A Kudla–Rapoport Formula for Exotic Smooth Models of Odd Dimension

Haodong Yao

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Abstract

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In this thesis, we prove a Kudla–Rapoport conjecture for $\mathcal{Y}$-cycles on exotic smooth unitary Rapoport–Zink spaces of odd arithmetic dimension, i.e. the arithmetic intersection numbers for $\mathcal{Y}$-cycles equals the derivatives of local representation density. We also compare $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles on these RZ spaces. The method is to relate both geometric and analytic sides to the even dimensional case and reduce the conjecture to the results in [LL22].

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Chapter 1: Introduction

1.1 Background

The classical *Siegel–Weil formula* ([Sie35, Sie51, Wei65]) relates special *values* of certain Eisenstein series with theta functions, which are generating series of representation numbers of quadratic forms. Later on, Kudla ([Kud97, Kud04]) proposed an influential program and introduced analogues of theta series in arithmetic geometry. One of the goals of the program is to prove the so-called *arithmetic Siegel–Weil formula* relating the *central derivative* of certain Eisenstein series with a certain arithmetic analogue of theta functions, which are generating series of arithmetic intersection numbers of n special divisors on Shimura varieties associated to $\mathrm{SO}(n-1,2)$ or $\mathrm{U}(n-1,1)$.

For $\mathrm{U}(n-1,1)$-Shimura varieties, Kudla and Rapoport ([KR11]) formulated a conjectural *local arithmetic Siegel–Weil formula* at an *unramified* place with hyperspecial level, now known as the *Kudla–Rapoport conjecture*. As a local analogue of the arithmetic Siegel–Weil formula, it relates the *central derivative* of local densities of hermitian forms with the arithmetic intersection number of special cycles on unitary Rapoport–Zink spaces. This conjectural identity was recently proved by Li and Zhang in [LZ22]. We refer the readers to the introduction of [LZ22] for more backgrounds and related results.

One of the distinguished features of the hyperspecial case [KR11] is that the corresponding Rapoport–Zink space has good reduction. Accordingly, the analytic side has a clear formulation. A natural and important question is to formulate and prove analogues of the Kudla–Rapoport conjecture when the level structure is nontrivial, where many unexpected new phenomena occur.

At a ramified place, there are two well-studied unitary Rapoport–Zink spaces with different level structures. One of them is the *exotic smooth model* which has good reduction, and the other one is the *Krämer model* which has bad (semistable) reduction. The analogue of Kudla–Rapoport conjecture for the even dimensional exotic smooth model was formulated and proved by Li and Liu in [LL22] using a strategy similar to [LZ22]. For the Krämer model, the analytic side is more involved. In fact, even the formulation of the conjecture is not clear and needs to be modified. This phenomenon in the presence of bad reduction was first discovered by Kudla and Rapoport in [KR00] via explicit computation in their study of the Drinfeld p-adic half plane. The Kudla–Rapoport conjecture for Krämer models, in general, was formulated in [HSY23] with conceptual formulation for the modification and proved in [HLSY23].

The present paper focuses on Kudla–Rapoport conjectures for the odd dimensional exotic smooth model. We propose and prove a Kudla–Rapoport conjecture for $\mathcal{Y}$-cycles. The main results we obtained can be used to relax the local assumption at ramified places in the arithmetic Siegel–Weil formula in [LZ22]. It should also be useful in extending the co-rank 1 arithmetic Siegel–Weil formula established in [Che24] to odd dimensional exotic smooth models. Lastly, it may be applied to the mixed arithmetic theta lifting and mixed arithmetic inner product formula proposed in [Liu21].

1.2 Main results

Let p be an odd prime. Let F_0 be a finite extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$. Let $\bar{k}$ be a fixed algebraic closure of k. Let F be a ramified quadratic extension of F_0. Denote by $a \mapsto \bar{a}$ the (nontrivial) Galois involution of F/F_0. Let π be a uniformizer of F such that $\bar{\pi} = -\pi$. Let $\pi_0 = \pi^2$, a uniformizer of F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $O_F, O_{\breve{F}}$ be the ring of integers of $F, \breve{F}$ respectively.

Let $n \geqslant 1$ be an integer. To define the exotic smooth unitary Rapoport–Zink space, we fix a hermitian formal O_F-module $\mathbb{X} = \mathbb{X}_n$ of signature $(1, n-1)$ over $\bar{k}$. The Rapoport–Zink space $\mathcal{N} = \mathcal{N}_n$ is the formal scheme over $\operatorname{Spf} O_{\breve{F}}$ parameterizing hermitian formal O_F-modules X of signature $(1, n-1)$ (see Definition 2.1) within the quasi-isogeny class of $\mathbb{X}$. The space $\mathcal{N}$ is formally locally of finite type, formally smooth of relative dimension $n-1$ over $\operatorname{Spf} O_{\breve{F}}$.

Let $\overline{\mathbb{E}}$ be the framing hermitian formal O_F-module of signature $(0,1)$ over $\bar{k}$. We define the *space of quasi-homomorphisms* to be $\mathbb{V} = \mathbb{V}_n := \mathrm{Hom}_{O_F}(\overline{\mathbb{E}}, \mathbb{X}) \otimes_{O_F} F$. We can associate $\mathbb{V}$ with a natural F/F_0-hermitian form h to make $(\mathbb{V}, h)$ a nonsplit nondegenerate F/F_0-hermitian space of dimension n. For any nonzero $x \in \mathbb{V}$, we define the *special cycle* $\mathcal{Z}(x) = \mathcal{Z}_n(x)$ (resp. $\mathcal{Y}(x) = \mathcal{Y}_n(x)$) (see Definition 2.9) to be the deformation locus of x (resp. $\lambda \circ x$) in $\mathcal{N}$.

Given an O_F-lattice $L \subset \mathbb{V}$ of full rank n, we can define integers: the *arithmetic intersection number* $\mathrm{Int}_{n,\mathcal{Z}}(L)$, $\mathrm{Int}_{n,\mathcal{Y}}(L)$ and the *derived local density* $\partial\mathrm{Den}(L)$.

Definition 1.1. Let $L \subset \mathbb{V}$ be an O_F-lattice and $x_1, \ldots, x_n$ be an O_F-basis of L. When n is even, we define the *arithmetic intersection number* for $\mathcal{Z}$-cycles

$$\mathrm{Int}_{n,\mathcal{Z}}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) \in \mathbb{Z}, \tag{1.1}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the special divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and χ denotes the Euler–Poincaré characteristic. By [LL22, Corollary 2.35], $\mathrm{Int}_{n,\mathcal{Z}}(L)$ is independent of the choice of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of L itself.

For general n we define similarly the *arithmetic intersection number* for $\mathcal{Y}$-cycles

$$\mathrm{Int}_{n,\mathcal{Y}}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Y}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Y}(x_n)}) \in \mathbb{Z}. \tag{1.2}$$

By Proposition 3.11 and Proposition 5.18, $\mathrm{Int}_{n,\mathcal{Y}}(L)$ is independent of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of L.

To define the *derived local density* $\partial\mathrm{Den}(L)$, we need to introduce local densities first. Let L be a hermitian O_F-lattice of rank n. Let M be another hermitian O_F-lattice (of arbitrary rank) and $\mathrm{Herm}_{L,M}$ denote the O_{F_0}-scheme of hermitian O_F-module homomorphisms from L to M. Then we define the corresponding *local density* to be

$$\mathrm{Den}(M,L) \coloneqq \lim_{d\to+\infty} \frac{|\mathrm{Herm}_{L,M}(O_{F_0}/\pi_0^d)|}{q^{d\cdot d_{L,M}}},$$

where $d_{L,M}$ is the dimension of $\mathrm{Herm}_{L,M}\otimes_{O_{F_0}} F_0$. Let H be the self-dual hermitian O_F-lattice of rank 2 (see Definition 3.3). It is well-known that there exists a *local density polynomial* $\mathrm{Den}(M,L,X)\in\mathbb{Q}[X]$ such that for any integer $k\geqslant 0$,

$$\mathrm{Den}(M,L,q^{-2k}) = \mathrm{Den}(H^k \obot M, L). \tag{1.3}$$

Here H^k denotes the orthogonal direct sum of k copies of H and $H^k\obot M$ denotes the orthogonal direct sum of H^k and M.

When M also has rank n and $M\otimes_{O_F} F$ is not isometric to $L\otimes_{O_F} F$, we have $\mathrm{Den}(M,L)=0$. In this case we write

$$\mathrm{Den}'(M,L) \coloneqq -2\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1}\mathrm{Den}(M,L,X),$$

and define the (normalized) *derived local density*

$$\partial\mathrm{Den}(L) \coloneqq \frac{\mathrm{Den}'(M_n,L)}{\mathrm{Den}(M_n,M_n)}\in\mathbb{Q}. \tag{1.4}$$

Here $M_n = H^{n/2}$ if n is even, and $M_n = H^{(n-1)/2}\obot I_1^1$ if n is odd where I_1^1 denotes a rank 1 hermitian lattice with moment matrix 1.

Finally we propose and prove the following Kudla–Rapoport conjecture for $\mathcal{Y}$-cycles. For any O_F-lattice $L\subset\mathbb{V}$, we view it as a hermitian O_F-lattice under the hermitian norm h unless otherwise specified.

Theorem 1.2 (Main Theorem, Theorem 3.8)**.** *For any O_F-lattice $L\subset\mathbb{V}$, we have*

$$\mathrm{Int}_{n,\mathcal{Y}}(L) = \partial\mathrm{Den}(L).$$

1.3 Strategy of proof

We first prove Theorem 1.2 for n even by reformulating [LL22, Theorem 2.11] in terms of $\mathcal{Y}$-cycles.

Theorem 1.3 ([LL22, Theorem 2.11]). *Let $n \geqslant 2$ be even. Normalize the hermitian form on $\mathbb{V}_n$ by $h' = \pi^{-2}h$. For any O_F-lattice $L \subset \mathbb{V}_n$ viewed as a hermitian O_F-lattice under the hermitian form h', we have*

$$\mathrm{Int}_{n,\mathcal{Z}}(L) = \partial\mathrm{Den}(L).$$

Theorem 1.2 for n even then follows from the fact that there is an isomorphism of hermitian O_F-lattices $(L, h) \simeq (\pi L, -h')$, and the following relation between $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles for n even.

Proposition 1.4 (Proposition 3.11). *Let $n \geqslant 2$ be even. For any nonzero $x \in \mathbb{V}_n$ we have natural identification $\mathcal{Y}(x) \simeq \mathcal{Z}(\pi x)$. As a corollary we get $\mathrm{Int}_{n,\mathcal{Y}}(L) = \mathrm{Int}_{n,\mathcal{Z}}(\pi L)$.*

Theorem 1.2 for $n = 1$ follows from direct computations as we shall explain in §6.

For the rest of the section we assume $n \geqslant 3$ is odd. We choose an O_F-isogeny ϕ_0 of degree $q \colon \mathbb{X}_{n+1} \to \mathbb{X}_n \times \overline{\mathbb{E}}$ as in [RSZ18, §9] (see (5.4)). This isogeny induces a natural orthogonal decomposition of hermitian spaces $\mathbb{V}_{n+1} = \mathbb{V}_n \obot \langle f \rangle_F$, where f has hermitian norm -1. For any O_F-lattice $L \subset \mathbb{V}_n$, view it as a hermitian lattice (of rank n) in $\mathbb{V}_{n+1}$ via the orthogonal decomposition, and set $L^\# = L \obot \langle f \rangle_{O_F}$. We show separately that

Proposition 1.5 (Analytic reduction, Proposition 4.5).

$$\partial\mathrm{Den}(L) = \frac{1}{2}\partial\mathrm{Den}(L^\#).$$

Proposition 1.6 (Geometric reduction, Proposition 5.18).

$$\mathrm{Int}_{n,\mathcal{Y}}(L) = \frac{1}{2}\mathrm{Int}_{n+1,\mathcal{Y}}(L^\#).$$

Theorem 1.2 for L then follows from the above reduction Propositions and Theorem 1.2 for $L^{\#}$.

For the analytic reduction, this is done by looking at the possible image of f in H^s and describing the orthogonal complement of the image in details (Proposition 4.2).

For the geometric reduction, we first follow the computations in [PR09, §5c] for local models to get a detailed description of the tangent space of $\mathcal{N}_m$ and $\mathcal{Z}$-cycles on $\mathcal{N}_m$ at geometric points for m even (Proposition 5.11). We then make use of the closed embedding $\delta^{\pm}\colon \mathcal{N}_n \to \mathcal{N}_{n+1}$ constructed in [RSZ18, Proposition 12.1]. Consider $u = \pi f \in \mathbb{V}_{n+1}$, we prove that

Theorem 1.7 (Theorem 5.5)**.** *The closed embedding $\delta^{+}\colon \mathcal{N}_n \to \mathcal{N}_{n+1}^{+}$ induces an isomorphism (still denoted by δ^{+})*

$$\delta^{+}\colon \mathcal{N}_n \xrightarrow{\sim} \mathcal{Z}(u)^{+} \coloneqq \mathcal{Z}(u) \cap \mathcal{N}_{n+1}^{+}.$$

The same is true for δ^{-}.

Theorems of this kind identifying special $\mathcal{Z}$-cycles with Rapoport–Zink spaces of lower dimension are proved in [Ter13, Lemma 2] (F/F_0 unramified, hyperspecial level), [Cho18, Proposition 5.10] (F/F_0 unramified, maximal parahoric level), [HSY23, Proposition 2.6] (F/F_0 ramified, Krämer model) and Proposition 5.15 (F/F_0 ramified, exotic smooth model, even to odd dimension). In these cases, the inverse morphism can be constructed directly, as the closed embedding of RZ spaces is straightforward. However in our setting, the definition of the morphism δ^{+} is much more involved (see (5.6)), and it is not clear what the inverse morphism looks like.

We adopt a different method and prove this theorem by three steps :

(1) Firstly we show that the closed embedding δ^{+} factors through $\mathcal{Z}(u)^{+}$(Proposition 5.13).

(2) Secondly we prove that δ^{+} induces a bijection on geometric points between $\mathcal{N}_n$ and $\mathcal{Z}(u)^{+}$(Proposition 5.14). This part involves a detailed description of the Dieudonné modules of the geometric points of the RZ spaces.

(3) As we know both $\mathcal{N}_n$ and $\mathcal{Z}(u)^{+}$ are formally smooth of same relative dimension (Corollary 5.12) we deduce that we have isomorphism $\mathcal{N}_n \xrightarrow{\sim} \mathcal{Z}(u)^{+}$ by formal algebraic geometry

argument.

To relate arithmetic intersection numbers on different exotic smooth models, we also study the pullback of special cycles. For any nonzero $x \in \mathbb{V}_n \subset \mathbb{V}_{n+1}$, there are natural morphisms $\delta_x^+ \colon \mathcal{Z}_{n+1}(x) \cap \mathcal{N}_n \to \mathcal{Z}_n(x)$ and $\gamma_x^+ \colon \mathcal{Y}_n(x) \to \mathcal{Y}_{n+1}(x) \cap \mathcal{N}_n$. By showing the surjectivity on geometric points and formal smoothness of these morphisms, we prove that

Proposition 1.8 (Proposition 5.16)**.** *The morphism δ_x^+ is an isomorphism if $h(x,x) \in \pi_0 O_{F_0}$.*

Proposition 1.9 (Proposition 5.17)**.** *The morphism γ_x^+ is an isomorphism for any nonzero $x \in \mathbb{V}_n$.*

In particular we get $\mathcal{Y}_n(x) \simeq \mathcal{Y}_{n+1}(x) \cap \mathcal{Z}_{n+1}(u)^+ \simeq \mathcal{Y}_{n+1}(x) \cap \mathcal{Y}_{n+1}(f)^+$. The same is true for δ^-. Combine all the results we get $\mathrm{Int}_{n,\mathcal{Y}}(L) = \frac{1}{2}\mathrm{Int}_{n+1,\mathcal{Y}}(L^\#)$ (Proposition 5.18).

Remark 1.10. It is not true that for $n \geqslant 3$ odd and every nonzero $x \in \mathbb{V}_n$ we have identifications $\mathcal{Y}_n(x) \simeq \mathcal{Z}_n(\pi x)$. This identification only holds for those x with $h(x,x) \in \pi_0 O_{F_0}$(Corollary 5.19). This phenomenon causes problem if we want a Kudla–Rapoport formula for $\mathcal{Z}$-cycles: we need nontrivial modification to the analytic side. This will be studied in a forthcoming paper.

1.4 Notation and terminology

- All rings are commutative and unital, and ring homomorphisms preserve units. For a ring R, we denote by $R^\times$ all its invertible elements.
- Let $R \to R'$ be a ring homomorphism and M be an R-module. We put $M_{R'} \coloneqq M \otimes_R R'$.
- Let p be an odd prime. Let F_0 be a finite extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$. Denote by $\bar{k}$ a fixed algebraic closure of k. Let F be a ramified quadratic extension of F_0. Denote by $a \mapsto \bar{a}$ the (nontrivial) Galois involution of F/F_0. Let π be a uniformizer of F such that $\bar{\pi} = -\pi$. Let $\pi_0 = \pi^2$, a uniformizer of F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $O_F, O_{\breve{F}}$ be the ring of integers of $F, \breve{F}$ respectively.

