in X_{\overline{k}}$, the map $H^0(X, \mathcal{L})_{\overline{k}} \longrightarrow \mathcal{L}_x / m_x^2 \mathcal{L}_x$ is surjective,*
4. $p_a(C) \leq \min\{n, n'\}$,
5. $\min\{n, n'\} - c_1(\mathcal{M})c_1(\mathcal{L}) + c_1(\mathcal{L})^2 - p_a(C) \geq 0$.

Then

$$\mathrm{Hilb}_X^n \sim_{\text{stably birational}} \mathrm{Hilb}_X^{n'}.$$

Effective results for Cubic Surfaces

We now come back to the original question, namely the stable birational type of Sym_X^n when X is a smooth cubic surface. By Chevalley-Warning [Che35]-[CE35], a cubic surface X over $\mathbb{F}_q$ has always an $\mathbb{F}_q$-rational point. Since the stable birational question here is motivated by the (CS) conjecture, we therefore assume k to be infinite.

We first compute an effective value of the n_0 defined in Theorem C.

Theorem F. *Let X be smooth cubic surface over an infinite field k. Then*

$$\mathrm{Hilb}_X^n \sim_{\text{stably birational}} \mathrm{Hilb}_X^{n+3} \quad \forall n \in \mathbb{N}_{\geq 10} \setminus \{13, 14, 15, 16, 17, 18, 25, 26, 27, 40, 41, 42\}.$$

Therefore, it is enough to focus on "small" values of n. Applying Theorem E we get the following enhancement of Theorem F.

Theorem G. *Let X be a smooth cubic surface over an infinite field k. Then*

$$\mathrm{Hilb}_X^n \sim_{\text{stably birational}} \mathrm{Hilb}_X^{n+3} \quad \forall n \in \mathbb{N}_{\geq 10} \setminus \{15, 18, 27, 42\}.$$

Moreover, for the following pairs (n,n')*, we have* $\mathrm{Hilb}^n_X \sim_{\text{stably birational}} \mathrm{Hilb}^{n'}_X$*:*

$$(1,2),(4,5),(5,7),(7,8).$$

Outline

In Chapter 1 we recall definitions and properties of objects that will be later used: results about the Abel-Jacobi map and about Brill-Noether theory. In the first Section of Chapter 2 we introduce a geometrical framework in order to compare Hilb^n_X with $\mathrm{Hilb}^{n'}_X$. The idea is to define an appropriate *incidence correspondence* $\mathrm{Inc}^Q_{X,\mathcal{L},n,n'}$ (Definition 2.0.1) together with two morphisms

$$\begin{tikzcd} & \mathrm{Inc}^Q_{X,\mathcal{L},n,n'} \arrow[dl, "\pi_n"'] \arrow[dr, "\pi_{n'}"] & \\ \mathrm{Hilb}^n_X & & \mathrm{Hilb}^{n'}_X. \end{tikzcd}$$

The rest of the Section is devoted to check that the functor $\mathrm{Inc}^Q_{X,\mathcal{L},n,n'}$ is representable by a quasi-projective k-scheme.

In the second Section of Chapter 2, we describe the general fiber of π_n and $\pi_{n'}$ respectively. We will see (in Proposition 2.1.1 and Proposition 2.1.5) that after restricting to appropriate opens of $\mathrm{Inc}^Q_{X,\mathcal{L},n,n'}$ the two maps are projective bundles.

In Chapter 3 we introduce assumptions 3.0.1 on a suitable triple $(X,\mathcal{L},Q)$, with the goal of ensuring that the intersection of the opens in $\mathrm{Inc}^Q_{X,\mathcal{L},n,n'}$ found in Chapter 2 is non-empty.

Chapters 3 and 4 are then devoted to prove that under those assumptions (with respect to n) we have that that Hilb^n_X is stably birational to $\mathrm{Hilb}^{n+\deg(Q)}_X$ (Theorem A). While doing so, we prove a Bertini's-like Theorem on surfaces for curves passing by a fixed effective 0-cycle (see Proposition 3.0.9).

In Chapter 4, Section 4.1, we first find sufficient and necessary conditions on n (Theorem B) for which we can apply Theorem A. Then we study the stable birational class of Hilb^n_X for $n >> 0$

and find a sufficient criterion on X in order to have only finitely many different stable birational classes (Corollary 4.1.11).

Using Iskovskikh classification of geometrically rational surfaces minimal over k (see [Isk79]), in Chapter 5 we prove that for geometrically rational surfaces there are always only finitely many different stable birational classes of Hilb^n_X (Theorem C).

In Chapter 6 we exploit the result of Chapter 5 in order to study the rationality of the motivic Hilbert zeta function (Proposition 6.0.11) and the motivic zeta function of geometrically rational surfaces (Theorem D).

Chapter 7 explores alternative definitions of the incidence correspondence defined in Chapter 2: in particular, it shows that changing the way of relating effective 0-cycle on curves leads to similar result of Theorem A but for different pairs (n, n') (Theorem E).

Finally, in Chapter 8, we apply Theorems A and E, in order to study the stable birational classes of Sym^n_X where X is a cubic surface (Theorems F and G).

Notation

In what follows, unadorned products are meant to be over k. By a variety over k we mean a reduced separated scheme over k of finite type.

Let T be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of O_T-modules. By $\mathbb{P}(\mathcal{E})$ we mean the T-scheme parametrizing the O_T-sheaf quotients of $g^*\mathcal{E}$ which are locally free of rank 1:

$$\mathbb{P}(\mathcal{E})(T' \xrightarrow{g} T) = \{(\mathcal{M}, \alpha) : \mathcal{M} \text{ is invertible on } T' \text{ and a quotient of } g^*\mathcal{E} \text{ via } \alpha : g^*\mathcal{E} \twoheadrightarrow \mathcal{M}\}/\text{isom.}.$$

Given a scheme Y, the unadorned tensor product $\mathcal{F} \otimes \mathcal{G}$ of O_Y-sheaves is meant over O_Y.

Chapter 1: Preliminaries

As stated in the introduction, we start with a triple $(X, \mathcal{L}, Q)$ where

- X is a smooth, geometrically connected, projective surface over a field k, whose $H^1(X, \mathcal{O}_X)$ is trivial,
- $\mathcal{L}$ is an ample line bundle on X, and
- Q is a 0-dimensional, geometrically reduced, closed subscheme of X of degree d.

Remark 1.0.1. Notice that any curve C in the linear system defined by $\mathcal{L}$ is geometrically connected.

Indeed, since $\mathcal{L}$ is ample, there is an integer m for which $H^1(X, \mathcal{L}^{\otimes m} \otimes \omega_X) = 0$. By Serre duality, $H^1(X, (\mathcal{L}^\vee)^{\otimes m}) = 0$, which implies $H^0(mC, \mathcal{O}_{mC}) = k$. Hence, C is a geometrically connected curve (see [Sta, Tag 0FD1]).

In the next section, we will define the *incidence correspondences* introduced for comparing Hilb^n_X and $\mathrm{Hilb}^{n'}_X$. So let us first recall the objects used in their definitions.

1.0.1 Relative effective Cartier divisors

An *effective Cartier divisor* $D \subset Y$ of a scheme Y is a closed subscheme whose ideal sheaf $\mathcal{I}_D$ (also denoted by $\mathcal{O}_Y(-D)$) is an invertible $\mathcal{O}_Y$-module.

Given a morphism of schemes $Y \longrightarrow S$, a *relative effective Cartier divisor* on Y/S is an effective Cartier divisor $D \subset Y$ such that D is flat over S. This notion is stable under base-change: in particular, the restrictions to the fibers $\{D_s \subset Y_s\}_{s \in S}$ are all effective Cartier divisor of Y_s (see [Sta, Lemma 056Q]).

We can define the functor

$$\mathrm{CDiv}_X : \mathrm{Sch}_k^{\mathrm{op}} \longrightarrow \mathrm{Set}, \qquad T \mapsto \{\text{relative effective Cartier divisors } D \text{ on } X_T/T\}.$$

Since X is projective and smooth, CDiv_X is representable by the disjoint union of connected projective schemes over k ([Bej, §13, Prop. 2]).

We have a natural transformation of functors into the relative Picard functor

$$AJ_X : \mathrm{CDiv}_X \longrightarrow \mathrm{Pic}_X, \quad D \subset X_T \mapsto [\mathcal{O}_{X_T}(D)]$$

called the *Abel-Jacobi morphism* (see [Bej, §14, Prop. 1]).

We refer to [Bej, §14] for a discussion of the relative Picard functor $\mathrm{Pic}_{X/k}$ and the Abel-Jacobi map (see also [Gro62, Thm. 3.1]). However, in the next subsection we recall some properties about its fibers.

1.0.2 Fibers of Abel-Jacobi map

Let $f : Y \longrightarrow S$ be a flat projective scheme over a quasi-compact scheme S. Assume that f has geometrically integral fibers.

Let T be an S-scheme and let $\mathcal{L}$ be a line bundle on the fiber product $Y_T \xrightarrow{f_T} T$. The class $[\mathcal{L}]$ in $\mathrm{Pic}_{Y/S}$ corresponds to a morphism $T \xrightarrow{[\mathcal{L}]} \mathrm{Pic}_{Y/S}$.

If $\mathcal{L}$ is *cohomologically flat of dimension zero* (namely, the formation of its direct image commutes with base change), then $f_{T*}\mathcal{L}$ is locally free and the bundle projection $\mathbb{P}((f_{T*}\mathcal{L})^\vee) \longrightarrow T$ fits in the cartesian diagram

$$\begin{tikzcd} \mathbb{P}((f_{T*}\mathcal{L})^\vee) \arrow[r] \arrow[d] & \mathrm{CDiv}_{Y/S} \arrow[d, "AJ_{Y/S}"] \\ T \arrow[r, "{[\mathcal{L}]}"] & \mathrm{Pic}_{Y/S} \end{tikzcd}$$

(see [BLR90, Prop. 8.2.7]).

Definition 1.0.2. In general, let us denote by $\mathrm{CDiv}^{\mathcal{L}}_{Y/S}$ the fiber product $\mathrm{CDiv}_{Y/S} \times_{\mathrm{Pic}_{Y/S}} T$ induced by $[\mathcal{L}]$.

In Section 2.1 we will need the description of the above mentioned isomorphism between the fiber product $\mathrm{CDiv}^{\mathcal{L}}_{Y/S}$ and $\mathbb{P}((f_{T*}\mathcal{L})^{\vee})$. Therefore let us recall how it is defined (for a complete proof see [BLR90, §8]).

Proposition 1.0.3. *Let $f : Y \longrightarrow S$ and $\mathcal{L}$ as above. The isomorphism between $\mathrm{CDiv}^{\mathcal{L}}_{Y/S}$ and $\mathbb{P}((f_{T*}\mathcal{L})^{\vee})$ is given by*

$$(D \subset Y_T) \mapsto (\mathcal{M}, (f_{T*}i_D)^{\vee})$$

where $i_D : \mathcal{O}_{Y_T} \hookrightarrow \mathcal{O}_{Y_T}(D)$ and $\mathcal{M}$ is such that $\mathcal{O}_{X_T}(D) \cong \mathcal{L} \otimes f_T^*(\mathcal{M})$.

Sketch of the proof. Since f is proper and has geometrically integral fibers, then the formation of the direct image $f_*\mathcal{O}_Y$ commutes with base change. Then we have the exact sequence

$$0 \longrightarrow \mathrm{Pic}(T) \xrightarrow{f_T^*} \mathrm{Pic}(Y_T) \longrightarrow \mathrm{Pic}_{Y/S}(T)$$

(see [BLR90, Prop. 8.1.4]).

Consider an element $D \subset Y_T$ in $\mathrm{CDiv}^{\mathcal{L}}_{Y/S}$: by definition of fiber product, the class $[\mathcal{O}_{Y_T}(D)]$ coincides with $[\mathcal{L}]$ in $\mathrm{Pic}_{Y/S}(T)$. Therefore, by the previous exact sequence this happens if and only if there exists a (unique) line bundle $\mathcal{M}$ on T such that

$$\mathcal{O}_{Y_T}(D) \cong \mathcal{L} \otimes f_T^*(\mathcal{M}).$$

Taking the pushforward of the inclusion $\mathcal{O}_{Y_T} \overset{i_D}{\hookrightarrow} \mathcal{O}_{X_T}(D)$ we still get an inclusion

$$f_{T*}\mathcal{O}_{Y_T} \cong \mathcal{O}_T \xhookrightarrow{f_{T*}i_D} f_{T*}\mathcal{O}_{Y_T}(D) \cong f_{T*}\mathcal{L} \otimes \mathcal{M},$$

where the last isomorphism is due to the projection formula.

Tensoring both sides by $\mathcal{M}^\vee$ and then taking the dual, we get a surjection

$$(f_{T*}i_D)^\vee : (f_{T*}\mathcal{L})^\vee \twoheadrightarrow \mathcal{M},$$

that is an element of $\mathbb{P}((f_{T*}\mathcal{L})^\vee)$. Conversely, given a surjective map

$$\alpha : (f_{T*}\mathcal{L})^\vee \twoheadrightarrow \mathcal{M}$$

onto a line bundle $\mathcal{M}$ on T, we can work backwards till we get

$$f_{T*}\mathcal{O}_{Y_T} \cong \mathcal{O}_T \hookrightarrow f_{T*}(\mathcal{L} \otimes f_T^*\mathcal{M}).$$

Then thanks to the pullback-pushforward adjunction we get a unique map

$$\mathcal{O}_{Y_T} \longrightarrow \mathcal{L} \otimes f_T^*\mathcal{M}.$$

This map is injective on each fiber Y_t (with $t \in T$) since its dual fits inside the factorization

$$\begin{tikzcd}
f_t^* f_{t*}(\mathcal{L} \otimes f_t^*\mathcal{M})^\vee \arrow[rr, two heads] \arrow[dr, dashed] & & \mathcal{O}_{Y_t}. \\
& (\mathcal{L}_t \otimes f_t^*\mathcal{M})^\vee \arrow[ur, two heads] &
\end{tikzcd}$$

This realises $(\mathcal{L} \otimes f_T^*\mathcal{M})^\vee$ as the ideal sheaf of a relative Cartier divisor. □

Let us apply the previous construction to the case $[\mathcal{L}] \in \mathrm{Pic}(X)$. The surface X is geometrically connected, smooth (hence flat and geometrically integral) and projective over k. By flat base change ([Sta, Tag 02KH]), any line bundle on a quasi-compact and quasi-separated scheme over a field is cohomologically flat of dimension zero. Therefore we have the following cartesian diagram

$$\begin{tikzcd} \mathbb{P}(H^0(X,\mathcal{L})^\vee) \arrow[r] \arrow[d] & \mathrm{CDiv}_{X/k} \arrow[d, "AJ_X"] \\ \mathrm{Spec}(k) \arrow[r, "{[\mathcal{L}]}"] & \mathrm{Pic}_{X/k}. \end{tikzcd}$$

1.0.3 Brill-Noether theory

For any positive m, consider the Abel-Jacobi map for a smooth, geometrically connected, projective $\overline{k}$-curve C:

$$\mathrm{Hilb}^m_{C/\overline{k}} \longrightarrow \mathrm{Pic}^m_{C/\overline{k}}.$$

Since C is geometrically connected, it is cohomologically flat in dimension zero ([Con, Cor. 1.3]). Moreover since $C(\overline{k}) \neq \varnothing$, then for any $\overline{k}$-schemes T we have the short exact sequence

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(C \times T) \longrightarrow \mathrm{Pic}_{C/\overline{k}}(T) \longrightarrow 0$$

(see [BLR90, Prop. 8.1.4]). In particular, all elements in $\mathrm{Pic}_{C/\overline{k}}$ are represented by the class of some line bundle.

Given an integer $r \geq 0$ there is a natural subfunctor of $\mathrm{Pic}^m_{C/k}$ whose support is the set of linear systems $|D|$ of degree m and dimension at least r. Precisely, we denote by $W^r_m(C/\overline{k})$ the functor

$$W^r_m(C/\overline{k}) : \mathrm{Sch}_k^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

$$T \mapsto \{[\mathcal{M}] \in \mathrm{Pic}^m_{C/\overline{k}}(T) : \dim_{k(t)} H^0(C_t, \mathcal{M}_t) \geq r+1 \; \forall\, t \in T\}.$$

By upper semicontinuity [Har77, Thm III.12.8], the subfunctor $W^r_m(C/\overline{k})$ is a closed subscheme of $\mathrm{Pic}^m_{C/\overline{k}}$ (see [ACH85, §IV.3] for a detailed description). Furthermore, we have the following fundamental property.

Fact 1.0.4 (Lemma IV.3.5 of [ACH85])**.** *If $p_a(C) + r - m \geq 0$, then no irreducible component of*

$W_m^r(C/\overline{k})$ is entirely contained in $W_m^{r+1}(C/\overline{k})$. In particular, $W_m^r(C/\overline{k}) \setminus W_m^{r+1}(C/\overline{k})$ is non-empty.

Chapter 2: The Incidence Correspondences

Fix an integer $n \geq 1$ and set $n' = n + d = n + \deg(Q)$.

Definition 2.0.1. Let us define the functor

$$\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'} : \mathrm{Sch}_k^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

sending T to the set of triples $(Z, Z', C) \in \mathrm{Hilb}^n_X(T) \times \mathrm{Hilb}^{n+d}_X(T) \times \mathrm{CDiv}^{\mathcal{L}}_X(T)$, such that:

(1) the subschemes Q_T, Z, and Z' are all contained in $C \subset X_T$,

(2) the scheme C is smooth over T,

(3) Q_T, Z, and Z' are relative effective Cartier divisors on C over T,

(4) the classes $[\mathcal{O}_C(Z + Q_T)]$ and $[\mathcal{O}_C(Z')]$ agree in $\mathrm{Pic}_{C/T}(T)$.

Definition 2.0.2. Analogously, one can define the functor

$$\mathrm{Inc}^{Q}_{X,\mathcal{L},n} : \mathrm{Sch}_k^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad T \longmapsto \{(Z, C) \in \mathrm{Hilb}^n_X(T) \times \mathrm{CDiv}^{\mathcal{L}}_X(T) : Z, Q_T \subset C \subset X_T\}.$$

The functors $\mathrm{Inc}^{Q}_{X,\mathcal{L},n}$ and $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}$ are the *incidence correspondences* we are interested in.

These functors fit in the following diagram

$$\begin{tikzcd}
\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}(T) \arrow[dr, hook] \arrow[drr, dashed, bend left, "{(Z,Z',C)\mapsto(Z',C)}"] \arrow[ddr, dashed, bend right, "{(Z,Z',C)\mapsto(Z,C)}"'] & & & \\
& \mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_X} \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}(T) \arrow[r] \arrow[d] & \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}(T) \arrow[r, "{(Z',C)\mapsto Z'}"] \arrow[d, "{(Z',C)\mapsto C}"] & \mathrm{Hilb}^{n+d}_X(T) \\
& \mathrm{Inc}^{Q}_{X,\mathcal{L},n}(T) \arrow[r, "{(Z,C)\mapsto C}"] \arrow[d, "{(Z,C)\mapsto Z}"] & \mathrm{CDiv}^{\mathcal{L}}_X(T) & \\
& \mathrm{Hilb}^{n}_X(T), & &
\end{tikzcd}$$

where all the maps are the forgetful functors. In particular we get two morphisms

$$\begin{tikzcd}
& \mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'} \arrow[dl, "\pi_n"'] \arrow[dr, "\pi_{n'}"] & \\
\mathrm{Hilb}^{n}_X & & \mathrm{Hilb}^{n'}_X.
\end{tikzcd}$$

Remark 2.0.3. Consider $C \in \mathrm{CDiv}^{\mathcal{L}}_X(T)$. Since the pullback of a relative effective Cartier divisor is still a relative effective Cartier divisor, then for any point t of T the pullback C_t is a relative effective Cartier divisor as well. In particular the map $C \longrightarrow T$ is surjective.

Remark 2.0.4. In general, if $Y \longrightarrow S$ is flat and $D \subset Y$ is a closed subscheme flat over S, then D is a relative effective Cartier divisor if and only if for any point s of S the subscheme $D_s \subset Y_s$ is (see [Bej, §13, Cor. 2]).

Therefore, since X_T is flat over T, if $C \subset X_T$ is a closed subscheme flat over T, then we can check whether $C \in \mathrm{CDiv}_X(T)$ on fibers.

Analogously, given a triple $(Z, Z', C) \in \mathrm{Hilb}^n(T) \times \mathrm{Hilb}^{n+d}(T) \times \mathrm{CDiv}_X(T)$, then since Q_T, Z, Z' and C are all flat over T, we can check whether Q_T, Z, Z' are relative effective Cartier divisor on C/T on fibers.

Remark 2.0.5. Since any 0-dimensional closed subscheme of a smooth curve over a field is an effective Cartier divisor, any triple $(Z, Z', C) \in \mathrm{Hilb}^n(T) \times \mathrm{Hilb}^{n+d}(T) \times \mathrm{CDiv}_X(T)$ satisfying (1) and (2) it satisfies (3) as well.

Remark 2.0.6. By Remark 1.0.1, for any $C \in \mathrm{CDiv}_X^{\mathcal{L}}(T)$ we know that C_t is geometrically connected. If C satisfies (2) then C_t is smooth and therefore it is geometrically integral.

This means that for any smooth element $C \in \mathrm{CDiv}_X^{\mathcal{L}}(T)$, the fibers of $C \longrightarrow T$ are geometrically integral.

Therefore since $C \longrightarrow T$ is projective, flat, surjective and has integral geometric fibers, the functor $\mathrm{Pic}_{C/T}$ is representable by a separated T-scheme (see [BLR90, Thm 8.2.1]).

2.0.1 Representability of the Incidence Correspondences

In the remaining part of this section, we want to check the representability of the functors just described.

We first focus on the representability of $\mathrm{Inc}^{Q}_{X,\mathcal{L},n}$. In order to do so, we need first a general lemma.

Lemma 2.0.7. *Let Y be a proper and flat scheme over a noetherian scheme S and let Z_1, Z_2 two closed subschemes of Y, both flat over S. Then the functor*

$$\delta_Z(T \xrightarrow{f} S) = \begin{cases} \{f\} & \text{if } Z_1 \times_f T \subseteq Z_2 \times_f T, \\ \varnothing & \text{otherwise.} \end{cases}$$

is representable by a closed subscheme of S.

Proof. Let f_T be the morphism $Y_T \longrightarrow T$ induced by f. Let $\mathcal{I}_i$ denote the ideal sheaf of $Z_i \overset{j_i}{\hookrightarrow} Y$: since $\mathcal{O}_Y$ and $j_{i\,*}\mathcal{O}_{Z_i}$ are all flat over S then $\mathcal{I}_i$ are flat over S as well. Therefore the kernels are stable under base-change:

$$\ker(\mathcal{O}_{Y_T} \longrightarrow j_{i\,*}\mathcal{O}_{Z_{i\,T}}) = \mathcal{I}_{i,T} = f_T^*\mathcal{I}_i.$$

Therefore we have a map $\sigma_T : \mathcal{I}_{2,T} \longrightarrow j_{1\,*}\mathcal{O}_{Z_{1\,T}}$ induced by

$$\begin{tikzcd} \mathcal{I}_{2,T} \arrow[dr, hook] \arrow[dd, dashed, "\sigma_T"'] & & \mathcal{I}_{1,T} \arrow[dl, hook'] \\ & \mathcal{O}_{Y_T} \arrow[dl, two heads] \arrow[dr, two heads] & \\ j_{1\,*}\mathcal{O}_{Z_{1\,T}} & & j_{2\,*}\mathcal{O}_{Z_{2\,T}}. \end{tikzcd}$$

Since the subscheme $Z_{1\,T}$ is contained in $Z_{2\,T}$ if and only if $\mathcal{I}_{2,T} \subseteq \mathcal{I}_{1,T}$, then $Z_{1\,T} \subseteq Z_{2\,T}$ if and only if $\sigma_T = 0_T$. Let σ be the one associated to $f = \mathrm{id}_S$.

Consider the Hom-functor

$$Hom_{\mathcal{O}_Y}(\mathcal{I}_2, j_{1\,*}\mathcal{O}_{Z_1}) : \mathrm{Sch}_S^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad T \mapsto \mathrm{Hom}_{\mathcal{O}_{Y_T}}(\mathcal{I}_{2,T}, j_{1\,*}\mathcal{O}_{Z_{1\,T}}).$$

Since Y is proper over the noetherian scheme S and $\mathcal{O}_{Z_1}$ is flat over S, then by [Sta, Tag 08JY] $Hom_{\mathcal{O}_Y}(\mathcal{I}_2, j_{1,*}\mathcal{O}_{Z_1})$ is an algebraic space affine and of finite presentation over S. Since affine morphisms of algebraic spaces are representable (see [Sta, Tag. 03WG]) then $Hom_{\mathcal{O}_Y}(\mathcal{I}_2, j_{1,*}\mathcal{O}_{Z_1})$ is a scheme affine and of finite presentation over S. In particular it is separated over S.

Therefore the global sections σ and 0 are both closed embedding of S into the functor. The fiber product of σ and 0 represents the S-functor

$$\begin{tikzcd}
 & S \arrow[dr, hook, "\sigma"] & & \\
\delta_Z \arrow[ur, hook] \arrow[dr, hook] & & Hom_{\mathcal{O}_Y}(\mathcal{I}_2, j_{1,*}\mathcal{O}_{Z_1}) \arrow[r] & S \\
 & S \arrow[ur, hook, "0"'] & &
\end{tikzcd}$$

which is then a closed subscheme of S.

$\square$

Proposition 2.0.8. $\mathrm{Inc}^{Q}_{X,\mathcal{L},n}$ *is a closed subscheme of* $\mathrm{Hilb}^n_X \times_k \mathrm{CDiv}^{\mathcal{L}}_X$*, so it is projective over* k*.*

Proof. Let $\mathcal{Z}$, C be the universal family associated to Hilb^n_X and $\mathrm{CDiv}^{\mathcal{L}}_X$ respectively. Therefore we have three closed subschemes

$$\begin{tikzcd}
\mathcal{Z}_{\mathrm{CDiv}^{\mathcal{L}}_X},\ Q_{\mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X},\ C_{\mathrm{Hilb}^n_X} \quad \subset & X \times \mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X \arrow[d] \\
 & \mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X
\end{tikzcd}$$

all flat over $\mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X$. Since X is proper and flat over k, Lemma 2.0.7 applies to the pairs

$$\mathcal{Z}_{\mathrm{CDiv}^{\mathcal{L}}_X} \subset C_{\mathrm{Hilb}^n_X} \quad \text{and} \quad Q_{\mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X} \subset C_{\mathrm{Hilb}^n_X}$$

producing two closed representable subfunctors of $\mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X$, whose intersection is $\mathrm{Inc}^{Q}_{X,\mathcal{L},n}$. Hence $\mathrm{Inc}^{Q}_{X,\mathcal{L},n}$ is a closed subscheme of $\mathrm{Hilb}^n_X \times \mathrm{CDiv}^{\mathcal{L}}_X$.

Finally both Hilb^n_X and $\mathrm{CDiv}^{\mathcal{L}}_X$ (which is isomorphic to $\mathbb{P}(H^0(X,\mathcal{L})^\vee)$) are projective over k. So $\mathrm{Inc}^{Q}_{X,\mathcal{L},n}$ is projective as well. $\square$

Corollary 2.0.9. *The functor* $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_X} \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}$ *is representable by a projective scheme over*

k.

Proof. Since the composite

$$\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \hookrightarrow \mathrm{Hilb}^{n}_{X} \times \mathrm{CDiv}^{\mathcal{L}}_{X} \longrightarrow \mathrm{CDiv}^{\mathcal{L}}_{X},$$

is projective, the fiber product $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_{X}} \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}$ is projective over k as well ($\mathrm{CDiv}^{\mathcal{L}}_{X}$ is projective over k). □

Notice that the projective scheme $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_{X}} \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}$ already parametrizes triples (Z, Z', C) which satisfy condition (1) of Definition 2.0.1.

Proposition 2.0.10. *The functor* $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}$ *is representable by a locally closed subscheme of* $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_{X}} \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}$. *In particular it is a quasi-projective scheme over* k.