- We say a scheme S is a $\mathrm{Spf}\, O_{\breve{F}}$-scheme, if S is a scheme over $\mathrm{Spec}\, O_{\breve{F}}$ and π is locally nilpotent in S. For a $\mathrm{Spf}\, O_{\breve{F}}$-scheme S, we denote its special fibre by

$$\overline{S} = S \times_{\mathrm{Spec}\, O_{\breve{F}}} \mathrm{Spec}\, \bar{k}.$$

- Throughout the paper, by an O_F-lattice L inside an F-vector space V, without mentioning the rank of L we always mean a lattice of full rank. A hermitian O_F-lattice L is a pair (L, h) where L is an O_F-lattice equipped with an O_{F_0}-bilinear pairing $h(-,-)\colon L \times L \to F$ such that the induced F-valued pairing on $L \otimes_{O_F} F$ is a nondegenerate hermitian pairing with respect to F/F_0. We shall simply write a hermitian O_F-lattice as L without mentioning the pairing h when it is clear from the context. We say a hermitian O_F-lattice L is split (nonsplit) if the corresponding F/F_0-hermitian space $L \otimes_{O_F} F$ is a split (nonsplit) hermitian space.

- Let R be a local ring. If L and M are two R-lattices of the same rank and $L \subset M$, we use the notation $L \subset^r M$ to indicate the co-length of L in M is r, and $L \subset^{\leqslant r} M$ to indicate that the co-length of L in M is smaller or equal to r.

- Let $\mathcal{O}$ be a complete discrete valuation ring over $\mathbb{Z}_p$. Let S be a scheme over $\mathrm{Spec}\, \mathcal{O}$. A pair (X, ι) consisting of a p-divisible group X over S and an action ι of $\mathcal{O}$ on X is called a strict $\mathcal{O}$-module if the action of $\mathcal{O}$ on $\mathrm{Lie}\, X$ is via the structure morphism $\mathcal{O} \to \mathcal{O}_S$. A strict $\mathcal{O}$-module is called formal if the p-divisible group is formal. For simplicity we will omit 'strict' and just call them formal $\mathcal{O}$-modules.

Chapter 2: Rapoport–Zink spaces and special cycles

2.1 Rapoport–Zink space

In this section, we review the definition and basic properties of Rapoport–Zink spaces and special cycles.

Definition 2.1. For any $\mathrm{Spf}\, O_{\breve{F}}$-scheme S, a hermitian formal O_F-module (X, ι, λ) of signature $(1, n-1)$ over S is the following data:

- X is a (strict) formal O_{F_0}-module over S of dimension n and relative height $2n$;
- ι is an action of O_F on X extending the O_{F_0}-action;
- λ is an ι-compatible polarization of X such that $\ker \lambda \subset X[\iota(\pi)]$ is of rank $q^{2[\frac{n}{2}]}$. Here ι-compatible means $\lambda \circ \iota(a) = \iota(\bar{a})^\vee \circ \lambda$ holds for every $a \in O_F$,

such that when n is even, the triple (X, ι, λ) satisfies the following conditions:

– (Kottwitz condition)

$$\text{the characteristic polynomial of } \iota(\pi) \text{ on the } \mathcal{O}_S\text{-module } \mathrm{Lie}(X) \text{ is } \\ (T-\pi)(T+\pi)^{n-1} \in \mathcal{O}_S[T], \tag{2.1}$$

– (Wedge condition)

$$\bigwedge^2 (\iota(\pi) + \pi \mid \mathrm{Lie}(X)) = 0, \tag{2.2}$$

– (Spin condition)

$$\text{for every geometric point } s \text{ of } S, \text{ the action of } \iota(\pi) \text{ on } \mathrm{Lie}(X_s) \text{ is nonzero;} \tag{2.3}$$

and when n is odd, we require the triple to satisfy condition 2.13 below: this condition is a little complicated to formulate and require some preparation; we postpone the formulation of the condition to §2.2.

An isomorphism of hermitian formal O_F-modules $(X, \iota, \lambda) \xrightarrow{\sim} (X', \iota', \lambda')$ is an O_F-linear isomorphism $\varphi \colon X \xrightarrow{\sim} X'$ such that $\varphi^*(\lambda') = \lambda$. To define the moduli problem, we fix an explicit choice of such triple $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ over $\bar{k}$ as the framing object in §2.3.

Definition 2.2. Let $\mathrm{Nilp}_{O_{\breve{F}}}$ be the category of $O_{\breve{F}}$-schemes S such that π is locally nilpotent on S. Then the Rapoport–Zink space associated with $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ is the functor

$$\mathcal{N}_n \to \mathrm{Spf}\, O_{\breve{F}}$$

sending $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$, where

- (X, ι, λ) is a hermitian formal O_F-module of signature $(1, n-1)$ over S;
- $\rho \colon X \times_S \overline{S} \to \mathbb{X}_n \times_{\bar{k}} \overline{S}$ is an O_F-linear quasi-isogeny of height 0 over $\overline{S}$ such that $\rho^*(\lambda_{\mathbb{X}_n, \overline{S}}) = \lambda_{\overline{S}}$.

Proposition 2.3 ([RSZ18, Theorem 6.5, Theorem 7.3])**.** *The functor $\mathcal{N}_n$ is (pro-)representable by a separated formal scheme, which is formally locally of finite type, essentially proper, and formally smooth of relative formal dimension $n-1$ over* $\mathrm{Spf}\, O_{\breve{F}}$*. In particular $\mathcal{N}_n$ is regular of formal dimension n.*

When n is even, there is a natural decomposition of $\mathcal{N}_n$. We will need this in §5. Let

$$\mathbb{M} = \mathbb{M}_n \quad \text{and} \quad \mathbb{N} = \mathbb{N}_n := \mathbb{M} \otimes_{O_{\breve{F}_0}} \breve{F}_0$$

denote the covariant relative Dieudonné module and rational Dieudonné module, respectively, of $\mathbb{X}_n$. The action $\iota_{\mathbb{X}_n}$ makes $\mathbb{M}$ into an $O_F \otimes_{O_{F_0}} O_{\breve{F}_0} = O_{\breve{F}}$-module. The polarization $\lambda_{\mathbb{X}_n}$ induces a nondegenerate alternating $\breve{F}_0$-bilinear form $\langle\ ,\ \rangle$ on $\mathbb{N}$ satisfying

$$\langle ax, y\rangle = \langle x, \bar{a}y\rangle \quad \text{for all} \quad x, y \in \mathbb{N},\ a \in \breve{F}.$$

The form

$$h(x,y) := \langle \pi x, y\rangle + \pi\langle x, y\rangle, \quad x, y \in \mathbb{N},$$

then makes $\mathbb{N}$ into an $\breve{F}/\breve{F}_0$-hermitian space of dimension n. By Dieudonné theory, for a perfect field extension K of $\bar{k}$, the set of K-points on $\mathcal{N}_n$ identifies with a certain subset $\mathcal{S}$ of π-modular (Definition 3.1) $O_{\breve{F}} \otimes_{O_{\breve{F}_0}} W(K)$-lattices $M \subset \mathbb{N} \otimes_{O_{\breve{F}_0}} W(K)$ (details in §5.2). For a lattice $M \in \mathcal{S}$, we say that the corresponding K-point on $\mathcal{N}_n$ lies in $\mathcal{N}_n^+$ or $\mathcal{N}_n^-$ according as the $O_{\breve{F}} \otimes_{O_{\breve{F}_0}} W(K)$-length of the module

$$\big(M + \mathbb{M} \otimes_{O_{\breve{F}_0}} W(K)\big)\big/\mathbb{M} \otimes_{O_{\breve{F}_0}} W(K) \tag{2.4}$$

is even or odd. The parity of this length may also be described as follows. Since the π-modular lattices in a hermitian space are all conjugate under the unitary group, and since $\mathbb{M} \otimes_{O_{\breve{F}_0}} W(K)$ is itself π-modular in $\mathbb{N} \otimes_{O_{\breve{F}_0}} W(K)$, there exists some $g \in \mathrm{U}(\mathbb{N})(\breve{F} \otimes_{O_{\breve{F}_0}} W(K))$ such that $g \cdot (\mathbb{M} \otimes_{O_{\breve{F}_0}} W(K)) = M$. The determinant $\det g$ is a norm one element in $\breve{F} \otimes_{O_{\breve{F}_0}} W(K)$, and hence lies in $O_{\breve{F}} \otimes_{O_{\breve{F}_0}} W(K)$ with reduction mod π equal to $\pm 1 \in K$. Then the length of (2.4) is even or odd according as this reduction is 1 or -1 (independent of the choice of g) by [RSZ17, Lemma 3.2].

Proposition 2.4 ([RSZ18, Proposition 6.4]). *$\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ define a decomposition*

$$\mathcal{N}_n = \mathcal{N}_n^+ \amalg \mathcal{N}_n^-$$

into open and closed formal subschemes.

2.2 Condition for n odd

In this section we specify the condition for the triple (X, ι, λ) when n is odd, and describe its relation with the Kottwitz condition, Wedge condition and Spin condition. We follow everything from [RSZ18, §7] and nothing new is proved here.

Since we will also need an analog of this condition in §5.1 for even n, for the moment let n be any positive integer. Let

$$m \coloneqq \lfloor n/2 \rfloor.$$

Let $e_1, \dots, e_n$ denote the standard basis in F^n, and let h be the standard split F/F_0-hermitian form on F^n with respect to this basis,

$$h(ae_i, be_j) \coloneqq a\bar{b}\delta_{i,n+1-j} \quad \text{(Kronecker delta)}. \tag{2.5}$$

Let $\langle\ ,\ \rangle$ and $(\ ,\ )$ be the respective alternating and symmetric O_{F_0}-bilinear forms $F^n \times F^n \to F_0$ defined by

$$\langle x, y\rangle \coloneqq \frac{1}{2}\operatorname{tr}_{F/F_0}\big(\pi^{-1}h(x,y)\big) \quad \text{and} \quad (x,y) \coloneqq \frac{1}{2}\operatorname{tr}_{F/F_0} h(x,y). \tag{2.6}$$

For $i = bn + c$ with $0 \leqslant c < n$, define the O_F-lattice

$$\Lambda_i \coloneqq \sum_{j=1}^{c} \pi^{-b-1} O_F e_j + \sum_{j=c+1}^{n} \pi^{-b} O_F e_j \subset F^n.$$

For each i, the form $\langle\ ,\ \rangle$ induces a perfect pairing

$$\Lambda_i \times \Lambda_{-i} \to O_{F_0}.$$

In this way, for fixed nonempty $I \subset \{0, \dots, m\}$, the set

$$\Lambda_I \coloneqq \{\, \Lambda_i \mid i \in \pm I + n\mathbb{Z} \,\}$$

forms a polarized chain of O_F-lattices over O_{F_0} in the sense of [RZ96, Definition 3.14].

Now define the $2n$-dimensional F-vector space

$$V := F^n \otimes_{F_0} F,$$

where F acts on the right tensor factor. The n-th wedge power ${}^nV := \bigwedge_F^n V$ admits a canonical decomposition

$${}^nV = \bigoplus_{\substack{r+s=n \\ \epsilon\in\{\pm 1\}}} {}^nV_\epsilon^{r,s} \tag{2.7}$$

which is described in [Smi15, §2.3, 2.5].[1] Let us briefly review it. The operator $\pi \otimes 1$ acts F-linearly on V with eigenvalues $\pm\pi$; let

$$V = V_\pi \oplus V_{-\pi}$$

denote the corresponding eigenspace decomposition. For a partition $r + s = n$, define[2]

$${}^nV^{r,s} := \bigwedge\nolimits_F^r V_\pi \otimes_F \bigwedge\nolimits_F^s V_{-\pi},$$

which is naturally a subspace of ${}^nV$. Furthermore, the symmetric form $(\ ,\ )$ splits after base change to V, and therefore there is a decomposition

$${}^nV = {}^nV_1 \oplus {}^nV_{-1} \tag{2.8}$$

as an $\mathrm{SO}((\ ,\ ))(F)$-representation. The subspaces ${}^nV_{\pm 1}$ have the property that for any Lagrangian (i.e. totally isotropic n-dimensional) subspace $\mathcal{F} \subset V$, the line $\bigwedge_F^n \mathcal{F} \subset {}^nV$ is contained in one of them, and in this way they distinguish the two connected components of the orthogonal Grassmannian $\mathrm{OGr}(n, V)$ over $\operatorname{Spec} F$. The subspaces ${}^nV_{\pm 1}$ are canonical up to labeling, and

[1] Here and below we replace the symbol W used in loc. cit. with ${}^nV$.

[2] Here and below we interchange r and s in the notation relative to [Smi15].

we will follow the labeling conventions in loc. cit., to which we refer the reader for details. The summands in the decomposition (2.7) are then given by

$$ {}^nV_\epsilon^{r,s} := {}^nV^{r,s} \cap {}^nV_\epsilon $$

(intersection in ${}^nV$) for $\epsilon \in \{\pm 1\}$.

Given an O_F-lattice $\Lambda \subset F^n$, now define

$$ {}^n\Lambda := \bigwedge\nolimits_{O_F}^n (\Lambda \otimes_{O_{F_0}} O_F), $$

which is naturally a lattice in ${}^nV$. For fixed r, s, and ϵ, define

$$ {}^n\Lambda_\epsilon^{r,s} := {}^n\Lambda \cap {}^nV_\epsilon^{r,s} \tag{2.9} $$

(intersection in ${}^nV$). Then ${}^n\Lambda_\epsilon^{r,s}$ is a direct summand of ${}^n\Lambda$, since the quotient ${}^n\Lambda/{}^n\Lambda_\epsilon^{r,s}$ is torsion-free. For an O_F-scheme S, define

$$ L_{i,\epsilon}^{r,s}(S) := \operatorname{im}\Big[{}^n(\Lambda_i)_\epsilon^{r,s} \otimes_{O_F} \mathcal{O}_S \to {}^n\Lambda_i \otimes_{O_F} \mathcal{O}_S\Big]. \tag{2.10} $$

For nonempty $I \subset \{0, \ldots, m\}$, let $\underline{\mathrm{Aut}}(\Lambda_I)$ denote the scheme of automorphisms of the polarized O_F-lattice chain Λ_I over $\operatorname{Spec} O_{F_0}$, in the sense of [RZ96, Theorem 3.16] or [Pap00, Page 581] (this is denoted by $\mathcal{P}$ in [Pap00]).

Lemma 2.5 ([RSZ18, Lemma 7.1]). *For any O_F-scheme S and $\Lambda_i \in \Lambda_I$, the submodule $L_{i,\epsilon}^{r,s}(S) \subset {}^n\Lambda_i \otimes_{O_F} \mathcal{O}_S$ is stable under the natural action of $\underline{\mathrm{Aut}}(\Lambda_I)(S)$ on ${}^n\Lambda_i \otimes_{O_F} \mathcal{O}_S$.*

This concludes our discussion for general n. We now formulate our condition on the triple (X, ι, λ) over a $\operatorname{Spf} O_{\breve{F}}$-scheme S in the case of odd n, which will make use of the above discussion in the case $I = \{m\}$. Let $M(X)$ and $M(X^\vee)$ denote the respective Lie algebras of the universal vector extensions of X and $X^\vee$. Since $\ker \lambda \subset X[\iota(\pi)]$, there is a unique (necessarily O_F-linear) isogeny λ' such that the composite

$$X \xrightarrow{\lambda} X^\vee \xrightarrow{\lambda'} X$$

is $\iota(\pi)$. Since $\ker \lambda$ furthermore has rank q^{n-1}, the induced diagram

$$M(X) \xrightarrow{\lambda_*} M(X^\vee) \xrightarrow{\lambda'_*} M(X)$$

extends periodically to a polarized chain of $O_F \otimes_{O_{F_0}} \mathcal{O}_S$-modules of type $\Lambda_{\{m\}}$, in the terminology of [RZ96]. By Theorem 3.16 in loc. cit., étale-locally on S there exists an isomorphism of polarized chains

$$[\cdots \xrightarrow{\lambda'_*} M(X) \xrightarrow{\lambda_*} M(X^\vee) \xrightarrow{\lambda'_*} \cdots] \xrightarrow{\sim} \Lambda_{\{m\}} \otimes_{O_{F_0}} \mathcal{O}_S, \tag{2.11}$$

which in particular gives an isomorphism of $O_F \otimes_{O_{F_0}} \mathcal{O}_S$-modules

$$M(X) \xrightarrow{\sim} \Lambda_{-m} \otimes_{O_{F_0}} \mathcal{O}_S. \tag{2.12}$$

The module $M(X)$ fits into the covariant Hodge filtration

$$0 \to \mathrm{Fil}^1 \to M(X) \to \mathrm{Lie}\, X \to 0$$

for X, and the condition we finally impose is that

> *upon identifying* Fil^1 *with a submodule of* $\Lambda_{-m} \otimes_{O_{F_0}} \mathcal{O}_S$ *via* (2.12)*, the line bundle*
> $$\textstyle\bigwedge^n_{\mathcal{O}_S} \mathrm{Fil}^1 \subset {}^n\Lambda_{-m} \otimes_{O_F} \mathcal{O}_S \tag{2.13}$$
> *is contained in* $L^{n-1,1}_{-m,-1}(S)$.

Note that Lemma 2.5 gives exactly what is needed to conclude that condition (2.13) is independent of the choice of chain isomorphism in (2.11).

Remark 2.6. The Kottwitz condition (2.1), Wedge condition (2.2), and Spin condition (2.3) all continue to make sense as written in the odd ramified setting. The first two of these conditions are implied by condition (2.13), cf. [Smi15, Lemma 5.1.2, Remark 5.2.2] (which shows that these implications hold on the local model). On the other hand, let $\mathcal{N}_n^\circ$ be the moduli space of quadruples $(X, \iota, \lambda, \rho)$ as in the definition of $\mathcal{N}_n$, except that instead of imposing condition (2.13), we impose conditions (2.2) and (2.3). Then (2.2) and (2.3) imply condition (2.13), and in this way $\mathcal{N}_n^\circ$ is an open formal subscheme of $\mathcal{N}_n$. Indeed, this statement follows from the analogous statement for the corresponding local models, which is explained in [Smi15, §3.3]. (More precisely, loc. cit. shows that the local model for $\mathcal{N}_n^\circ$ is the complement of the "worst point" in the local model for $\mathcal{N}_n$.)

Remark 2.7. There is a natural analog of condition (2.13) on $\mathcal{N}_n$ when n is even (still with F/F_0 ramified). However this analog is automatically satisfied on the whole space, which follows from the fact that this condition is automatically satisfied in the generic fiber of the local model for $\mathcal{N}_n$, and this local model is already flat (in fact smooth).

2.3 Framing objects

To complete the definition of $\mathcal{N}_n$, it remains to specify a framing object $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ for this moduli problem over $\bar{k}$.

We denote by $\mathbb{E}$ the unique formal O_{F_0}-module of dimension 1 and relative height 2 over $\bar{k}$. The Dieudonné module $\mathbb{M}_0$ of $\mathbb{E}$ can be identified with $W_{O_{F_0}}(\bar{k})^2 = O_{\breve{F}_0}^2$ endowed with the Frobenius and Verschiebung operator given in matrix form by

$$F = \begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix} \sigma \, , V = \begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix} \sigma^{-1} \tag{2.14}$$

where σ denotes the usual Frobenius homomorphism on the Witt vectors. To give a polarization on $\mathbb{E}$ is to give an alternating bilinear pairing on $\mathbb{M}_0$ with associated matrix of the form

$$\begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}$$

for $\delta \in O_{\breve{F}_0}$ satisfying $\sigma(\delta) = -\delta$. We define the principal polarization $\lambda_{\mathbb{E}}$ by fixing any such $\delta \in O_{\breve{F}_0}^{\times}$ once and for all. Note that any other principal polarization of $\mathbb{E}$ differs from $\lambda_{\mathbb{E}}$ by an $O_{F_0}^{\times}$-multiple.

Let $\iota_{\mathbb{E}}$ be an embedding

$$\iota_{\mathbb{E}} \colon O_F \hookrightarrow \mathrm{End}_{O_{F_0}}(\mathbb{E})$$

which makes $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ into a hermitian formal O_F-module of signature $(1,0)$ over $\bar{k}$. Explicitly we take

$$\iota_{\mathbb{E}}(a + b\pi) := \begin{pmatrix} a & b\pi_0 \\ b & a \end{pmatrix}, \quad a, b \in O_{F_0}$$

acting on the Dieudonné module.

We denote by $(\overline{\mathbb{E}}, \iota_{\overline{\mathbb{E}}}, \lambda_{\overline{\mathbb{E}}})$ the same object as $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$, except where the O_F-action $\iota_{\overline{\mathbb{E}}}$ is given by the composition of $\iota_{\mathbb{E}}$ and the Galois involution on F.

- When n is even, up to O_F-linear quasi-isogeny compatible with polarizations, there is a unique hermitian formal O_F-module $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ over $\bar{k}$, which follows from Proposition 3.1 and its proof in [RSZ17].

  When $n = 2$, we take

  $$\mathbb{X}_2 := \mathbb{E} \times \mathbb{E}$$

  as a formal O_{F_0}-module, and we define $\iota_{\mathbb{X}_2}$ by

  $$\iota_{\mathbb{X}_2}(a + b\pi) := \begin{pmatrix} a & b\pi_0 \\ b & a \end{pmatrix}, \quad a, b \in O_{F_0}.$$

  (This identifies $\mathbb{X}_2$ with the Serre tensor construction $O_F \otimes_{O_{F_0}} \mathbb{E}$.) For the polarization, we take

$$\lambda_{\mathbb{X}_2} := \begin{pmatrix} -2\lambda_{\mathbb{E}} & \\ & 2\pi_0\lambda_{\mathbb{E}} \end{pmatrix}. \tag{2.15}$$

We then take

$$\begin{aligned}
\mathbb{X}_n &:= \mathbb{X}_2 \times \overline{\mathbb{E}}^{n-2},\\
\iota_{\mathbb{X}_n} &:= \iota_{\mathbb{X}_2} \times \iota_{\overline{\mathbb{E}}}^{n-2},\\
\lambda_{\mathbb{X}_n} &:= \lambda_{\mathbb{X}_2} \times \mathrm{diag}\Bigg(\underbrace{\begin{pmatrix} 0 & \lambda_{\overline{\mathbb{E}}}\,\iota_{\overline{\mathbb{E}}}(\pi) \\ -\lambda_{\overline{\mathbb{E}}}\,\iota_{\overline{\mathbb{E}}}(\pi) & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_{\overline{\mathbb{E}}}\,\iota_{\overline{\mathbb{E}}}(\pi) \\ -\lambda_{\overline{\mathbb{E}}}\,\iota_{\overline{\mathbb{E}}}(\pi) & 0 \end{pmatrix}}_{(n-2)/2 \text{ times}}\Bigg).
\end{aligned} \tag{2.16}$$

It is straightforward to see that the framing object satisfies the Kottwitz condition, Wedge condition and Spin condition.

- When n is odd, in contrast to the previous case, a triple $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ of hermitian formal O_F-module over $\bar{k}$ is *not unique up to quasi-isogeny*. In fact, there are two isogeny classes (as always, up to O_F-linear quasi-isogeny compatible with the polarizations), corresponding to the two possible isometry classes of the hermitian space C in the proof of [RSZ17, Proposition 3.1].[3]

  As an explicit representative for which C is nonsplit, we take the same framing object as in [RSZ17]. Thus when $n = 1$ we define

$$\mathbb{X}_1^{(1)} := \mathbb{E}, \quad \iota_{\mathbb{X}_1^{(1)}} := \iota_{\mathbb{E}}, \quad \text{and} \quad \lambda_{\mathbb{X}_1^{(1)}} := -\lambda_{\mathbb{E}}. \tag{2.17}$$

  When $n \geqslant 3$, we take

$$\mathbb{X}_n^{(1)} := \mathbb{X}_{n-1} \times \overline{\mathbb{E}}, \quad \iota_{\mathbb{X}_n^{(1)}} := \iota_{\mathbb{X}_{n-1}} \times \iota_{\overline{\mathbb{E}}}, \quad \text{and} \quad \lambda_{\mathbb{X}_n^{(1)}} := \lambda_{\mathbb{X}_{n-1}} \times \lambda_{\overline{\mathbb{E}}}; \tag{2.18}$$

  here $n-1$ is even and $(\mathbb{X}_{n-1}, \iota_{\mathbb{X}_{n-1}}, \lambda_{\mathbb{X}_{n-1}})$ is as defined in (2.16).

[3] The hermitian space C is isomorphic to the hermitian space of quasi-homomorphisms $\mathbb{V}_n$ defined in (2.20).

To fix a framing object in the other isogeny class, we fix $\epsilon \in O_{F_0}^\times \smallsetminus \mathrm{Nm}\, F^\times$ and define

$$\mathbb{X}_n^{(0)} := \mathbb{X}_n^{(1)}, \quad \iota_{\mathbb{X}_n^{(0)}} := \iota_{\mathbb{X}_n^{(1)}}, \quad \text{and} \quad \lambda_{\mathbb{X}_n^{(0)}} := \epsilon\lambda_{\mathbb{X}_n^{(1)}}. \tag{2.19}$$

Note that such an ϵ exists since F/F_0 is ramified.

Taking $\mathbb{X}_n^{(0)}$ and $\mathbb{X}_n^{(1)}$ as the framing objects, we obtain respective moduli spaces $\mathcal{N}_n^{(0)}$ and $\mathcal{N}_n^{(1)}$. However, these spaces are isomorphic via the map

$$\begin{aligned} \mathcal{N}_n^{(1)} &\xrightarrow{\ \sim\ } \mathcal{N}_n^{(0)} \\ (X, \iota, \lambda, \rho) &\longmapsto \big(X, \iota, \lambda \circ \iota(\epsilon), \rho\big). \end{aligned}$$

To simplify notation in the rest of the paper, from now on we set

$$\mathcal{N}_n := \mathcal{N}_n^{(1)} \quad \text{and} \quad \mathbb{X}_n := \mathbb{X}_n^{(1)}.$$

We note that when $n \geqslant 3$, the framing objects $\mathbb{X}_n$ obviously satisfies (2.2) and (2.3) (because $\mathbb{X}_{n-1}$ does), and therefore they indeed satisfy (2.13); and it is trivial to check directly that $\mathbb{X}_1$ satisfies (2.13) when $n = 1$.

Example 2.8 ($n = 1$). When $n = 1$, the moduli problem for $\mathcal{N}_1$ is just the moduli problem of lifting $\mathbb{E}$ as a formal O_F-module. By the theory of canonical lifting ([Gro86]), we have $\mathcal{N}_1 = \mathrm{Spf}\, O_{\breve{F}}$, with universal object the canonical lift $(\mathcal{E}, \iota_{\mathcal{E}}, -\lambda_{\mathcal{E}}, \rho_{\mathcal{E}})$. (In this case condition (2.13) is redundant in the moduli problem.)

2.4 Special cycles

We define the space of *special quasi-homomorphisms*

$$\mathbb{V}_n = \mathbb{V}(\mathbb{X}_n) := \mathrm{Hom}_{O_F}(\overline{\mathbb{E}}, \mathbb{X}_n) \otimes_{O_F} F \tag{2.20}$$

and for $x, y \in \mathbb{V}_n$, define $h(x, y)$ to be the composition

$$\overline{\mathbb{E}} \xrightarrow{x} \mathbb{X}_n \xrightarrow{\lambda_{\mathbb{X}_n}} \mathbb{X}_n^\vee \xrightarrow{y^\vee} \overline{\mathbb{E}}^\vee \xrightarrow{\lambda_{\overline{\mathbb{E}}}^{-1}} \overline{\mathbb{E}}$$

which lies in

$$\mathrm{End}_{O_F}(\overline{\mathbb{E}}) \otimes_{O_F} F \overset{\iota_{\overline{\mathbb{E}}}^{-1}}{\simeq} F.$$

It turns out that h defines a hermitian form on $\mathbb{V}_n$ which makes $(\mathbb{V}_n, h)$ a nondegenerate and nonsplit hermitian space. ([RSZ17, Lemma 3.5], note that for n odd we take $\mathbb{X}_n = \mathbb{X}_n^{(1)}$.)

We denote by $\mathcal{E}$ the canonical lift of $\mathbb{E}$ over $\mathrm{Spf}\, O_{\breve{F}}$ with respect to $\iota_{\mathbb{E}}$, equipped with its O_F-action $\iota_{\mathcal{E}}$, O_F-linear framing isomorphism $\rho_{\mathcal{E}} \colon \mathcal{E}_{\bar{k}} \xrightarrow{\sim} \mathbb{E}$ and principal polarization $\lambda_{\mathcal{E}}$ lifting $\rho_{\mathcal{E}}^*(\lambda_{\mathbb{E}})$. We denote by $(\overline{\mathcal{E}}, \iota_{\overline{\mathcal{E}}}, \lambda_{\overline{\mathcal{E}}}, \rho_{\overline{\mathcal{E}}})$ the same object but with O_F-action twisted by the Galois involution on F.

Definition 2.9. For every nonzero $x \in \mathbb{V}_n$, we define the special $\mathcal{Z}$-cycle $\mathcal{Z}(x) = \mathcal{Z}_n(x)$ of $\mathcal{N}_n$ to be the maximal closed formal subscheme over which the quasi-homomorphism

$$\rho^{-1} \circ x \colon \overline{\mathbb{E}} \times_{\bar{k}} \overline{S} \to X \times_S \overline{S}$$

extends to a homomorphism $\overline{\mathcal{E}}_S \to X$.

We also define the special $\mathcal{Y}$-cycle $\mathcal{Y}(x) = \mathcal{Y}_n(x)$ of $\mathcal{N}_n$ to be the maximal closed formal subscheme over which the quasi-homomorphism

$$\lambda \circ \rho^{-1} \circ x \colon \overline{\mathbb{E}} \times_{\bar{k}} \overline{S} \to X \times_S \overline{S} \to X^\vee \times_S \overline{S}$$

extends to a homomorphism $\overline{\mathcal{E}}_S \to X^\vee$.

By Grothendieck–Messing theory, $\mathcal{Z}(x)$ and $\mathcal{Y}(x)$ are closed formal subschemes. It is proved that when n is even, $\mathcal{Z}(x)$ is a relative divisor ([LL22, Lemma 2.40]).

Definition 2.10. For any O_F-lattice $L \subset \mathbb{V}_n$, we define the arithmetic intersection numbers for

$\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles.

For n even, the Serre intersection multiplicity

$$\chi\left(\mathcal{O}_{\mathcal{Z}_n(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} \mathcal{O}_{\mathcal{Z}_n(x_n)}\right)$$

does not depend on the choice of a basis $\{x_1, \ldots, x_n\}$ of L by [LL22, Corollary 2.35], which we define to be $\mathrm{Int}_{n,\mathcal{Z}}(L)$.

For general n, the Serre intersection multiplicity

$$\chi\left(\mathcal{O}_{\mathcal{Y}_n(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} \mathcal{O}_{\mathcal{Y}_n(x_n)}\right)$$

does not depend on the choice of a basis $\{x_1, \ldots, x_n\}$ of L by Proposition 3.11 and Proposition 5.18, which we define to be $\mathrm{Int}_{n,\mathcal{Y}}(L)$.

For $n \geqslant 1$, let $g \mapsto g^\dagger$ denote the Rosati involution on $\mathrm{End}^\circ_{O_F}(\mathbb{X}_n)$ induced by $\lambda_{\mathbb{X}_n}$. Define

$$U(\mathbb{X}_n) \coloneqq \left\{g \in \mathrm{End}^\circ_{O_F}(\mathbb{X}_n) \mid gg^\dagger = \mathrm{id}_{\mathbb{X}_n}\right\}.$$

Thus $U(\mathbb{X}_n)$ is the group of O_F-linear self-quasi-isogenies of $\mathbb{X}_n$ which preserve $\lambda_{\mathbb{X}_n}$ on the nose.

The group $U(\mathbb{X}_n)$ acts naturally from the left on $\mathbb{V}_n$, and in this way identifies with the unitary group $U(\mathbb{V}_n, h)$. For any $\beta \in O_{F_0}^\times$, the group $U(\mathbb{X}_n) \simeq U(\mathbb{V}_n)$ acts transitively on the subset of elements in $\mathbb{V}_n$ whose hermitian norm is β.

On the other hand, each g in $U(\mathbb{X}_n)$ is a quasi-isogeny of height 0, and therefore $U(\mathbb{X}_n)$ acts naturally on $\mathcal{N}_n$ on the left via the rule $g \cdot (X, \iota, \lambda, \rho) = (X, \iota, \lambda, g\rho)$. It is easy to see that under this action, for any $x \in \mathbb{V}_n$ and $g \in U(\mathbb{X}_n) \simeq U(\mathbb{V}_n)$ we have

$$g \cdot \mathcal{Z}_n(x) = \mathcal{Z}_n(gx) \quad \text{and} \quad g \cdot \mathcal{Y}_n(x) = \mathcal{Y}_n(gx). \tag{2.21}$$

Chapter 3: Local density and Kudla–Rapoport conjecture

3.1 Local density

In this section we study local representation densities of hermitian lattices. We state all results for F/F_0-hermitian forms, although for $\breve{F}/\breve{F}_0$-hermitian forms everything works as well. We first introduce some notions about hermitian O_F-lattices.

Definition 3.1. Let (V, h) be a hermitian space over F of dimension m.

(1) For an O_F-lattice L of V, define

$$L^\vee \coloneqq \{x \in V \mid h(x, y) \in \pi^{-1} O_F \text{ for every } y \in L\},$$
$$L^* \coloneqq \{x \in V \mid h(x, y) \in O_F \text{ for every } y \in L\}.$$

We have $L^* = \pi L^\vee$. We say L is integral if $L \subset L^\vee$ and self-dual if $L = L^\vee$. When m is even, we say L is π-modular if $L^* = \pi^{-1} L$, and when m is odd, we say L is almost π-modular if $L \subset L^* \subset^1 \pi^{-1} L$.

(2) For an integral O_F-lattice L of V, we define

- the fundamental invariants of L to be the unique integers $0 \leqslant a_1 \leqslant \cdots \leqslant a_m$ such that $L^\vee / L \simeq O_F/(\pi^{a_1}) \oplus \cdots \oplus O_F/(\pi^{a_m})$ as O_F-modules;
- the valuation of L to be $\mathrm{val}(L) = \sum_{i=1}^m a_i$.