Proof. Let $\mathcal{S}$ denote the scheme $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_{X}} \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}$. Let $\mathcal{Z}, \mathcal{Z}', Q_{\mathcal{S}}, C$ be the closed subschemes of $X \times \mathcal{S}$ corresponding to the universal families of Z, Z', Q and C respectively. By openness of the smooth locus the locus where the composite

$$p : C \subset X \times \mathcal{S} \longrightarrow \mathcal{S}$$

is smooth (and hence cohomologically flat of dimension zero) is open. Let $\mathcal{S}^{\circ}$ denote such open and let $p^{\circ} : C^{\circ} \longrightarrow \mathcal{S}^{\circ}$ the restriction.

By Remark 2.0.5, on the open $\mathcal{S}^{\circ}$ the closed subschemes $\mathcal{Z}^{\circ}, \mathcal{Z}'^{\circ}, Q_{\mathcal{S}^{\circ}}$ are all relative effective divisors. Therefore we can consider the line bundles $O_{C^{\circ}}(\mathcal{Z}'^{\circ})$ and $O_{C^{\circ}}(\mathcal{Z}^{\circ}+Q^{\circ})$ and their induced maps

$$\begin{tikzcd}[column sep=small]
 & \mathcal{S}^{\circ} \arrow[dr, hook, "{[\mathcal{O}_{C^{\circ}}(\mathcal{Z}'^{\circ})]}"] & & \\
\mathcal{P} \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Pic}_{C^{\circ}/\mathcal{S}^{\circ}} \arrow[r] & \mathcal{S}^{\circ}. \\
 & \mathcal{S}^{\circ} \arrow[ur, hook, "{[\mathcal{O}_{C^{\circ}}(\mathcal{Z}^{\circ}+Q^{\circ})]}"'] & &
\end{tikzcd}$$

Notice that the fiber product $\mathcal{P}$ is exactly the functor $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}$. Due to cohomological flatness, as mentioned before the functor $\mathrm{Pic}_{C^{\circ}/\mathcal{S}^{\circ}}$ is represented by an algebraic space. Therefore the fiber product $\mathcal{P}$ is represented by a scheme. Since the global sections induced by the line bundles are locally closed embeddings, $\mathcal{P}$ is locally closed in $\mathcal{S}^{\circ}$. Therefore $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}$ is a locally closed subscheme of $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \times_{\mathrm{CDiv}^{\mathcal{L}}_{X}} \mathrm{Inc}_{X,\mathcal{L},n'}$. □

2.1 Description of the General fibers in the Incidence Correspondences

Our next step is to study the general fibers of the projection maps $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \longrightarrow \mathrm{Hilb}^{n}_{X}$ and $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'} \longrightarrow \mathrm{Inc}^{Q}_{X,\mathcal{L},n}$.

2.1.1 The general fiber on $\mathrm{Hilb}^{\mathbf{n}}_{\mathbf{X}}$

We focus first on $\mathrm{Inc}^{Q}_{X,\mathcal{L},n} \longrightarrow \mathrm{Hilb}^{n}_{X}$. We want to prove the following.

Proposition 2.1.1 (Definition of U_n and U_{n+d})**.** *Define U_n to be the subfunctor of* $\mathrm{Hilb}^{n}_{X\setminus Q}$ *describing the points y for which*

$$H^0(X_y, (pr_X^*\mathcal{L})_y) \longrightarrow H^0(X_y, (pr_X^*\mathcal{L})_y|_{\mathcal{Z}_y \sqcup Q_y})$$

is surjective and

$$pr_{H*}(pr_X^*\mathcal{L}) \otimes_{\mathcal{O}_{\mathrm{Hilb}^n_X, y}} k(y) \longrightarrow H^0(X_y, (pr_X^*\mathcal{L})_y)$$

is an isomorphism.

For notation sake, let us denote by $H^n_{\backslash Q}$ *the open subscheme* $\mathrm{Hilb}^n_{X\backslash Q}$ *of* Hilb^n_X. *Then consider the universal families* $\mathcal{Z}$ *and* $Q_{\mathrm{Hilb}^n_X}$ *restricted to this open:*

$$\mathcal{Z}_{H^n_{\backslash Q}} \sqcup Q_{H^n_{\backslash Q}} \subset X \times H^n_{\backslash Q} \xrightarrow{pr_{\mathrm{Hilb}^n_{X\backslash Q}}} H^n_{\backslash Q}.$$

Because of they are disjoint, we can consider the ideal sheaf $\mathcal{I}_{\mathcal{Z}_{H^n_{\backslash Q}} \sqcup Q_{H^n_{\backslash Q}}}$.

The following hold:

1. *the subfunctor* U_n *is an open subscheme of* $H^n_{\backslash Q}$,
2. *on* U_n *the sheaves* $pr_{H^n_{\backslash Q}*}(pr_X^*\mathcal{L})$ *and* $pr_{H^n_{\backslash Q}*}(pr_X^*\mathcal{L}\otimes\mathcal{I}_{\mathcal{Z}_{H^n_{\backslash Q}} \sqcup Q_{H^n_{\backslash Q}}})$ *are locally free and their formation commutes with base change,*
3. *denote by* $\mathcal{E}_n$ *the dual sheaf*

$$(pr_{H^n*}(pr_X^*\mathcal{L}\otimes\mathcal{I}_{\mathcal{Z}_{H^n_{\backslash Q}} \sqcup Q_{H^n_{\backslash Q}}})\,|_{U_n})^{\vee}.$$

Then the projection map $\mathbb{P}(\mathcal{E}_n) \longrightarrow U_n$ *is the fiber product in the cartesian diagram*

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}_n) \arrow[r, hook] \arrow[d] & \mathrm{Inc}^Q_{X,\mathcal{L},n} \arrow[d] \\ U_n \arrow[r, hook] & \mathrm{Hilb}^n_X. \end{tikzcd}$$

Same definition and properties hold for $U_{n+d} \subseteq \mathrm{Hilb}^{n+d}_{X\backslash Q}$ *and* $\mathcal{E}_{n+d}$.

The idea is to use the result about the fibers of Abel-Jacobi map with respect to the diagram

$$\begin{tikzcd} & \mathrm{CDiv}_X \arrow[d, "AJ_X"] \\ \mathrm{Hilb}^n_X \arrow[r, "{[pr_X^*\mathcal{L}]}"] & \mathrm{Pic}_X \end{tikzcd}$$

where pr_X is the projection $X \times \mathrm{Hilb}^n_X \longrightarrow X$. The issue is that at priori we do not know that formation of direct image of $pr_{H*}pr^*_X\mathcal{L}$ commutes with base change.

Furthermore $\mathrm{Inc}^Q_{X,\mathcal{L},n}$ is a closed subscheme of the above fiber product: therefore we need to find an appropriate subbundle of $pr_{H*}pr^*_X\mathcal{L}$ which is still cohomologically flat of dimension zero.

Proposition 2.1.1 tells us that both issues can be overcome after restricting to a suitable open subscheme of Hilb^n_X.

Remark 2.1.2. Let $f : Y \longrightarrow S$ be a proper, flat morphism with S locally noetherian. Let $W \subseteq Y$ be a closed subscheme, flat over S. Let $\mathcal{L}$ be an invertible sheaf on Y. Since the sheaf $\mathcal{L} \otimes \mathcal{I}_W$ is flat over S as well, by Cohomology and base change [Har77, Thm. III.12.11], the locus where the formation of direct image $f_*(\mathcal{I}_W \otimes \mathcal{L})$ commutes with base change is open.

Remark 2.1.3. The cited *Cohomology and Base Change* result from Hartshorne's *Algebraic Geometry* is proven under projectivity assumptions on f. However, properness of the map is enough (see [Con]).

We first want to prove the first two items of Proposition 2.1.1.

Lemma 2.1.4. *Let $Y, S, W, \mathcal{L}$ be as in Remark 2.1.2. Let U be the set of points $s \in S$ where*

$$H^0(Y_s, \mathcal{L}_s) \longrightarrow H^0(Y_s, \mathcal{L}_s|_{W_s})$$

is surjective and

$$f_*\mathcal{L} \otimes_{\mathcal{O}_{S,s}} k(s) \longrightarrow H^0(Y_s, \mathcal{L}_s)$$

is an isomorphism (we denote by $\mathcal{L}_s$ the pullback of $\mathcal{L}$ to Y_s).

Then the set U is open and formation of direct images of $f_*\mathcal{L}$ and $f_*(\mathcal{I}_W \otimes \mathcal{L})$ commutes with base change on U.

Proof. Again by Cohomology and base change [Har77, Thm. III.12.11] we know that the set S° where the map

$$f_*\mathcal{L} \otimes_{\mathcal{O}_{S,s}} k(s) \longrightarrow H^0(Y_s, \mathcal{L}_s)$$

is an isomorphism is open. In particular, over S° the sheaf $f_*\mathcal{L}$ is locally free and its formation commutes with base change.

In particular on each connected component of $\mathcal{S}^\circ$ the rank of $f_*\mathcal{L}$ is constant: hence we can assume that $H^0(Y_s, \mathcal{L}_s)$ is constant.

Surjectivity of $H^0(Y_s, \mathcal{L}_s) \longrightarrow H^0(Y_s, \mathcal{L}_s|_{W_s})$ is equivalent to have dimension of the kernel $\dim_{k(s)} H^0(Y_s, \mathcal{L}_s \otimes \mathcal{I}_{W_s})$ equal to

$$\dim_{k(s)} H^0(Y_s, \mathcal{L}_s) - \dim_{k(s)} H^0(Y_s, \mathcal{L}_s|_{W_s}).$$

By upper-semicontinuity [Har77, Thm. III.12.8], for any integer m the locus where

$$\dim_{k(s)} H^0(Y_s, \mathcal{L}_s|_{W_s}) \leq m$$

is open. Since (again by upper-semicontinuity) the locus where

$$\dim_{k(s)} H^0(Y_s, \mathcal{L}_s \otimes \mathcal{I}_{W_s}) \leq \dim_{k(s)} H^0(Y_s, \mathcal{L}_s) - m$$

is open then surjectivity is achieved on an open. Therefore the set U is open.

Last, we need to check the formation of direct image of $\mathcal{L} \otimes \mathcal{I}_W$ on U. Since $\mathcal{I}_W \otimes \mathcal{L}$, $\mathcal{L}$ and $\mathcal{O}_W \otimes \mathcal{L}$ are all flat over S, the exact sequence

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_W \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_W \longrightarrow 0$$

remains exact after we base change along $s : \mathrm{Spec}(k(s)) \longrightarrow S$. By definition of U, for any point $s \in U$ we get the exact sequence

$$0 \longrightarrow H^0(Y_s, \mathcal{L}_s \otimes \mathcal{I}_{W_s}) \longrightarrow H^0(Y_s, \mathcal{L}_s) \longrightarrow H^0(Y_s, \mathcal{L}_s|_{W_s}) \longrightarrow 0$$

If we now restrict to U the pushforwarded sequence,

$$0 \longrightarrow f_*(\mathcal{L} \otimes \mathcal{I}_W)|_U \longrightarrow f_*\mathcal{L}|_U \longrightarrow f_*(\mathcal{L}|_W)|_U, \tag{2.1}$$

and we tensor it by $- \otimes_{\mathcal{O}_{U,s}} k(s)$,

$$f_*(\mathcal{L} \otimes \mathcal{I}_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s) \longrightarrow f_*\mathcal{L}|_U \otimes_{\mathcal{O}_{U,s}} k(s) \longrightarrow f_*(\mathcal{L}|_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s),$$

we get to the following commutative diagram:

$$\begin{tikzcd} & f_*(\mathcal{L} \otimes \mathcal{I}_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s) \arrow[r] \arrow[d] & f_*\mathcal{L}|_U \otimes_{\mathcal{O}_{U,s}} k(s) \arrow[r] \arrow[d, "\wr"] & f_*(\mathcal{L}|_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s) \arrow[d] & \\ 0 \arrow[r] & H^0(Y_s, \mathcal{L}_s \otimes \mathcal{I}_{W_s}) \arrow[r] & H^0(Y_s, \mathcal{L}_s) \arrow[r] & H^0(Y_s, \mathcal{L}_s|_{W_s}) \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

where the vertical maps are the natural maps given by the base change. Since $s \in U$ the vertical map in the middle of the diagram 2.2 is an isomorphism which implies that the one on the right is surjective, and hence an isomorphism as well. Therefore by Cohomology and base change [Har77, Thm. III.12.11], the sheaf $f_*(\mathcal{L}|_W)|_U$ is locally free and its formation commutes with base change.

Furthermore, by composition, the morphism $f_*\mathcal{L}|_U \otimes_{\mathcal{O}_{U,s}} k(s) \longrightarrow f_*(\mathcal{L}|_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s)$ is surjective as well. Since both $f_*\mathcal{L}|_U$ and $f_*(\mathcal{L}|_W)|_U$ are locally free, the surjectivity of the map lifts to

$$f_*\mathcal{L}|_U \longrightarrow f_*(\mathcal{L}|_W)|_U.$$

Therefore the pushforwarded sequence 2.1 is exact on the right as well:

$$0 \longrightarrow f_*(\mathcal{L} \otimes \mathcal{I}_W)|_U \longrightarrow f_*\mathcal{L}|_U \longrightarrow f_*(\mathcal{L}|_W)|_U \longrightarrow 0.$$

Therefore for any $s \in U$ we get the exact sequence

$$f_*(\mathcal{L} \otimes \mathcal{I}_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s) \longrightarrow f_*\mathcal{L}|_U \otimes_{\mathcal{O}_{U,s}} k(s) \longrightarrow f_*(\mathcal{L}|_W)|_U \otimes_{\mathcal{O}_{U,s}} k(s) \longrightarrow 0.$$

By Snake lemma, the vertical map on the left in 2.2 is then surjective, hence an isomorphism, which implies that $f_*(\mathcal{L} \otimes \mathcal{I}_W)|_U$ is locally free and its formation commutes with base change. $\square$

Finally, we focus on the third item of Proposition 2.1.1.

Proof of Proposition 2.1.1. Consider the diagram

$$\begin{tikzcd}
\mathbb{P}(\mathcal{E}_n) \arrow[r, hook] & \mathbb{P}((pr_{U_n *}(pr_X^*\mathcal{L})|_{U_n})^\vee) \arrow[r] \arrow[d] & \mathrm{CDiv}_X \arrow[d] \\
& U_n \arrow[r, "{[pr_X^*\mathcal{L}]}"] & \mathrm{Pic}_X .
\end{tikzcd}$$

The fiber product $\mathbb{P}((pr_{U_n *}(pr_X^*\mathcal{L})|_{U_n})^\vee)$ coincides with $U_n \times \mathrm{CDiv}_X^{pr_X^*\mathcal{L}}$ because $[pr_X^*\mathcal{L}]$ factors as

$$U_n \longrightarrow \mathrm{Spec}(k) \xrightarrow{[\mathcal{L}]} \mathrm{Pic}_X .$$

So both $\mathbb{P}(\mathcal{E}_n)$ and $\mathrm{Inc}^Q_{X,\mathcal{L},n}|_{U_n}$ are closed subschemes of $\mathbb{P}((pr_{U_n *}(pr_X^*\mathcal{L})|_{U_n})^\vee)$. We want to prove that they coincide.

In order to do so, we check that they correspond via the isomorphism between

$$\mathrm{CDiv}_X^{pr_X^*\mathcal{L}} \cong \mathbb{P}((pr_{U_n *}(pr_X^*\mathcal{L})|_{U_n})^\vee)$$

described in Proposition 1.0.3. Fix $t : T \longrightarrow U_n$ so that we have

$$\begin{tikzcd}
X \times T \arrow[r] \arrow[d, "pr_T"] & X \times U_n \arrow[r, "pr_X"] \arrow[d, "pr_{U_n}"] & X \arrow[d] \\
T \arrow[r, "t"] & U_n \arrow[r] & \mathrm{Spec}(k).
\end{tikzcd}$$

On one hand, the subbundle $\mathbb{P}(\mathcal{E}_n)$ parametrizes line bundles $(\mathcal{M}, \alpha)$ on $T \xrightarrow{t} U_n$ whose quotient map $\alpha : t^*(pr_{U_n*}(pr_X^*\mathcal{L})|_{U_n})^\vee \longrightarrow \mathcal{M}$ factors via the inclusion $\mathcal{I}_{\mathcal{Z}_{U_n}\sqcup Q_{U_n}} \hookrightarrow \mathcal{O}_{X\times U_n}$

$$\begin{tikzcd} t^*(pr_{U_n*}(pr_X^*\mathcal{L})|_{U_n})^\vee \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{M} \\ & t^*(\mathcal{E}_n) \arrow[ur] & \end{tikzcd} \tag{2.3}$$

(see [Gro61, Rmk. 4.2.9]).

On the other hand, a Cartier divisor $D \subset X_T$ whose ideal sheaf is isomorphic to

$$\mathcal{O}_{X_T}(-D) \cong (id\times t)^* pr_X^*\mathcal{L}^\vee \otimes pr_T^*\mathcal{M}^\vee$$

(where $\mathcal{M}$ a line bundle on T) contains $\mathcal{Z}_T \sqcup Q_T$ if and only if the inclusion $\mathcal{O}_{X_T} \hookrightarrow \mathcal{O}_X(D)$ factors as

$$\begin{tikzcd} \mathcal{O}_{X_T} \arrow[rr, hook] \arrow[dr, hook] & & (id\times t)^* pr_X^*\mathcal{L} \otimes pr_T^*\mathcal{M} \\ & \mathcal{I}_{\mathcal{Z}_T\sqcup Q_T} \otimes (id\times t)^* pr_X^*\mathcal{L} \otimes pr_T^*\mathcal{M}. \arrow[ur, hook] & \end{tikzcd}$$

Taking the pushforward with respect to pr_{T*} and then tensoring by $\mathcal{M}^\vee$, by Proposition 1.0.3 we have this factorization if and only if

$$\begin{tikzcd} \mathcal{M}^\vee \arrow[rr, hook] \arrow[dr, hook] & & pr_{T*}(id\times t)^* pr_X^*\mathcal{L} \\ & pr_{T*}(\mathcal{I}_{\mathcal{Z}_T\sqcup Q_T} \otimes (id\times t)^* pr_X^*\mathcal{L}). \arrow[ur, hook] & \end{tikzcd} \tag{2.4}$$

By diagram chasing the sheaf $pr_{T*}(id\times t)^* pr_X^*\mathcal{L}$ coincides with $t^*(pr_{U_n*}(pr_X^*\mathcal{L}|_{U_n}))$. Since $\mathcal{I}_{\mathcal{Z}_T\sqcup Q_T}$ is the pullback via $(id\times t)$ of the ideal sheaf $\mathcal{I}_{\mathcal{Z}_{U_n}\sqcup Q_{U_n}}$ then

$$pr_{T*}(id\times t)^*(\mathcal{I}_{\mathcal{Z}_{U_n}\sqcup Q_{U_n}} \otimes pr_X^*\mathcal{L}) \cong t^*(pr_{U_n*}(pr_X^*\mathcal{L} \otimes \mathcal{I}_{\mathcal{Z}_{U_n}\sqcup Q_{U_n}})|_{U_n}) \cong t^*\mathcal{E}_n.$$

Taking the dual of diagram 2.4 we get a factorization like in diagram 2.3 and viceversa. Therefore the subbundle $\mathbb{P}(\mathcal{E}_n)$ coincides with $\mathrm{Inc}^Q_{X,\mathcal{L},n}|_{U_n}$. $\square$

2.1.2 The general fiber on $\mathrm{Inc}^{\mathbf{Q}}_{\mathbf{X},\mathcal{L},\mathbf{n}}$

It is now time to turn our attention to the projection maps

$$\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'} \longrightarrow \mathrm{Inc}^{Q}_{X,\mathcal{L},n} \quad \text{and} \quad \mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'} \longrightarrow \mathrm{Inc}^{Q}_{X,\mathcal{L},n'}$$

(where n' is always $n+d$). Since we are interested in the general fiber on a open of Hilb^n_X (respectively of Hilb^{n+d}_X), we study their restriction to U_n and U_{n+d}

$$\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}|_{U_n} \longrightarrow \mathbb{P}(\mathcal{E}_n) \quad \text{and} \quad \mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}|_{U_{n+d}} \longrightarrow \mathbb{P}(\mathcal{E}_{n+d}).$$

We aim for a statement analogous to Proposition 2.1.1.

The strategy is similar to the one adopted in the previous subsection. However now we want to use the Abel-Jacobi map for

$$AJ_{C/\mathbb{P}(\mathcal{E}_n)} : \mathrm{CDiv}_{C/\mathbb{P}(\mathcal{E}_n)} \longrightarrow \mathrm{Pic}_{C/\mathbb{P}(\mathcal{E}_n)}$$

where C is the universal curve

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "g"'] & X \times \mathbb{P}(\mathcal{E}_n) \arrow[d] \\ & \mathbb{P}(\mathcal{E}_n). \end{tikzcd} \tag{2.5}$$

Let $\mathcal{Z}$ be the universal family (contained in C) with respect to the subscheme Z.

Proposition 2.1.5 (Definition of V_n and V'_{n+d})**.** *Let V be the open of $\mathbb{P}(\mathcal{E}_n)$ where g is smooth (and hence cohomologically flat of dimension zero).*

In particular, all the fibers of g on V are geometrically integral (see Remark 2.0.6). Moreover restricted to V the subschemes $\mathcal{Z}_V$ and Q_V are relative effective Cartier divisor on C_V (see Remark 2.0.5).

Consider now the subfunctor V_n of the subscheme $V \subseteq \mathbb{P}(\mathcal{E}_n)$ made of the points $y \in V$ such *that*

$$g_*\mathcal{O}_{C_V}(\mathcal{Z}_V + Q_V) \otimes \mathcal{O}_{V,y}\, k(y) \longrightarrow H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y))$$

is surjective.

Then the following hold:

1. *the subfunctor V_n is an open subscheme of V,*

2. *the sheaf $g_*\mathcal{O}_{C|_{V_n}}(\mathcal{Z}_{V_n} + Q_{V_n})$ is locally free and its formation commutes with base change,*

3. *denote by $\mathcal{F}_n$ the dual of $g_*\mathcal{O}_{C|_{V_n}}(\mathcal{Z}_{V_n} + Q_{V_n})$: then the projection map $\mathbb{P}(\mathcal{F}_n) \longrightarrow V_n$ fits in the cartesian diagram*

$$\begin{tikzcd} \mathbb{P}(\mathcal{F}_n) \arrow[r, hook, "\circ" marking] \arrow[d] & \mathrm{Inc}^Q_{X,\mathcal{L},n,n'}|_{U_n} \arrow[d] \\ V_n \arrow[r, hook, "\circ" marking] & \mathbb{P}(\mathcal{E}_n). \end{tikzcd}$$

Analogously, let V' be the open where g' is smooth: then same definition and properties go for $\mathrm{Inc}^Q_{X,\mathcal{L},n,n'}|_{U_{n+d}} \longrightarrow \mathbb{P}(\mathcal{E}_{n+d})$: but in this case the sheaf considered is $g'_*\mathcal{O}_{C_{V'}}(\mathcal{Z}'_{V'} - Q_{V'})$. We denote the open and the dual sheaf by V'_{n+d} and $\mathcal{F}'_{n+d}$ respectively.

Proof. Since g is flat and projective over a noetherian scheme, by [Bej, §13 Prop. 2] the functor $\mathrm{CDiv}_{C/\mathbb{P}(\mathcal{E}_n)}$ is representable by a scheme. As remarked in 2.0.6, by [BLR90, §Thm. 8.2.1] the functor $\mathrm{Pic}_{C_V/V}$ is representable by a scheme.

Consider the diagram induced by the line bundle $\mathcal{O}_{C_V}(\mathcal{Z}_V + Q_V)$:

$$\begin{tikzcd} \mathrm{CDiv}^{\mathcal{O}_{C_V}(\mathcal{Z}+Q_V)}_{C_V/V} \arrow[r] \arrow[d] & \mathrm{CDiv}_{C_V/V} \arrow[d, "AJ_{C_V/V}"] \\ V \arrow[r, "{[\mathcal{O}_{C_V}(\mathcal{Z}_V+Q_V)]}"] & \mathrm{Pic}_{C_V/V}. \end{tikzcd}$$

By Cohomology and base change [Har77, Thm. III.12.11], the set where

$$g_*\mathcal{O}_{C_V}(\mathcal{Z}_V + Q_V) \otimes \mathcal{O}_{V,y}\, k(y) \longrightarrow H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y))$$

is surjective is an open V_n of V over which the direct image $g_*(\mathcal{O}_{C_V}(\mathcal{Z}_V + Q_V))$ is locally free and its formation commutes with base-change.

Then the projection $\mathbb{P}(\mathcal{F}_n) \longrightarrow V_n$ is the fiber product in

$$\begin{tikzcd} \mathbb{P}(\mathcal{F}_n) \arrow[r] \arrow[d] & \mathrm{CDiv}_{C_{V_n}/V_n} \arrow[d, "AJ_{C_{V_n}/V_n}"] \\ V_n \arrow[r, "{[\mathcal{O}_{C_{V_n}}(\mathcal{Z}_{V_n}+Q_{V_n})]}"] & \mathrm{Pic}_{C_{V_n}/V_n}. \end{tikzcd}$$

Since the fiber product $\mathrm{CDiv}_{C_{V_n}/V_n}^{\mathcal{O}_{C_{V_n}}(\mathcal{Z}_{V_n}+Q_{V_n})}$ is the same as $\mathrm{Inc}^Q_{X,\mathcal{L},n,n'}$ we get the wanted cartesian diagram:

$$\begin{tikzcd} \mathbb{P}(\mathcal{F}_n) \arrow[r, hook] \arrow[d] & \mathrm{Inc}^Q_{X,\mathcal{L},n,n'}|_{U_n} \arrow[d] \\ V_n \arrow[r, hook] & \mathbb{P}(\mathcal{E}_n). \end{tikzcd}$$

The proof for V'_{n+d} goes in the same way. $\square$

Chapter 3: The General fiber is non-empty

The aim of this Chapter is to give sufficient conditions on the triple $(X, \mathcal{L}, Q)$ so that we can ensure that the opens defined in the previous Chapter are all non-empty.

Assumptions 3.0.1. The five assumptions *with respect to n* are the following:

(1) the restriction map $H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}|_Q)$ is surjective,

(2) $\dim_k H^0(X, \mathcal{L}) \geq n + 1 + 2d$,

(3) the sheaf $\mathcal{L} \otimes \mathcal{I}_Q$ is generated by global sections,

(4) for any $\overline{k}$-point $x \in X_{\overline{k}} \setminus Q_{\overline{k}}$, we have that

$$H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)_{\overline{k}} \longrightarrow \mathcal{L}_x / m_x^2 \mathcal{L}_x$$

is surjective,

(5) for any curve C in the linear system $|\mathcal{L}|$, the arithmetic genus $p_a(C) \leq n$.

Remark 3.0.2. Notice that assumption (5) makes sense since C is a geometrically connected curve (see Remark 1.0.1).

Lemma 3.0.3. *Assumptions (1) and (3) imply that $\mathcal{L}$ is generated by global sections.*

Proof. A line bundle $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$ there is a section of $\mathcal{L}$ not vanishing at x. Therefore, it is enough to check this property on closed points.

Let x be a closed point. If $x \notin Q$ then $\mathcal{L}_x \cong \mathcal{L}_x \otimes \mathcal{I}_{Q,x}$ and it is enough to pick a right section in $H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)$, which exists by (3). If $x \in Q$ then by (1) it is enough to pick a lift of a section of $H^0(X, \mathcal{L}|_Q)$ not vanishing at x. □

Remark 3.0.4. By assumption (3), (4) is equivalent to the fact that for each $\overline{k}$-point $x \in X_{\overline{k}} \setminus Q_{\overline{k}}$ the sections $H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)_{\overline{k}}$ separate tangent vectors in $X_{\overline{k}} \setminus Q_{\overline{k}}$. Namely, for each $\overline{k}$-point $x \in X_{\overline{k}} \setminus Q_{\overline{k}}$, the set

$$\{s \in H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)_{\overline{k}} : s_x \in m_x \mathcal{L}_x\}$$

spans the $\overline{k}$-vector space $m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x$,

Remark 3.0.5. By Riemann-Roch for surfaces, for any element C of the linear system $|\mathcal{L}|$ we have

$$p_a(C) = \frac{1}{2} C \cdot (C + K_X) + 1 = \frac{1}{2} c_1(\mathcal{L}) \cdot (c_1(\mathcal{L}) + K_X) + 1.$$

Therefore assumption (5) is equivalent to

$$\frac{c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X}{2} + 1 \leq n.$$

3.0.1 Non-emptiness of $\mathbf{U_n}$ and $\mathbf{U_{n+d}}$

Let us consider the opens U_n and U_{n+d} defined in Proposition 2.1.1.

Proposition 3.0.6. *The opens $U_n \subseteq \mathrm{Hilb}_X^n$ and $U_{n+d} \subseteq \mathrm{Hilb}_X^{n+d}$ are non-empty.*

Let us focus on U_n. In order to show that it is non-empty, we can assume that the field k is algebraically closed. Hence, by definition of U_n 2.1.1, it is enough to find a closed 0-dimensional subscheme Z of length n (defined over $\overline{k}$ such that

$$H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}|_{Z \sqcup Q})$$

is surjective and

$$pr_{H*}(pr_X^*\mathcal{L}) \otimes_{O_{\mathrm{Hilb}^n_X, Z}} k \longrightarrow H^0(X, \mathcal{L})$$

is an isomorphism.

The latter condition is true for any point Z, due to flat base change (see [Sta, Tag 02KH]). So we need to check only the former. In order to do so, we prove a general lemma.

Lemma 3.0.7. *Let k be an algebraically closed field. Assume that a triple $(X, \mathcal{L}, Q)$ satisfies assumption (1). Let N denote $\dim_k H^0(X, \mathcal{L})$. Then for any $0 \leq m \leq N - d$ there exists a degree m effective 0-cycle Z_m such that*

1. *it is reduced,*
2. *it is disjoint from Q, and*
3. *the map*

$$H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}|_{Z_m \sqcup Q})$$

is surjective.

Proof. By assumption (1) we have d-global sections $s_1, \dots, s_d$ of $H^0(X, \mathcal{L})$ such that they form a basis of $H^0(Q, \mathcal{L}|_Q)$. So the statement is true for $m = 0$.

Consider now the greatest non-negative integer m such that for any $m' \leq m$ we have

- a degree m' effective reduced 0-cycle $Z_{m'} \subset X$, disjoint from Q, and
- m'-sections $s_{d+1}, \dots, s_{d+m'}$ of $H^0(X, \mathcal{L})$ such that $\{s_d, \dots, s_d, s_{d+1}, \dots, s_{d+m'}\}$ is a basis of $H^0(X, \mathcal{L}|_{Z_{m'} \sqcup Q})$.

Assume by contradiction that $m < N - d$. Then there exists a section $t \in H^0(X, \mathcal{L})$ linearly independent of $s_1, \dots, s_{d+m}$. Since $\{s_1, \dots, s_{d+m}\}$ is a basis of $H^0(Z_m \sqcup Q, \mathcal{L}|_{Z_m \sqcup Q})$, we can write

$$t|_{Z_m \sqcup Q} = \sum_{i=1}^{d+m} a_i s_i|_{Z_m \sqcup Q} \quad \text{with } a_i \in k.$$

Hence, changing t with $t - \sum_{i=1}^{d+m} a_i s_i$ we can assume that $t|_{Z_m \sqcup Q} = 0$.

Consider the open $X \setminus \{x \in X : t(x) = 0 \in k(x)\}$: it is non-empty, and therefore it has a k-point z. Then set $Z_{m+1} = Z_m \sqcup z$: since $Z_m \sqcup Q \subseteq \{x \in X : t(x) = 0 \in k(x)\}$, Z_{m+1} is a well-defined effective 0-cycle of degree $m+1$, disjoint from Q.

Consider the restriction map

$$H^0(X, \mathcal{L}) \longrightarrow H^0(Z_{m+1} \sqcup Q, \mathcal{L}|_{Z_{m+1} \sqcup Q}). \tag{3.1}$$

Then the elements $s_1|_{Z_{m+1} \sqcup Q}, \ldots, s_{d+m}|_{Z_{m+1} \sqcup Q}, t|_{Z_{m+1} \sqcup Q}$ form a basis of $H^0(Z_{m+1} \sqcup Q, \mathcal{L}|_{Z_{m+1} \sqcup Q})$. Indeed the matrix corresponding to 3.1 looks like

$$\left[\begin{array}{ccc|cccc} & & & a_1 & * & \ldots & * \\ & Id_{d+m} & & \vdots & \vdots & \ddots & \vdots \\ & & & a_{d+m} & * & \ldots & * \\ \hline 0 & \ldots & 0 & b & * & \ldots & * \end{array}\right].$$

However since the element

$$t|_{Z_m \sqcup Q} = \sum_{i=1}^{d+m} a_i s_i|_{Z_m \sqcup Q}$$

is trivial and the elements $s_i|_{Z_m \sqcup Q}$'s are linearly independent, then $a_i = 0$ for any $i = 1, \ldots, d+m$. Since $t|_z = t(z) \neq 0$ then $b \neq 0$ and the above matrix has full rank. Therefore we can increase m by 1 and hence $m \geq N - d$. $\square$

Proof of Proposition 3.0.6. Let us recall that we are still working under the assumption that k is algebraically closed. By assumption (2) the dimension N is greater than $n + 2d$. Then by Lemma 3.0.7 we can find a closed subscheme Z for which the restriction map is surjective. Same proof goes for U_{n+d}: so U_n and U_{n+d} are both non-empty. $\square$

3.0.2 Non-emptiness of $\mathbf{P}(\mathcal{E}_\mathbf{n})$ and $\mathbf{P}(\mathcal{E}_\mathbf{n+d})$

Lemma 3.0.8. *The scheme $\mathbb{P}(\mathcal{E}_n)$ (resp. $\mathbb{P}(\mathcal{E}_{n+d})$) is integral of dimension at least*

$$n + \dim_k H^0(X, \mathcal{L}) - (d+1).$$

(resp. $(n+d) + \dim_k H^0(X, \mathcal{L}) - (d+1)$). In particular, it is non-empty (same holds for $\mathbb{P}(\mathcal{E}_{n+d})$).

Proof. Consider the projective-bundle $\mathbb{P}(\mathcal{E}_n) \longrightarrow U_n$ (for $\mathbb{P}(\mathcal{E}_{n+d})$ it is analogous). For any point $y \in U_n$ we have

$$\dim_{k(y)} H^0(X_y, \mathcal{L}_y \otimes \mathcal{I}_{\mathcal{Z}_y \sqcup Q_y}) \geq \dim_{k(y)} H^0(X_y, \mathcal{L}_y) - (n+d)$$

Therefore, the dimension of $\mathbb{P}(\mathcal{E}_n)$ is at least

$$\dim U_n + \min_{y \in U_n} \dim_{k(y)} H^0(X_y, \mathcal{L}_y \otimes \mathcal{I}_{\mathcal{Z}_y \sqcup Q_y}) - 1 \geq 2n + \dim_k H^0(X, \mathcal{L}) - (n+d+1).$$

By assumption (2), $\dim_k H^0(X, \mathcal{L}) \geq n + 1 + 2d$: so we get

$$\dim \mathbb{P}(\mathcal{E}_n) \geq 2n + d$$

In particular it is non-empty.

For what concerns being integral, let us notice first that U_n is integral, because it is an open of the integral scheme Hilb^n_X. Therefore, since $\mathcal{E}_n$ is locally free, $\mathrm{Sym}^*(\mathcal{E}_n)$ is a sheaf of integral domains on U_n: so $\mathbb{P}(\mathcal{E}_n) = \underline{\mathrm{Proj}}(\mathrm{Sym}^*(\mathcal{E}_n))$ is integral as well. □

3.0.3 Non-emptiness of $\mathbf{V_n}$ and $\mathbf{V'_{n+d}}$

It is now time to turn our attention to the opens $V_n \subseteq \mathbb{P}(\mathcal{E}_n)$ and $V'_{n+d} \subseteq \mathbb{P}(\mathcal{E}_{n+d})$. Recall that by definition, V_n is the open made of points $y \in \mathbb{P}(\mathcal{E}_n)$ for which

1. the map g in 2.5 is smooth (call the open where this happens V), and
2. Cohomology and base change holds for the map

$$g_*\mathcal{O}_{C_V}(\mathcal{Z}_V + Q_V) \otimes_{\mathcal{O}_{V,y}} k(y) \longrightarrow H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y)). \tag{3.2}$$

Similarly, V'_{n+d} is the open made of points $y \in \mathbb{P}(\mathcal{E}_{n+d})$ for which

1. the map g' is smooth (call the open where this happens V'), and
2. Cohomology and base change holds for the map

$$g'_*\mathcal{O}_{C_{V'}}(\mathcal{Z}'_{V'} - Q_{V'}) \otimes_{\mathcal{O}_{V',y}} k(y) \longrightarrow H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}'_y - Q_y)). \tag{3.3}$$

We proceed to first check that the opens V and V' are non-empty, and only subsequently we focus on V_n and V'_{n+d}.

Proposition 3.0.9 (Bertini's theorem with a fixed closed)**.** *The open locus V (resp. V') where g (resp. g') is smooth are non-empty (and integral).*

Proof. - The proof is a modification of the proof of classical Bertini's theorem ([Har77, Thm. II.8.18]) and de Jong's proof ([Sta, Tag. 0FD6]). Let Σ be the singular locus in C, namely the closed subscheme of points in C where g is not smooth. We need to check that Σ is not the whole C.

The image $g(\Sigma)$ is a closed subset of the integral scheme $\mathbb{P}(\mathcal{E}_n)$. Since g is surjective, in order to show that $\Sigma \neq C$ it is enough to prove that

$$\dim g(\Sigma) < \dim \mathbb{P}(\mathcal{E}_n).$$

Dimensions are unaltered by passing to the algebraic closure of k: so we can assume that $k = \overline{k}$.

Claim. Let x be a k-point of X. Consider the fiber Σ_x via the map

$$\Sigma \subseteq C \subset X \times \mathbb{P}(\mathcal{E}_n) \xrightarrow{prx} X.$$

Then

1. for all but finitely many k-points $x \in X$ we have $\dim \Sigma_x \leq \dim \mathbb{P}(\mathcal{E}_n) - 3$, and

2. for the remaining k-points we have $\dim \Sigma_x \leq \dim \mathbb{P}(\mathcal{E}_n) - 1$:

Proof of the Claim. By definition of Σ the set $\Sigma_x(k)$ coincides with the pairs

$$\{(Z, C) \in U_n(k) \times \mathbb{P}(H^0(X, \mathcal{L})^\vee)(k) : Z \sqcup Q \subset C, \text{ and } x \in C \text{ is singular}\}.$$

In terms of sections $f \in H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)^*$, $\Sigma_x(k)$ can be described as

$$\{(Z, f) : Z \sqcup Q \subset V(f) \subset X, \text{ and } x \in V(f) \text{ is singular}\}/f \sim \lambda f'.$$

Now $x \in V(f)$ if and only if $f_x \in m_x\mathcal{L}_x$, and $x \in V(f)$ is singular if and only if $f_x \in m_x^2\mathcal{L}_x$. Indeed being singular at x means

$$\dim_k m_x/(m_x^2, f_x) > 1 = \dim \mathcal{O}_{V(f),x}.$$

If $f_x \in m_x^2$, then the dimension on the right is 2 and if $f_x \in m_x \setminus m_x^2$ the dimension drops to 1. So the sections f for which x is singular (together with the zero-section) correspond to the kernel

$$\phi_x : H^0(X, \mathcal{L} \otimes \mathcal{I}_Q) \longrightarrow \mathcal{L}_x / m_x^2 \mathcal{L}_x.$$

Therefore the set $\Sigma_x(k)$ coincides with

$$\{(Z, f) \in U_n(k) \times \mathbb{P}(\ker(\phi_x)^\vee)(k) : Z \in \mathrm{Hilb}^n_{V(f)}\}.$$

This implies that the dimension of Σ_x is smaller or equal to (we use assumption (1))

$$\dim \mathbb{P}(\ker(\phi_x)^\vee) + \max_{f \neq 0} \dim \mathrm{Hilb}^n_{V(f)} = \dim H^0(X, \mathcal{L} \otimes \mathcal{I}_Q) - 1 - \dim \mathrm{im}(\phi_x) + n =$$
$$= n + \dim_k H^0(X, \mathcal{L}) - (d+1) - \dim \mathrm{im}(\phi_x) \leq \dim \mathbb{P}(\mathcal{E}_n) - \dim \mathrm{im}(\phi_x).$$

If x does not belong to Q then by assumptions (3) and (4) we have that ϕ_x is surjective (see Remark 3.0.4). So in this case

$$\dim \Sigma_x \leq \dim \mathbb{P}(\mathcal{E}_n) - 3.$$

If $x \in Q$ then $\mathcal{L}_x \otimes \mathcal{I}_{Q,x} = \mathcal{L}_x \otimes m_x$ (Q is reduced at x). Since $\mathcal{L} \otimes \mathcal{I}_Q$ is generated by global sections, the map ϕ_x is not the zero-map, and so the image has dimension at least 1. Therefore

$$\dim \Sigma_x \leq \dim \mathbb{P}(\mathcal{E}_n) - 1.$$

□

From the bounds of the claim it follows directly that $\dim g(\Sigma)$ is smaller that $\dim \mathbb{P}(\mathcal{E}_n)$ (see for instance [Sta, Tag 0B2L]). The proof for V' is analogous. □

We can finally focus on the opens V_n and V'_{n+d}. In order to prove that they are non-empty it remains to check that there is a point y (resp. y') for which equation 3.2 (resp. 3.3) holds. In order to do so, we prove a sufficient condition first.

Lemma 3.0.10. *Let $y \in V$: if $H^1(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y))$ is trivial, then equation 3.2 holds.*

Analogously, let $y' \in V'$: if $H^1(C_{y'}, \mathcal{O}_{C_{y'}}(\mathcal{Z}'_{y'} - Q_{y'})) = 0$ then equation 3.3 holds.

Proof. This is a standard Cohomology and base change argument.

By upper semi-continuity in y of $\dim_{k(y)} H^1(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y))$, the set where the H^1 is trivial is open ([Har77, Thm. III.12.8]). Then on a (connected) open neighbourhood U_y of y (which is integral because open of V, integral scheme) the sheaf $R^1 g_* \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y)$ is identically zero (see [Har77, Cor. III.12.9]).

By Cohomology and base change [Har77, Thm. III.12.11(b)], on U_y the base change map at the level of H^0 is an isomorphism. □

Proposition 3.0.11. *Thanks to assumption (5), the locus of points $y \in V$ such that $H^1(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y))$ is trivial, is non-empty. Analogously, the locus of points $y \in V'$ such that $H^1(C_{y'}, \mathcal{O}_{C_{y'}}(\mathcal{Z}'_{y'} - Q_{y'})) = 0$ is non-empty.*

In particular, the opens V_n and V'_{n+d} are non-empty.

Proof. Denote by $\mathrm{CDiv}_X^{Q\subset\mathcal{L}}$ the subfunctor of $\mathrm{CDiv}_X^{\mathcal{L}}$ of the divisors in $\mathrm{CDiv}_X^{\mathcal{L}}$ containing Q. By Lemma 2.0.7 it is closed subscheme of $\mathrm{CDiv}_X^{\mathcal{L}}$.

Denote by $\mathrm{SmCDiv}_X^{Q\subset\mathcal{L}}$ the open subscheme of $\mathrm{CDiv}_X^{Q\subset\mathcal{L}}$ consisting of divisors containing Q and smooth over the base. It is non empty because V is non-empty. Then we have the following diagram (all the squares are cartesian):

$$\begin{tikzcd}
V \arrow[rr] \arrow[d, hook] & & \mathrm{SmCDiv}_X^{Q\subset\mathcal{L}} \arrow[d, hook] & \\
\mathbb{P}(\mathcal{E}_n) \arrow[r, hook, "\sim"] & \mathcal{P} \arrow[r] \arrow[d, hook] & \mathrm{CDiv}_X^{Q\subset\mathcal{L}} \arrow[d, hook] & \\
& \mathbb{P}((pr_{H*}(pr_X^*\mathcal{L})|_{U_n})^\vee) \arrow[r] \arrow[d] & \mathrm{CDiv}_X^{\mathcal{L}} \arrow[r] \arrow[d] & \mathrm{CDiv}_X \arrow[d, "AJ_X"] \\
& U_n \arrow[r] & \mathrm{Spec}(k) \arrow[r, "{[\mathcal{L}]}"] & \mathrm{Pic}_X
\end{tikzcd}$$

Let $y \in \mathrm{SmCDiv}_X^{Q\subset\mathcal{L}}$ be a geometric point such that the fiber V_y is non-empty. Namely, y corresponds to a smooth curve $C \subset X_{k(y)}$ containing $Q_{k(y)}$ (and $k(y) = \overline{k(y)}$).

The $k(y)$-points of V_y are exactly those closed subscheme $Z \subset C$ of degree n such that $Z \in U_n(k(y))$. More generally, the T-points of V_y are the closed subschemes $Z \subset C_T$, flat and finite of degree n over T, such that $Z \in U_n(T)$. In other words, the fiber V_y coincides with $U_n \times_{\mathrm{Hilb}_X^n} \mathrm{Hilb}^n_{C/k(y)}$, an open subscheme of $\mathrm{Hilb}^n_{C/k(y)}$.

Consider now the Abel-Jacobi map for the curve C

$$AJ_C : \mathrm{Hilb}^n_{C/k(y)} \longrightarrow \mathrm{Pic}^n_{C/k(y)}. \tag{3.4}$$

By assumption (5) the genus of C is less or equal than the degree n: by Riemann-Roch, this means that the Abel-Jacobi map 3.4 is surjective. Therefore, since $\mathrm{Hilb}^n_{C/k(y)}$ is integral, the composite

$$V_y \hookrightarrow \mathrm{Hilb}^n_{C/k(y)} \xrightarrow{AJ_C} \mathrm{Pic}^n_{C/k(y)}$$

is dominant. Further composing with the isomorphism

$$\mathrm{Pic}^n_{C/k(y)} \xrightarrow{\sim} \mathrm{Pic}^{n+d}_{C/k(y)}, \quad \mathcal{M} \mapsto \mathcal{M} \otimes \mathcal{O}_C(Q_{k(y)}),$$

we get another dominant map

$$V_y \hookrightarrow \mathrm{Hilb}^n_{C/k(y)} \longrightarrow \mathrm{Pic}^n_{C/k(y)} \longrightarrow \mathrm{Pic}^{n+d}_{C/k(y)}.$$

Consider the closed subschemes $W^{n+d-p_a(C)}_{n+d}(C/k(y))$ and $W^{n+d-p_a(C)+1}_{n+d}(C/k(y))$ inside $\mathrm{Pic}^{n+d}_{C/k(y)}$. By Riemann-Roch $W^{n+d-p_a(C)}_{n+d}(C/k(y))$ coincides with $\mathrm{Pic}^{n+d}_{C/k(y)}$. Since $p_a(C) + n + d - p_a(C) - n - d \geq 0$, by the Brill-Noether theory Fact 1.0.4, we have

$$\mathrm{Pic}^{n+d}_{C/k(y)} \setminus W^{n+d-p_a(C)+1}_{n+d}(C/k(y))$$

is non-empty, open and hence integral (because $\mathrm{Pic}^{n+d}_{C_y/k(y)}$ is integral).

The K-points of $\mathrm{Pic}^{n+d}_{C/k(y)} \setminus W^{n+d-p_a(C)+1}_{n+d}(C/k(y))$ are exactly those classes of line bundles $\mathcal{O}_{C_K}(Z+Q_K) \in \mathrm{Pic}^{n+d}_{C/k(y)}(K)$ such that $H^0(C_K, \mathcal{O}_{C_K}(Z+Q_K)) = n+d-p_a(C)+1$ or equivalently $H^1(C_K, \mathcal{O}_{C_K}(Z+Q_K)) = 0$.

The preimage of $\mathrm{Pic}^{n+d}_{C/k(y)} \setminus W^{n+d-p_a(C)+1}_{n+d}(C/k(y))$ in V_y is a non-empty open: by Lemma 3.0.10 V_n is non-empty.

The proof for the open V'_{n+d} is the same: in this case we set $m = n$ and $r = n - p_a(C)$ (still non negative by assumption (5)). Again by Fact 1.0.4, the open $\mathrm{Pic}^{n}_{C/k(y')} \setminus W^{n-p_a(C)+1}_{n}(C/k(y'))$ is non-empty which gives a non-empty open of $V'_{y'}$ via the map

$$V'_{y'} \hookrightarrow \mathrm{Hilb}^{n+d}_{C/k(y')} \longrightarrow \mathrm{Pic}^{n+d}_{C/k(y')} \xrightarrow{-\otimes \mathcal{O}_C(-Q_{k(y')})} \mathrm{Pic}^{n}_{C/k(y')}.$$

Lemma 3.0.10 concludes the proof as before. $\square$

Chapter 4: Stable Birational classes of Hilbert schemes

As checked in the previous Chapter, we have four non-empty integral opens

$$U_n,\ U_{n+d},\ V_n \text{ and } V'_{n+d}$$

such that they fit in the cartesian diagram:

$$\begin{tikzcd}
 & & \mathbb{P}(\mathcal{F}'_{n+d}) \arrow[r, "p'_{n+d}"] \arrow[dd, hook, "\circ" marking] & V'_{n+d} \arrow[d, hook, "\circ" marking] & \\
 & & & \mathbb{P}(\mathcal{E}_{n+d}) \arrow[r, two heads] \arrow[d, hook, "\circ" marking] & U_{n+d} \arrow[d, hook, "\circ" marking] \\
\mathbb{P}(\mathcal{F}_n) \arrow[rr, hook, "\circ" marking] \arrow[d, "p_n"'] & & \mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'} \arrow[r] \arrow[d] & \mathrm{Inc}^{Q}_{X,\mathcal{L},n'} \arrow[r] & \mathrm{Hilb}^{n+d}_X \\
V_n \arrow[r, hook, "\circ" marking] & \mathbb{P}(\mathcal{E}_n) \arrow[r, hook, "\circ" marking] \arrow[d, two heads] & \mathrm{Inc}^{Q}_{X,\mathcal{L},n} \arrow[d] & & \\
 & U_n \arrow[r, hook, "\circ" marking] & \mathrm{Hilb}^n_X. & &
\end{tikzcd}$$

Remark 4.0.1. The projections $p_n : \mathbb{P}(\mathcal{F}_n) \longrightarrow V_n$ and $p'_{n+d} : \mathbb{P}(\mathcal{F}'_{n+d}) \longrightarrow V'_{n+d}$ are surjective as well. Indeed the rank of $\mathcal{F}_n$ at $y \in V_n$ is

$$\dim_{k(y)} H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y + Q_y)) \geq \deg(\mathcal{Z}_y + Q_y) - p_a(C_y) + 1 = n + d - p_a(C) + 1 \geq 1,$$

and the rank of $\mathcal{F}'_{n+d}$ at $y \in V'_{n+d}$ is

$$\dim_{k(y)} H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}'_y - Q_y)) \geq \deg(\mathcal{Z}'_y - Q_y) - p_a(C_y) + 1 = n - p_a(C) + 1 \geq 1,$$

Now we want to check that the two (integral) opens $\mathbb{P}(\mathcal{F}_n)$ and $\mathbb{P}(\mathcal{F}'_{n+d})$ intersect.

Proposition 4.0.2. *The two non-empty open subschemes* $\mathbb{P}(\mathcal{F}_n)$ *and* $\mathbb{P}(\mathcal{F}'_{n+d})$ *intersect in* $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n+d}$.

Proof. Consider a geometric point $y \in V'_{n+d}$ for which the group $H^1(C_y, \mathcal{O}_{C_y}(\mathcal{Z}'_y - Q_y))$ is trivial. Pick a point $(Z, \mathcal{Z}'_y, C_y) \in \mathbb{P}(\mathcal{F}'_{n+d})(k(y))$ (it exists because p'_{n+d} is surjective). By definition, Z is an effective divisor on C_y equivalent to $\mathcal{Z}'_y - Q_y$ and disjoint from Q_y. We claim that $(Z, \mathcal{Z}_y, C_y) \in \mathbb{P}(\mathcal{F}_n)(k(y))$: equivalently $(Z, C_y) \in V_n(k(y))$.

Let us take the cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_{C_y}(\mathcal{Z}'_y - Q_y) \longrightarrow \mathcal{O}_{C_y}(\mathcal{Z}'_y) \longrightarrow \mathcal{O}_{C_y}(\mathcal{Z}'_y)|_{Q_y} \longrightarrow 0.$$

We obtain

$$H^0(C_y, \mathcal{O}_{C_y}(\mathcal{Z}'_y)|_{Q_y}) \longrightarrow 0 \longrightarrow H^1(C_y, \mathcal{O}_{C_y}(\mathcal{Z}'_y)) \longrightarrow 0.$$

Therefore $H^1(C_y, \mathcal{O}_{C_y}(Z + Q_y)) = 0$. In particular Cohomology and Base change holds true for $\mathcal{O}_{C_y}(Z + Q_y)$.

It remains to prove that

$$H^0(X_y, \mathcal{L}_y) \longrightarrow H^0(X_y, \mathcal{L}_y|_{Z \sqcup Q_y})$$

is surjective. In order to show this, we consider the diagram

$$\begin{tikzcd}
H^0(X_y, \mathcal{L}_y) \arrow[r] & H^0(C_y, \mathcal{L}_y|_{C_y}) \arrow[r] \arrow[d] & H^0(Z \sqcup Q_y, \mathcal{L}_y|_{Z \sqcup Q_y}) \\
& H^0(\mathcal{Z}'_y \sqcup Q_y, \mathcal{L}_y|_{\mathcal{Z}'_y \sqcup Q_y}). &
\end{tikzcd}$$

Since the restriction from X_y to $\mathcal{Z}'_y \sqcup Q_y$ is surjective, then the restriction from C_y

$$H^0(C_y, \mathcal{L}_y|_{C_y}) \longrightarrow H^0(\mathcal{Z}'_y \sqcup Q_y, \mathcal{L}_y|_{\mathcal{Z}'_y \sqcup Q_y})$$

is surjective as well. Moreover the map

$$H^0(X_y, \mathcal{L}_y) \longrightarrow H^0(C_y, \mathcal{L}_y|_{C_y})$$

is always surjective because it is induced by the short exact sequence

$$0 \longrightarrow \mathcal{I}_{C_y} \otimes \mathcal{L}_y \longrightarrow \mathcal{L}_y \longrightarrow \mathcal{L}_y|_{C_y} \longrightarrow 0$$

and $H^1(X_y, \mathcal{I}_{C_y} \otimes \mathcal{L}_y) \cong H^1(X_y, \mathcal{O}_{X_y}) = 0$ by hypothesis on X. Therefore we just have to prove that

$$H^0(C_y, \mathcal{L}_y|_{C_y}) \longrightarrow H^0(Z \sqcup Q_y, \mathcal{L}_y|_{Z \sqcup Q_y})$$

is surjective. Equivalently we have to prove that

$$H^1(C_y, \mathcal{L}_y|_{C_y}(-Z - Q_y)) \longrightarrow H^1(C_y, \mathcal{L}_y|_{C_y}) \longrightarrow 0 \tag{4.1}$$

is injective. Since this map is always surjective, injectivity is equivalent to have the following inequality:

$$\dim_{k(y)} H^1(C_y, \mathcal{L}_y|_{C_y}) \geq \dim_{k(y)} H^1(C_y, \mathcal{L}_y|_{C_y}(-Z - Q_y)).$$

Since the map $H^0(C_y, \mathcal{L}_y|_{C_y}) \longrightarrow H^0(\mathcal{Z}'_y \sqcup Q_y, \mathcal{L}_y|_{\mathcal{Z}'_y \sqcup Q_y})$ is surjective, then the left-hand side is equal to

$$\dim_{k(y)} H^1(C_y, \mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y - Q_y)).$$

Using the short exact sequence

$$0 \longrightarrow \mathcal{O}_{C_y}(-Q_y) \longrightarrow \mathcal{O}_{C_y} \longrightarrow \mathcal{O}_{Q_y} \longrightarrow 0$$

tensored by $\mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y)$, we get at the cohomology level the following exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(C_y, \mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y - Q_y)) \longrightarrow H^0(C_y, \mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y)) \longrightarrow H^0(Q_y, \mathcal{O}_{Q_y}) \longrightarrow \\ \longrightarrow H^1(C_y, \mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y - Q_y)) \longrightarrow H^1(C_y, \mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y)) \longrightarrow 0. \end{aligned}$$

In particular we get that

$$\dim_{k(y)} H^1(C_y, \mathcal{L}_y|_{C_y}(-\mathcal{Z}'_y - Q_y)) \geq \dim_{k(y)} H^1(C_y, \mathcal{L}_y|_{C_y}(-Z - Q_y)),$$

which is the wanted inequality. Therefore the point $(Z, C_y) \in V_n(k(y))$ and the two opens intersect. □

Corollary 4.0.3 (Theorem A)**.** *If a suitable triple $(X, \mathcal{L}, Q)$ satisfies assumptions (1) to (5) with respect to the integer n, then* Hilb^n_X *is stably birational to* Hilb^{n+d}_X.