Remark 3.2. For a hermitian O_F-lattice L, we say a basis $e_1, \ldots, e_m$ of L is a *normal basis* if its moment matrix $T = (h(e_i, e_j))_{i,j=1}^m$ is conjugate to

$$(\beta_1\pi^{2b_1}) \oplus \cdots \oplus (\beta_s\pi^{2b_s}) \oplus \begin{pmatrix} 0 & \pi^{2c_1-1} \\ -\pi^{2c_1-1} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \pi^{2c_t-1} \\ -\pi^{2c_t-1} & 0 \end{pmatrix}$$

by a permutation matrix, for some $\beta_1, \ldots, \beta_s \in O_{F_0}^\times$ and $b_1, \ldots, b_s, c_1, \ldots, c_t \in \mathbb{Z}$. We have ([LL22, Lemma 2.12])

(1) normal basis exists;

(2) the invariants s, t and $b_1, \ldots, b_s, c_1, \ldots, c_t$ depend only on L;

(3) when L is integral, the fundamental invariants of L are the unique nondecreasing rearrangement of $(2b_1 + 1, \ldots, 2b_s + 1, 2c_1, \ldots, 2c_t)$.

Definition 3.3. Denote by H the standard hyperbolic hermitian O_F-lattice of rank 2 given by the matrix $\left(\begin{smallmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{smallmatrix}\right)$. For an integer $s \geqslant 0$, let $H^s := H^{\oplus s}$ be the orthogonal direct sum of s copies of H. Then H^s is a self-dual hermitian O_F-lattice of rank $2s$.

Let $\epsilon \in O_{F_0}^\times$. Denote by I_1^ϵ the rank 1 hermitian O_F-lattice $\langle x \rangle$ with $h(x, x) = \epsilon$.

We also use the same notations for $O_{\breve{F}}$-lattices.

Let L be an integral hermitian O_F-lattice. For any $d \geqslant 1$, the hermitian form h on L induces a pairing $\bar{h}$ valued in $O_F/\pi^{2d-1}O_F$ on $L/\pi^{2d}L$ by

$$\bar{h}(x, y) := h(\tilde{x}, \tilde{y}) \mod \pi^{2d-1}O_F, \quad \text{for any } x, y \in L/\pi^{2d}L$$

where $\tilde{x}, \tilde{y} \in L$ is any lift of $x, y \in L/\pi^{2d}L$. The definition is independent of choices of lifts as L is integral.

Definition 3.4. Let (M, h_M) and (L, h_L) be two integral hermitian O_F-lattices. We denote by $\mathrm{Herm}_{L,M}$ the scheme of hermitian O_F-module homomorphisms from L to M, which is a scheme of finite type over $\operatorname{Spec} O_{F_0}$. We define the local density to be

$$\operatorname{Den}(M, L) \coloneqq \lim_{d \to +\infty} \frac{|\operatorname{Herm}_{L,M}(O_{F_0}/(\pi_0^d))|}{q^{d \cdot d_{L,M}}}$$

where $d_{L,M}$ is the dimension of $\operatorname{Herm}_{L,M} \otimes_{O_{F_0}} F_0$. Here $\operatorname{Herm}_{L,M}(O_{F_0}/\pi_0^d)$ is given by the set

$$\left\{ \phi \in \operatorname{Hom}_{O_F}(L/\pi^{2d}L, M/\pi^{2d}M) \;\middle|\; \bar{h}_M(\phi(x), \phi(y)) = \bar{h}_L(x, y) \text{ for any } x, y \in L/\pi^{2d}L \right\}.$$

Remark 3.5. Let M and L be two hermitian O_F-lattices of rank m and n and assume $\operatorname{Herm}_{L,M} \otimes_{O_{F_0}} F_0 \neq \emptyset$. Then

$$d_{L,M} = \dim U_m - \dim U_{m-n} = n(2m - n).$$

It is well-known that there is a local density polynomial $\operatorname{Den}(M, L, X) \in \mathbb{Q}[X]$ such that

$$\operatorname{Den}(M, L, q^{-2k}) = \operatorname{Den}(M \oplus H^k, L).$$

Definition 3.6. Let L be a nonsplit hermitian O_F-lattice of rank n. Define

$$M_n \coloneqq \begin{cases} H^r, & \text{if } n = 2r \text{ is even,} \\ H^r \oplus I_1^1, & \text{if } n = 2r + 1 \text{ is odd.} \end{cases}$$

Then M_n is always a split hermitian O_F-lattice and $\operatorname{Den}(M_n, L) = 0$. We define

$$\operatorname{Den}'(L) \coloneqq -2 \left.\frac{\mathrm{d}}{\mathrm{d}X}\right|_{X=1} \operatorname{Den}(M_n, L, X) \text{ and } \partial \operatorname{Den}(L) \coloneqq \frac{\operatorname{Den}'(L)}{\operatorname{Den}(M_n, M_n)}.$$

Remark 3.7. Let $L \subset \mathbb{V}_n$ be an O_F-lattice. Let $T \in \mathrm{GL}_n(F)$ be a representing moment matrix of L, and consider the T-th Whittaker function $W_T(s, 1_{2n}, \mathbb{1}_{M_n^n})$ of the Schwartz function $\mathbb{1}_{M_n^n}$ at the identity element 1_{2n}. By [KR14, Proposition 10.1],[1] we have

$$W_T(s, 1_{2n}, \mathbb{1}_{M_n^n}) = \mathrm{Den}(M_n \oplus H^s, L)$$

for every integer $s \geqslant 0$. Thus, we obtain

$$\log q \cdot \partial\mathrm{Den}(L) = \frac{W_T'(0, 1_{2n}, \mathbb{1}_{M_n^n})}{\mathrm{Den}(M_n, M_n)}$$

by Definition 3.6.

3.2 Kudla–Rapoport conjecture

The main result of this paper is the following theorem.

Theorem 3.8. *For any hermitian O_F-lattice $L \subset \mathbb{V}_n$ we have*

$$\mathrm{Int}_{n,\mathcal{Y}}(L) = \partial\mathrm{Den}(L).$$

The rest of the section will prove this for n even by reformulating the result in [LL22].

In [LL22], Li and Liu defined a normalized hermitian form h' on $\mathbb{V}_n$:

$$\text{for any } x, y \in \mathbb{V}_n, h'(x, y) := \pi^{-2} h(x, y).$$

For any O_F-lattice $L \subset \mathbb{V}_n$, we shall use the notation $\partial\mathrm{Den}_{h'}(L)$ to emphasis that L is viewed as a hermitian O_F-lattice via h' (instead of h). When n is even, normalizing the hermitian form on L by -1 does not change $\partial\mathrm{Den}(L)$ by the formula given in [LL22, Lemma 2.15]. We then have relation $\partial\mathrm{Den}_{h'}(L) = \partial\mathrm{Den}_{-h'}(L) = \partial\mathrm{Den}(\pi^{-1}L)$.

The following theorem is one of the main results in [LL22].

[1]In [KR14, Proposition 10.1] and its proof, the lattice $L_{r,r}$ should be replaced by M_n.

Theorem 3.9 ([LL22, Theorem 2.11])**.** *Let $n \geqslant 2$ be even. For any O_F-lattice $L \subset \mathbb{V}_n$ we have*

$$\mathrm{Int}_{n,\mathcal{Z}}(L) = \partial\mathrm{Den}_{h'}(L).$$

Remark 3.10. Suppose $n \geqslant 2$ is even. For any hermitian formal O_F-module (X, ι_X, λ_X) of signature $(1, n-1)$, the polarization λ_X of X satisfies $\ker(\lambda_X) = X[\iota_X(\pi)]$. Then there is a unique ι_X-compatible morphism $\sigma_X \colon X \to X^\vee$ satisfying $\lambda_X = \sigma_X \circ \iota_X(\pi)$, which is in fact a symmetrization, i.e. an isomorphism with $\sigma_X^\vee = \sigma_X$, . Conversely, given a ι_X-compatible symmetrization σ_X of X, we may recover λ_X as $\sigma_X \circ \iota_X(\pi)$. In what follows, we call σ_X the symmetrization of λ_X.

Proposition 3.11. *Let $n \geqslant 2$ be even. For any nonzero $x \in \mathbb{V}_n$, we have a natural morphism*

$$\begin{array}{ccc} \mathcal{Y}(x) & \longrightarrow & \mathcal{Z}(\pi x) \\ (X, \overline{\mathcal{E}} \to X^\vee) & \longmapsto & (X, \overline{\mathcal{E}} \to X^\vee \xrightarrow{\sigma_X^{-1}} X) \end{array}$$

which turns out to be an isomorphism. Consequently, for any O_F-lattice $L \subset \mathbb{V}_n$ we have

$$\mathrm{Int}_{n,\mathcal{Y}}(L) = \mathrm{Int}_{n,\mathcal{Z}}(\pi L).$$

In particular $\mathrm{Int}_{n,\mathcal{Y}}(L)$ is well-defined, independent of a choice of basis of L.

Proof. The statements are all clear and easy to verify : the inverse morphism is just given by

$$\begin{array}{ccc} \mathcal{Z}(\pi x) & \longrightarrow & \mathcal{Y}(x) \\ (X, \overline{\mathcal{E}} \to X^{)} & \longmapsto & (X, \overline{\mathcal{E}} \to X \xrightarrow{\sigma_X} X^\vee). \end{array}$$

□

Corollary 3.12. *Theorem 3.8 holds for any n even.*

Proof. When n is even, for any O_F-lattice $L \subset \mathbb{V}_n$, the h-hermitian local density $\partial\mathrm{Den}(L)$ is the same as the h'-hermitian local density $\partial\mathrm{Den}_{h'}(\pi L)$. Combine Theorem 3.9 and Proposition 3.11

we get

$$\mathrm{Int}_{n,\mathcal{Y}}(L) = \mathrm{Int}_{n,\mathcal{Z}}(\pi L) = \partial\mathrm{Den}_{h'}(\pi L) = \partial\mathrm{Den}(L).$$

$\square$

Chapter 4: Reduction at analytic side

To simplify the notations, we first define *standard normal base* for H^s and $H^s \obot I_1^\epsilon$.

Definition 4.1. We say an O_F-basis $e_1, f_1, \ldots, e_s, f_s$ (or an $O_{\breve{F}}$-basis if we view H^s as an $O_{\breve{F}}$-lattice) of H^s is a *standard normal basis* if the only nonzero hermitian pairing between them is

$$h(e_i, f_i) = -h(f_i, e_i) = \pi^{-1} \text{ for each } i.$$

In other words, for each i, e_i, f_i is a normal basis for H, and for $i \neq j$, these H are orthogonal to each other.

We say a basis $e_1, f_1, \ldots, e_s, f_s, \varphi$ of $H^s \obot I_1^\epsilon$ is a *standard normal basis* if $e_1, f_1, \ldots, e_s, f_s$ is a standard normal basis for H^s, φ is a basis for I_1^ϵ orthogonal to all e_i, f_j with $h(\varphi, \varphi) = \epsilon$.

Our strategy to deal with the analytic side is to relate it to the even dimensional case which is one dimensional higher. We first prove a key lemma which will be used repeatedly later.

Lemma 4.2. *Let $M = H^s$ and $\phi \in M$ be any element such that $\phi \notin \pi M$. Then the submodule*

$$M(\phi) \coloneqq \{x \in M \mid h(x, \phi) = 0\}$$

can be described as

$$M(\phi) = H^{s-1} \obot I$$

where $I = \langle \phi' \rangle_{O_F}$ is a rank one O_F-lattice such that $h(\phi', \phi') = -h(\phi, \phi)$ and $\phi' - \phi \in \pi M$.

Proof. Choose a standard normal basis $e_1, f_1, e_2, f_2, \ldots, e_s, f_s$ of M. Write

$$\phi = a_1 e_1 + b_1 f_1 + a_2 e_2 + b_2 f_2 + \cdots + a_s e_s + b_s f_s.$$

As $\phi \notin \pi M$, without loss of generality we may assume $a_1 \in O_F^\times$. Then divide ϕ by a_1 does not change the submodule $M(\phi)$ so we may assume $a_1 = 1$.

For $i \geqslant 2$ we define

$$\sigma_i := e_i + \bar{b}_i f_1, \quad \tau_i := f_i - \bar{a}_i f_1. \tag{4.1}$$

It is a direct computation to verify that

$$\begin{aligned}
h(\phi, \sigma_i) &= h(\phi, \tau_i) = 0, \\
h(\sigma_i, \sigma_j) &= h(\tau_i, \tau_j) = 0, \\
h(\sigma_i, \tau_j) &= -h(\tau_i, \sigma_j) = \pi^{-1}\delta_{i,j}.
\end{aligned}$$

In particular the O_F-span of $\{\sigma_i, \tau_i\}$ for $i \geqslant 2$ is isomorphic to $H^{s-1} \subset M(\phi)$.

Now we let

$$\phi' := \phi + h(\phi, \phi)\pi f_1 = e_1 + (b_1 + h(\phi, \phi)\pi) f_1 + \sum_{i=2}^{s}(a_i e_i + b_i f_i). \tag{4.2}$$

Then it is easy to compute

$$\begin{aligned}
h(\phi', \phi) &= h(\phi', \sigma_i) = h(\phi', \tau_i) = 0, \\
h(\phi', \phi') &= -h(\phi, \phi).
\end{aligned}$$

Let $I := \langle \phi' \rangle_{O_F}$ and we have

$$\langle \phi', \sigma_2, \tau_2, \ldots, \sigma_s, \tau_s \rangle_{O_F} \simeq I \oplus H^{s-1} \subset M(\phi).$$

For any $x \in M(\phi)$, we may use ϕ' to get rid of e_1 in the expression of x, σ_i to remove e_i, and τ_i to remove f_i in x for all $i \geqslant 2$. Then $x \in M(\phi)$ is just a multiple of f_1, which must be 0. We have shown that indeed $M(\phi) \simeq I \oplus H^{s-1}$ as described. □

Corollary 4.3. *If $\phi \in M = H^s$ and $\phi \notin \pi M$ with $h(\phi, \phi) = \beta$, then there is a standard* *normal basis σ_i, τ_i of M such that $\sigma_1 + \frac{\pi\beta}{2}\tau_1, \sigma_2, \tau_2, \ldots, \sigma_s, \tau_s$ is a normal basis of $M(\phi)$ while $\phi = \sigma_1 - \frac{\pi\beta}{2}\tau_1$. Consequently if $\beta \neq 0$, i.e. ϕ is not isotropic, then*

$$M(\phi) \ominus \langle\phi\rangle_{O_F} \subset^b M \quad \textit{and} \quad M(\phi)_F \ominus \langle\phi\rangle_F \simeq M \otimes_{O_F} F$$

where $b = 1 + \mathrm{val}_\pi(\beta)$.

Proof. As in the previous lemma, choose a standard normal basis e_i, f_i of M. Write $\phi = a_1e_1 + b_1f_1 + \cdots + a_se_s + b_sf_s$ and we may assume $a_1 \in O_F^\times$ and for simplicity we shall just assume it is already 1. Then by the proof of the previous lemma, we can construct $\sigma_2, \tau_2, \ldots, \sigma_s, \tau_s$. All we need to do is to find σ_1 and τ_1.

Use the same notation, $\phi' = \phi + \beta\pi f_1$. We just set

$$\sigma_1 := \frac{\phi + \phi'}{2} = \phi + \frac{\pi\beta}{2} f_1 \text{ and } \tau_1 := \frac{\phi' - \phi}{\pi\beta} = f_1. \tag{4.3}$$

It is easy to verify σ_1, τ_1 is a normal basis of H.

Then $M(\phi) \ominus \langle\phi\rangle_{O_F} = \langle\sigma_1, \beta\pi\tau_1, \sigma_2, \tau_2, \ldots, \sigma_s, \tau_s\rangle_{O_F}$ has co-length $1 + \mathrm{val}_\pi(\beta)$ in M if $\beta \neq 0$. □

Lemma 4.4. *Let $M_n = H^{n/2}$ if n is even, and $M_n = H^{(n-1)/2} \ominus I_1^1$ if n is odd. Then*

$$\mathrm{Den}(M_n, M_n) = \begin{cases} \prod_{i=1}^{r}(1 - q^{-2i}), & \textit{if } n = 2r \textit{ even}, \\ 2\prod_{i=1}^{r}(1 - q^{-2i}), & \textit{if } n = 2r+1 \textit{ odd}. \end{cases}$$

Proof. By [LL22, Lemma 2.15], if $n = 2r$ is even, we get $\mathrm{Den}(H^r, H^r) = \prod_{i=1}^{r}(1 - q^{-2i})$.

Now assume $n = 2r + 1$ is odd. By [HLSY23, Lemma 5.13] we have

$$\begin{aligned}\mathrm{Den}(M_n, M_n) &= \mathrm{Den}(I_1^1, M_n, q^{-2r}) \\ &= \left(\prod_{\ell=0}^{r-1}(1 - q^{2\ell-2r})\right)\mathrm{Den}(I_1^1, I_1^1, 1) = \left(\prod_{i=1}^{r}(1 - q^{-2i})\right)\mathrm{Den}(I_1^1, I_1^1)\end{aligned}$$

and by [HLSY23, Corollary 5.17] we have

$$\mathrm{Den}(I_1^1, I_1^1) = 2.$$

□

Proposition 4.5. *Let $n+1 = 2r+2 \geqslant 2$ be even. Let $L^\# \subset \mathbb{V}_{n+1}$ be a hermitian O_F-lattice and assume $L^\#$ has a hermitian lattice decomposition $L^\# = L \obot I_1^{-1}$. Then*

$$\partial \mathrm{Den}(L) = \frac{1}{2} \partial \,\mathrm{Den}(L^\#).$$

Proof. Let $M = H^s$ and choose a standard normal basis $e_1, f_1, e_2, f_2, \ldots, e_s, f_s$ of M.