Proof. Pick a point $y = (Z, Z', C)$ in the intersection of $\mathbb{P}(\mathcal{F}_n)$ and $\mathbb{P}(\mathcal{F}'_{n+d})$. Consider the images $p_n(y) = (Z, C)$ and $p'_{n+d}(y) = (Z', C)$ in V_n and V'_{n+d}. Let U_Z be an integral open neighbourhood of $Z \in U_n$, which trivializes the projective bundle $\mathbb{P}(\mathcal{E}_n)$:

$$\begin{tikzcd} U_Z \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}_n)-1} \arrow[r, hook, "\circ"] \arrow[d] & \mathbb{P}(\mathcal{E}_n) \arrow[d] \\ U_Z \arrow[r, hook, "\circ"] & U_n. \end{tikzcd}$$

Since $\mathbb{P}(\mathcal{E}_n)$ is integral, the open $U_Z \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}_n)-1}$ is integral as well. Define $U_{Z'}$ in the same way.

Let $V_{(Z,C)}$ be an integral open neighbourhood of $p_n(y)$ contained in $V_n \cap U_Z \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}_n)-1}$, which trivializes the projective bundle $\mathbb{P}(\mathcal{F}_n)$:

$$\begin{tikzcd}
V_{(Z,C)} \times \mathbb{P}_k^{\mathrm{rk}_{(Z,C)}(\mathcal{F}_n)-1} \arrow[rr, hook, "\circ" marking] \arrow[d, "p_n|_{V_{(Z,C)}}"'] & & \mathbb{P}(\mathcal{F}_n) \arrow[d, "p_n"] \\
V_{(Z,C)} \arrow[r, hook, "\circ" marking] & V_n \cap U_Z \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}_n)-1} \arrow[r, hook, "\circ" marking] & V_n
\end{tikzcd}$$

Since $V_{(Z,C)}$ and $U_Z \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}_n)-1}$ are both integral neighbourhoods of $p_n(y)$, they are birational. Define $V'_{(Z',C)}$ in the same way. Therefore, we have that

$$U_Z \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}_n)+\mathrm{rk}_{(Z,C)}(\mathcal{F}_n)-2} \sim_{\mathrm{bir}} V_{(Z,C)} \times \mathbb{P}_k^{\mathrm{rk}_{(Z,C)}(\mathcal{F}_n)-1}$$

and

$$U_{Z'} \times \mathbb{P}_k^{\mathrm{rk}_Z(\mathcal{E}'_{n+d})+\mathrm{rk}_{(Z',C)}(\mathcal{F}'_{n+d})-2} \sim_{\mathrm{bir}} V'_{(Z',C)} \times \mathbb{P}_k^{\mathrm{rk}_{(Z',C)}(\mathcal{F}'_{n+d})-1}.$$

Since $V_{(Z,C)}, V'_{(Z',C')}$ are integral, then $V_{(Z,C)} \times \mathbb{P}_k^{\mathrm{rk}_{(Z,C)}(\mathcal{F}_n)-1}$ and $V'_{(Z',C)} \times \mathbb{P}_k^{\mathrm{rk}_{(Z',C)}(\mathcal{F}'_{n+d})-1}$ are integral as well. Since they are both neighbourhoods of y they are birational.

Therefore, the two opens U_Z and $U_{Z'}$ are stably-birational, and so are Hilb_X^n and Hilb_X^{n+d}. □

4.1 Asymptotic behavior of Stable Birational classes of Hilb_X^n

In this section we are interested in finding sufficient and necessary criteria on n and on X so that we find suitable triples satisfying the assumptions 3.0.1 with respect to n (in order to apply Corollary 4.0.3).

Given a suitable triple $(X, \mathcal{L}, Q)$ as in the introduction, we want to check for which positive integers (e, n) the triple $(X, \mathcal{L}^{\otimes e}, Q)$ satisfies the assumptions with respect to n.

Theorem 4.1.1 (Theorem B)**.** *Let $(X, \mathcal{L}, Q)$ be a suitable triple. Then there exists an integer $e_0 = e_0(X, \mathcal{L}, Q)$ such that for any $e \geq e_0$ the triple $(X, \mathcal{L}^{\otimes e}, Q)$ satisfies the assumptions 3.0.1* *with respect to n if and only if n is in the interval*

$$I_e := \left[\frac{e^2 c_1(\mathcal{L})^2 + e\, c_1(\mathcal{L}) \cdot K_X}{2} + 1, \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + \dim_k H^2(X, \mathcal{O}_X) - 2\deg(Q) \right],$$

in which case the variety Hilb^n_X *is stably birational to* $\mathrm{Hilb}^{n+\deg(Q)}_X$.

Proof. Let us consider one assumption at a time.

Assumption 1. By Serre vanishing theorem, the group $H^1(X, \mathcal{L}^{\otimes e} \otimes \mathcal{I}_Q) = 0$ for any $e >> 0$. Therefore, the restriction map

$$H^0(X, \mathcal{L}^{\otimes e}) \longrightarrow H^0(X, \mathcal{L}^{\otimes e}|_Q) \longrightarrow 0$$

is surjective for any $e >> 0$.

Assumption 2. Since $\mathcal{L}$ is ample, again by Serre vanishing theorem $H^i(X, \mathcal{L}^{\otimes e}) = 0$ for any $i > 0$ and $e >> 0$. Then, by Riemann Roch, we have that for any $e >> 0$

$$\dim_k H^0(X, \mathcal{L}^{\otimes e}) = \chi(X, \mathcal{L}^{\otimes e}) = \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + 1 + \dim_k H^2(X, \mathcal{O}_X).$$

Assumption 3. Since $\mathcal{L}$ is ample, the sheaf $\mathcal{L}^{\otimes e} \otimes \mathcal{I}_Q$ is generated by global sections for any $e >> 0$.

Assumption 4. We need to check that for each $\overline{k}$-point $x \in X_{\overline{k}} \setminus Q_{\overline{k}}$ the set

$$\{s \in H^0(X, \mathcal{L} \otimes \mathcal{I}_Q)_{\overline{k}} : s_x \in m_x \mathcal{L}_x\}$$

spans the $\overline{k}$-vector space $m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x$. As seen in the proof of [Sta, Tag 0FD5], if we pick $e \geq e_g + e_{va}$ where $\mathcal{L}^{e_g} \otimes \mathcal{I}_Q$ is globally generated and $\mathcal{L}^{e_{va}}$ is very ample then we have a locally closed immersion

$$X \setminus Q \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{I}_Q)^\vee).$$

This is enough to ensure that assumption (4) is satisfied (see [Har77, Prop. II.7.3]).

Assumption 5. Finally, the arithmetic genus $p_a(C)$ of any curve $C \in |\mathcal{L}^{\otimes e}|$ is equal to

$$\frac{c_1(\mathcal{L}^{\otimes e})^2 + (c_1(\mathcal{L}^{\otimes e}) \cdot K_X)}{2} + 1 = \frac{e^2 c_1(\mathcal{L})^2 + e\, c_1(\mathcal{L}) \cdot K_X}{2} + 1.$$

Therefore for $e >> 0$ the only restrictions given by the assumptions are the inequalities about n. □

Notice that the theorem has a meaning only after one checks that some predicted intervals I_e are non-empty (see subsection 4.1.2).

4.1.1 Blow-ups of surfaces

Let $\widetilde{X}$ be the blow up of X along some geometrically reduced effective 0-cycle Q'. Then by Fact we have that Hilb^n_X is stably birational to $\mathrm{Hilb}^n_{\widetilde{X}}$.

We aim to exploit theorem 4.1.1 for $\widetilde{X}$ instead of X, and gain information on the stable birationality classes of Hilb^n_X. Let us fix the following general framework.

Framework. Let $(X, \mathcal{L}, Q)$ be a suitable triple. Let us furthermore fix a geometrically reduced effective 0-cycle Q' of degree d', disjoint from Q.

Consider the blow-up $\widetilde{X} = Bl_{Q'}X$ of X at Q'. Call E the exceptional divisor above Q'. Then we have the cartesian diagram

$$\begin{tikzcd} E \arrow[r, hook, "i_E"] \arrow[d, "b_E"] & \widetilde{X} \arrow[d, "b"] \\ Q' \arrow[r, hook, "i'"] & X. \end{tikzcd}$$

Denote by $\widetilde{\mathcal{L}}_e$ the line bundle $\widetilde{\mathcal{L}}_e = b^*(\mathcal{L}^{\otimes e}) \otimes \mathcal{O}_{\widetilde{X}}(-2E)$ on $\widetilde{X}$. Finally, let $\widetilde{Q}$ be the pullback

$b^{-1}(Q)$ in $\widetilde{X}$.

Now we want to study the triples $(\widetilde{X}, \widetilde{\mathcal{L}}_e, \widetilde{Q})$. These notations will be used consistently throughout this subsection.

Before proceeding further, let us now see that given an effective 0-cycle Z on any (smooth, projective, geometrically connected) surface X there is always a smooth effective 0-cycle Q' of degree $d' \geq 2\deg(Q)$ disjoint from Z. Recall first the usual avoidance lemma.

Lemma 4.1.2. *Given an effective* 0*-cycle* Z *in* $\mathbb{P}^m_k$*, there exists a non-empty (and hence dense) open* $U \subseteq (\mathbb{P}^m_k)^*$ *of hyperplanes of* $\mathbb{P}^m_k$ *avoiding* Z*.*

If k *is infinite, this means that there exist infinitely many* k*-hyperplanes in* $\mathbb{P}^m_k$ *avoiding* Z*.*

Proof. Avoiding Z is the same as avoiding its reduction Z_{red}: then we can assume Z to be reduced. Write Z as the disjoint union of its integral (disjoint) components $\sqcup Z_i$. If an hyperplane H intersect Z_i then $Z_i \subset H$: hence the locus U avoiding Z is open.

Notice that U is non-empty if and only if $U_{\overline{k}}$ is non-empty. Therefore, without loss of generality we can work on the algebraic closure $\overline{k}$.

Let P_i the homogeneous prime ideal associated to Z_i. Assume by contradiction that U is empty: that means that any homogeneous polynomial F of degree 1 is contained in at least one of the P_i. Namely $k[x_0, \ldots, x_m]_1 \subseteq \cup_i P_i$. Hence

$$k[x_0, \ldots, x_m]_1 = \cup_i (k[x_0, \ldots, x_m]_1 \cap P_i) = \cup_i \mathrm{Span}_k \langle k[x_0, \ldots, x_m]_1 \cap P_i \rangle.$$

Since $k[x_0, \ldots, x_m]_1$ is an infinite dimensional vector space and k is infinite (because k is algebraically closed), it cannot be a finite union of proper subspaces of itself.

Then there is a P_i such that

$$k[x_0, \ldots, x_m]_1 = \mathrm{Span}_k \langle k[x_0, \ldots, x_m]_1 \cap P_i \rangle \subseteq P_i.$$

Therefore, $k[x_0, \dots, x_m]_+ \subseteq P_i$, which is not possible. So we have a dense open $U \subseteq (\mathbb{P}^m_k)^*$: if k is infinite, the open U has infinitely many k-rational points. $\square$

Proposition 4.1.3. *Let X be a (smooth, projective and geometrically connected) surface and let Z be a 0-cycle: then there exists a geometrically reduced effective 0-cycle Q' of degree $d' \geq 2\deg(Z)$ disjoint from Z.*

Proof. Consider a very ample divisor H_X and the closed embeddings $X \hookrightarrow \mathbb{P}^{N_m}_k$ induced by its positive multiples mH_X. Call d_X the degree of the embedding defined by H_X.

If k is infinite, let U_Z be the open dense subset of hyperplanes avoiding Z given by Lemma 4.1.2. By Bertini's theorem [Har77, Thm. II.8.18] there is an hyperplane $H \subset \mathbb{P}^N_k$ defined over k such that $X \cap H$ is a smooth curve. Let $V_{X\cap H}$ be the dense open set of hyperplanes in $\mathbb{P}^{N_m}_k$ whose intersection with $X \cap H$ is smooth (provided again by Bertini's). Then the intersection $U_Z \cap V_{X\cap H}$ is still an open dense subset and (since k is infinite) it contains infinitely many k-points. Pick then $H' \in U_Z \cap V_{X\cap H}(k)$. This gives a geometrically reduced effective 0-cycle $Q' = X \cap H \cap H'$ of degree $m^2 d_X$ disjoint from Z.

If k is finite, by Poonen's version of Bertini's for smooth quasi-projective varieties over finite fields [Poo04, Thm. 1.1] we can pick two hypersurfaces D_m and D'_m of degree d_m and d'_m such that $(X \setminus Z) \cap D_m \cap D'_m$ is a smooth effective 0-cycle Q' of degree $(m^2 \cdot d_m \cdot d'_m)d_X$. $\square$

Now, back to framework 4.1.1, let us check first whether a triple $(\widetilde{X}, \widetilde{\mathcal{L}}_e, \widetilde{Q})$ is indeed suitable.

Lemma 4.1.4. *For $e >> 0$ the triple $(\widetilde{X}, \widetilde{\mathcal{L}}_e, \widetilde{Q})$ is suitable.*

Proof. The surface $\widetilde{X}$ is a suitable surface (it is geometrically integral since it is the blow-up of a geometrically integral scheme, see [Sta, Tag 02ND]) Indeed its $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is still trivial (see [Har77, Prop. V.3.4]).

Since Q is disjoint from Q', the scheme $\widetilde{Q}$ is isomorphic to Q and in particular it is a geometrically reduced effective 0-cycle of degree d on $\widetilde{X}$.

Therefore, to conclude that $(\widetilde{X}, \widetilde{\mathcal{L}}_e, \widetilde{Q})$ is suitable it is enough to check that for $e >> 0$ the line bundle $\widetilde{\mathcal{L}}_e$ is ample.

Let e be big enough so that $\mathcal{I}_{Q'} \otimes \mathcal{L}^{\otimes e}$ is globally generated. Denote by $\mathcal{O}_{\widetilde{X}/X}(1)$[1] the invertible sheaf on the relative projective scheme $\widetilde{X}$ (over X). Let r_e+1 be a set of global sections $\{s_0, \ldots, s_{r_e}\}$ which generate the sheaf. They define a surjective map

$$\mathcal{O}_X^{r_e+1} \longrightarrow \mathcal{I}_{Q'} \otimes \mathcal{L}^{\otimes e} \longrightarrow 0.$$

This in turns induces a surjective morphism

$$\mathcal{O}_X[s_0, \ldots, s_{r_e}] \longrightarrow \mathrm{Sym}^*(\mathcal{I}_{Q'} \otimes \mathcal{L}^{\otimes e}) \longrightarrow 0$$

which means a closed immersion $\widetilde{X} \hookrightarrow \mathbb{P}^{r_e}_X$. Let e_{va} be an integer such that $\mathcal{L}^{\otimes e_{va}}$ is very ample. Then we have the sequence of closed embeddings

$$\widetilde{X} \hookrightarrow \mathbb{P}^{r_e}_X = \mathbb{P}^{r_e}_k \times X \xhookrightarrow{\mathrm{id} \times |\mathcal{L}^{\otimes e_{va}}|} \mathbb{P}^{r_e}_k \times \mathbb{P}(H^0(X, \mathcal{L}^{\otimes e_{va}})^\vee) \hookrightarrow \mathbb{P}^N_k$$

(the last one is the Segre embedding). The (very) ample line bundle associated to the composite embedding is

$$\mathcal{O}_{\widetilde{X}/X}(1) \otimes b^*\mathcal{L}^{\otimes e} \otimes b^*\mathcal{L}^{\otimes e_{va}} \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes b^*\mathcal{L}^{\otimes e+e_{va}}.$$

Therefore for $e >> 0$ the line bundle $\widetilde{\mathcal{L}}_e \cong b^*\mathcal{L}^{\otimes e} \otimes \mathcal{O}_{\widetilde{X}}(-2E)$ is (very) ample. $\square$

Theorem 4.1.5. *Within the notation of framework 4.1.1, for $e >> 0$ the suitable triple $(\widetilde{X}, \widetilde{\mathcal{L}}_e, \widetilde{Q})$ satisfies the assumptions 3.0.1 with respect to n if and only if n belongs to the following interval*

$$\widetilde{I}_e := \left[\frac{e^2 c_1(\mathcal{L})^2 + e\, c_1(\mathcal{L}) \cdot K_X}{2} + 1 - d', \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + \dim_k H^2(X, \mathcal{O}_X) - 2d - 3d'\right],$$

[1] Usually it is denoted by $\mathcal{O}_{\widetilde{X}}(1)$. See [Har77, Construction p. 160].

in which case Hilb^n_X *is stably birational to* $\mathrm{Hilb}^{n+\deg(Q)}_X$.

The proof of Theorem 4.1.5 will be conceptually the same as the proof of Theorem 4.1.1. But before checking the assumptions one by one, we compute some higher direct images of sheaves and cohomology groups, which will be needed in proof of 4.1.5.

The following are standard computations.

Lemma 4.1.6. *In the blow-up framework 4.1.1 we have:*

1. $R^i b_*(i_{E*}\mathcal{O}_E) = \begin{cases} i'_*\mathcal{O}_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$;
2. $R^i b_*(i_{E*}\mathcal{O}_E(1)) = \begin{cases} \mathcal{I}_{Q'}/\mathcal{I}^2_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$
3. $R^i b_*\mathcal{O}_{\widetilde{X}}(-E) = \begin{cases} \mathcal{I}_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$;
4. $R^i b_*\mathcal{O}_{\widetilde{X}}(-2E) = \begin{cases} \mathcal{I}^2_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$
5. $R^1 b_*(\mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}}) = 0$;
6. $\dim_k H^0(Q'(2), \mathcal{O}_{Q'(2)}) = 3\deg(Q')$.

Proof. For any $i \geq 0$ we have

$$i'_* R^i(b_E)_*\mathcal{O}_E \cong R^i(b \circ i_E)_*\mathcal{O}_E \cong R^i b_*(i_{E*}\mathcal{O}_E).$$

This, combined with the following isomorphisms

$$R^i(b_E)_*\mathcal{O}_E \cong H^i(E, \mathcal{O}_E)^\sim \cong H^i(\mathbb{P}^1_{Q'}, \mathcal{O}_{\mathbb{P}^1_{Q'}})^\sim,$$

implies that

$$R^i b_*(i_{E*}\mathcal{O}_E) = \begin{cases} i'_*\mathcal{O}_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Similarly for any $i \geq 0$ we have

$$i'_* R^i(b_E)_*\mathcal{O}_E(1) \cong R^i(b \circ i_E)_*\mathcal{O}_E(1) \cong R^i b_*(i_{E*}\mathcal{O}_E(1)),$$

which, combined with

$$R^i(b_E)_*\mathcal{O}_E(1)) \cong H^i(E, \mathcal{O}_E(1))^{\sim} \cong H^i(\mathbb{P}^1_{Q'}, \mathcal{O}_{\mathbb{P}^1_{Q'}}(1))^{\sim},$$

gives

$$R^i b_*(i_{E*}\mathcal{O}_E(1)) = \begin{cases} \mathcal{I}_{Q'}/\mathcal{I}^2_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

(the result for $i = 0$ follows from the direct computation of the stalks at each point of X).

Consider the push-forward of the exact sequence defining E: thanks to what we have just proven about $R^i b_*\mathcal{O}_E$, it looks as follows:

$$0 \longrightarrow b_*\mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \mathcal{O}_X \longrightarrow i'_*\mathcal{O}_{Q'} \longrightarrow R^1 b_*\mathcal{O}_{\widetilde{X}}(-E) \longrightarrow 0 \longrightarrow 0 \longrightarrow R^2 b_*\mathcal{O}_{\widetilde{X}}(-E) \longrightarrow 0$$

(the vanishing of $R^2 b_*\mathcal{O}_{\widetilde{X}} = 0$ is proved in [CR15, Thm. 1.1]). Since $\mathcal{O}_X \longrightarrow i'_*\mathcal{O}_{Q'}$ is surjective we have

$$R^i b_*\mathcal{O}_{\widetilde{X}}(-E) = \begin{cases} \mathcal{I}_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Finally, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-2E) \longrightarrow \mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \mathcal{O}_{\widetilde{X}}(-E)|_E \longrightarrow 0.$$

Consider the connected (and hence integral) components $Q'_j = \mathrm{Spec}(k'_j)$ of Q': it follows that $E = \sqcup E_j \cong \sqcup \mathbb{P}^1_{k'_j}$. Since $E_j^2 = -\deg(k'_j)$ and $\mathrm{Pic}(E) \cong \deg(k'_j)\mathbb{Z}$ we have that $\mathcal{O}_{\widetilde{X}}(-E)|_E \cong \mathcal{O}_E(1)$. Therefore we get that

$$0 \longrightarrow b_*\mathcal{O}_{\widetilde{X}}(-2E) \longrightarrow b_*\mathcal{O}_{\widetilde{X}}(-E) \longrightarrow b_*i_{E*}\mathcal{O}_E(1) \longrightarrow R^1b_*\mathcal{O}_{\widetilde{X}}(-2E) \longrightarrow 0$$

and that $R^2b_*\mathcal{O}_{\widetilde{X}}(-2E)$ is trivial. Therefore

$$R^ib_*\mathcal{O}_{\widetilde{X}}(-2E) = \begin{cases} \mathcal{I}^2_{Q'} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Let $E(2)$ be the thickening of E defined by $\mathcal{O}_{\widetilde{X}}(-2E)$ and let $Q'(2)$ be the corresponding thickening of Q' defined by the ideal $\mathcal{I}^2_{Q'}$.

From the discussion above we have that the sheaf $b_*\mathcal{O}_{E(2)}$ coincides $\mathcal{O}_{Q'(2)}$. Therefore, since Q is disjoint from Q' we get that $b_*\mathcal{O}_{E(2)\sqcup\widetilde{Q}} \cong \mathcal{O}_{Q'(2)\sqcup Q}$ as well.

In particular we get the following exact sequences

$$0 \longrightarrow b_*(\mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}}) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{Q'(2)\sqcup Q} \longrightarrow R^1b_*(\mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}}) \longrightarrow 0.$$

Since the map $\mathcal{O}_X \longrightarrow \mathcal{O}_{Q'(2)\sqcup Q}$ is surjective, the sheaf $R^1b_*(\mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}}) = 0$.

Let us now compute the dimension of $H^0(Q'(2), \mathcal{O}_{Q'_2(2)})$: in order to do so we can pass to the algebraic closure $\overline{k}$. Over $\overline{k}$ the scheme $Q'(2)$ is the disjoint union of $\deg(Q')$-copies of $\mathrm{Spec}(\overline{k})$: call m_i the maximal ideal corresponding to the i-th connected component. Then the thickening is given by m_i^2. So for each of them, the dimension of the global sections is

$$\dim_{\overline{k}}(A_i/m_i^2) = \dim_{\overline{k}} A_i/m_i \oplus m_i/m_i^2 = 3,$$

where $\mathrm{Spec}(A_i)$ is an open of $X_{\overline{k}}$ around the i-th connected component of Q'. Therefore

$$\dim_k H^0(Q'(2), \mathcal{O}_{Q'(2)}) = 3\deg(Q').$$

$\square$

Proof of Theorem 4.1.5. As in the proof of Theorem 4.1.1, let us check the assumptions one at a time.

Assumption 1. It is enough to prove that $H^1(\widetilde{X}, \widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}) = 0$ for $e >> 0$. The Leray Spectral Sequence for the sheaf $\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}$ with respect to the map b tells us

$$H^1(X, b_*(\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}})) \longrightarrow H^1(\widetilde{X}, \widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}) \longrightarrow H^0(X, R^1 b_*(\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}})).$$

Using the projection formula

$$R^i b_*(\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}) \cong \mathcal{L}^{\otimes e} \otimes R^i b_*(\mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}}),$$

and Serre vanishing theorem, we get that the cohomology group $H^1(X, b_*(\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}))$ vanish for $e >> 0$. Since the sheaf $R^1 b_*(\mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}})$ is already trivial (by Lemma 4.1.6), the cohomology group $H^0(X, R^1 b_*(\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}))$ is trivial as well. Hence we get the vanishing of $H^1(\widetilde{X}, \widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}) = 0$ for $e >> 0$.

Assumption 2. By Lemma 4.1.6 the sheaves $R^i b_*(\mathcal{O}_{\widetilde{X}}(-2E))$ are trivial for $i \geq 1$. Hence, the cohomology groups $H^j(X, \mathcal{L}^{\otimes e} \otimes R^i b_*(\mathcal{O}_{\widetilde{X}}(-2E)))$ are all zeroes (for $i \geq 1$). Therefore, by Leray Spectral sequence

$$H^i(X, b_* \widetilde{\mathcal{L}}_e) \cong H^i(\widetilde{X}, \widetilde{\mathcal{L}}_e), \quad \forall i \geq 0.$$

By projection formula, they are all trivial for $i \geq 1$ and $e >> 0$. So for $e >> 0$ we get

$\dim_k H^0(\widetilde{X}, \widetilde{\mathcal{L}}_e) = \chi(\widetilde{X}, \widetilde{\mathcal{L}}_e)$. By Riemann-Roch we then have

$$\dim_k H^0(\widetilde{X}, \widetilde{\mathcal{L}}_e) = \frac{c_1(\widetilde{\mathcal{L}}_e)^2 - c_1(\widetilde{\mathcal{L}}_e) \cdot K_{\widetilde{X}}}{2} + \chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

By [Har77, Prop. V.3.4] we have $\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}} = 1 + \dim_k H^2(X, \mathcal{O}_X)$. Using the structure of the intersection theory on the blow-up (see [Har77, Prop. V.3.2]), we get

$$\dim_k H^0(\widetilde{X}, \widetilde{\mathcal{L}}_e) = \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X)}{2} - 3d' + 1 + \dim_k H^2(X, \mathcal{O}_X).$$

So for $e >> 0$ assumption (2) is true with respect to n if and only if

$$n \leq \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + \dim_k H^2(X, \mathcal{O}_X) - 2d - 3d'.$$

Assumption 3. In order to show that the sheaf $\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated for $e >> 0$, we write it as

$$\widetilde{\mathcal{L}}_e \otimes \mathcal{I}_{\widetilde{Q}} \cong b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}} \otimes b^* \mathcal{L}^{\otimes e_1} \otimes \mathcal{O}_{\widetilde{X}}(-2E),$$

where $e = e_1 + e_2$ and e_1 is an integer such that $b^* \mathcal{L}^{\otimes e_1} \otimes \mathcal{O}_{\widetilde{X}}(-2E)$ is very ample. In this way it is sufficient to check that $b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated.

The counit map $b^* b_* (b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}) \longrightarrow b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}$ gives the following commutative diagram

$$\begin{tikzcd} H^0(\widetilde{X}, b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}) \otimes b^* b_* \mathcal{O}_{\widetilde{X}} \arrow[r] \arrow[d] & H^0(\widetilde{X}, b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}) \otimes \mathcal{O}_{\widetilde{X}} \arrow[d] \\ b^* b_* (b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}) \arrow[r] & b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}. \end{tikzcd} \tag{4.2}$$

Recall that, since Q is disjoint from Q', then

$$b_* \mathcal{I}_{\widetilde{Q}} \cong \mathcal{I}_Q \quad \text{and} \quad b^* \mathcal{I}_Q \cong \mathcal{I}_{\widetilde{Q}}.$$

Therefore, by projection formula we get $b^* b_* (b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}) \cong b^* \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_{\widetilde{Q}}$. So the bottom arrow in 4.2 is an isomorphism.