Let $d \geqslant 1$. Consider the restriction map

$$R\colon\ \mathrm{Herm}_{L^\#, M}(O_{F_0}/\pi_0^d) \to \mathrm{Herm}_{I_1^{-1}, M}(O_{F_0}/\pi_0^d).$$

We claim that the fibre of this map can be identified with $\mathrm{Herm}_{L,M'}(O_{F_0}/\pi_0^d)$ where $M' := M_{2s-1} = I_1^1 \obot H^{s-1}$.

Let $\phi \in \mathrm{Herm}_{I_1^{-1}, M}(O_{F_0}/\pi_0^d)$. Then ϕ is determined by $\bar{x} \in M/\pi^{2d}M$ with $\bar{h}(\bar{x}, \bar{x}) = -1$. For any lift $x \in M$ we have $h(x,x) \equiv -1 \mod \pi^{2d} O_F \in O_{F_0}$. We first modify the lift such that $M(x) \simeq M'$.

Write $h(x,x) = -1 + \pi^{2d}\alpha$ for some $\alpha \in O_{F_0}$. From Lemma 4.2 we can find $y \in M(x)$ with $h(y,y) = -h(x,x)$. Then $x' = x + \pi^{2d}(ax + ay)$ for $a = -\frac{\alpha}{2}$ is also a lift of $\bar{x}$. Now we compute

$$\begin{aligned} h(x', x') &= h((1+\pi^{2d}a)x + \pi^{2d}ay, (1+\pi^{2d}a)x + \pi^{2d}ay) \\ &= (1+\pi^{2d}a)^2 h(x,x) + (\pi^{2d}a)^2 h(y,y) \\ &= (1 + 2\pi^{2d}a) h(x,x) = -h(x,x)^2. \end{aligned}$$

As $h(x,x)$ is in $O_{F_0}^\times$, the submodule $M(x')$ is the same as $M(x'/h(x,x))$, and the latter is just M'.

We shall fix such a lift and still denote it by x from now on.

Choose an O_F-basis $\ell_1, \ldots, \ell_n$ for L. The fibre over ϕ is given by

$$R^{-1}(\phi) = \left\{ \bar{x}_1, \ldots, \bar{x}_n \in M/\pi^{2d}M \;\middle|\; \begin{array}{c} \bar{h}(\bar{x}_i, \bar{x}) = 0 \text{ for any } i, \\ \bar{h}(\bar{x}_i, \bar{x}_j) = h(\ell_i, \ell_j) \mod \pi^{2d-1}O_F \text{ for any } i, j \end{array} \right\}.$$

Let $K = \{\bar{y} \in M/\pi^{2d}M \mid \bar{h}(\bar{y}, \bar{x}) = 0\}$. Clearly there is an injective map

$$M(x)/\pi^{2d}M(x) \to M/\pi^{2d}M$$

and the image lies in K. We next show that the image is exactly K.

For any $\bar{y} \in K$ we need to find a lift of $\bar{y}$ in $M(x)$. Choose any lift $y \in M$, then $h(y, x) \in \pi^{2d-1}O_F$. As $h(x, x) \neq 0$, by Corollary 4.3 we have $M \otimes_{O_F} F \simeq \langle x \rangle_F \oplus M(x)_F$. So we may write $y = y_x + y_r$ according to the decomposition. Then $h(y, x) = h(y_x, x) \in \pi^{2d-1}O_F$. Write $y_x = a' \cdot x$ for some $a' \in F$, we must have $a' \cdot h(x, x) \in \pi^{2d-1}O_F$. As $h(x, x)$ is a unit, $a' \in \pi^{2d-1}O_F$ and $y_x \in \pi^{2d-1}M$. Thus $y_r = y - y_x \in M(x)$. If $a' \in \pi^{2d}O_F$ then y_r is a lift of $\bar{y}$ in $M(x)$. Otherwise suppose $a' = \pi^{2d-1}\lambda$ for some $\lambda \in O_F^\times$ and write $h(x, x) = \beta \in O_{F_0}^\times$. By Corollary 4.3, we can find a standard normal basis σ_i, τ_i of M such that $x = \sigma_1 - \frac{\pi\beta}{2}\tau_1$. Consider $y' = y + \pi^{2d}z$ where $z = \lambda\beta\tau_1$. Then y' is a lift of $\bar{y}$ and

$$h(y', x) = h(y_x + \pi^{2d}z, x) = \pi^{2d-1}\lambda\beta + \pi^{2d}h(z, x) = \pi^{2d-1}(\lambda\beta + \pi \cdot h(\lambda\beta\tau_1, \sigma_1 - \frac{\pi\beta}{2}\tau_1)) = 0.$$

Thus y' is a lift of $\bar{y}$ in $M(x)$. We have shown that

$$K = M(x)/\pi^{2d}M(x) \simeq M'/\pi^{2d}M'$$

and consequently

$$R^{-1}(\phi) \simeq \mathrm{Herm}_{L,M'}(O_{F_0}/\pi_0^d).$$

As a corollary

$$|\operatorname{Herm}_{L^{\#},M_{2s}}(O_{F_0}/\pi_0^d)| = |\operatorname{Herm}_{L,M_{2s-1}}(O_{F_0}/\pi_0^d)| \cdot |\operatorname{Herm}_{I_1^{-1},M_{2s}}(O_{F_0}/\pi_0^d)|.$$

Then by Definition 3.4 and Remark 3.5,

$$\begin{aligned}
\operatorname{Den}(M_{n+1}, L^{\#}, q^{-2k}) &= \operatorname{Den}(M_{n+1} \oplus H^k, L^{\#}) \\
&= \lim_{d\to+\infty} \frac{|\operatorname{Herm}_{L^{\#},M_{n+1+2k}}(O_{F_0}/\pi_0^d)|}{q^{d\cdot(n+1)(2(n+1+2k)-n-1)}} \\
&= \lim_{d\to+\infty} \frac{|\operatorname{Herm}_{L,M_{n+2k}}(O_{F_0}/\pi_0^d)| \cdot |\operatorname{Herm}_{I_1^{-1},M_{n+1+2k}}(O_{F_0}/\pi_0^d)|}{q^{d\cdot(n+1)(n+1+4k)}} \\
&= \lim_{d\to+\infty} \frac{|\operatorname{Herm}_{L,M_{n+2k}}(O_{F_0}/\pi_0^d)|}{q^{d\cdot n(2(n+2k)-n)}} \cdot \frac{|\operatorname{Herm}_{I_1^{-1},M_{n+1+2k}}(O_{F_0}/\pi_0^d)|}{q^{d\cdot 1\cdot(2(n+1+2k)-1)}} \\
&= \operatorname{Den}(M_n \oplus H^k, L) \cdot \operatorname{Den}(M_{n+1} \oplus H^k, I_1^{-1}) \\
&= \operatorname{Den}(M_n, L, q^{-2k}) \cdot (1 - q^{-1-n-2k})
\end{aligned}$$

where the last equality $\operatorname{Den}(M_{n+1} \oplus H^k, I_1^{-1}) = 1 - q^{-1-n-2k}$ comes from [LL22, Lemma 2.15].

Thus

$$\begin{aligned}
\operatorname{Den}(M_{n+1}, L^{\#}, X) &= \operatorname{Den}(M_n, L, X) \cdot (1 - q^{-1-n}X) \\
\operatorname{Den}'(L^{\#}) &= \operatorname{Den}'(L) \cdot (1 - q^{-1-n}) + 2\operatorname{Den}(M_n, L, 1) \cdot q^{-1-n} \\
&= (1 - q^{-1-n})\operatorname{Den}'(L). \qquad \text{the second term vanishes as } L \text{ is nonsplit.}
\end{aligned}$$

Divide both sides by $\prod_{i=1}^{r+1}(1-q^{-2i})$. By the computation result from Lemma 4.4, we get

$$\partial\operatorname{Den}(L^{\#}) = \frac{\operatorname{Den}'(L^{\#})}{\operatorname{Den}(M_{n+1}, M_{n+1})} = \frac{(1-q^{-1-n})\operatorname{Den}'(L)}{\prod_{i=1}^{r+1}(1-q^{-2i})} \overset{n+1=2r+2}{=\!=\!=} \frac{\operatorname{Den}'(L)}{\prod_{i=1}^{r}(1-q^{-2i})} = 2\partial\operatorname{Den}(L).$$

□

Chapter 5: Reduction at geometric side

Throughout the chapter we let $n = 2m \geqslant 4$ be even.

5.1 Auxiliary Rapoport–Zink spaces

Following [RSZ18, §9], to describe the relation between $\mathcal{N}_{n-1}$ and $\mathcal{N}_n$ we first introduce some auxiliary spaces.

Definition 5.1. We consider the moduli space $\mathcal{P}_n$ over $\operatorname{Spf} O_{\breve{F}}$, sending $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ to the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ where

- X is a (strict) formal O_{F_0}-module over S of dimension n and relative height $2n$;
- ι is an action of O_F on X extending the O_{F_0}-action;
- λ is an ι-compatible polarization of X;
- $\rho\colon X \times_S \overline{S} \to \mathbb{X}'_n \times_{\bar{k}} \overline{S}$ is an O_F-linear quasi-isogeny of height 0 over $\overline{S}$ such that $\rho^*(\lambda_{\mathbb{X}'_n,\overline{S}}) = \lambda_{\overline{S}}$,

with the framing object $(\mathbb{X}'_n, \iota_{\mathbb{X}'_n}, \lambda_{\mathbb{X}'_n})$ to be specified below. Here we impose on the polarization λ that

$$\ker \lambda \subset X[\iota(\pi)] \text{ is of rank } q^{n-2},$$

and we require the triple (X, ι, λ) to satisfy condition (5.2) below, which is the natural analog of condition (2.13).[1]

[1]In [RSZ18, §9], the triple is also required to satisfy the Wedge condition (2.2). It is shown in [Yu19, Remark 1] that condition (5.2) implies Wedge condition.

As in §2.2, let $M(X)$ and $M(X^\vee)$ denote the respective Lie algebras of the universal vector extensions of X and $X^\vee$. Since $\ker \lambda$ is contained in $X[\iota(\pi)]$ and of rank q^{n-2}, there is a unique (necessarily O_F-linear) isogeny λ' such that the composite

$$X \xrightarrow{\lambda} X^\vee \xrightarrow{\lambda'} X$$

is $\iota(\pi)$, and the induced diagram

$$M(X) \xrightarrow{\lambda_*} M(X^\vee) \xrightarrow{\lambda'_*} M(X)$$

then extends periodically to a polarized chain of $O_F \otimes_{O_{F_0}} \mathcal{O}_S$-modules of type $\Lambda_{\{m-1\}}$. By [RZ96, Theorem 3.16], étale-locally on S there exists an isomorphism of polarized chains

$$[\cdots \xrightarrow{\lambda'_*} M(X) \xrightarrow{\lambda_*} M(X^\vee) \xrightarrow{\lambda'_*} \cdots] \xrightarrow{\sim} \Lambda_{\{m-1\}} \otimes_{O_{F_0}} \mathcal{O}_S,$$

which in particular gives an isomorphism of $O_F \otimes_{O_{F_0}} \mathcal{O}_S$-modules

$$M(X) \xrightarrow{\sim} \Lambda_{-(m-1)} \otimes_{O_{F_0}} \mathcal{O}_S. \tag{5.1}$$

Denoting by $\mathrm{Fil}^1 \subset M(X)$ the covariant Hodge filtration for X, the analog of (2.13) we impose is that

> *upon identifying* Fil^1 *with a submodule of* $\Lambda_{-(m-1)} \otimes_{O_{F_0}} \mathcal{O}_S$ *via* (5.1)*, the line bundle*
>
> $$\textstyle\bigwedge^n_{\mathcal{O}_S} \mathrm{Fil}^1 \subset {}^n\Lambda_{-(m-1)} \otimes_{O_F} \mathcal{O}_S \tag{5.2}$$
>
> *is contained in* $L^{n-1,1}_{-(m-1),-1}(S)$*, cf.* (2.10)*.*

Just as before, condition (5.2) is independent of the above choice of chain isomorphism by Lemma 2.5.

As in §2.3, over $\bar{k}$ there are *two* supersingular isogeny classes of framing objects for this moduli problem, distinguished by the splitness of the hermitian space $\mathbb{V}$ in (2.20). To complete the definition of $\mathcal{P}_n$, we will choose a framing object $\mathbb{X}_n'$ for which $\mathbb{V}(\mathbb{X}_n')$ is nonsplit. We take $\mathbb{X}_n'$ to be the product of the framing object $\mathbb{X}_{n-1} = \mathbb{X}_{n-1}^{(1)}$ (cf. (2.18)) and $\overline{\mathbb{E}}$,

$$\left(\mathbb{X}_n', \iota_{\mathbb{X}_n'}, \lambda_{\mathbb{X}_n'}\right) := \left(\mathbb{X}_{n-1} \times \overline{\mathbb{E}}, \iota_{\mathbb{X}_{n-1}} \times \iota_{\overline{\mathbb{E}}}, \lambda_{\mathbb{X}_{n-1}} \times (-\lambda_{\overline{\mathbb{E}}})\right). \tag{5.3}$$

Since $\mathbb{V}(\mathbb{X}_{n-1})$ is nonsplit and n is even, it follows from definition that $\mathbb{V}(\mathbb{X}_n')$ is indeed the nonsplit space of dimension n. Furthermore, it is easy to see that $\mathbb{X}_n'$ satisfies (5.2) because $\mathbb{X}_{n-1}$ satisfies (2.13).

It follows from the general theory of Rapoport–Zink spaces [RZ96] that $\mathcal{P}_n$ is represented by a formal scheme which is formally locally of finite type and essentially proper over $\mathrm{Spf}\, O_{\breve{F}}$. However it is not regular.

To define the second moduli space we first fix an O_F-linear isogeny of degree q,

$$\phi_0 \colon \mathbb{X}_n \longrightarrow \mathbb{X}_n'$$

such that $\phi_0^*(\lambda_{\mathbb{X}_n'}) = \lambda_{\mathbb{X}_n}$. Since $n \geqslant 4$, we have

$$\mathbb{X}_n = \mathbb{X}_n' = \mathbb{X}_{n-2} \times \overline{\mathbb{E}} \times \overline{\mathbb{E}}$$

as O_F-modules, and we define

$$\phi_0 := \mathrm{id}_{\mathbb{X}_{n-2}} \times \begin{pmatrix} 1 & \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} \\ 1 & -\frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} \end{pmatrix}. \tag{5.4}$$

It is easy to check that $\phi_0^*(\lambda_{\mathbb{X}_n'}) = \lambda_{\mathbb{X}_n}$.

Definition 5.2. We consider the moduli space $\mathcal{P}'_n$ for tuples

$$(X, \iota, \lambda, \rho, X', \iota', \lambda', \rho', \phi\colon X \to X')$$

where $(X, \iota, \lambda, \rho)$ is a point on $\mathcal{N}_n$, $(X', \iota', \lambda', \rho')$ is a point on $\mathcal{P}_n$, and ϕ is an O_F-linear isogeny of degree q lifting ϕ_0 in the sense that the following diagram commutes

$$\begin{tikzcd} X_{\overline{S}} \arrow[r, "\bar{\phi}"] \arrow[d, "\rho"'] & X'_{\overline{S}} \arrow[d, "\rho'"] \\ \mathbb{X}_{n,\overline{S}} \arrow[r, "\phi_0"] & \mathbb{X}'_{n,\overline{S}}. \end{tikzcd}$$

The notion of isomorphism between tuples as above is the obvious one.

By definition, there are tautological projection maps

$$\begin{tikzcd} & \mathcal{P}'_n \arrow[dl] \arrow[dr, "\varphi"] & \\ \mathcal{P}_n & & \mathcal{N}_n. \end{tikzcd}$$

By Proposition 2.4, $\mathcal{N}_n$ decomposes into a disjoint union $\mathcal{N}_n = \mathcal{N}_n^+ \amalg \mathcal{N}_n^-$. Pulling back along φ, we obtain a decomposition

$$\mathcal{P}'_n = (\mathcal{P}'_n)^+ \amalg (\mathcal{P}'_n)^-. \tag{5.5}$$

Theorem 5.3 ([RSZ18, Theorem 9.3])**.** *Writing $(\mathcal{P}'_n)^\pm$ for either of the summands in (5.5), the projection $\mathcal{P}'_n \to \mathcal{P}_n$ induces an isomorphism*

$$(\mathcal{P}'_n)^\pm \xrightarrow{\sim} \mathcal{P}_n.$$

We denote by $\psi^\pm\colon \mathcal{P}_n \xrightarrow{\sim} (\mathcal{P}'_n)^\pm$ the inverse isomorphism.

There is a natural closed embedding

$$\widetilde{\delta}\colon \mathcal{N}_{n-1} \longrightarrow \mathcal{P}_n$$

sending a quadruple $(X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S)$ for $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ to

$$(X' \times \overline{\mathcal{E}}_S, \iota' \times \iota_{\overline{\mathcal{E}}}, \lambda' \times (-\lambda_{\overline{\mathcal{E}}}), \rho' \times \rho_{\overline{\mathcal{E}}}) \in \mathcal{P}_n(S).$$

It is straightforward to verify that $\widetilde{\delta}$ is well-defined.