Now, since $b_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ (see [Har77, Prop. V.3.4]), the left vertical arrow in 4.2 coincides with

$$b^* \left(H^0(X, \mathcal{L}^{e_2} \otimes \mathcal{I}_Q) \otimes \mathcal{O}_X \longrightarrow \mathcal{L}^{\otimes e_2} \otimes \mathcal{I}_Q \right).$$

Since $\mathcal{L}$ is ample, the sheaf $\mathcal{L}^{\otimes e-e_1} \otimes \mathcal{I}_Q$ is globally generated for $e >> 0$. Therefore the left vertical arrow in 4.2 is surjective (for $e >> 0$), which implies that $b^*\mathcal{L}^{\otimes e-e_1} \otimes \mathcal{I}_{\widetilde{Q}}$ is generated by global sections as well for $e >> 0$.

Assumption 4. As done in the proof of Theorem 4.1.1 (when we were checking assumption (4)), it is enough to pick $e \geq e_g + e_{va}$ where $b^*\mathcal{L}^{\otimes e_g} \otimes \mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated and e_{va} is such that $b^*\mathcal{L}^{\otimes e_{va}} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample.

Assumption 5. Using again the intersection theory on $\widetilde{X}$ we can compute

$$\frac{c_1(\widetilde{\mathcal{L}}_e)^2 + c_1(\widetilde{\mathcal{L}}_e) \cdot K_{\widetilde{X}}}{2} + 1.$$

It is equal to

$$\frac{e^2 c_1(\mathcal{L})^2 + e\, c_1(\mathcal{L}_e) \cdot K_X}{2} + 1 - d'.$$

Hence the statement of the theorem. □

4.1.2 Necessary and sufficient conditions for Theorems 4.1.1 and 4.1.5

Theorems 4.1.1 and 4.1.5 define intervals I_e and $\widetilde{I}_e$ for $e >> 0$. However, as mentioned previously, these theorems have a meaning only when we check that those intervals are non-empty.

In this subsection, we study when I_e and $\widetilde{I}_e$ are non-empty for infinitely many values of e.

Proposition 4.1.7. *Let $(X, \mathcal{L}, Q)$ be a suitable triple. The interval I_e is non-empty for infinitely many e if and only if $c_1(\mathcal{L}) \cdot K_X < 0$. The same holds true for $\widetilde{I}_e$.*

If this happens, the surface X has Kodaria dimension $\kappa(X) = -\infty$. In particular $H^2(X, \mathcal{O}_X) = 0$ and $H^2(X, 2K_X) = 0$. Since we are assuming $H^1(X, \mathcal{O}_X) = 0$, the surface X is geometrically rational.

Proof. The interval defined in Theorem 4.1.1 is non-empty if and only if the difference

$$\frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + \dim_k H^2(X, \mathcal{O}_X) - 2\deg(Q) - \frac{e^2 c_1(\mathcal{L})^2 + e\, c_1(\mathcal{L}) \cdot K_X}{2} - 1 =$$
$$= -e\, c_1(\mathcal{L}) \cdot K_X + \dim_k H^2(X, \mathcal{O}_X) - 2\deg(Q) - 1$$

is non-negative. If $c_1(\mathcal{L}) \cdot K_X > 0$, then for $e >> 0$ the difference is negative. If $c_1(\mathcal{L}) \cdot K_X \leq 0$, then by Nakai-Moishezon Criterion of ampleness, the divisor K_X has no global section (otherwise K_X would be equivalent to an effective divisor $E > 0$ and $c_1(\mathcal{L}) \cdot K_X > 0$ or $K_X \sim 0$ and the difference is $-2\deg(Q) < 0$). Therefore by Serre duality $\dim_k H^2(X, \mathcal{O}_X) = \dim_k H^0(X, K_X) = 0$ and the difference is $-e\, c_1(\mathcal{L}) \cdot K_X - 2\deg(Q) - 1$, which is non-negative for $e >> 0$ if and only if $c_1(\mathcal{L}) \cdot K_X < 0$.

By the same argument we get that $H^0(X, mK_X) = 0$ for all $m \geq 1$. Therefore $\kappa(X) = \kappa(X_{\overline{k}}) = -\infty$. Since $H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ and $H^0(X_{\overline{k}}, 2K_{X_{\overline{k}}})$ are both trivial, by Castelnuovo's rationality criterion $X_{\overline{k}}$ is rational (see [Pie, Thm. 1.53]). □

Remark 4.1.8. For instance, if X is a K3 surface, then K_X is trivial and therefore $c_1(\mathcal{L}) \cdot K_X = 0$. So in this case, we cannot apply Theorem 4.1.1 for large e. This was already predicted by Litt's theorem [Lit14, Thm. 19].

Given Proposition 4.1.7, let us restrict our attention to those surfaces with an ample line bundle $\mathcal{L}$ for which the intersection $c_1(\mathcal{L}) \cdot K_X$ is negative.

The intervals $I_e = [\min_e, \max_e]$ leave gaps $\mathrm{gap}_e := [\max_e +1, \min_{e+1} -1]$: they have width

$$\min_{e+1} - 1 - \max_e = \frac{(e+1)^2 c_1(\mathcal{L})^2 + (e+1)c_1(\mathcal{L}) \cdot K_X}{2} - \frac{e^2 c_1(\mathcal{L})^2 - e\, c_1(\mathcal{L}) \cdot K_X}{2} + 2\deg(Q)$$

$$\min_{e+1} - 1 - \max_e = (2e+1)\frac{c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X}{2} + 2\deg(Q). \tag{4.3}$$

Therefore the growth of the gaps depend on the sign of $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X$.

Remark 4.1.9. Since $\mathcal{L}$ is ample, by Nakai-Moishezon Criterion, $c_1(\mathcal{L})^2 > 0$: so if $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X \leq 0$ then the intersection $c_1(\mathcal{L}) \cdot K_X$ is negative. So the intervals I_e and $\widetilde{I_e}$ are growing in width.

Moreover, by equation 4.3, if $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X < 0$ then for $e >> 0$ we have that $\min_{e+1} -1 \leq \max_e +1$. Namely, the intervals I_e's contain all the integers $n >> 0$. If $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X = 0$ then the gaps have fixed width equal to $2d$. If $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X > 0$ then the gaps are growing in width.

Therefore, for $e >> 0$ the gaps happen only in the case $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X \geq 0$: let us see when the intervals $\widetilde{I_e}$ can help fill these gaps.

Lemma 4.1.10. *Let $(X, \mathcal{L}, Q)$ be a suitable triple where $c_1(\mathcal{L}) \cdot K_X < 0$ and $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X \geq 0$. Let Q' be as in 4.1.1. Then the following are equivalent:*

1. *for any $e >> 0$ there exists an $f = f(e)$ such that $\mathrm{gap}_e \subseteq \widetilde{I_f}$;*

2. *we have $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X = 0$ and $d' \geq 2d$.*

Proof. The interval gap_e is contained in $\widetilde{I_f} = [\widetilde{\min}_f, \widetilde{\max}_f]$ if and only if

$$\widetilde{\min}_f \leq \max_e +1 \text{ and } \min_{e+1} -1 \leq \widetilde{\max}_f$$

or equivalently

$$\frac{f^2c_1(\mathcal{L})^2 + f\,c_1(\mathcal{L})\cdot K_X}{2} + 1 - d' \le \frac{e^2c_1(\mathcal{L})^2 - e\,c_1(\mathcal{L})\cdot K_X}{2} + 1 - 2d,$$
$$\frac{(e+1)^2c_1(\mathcal{L})^2 + (e+1)c_1(\mathcal{L})\cdot K_X}{2} \le \frac{f^2c_1(\mathcal{L})^2 - f\,c_1(\mathcal{L})\cdot K_X}{2} - 2d - 3d'$$

Using $c_1(\mathcal{L})\cdot K_X \ge -c_1(\mathcal{L})^2$: then the above inequalities imply (for $e >> 0$)

$$\frac{(f^2-f)}{2}c_1(\mathcal{L})^2 \le \frac{(e^2+e)}{2}c_1(\mathcal{L})^2 + d' - 2d \implies f \le e+1$$
$$\frac{(e^2+e)}{2}c_1(\mathcal{L})^2 \le \frac{(f^2+f)}{2}c_1(\mathcal{L})^2 - 2d - 3d' \implies f \ge e+1.$$

Therefore for $e >> 0$, $\mathrm{gap}_e \subseteq \widetilde{\mathcal{I}}_f$ forces f to be equal to $e+1$. Therefore, writing the initial inequalities for $\mathrm{gap}_e \subseteq \widetilde{\mathcal{I}}_{e+1}$, we get

$$(2e+1)\frac{c_1(\mathcal{L})^2 + c_1(\mathcal{L})\cdot K_X}{2} \le d' - 2d \;\text{ and }\; (e+1)(-c_1(\mathcal{L})\cdot K_X) \ge 2d + 3d',$$

which implies $c_1(\mathcal{L})^2 + c_1(\mathcal{L})\cdot K_X = 0$ and $d' \ge 2d$.

The same computations show that if $c_1(\mathcal{L})^2 + c_1(\mathcal{L})\cdot K_X = 0$ and $d' \ge 2d$ then for $e >> 0$ $\mathrm{gap}_e \subseteq \widetilde{\mathcal{I}}_{e+1}$. □

This comparison between $\mathcal{I}_e$ and $\widetilde{\mathcal{I}}_e$ leads to the following corollary of Theorems 4.1.1 and 4.1.5.

Corollary 4.1.11. *Let $(X, \mathcal{L}, Q)$ be a suitable triple where $\mathcal{L}$ is such that*

$$c_1(\mathcal{L})^2 + c_1(\mathcal{L})\cdot K_X \le 0. \tag{4.4}$$

Then X is geometrically rational. Furthermore, for $n >> 0$ the Hilbert schemes Hilb^n_X is stably birational to Hilb^{n+d}_X.

Proof. By Remark 4.1.9, if $c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X \leq 0$ then $c_1(\mathcal{L}) \cdot K_X$ is negative. By Proposition 4.1.7 X is geometrically rational. Meanwhile, Theorem 4.1.1 tells us that Hilb_X^n is stably birational to Hilb_X^{n+d} for any $n \in I_e$. If some gaps gap_e are left, then pick a smooth effective 0-cycle Q' disjoint from Q and of degree $d' \geq 2\deg(Q)$ (we can do it by Proposition 4.1.3). Then for $e >> 0$, those eventual gaps are covered by the intervals $\widetilde{I}_{e+1}$ as seen in Lemma 4.1.10. We conclude using Theorem 4.1.5. □

Chapter 5: Geometrically rational surfaces

As seen in the previous chapter, Theorems 4.1.1 and 4.1.5 work best for geometrically rational surfaces. In particular, Corollary 4.1.11 tells us sufficient conditions on X in order to have Hilb^n_X stably birational to Hilb^{n+d}_X. In this chapter, we want to complete the study of the stable birational classes of Hilb^n_X for all geometrically rational surfaces.

Since the stable birational class of Hilb^n_X depends only on the stable birational class of X (see Fact), we can assume without loss of generality that X is minimal (relatively over k).

Definition 5.0.1. Recall that X is said to be *minimal over k* if any birational morphism $X \longrightarrow X'$ to a smooth geometrically connected projective surface X' over k is an isomorphism.

Iskovskih in [Isk79] classified minimal geometrically rational surfaces over arbitrary fields.

Theorem 5.0.2 (Thm. 1 of [Isk79])**.** *Let X be a geometrically rational, suitable surface over an arbitrary field k.*

If X is minimal over k then it is either:

1. *isomorphic to the projective plane $\mathbb{P}^2_k$;*

2. *isomorphic to a quadric $Q \subset \mathbb{P}^3_k$ such that* $\mathrm{Pic}(Q) \cong \mathbb{Z}$*;*

3. *isomorphic to a del Pezzo surface; or*

4. *the Picard group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and there exists a smooth, geometrically connected, projective, genus zero curve C and a surjective map $f : X \longrightarrow C$ whose generic fiber $F_{\eta(C)}$ is a smooth curve of genus zero.*

Definition 5.0.3. If X is of the fourth type, we say that X is a *rational conic bundle*.

Notice that Theorem 5.0.2 is not a classification up to birational maps: for instance, some del Pezzo surfaces are already rational over k, a smooth quadric Q in $\mathbb{P}^3_k$ is a del Pezzo (by adjunction formula the canonical ω_Q is isomorphic to $O_Q(-2)$ and hence anti-ample) and some rational conic bundles are del Pezzo (see [Isk79, Thm. 5]).

Therefore, since we are interested in X up to birational equivalence, we have two families to consider.

Corollary 5.0.4. *Let X be a geometrically rational surface minimal over k. Then either:*

Type I. *X is birational to a del Pezzo surface, or*

Type II. *X is a rational conic bundle as in Theorem 5.0.2.*

5.0.1 Del Pezzo Surfaces

Let us first focus on del Pezzo surfaces X of degree $K_X^2 = d_X$. The aim of this subsection is to prove the following.

Theorem 5.0.5. *Let X be a del Pezzo surface and let* $\operatorname{ind}(X)$ *be the index of X: then for $n >> 0$ the Hilbert scheme* Hilb^n_X *is stably birational to* $\operatorname{Hilb}^{n+\operatorname{ind}(X)}_X$.

Proof. As line bundle $\mathcal{L}$, pick the anticanonical $\omega_X^\vee$: it is ample (X is del Pezzo) and satisfies condition 4.4 of Corollary 4.1.11

$$c_1(\mathcal{L})^2 + c_1(\mathcal{L}) \cdot K_X = K_X^2 - K_X^2 = 0.$$

Now recall that for a smooth variety X the index is equal to the greatest common divisor of closed points $x \in X$ such that $k(x)$ is a separable extension of k (see [GLL12, Thm. 9.2]). If $k(x)/k$ separable, then $\operatorname{Spec}(k(x))$ is geometrically reduced (see [Sta, tag. 030W]): then there exist geometrically reduced closed points $Q_1, \ldots, Q_r$ on X such that $\gcd\{\deg(Q_i)\}_{i=1}^r = \operatorname{ind}(X)$. Setting

$\deg(Q_i) = d_i$ we can assume

$$\mathrm{ind}(X) = \sum_{i=1}^{s} a_i d_i - \sum_{i=s+1}^{r} a_i d_i$$

where the a_i's are positive integers.

Then $(X, \mathcal{L}, Q_i)$ satisfies the hypotheses of Corollary 4.1.11. Therefore, for $n \geq N_i$ the variety Hilb_X^n is stably birational to $\mathrm{Hilb}_X^{n+d_i}$.

This means that for any $n \geq \max_i N_i$ we have that Hilb_X^n is stably birational to $\mathrm{Hilb}_X^{n+\sum_{i=1}^{s} a_i d_i}$ and that $\mathrm{Hilb}_X^{n+\mathrm{ind}(X)}$ is stably birational to $\mathrm{Hilb}_X^{n+\mathrm{ind}(X)+\sum_{i=s+1}^{r} a_i d_i} = \mathrm{Hilb}_X^{n+\sum_{i=1}^{s} a_i d_i}$.

Therefore, for $n >> 0$ we always have Hilb_X^n stably birational to $\mathrm{Hilb}_X^{n+\mathrm{ind}(X)}$. □

5.0.2 Minimal rational conic bundles

We focus now on Type II surfaces: namely, minimal rational conic bundles. In [Isk79], Iskovskih gives a detailed description of rational conic bundles over arbitrary fields.

Theorem 5.0.6 (Thm. 3 of [Isk79])**.** *Let $f : X \longrightarrow C$ be a rational conic bundle as in Theorem 5.0.2. The following assertions hold:*

1. *The sheaf $f_*\omega_X^{-1}$ is locally free of rank 3 and induces a closed embedding*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}_C(f_*\omega_X^{-1}) \arrow[d] \\ & C \end{tikzcd}$$

   mapping each fiber F_t, $t \in C$, into a (not necessarily smooth) conic of $\mathbb{P}^2_{k(t)}$.

2. *If f is smooth then either X is isomorphic to a product of curves $C \times C'$, where C' is a smooth curve of genus 0 without k-points, or X is isomorphic to $\mathbb{P}_C(\mathcal{E})$ for some rank 2 vector bundle $\mathcal{E}$ on C.*

3. *if f is not smooth, then*

$$\mathrm{Pic}(X) = f^*\mathrm{Pic}(C) + \mathbb{Z}[-K_X].$$

Let r be the number of non-smooth fibers: then $K_X^2 = 8 - r$.

In this subsection we want to use the above properties to prove an analogous Theorem 5.0.5 but for minimal rational conic bundles.

Theorem 5.0.7. *Let X be a rational conic bundle as in Theorem 5.0.2. Let* $\mathrm{ind}(X)$ *be the index of X. Then for $n >> 0$ the variety* Hilb_X^n *is stably birational to* $\mathrm{Hilb}_X^{n+\mathrm{ind}(X)}$.

We split the proof in two parts, depending on whether the map f is smooth.

Case f smooth. The projective bundle $\mathbb{P}_C(\mathcal{E})$ is birational to $C \times \mathbb{P}_k^1$. Since the index is a birational invariant for projective smooth varieties (see [Mer, Prop. 1.4] or [GLL12, Prop. 6.8]), without loss of generality we can therefore study the case when X is the product of two smooth curves of genus 0. However, any surface of this type is a del Pezzo surface since the surface $X_{\overline{k}} \cong \mathbb{P}_{\overline{k}}^1 \times \mathbb{P}_{\overline{k}}^1$ is. In particular, Theorem 5.0.5 applies. □

Therefore, we are left with the case f singular.

Remark 5.0.8. In this case, we cannot automatically apply the previous strategy since there are rational conic bundles which are not birational to any del Pezzo surface.

For instance, let k be an (infinite perfect) field and X be a minimal rational conic bundle of negative degree $K_X^2 < 0$. Assume by contradiction that X is birational to a del Pezzo X'. Then consider a birational morphism $f : X' \longrightarrow X'_{\mathrm{min}}$ where X'_{min} is minimal over k. Expressing the morphism f as a sequence of blow-ups, by [Has09, Cor. 2.8] we have that X'_{min} is still del Pezzo. By [Sko86, Prop. 2.4] the degree $K^2_{X'_{\mathrm{min}}}$ is equal to K_X^2, so in particular it is negative, which is not possible.

For del Pezzo's we have used blow-ups of the surface in order to conclude. Now we can use the information on the structure of X and on its Picard group $\mathrm{Pic}(X)$ given by Iskovskih theorem.

We aim to find a family of line bundles $\{\mathcal{L}_m\}_m$ on X (and a 0-cycle Q) such that for any $n >> 0$ there exists an $m(n)$ such that $(X, \mathcal{L}_{m(n)}, Q)$ is a suitable triple satisfying the assumptions with respect to that n.

Remark 5.0.9. As for del Pezzo surfaces, the index of a rational conic bundle X is the greatest common divisor of its geometrically reduced closed points Q_i (see again [GLL12, Thm. 9.2]). In what follows, Q will be any geometrically reduced closed point on X, so that the pair (X, Q) is suitable.

Let us find suitable line bundles, namely ample line bundles on X in the case when f is singular.

Proposition 5.0.10. *Let X be a rational conic bundle with singular f as in Theorem 5.0.2. Let F be a smooth fiber over a closed point $t \in C$ of minimal degree. There exists a positive integer m such that for any $a >> 0$ the divisor $-mK_X + aF$ is ample.*

Before dealing with the proof, let us prove the following lemma.

Lemma 5.0.11. *Let X be a rational conic bundle as in Theorem 5.0.2. If D is an ample divisor, then $D + F$ is ample as well.*

Proof. The sum of an ample divisor D with a globally generated divisor is still ample: therefore it is enough to prove that $\mathcal{O}_X(F)$ is generated by global sections.

The degree of $\mathcal{O}_X(F)|_F$ is F^2 which is 0 being a fiber of f. Therefore, since $\mathrm{Pic}(F) \cong \mathbb{Z}$ (because F is a plane conic), the sheaf $\mathcal{O}_X(F)|_F$ is trivial.

From the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(F) \longrightarrow i_{F,*}(\mathcal{O}_X(F)|_F) \longrightarrow 0$$

we have

$$H^0(X, \mathcal{O}_X(F)) \cong H^0(X, \mathcal{O}_X) \oplus H^0(F, \mathcal{O}_X(F)|_F) \cong H^0(X, \mathcal{O}_X) \oplus H^0(F, \mathcal{O}_F)$$

(recall that $H^1(X, \mathcal{O}_X) = 0$). Since both $\mathcal{O}_X$ and $\mathcal{O}_F$ are generated by global sections, the sheaf $\mathcal{O}_X(F)$ is globally generated as well. □

Proof of Proposition 5.0.10. Since X is projective, there exists an ample line bundle $\mathcal{L}$ on it. Since any closed point of minimal degree generates $\operatorname{Pic}(C)$, by item (3) of Theorem 5.0.6 we have that F and $-K_X$ generates $\operatorname{Pic}(X)$. Therefore $c_1(\mathcal{L})$ has the form $-mK_X + aF$ with m and a integers.

By Nakai-Moishezon Criterion (see [Har77, Thm. V.1.10]), a necessary condition for $\mathcal{L}$ to be ample is that $c_1(\mathcal{L}) \cdot F > 0$. Since $F^2 = 0$ then this is equivalent to have

$$(-mK_X + aF) \cdot F = -mK_X \cdot F > 0.$$

By adjunction formula for $F \subset X$

$$\deg(K_F) = K_X \cdot F + F^2 = K_X \cdot F.$$

Because F is a conic (over $k(t)$) the degree of K_F (over k) is $-2 \deg k(t)$. Thus we must have

$$-mK_X \cdot F = 2m \deg k(t) > 0.$$

Hence the integer m must be positive.

Therefore there exists an a_0 such that $-mK_X + a_0 F$ is ample and $m > 0$. By Lemma 5.0.11, we get that for any $a \geq a_0$ the divisor $-mK_X + aF$ is ample as well. □

Now that we have many ample line bundles, let us check which of them can lead to a triple which satisfies the assumptions.

Corollary 5.0.12. *Let $\mathcal{L}$ be an ample line bundle $\mathcal{O}_X(-mK_X + aF)$ such that $a > \frac{m(r-8)}{2 \deg k(t)}$. Then for $e >> 0$ the triple $(X, \mathcal{O}_X(-mK_X + aF)^{\otimes e}, Q)$ satisfies the assumptions with respect to any $n \in I_e$ (defined in Theorem 4.1.1).*

Proof. By Proposition 4.1.7, it is enough to find an ample line bundle $\mathcal{L}$ on X such that $c_1(\mathcal{L}) \cdot$

$K_X < 0$. By item (3) in Theorem 5.0.6 we have $K_X^2 = (8-r)$. Therefore

$$c_1(\mathcal{L}) \cdot K_X = (-mK_X + aF) \cdot K_X = -m(8-r) - 2a \deg k(t)$$

This is negative if and only if $a > \frac{m(r-8)}{2 \deg k(t)}$. $\square$

Proposition 5.0.13. *Let $\mathcal{L}$ be an ample line bundle $\mathcal{O}_X(-mK_X + aF)$ such that $a > \frac{m(r-8)}{2 \deg k(t)}$. Assume that*

1. *the triple $(X, \mathcal{L}^{\otimes e}, Q)$ satisfies the assumptions with respect to $n_0 \in I_e$,*
2. *$e \geq e_g + e_{va}$ where $\mathcal{L}^{\otimes e_g} \otimes \mathcal{I}_Q$ is globally generated and $\mathcal{L}^{\otimes e_{va}}$ very ample, and*
3. *$e >> 0$ so that $H^i(X, \mathcal{L}^{\otimes e}) = 0\ \forall i > 0$ and $-e\, c_1(\mathcal{L}) \cdot K_X - 1 - 2\deg(Q) \geq 0$.*

Then for any $b \in \mathbb{N}$ the triple $(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF), Q)$ satisfies the assumptions with respect to any integer n_b inside the non-empty interval

$$I_{(e,b)} := \left[\frac{(e\, c_1(\mathcal{L}) + bF)^2 + (e\, c_1(\mathcal{L}) + bF) \cdot K_X}{2} + 1, \dim_k H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) - 1 - 2\deg(Q) \right].$$

Proof. Let us check the assumptions one by one.

Assumption 1. Since the composite map

$$H^0(X, \mathcal{L}^{\otimes e}) \hookrightarrow H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) \longrightarrow H^0(Q, \mathcal{O}_Q)$$

is surjective, then

$$H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) \longrightarrow H^0(Q, \mathcal{O}_Q)$$

is surjective as well.

Assumption 2.

$$n_b \le \dim_k H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) - 1 - 2\deg(Q). \tag{5.1}$$

Assumption 3. Since both $\mathcal{L}^{\otimes e} \otimes \mathcal{I}_Q$ and $\mathcal{O}_X(bF)$ are generated by global sections (see proof of Lemma 5.0.11), the product $\mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF) \otimes \mathcal{I}_Q$ is globally generated as well.

Assumption 4. Since $\mathcal{L}^{\otimes e_{va}}$ is very ample and $\mathcal{O}_X(bF)$ is globally generated, then $\mathcal{L}^{\otimes e_{va}} \otimes \mathcal{O}_X(bF)$ is very ample as well. Since $\mathcal{L}^{\otimes e_g} \otimes \mathcal{O}_X(bF) \otimes \mathcal{I}_Q$ is globally generated, we can do as done in proof of Theorem 4.1.1 to infer that assumption (4) is satisfied as well.

Assumption 5.

$$\frac{(e\, c_1(\mathcal{L}) + bF)^2 + (e\, c_1(\mathcal{L}) + bF) \cdot K_X}{2} + 1 \le n_b. \tag{5.2}$$

Therefore it remains to check that the intervals $I_{(e,b)}$ are non-empty. Equivalently, we need to prove that the difference

$$\dim_k H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) - 1 - 2\deg(Q) - \frac{(e\, c_1(\mathcal{L}) + bF)^2 + (e\, c_1(\mathcal{L}) + bF) \cdot K_X}{2} - 1 \tag{5.3}$$

is non-negative.

In order to do so, we want to express $\dim_k H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF))$ in terms of the Euler-characteristic $\chi(\mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF))$.

Claim. Let $b \in \mathbb{N}$. If the groups $H^i(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF))$ are trivial for any $i \ge 1$, then $H^i(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X((b+1)F))$ are trivial as well.

Proof of the Claim. Thanks to the exact sequence

$$0 \longrightarrow \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF) \longrightarrow \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X((b+1)F) \longrightarrow \mathcal{L}^{\otimes e}|_F \otimes \mathcal{O}_F((b+1)F) \longrightarrow 0$$

we get (for $i \geq 2$)

$$\begin{aligned} 0 \longrightarrow H^1(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X((b+1)F)) \longrightarrow & H^1(F, \mathcal{L}^{\otimes e}|_F \otimes \mathcal{O}_F((b+1)F)) \longrightarrow 0 \dots \\ & \cdots \longrightarrow 0 \longrightarrow H^i(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X((b+1)F)) \longrightarrow 0 \longrightarrow \cdots . \end{aligned}$$

In particular, $H^i(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X((b+1)F))$ is trivial for $i \geq 2$.

The sheaf $\mathcal{L}^{\otimes e}|_F$ is isomorphic to

$$\mathcal{O}_X(-emK_X + eaF)|_F \cong (\omega_X^{-1}|_F)^{\otimes me} \otimes (\mathcal{O}_X(F)|_F)^{\otimes ea} \cong \omega_F^{-\otimes me} \cong \mathcal{O}_F(2me)$$

(because $\mathcal{O}_F(F)$ is trivial: see proof of Lemma 5.0.11). Then the cohomology group $H^1(F, \mathcal{L}^{\otimes e}|_F \otimes \mathcal{O}_F((b+1)F))$ is trivial as well. □

Thanks to the claim, for any positive $b \in \mathbb{N}$ we have

$$\dim_k H^0(X, \mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) = \chi(\mathcal{L}^{\otimes e} \otimes \mathcal{O}_X(bF)) = \frac{(e\, c_1(\mathcal{L}) + bF)^2 - (e\, c_1(\mathcal{L}) + bF) \cdot K_X}{2} + 1.$$

Hence the difference 5.3 becomes

$$\frac{(e\, c_1(\mathcal{L}) + bF)^2 - (e\, c_1(\mathcal{L}) + bF) \cdot K_X}{2} - 2 \deg Q - \frac{(e\, c_1(\mathcal{L}) + bF)^2 + (e\, c_1(\mathcal{L}) + bF) \cdot K_X}{2} - 1$$

which is equal to

$$-(e\, c_1(\mathcal{L}) + bF) \cdot K_X - 1 - 2 \deg Q = -e\, c_1(\mathcal{L}) \cdot K_X + 2b \deg k(t) - 1 - 2 \deg(Q).$$

Therefore as soon as $-e\, c_1(\mathcal{L}) \cdot K_X - 1 - 2 \deg(Q)$ is non-negative, then the difference is always non-negative. $\square$

Proof of Theorem 5.0.7 with f singular. Fix (m, a) a pair of integers such that m is positive, $a > \frac{m(r-8)}{2 \deg k(t)}$ and $\mathcal{L} = O_X(-mK_X + aF)$ is ample. Let e be any integer big enough such that the triple $(X, O_X(-emK_X + eaF), Q)$ satisfies the assumptions (with respect to some n_0) and such that e satisfies the hypotheses of Proposition 5.0.13.