Consider the composite morphism

$$\delta^{\pm}\colon \mathcal{N}_{n-1} \xrightarrow{\widetilde{\delta}} \mathcal{P}_n \xrightarrow{\psi^{\pm}} (\mathcal{P}'_n)^{\pm} \xrightarrow{\varphi} \mathcal{N}_n. \tag{5.6}$$

Explicitly, the morphism $\delta^{\pm}$ sends $(X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S)$ for $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ to $(X, \iota, \lambda, \rho) \in \mathcal{N}_n^{\pm}(S)$ such that there exists an O_F-isogeny $\phi\colon X \to X' \times \overline{\mathcal{E}}_S$ of degree q lifting ϕ_0.

Proposition 5.4 ([RSZ18, Proposition 12.1])**.** *The morphism $\delta^{\pm}$ is a closed embedding.*

The isogeny $\phi_0\colon \mathbb{X}_n \to \mathbb{X}'_n = \mathbb{X}_{n-1} \times \overline{\mathbb{E}}$ gives an isomorphism of hermitian spaces

$$\phi_0\colon \mathbb{V}_n \xrightarrow{\sim} \mathbb{V}_{n-1} \oplus \langle f \rangle_F \tag{5.7}$$

where $f = \mathrm{id}_{\overline{\mathbb{E}}}$ has hermitian norm -1. This allows us to have an embedding of hermitian spaces $\mathbb{V}_{n-1} \longrightarrow \mathbb{V}_n$. Consider $u = (0_{n-2}, \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2}, -1) \in \mathbb{V}_n$ which is of hermitian norm π^2. Then under ϕ_0, u is mapped to πf.

The first main result of the section is the following theorem.

Theorem 5.5. *The closed embedding $\delta^{\pm}$ identifies $\mathcal{N}_{n-1}$ with $\mathcal{Z}_n(u) \cap \mathcal{N}_n^{\pm}$.*

For the rest of the section we will state everything for the morphism δ^+, although the same argument applies to δ^- as well. For simplicity we set $\mathcal{Z}_n(u)^+ := \mathcal{Z}_n(u) \cap \mathcal{N}_n^+$.

5.2 Description of Dieudonné modules

Before going into the proof of Theorem 5.5, we need to do some preparations. Let $r \geqslant 1$ be any integer. Firstly we give a description of $\bar{k}$-points on $\mathcal{N}_r$ using Dieudonné theory.

Let $\mathbb{N}_r, \overline{\mathbb{N}}_1$ denote the covariant rational Dieudonné modules of $\mathbb{X}_r$ and $\overline{\mathbb{E}}$ respectively. Let F and V be the Frobenius operator and Verschiebung on $\mathbb{N}_r$. The polarization $\lambda_{\mathbb{X}_r}$ induces a nondegenerate alternating $\breve{F}_0$-bilinear form $\langle -, - \rangle$ on $\mathbb{N}_r$ satisfying

$$\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma \text{ for all } x, y \in \mathbb{N}_r$$

where σ denotes the extension of Frobenius operator on $W_{O_{F_0}}(\bar{k}) = O_{\breve{F}_0}$ to $\breve{F}_0$. The action $\iota_{\mathbb{X}_r}$ makes $\mathbb{N}_r$ into an $\breve{F}$-vector space such that the $\breve{F}$-action commutes with F and V and

$$\langle ax, y \rangle = \langle x, \bar{a}y \rangle \text{ for all } x, y \in \mathbb{N}_r, a \in \breve{F}.$$

The form

$$h(x, y) \coloneqq \langle \pi x, y \rangle + \pi \langle x, y \rangle, x, y \in \mathbb{N}_r$$

then makes $\mathbb{N}_r$ into an $\breve{F}/\breve{F}_0$-hermitian space of dimension r. Similarly $\overline{\mathbb{N}}_1$ is made into an $\breve{F}/\breve{F}_0$-hermitian space.

By Dieudonné theory, for a perfect extension K of $\bar{k}$, the set of K-points on $\mathcal{N}_r$ identifies with the set of $O_{\breve{F}} \otimes_{O_{\breve{F}_0}} W_{O_{F_0}}(K)$-lattices $M \subset \mathbb{N}_r \otimes_{O_{\breve{F}_0}} W_{O_{F_0}}(K)$ such that:

- if r is even, then M is π-modular, and we have

$$\pi_0 M \subset VM \subset M, VM \subset^1 VM + \pi M; \tag{5.8}$$

- if r is odd, then M is almost π-modular, and we have

$$\pi_0 M \subset VM \subset M, VM \subset^{\leqslant 1} VM + \pi M. \tag{5.9}$$

In what follows, for simplicity, we only state our results for $K = \bar{k}$, although all results hold for general perfect extension K of $\bar{k}$ as well.

Note that if r is even and M is π-modular, then $M \simeq \pi H^{r/2}$. In this case we call an $O_{\breve{F}}$-basis $e_1, f_1, \ldots, e_{r/2}, f_{r/2}$ a *standard normal basis* of M if $\pi^{-1}e_1, \pi^{-1}f_1, \ldots, \pi^{-1}e_{r/2}, \pi^{-1}f_{r/2}$ is a standard normal basis of $H^{r/2}$ as in Definition 4.1; if r is odd and M is almost π-modular, then $M \simeq \pi H^{(r-1)/2} \oplus I_1^1$. In this case we call an $O_{\breve{F}}$-basis $e_1, f_1, \ldots, e_{(r-1)/2}, f_{(r-1)/2}, \varphi$ a *standard normal basis* of M if $e_1, f_1, \ldots, e_{(r-1)/2}, f_{(r-1)/2}$ is a standard normal basis of $\pi H^{(r-1)/2}$ and φ is an $O_{\breve{F}}$-basis of I_1^1 with hermitian norm 1 and orthogonal to all other e_i, f_j.

Let $x \in \mathbb{V}_r$ and set x^* to be the quasi-morphism

$$\mathbb{X}_r \xrightarrow{\lambda_{\mathbb{X}_r}} \mathbb{X}_r^\vee \xrightarrow{x^\vee} \overline{\mathbb{E}}^\vee \xrightarrow{\lambda_{\overline{\mathbb{E}}}^{-1}} \overline{\mathbb{E}}.$$

Then x^* is the adjoint of x with respect to the polarizations, and

$$\langle x\alpha, \beta\rangle = \langle \alpha, x^*\beta\rangle \text{ for any } \alpha \in \overline{\mathbb{N}}_1 \text{ and } \beta \in \mathbb{N}_r.$$

From this we can relate the hermitian form on $\mathbb{N}_r$ and $\mathbb{V}_r$. If $x, y \in \mathbb{V}_r$ then

$$h(xe, ye) = \delta \cdot h(x, y). \tag{5.10}$$

Lemma 5.6. *Let $x, y \in \mathbb{N}_r$. Then we have*

(1) $h(Vx, Vy) = 0$ if and only if $h(Fx, Fy) = 0$, if and only if $h(x, y) = 0$;

(2) $h(Vx, Vy) \in \pi_0^a O_{\breve{F}}$ if and only if $h(Fx, Fy) \in \pi_0^a O_{\breve{F}}$, if and only if $h(x, y) \in \pi_0^{a-1} O_{\breve{F}}$.

Proof. We have

$$
\begin{aligned}
h(Vx,Vy) &= \langle \pi Vx, Vy\rangle + \pi\langle Vx,Vy\rangle = \langle F\pi Vx, y\rangle^{\sigma^{-1}} + \pi\langle FVx, y\rangle^{\sigma^{-1}} \\
&= \langle \pi\pi_0 x, y\rangle^{\sigma^{-1}} + \pi\langle \pi_0 x, y\rangle^{\sigma^{-1}} = \pi_0\langle \pi x, y\rangle^{\sigma^{-1}} + \pi_0\pi\langle x, y\rangle^{\sigma^{-1}}, \\
h(Fx,Fy) &= \langle \pi Fx, Fy\rangle + \pi\langle Fx,Fy\rangle = \langle \pi x, VFy\rangle^{\sigma} + \pi\langle x, VFy\rangle^{\sigma} \\
&= \langle \pi x, \pi_0 y\rangle^{\sigma} + \pi\langle x, \pi_0 y\rangle^{\sigma} = \pi_0\langle \pi x, y\rangle^{\sigma} + \pi_0\pi\langle x, y\rangle^{\sigma}, \\
h(x,y) &= \langle \pi x, y\rangle + \pi\langle x,y\rangle.
\end{aligned}
$$

For (1), $h(Vx,Vy) = 0$, $h(Fx,Fy) = 0$ and $h(x,y) = 0$ are all equivalent to $\langle \pi x, y\rangle = \langle x, y\rangle = 0$.

For (2), $h(Vx,Vy) \in \pi_0^a O_{\breve{F}}$, $h(Fx,Fy) \in \pi_0^a O_{\breve{F}}$ and $h(x,y) \in \pi_0^{a-1} O_{\breve{F}}$ are all equivalent to $\langle \pi x, y\rangle$ and $\langle x, y\rangle$ lie in $\pi_0^{a-1} O_{\breve{F}}$. □

Corollary 5.7. *Let $L \subset \mathbb{N}_r$ be a hermitian lattice. Then we have*

(1) $\pi L^* = (\pi^{-1}L)^*$*;*

(2) $VL^* = \pi_0 (VL)^*$.

Proof. By definition,

$$
\begin{aligned}
(\pi^{-1}L)^* &= \{x \in \mathbb{N}_r \mid h(x, \pi^{-1}y) \in O_{\breve{F}} \text{ for any } y \in L\} \\
&= \{x \in \mathbb{N}_r \mid h(\pi^{-1}x, y) \in O_{\breve{F}} \text{ for any } y \in L\} \\
&= \{x \in \mathbb{N}_r \mid \pi^{-1}x \in L^*\} \\
&= \pi L^*,
\end{aligned}
$$

$$\begin{aligned}
(VL)^* &= \{x \in \mathbb{N}_r \mid h(x, Vy) \in O_{\breve{F}} \text{ for any } y \in L\} \\
&= \{x \in \mathbb{N}_r \mid h(V^{-1}x, y) \in \pi_0^{-1} O_{\breve{F}} \text{ for any } y \in L\} \qquad \text{by Lemma 5.6} \\
&= \{x \in \mathbb{N}_r \mid h(\pi_0 V^{-1}x, y) \in O_{\breve{F}} \text{ for any } y \in L\} \\
&= \pi_0^{-1} V L^*.
\end{aligned}$$

□

We have the following detailed description of the Dieudonné module and Hodge filtration of $\overline{\mathbb{E}}$.

Lemma 5.8. *The Dieudonné module $\overline{\mathbb{M}}_0$ of $\overline{\mathbb{E}}$ can be identified with $W_{O_{F_0}}(\bar{k})^2 = O_{\breve{F}_0}^2$ endowed with the Frobenius and Verschiebung operator given in matrix form by*

$$F = \begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix} \sigma\,, V = \begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix} \sigma^{-1}. \tag{5.11}$$

Moreover we can find an $O_{\breve{F}_0}$-basis e, f of $\overline{\mathbb{M}}_0$ such that

(1) $f = -\pi e$ and $\overline{\mathbb{M}}_0 = O_{\breve{F}} e$;

(2) $Ve = Fe = f, Vf = Ff = \pi_0 e$;

(3) The Hodge filtration of $\overline{\mathbb{E}}$ is given by

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Fil}^1 \arrow[r] \arrow[d, equal] & \mathbb{D}(\overline{\mathbb{E}}) \arrow[r] \arrow[d, equal] & \mathrm{Lie}\,\overline{\mathbb{E}} \arrow[r] \arrow[d, equal] & 0. \\ & \langle f \rangle_{\bar{k}} & \langle e, f \rangle_{\bar{k}} & \langle e \rangle_{\bar{k}} & \end{tikzcd} \tag{5.12}$$

Proof. The description of the Dieudonné module is the same as $\mathbb{E}$, so we get the matrix form of F and V as in (2.14). Take

$$e = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \text{ and } f = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right).$$

Note that the action of π on $\overline{\mathbb{M}}_0$ is given by $\left(\begin{smallmatrix} 0 & -\pi_0 \\ -1 & 0 \end{smallmatrix}\right)$ as we twist the O_F-action by Galois conjugation. All the other results follow directly from computation. □

Lemma 5.9. *Let $z \in \mathcal{N}_n(\bar{k})$ be any point and M its Dieudonné module. Then we can always find a standard normal basis $\sigma_1, \tau_1, \dots, \sigma_m, \tau_m$ of M with $\sigma_1 \in VM$.*

Proof. The Dieudonné module M is an $O_{\breve{F}}$-lattice in $\mathbb{N}_n$ such that $M^* = \pi^{-1}M$ and $\pi_0 M \subset VM \subset M$, $VM \subset^1 VM + \pi M$. Since the isocrystal $\mathbb{N}_n$ is supersingular, the lattices M and $\pi^{-1}VM$ have the same co-length in any lattice that contains them both. The condition $VM \subset^1 VM + \pi M$ then implies that $VM \cap \pi M \subset^1 VM$.

Pick any $e \in VM$ and $e \notin VM \cap \pi M$. Write $e = Ve'$ for some $e' \in M$. Write $n = 2m$. As M is π-modular, $M \simeq \pi H^m$. As we assume $e \notin \pi M$, we have $e' \notin \pi M$. Thus we can apply Lemma 4.2[2] to see that the orthogonal complement of e' in M is just

$$M(e') = \pi H^{m-1} \oplus I$$

where $I = \langle e'' \rangle_{O_{\breve{F}}}$ with $h(e'', e'') = -h(e', e')$ and $e'' - e' \in \pi M$. Now consider the element $e + Ve'' = Ve' + Ve'' \in VM$. It is isotropic by Lemma 5.6 as

$$h(e' + e'', e' + e'') = h(e', e') + h(e'', e'') = 0.$$

Moreover $Ve' + Ve'' \notin \pi M$ as we already have $Ve'' - Ve' \in V\pi M \subset \pi M$ and $e = Ve' \notin \pi M$. Thus we may just take e at the beginning to be isotropic.

As $e \notin \pi M$, by Lemma 4.2 again we see that the orthogonal complement of e in M contains πH^{m-1}. Choose a standard normal basis $\sigma_2, \tau_2, \dots, \sigma_m, \tau_m$ for this πH^{m-1} and extend it to a standard normal basis $\sigma_1, \tau_1, \sigma_2, \tau_2, \dots, \sigma_m, \tau_m$ of M. Then $e = a\sigma_1 + b\tau_1$. As $e \notin \pi M$ we may assume $a \in O_{\breve{F}}^\times$ and by normalizing σ_1, τ_1, we assume $a = 1$. We may replace σ_1 by e and still get a standard normal basis of M, with $\sigma_1 = e \in VM$. $\square$

Using the basis provided in Lemma 5.9, we have the following detailed description of the Dieudonné module and Hodge filtration of $\bar{k}$-points on $\mathcal{N}_n$. This Lemma will be used repeatedly

[2] To be precise, apply to the orthogonal complement of $\pi^{-1}e'$ in π^{-1}M first, then multiply the complement by π. The same remark applies to all following cases when applying Lemma 4.2 or Corollary 4.3 to π-modular lattices.

when showing surjectivity of morphisms between RZ spaces on $\bar{k}$-points.

Lemma 5.10. *Let $z = (X_z, \iota_z, \lambda_z, \rho_z) \in \mathcal{N}_n(\bar{k})$ be any point. Let M be the Dieudonné module of X_z and $\mathrm{Fil}^1_z \subset \mathbb{D}_z$ be the Hodge Filtration of X_z. Suppose we have a standard normal basis $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m$ of M (or M^*) such that $\sigma_1 \in VM$ (resp. $\sigma_1 \in VM^*$). Then*

(1) VM (resp. VM^*) is the $O_{\breve{F}}$-span of

$$\sigma_1, \pi\sigma_2, \ldots, \pi\sigma_m, \pi_0\tau_1, \pi\tau_2, \pi\tau_3, \ldots, \pi\tau_m;$$

(2) Fil^1_z is the $\bar{k}$-span of

$$\sigma_1, \pi\sigma_1, \pi\sigma_2, \ldots, \pi\sigma_m, \pi\tau_2, \pi\tau_3, \ldots, \pi\tau_m;$$

(3) $\mathrm{Lie}\, X_z$ (resp. $\mathrm{Lie}\, X_z^\vee$) is the $\bar{k}$-span of

$$\sigma_2, \sigma_3, \ldots, \sigma_m, \tau_1, \pi\tau_1, \tau_2, \tau_3, \ldots, \tau_m.$$

Proof. Write $n = 2m$. The Dieudonné module M is an $O_{\breve{F}}$-lattice in $\mathbb{N}_n$ such that $M^* = \pi^{-1}M$ and $\pi_0 M \subset VM \subset M$, $VM \subset^1 VM + \pi M$. We first assume $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m$ is a standard normal basis of M.

As λ_z is π-modular, there exists a unique morphism $\sigma_z \colon X_z \to X_z^\vee$ such that $\lambda_z = \sigma_z \iota_z(\pi)$. The morphism σ_z is an isomorphism and symmetric in the sense that $\sigma_z^\vee = \sigma_z$. This symmetrization gives a perfect symmetric pairing on the Dieudonné module M (viewed as a rank $2n = 4m$ $O_{\breve{F}_0}$-lattice)

$$(-,-)\colon M \times M \longrightarrow O_{\breve{F}_0}$$

and under the $O_{\breve{F}_0}$-basis

$$\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_m, \pi\sigma_1, \ldots, \pi\sigma_m, \pi\tau_1, \ldots, \pi\tau_m$$

the matrix of the symmetric pairing is of the form

$$\begin{pmatrix} 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ -I_m & 0 & 0 & 0 \end{pmatrix}. \tag{5.13}$$

Now the condition $\pi_0 M \subset VM$ implies

$$\pi_0\sigma_1, \dots, \pi_0\sigma_m, \pi_0\tau_1, \dots, \pi_0\tau_m \in VM$$

and by our assumption $\sigma_1 \in VM$.