In order to conclude the proof, it is enough to check that the intervals $I_{(e,b)} = [\min_{(e,b)}, \max_{(e,b)}]$'s for $\mathcal{L}^{\otimes e} \otimes O_X(bF)$

$$\left[\frac{(e\,c_1(\mathcal{L}) + bF)^2 + (e\,c_1(\mathcal{L}) + bF) \cdot K_X}{2} + 1, \frac{(e\,c_1(\mathcal{L}) + bF)^2 - (e\,c_1(\mathcal{L}) + bF) \cdot K_X}{2} - 2\deg(Q)\right],$$

provided by Proposition 5.0.13, contain all the integers bigger than some integer n_Q. The width of the interval $I_{(e,b)}$

$$-e\,c_1(\mathcal{L}) \cdot K_X + 2b \deg k(t) - 1 - 2\deg(Q)$$

grows when b grows. Therefore, it is enough to check that for b big enough we have $\max_{(e,b)} +1 \in I_{(e,b+1)}$.

For what concerns the inequality $\max_{(e,b)} +1 \leq \max_{(e,b+1)}$, it is always true. Indeed it reads as follows:

$$\frac{(e\,c_1(\mathcal{L}) + bF)^2 - (e\,c_1(\mathcal{L}) + bF) \cdot K_X}{2} + 1 \leq \frac{(e\,c_1(\mathcal{L}) + (b+1)F)^2 - (e\,c_1(\mathcal{L}) + (b+1)F) \cdot K_X}{2}$$

which is equivalent to

$$e\,c_1(\mathcal{L}) \cdot F - e\frac{K_X \cdot F}{2} - 1 = 2em \deg k(t) + e \deg k(t) - 1 \geq 0.$$

The other inequality, $\min_{(e,b+1)} \leq \max_{(e,b)} +1$, is equivalent to

$$\frac{(e\,c_1(\mathcal{L}) + (b+1)F)^2 + (e\,c_1(\mathcal{L}) + (b+1)F) \cdot K_X}{2} \leq \frac{(e\,c_1(\mathcal{L}) + bF)^2 - (e\,c_1(\mathcal{L}) + bF) \cdot K_X}{2} - 2\deg(Q).$$

After carrying on the computations on both sides, we get

$$0 \geq e\,(-mK_X + aF) \cdot (K_X + F) + \frac{2b+1}{2} F \cdot K_X + 2\deg(Q) =$$
$$= e(m(r-8) - 2a \deg k(t) + 2m \deg k(t)) - (2b+1)\deg k(t) + 2\deg(Q).$$

Therefore it is enough to pick b greater or equal than

$$b \geq \frac{\deg(Q)}{\deg k(t)} + e\left(\frac{m(r-8)}{2\deg k(t)} - a + m\right) - \frac{1}{2}.$$

So we get that for any $n \geq n_Q$, we have Hilb^n_X stably birational to $\mathrm{Hilb}^{n+\deg(Q)}_X$.

Let $Q_1, \ldots, Q_r$ be geometrically reduced closed points of X such that

$$\mathrm{ind}(X) = \sum_{i=1}^{s} c_i \deg(Q_i) - \sum_{i=s+1}^{r} c_i \deg(Q_i)$$

and $c_i > 0$. Picking n greater or equal than $\max_i n_{Q_i}$, we conclude the proof like in Theorem 5.0.5. □

In particular we have treated all the minimal geometrically rational surfaces. This can be summarised in the following corollary.

Corollary 5.0.14 (Theorem C)**.** *Let X be a geometrically rational suitable surface. Let* $\mathrm{ind}(X)$ *be the index of X. Then there exists an integer n_0 such that for any $n \geq n_0$ the variety* Hilb^n_X *is stably birational to* $\mathrm{Hilb}^{n+\mathrm{ind}(X)}_X$.

Chapter 6: Rationality of the Motivic Zeta function

Let us now focus on $K_0(Var/k)$, the Grothendieck ring of varieties over k: it is the commutative ring generated by isomorphism classes $[Y]$ of separated, reduced schemes of finite type over k modulo the *cut-and-paste* relations

$$[Y] = [Y \setminus Z] + [Z_{\text{red}}] \quad \text{and } Z \hookrightarrow Y \text{ closed immersion.}$$

The product operation is the one induced by the product of schemes

$$[Y] \cdot [W] = [(Y \times W)_{\text{red}}].$$

Let R be a commutative ring.

Definition 6.0.1. An R-valued *motivic measure* on $K_0(Var/k)$ is a ring homomorphism

$$\mu : K_0(Var/k) \longrightarrow R.$$

If $R = K_0(Var/k)$, $\mu = id$ is called the *universal motivic measure*.

Given a motivic measure $\mu : K_0(Var/k) \longrightarrow R$, Kapranov defined in [Kap] the μ-*motivic zeta function* of an irreducible quasi-projective variety Y to be the formal series

$$\zeta_\mu(Y, t) = 1 + \sum_{n=1}^{\infty} \mu([(\mathrm{Sym}^n_Y)_{\text{red}}])t^n \in R[[t]].$$

If μ is the universal motivic measure, we get the so-called *motivic zeta function* $\zeta_{\mathrm{id}}(Y,t)$, denoted by $\zeta_{\mathrm{mot}}(Y,t)$. It is a generalization of the Hasse-Weil zeta function.

Proposition 6.0.2 (Prop. 2.9, [Mus]). *Let $k = \mathbb{F}_q$ be a finite field. Consider the motivic measure*

$$|\cdot| : K_0(Var/k) \longrightarrow \mathbb{Z}, \quad [Y] \mapsto |Y(\mathbb{F}_q)|.$$

Then $\zeta_{|\cdot|}(Y,t)$ coincides with the Hasse-Weil zeta function

$$Z(Y,t) = \exp\left(\sum_{m\geq 1} \frac{|Y(\mathbb{F}_{q^m})|}{m} t^m\right).$$

The Hasse-Weil zeta function $Z(Y,t)$ is known to be a rational function (see [Dwo60]). Therefore, it is natural to ask for which motivic measures μ and variety Y, the zeta function $\zeta_\mu(Y,t)$ is rational as well.

Definition 6.0.3. Let R be a commutative ring: we say that $f \in R[[t]]$ is rational if there exist polynomials $p(t), q(t) \in R[t]$ such that $q(t)$ is invertible in $R[[t]]$ and $f(t) = \frac{p(t)}{q(t)}$.

For curves we have the following rationality result.

Theorem 6.0.4. *[Kap, Thm. 1.1.9] If C is a smooth, geometrically connected, projective curve with index 1, then $\zeta_{\mathrm{mot}}(C,t)$ is a rational function.*

In [Kap] it was conjectured that the motivic zeta function was rational for any algebraic variety Y. However Larsen and Lunts provided a counterexample to this conjecture for $k = \mathbb{C}$ in [LL03] and they actually gave a sufficient and necessary condition for rationality in the case of complex surfaces (in [LL04]).

Theorem 6.0.5. *[LL04, Thm. 1.1] Let $k = \mathbb{C}$ and let X be a smooth connected projective complex surface. Then the motivic zeta function $\zeta_{\mathrm{mot}}(X,t)$ is rational if and only if X has negative Kodaira dimension.*

It turns out that having negative Kodaira dimension is a sufficient condition on any algebraically closed field of characteristic zero.

Proposition 6.0.6. *[Mus, Prop. 4.3] Let X be a smooth connected projective surface over an algebraically closed field k of characteristic zero. If X has negative Kodaira dimension, then $\zeta_{\mathrm{mot}}(X,t)$ is a rational function.*

We now aim to study what happens when we drop the assumption on the field to be algebraically closed.

In order to do so, let us first study the rationality of a closely related zeta function, the *μ-motivic Hilbert zeta function*.

Definition 6.0.7. Let μ be an R-valued motivic measure. Let X be a geometrically rational surface. The *μ-motivic Hilbert zeta function* of X is the power series

$$\zeta_{\mu}^{\mathrm{Hilb}}(X,t) = 1 + \sum_{n=1}^{\infty} \mu([\mathrm{Hilb}_X^n])t^n \ \in R[[t]].$$

Denote by $\mathbb{L}$ the class of $[\mathbb{A}_k^1]$ in $K_0(Var/k)$. Consider the motivic measure defined modding out by the ideal generated by $\mathbb{L}$:

$$\mu_{\mathbb{L}} : K_0(Var/k) \longrightarrow K_0(Var/k)/(\mathbb{L}).$$

Hypothesis 6.0.8 (Weak Factorization property)**.** *Let f be a birational map $Y \overset{f}{\dashrightarrow} Y'$ between smooth projective connected varieties: then it can be realized as a composition of blow-ups and blow-downs along smooth irreducible centers on smooth projective varieties.*

Remark 6.0.9. If $\mathrm{char}(k) = 0$ then this hypothesis is satisfied [Wło09, Thm. 0.0.1].

Here onward, let us assume that the field k satisfies the weak factorization property. The usefulness of assuming the weak factorization property is that birational smooth projective varieties are identified in the quotient $K_0(Var/k)/(\mathbb{L})$.

Remark 6.0.10. Let f be a birational map $Y \overset{f}{\dashrightarrow} Y'$ between smooth projective varieties: then $[Y] = [Y']$ in $K_0(Var/k)/(\mathbb{L})$.

Proof. Decomposing f into a sequence of blow-ups and downs, we can assume that f is a blow-up along a smooth centre. Since $\mathbb{L} = 0$ then

$$[\mathbb{A}^n_k] = \mathbb{L}^n = 0, \quad \text{and } [\mathbb{P}^n_k] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n = 1.$$

Therefore if $\mathcal{E}$ is a vector bundle on Y of rank n then $[\mathbb{P}(\mathcal{E})] = [Y]$. If we blow-up a smooth connected projective variety Y along a smooth connected closed subvariety Z then

$$[Bl_Z Y] - [Y] = [\mathbb{P}(N_Z Y)] - [Z] = 0 \implies [Bl_Z Y] = [Y],$$

where $N_Z Y$ is the normal bundle of Z in Y. $\square$

Proposition 6.0.11. *Let k be a field for which weak resolution theorem holds true. Let X be a geometrically rational, connected smooth projective surface. Then the $\mu_{\mathbb{L}}$-motivic Hilbert zeta function*

$$\zeta^{\mathrm{Hilb}}_{\mu}(X,t) = 1 + \sum_{n=1}^{\infty} \mu_{\mathbb{L}}([\mathrm{Hilb}^n_X])t^n \in K_0(Var/k)/(\mathbb{L})[[t]]$$

is a rational function.

Proof. By Corollary 5.0.14, since X is geometrically rational there is an n_0 (which depends only on the birational type of X) such that for any $n \geq n_0$ we have Hilb^n_X stably birational to $\mathrm{Hilb}^{n+\mathrm{ind}(X)}_X$. Therefore the coefficients of $\zeta^{\mathrm{Hilb}}_{\mathbb{L}}(X,t)$ are eventually periodic of period a divisor of $\mathrm{ind}(X)$, which means that the series can be written as

$$
\begin{aligned}
1+\sum_{n=1}^{\infty}\mu_{\mathbb{Z}}([\mathrm{Hilb}_X^n])t^n &= 1+\mu_{\mathbb{L}}([X])t+\cdots+\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0-1}])t^{n_0-1}+\sum_{j=0}^{\infty}\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0+j}])t^{n_0+j}\\
&= P_X(t)+t^{n_0}\sum_{j=0}^{\infty}t^{j\cdot\mathrm{ind}(X)}\left(\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0}])+\cdots+\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0+\mathrm{ind}(X)-1}])t^{\mathrm{ind}(X)-1}\right)\\
&= P_X(t)+t^{n_0}\frac{\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0}])+\cdots+\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0+\mathrm{ind}(X)-1}])t^{\mathrm{ind}(X)-1}}{1-t^{\mathrm{ind}(X)}}
\end{aligned}
$$

where $P_X(t)$ is the polynomial $1+\mu_{\mathbb{L}}([X])t+\cdots+\mu_{\mathbb{L}}([\mathrm{Hilb}_X^{n_0-1}])t^{n_0-1}$. □

In order to recover information on the $\mu_{\mathbb{L}}$-motivic zeta function, we now need to compare the classes $\mu_{\mathbb{L}}([\mathrm{Sym}_X^n])$ with $\mu_{\mathbb{L}}([\mathrm{Hilb}_X^n])$ in $K_0(Var/k)/(\mathbb{L})$.

The following theorem is due to Göttsche. It was proven under the assumption that k is a field of characteristic zero and algebraically closed, but the latter assumption is not used and can be dropped.

Theorem 6.0.12. *[Göt01, Thm. 1.1] Let k be a field of characteristic zero. Let $\alpha=(m_1,\ldots,m_r)$ be a partition of n. Denote by a_i is the number of times for which $m_j=i$. Let $|\alpha|=\sum_{i=1}^n a_i$. Then the class $[\mathrm{Hilb}_X^n]$ in $K_0(Var/k)$ is equal to*

$$\sum_{\alpha\dashv n}[\mathrm{Sym}_X^{a_1}\times\cdots\times\mathrm{Sym}_X^{a_n}\times\mathbb{A}_k^{n-|\alpha|}].$$

Therefore, here onwards, we will assume that k has characteristic zero. In particular, k satisfies the weak factorization theorem.

Corollary 6.0.13. *Let k be a field of characteristic zero: then in the quotient ring $K_0(Var/k)/(\mathbb{L})$ we have that*

$$\mu_{\mathbb{L}}([\mathrm{Sym}_X^n])=\mu_{\mathbb{L}}([\mathrm{Hilb}_X^n]).$$

Proof. Since $\mathbb{L}$ is sent to zero, then $\mu_{\mathbb{L}}([\mathbb{A}_k^{n-|\alpha|}])$ is zero if $|\alpha|$ is not n. So the only non-zero term of the Göttsche's formula in $K_0(Var/k)/(\mathbb{L})$ is the one associated to the partition $\alpha = (1, \dots, 1)$ (so $a_1 = n$). Hence

$$\mu_{\mathbb{L}}([\mathrm{Sym}^n_X]) = \mu_{\mathbb{L}}([\mathrm{Hilb}^n_X]).$$

□

In particular in $K_0(Var/k)/(\mathbb{L})$, the motivic Hilbert zeta function coincides with the motivic zeta function

$$1 + \sum_{n=1}^{\infty} \mu_{\mathbb{L}}([\mathrm{Sym}^n_X])t^n.$$

Corollary 6.0.14 (Theorem D)**.** *Let k be a field of characteristic zero: if X is a geometrically rational connected smooth projective surface, then the $\mu_{\mathbb{L}}$-motivic zeta function is rational.*

Remark 6.0.15. By the proof above, notice that in order to conclude the rationality of $\zeta_{\mathbb{L}}(X, t)$ (for a geometrically rational surface X), it is sufficient for k to have the weak factorization property and to be such that $\mu_{\mathbb{L}}([\mathrm{Sym}^n_X]) = \mu_{\mathbb{L}}([\mathrm{Hilb}^n_X])$ for any $n > 0$.

Chapter 7: An alternative way to relate Hilb_X^n and $\mathrm{Hilb}_X^{n'}$

We want to generalize the incidence correspondences defined in Chapter 2 relaxing the last condition (namely (4)) in order to relate Hilb_X^n and $\mathrm{Hilb}_X^{n'}$ for different pairs (n, n'). The idea is still the same: to choose an appropriate curve C and relate the effective 0-cycles of X looking at the effective 0-cycles on C.

Let a be an integer, and $\mathcal{L}, \mathcal{M}$ be two line bundles on X such that $\mathcal{L}$ is ample. Let Q be a geometrically reduced effective 0-cycle on X. We can then generalize Definition 2.0.1 in the following way.

Definition 7.0.1. Let us define the functor

$$\mathrm{Inc}_{X,\mathcal{L},n,n'}^{a,\mathcal{M},\pm} : (\mathrm{Sch}/k)^{op} \longrightarrow \mathrm{Set},$$

sending T to the set of triples $(Z, Z', C) \in \mathrm{Hilb}_X^n(T) \times \mathrm{Hilb}_X^{n'}(T) \times \mathrm{CDiv}_X^{\mathcal{L}}(T)$, such that:

(1) the subschemes Z, Z' and, if $a \neq 0$, Q_T are contained in $C \subseteq X_T$,

(2) the scheme C is smooth over T,

(3) Z, Z' are relative effective Cartier divisors on C over T,

(4) the classes $[\mathcal{O}_C(Z' \pm Z)]$ and $[\mathcal{M}_T|_C + \mathcal{O}_C(aQ)]$ agree in $Pic_{C/T}(T)$ (where $\mathcal{M}_T$ is the pullback of $\mathcal{M}$ to X_T).

Definition 7.0.2. Let $\mathrm{Inc}^{a}_{X,\mathcal{L},n} : (\mathrm{Sch}/k)^{op} \longrightarrow \mathrm{Set}$ be the functor

$$T \mapsto \{(Z, C) \in \mathrm{Hilb}^n_X(T) \times \mathrm{CDiv}^{\mathcal{L}}_X(T) : Z \subset C \text{ and, if } a \neq 0, Q_T \subset C\}.$$

Remark 7.0.3. As noted in the Remarks following Definition 2.0.1,

1. due to the smoothness condition, conditions (1) and (2) imply condition (3), and
2. since C/T is surjective, smooth, proper, cohomologically flat of dimension 0 and with geometrically integral fibers, the functor $Pic_{C/T}$ is represented by a separated T-scheme [BLR90, Thm. 8.2.1].

The same proofs that led to Propositions 2.0.8 and 2.0.10 show that $\mathrm{Inc}^{a,\mathcal{M},\pm}_{X,\mathcal{L},n,n'}$ and $\mathrm{Inc}^{a}_{X,\mathcal{L},n}$ are representable as well.

Remark 7.0.4. Note that $\mathcal{M}_T|C$ recovers many natural choices of 0-cycles on C. For instance

1. it recovers ω_C, thanks to adjunction formula (in this case $\mathcal{M} = \omega_X \otimes \mathcal{L}$),
2. it recovers $\mathcal{O}_C(1)$, embedding C using the embedding of $X \hookrightarrow \mathbb{P}^N_k$) (in this case $\mathcal{M} = \mathcal{O}_X(1)$),

Chapter 2 works with the case when $\mathcal{M} = \mathcal{O}_X$, the sign chosen is $-$ and $a = 1$. Among the possible choices, we now consider the "orthogonal case" of $(1, \mathcal{O}_X, -)$: namely, we assume $a = 0$ (so no fixed Q), positive sign (so $\mathcal{O}_C(Z' + Z)$) and a non trivial $\mathcal{M}$. We denote the incidence correspondences respectively as $\mathrm{Inc}^{\mathcal{M},+}_{X,\mathcal{L},n,n'}$ and $\mathrm{Inc}_{X,\mathcal{L},n}$.

As before, we start by checking that the general fibers of

$$\mathrm{Inc}^{\mathcal{M},+}_{X,\mathcal{L},n,n'} \longrightarrow \mathrm{Inc}_{X,\mathcal{L},n}, \quad \text{and} \quad \mathrm{Inc}_{X,\mathcal{L},n} \longrightarrow \mathrm{Hilb}^n_X$$

are projective spaces, finding appropriate (integral) opens where this is true.

Remark 7.0.5. Note that the role of n (and Z) is completely symmetrical to the one of n' (and Z'): so many statements and proofs will be given only in terms of n (and Z).

As noticed in proof of Lemma 2.1.4:

1. as soon as

$$pr_{H*}(pr_X^*\mathcal{L}) \otimes_{\mathcal{O}_{\mathrm{Hilb}^n_X,y}} k(y) \longrightarrow H^0(X_y, (pr_X^*\mathcal{L})_y)$$

   is an isomorphism, then $pr_{H,*}(pr_X^*\mathcal{L})$ is locally free and its formation commutes with base change,

2. the former condition is true everywhere due to flat base change [Sta, Tag 02KH]).

We can therefore drop this condition from the definition of U_n of Proposition 2.1.1 and get the following simplified one.

Proposition 7.0.6 (Definition of U_n and $U_{n'}$)**.** *Define U_n to be the subfunctor of* Hilb^n_X *made of points $y \in \mathrm{Hilb}^n_X$ such that*

$$H^0(X_y, (pr_X^*\mathcal{L})_y) \longrightarrow H^0(X_y, (pr_X^*\mathcal{L})_y|_{\mathcal{Z}_y})$$

is surjective. The following hold:

1. *the subfunctor U_n is an open subscheme of* Hilb^n_X *(and hence integral),*

2. *on U_n the sheaf $pr_{H,*}(pr_X^*\mathcal{L} \otimes \mathcal{I}_{\mathcal{Z}})$ is locally free and its formation commutes with base change, and*

3. *if $\mathcal{E}_n$ is the dual sheaf $(pr_{H,*}(pr_X^*\mathcal{L}\otimes\mathcal{I}_{\mathcal{Z}})\,|_{U_n})^\vee$, then we have the following cartesian square:*

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}_n) \arrow[r, hook, "\circ"] \arrow[d] & \mathrm{Inc}_{X,\mathcal{L},n} \arrow[d] \\ U_n \arrow[r, hook, "\circ"] & \mathrm{Hilb}^n_X. \end{tikzcd}$$

Moreover, if $h^0(X,\mathcal{L}) = \dim_k H^0(X,\mathcal{L}) \geq n$, then U_n is non-empty.

Proof. The first part is the analogue of Proposition 2.1.1. The part regarding the non-emptiness follows from Lemma 3.0.7. ◻

Remark 7.0.7 (Non emptiness of $\mathbb{P}(\mathcal{E}_n)$ and $\mathbb{P}(\mathcal{E}_{n'})$). Let $y \in U_n$: we have

$$\dim_{k(y)} H^0(X_y, \mathcal{L}_y \otimes \mathcal{I}_{Z_y}) \geq \dim_{k(y)} H^0(X_y, \mathcal{L}_y) - n.$$

Therefore,

$$\dim \mathbb{P}(\mathcal{E}_n) \geq 2n + \min_{y \in U_n} \left(\dim_{k(y)} H^0(X_y, \mathcal{L}_y) - n - 1 \right).$$

In particular, if $h^0(X, \mathcal{L}) \geq n+1$, then both U_n and $\mathbb{P}(\mathcal{E}_n)$ are non-empty (integral) opens.

Consider the universal curve

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "g"'] & X \times \mathbb{P}(\mathcal{E}_n) \arrow[d] \\ & \mathbb{P}(\mathcal{E}_n). \end{tikzcd}$$

Let $\mathcal{Z}$ be the universal family parametrizing the subscheme $Z \in \mathrm{Hilb}^n_X$ (which is then contained in C).

Definition 7.0.8 (Definition of V and V')**.** Let V be the (integral) open of $\mathbb{P}(\mathcal{E}_n)$ where g is smooth. Analogously, let V' be the (integral) open of $\mathbb{P}(\mathcal{E}_{n'})$ where $g' : C' \longrightarrow \mathbb{P}(\mathcal{E}_{n'})$ is smooth.

Remark 7.0.9 (Non emptiness of V and V'). Assume $\mathbb{P}(\mathcal{E}_n) \neq \varnothing$. By Proposition 3.0.9, if $\mathcal{L}$ is globally generated, and for each $\overline{k}$-point $x \in X_{\overline{k}}$ the map

$$H^0(X, \mathcal{L})_{\overline{k}} \longrightarrow \mathcal{L}_x / m_x^2 \mathcal{L}_x$$

is surjective, then V is non-empty.

On V the map $g|_V$ is cohomologically flat of dimension zero, it has integral fibers, and $\mathcal{Z}|_V$ is a relative effective Cartier divisor on $C|_V$.

Proposition 7.0.10 (Definition of V_n and $V_{n'}$)**.** *Define V_n to be the subfunctor of $V \subset \mathbb{P}(\mathcal{E}_n)$ made* *of points $y \in V$ such that*

$$g_*(\mathcal{O}_{C_V}(-\mathcal{Z}_V) \otimes \mathcal{M}_V|_{C_V}) \otimes \mathcal{O}_{V,y} k(y) \longrightarrow H^0(C_y, \mathcal{O}_{C_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{C_y})$$

is surjective. Then the following hold:

1. *the subfunctor V_n is an (integral) open subscheme of V,*

2. *the sheaf $g_*(\mathcal{O}_{C_{V_n}}(-\mathcal{Z}_{V_n}) \otimes \mathcal{M}_{V_n}|_{C_{V_n}})$ is locally free and its formation commutes with base change, and*

3. *if $\mathcal{F}_n$ is the dual sheaf $(g_*(\mathcal{O}_{C_{V_n}}(-\mathcal{Z}_{V_n}) \otimes \mathcal{M}_{V_n}|_{C_{V_n}}))^\vee$, then we have the following cartesian square:*

$$\begin{tikzcd}
\mathbb{P}(\mathcal{F}_n) \arrow[r, hook, "\circ" marking] \arrow[d] & \mathrm{Inc}^{\mathcal{M},+}_{X,\mathcal{L},n,n'}|_{U_n} \arrow[d] \\
V_n \arrow[r, hook, "\circ" marking] & \mathbb{P}(\mathcal{E}_n).
\end{tikzcd}$$

Proof. The proof is the analogue of the one for Proposition 2.1.5. ◻

Remark 7.0.11. Let $y \in V$ be a point such that $H^1(C_y, \mathcal{O}_{C_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{C_y}) = 0$. Then there is an open neighborhood U_y of y such that $U_y \subseteq V_n$ (this is due to Cohomology and Base change Theorem; see Lemma 3.0.10).

7.0.1 Non-emptiness of the opens V_n and $\mathbb{P}(\mathcal{F}_n)$

Proposition 7.0.12 (Non emptiness of V_n and $V_{n'}$)**.** *Assume V is non-empty. If*

$$n \geq p_a(C) \quad \text{and} \quad c_1(\mathcal{L})c_1(\mathcal{M}) - n - p_a(C) \geq 0,$$

then the locus of points in V such that $H^1(C_y, \mathcal{O}_{C_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{C_y}) = 0$ is open and non-empty, which means that V_n is non-empty.

Proof. It is open due to the upper semicontinuity of the function

$$h^1(C_y, \mathcal{O}_{C_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{C_y})$$

(see [Har77, Thm. III.12.8]). Let us check then it is non-empty by a very similar proof to the one of Proposition 3.0.11. Denote by $\mathrm{SmCDiv}_X^{\mathcal{L}}$ the open subscheme of $\mathrm{CDiv}_X^{\mathcal{L}}$ consisting of divisors smooth over the base. Then we have the following diagram (all the squares are cartesian):

$$\begin{tikzcd}
V \arrow[rr] \arrow[d, hook] & & \mathrm{SmCDiv}_X^{\mathcal{L}} \arrow[d, hook] \\
\mathbb{P}(\mathcal{E}_n) \arrow[r, hook] & \mathbb{P}((pr_{H*}(pr_X^*\mathcal{L})|_{U_n})^\vee) \arrow[r] \arrow[d] & \mathrm{CDiv}_X^{\mathcal{L}} \arrow[d] \\
& U_n \arrow[r] & \mathrm{Spec}(k).
\end{tikzcd}$$

$\mathrm{SmCDiv}_X^{\mathcal{L}}$ is non empty because V is non-empty. Let $y \in \mathrm{SmCDiv}_X^{\mathcal{L}}$ be a geometric point (so $k(y)$ is algebraically closed) such that the fiber V_y is non-empty. The point y corresponds to a smooth $k(y)$-curve $C \subset X_{k(y)}$: therefore the fiber V_y coincides with $U_n \times_{\mathrm{Hilb}_X^n} \mathrm{Hilb}_{C/k(y)}^n$, open subscheme of $\mathrm{Hilb}_{C/k(y)}^n$.