Moreover the point z also gives a Hodge filtration of its Dieudonné crystal as $\bar{k}$-vector spaces

$$0 \longrightarrow \mathrm{Fil}^1_z \longrightarrow \mathbb{D}_z \longrightarrow \mathrm{Lie}\, X_z \longrightarrow 0$$

where $\mathbb{D}_z = M/\pi_0 M$ and $\mathrm{Fil}^1_z = VM/\pi_0 M$. The induced perfect pairing

$$(-,-)\colon \mathbb{D}_z \times \mathbb{D}_z \longrightarrow \bar{k}$$

makes Fil^1_z a maximal totally isotropic subspace. From this we see that, as we already have $\sigma_1 \in VM$ and $(\sigma_1, \pi\tau_1) = -1$, we have $\pi\tau_1 \notin VM$. Now the condition

$$VM \subset^1 VM + \pi M$$

would imply that $\pi\sigma_1, \pi\sigma_2, \dots, \pi\sigma_m, \pi\tau_2, \pi\tau_3, \dots, \pi\tau_m \in VM$. In particular we have

$$V_n \coloneqq \langle \sigma_1, \pi\sigma_2, \dots, \pi\sigma_m, \pi\tau_2, \pi\tau_3, \dots, \pi\tau_m, \pi_0\tau_1 \rangle_{O_{\breve{F}}} \subset VM. \tag{5.14}$$

Since the isocrystal $\mathbb{N}_n$ is supersingular, the co-length of VM in M is n. As the co-length of V_n in

M is already n, the inclusion (5.14) is an equality. Consequently

$$\mathrm{Fil}^1_z = \langle \sigma_1, \pi\sigma_1, \pi\sigma_2, \ldots, \pi\sigma_m, \pi\tau_2, \pi\tau_3, \ldots, \pi\tau_m\rangle_{\bar{k}}$$

and

$$\mathrm{Lie}\, X_z = M/VM = \langle \sigma_2, \sigma_3, \ldots, \sigma_m, \tau_1, \pi\tau_1, \tau_2, \tau_3, \ldots, \tau_m\rangle_{\bar{k}}.$$

Now suppose $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m$ is a standard normal basis of $M^* = \pi^{-1}M$. Then $\sigma_i' := \pi\sigma_i, \tau_j' := \pi\tau_j$ is a standard normal basis of M. Thus we have

$$\begin{aligned} VM^* = V(\pi^{-1}M) = \pi^{-1}VM &= \pi^{-1}\langle \sigma_1', \pi\sigma_2', \ldots, \pi\sigma_m', \pi_0\tau_1', \pi\tau_2', \pi\tau_3', \ldots, \pi\tau_m'\rangle_{O_{\breve{F}}} \\ &= \langle \sigma_1, \pi\sigma_2, \ldots, \pi\sigma_m, \pi_0\tau_1, \pi\tau_2, \pi\tau_3, \ldots, \pi\tau_m\rangle_{O_{\breve{F}}} \end{aligned}$$

and

$$\mathrm{Lie}\, X_z^\vee = M^*/VM^* = \langle \sigma_2, \sigma_3, \ldots, \sigma_m, \tau_1, \pi\tau_1, \tau_2, \tau_3, \ldots, \tau_m\rangle_{\bar{k}}.$$

□

Next we prove a lemma, which shows that $\mathcal{Z}_n(u)$ is formally smooth of relative dimension $n-2$ over $\mathrm{Spf}\, O_{\breve{F}}$.

Lemma 5.11. *Let $z \in \mathcal{N}_n(\bar{k})$ be any point. If there is nonzero $x \in \mathbb{V}_n$ such that $z \in \mathcal{Z}_n(x)(\bar{k})$ and $z \notin \mathcal{Z}_n(\pi^{-1}x)(\bar{k})$ then the tangent space of $\mathcal{Z}_n(x)$ at z is of dimension $n-2$.*

Proof. We shall first recall the computations in [PR09, §5.c]. To be precise, let V be an F-vector space of dimension n and let $h\colon V \times V \to F$ be a split F/F_0-hermitian form. Choose a basis $e_1, \ldots, e_n$ of V such that

$$h(e_i, e_{n+1-j}) = \delta_{i,j}$$

and for $i = 0, \ldots, n-1$ set

$$\Lambda_i := \langle \pi^{-1}e_1, \ldots, \pi^{-1}e_i, e_{i+1}, \ldots, e_n\rangle_{O_F}.$$

Attached to the hermitian form h we also have two F_0-bilinear form

$$(x, y) := \frac{1}{2}\operatorname{Tr}_{F/F_0}(h(x, y)) \quad \text{and} \quad \langle x, y\rangle := \frac{1}{2}\operatorname{Tr}_{F/F_0}(\pi^{-1}h(x, y))$$

so the form $(\ ,\ )$ is symmetric and $\langle\ ,\ \rangle$ is alternating.

Now Λ_m is equipped with the perfect symmetric pairing $(\ ,\ )$. Let $f_1 = -\pi^{-1}e_1, \ldots, f_m = -\pi^{-1}e_m$ and $f_{m+1} = e_{m+1}, \ldots, f_n = e_n$. Consider the O_{F_0}-basis of Λ_m given by

$$f_1, \ldots, f_n, -\pi f_1, \ldots, -\pi f_m, \pi f_{m+1}, \ldots, \pi f_n.$$

Reorder the basis, set

$$\Lambda' := \langle f_1, \pi f_1, \pi f_2, \pi f_3, \ldots, \pi f_{n-1}\rangle_{O_{F_0}},$$
$$\Lambda'' := \langle f_n, \pi f_n, f_2, f_3, \ldots, f_{n-1}\rangle_{O_{F_0}}.$$

Then $\Lambda_m = \Lambda' \oplus \Lambda''$.

Let M be the Dieudonné module of $z = (X_z, \iota_z, \lambda_z, \rho_z)$. By Lemma 5.9 and Lemma 5.10 we see that there is a standard normal basis $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m$ of M with $\sigma_1 \in VM$ and symmetric pairing matrix (5.13). Then we have an isomorphism of $O_{\breve{F}}$-lattices preserving perfect symmetric pairing

$$\Lambda_m \otimes_{O_{F_0}} O_{\breve{F}_0} \simeq M \tag{5.15}$$

sending $f_i \otimes 1$ to σ_i for $1 \leqslant i \leqslant m$ and $f_{n+1-j} \otimes 1$ to $-\tau_j$ for $1 \leqslant j \leqslant m$. Under this isomorphism, the Hodge filtration

$$\mathrm{Fil}^1_z = \langle \sigma_1, \pi\sigma_1, \pi\sigma_2, \ldots, \pi\sigma_m, \pi\tau_2, \pi\tau_3, \ldots, \pi\tau_m\rangle_{\bar{k}} \subset \mathbb{D}_z$$

is identified with

$$\mathcal{F}_1 \coloneqq \Lambda' \otimes_{O_{F_0}} \bar{k} = \langle f_1, \pi f_1, \pi f_2, \pi f_3, \ldots, \pi f_{n-1} \rangle_{\bar{k}} \subset \Lambda_m \otimes_{O_{F_0}} \bar{k}.$$

Now we want to look at the tangent space of $\mathcal{N}_n$ at z, which can be identified with $\mathcal{N}_{n,z}(\bar{k}[\varepsilon])$ where $\bar{k}[\varepsilon] = \bar{k}[x]/(x^2)$.

Any lift z' of z to $\bar{k}[\varepsilon]$ gives rise to a Hodge filtration $\mathcal{F} = \mathrm{Fil}^1_{z'} \subset \Lambda_m \otimes_{O_{F_0}} \bar{k}[\varepsilon]$ lifting $\mathcal{F}_1$. We may find a linear map $X \colon \Lambda'_{\bar{k}[\varepsilon]} \to \Lambda''_{\bar{k}[\varepsilon]}$ such that $\mathcal{F} = \{v + X \cdot v \mid v \in \Lambda'_{\bar{k}[\varepsilon]}\}$. As we want $\mathcal{F}$ to be a lift of $\mathcal{F}_1$, $X \in \varepsilon M_n(\bar{k})$.

The matrix of the symmetric form on Λ_m under the basis

$$f_1, \pi f_1, \pi f_2, \pi f_3, \ldots, \pi f_{n-1}, f_n, \pi f_n, f_2, f_3, \ldots, f_{n-1}$$

is of the form

$$\begin{pmatrix} 0 & S \\ S^t & 0 \end{pmatrix}$$

where S is the skew matrix of size n,

$$S = \begin{pmatrix} J_2^t & 0 \\ 0 & J_{n-2} \end{pmatrix}, J_{2s} = \begin{pmatrix} 0 & -H_s \\ H_s & 0 \end{pmatrix}$$

where H_s is the unit antidiagonal matrix of size s.

We may write the map X as a matrix

$$X = \begin{pmatrix} T & B \\ C & Y \end{pmatrix}$$

where each block has size 2×2 for T, $2 \times (n-2)$ for B, $(n-2) \times 2$ for C and $(n-2) \times (n-2)$

for Y. Then the condition that $\mathcal{F}$ is isotropic translates into

$$SX^t = XS$$

which is just $J_2 \cdot T^t = T \cdot J_2$, $J_{n-2} \cdot Y^t = Y \cdot J_{n-2}$ and $C \cdot J_2^t = J_{n-2} \cdot B^t$. The first condition implies that T is diagonal, $T = \mathrm{diag}(x, x)$, the last condition shows that C is determined by B via $C = J_{n-2} B^t J_2$.

The condition that $\mathcal{F}$ is π-stable translates to[3]

$$Y^2 = \pi_0 \cdot I_{n-2} \, , B_2 = -B_1 \cdot Y.$$

As we have signature $(1, n-1)$ in our setting, and $\pi = 0$ in $\bar{k}[\varepsilon]$, the Wedge condition implies

$$Y = \sqrt{\pi_0} \cdot I_{n-2} = 0.$$

So the matrix X looks like

$$X = \begin{pmatrix} x & 0 & b_1 & b_2 & \dots & b_{n-2} \\ 0 & x & 0 & 0 & \dots & 0 \\ 0 & b_{n-2} & 0 & 0 & \dots & 0 \\ & \dots & \dots & & \dots & \\ 0 & b_{(n-2)/2} & 0 & 0 & \dots & 0 \\ 0 & -b_{(n-2)/2-1} & 0 & 0 & \dots & 0 \\ & \dots & \dots & & \dots & \\ 0 & -b_1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

[3]The computation from [PR09, §5.c] says $B_1 = B_2 \cdot Y$, which is incorrect. The correct computation shows that it should be $B_2 = -B_1 \cdot Y$, as can be shown from the description below.

Thus $\mathcal{F}$ is given by $\bar{k}[\varepsilon]$-span of

$$
\begin{aligned}
t_1 &= f_1 + x f_n + b_1 f_2 + \cdots + b_{n-2} f_{n-1}, \\
t_2 &= \pi f_1 + x\pi f_n, \\
t_3 &= \pi f_2 + b_{n-2}\pi f_n, \\
&\cdots \\
t_n &= \pi f_{n-1} - b_1 \pi f_n
\end{aligned}
$$

with all $x, b_1, \ldots, b_{n-2} \in \varepsilon \cdot \bar{k}$. One checks easily that

$$
\pi t_1 = t_2 + b_1 t_3 + b_2 t_4 + \cdots + b_{n-2} t_n
$$

and $\pi t_i = 0$ for $i = 2, \ldots, n$ (so indeed $\mathcal{F}$ is π-stable). In particular this implies that the tangent space of $\mathcal{N}_n$ at z is of $\bar{k}$-dimension $n-1$.

Now suppose $x \in \mathbb{V}_n$ such that $z \in \mathcal{Z}_n(x)(\bar{k})$ but $z \notin \mathcal{Z}_n(\pi^{-1}x)(\bar{k})$. This means we have a morphism of p-divisible groups

$$
x\colon \overline{\mathbb{E}} \longrightarrow X_z
$$

where X_z is the p-divisible group corresponding to the $\bar{k}$-point z. Write the Hodge filtration of $\overline{\mathbb{E}}$ as

$$
0 \longrightarrow \mathrm{Fil}^1 \longrightarrow \mathbb{D}(\overline{\mathbb{E}}) \longrightarrow \mathrm{Lie}\,\overline{\mathbb{E}} \longrightarrow 0
$$

where Fil^1 is a 1-dimension $\bar{k}$-vector space. Choose a $\bar{k}$-basis f of Fil^1 as in Lemma 5.8 and consider its image in $\mathbb{D}(X_z) \simeq \Lambda_{m,\bar{k}}$ under the map $\mathbb{D}(x)$. As we assume $z \in \mathcal{Z}_n(x)(\bar{k})$, the image lies in $\mathcal{F}_1$. The map $\mathbb{D}(x)$ also commutes with π-action and $\pi \cdot f = -\pi_0 e = 0$. So we may write its image as

$$
a_1 \pi f_1 + a_2 \pi f_2 + \cdots + a_{n-1} \pi f_{n-1} \in \mathcal{F}_1.
$$

Similarly we can write the image of f in $\mathbb{D}(X_z)_{\bar{k}[\varepsilon]}$ under the map $\mathbb{D}(x)_{\bar{k}[\varepsilon]}$ as

$$f' \coloneqq a_1\pi f_1 + \cdots + a_{n-1}\pi f_{n-1} + \varepsilon(\mu_1\pi f_1 + \cdots + \mu_n\pi f_n).$$

Then a lift z' of z still lies in $\mathcal{Z}_n(x)(\bar{k}[\varepsilon])$ is equivalent to $f' \in \mathcal{F}$. We can first use $\varepsilon \cdot t_i = \varepsilon\pi f_{i-1}$ for $i = 2, \ldots, n$ to get rid of $\varepsilon\mu_i\pi f_i$ for $i = 1, \ldots, n-1$, then the only obstruction for $f' \in \mathcal{F}$ is to require

$$a_1 x + a_2 b_{n-2} + \cdots + a_{\frac{n+2}{2}} b_{\frac{n-2}{2}} - a_{\frac{n+2}{2}+1} b_{\frac{n-2}{2}-1} - \cdots - a_{n-1}b_1 = \mu_n.$$

In particular if not all the a_i are zero, then the tangent space of $\mathcal{Z}_n(x)$ at z is of dimension $n - 1 - 1 = n - 2$.

Suppose all the a_i are zero, which means the image of f in $\mathcal{F}_1$ is already zero. Follow the notation in Lemma 5.8 we let $\overline{\mathbb{M}}_0$ be the Dieudonné module of $\overline{\mathbb{E}}$. The morphism $x \colon \overline{\mathbb{E}} \to X_z$ gives rise to a morphism of Dieudonné modules $x \colon \overline{\mathbb{M}}_0 = O_{\breve{F}}e \to M$. The vanishing of all the a_i is equivalent to say that

$$xV\overline{\mathbb{M}}_0 \subset VM \cap \pi_0 M = \pi_0 M.$$

As $\overline{\mathbb{M}}_0 = \langle e, f\rangle_{O_{\breve{F}_0}}$ and explicitly $Ve = f$ and $Vf = \pi_0 e$, we have

$$xf \in \pi_0 M \text{ and } x\pi_0 e \in \pi_0 M.$$

But also we have $\pi e = -f$ and so from $xf \in \pi_0 M$ we get

$$\pi^{-1}xf \in \pi M \subset M \text{ and } x\pi e \in \pi_0 M \text{ , } \pi^{-1}xe \in M.$$

In particular we show that $\pi^{-1}x$ is also a morphism of Dieudonné modules $\overline{\mathbb{M}}_0 \to M$ hence z is in $\mathcal{Z}_n(\pi^{-1}x)(\bar{k})$, contradiction! Thus we can not have all the a_i to be zero. The Lemma is proved. $\square$

Corollary 5.12. *If $x \in \mathbb{V}_n$ has hermitian norm $h(x,x) \in \pi^2 O_{F_0}^{\times}$ then $\mathcal{Z}_n(x)$ is formally smooth of relative dimension $n-2$ over* $\operatorname{Spf} O_{\breve{F}}$.

Proof. If $h(x,x) \in \pi^2 O_{F_0}^\times$ then $h(\pi^{-1}x, \pi^{-1}x) \in O_{F_0}^\times$ and $\mathcal{Z}_n(\pi^{-1}x)$ is empty. By Lemma 5.11 we see that the tangent space of $\mathcal{Z}_n(x)$ at every geometric point has dimension $n-2$. As $\mathcal{Z}_n(x)$ is a relative divisor in $\mathcal{N}_n$ ([LL22, Lemma 2.40]) we conclude that $\mathcal{Z}_n(x)$ is formally smooth of relative dimension $n-2$ over $\mathrm{Spf}\, O_{\breve{F}}$. □

5.3 Proof of Theorem 5.5

Now let us prove Theorem 5.5. We shall consider δ^+. All arguments apply to δ^- as well. The structure of the proof is as follows:

(1) We first show δ^+ factors through $\mathcal{Z}_n(u)^+$.

(2) We then show that δ^+ is surjective.

(3) We finally show that δ^+ is formally étale, and deduce that it is an isomorphism.