Consider the Abel-Jacobi map for the curve C

$$AJ_C^n : \mathrm{Hilb}_{C/k(y)}^n \longrightarrow \mathrm{Pic}_{C/k(y)}^n.$$

By assumption, $p_a(C) \leq n$ so that AJ_C^n surjective and the composite

$$V_y \hookrightarrow \mathrm{Hilb}_{C/k(y)}^n \xrightarrow{AJ_C^n} \mathrm{Pic}_{C/k(y)}^n$$

dominant. Further composing with the isomorphism

$$\mathrm{Pic}_{C/k(y)}^n \longrightarrow \mathrm{Pic}_{C/k(y)}^{-n+\deg(\mathcal{M}_y|_C)}, \quad \mathcal{J} \mapsto \mathcal{J}^\vee \otimes \mathcal{M}_y|_C,$$

we get another dominant map

$$V_y \hookrightarrow\!\!\circ\!\!\rightarrow \mathrm{Hilb}^n_{C/k(y)} \xrightarrow{AJ^n_C} \mathrm{Pic}^n_{C/k(y)} \xrightarrow{(-)^\vee \otimes \mathcal{M}_y|_C} \mathrm{Pic}^{-n+\deg(\mathcal{M}_y|_C)}_{C/k(y)}.$$

Consider the closed subschemes

$$W^{-n+\deg(\mathcal{M}_y|_C)-p_a(C)+1}_{-n+\deg(\mathcal{M}_y|_C)}(C/k(y)) \subseteq W^{-n+\deg(\mathcal{M}_y|_C)-p_a(C)}_{-n+\deg(\mathcal{M}_y|_C)}(C/k(y)).$$

By Riemann-Roch, $W^{-n+\deg(\mathcal{M}_y|_C)-p_a(C)}_{-n+\deg(\mathcal{M}_y|_C)}(C/k(y)) = \mathrm{Pic}^{-n+\deg(\mathcal{M}_y|_C)}_{C/k(y)}$. Note that by assumption

$$r = -n + \deg(\mathcal{M}_y|_C) - p_a(C) = c_1(\mathcal{L})c_1(\mathcal{M}) - n - p_a(C) \geq 0.$$

Then by Fact 1.0.4, since $p_a(C) - n + \deg(\mathcal{M}_y|_C) - p_a(C) + n - \deg(\mathcal{M}_y|_C) \geq 0$, we have

$$\mathrm{Pic}^{-n+\deg(\mathcal{M}_y|_C)}_{C/k(y)} \setminus W^{-n+\deg(\mathcal{M}_y|_C)-p_a(C)+1}_{-n+\deg(\mathcal{M}_y|_C)}(C/k(y))$$

is a non-empty integral open. Moreover, the K-points of this open corresponds precisely to $[\mathcal{O}_{C_K}(-Z)\otimes\mathcal{M}_K|_{C_K}] \in \mathrm{Pic}^{-n+\deg(\mathcal{M}_y|_C)}_{C/k(y)}(K)$ such that $H^1(C_K, \mathcal{O}_{C_K}(-Z)\otimes\mathcal{M}_K|_{C_K}) = 0$. Therefore, the preimage of $\mathrm{Pic}^{-n+\deg(\mathcal{M}_y|_C)}_{C/k(y)} \setminus W^{-n+\deg(\mathcal{M}_y|_C)-p_a(C)+1}_{-n+\deg(\mathcal{M}_y|_C)}(C/k(y))$ in V_y is the wanted non-empty open: hence the statement. □

Similarly we have the following.

Proposition 7.0.13. *Assume V is non-empty. If*

$$n \geq p_a(C) \quad \text{and} \quad n + c_1(\mathcal{L})^2 - c_1(\mathcal{M})c_1(\mathcal{L}) - p_a(C) \geq 0$$

then the locus of points $y = (\mathcal{Z}_y, \mathcal{C}_y) \in V$ such that

$$H^1(\mathcal{C}_y, \mathcal{O}_{\mathcal{C}_y}(\mathcal{Z}_y)) = H^1(\mathcal{C}_y, \mathcal{O}_{\mathcal{C}_y}(\mathcal{Z}_y) \otimes \mathcal{L}_y|_{\mathcal{C}_y} \otimes \mathcal{M}_y^\vee|_{\mathcal{C}_y}) = 0$$

is open and non-empty.

Proof. It is enough to check that each vanishing condition defines a non-empty open of V. Indeed V is integral, and therefore two non-empty opens will always intersect. For the vanishing of $H^1(C_y, \mathcal{O}_{C_y}(\mathcal{Z}_y))$, consider again the dominant map

$$V_y \hookrightarrow \circ \rightarrow \mathrm{Hilb}^n_{C/k(y)} \xrightarrow{AJ^n_C} \mathrm{Pic}^n_{C/k(y)}$$

and the closed subschemes

$$W_n^{n-p_a(C)+1}(C/k(y)) \subseteq W_n^{n-p_a(C)}(C/k(y)) = \mathrm{Pic}^n_{C_y/k(y)}$$

inside the target. By Fact 1.0.4, their difference is a non-empty integral open corresponding to the $Z \subset C$ such that $H^1(C, \mathcal{O}_C(Z)) = 0$.

For the other vanishing, consider the dominant map

$$V_y \hookrightarrow\!\!\circ\!\!\longrightarrow \mathrm{Hilb}^n_{C/k(y)} \xrightarrow{AJ^n_{C/k(y)}} \mathrm{Pic}^n_{C/k(y)} \xrightarrow{(-)\otimes\mathcal{L}_y|_C\otimes\mathcal{M}_y^\vee|_C} \mathrm{Pic}^{n+\deg(\mathcal{L}_y|_C)-\deg(\mathcal{M}_y|_C)}_{C/k(y)}$$

and the closed subschemes

$$W^{n+\deg(\mathcal{L}_y|_C)-\deg(\mathcal{M}_y|_C)-p_a(C)+1}_{n+\deg(\mathcal{L}_y|_C)-\deg(\mathcal{M}_y|_C)} \subseteq W^{n+\deg(\mathcal{L}_y|_C)-\deg(\mathcal{M}_y|_C)-p_a(C)}_{n+\deg(\mathcal{L}_y|_C)-\deg(\mathcal{M}_y|_C)} = \mathrm{Pic}^{n+\deg(\mathcal{L}_y|_C)-\deg(\mathcal{M}_y|_C)}_{C/k(y)}.$$

Since $n + \deg(\mathcal{L}_y|_C) - \deg(\mathcal{M}_y|_C) - p_a(C) \geq 0$, then by Fact 1.0.4, their difference is a non-empty integral open corresponding to the $Z \subset C$ such that $H^1(C, \mathcal{O}_C(Z) \otimes \mathcal{L}_y|_C \otimes \mathcal{M}_y^\vee|_C) = 0$. $\square$

Remark 7.0.14 (Non emptiness of $\mathbb{P}(\mathcal{F}_n)$ and $\mathbb{P}(\mathcal{F}_{n'})$). Assume V_n is non-empty. For any $y \in V_n$ we have

$$h^0(C_y, \mathcal{O}_{C_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{C_y}) \geq -n + \deg(\mathcal{M}_y|_{C_y}) - p_a(C_y) + 1 = -n + c_1(\mathcal{L})c_1(\mathcal{M}) - p_a(C) + 1$$

Therefore,

$$\dim \mathbb{P}(\mathcal{F}_n) \geq \dim V_n + \min_{y \in V_n} (-n + c_1(\mathcal{L})c_1(\mathcal{M}) - p_a(C))$$

In particular, if V_n is non-empty and

$$-n + c_1(\mathcal{L})c_1(\mathcal{M}) - p_a(C) \geq 0,$$

then $\mathbb{P}(\mathcal{F}_n)$ is non-empty.

It remains to compare $\mathbb{P}(\mathcal{F}_n)$ and $\mathbb{P}(\mathcal{F}_{n'})$ in order to see whether they intersect in $\mathrm{Inc}^{\mathcal{M},+}_{X,\mathcal{L},n,n'}$.

Lemma 7.0.15. *Assume $\mathbb{P}(\mathcal{F}_n)$ is non-empty and let $y = (Z, Z', C)$ be a point of $\mathbb{P}(\mathcal{F}_n)$. Assume V' is non-empty. Then if the point y satisfies*

$$H^1(C, \mathcal{O}_C(Z)) = H^1(C, \mathcal{O}_C(-Z') \otimes \mathcal{L}_y|_C) = 0$$

then $y \in \mathbb{P}(\mathcal{F}_{n'})$.

Proof. The point y is in $\mathbb{P}(\mathcal{F}_{n'})$ if and only if $(Z', C) \in V_{n'}$. Since C is smooth and $H^1(C, \mathcal{O}_C(-Z') \otimes \mathcal{M}_y|_C) = H^1(C_y, \mathcal{O}_C(Z)) = 0$, by Remark 7.0.11 then this is true if and only if $Z' \in U_{n'}$. Namely, we need to check that

$$H^0(X_y, \mathcal{L}_y) \longrightarrow H^0(Z', \mathcal{L}_y|_{Z'}) \tag{7.1}$$

is surjective. Consider the maps

$$\begin{tikzcd}
& & & H^1(C, \mathcal{O}_C(-Z') \otimes \mathcal{L}_y|_C) \\
H^0(X_y, \mathcal{L}_y) \arrow[rr] \arrow[dr] & & H^0(Z', \mathcal{L}_y|_{Z'}) \arrow[ur] & \\
& H^0(C, \mathcal{L}_y|_C) \arrow[ur] \arrow[dr] & & \\
& & H^1(X_y, \mathcal{O}_{X_y}). &
\end{tikzcd}$$

Since $H^1(X_y, \mathcal{O}_{X_y}) = 0$, the map $H^0(X_y, \mathcal{L}_y) \longrightarrow H^0(C, \mathcal{L}_y|_C)$ is surjective. Since by assumption $H^1(C, \mathcal{O}_C(-Z') \otimes \mathcal{L}_y|_C) = 0$, then also $H^0(C, \mathcal{L}_y|_C) \longrightarrow H^0(Z', \mathcal{L}_y|_{Z'})$ is surjective, making (7.1) surjective as well. □

Remark 7.0.16. Note that not all the suitable pairs $(X, \mathcal{L})$ will satisfy the previous lemma. Indeed, if we have a y such that $H^1(C, \mathcal{O}_C(-Z') \otimes \mathcal{L}_y|_C) = 0$ we need $H^1(C, \mathcal{L}_y|_C) = 0$ (because $H^1(Z', \mathcal{L}_y|_{Z'}) = 0$). The $k(y)$-dimension of this cohomology group coincides with $h^1(C, \mathcal{L}|_C)$ where C is a curve in the linear system $|\mathcal{L}|$.

For instance, this is automatically zero if $h^1(X, \mathcal{L}) = h^2(X, \mathcal{O}_X) = 0$ due to the exact sequence defining C.

Another necessary condition can be seen when C is smooth: indeed by Serre duality and adjunction formula,

$$H^1(C, \mathcal{L}|_C) \cong H^0(C, \omega_C \otimes \mathcal{L}^\vee|_C) \cong H^0(C, \omega_X|_C)$$

and now by Riemann-Roch

$$\deg(\mathcal{L}|_C) + 1 - p_a(C) \leq 0 \implies c_1(\mathcal{L})^2 \geq c_1(\mathcal{L})K_X.$$

Theorem 7.0.17. *Let $(X, \mathcal{L})$ be a suitable pair. Let $\mathcal{M}$ be a line bundle on X. Let n be a positive integer such that $n' = c_1(\mathcal{L})c_1(\mathcal{M}) - n$ is positive. Assume that $\mathcal{L}$ satisfies the following assumptions with respect to $(n, \mathcal{M})$:*

1. $h^0(X, \mathcal{L}) \geq \max\{n, n'\} + 1$,
2. *the bundle $\mathcal{L}$ is globally generated,*
3. *for any $\overline{k}$-point $x \in X_{\overline{k}}$, the map $H^0(X, \mathcal{L})_{\overline{k}} \longrightarrow \mathcal{L}_x / m_x^2 \mathcal{L}_x$ is surjective,*
4. $p_a(C) \leq \min\{n, n'\}$,
5. $\min\{n, n'\} + c_1(\mathcal{L})^2 - c_1(\mathcal{M})c_1(\mathcal{L}) - p_a(C) \geq 0$.

Then

$$\mathrm{Hilb}^n_X \sim_{\text{stably birational}} \mathrm{Hilb}^{n'}_X.$$

Proof. Thanks to the first assumption, the opens $U_n, U_{n'}, \mathbb{P}(\mathcal{E}_n)$ and $\mathbb{P}(\mathcal{E}_{n'})$ are all non-empty (as observed in Remark 7.0.7). Thanks to second and third assumption, the opens V and V' are non-empty (see Remark 7.0.9).

By Propositions 7.0.12, since

$$n \geq p_a(C), \quad c_1(\mathcal{L})c_1(\mathcal{M}) - n - p_a(C) = n' - p_a(C) \geq 0,$$

the locus in V where $H^1(\mathcal{C}_y, \mathcal{O}_{\mathcal{C}_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{\mathcal{C}_y}) = 0$ is non-empty, making V_n non-empty as well. The analogous is true for $V_{n'}$.

By Proposition 7.0.13, since

$$n \geq p_a(C), \quad n + c_1(\mathcal{L})^2 - c_1(\mathcal{L})c_1(\mathcal{M}) - p_a(C) \geq 0$$

then we can further restrict to the non-empty open of V_n made of points $y \in V_n$ such that

$$H^1(\mathcal{C}_y, \mathcal{O}_{\mathcal{C}_y}(-\mathcal{Z}_y) \otimes \mathcal{M}_y|_{\mathcal{C}_y}) = H^1(\mathcal{C}_y, \mathcal{O}_{\mathcal{C}_y}(\mathcal{Z}_y)) = H^1(\mathcal{C}_y, \mathcal{O}_{\mathcal{C}_y}(\mathcal{Z}_y) \otimes \mathcal{L}_y|_{\mathcal{C}_y} \otimes \mathcal{M}^\vee_y|_{\mathcal{C}_y}) = 0.$$

So pick $(Z, C) \in V_n$ in this locus: then, by Remark 7.0.14, you can pick (Z, Z', C) in $\mathbb{P}(\mathcal{F}_n)$ above it (because $n' \geq p_a(C)$). By Lemma 7.0.15, (Z, Z', C) belongs to $\mathbb{P}(\mathcal{F}_{n'})$. □

Chapter 8: Effective results for Cubic Surfaces

As mentioned in the Introduction, since we are ultimately interested in the (CS) conjecture, we can discard the finite-field case by Chevalley-Warning [Che35]-[CE35]. Therefore we assume here onward k to be infinite.

Let Q be a smooth effective 0-cycle of degree 3 on X obtained by intersecting two hyperplanes H_1, H_2 with X in $\mathbb{P}^3_k$.

8.1 Using the incidence correspondence $\mathrm{Inc}^{Q}_{X,\mathcal{L},n,n'}$

Let us set $\mathcal{L} = \mathcal{O}_X(e)$. We are first of all interested to see for which $e \geq 1$ the suitable triple $(X, \mathcal{O}_X(e), Q)$ satisfies the assumptions of Theorem A.

8.1.1 Necessary conditions on e

Proposition 8.1.1. *Consider $(X, \mathcal{O}_X(e))$: if there exists Z smooth effective 0-cycle of degree 3 such that the triple $(X, \mathcal{O}_X(e), Z)$ satisfies assumptions of Theorem A then $e \geq 3$.*

Proof. Assumption (2) reads as

$$h^0(X, \mathcal{O}_X(e)) \geq n + 7.$$

where the dimension $h^0(X, \mathcal{O}_X(e))$ is equal to $\frac{3e(e+1)}{2} + 1$ for $e \geq 1$. On the other hand, assumption (5) is equivalent to $\frac{3e(e-1)}{2} + 1 \leq n$. Combining this two inequalities, we get the necessary conditions on e

$$\frac{3e(e-1)}{2}+8 \leq n+7 \leq \frac{3e(e+1)}{2}+1 \implies e \geq 3.$$

$\square$

We now would like to consider the blow-up case.

Framework 8.1.2 (Blow-up framework)**.** Let Q' be a smooth effective 0-cycle of degree 3 (resp. 6) defined as a smooth complete intersection $Q' = H_1' \cap H_2' \cap X$ (respectively $Q' = H_1' \cap S \cap X$ where S is a quadric) and let us assume that the hyperplanes involving the definition of Q and Q' are all distinct (k is infinite). In particular Q and Q' are disjoint. Consider $b : \widetilde{X} \longrightarrow X$ the blow-up in Q' and let E be the exceptional divisor.

Let us start with a computational Lemma that will be useful in order to deal with this case.

Lemma 8.1.3. *Consider the blow-up framework. Let e be a positive integer. Then*

$$h^0(X, \mathcal{I}_{Q'}^2(e)) = \begin{cases} 3h^0(X, \mathcal{O}_X(e-2)) - 2h^0(X, \mathcal{O}_X(e-3)) & \text{if } \deg(Q') = 3 \\ h^0(X, \mathcal{O}_X(e-3)) + h^0(X, \mathcal{O}_X(e-2)) - h^0(X, \mathcal{O}_X(e-5)) & \text{if } \deg(Q') = 6. \end{cases}$$

Moreover

$$h^1(X, \mathcal{I}_{Q'}^2(e)) = 0 \text{ for } \begin{cases} e \geq 3 \text{ if } \deg(Q') = 3 \\ e \geq 5 \text{ if } \deg(Q') = 6. \end{cases} \quad \text{and} \quad h^2(X, \mathcal{I}_{Q'}^2(e)) = 0 \text{ for } \begin{cases} e \geq 2 \text{ if } \deg(Q') = 3 \\ e \geq 4 \text{ if } \deg(Q') = 6. \end{cases}$$

Proof. Let us start with the degree 3 case. Since Q' is of complete intersection, the Koszul complex resolves the ideal $\mathcal{I}_{Q'}$ in X. So we have an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_X(-2) \longrightarrow \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1) \longrightarrow \mathcal{I}_{Q'} \longrightarrow 0.$$

Tensor it by $\mathcal{I}_{Q'}$ we get:

$$\mathcal{I}_{Q'}(-2) \longrightarrow \mathcal{I}_{Q'}(-1)^{\oplus 2} \longrightarrow \mathcal{I}_{Q'}^{\otimes 2} \longrightarrow 0. \tag{8.1}$$

We claim that the injectivity on the left is preserved in (8.1) as well. We can check it locally on X. Then X is the spectrum of the ring $A = k[x_1, x_2, x_3]/(f)$ with f an irreducible polynomial of degree 3, and the ideal $I_{Q'}$ is $(h'_1, h'_2, f)/(f)$ where h'_i are polynomials of degree 1. Therefore the map on the left in (8.1) is

$$(h'_1, h'_2, f)/(f) \xrightarrow{a \mapsto (ah'_1, -ah'_2)} (h'_1, h'_2, f)/(f) \oplus (h'_1, h'_2, f)/(f).$$

Since A is a domain, neither h'_1 or h'_2 are 0 divisors in A: so we get

$$0 \longrightarrow \mathcal{I}_{Q'}(-2) \longrightarrow \mathcal{I}_{Q'}(-1)^{\oplus 2} \longrightarrow \mathcal{I}_{Q'}^{\otimes 2} \longrightarrow 0. \tag{8.2}$$

Let us now consider the surjective map

$$\varphi : \mathcal{I}_{Q'}(-1)^{\oplus 2} \longrightarrow \mathcal{I}_{Q'}^{\otimes 2} \longrightarrow \mathcal{I}_{Q'}^{2}$$

sending $(s, t) \mapsto sh'_2 + th'_1$. We get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{Q'}(-2) \arrow[r] \arrow[d, dashed] & \mathcal{I}_{Q'}^{\oplus 2}(-1) \arrow[r] \arrow[d, equal] & \mathcal{I}_{Q'}^{\otimes 2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] & \mathcal{I}_{Q'}^{\oplus 2}(-1) \arrow[r, "\varphi"] & \mathcal{I}_{Q'}^{2} \arrow[r] & 0.
\end{tikzcd}$$

Note that $\mathcal{O}_X(-2) \subseteq \ker(\varphi)$. Let us check that this inclusion is an equality. Consider $(s, t) \in \ker(\varphi)$ so such that $h'_2 s + h'_1 t = 0$. Then $h'_2 s = 0$ in $A/(h'_1)$. Since Q' is smooth, the sequence (h'_1, h'_2) is regular in A and therefore h'_2 is not a zero divisor in $A/(h'_1)$. This implies that $s = h'_1 a_1$ and $a_1 \in A$. Analogously one gets that $t = h'_2 a_2$ and $a_2 \in A$. In particular $\ker\varphi$ coincides with

$\mathcal{O}_X(-2)$. Hence we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2) \longrightarrow \mathcal{I}_{Q'}(-1)^{\oplus 2} \longrightarrow \mathcal{I}_{Q'}^2 \longrightarrow 0. \tag{8.3}$$

In particular, tensoring 8.3 by $\mathcal{O}_X(e)$ and then taking cohomology, we get

$$0 \longrightarrow H^0(X, \mathcal{O}_X(e-2)) \longrightarrow H^0(X, \mathcal{I}_{Q'}(e-1))^{\oplus 2} \longrightarrow H^0(X, \mathcal{I}_{Q'}^2(e)) \longrightarrow 0$$
$$0 \longrightarrow H^1(X, \mathcal{I}_{Q'}(e-1))^{\oplus 2} \longrightarrow H^1(X, \mathcal{I}_{Q'}^2(e)) \longrightarrow H^2(X, \mathcal{O}_X(e-2)) \longrightarrow$$
$$\longrightarrow H^2(X, \mathcal{I}_{Q'}(e-1))^{\oplus 2} \longrightarrow H^2(X, \mathcal{I}_{Q'}^2(e)) \longrightarrow 0$$

Similarly, tensoring the Koszul resolution of $\mathcal{I}_{Q'}$ by $\mathcal{O}_X(e-1)$ and taking cohomology we get

$$0 \longrightarrow H^0(X, \mathcal{O}_X(e-3)) \longrightarrow H^0(X, \mathcal{O}_X(e-2))^{\oplus 2} \longrightarrow H^0(X, \mathcal{I}_{Q'}(e-1)) \longrightarrow 0.$$
$$0 \longrightarrow H^1(X, \mathcal{I}_{Q'}(e-1)) \longrightarrow H^2(X, \mathcal{O}_X(e-3)) \longrightarrow H^2(X, \mathcal{O}_X(e-2))^{\oplus 2} \longrightarrow$$
$$\longrightarrow H^2(X, \mathcal{I}_{Q'}(e-1)) \longrightarrow 0.$$

In particular then

$$h^0(X, \mathcal{I}_{Q'}^2(e)) = 2h^0(X, \mathcal{I}_{Q'}(e-1)) - h^0(X, \mathcal{O}_X(e-2)) = 3h^0(X, \mathcal{O}_X(e-2)) - 2h^0(X, \mathcal{O}_X(e-3))$$

and

$$h^1(X, \mathcal{I}_{Q'}(e-1)) = h^1(X, \mathcal{I}_{Q'}^2(e)) = 0 \text{ for } e \geq 3,$$
$$h^2(X, \mathcal{I}_{Q'}(e-1)) = h^2(X, \mathcal{I}_{Q'}^2(e)) = 0 \text{ for } e \geq 2.$$

Let us now focus on the degree 6 case. The proof is the same one but with a Koszul complex of the form

$$0 \longrightarrow \mathcal{O}_X(-3) \longrightarrow \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1) \longrightarrow \mathcal{I}_{Q'} \longrightarrow 0.$$

Then we get for any $m \in \mathbb{Z}$

$$h^0(X, \mathcal{I}_{Q'}(m)) = h^0(X, \mathcal{O}_X(m-2)) + h^0(X, \mathcal{O}_X(m-1)) - h^0(X, \mathcal{O}_X(m-3))$$

and therefore

$$\begin{aligned} h^0(X, \mathcal{I}^2_{Q'}(e)) &= h^0(X, \mathcal{I}_{Q'}(e-2)) + h^0(X, \mathcal{I}_{Q'}(e-1)) - h^0(X, \mathcal{O}_X(e-3)) \\ &= h^0(X, \mathcal{O}_X(e-3)) + h^0(X, \mathcal{O}_X(e-2)) - h^0(X, \mathcal{O}_X(e-5)). \end{aligned}$$

□

Proposition 8.1.4. *Consider the blow-up of X in Q' as described in 8.1.2. Consider $(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E))$: if there exists Z a smooth effective 0-cycle of degree 3, such that $(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E), Z)$ satisfies assumptions of Theorem A then $e \geq 5$ if $\deg(Q') = 3$ and respectively $e \geq 7$ if $\deg(Q') = 6$.*

Proof. Assumption (5) reads as

$$\frac{3e^2 - 3e - 2\deg(Q')}{2} + 1 \leq n.$$

Moreover, since the number $\frac{3e^2-3e-2\deg(Q')}{2} + 1$ represents the arithmetic genus of a geometrically connected curve, it must be non negative. Therefore

$$\frac{3e^2 - 3e - 2\deg(Q')}{2} + 1 \geq 0 \implies e(e-1) \geq \frac{2(\deg(Q') - 1)}{3} \implies \begin{cases} e \geq 2 & \text{if } \deg(Q') = 3, \\ e \geq 3 & \text{if } \deg(Q') = 6. \end{cases}$$

In order to have assumption (2) satisfied we need:

$$h^0(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E)) = h^0(X, \mathcal{O}_X(e) \otimes \mathcal{I}^2_{Q'}) \geq n + 7.$$

Thanks to Lemma 8.1.3, we can compute $h^0(X, \mathcal{I}^2_{Q'}(e))$. We have

$$h^0(X, \mathcal{O}_X(e-2)) = \frac{3(e-2)(e-1)}{2} + 1 \text{ if } e \geq 2, \quad h^0(X, \mathcal{O}_X(e-3)) = \begin{cases} 0 & \text{if } e = 2, \\ \frac{3(e-3)(e-2)}{2} + 1 & \text{if } e \geq 3, \end{cases}$$

$$\text{and} \quad h^0(X, \mathcal{O}_X(e-5)) = \begin{cases} 0 & \text{if } e \leq 4, \\ \frac{3(e-5)(e-4)}{2} + 1 & \text{if } e \geq 5. \end{cases}$$

So if $\deg(Q') = 3$, then $e \geq 2$ and we have

$$h^0(X, \mathcal{I}^2_{Q'}(e)) = \begin{cases} 3 & \text{if } e = 2, \\ \frac{3(e-2)(e+3)}{2} + 1 = \frac{3e(e+1)}{2} + 1 - 9 & \text{if } e \geq 3. \end{cases}$$

In particular, in this case in order to have both assumptions (2) and (5) satisfied, we need

$$\frac{3e(e-1)}{2} - 3 + 1 + 7 \leq h^0(X, \mathcal{I}_{Q'}(e))$$

which is then possible if and only if $e \geq 5$.

If now $\deg(Q') = 6$, then $e \geq 3$ and we have

$$h^0(X, \mathcal{I}^2_{Q'}(e)) = \begin{cases} 3(e-2)^2 + 2 & \text{if } e \leq 4, \\ \frac{3}{2}[(e-2)(e-1) + (e-3)(e-2) - (e-5)(e-4)] + 1 = \frac{3e(e+1)}{2} + 1 - 18 & \text{if } e \geq 5. \end{cases}$$

In particular, in this case in order to have both assumptions (2) and (5) satisfied, we need

$$\frac{3e(e-1)}{2} - 6 + 1 + 7 \leq h^0(X, \mathcal{I}_{Q'}(e))$$

which is possible if and only if $e \geq 7$. □

Remark 8.1.5. Note that we have other choices for the 0-cycle Q': for instance we could pick a union of two complete intersection (smooth) effective 0-cycles of degree 3. However, the necessary conditions on e in order to have Theorem A get worse.

Same applies for the case $\mathcal{L} = b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-3E)$ is worse (for instance if $\deg(Q') = 3$ we would get $e \geq 7$).

For what concerns $\mathcal{L} = b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-E)$, the genus $p_a(C)$ does not change but we have less global sections of $\mathcal{L}$. So also this choice does not improve the necessary conditions.

8.1.2 Remark on Ampleness

Lemma 8.1.6. *Consider the blow-up of X in Q' of degree 3 as in 8.1.2. Then*

$b^*\mathcal{O}_X(2) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample and $b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is globally generated.

Proof. Thanks to the Koszul resolution of $\mathcal{I}_{Q'}$ we have

$$\mathcal{O}_X^{\oplus 2} \longrightarrow \mathcal{I}_{Q'}(1) \longrightarrow 1.$$

In particular we get a surjection of algebras

$$\mathcal{O}_X[s_0, s_1] \longrightarrow Sym^*(\mathcal{I}_{Q'}(1)) \longrightarrow 0$$

which in turn induces a closed embedding over X

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "|b^*\mathcal{O}_X(1)\otimes\mathcal{O}_{\widetilde{X}/X}(1)|"] \arrow[d, "b"] & \mathbb{P}^1_X = \mathbb{P}^1_k \times_k X \arrow[r, hook, "id\times|\mathcal{O}_X(1)|"] \arrow[dl] & \mathbb{P}^1_k \times_k \mathbb{P}^3_k \\ X. & & \end{tikzcd}$$

This means that the sheaf $b^*\mathcal{O}_X(1) \otimes b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}/X}(1) \cong b^*\mathcal{O}_X(2) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample.

Moreover, since $b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample over X, the counit map

$$b^*\mathcal{O}_X(1) \otimes b^*\mathcal{I}_{Q'} = b^*b_*\left(b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)\right) \longrightarrow b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E) \tag{8.4}$$

is surjective (see [Sta, Tag. 01VR]). This implies that $b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is globally generated. Indeed, taking the push-forward of the map

$$H^0(\widetilde{X}, b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)) \otimes \mathcal{O}_{\widetilde{X}} \longrightarrow b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E) : \tag{8.5}$$

one gets the surjective map

$$H^0(X, \mathcal{I}_{Q'}(1)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_{Q'}(1).$$

Pulling back along b and post-composing with the counit map (8.4)

$$H^0(\widetilde{X}, b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)) \otimes \mathcal{O}_{\widetilde{X}} \twoheadrightarrow b^*\mathcal{O}_X(1) \otimes b^*\mathcal{I}_{Q'} \twoheadrightarrow b^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E)$$

we still get a surjective map which coincides with (8.5). □

Lemma 8.1.7. *Consider the blow-up of X in Q' of degree 6 as in 8.1.2. Then*

$$b^*\mathcal{O}_X(3) \otimes \mathcal{O}_{\widetilde{X}}(-E) \text{ is very ample and } b^*\mathcal{O}_X(2) \otimes \mathcal{O}_{\widetilde{X}}(-E) \text{ is globally generated.}$$

Proof. Same proof as in Lemma 8.1.6. □

Remark 8.1.8. Regardless of the degree, the counit map $b^*b_*b^*\mathcal{O}_X(1) \longrightarrow b^*\mathcal{O}_X(1)$ is the identity (by projection formula and the fact that $b_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$). Hence we have that

$$H^0(\widetilde{X}, b^*\mathcal{O}_X(1)) \otimes \mathcal{O}_{\widetilde{X}} = H^0(X, \mathcal{O}_X(1)) \otimes b^*\mathcal{O}_X \longrightarrow b^*\mathcal{O}_X(1)$$

is surjective: namely $b^*\mathcal{O}_X(1)$ is globally generated.

Lemma 8.1.9. *Consider the blow-up of X in Q' as in 8.1.2. Then*

$$b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E) \text{ is very ample for } \begin{cases} e \geq 3 & \text{if } \deg(Q') = 3 \\ e \geq 5 & \text{if } \deg(Q') = 6. \end{cases}$$

Proof. Combining Remark 8.1.8 and Lemmas 8.1.6 and 8.1.7, we have the statement. □

8.1.3 Sufficient conditions on e

Proposition 8.1.10. *If $e \geq 3$, the triple $(X, \mathcal{O}_X(e), Q)$ satisfies assumptions (1)-(5) of Theorem A with respect to any integer n in*

$$I_{e,\varnothing} = \left[\frac{3e(e-1)}{2} + 1, \frac{3e(e+1)}{2} - 6\right].$$

Therefore, for any $n \in I_{e,\varnothing}$ (which implies $n \geq n_{0,\varnothing} = 10$) the Hilbert scheme Hilb^n_X is stably-birational to Hilb^{n+3}_X.

Proof. **Assumption (1).** We need to check that the restriction map

$$H^0(X, \mathcal{O}_X(e)) \longrightarrow H^0(Q, \mathcal{O}_X(e)|_Q)$$

is surjective. Since $H^1(X, \mathcal{O}_X(e))$ is trivial, surjectivity is equivalent to the triviality of $H^1(X, \mathcal{I}_Q(e))$. By the Koszul resolution we have

$$0 \longrightarrow H^1(X, \mathcal{I}_Q(e)) \longrightarrow H^2(X, \mathcal{O}_X(e-2))$$

which is trivial for any $e \geq 2$.

Assumption (2). It is automatically true by the inequality

$$n \leq \frac{3e(e+1)}{2} - 6 = h^0(X, \mathcal{O}_X(e)) - 1 - 2\deg(Q).$$

Assumption (3). Again by the Koszul resolution we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \longrightarrow \mathcal{I}_Q(1) \longrightarrow 0$$

and consequently

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \oplus H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{I}_Q(1)) \longrightarrow H^1(X, \mathcal{O}_X(-1)) = 0.$$

This means that

$$H^0(X, \mathcal{O}_X) \oplus H^0(X, \mathcal{O}_X) \cong H^0(X, \mathcal{I}_Q(1)).$$

Hence, the global sections that generate $\mathcal{O}_X \oplus \mathcal{O}_X$ come from $H^0(X, \mathcal{I}_Q(1))$, which means that $\mathcal{I}_Q(1)$ is globally generated. Therefore $\mathcal{I}_Q(e)$ is globally generated for any $e \geq 1$.

Assumption (4). If $e \geq 2$ then [Sta, Tag 0FD5] implies that the sections in $H^0(X, \mathcal{I}_Q(e))$ define a locally closed immersion

$$X \setminus Q \longrightarrow \mathbb{P}(H^0(X, \mathcal{I}_Q(e))) \tag{8.6}$$

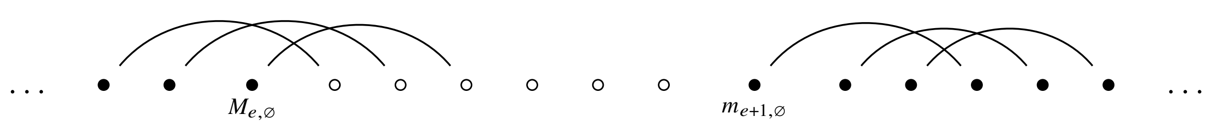
(enough to set $e_1 = e_2 = 1$ in the notation of [Sta, Tag 0FD5]). This means that assumption (4) is satisfied for any $e \geq 2$.

Assumption (5). It is automatically satisfied since

$$e^2 K_X^2 - e K_X^2 = 3e^2 - 3e \leq 2n - 2.$$

□

Denote by $M_{e,\emptyset}$ the number $\frac{3e(e+1)}{2} - 6$ and by $m_{e,\emptyset}$ the number $\frac{3e(e-1)}{2} + 1$. Note that $m_{e+1,\emptyset}$ is equal to $M_{e,\emptyset} + 7$. The intervals $I_{e,\emptyset}$'s leave then "gaps" of length 6, which look like this:



As done for the general case, let us cover those gaps using blow-ups of X.

Proposition 8.1.11 (Degree 3 case). *If $e \geq 5$, the triple $(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E), b^{-1}Q)$ satisfies* *assumptions (1)-(5) of Theorem A for any integer n in*

$$I_{e,3} = \left[\frac{3e(e-1)}{2} - 2, \frac{3e(e+1)}{2} - 15\right].$$

Therefore, for any $n \in I_{e,3}$ (which implies $n \geq n_{0,3} = 28$) the Hilbert scheme Hilb^n_X is stably-birational to Hilb^{n+3}_X.

Proposition 8.1.12 (Degree 6 case)**.** *If $e \geq 7$, the triple $(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E), b^{-1}Q)$ satisfies assumptions (1)-(5) of Theorem A for any integer n in*

$$I_{e,6} = \left[\frac{3e(e-1)}{2} - 5, \frac{3e(e+1)}{2} - 24\right].$$

Therefore, for any $n \in I_{e,6}$ (which implies $n \geq n_{0,6} = 58$) the Hilbert scheme Hilb^n_X is stably-birational to Hilb^{n+3}_X.

The structure of the proof is the same for both. So we do them together.

Proof. **Assumptions (2) and (5).** They are automatically satisfied thanks to n belonging to $I_{e,\deg(Q')}$.

Assumption (1). Set $\widetilde{Q} = b^{-1}Q$. We need to check that the restriction map

$$\begin{array}{ccc} H^0(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E)) & \longrightarrow & H^0(\widetilde{Q}, (b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E))|_{\widetilde{Q}}) \\ \| \wr\cong & & \| \\ H^0(X, \mathcal{O}_X(e) \otimes \mathcal{I}^2_{Q'}) & & H^0(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}) \end{array}$$

is surjective. Note that $H^0(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}) = H^0(Q, \mathcal{O}_Q)$ because Q is disjoint from the center of the blow-up Q' and therefore the structure sheaf $\mathcal{O}_Q$ coincides with $b_*\mathcal{O}_{\widetilde{Q}}$.

Denote by $Q'(2)$ the subscheme defined by $\mathcal{I}^2_{Q'}$. Consider the short exact sequence defining Q in X and tensor it by $\mathcal{O}_X(e) \otimes \mathcal{I}_{Q'(2)}$:

$$\mathcal{O}_X(e) \otimes \mathcal{I}_{Q'(2)} \otimes \mathcal{I}_Q \longrightarrow \mathcal{O}_X(e) \otimes \mathcal{I}_{Q'(2)} \longrightarrow (\mathcal{O}_X(e) \otimes \mathcal{I}_{Q'(2)})|_Q \longrightarrow 0.$$

Again, since Q and Q' are disjoint,

$$(\mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)})|_Q \cong i_{Q,*}\mathcal{O}_Q \text{ and } \mathrm{Im}(\mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)}\otimes\mathcal{I}_Q \longrightarrow \mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)}) = \mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)\sqcup Q}.$$

Therefore we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)\sqcup Q} \longrightarrow \mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)} \longrightarrow i_{Q,*}\mathcal{O}_Q \longrightarrow 0$$

and the original restriction map coincides with the restriction map

$$H^0(X,\mathcal{O}_X(e)\otimes\mathcal{I}_{Q'(2)}) \longrightarrow H^0(Q,\mathcal{O}_Q).$$

Therefore, in order to ensure surjectivity, it is enough to check that $H^1(X,\mathcal{I}_{Q'(2)\sqcup Q}(e))$ vanishes.

From tensoring the Koszul resolution of $\mathcal{I}_Q$ by $\mathcal{I}_{Q'(2)}(e)$ we get the sequence

$$\mathcal{I}_{Q'(2)}(e-2) \xrightarrow{(h_1,-h_2)} \mathcal{I}_{Q'(2)}(e-1)^{\oplus 2} \longrightarrow \mathcal{I}_{Q'(2)\sqcup Q}(e) \longrightarrow 0. \tag{8.7}$$

Outside Q', the sequence coincides with the original one twisted by $\mathcal{O}_X(e)$, so it is exact. To check exactness at the points of $Q'(2)$, since the hyperplanes $H_1 = V(h_1), H_2 = V(h_2)$ involved in the definition of Q are different from the ones defining Q', we can assume h_1, h_2 both invertible. On this open (containing $Q'(2)$) the map $(h_1,-h_2)$ is injective. Therefore 8.7 is exact on the left as well and the cohomology group $H^1(X,\mathcal{I}_{Q'(2)\sqcup Q}(e-1))$ sits between

$$H^1(X,\mathcal{I}^2_{Q'}(e-1))^{\oplus 2} \longrightarrow H^1(X,\mathcal{I}_{Q'(2)\sqcup Q}(e-1)) \longrightarrow H^2(X,\mathcal{I}^2_{Q'}(e-2)).$$

By Lemma 8.1.3 the groups on the left and on the right are trivial if $e \geq 4$ and $\deg(Q') = 3$, and respectively if $e \geq 6$ and $\deg(Q') = 6$.

Assumption (3). In order to check that $b^*\mathcal{O}_X(e)\otimes\mathcal{O}_{\widetilde{X}}(-2E)\otimes\mathcal{I}_{\widetilde{Q}}$ is globally generated we write the sheaf as

$$b^*\mathcal{O}_X(2) \otimes \mathcal{O}_{\widetilde{X}}(-2E) \otimes b^*\mathcal{O}_X(e-2) \otimes \mathcal{I}_{\widetilde{Q}}$$

if the degree of Q' is 3 and in the other case

$$b^*\mathcal{O}_X(4) \otimes \mathcal{O}_{\widetilde{X}}(-2E) \otimes b^*\mathcal{O}_X(e-4) \otimes \mathcal{I}_{\widetilde{Q}}.$$

Since Q is disjoint from Q' then the ideal sheaf $\mathcal{I}_{\widetilde{Q}}$ coincides with the pullback $b^*\mathcal{I}_Q$. Pull-backing the Koszul resolution of $\mathcal{I}_Q(e-2)$ we get

$$\begin{tikzcd}
H^0(\widetilde{X}, b^*\mathcal{O}_X(e-3))^{\oplus 2} \otimes \mathcal{O}_{\widetilde{X}} \arrow[r] \arrow[d] & H^0(\widetilde{X}, b^*\mathcal{O}_X(e-2) \otimes \mathcal{I}_{\widetilde{Q}}) \otimes \mathcal{O}_{\widetilde{X}} \arrow[d] & \\
b^*\mathcal{O}_X(e-3)^{\oplus 2} \arrow[r] & b^*\mathcal{O}_X(e-2) \otimes \mathcal{I}_{\widetilde{Q}} \arrow[r] & 0.
\end{tikzcd}$$

If $e \geq 3$, then the left vertical arrow is surjective, implying that $b^*\mathcal{O}_X(e-2) \otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated. Similarly, if $e \geq 5$, $b^*\mathcal{O}_X(e-4) \otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated.

By Lemmas 8.1.6 and 8.1.7, the sheaves $b^*\mathcal{O}_X(2)\otimes\mathcal{O}_{\widetilde{X}}(-2E)$ (for $\deg(Q') = 3$) and $b^*\mathcal{O}_X(4)\otimes \mathcal{O}_{\widetilde{X}}(-2E))$ (for $\deg(Q') = 6$) are globally generated. Therefore we have that $b^*\mathcal{O}_X(e)\otimes\mathcal{O}_{\widetilde{X}}(-2E)\otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated for $e \geq 3$ and $\deg(Q') = 3$ and respectively $e \geq 5$ and $\deg(Q') = 6$.

Assumption (4). By the proof of [Sta, Tag. 0FD5], it is enough to pick $e \geq e_1 + e_2$ such that

1. $b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E) \otimes \mathcal{I}_{\widetilde{Q}}$ is globally generated (enough $e \geq 3$ for degree 3 case and $e \geq 5$ for degree 6 case),

2. $b^*\mathcal{O}_X(e_1)\otimes\mathcal{O}_{\widetilde{X}}(-E)\otimes\mathcal{I}_{\widetilde{Q}}$ is globally generated (enough $e_1 \geq 2$ for degree 3 case and $e_1 \geq 3$ for degree 6 case), and

3. $b^*\mathcal{O}_X(e_2) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample (enough $e_2 \geq 2$ for degree 3 case and $e_2 \geq 3$ for degree 6 case).

Therefore any $e \geq 4$ (for $\deg(Q') = 3$) and $e \geq 6$ (for $\deg(Q') = 6$) work fine. $\square$

Corollary 8.1.13. *Let X be a smooth cubic surface over an infinite field k. Then* Hilb^n_X *is stably birational to* Hilb^{n+3}_X *for all*

$$n \in \mathbb{N}_{\geq 10} \setminus \{13, 14, 15, 16, 17, 18, 25, 26, 27, 40, 41, 42\}.$$

Proof. This is obtained combining the intervals $I_{e,\emptyset}, I_{e,3}$ and $I_{e,6}$ for $e \geq 3$, $e \geq 5$ and $e \geq 7$ respectively. □

8.2 Using the incidence correspondence $\mathrm{Inc}^{\mathcal{M},+}_{X,\mathcal{L},n,n'}$

Let $e \geq 1$, so that $\mathcal{L} = \mathcal{O}_X(e)$ is both ample and globally generated. Let us set $\mathcal{M} = \mathcal{O}_X(m)$. Note that in order to have $n + n' = c_1(\mathcal{L})c_1(\mathcal{M}) = 3em$, the integer m must be positive as well. Then we have the following.

Proposition 8.2.1. *Let X be a smooth cubic surface over an infinite field k. Let e, m be positive integers. Set $\mathcal{L} = \mathcal{O}_X(e)$ and $\mathcal{M} = \mathcal{O}_X(m)$. Then the suitable pair $(X, \mathcal{L})$ satisfies the 5 conditions in Theorem E with respect to $(n, \mathcal{O}_X(m))$ if and only if*

$$m = e, \quad \text{and} \quad n \in I^{e,+}_e = \left[\frac{3e(e-1)}{2} + 1, \frac{3e(e+1)}{2} - 1\right].$$

For these values of n we have then

$$\mathrm{Hilb}^n_X \sim_{\text{stably birational}} \mathrm{Hilb}^{3e^2-n}_X.$$

Proof. Let us check each of the 5 assumptions, one by one.

Assumption (1). The first assumption is

$$\frac{3e(e+1)}{2} \geq \max\{n, 3em - n\}.$$

Assumption (2) & (3). The second and the third one are satisfied for any $e \geq 1$ (this is in accordance with classical Bertini's Theorem over infinite fields).

Assumption (4). The fourth assumption is

$$\frac{3e(e-1)}{2} + 1 \leq \min\{n, 3em - n\}.$$

Assumption (5). Finally, the fifth one is

$$\min\{n, 3em - n\} - \left(\frac{3e(e-1)}{2} + 1\right) + 3e(e-m) \geq 0,$$

which is equivalent to

$$n \geq 3e\left(m - \frac{e+1}{2}\right) + 1 \text{ and } n \leq \frac{3e(e+1)}{2} - 1.$$

In particular we get the following

$$n \in I_e^{m,+} = \left[\max\left\{3e\left(m - \frac{e+1}{2}\right) + 1, \frac{3e(e-1)}{2} + 1\right\}, \min\left\{\frac{3e(e+1)}{2} - 1, 3e\left(m - \frac{e-1}{2}\right) - 1\right\}\right].$$

In order to have a non-empty $I_e^{m,+}$ we need $m = e$ (otherwise the left extreme of $I_e^{m,+}$ is greater than the right one). Plugging-in $m = e$, we get

$$n \in I_e^{e,+} = \left[\frac{3e(e-1)}{2} + 1, \frac{3e(e+1)}{2} - 1\right].$$

□

Remark 8.2.2. Since we are interested in "small n" (in particular in $n \leq 42$), let us start plug-in the first values of e:

$e = 1$: then $1 \leq n \leq 2$ and hence we have the pair $(1, 2)$;

$e = 2$: then $4 \leq n \leq 8$ and hence we have the pairs $(4, 8)$, $(5, 7)$;

$e = 3$: then $10 \leq n \leq 17$ and hence we have the pairs $(10, 17), (11, 16), (12, 15), (13, 14)$;

$e = 4$: then $19 \leq n \leq 29$ and hence we have the pairs $(19, 29), (20, 28), (21, 27), (22, 26), (23, 25)$;

$e = 5$: then $31 \leq n \leq 44$ and hence we have the pairs $(31, 44), (32, 43), (33, 42), (34, 41), (35, 40), (36, 39), (37, 38)$;

$e = 6$: then $n \geq 45$, therefore it is not anymore of our interest.

In particular, combining the above pairs with Corollary 8.1.13, we get

$$\begin{aligned}
43 \sim 32 \sim 35 \sim 40 &\implies \mathrm{Hilb}^{40}_X \sim_{\text{stably birational}} \mathrm{Hilb}^{43}_X,\\
41 \sim 34 \sim 31 \sim 44 &\implies \mathrm{Hilb}^{41}_X \sim_{\text{stably birational}} \mathrm{Hilb}^{44}_X,\\
25 \sim 23 \sim 20 \sim 28 &\implies \mathrm{Hilb}^{25}_X \sim_{\text{stably birational}} \mathrm{Hilb}^{28}_X,\\
26 \sim 22 \sim 19 \sim 29 &\implies \mathrm{Hilb}^{26}_X \sim_{\text{stably birational}} \mathrm{Hilb}^{29}_X,\\
17 \sim 10 \sim 13 \sim 14 &\implies \mathrm{Hilb}^{14}_X \sim_{\text{stably birational}} \mathrm{Hilb}^{17}_X,\\
16 \sim 11 \sim 14 \sim 13 &\implies \mathrm{Hilb}^{13}_X \sim_{\text{stably birational}} \mathrm{Hilb}^{16}_X,
\end{aligned}$$

In particular the integers $n \geq 10$ which are not yet connected to $n + 3$ are $15, 16, 17, 18, 27$ and 42.

Theorem 8.2.3. *Let k be an infinite field. Let $\widetilde{X}$ be the blow-up as in 8.1.2 with $\deg(Q') = 3$. Let e, m, b be integers. Set $\widetilde{\mathcal{L}} = b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E)$ and $\widetilde{\mathcal{M}} = b^*\mathcal{O}_X(m) \otimes \widetilde{\mathcal{O}}_X(bE)$. If $e \geq 3$ (see Lemma 8.1.6) the pair $(\widetilde{X}, \widetilde{\mathcal{L}})$ is suitable. It satisfies the five conditions in Theorem E with respect to $(n, b^*\mathcal{O}_X(m) \otimes \mathcal{O}_{\widetilde{X}}(bE))$ if and only if n belongs to the interval*

$$\left[\max\left\{\frac{3e(e-1)}{2} - 2, 3em + 6b - \frac{3e(e+1)}{2} + 10\right\}, \min\left\{\frac{3e(e+1)}{2} - 10, 3em + 6b - \frac{3e(e-1)}{2} + 2\right\}\right].$$

This interval is non-empty if and only if $3e(e-1) - 4 \leq 3em + 6b \leq 3e(e+1) - 20$. For these values of n we have then

$$\mathrm{Hilb}^n_{\widetilde{X}} \sim_{\text{stably birational}} \mathrm{Hilb}^{3em+6b-n}_{\widetilde{X}}.$$

Proof. First of all, in this case

$$c_1(\widetilde{\mathcal{L}}) \cdot c_1(\widetilde{\mathcal{M}}) = (e\, b^* c_1(\mathcal{O}_X(1)) - 2E)\, (m\, b^* c_1(\mathcal{O}_X(1)) + bE) = 3em + 6b = n + n'.$$

Assumption (1). By Lemma 8.1.3, since $e \geq 3$,

$$h^0(\widetilde{X}, b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E)) = \frac{3e(e+1)}{2} + 1 - 9 \geq \max\{n, 3em + 6b - n\} + 1.$$

Assumption (2). By Lemma 8.1.9, $b^*\mathcal{O}_X(e) \otimes \mathcal{O}_{\widetilde{X}}(-2E)$ is very ample for any $e \geq 3$, so it is globally generated.

Assumption (3). As done for 8.1.3, by the proof of [Sta, Tag. 0FD5] it is enough that

- $\widetilde{\mathcal{L}}$ is globally generated for $e \geq 2$,
- $b^*\mathcal{O}_X(e_1) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is globally generated for any $e_1 \geq 1$, and
- $b^*\mathcal{O}_X(e_2) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample for any $e_2 \geq 2$.

Then for any $e \geq 3 = 1 + 2$ we get the surjectivity statement.

Assumption (4). This one reads as

$$p_a(C) = \frac{3e(e-1)}{2} - 3 + 1 \leq \min\{n, 3em + 6b - n\}.$$

Assumption (5). Let us develop the expression

$$\min\{n, 3em + 6b - n\} - \left[\frac{3e(e-1)}{2} - 2\right] + c_1(\widetilde{\mathcal{L}}) \left[c_1(\widetilde{\mathcal{L}}) - c_1(\widetilde{\mathcal{M}})\right] \geq 0. \tag{8.8}$$

The term $c_1(\widetilde{\mathcal{L}}) - c_1(\widetilde{\mathcal{M}})$ is equal to $(e - m)b^*c_1(\mathcal{O}_X(1)) - (b + 2)E$, making

$$c_1(\widetilde{\mathcal{L}}) \left[c_1(\widetilde{\mathcal{L}}) - c_1(\widetilde{\mathcal{M}})\right] = 3e(e - m) - 6(b + 2).$$

Therefore, the expression (8.8) is equal to

$$\min\{n, 3em + 6b - n\} + 3e\left(\frac{e+1}{2} - m\right) - 6b - 10 \geq 0.$$

Therefore, for $e \geq 3$, the five assumptions are satisfied with respect to $(n, \widetilde{\mathcal{M}})$ if and only if n belongs to

$$\left[\max\left\{\frac{3e(e-1)}{2} - 2, 3em + 6b - \frac{3e(e+1)}{2} + 10\right\}, \min\left\{\frac{3e(e+1)}{2} - 10, 3em + 6b - \frac{3e(e-1)}{2} + 2\right\}\right].$$

Furthermore, this interval is non empty if and only if

$$3e(e-1) - 4 \leq 3em + 6b \leq 3e(e+1) - 20, \text{ and } \frac{3e(e-1)}{2} - 2 \leq \frac{3e(e+1)}{2} - 10,$$

where the second inequality is always true for $e \geq 3$. □

Corollary 8.2.4. *Let X be a smooth cubic surface over an infinite field k. Let $e \geq 3$. Then for any $n \in \left[\frac{3e(e-1)}{2} - 2, \frac{3e(e+1)}{2} - 10\right]$ we have*

$$\mathrm{Hilb}^n_X \sim_{\text{stably birational}} \mathrm{Hilb}^{3e^2-12-n}_X.$$

Proof. Consider the same setting as in Theorem 8.2.3. Set $e = m$ and $b = -2$. Then the inequalities $3e(e-1) - 4 \leq 3e^2 - 12 \leq 3e(e+1) - 20$ are always true for $e \geq 3$ and

$$3e^2 - 12 - \frac{3e(e+1)}{2} + 10 = \frac{3e(e-1)}{2} - 2, \text{ and } 3e^2 - 12 - \frac{3e(e-1)}{2} + 2 = \frac{3e(e+1)}{2} - 10.$$

Therefore by Theorem 8.2.3 we have the statement. □

Remark 8.2.5. As we did before, we plug-in the first values of e to see if we can cover new pairs:

$e = 3$: then $7 \leq n \leq 8$ and we get the pair $(7, 8)$.

$e = 4$: then $16 \leq n \leq 20$ and we get the pairs $(16, 20), (17, 19)$. In particular, combining with the pairs found in Remark 8.2.2

$$16 \sim 20 \sim 28 \sim 25 \sim 19 \implies \mathrm{Hilb}_X^{16} \sim_{\text{stably birational}} \mathrm{Hilb}_X^{19}$$
$$17 \sim 19 \sim 29 \sim 26 \sim 20 \implies \mathrm{Hilb}_X^{17} \sim_{\text{stably birational}} \mathrm{Hilb}_X^{20};$$

$e = 5$: then $28 \leq n \leq 35$ which is an interval where n is already connected to $n + 3$;

$e \geq 6$: then $n \geq 43$, so it is not of our interest.

Remark 8.2.6. Similar computations can be done for the blow-up in degree 6 smooth effective 0-cycle Q': however this does not lead to cover new pairs $(n, n + 3)$.

Therefore we have the following corollary.

Corollary 8.2.7. *Let X be a smooth cubic surface over an infinite field k. Then*

$$\mathrm{Hilb}_X^n \sim_{\text{stably birational}} \mathrm{Hilb}_X^{n+3} \quad \forall n \in \mathbb{N}_{\geq 10} \setminus \{15, 18, 27, 42\}.$$

Moreover we have

$$X \sim_{\text{stably birational}} \mathrm{Hilb}_X^2, \quad \mathrm{Hilb}_X^4 \sim_{\text{stably birational}} \mathrm{Hilb}_X^5 \sim_{\text{stably birational}} \mathrm{Hilb}_X^7 \sim_{\text{stably birational}} \mathrm{Hilb}_X^8.$$

Proof. This is obtained combining the results of Corollary 8.1.13 and Remarks 8.2.2 and 8.2.5. $\square$

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