Proposition 5.13. *The morphism δ^+ factors through $\mathcal{Z}_n(u)^+$.*

Proof. For $S \in \mathrm{Nilp}_{O_{\breve{F}}}$ and $(X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S)$, the morphism δ^+ provides a quadruple $(X, \iota, \lambda, \rho) \in \mathcal{N}_n^+(S)$ together with an O_F-isogeny of degree q lifting ϕ_0, which we denote by $\phi\colon X \to X' \times \overline{\mathcal{E}}_S$.

Since $\ker(\phi)$ is π-power torsion and of rank q, it is killed by π. Hence there exists a unique (necessarily O_F-linear) isogeny $\phi'\colon X' \times \overline{\mathcal{E}}_S \to X$ such that the composite $\phi' \circ \phi\colon X \to X$ is $\iota(\pi)$.

Let $\tilde{u}$ be the composite

$$\overline{\mathcal{E}}_S \xrightarrow{(0,\mathrm{id})} X' \times \overline{\mathcal{E}}_S \xrightarrow{\phi'} X$$

which is clearly a morphism. Claim that $\tilde{u}$ lifts $\rho^{-1} \circ u$ so that $(X, \iota, \lambda, \rho) \in \mathcal{Z}_n(u)^+$.

Reduce $\tilde{u}$ from S to its special fibre $\overline{S}$ we get

$$\overline{\mathbb{E}}_{\overline{S}} \xrightarrow{(0,\mathrm{id})} X'_{\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \xrightarrow{\bar{\phi}'} X_{\overline{S}}$$

and by uniqueness of ϕ' we have a commutative diagram

$$\begin{tikzcd}
X'_{\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \arrow[r, "\bar{\phi}'"] \arrow[d, "\rho' \times \mathrm{id}"] & X_{\overline{S}} \arrow[d, "\rho"] \\
\mathbb{X}_{n-1,\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \arrow[r, "\phi'_0"] & \mathbb{X}_{n,\overline{S}}
\end{tikzcd}$$

where ϕ'_0 is given by

$$\phi'_0 \coloneqq \iota_{\mathbb{X}_{n-2}}(\pi) \times \begin{pmatrix} \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} & \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} \\ 1 & -1 \end{pmatrix}$$

so that $\phi'_0 \circ \phi_0 = \iota_{\mathbb{X}_{n,\overline{S}}}(\pi)$.

Hence it suffices to show that

$$\overline{\mathbb{E}}_{\overline{S}} \xrightarrow{(0,\mathrm{id})} X'_{\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \xrightarrow{\rho' \times \mathrm{id}} \mathbb{X}_{n-1,\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \xrightarrow{\phi'_0} \mathbb{X}_{n,\overline{S}}$$

is just u, which is an easy computation:

$$\left(\iota_{\mathbb{X}_{n-2}}(\pi) \times \begin{pmatrix} \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} & \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} \\ 1 & -1 \end{pmatrix}\right) \cdot \begin{pmatrix} 0_{n-2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0_{n-2} \\ \frac{\iota_{\overline{\mathbb{E}}}(\pi)}{2} \\ -1 \end{pmatrix}.$$

□

Now we have a morphism still denoted by $\delta^+ \colon \mathcal{N}_{n-1} \to \mathcal{Z}_n(u)^+$, which is universally injective and formally unramified as it is a closed embedding. To show it is an isomorphism we need to show it is also surjective and formally étale.

Proposition 5.14. *The morphism $\delta^+ \colon \mathcal{N}_{n-1} \to \mathcal{Z}_n(u)^+$ is surjective.*

Proof. We will just show that δ^+ is a surjection on $\bar{k}$-points and the argument for points in an arbitrary algebraically closed field is the same.

The isogeny ϕ_0 given in (5.4) allows us to identify $\mathbb{N}_n$ with $\mathbb{N}_{n-1} \oplus \overline{\mathbb{N}}_1$. Follow Lemma 5.8 we let $O_{\breve{F}}e$ be the Dieudonné module of $\overline{\mathbb{E}}$. Then e is also an $\breve{F}$-basis of $\overline{\mathbb{N}}_1$. Using Dieudonné theory, the $\bar{k}$-points of $\mathcal{N}_{n-1}$ and $\mathcal{N}_n^+$ are related as follows: let $(X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(\bar{k})$ with Dieudonné module $\mathbb{L} \subset \mathbb{N}_{n-1}$ and suppose its image in $\mathcal{N}_n^+(\bar{k})$ has Dieudonné module $\mathbb{M}$, then

$$\pi\mathbb{L}^* \oplus \pi O_{\breve{F}}e \subset^1 \mathbb{M} \subset^1 \mathbb{L} \oplus O_{\breve{F}}e. \tag{5.16}$$

Suppose there is a point in $\mathcal{N}_n^+(\bar{k})$ with Dieudonné module $\mathbb{M}$. Then the point is in $\mathcal{Z}_n^+(u)$ is equivalent to that there is a morphism of Diendonné modules

$$u\colon O_{\breve{F}}e \longrightarrow \mathbb{M}$$

such that when viewing $\mathbb{M}$ in $\mathbb{N}_{n-1} \oplus \overline{\mathbb{N}}_1$ via ϕ_0, the morphism u sends e to πe (as $\phi_0 u\colon \overline{\mathbb{E}} \to \mathbb{X}_n = \mathbb{X}_{n-1} \times \overline{\mathbb{E}}$ is just $(0_{n-1}, \iota_{\overline{\mathbb{E}}}(\pi))$). This turns out to be a simple condition that $\pi e \in \mathbb{M}$.

It is enough to show that given π-modular lattice $\mathbb{M} \subset \mathbb{N}_n$ satisfying condition (5.8) which contains πe, we can recover the almost π-modular lattice $\mathbb{L} \subset \mathbb{N}_{n-1}$ satisfying condition (5.9) and (5.16). We set

$$(\pi e)^{\perp} := \{a \in \mathbb{M} \mid h(a, \pi e) = 0\} \text{ and } \mathbb{L} := (\pi^{-1}(\pi e)^{\perp})^* = \pi((\pi e)^{\perp})^*. \tag{5.17}$$

Check condition (5.16): As $\mathbb{M}$ is π-modular, $\mathbb{M} \simeq \pi H^{n/2}$ and under a standard normal basis $\mathbb{M}$ has moment matrix

$$\begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}^{\oplus n/2}.$$

By (5.10), $\pi e \in \mathbb{M}$ is of hermitian norm $\delta \cdot h(u,u) = \delta\pi^2$. If we divide everything by π and apply Lemma 4.2 and Corollary 4.3, we see that $(\pi e)^{\perp}$ will have the moment matrix under a normal basis

$$\begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}^{\oplus n/2-1} \oplus (-\delta\pi^2)$$

and $(\pi e)^{\perp} \oplus \pi O_{\breve{F}} e \subset^1 \mathbb{M}$. Correspondingly $\mathbb{L}$ will have the moment matrix under a normal basis

$$\begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}^{\oplus n/2-1} \oplus (\delta).$$

In particular $\mathbb{L}$ is almost π-modular and $\pi\mathbb{L}^* \oplus \pi O_{\breve{F}} e \subset^1 \mathbb{M}$. Then from the fact that $\mathbb{M}^* = \pi^{-1}\mathbb{M}$, $\mathbb{L}$ is almost π-modular and $(O_{\breve{F}} e)^* = O_{\breve{F}} e$, we get $\mathbb{M} \subset^1 \mathbb{L} \oplus O_{\breve{F}} e$.

Check condition (5.9): We need to show $\mathbb{L}$ satisfies condition (5.9). We first claim that $F\mathbb{L}^* \subset \mathbb{L}^*$. This is equivalent to $F(\pi e)^{\perp} \subset (\pi e)^{\perp}$. By Lemma 5.8 we have $Ve = Fe = -\pi e$. Take $z \in (\pi e)^{\perp}$, we have

$$h(VFz, V\pi e) = h(\pi_0 z, \pi V e) = \pi_0 h(z, \pi(-\pi e)) = \pi_0 \pi h(z, \pi e) = 0.$$

Thus by Lemma 5.6 we have $h(Fz, \pi e) = 0$ and $F(\pi e)^{\perp} \subset (\pi e)^{\perp}$. The claim is proved. From $F\mathbb{L}^* \subset \mathbb{L}^*$ we get $V^{-1}\mathbb{L}^* \subset \pi_0^{-1}\mathbb{L}^*$.

To show $V\mathbb{L} \subset \mathbb{L}$, we claim that

$$h(x, y) \in O_{\breve{F}} \text{ for any } x \in \mathbb{L}^*, y \in V\mathbb{L}.$$

We let $y = Vy' \in V\mathbb{L}$ and $x \in \mathbb{L}^*$. Then $V^{-1}x \in V^{-1}\mathbb{L}^* \subset \pi_0^{-1}\mathbb{L}^*$. Now

$$h(V^{-1}x, V^{-1}y) = h(V^{-1}x, y') \in h(\pi_0^{-1}\mathbb{L}^*, \mathbb{L}) \subset \pi_0^{-1} O_{\breve{F}}$$

and by Lemma 5.6 we have $h(x, y) \in O_{\breve{F}}$. The claim is proved and $V\mathbb{L} \subset \mathbb{L}$.

Similarly to show $\pi_0\mathbb{L} \subset V\mathbb{L}$, it is enough to show $F\mathbb{L} \subset \mathbb{L}$. and essentially to show $V\mathbb{L}^* \subset \mathbb{L}^*$, which can be proved the same way using $Fe = -\pi e$ and Lemma 5.6.

The only thing left in condition (5.9) is to show $V\mathbb{L} \subset^{\leqslant 1} \pi\mathbb{L} + V\mathbb{L}$. Write $n = 2m$ and pick a standard normal basis $e_1, f_1, \ldots, e_m, f_m$ of $\mathbb{M}$ such that $e_1 \in V\mathbb{M}$. This is possible by Lemma 5.9

and moreover by Lemma 5.10 we have

$$V\mathbb{M} = \langle e_1, \pi_0 f_1, \pi e_2, \pi f_2, \dots, \pi e_m, \pi f_m\rangle_{O_{\breve{F}}}.$$

Consider $\pi e \in \mathbb{M}$ under this basis and write

$$\pi e = a_1 e_1 + b_1 f_1 + \cdots + a_m e_m + b_m f_m.$$

If $\pi e \in V\mathbb{M}$ then $F\pi e \in FV\mathbb{M} = \pi_0\mathbb{M}$, but $F\pi e = \pi F e = \pi(-\pi e) = -\pi_0 e$ so $e \in \mathbb{M}$ contradiction! Thus $\pi e \notin V\mathbb{M}$. Also $\pi \cdot \pi e = \pi(-Ve) = -V(\pi e) \in V\mathbb{M}$. Hence we must have $b_1 \in (\pi)$ and either one of $a_2, b_2, \dots, a_m, b_m$ is in $O_{\breve{F}}^{\times}$, or $a_2, b_2, \dots, a_m, b_m \in (\pi)$ and $b_1 \notin (\pi_0)$.

We then have two cases. We will see they correspond to the co-length of $V\mathbb{L}$ in $\pi\mathbb{L} + V\mathbb{L}$ is 0 or 1, by finding a standard normal basis of $\mathbb{M}$ and describing $\mathbb{L}, V\mathbb{L}$ in terms of this basis.

Case 1: Suppose one of $a_2, b_2, \dots, a_m, b_m$ is in $O_{\breve{F}}^{\times}$. We may assume $a_m = 1$. Then by the proof of Corollary 4.3 we can find a standard normal basis $\sigma_1, \tau_1, \dots, \sigma_m, \tau_m$ of $\mathbb{M}$ such that $\pi e = \sigma_m + \frac{\pi\delta}{2}\tau_m$ and

$$\langle \pi e\rangle^{\perp} = \langle \sigma_1, \tau_1, \dots, \sigma_{m-1}, \tau_{m-1}, \sigma_m - \frac{\pi\delta}{2}\tau_m\rangle_{O_{\breve{F}}}.$$

Moreover as $b_1 \in (\pi)$ and $\pi f_m \in V\mathbb{M}$, by construction (4.1) the element $\sigma_1 = e_1 + \bar{b}_1 f_m$ is still in $V\mathbb{M}$. Thus by Lemma 5.10 again we have

$$V\mathbb{M} = \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}}.$$

As $V(\pi e) = -\pi\pi e$ we have $V(\sigma_m + \frac{\pi\delta}{2}\tau_m) = -\pi(\sigma_m + \frac{\pi\delta}{2}\tau_m)$. As $V\mathbb{L}^* \subset \mathbb{L}^*$, we have $V\langle \pi e\rangle^{\perp} \subset \langle \pi e\rangle^{\perp}$. We may then write

$$V(\sigma_m - \frac{\pi\delta}{2}\tau_m) = \alpha \cdot (\sigma_m - \frac{\pi\delta}{2}\tau_m) + \alpha_1\sigma_1 + \beta_1\tau_1 + \cdots + \alpha_{m-1}\sigma_{m-1} + \beta_{m-1}\tau_{m-1} \in V\mathbb{M}.$$

Hence $\alpha \in (\pi)$ as $\sigma_m \notin V\mathbb{M}$ and $\pi\sigma_m \in V\mathbb{M}$. Thus

$$2V\sigma_m = V(\sigma_m - \frac{\pi\delta}{2}\tau_m) + V(\sigma_m + \frac{\pi\delta}{2}\tau_m) \in (\pi_0)\tau_m + \langle \sigma_1, \pi_0\tau_1, \ldots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \pi\sigma_m \rangle_{O_{\breve{F}}}. \tag{5.18}$$

Now let γ be one of $\sigma_1, \tau_1, \ldots, \sigma_{m-1}, \tau_{m-1}$. From $h(\gamma, \sigma_m + \frac{\pi\delta}{2}\tau_m) = 0$ and Lemma 5.6 we get

$$h(V\gamma, V(\sigma_m + \frac{\pi\delta}{2}\tau_m)) = 0.$$

If we write $V\gamma \in a\pi\sigma_m + b\pi\tau_m + \langle \sigma_1, \pi_0\tau_1, \ldots, \pi\sigma_{m-1}, \pi\tau_{m-1} \rangle_{O_{\breve{F}}}$ then

$$h(a\pi\sigma_m + b\pi\tau_m, -\pi(\sigma_m + \frac{\pi\delta}{2}\tau_m)) = \frac{a\pi_0^2\delta}{2} + b\pi\pi_0 = 0.$$

Hence $b \in (\pi)$ and

$$V\gamma \in (\pi_0)\tau_m + \langle \sigma_1, \pi_0\tau_1, \ldots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \pi\sigma_m \rangle_{O_{\breve{F}}}. \tag{5.19}$$

Consider $N = \pi O_{\breve{F}} e \oplus \langle \pi e \rangle^{\perp} \subset^1 \mathbb{M}$. Then $N = \langle \sigma_1, \tau_1, \ldots, \sigma_{m-1}, \tau_{m-1}, \sigma_m, \pi\tau_m \rangle_{O_{\breve{F}}}$ and $\pi N^* = \mathbb{L} \oplus O_{\breve{F}} e$. As $V\tau_m \in V\mathbb{M}$, we have

$$V(\pi\tau_m) \in (\pi_0)\tau_m + \langle \sigma_1, \pi_0\tau_1, \ldots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \pi\sigma_m \rangle_{O_{\breve{F}}}.$$

Combine (5.18) and (5.19) we have shown that

$$VN \subset \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \ldots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \pi\sigma_m, \pi_0\tau_m \rangle_{O_{\breve{F}}} \subset^1 V\mathbb{M}.$$

As $VN \subset^1 V\mathbb{M}$, we conclude that

$$VN = \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \ldots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \pi\sigma_m, \pi_0\tau_m \rangle_{O_{\breve{F}}}. \tag{5.20}$$

By construction (5.17), we have

$$\mathbb{L} = \pi(\langle \pi e\rangle^{\perp})^* = \langle \sigma_1, \tau_1, \dots, \sigma_{m-1}, \tau_{m-1}, \pi^{-1}\sigma_m - \frac{\delta}{2}\tau_m\rangle_{O_{\breve{F}}}. \tag{5.21}$$

We have shown that $V\mathbb{L} \subset \mathbb{L}$ and $V(\mathbb{L} \oplus O_{\breve{F}} e) = V(\pi N^*) = \pi V(N^*) = \pi\pi_0(VN)^*$ by Corollary 5.7. From (5.20) we get

$$\begin{aligned} V\mathbb{L} \oplus \pi O_{\breve{F}} e &= \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}}, \\ V\mathbb{L} &= \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_{m-1}, \pi\tau_{m-1}, \sigma_m - \frac{\pi\delta}{2}\tau_m\rangle_{O_{\breve{F}}}. \end{aligned} \tag{5.22}$$

It is then clear that $V\mathbb{L} \subset^1 \pi\mathbb{L} + V\mathbb{L}$.

Case 2: Suppose $b_1, a_2, b_2, \dots, a_m, b_m \in (\pi)$. As

$$h(\pi e, \pi e) = \delta\pi^2 = \sum_{i=1}^{m} (-a_i\bar{b}_i + \bar{a}_i b_i)\pi \in \pi^2 O_{\breve{F}}^{\times}$$

we must have $a_1 \in O_{\breve{F}}^{\times}$ and we may assume $a_1 = 1$ after normalizing e_1 and f_1 by a unit. Again by Corollary 4.3 we can find a standard normal basis $\sigma_1, \tau_1, \dots, \sigma_m, \tau_m$ of $\mathbb{M}$ such that $\pi e = \sigma_1 + \frac{\pi\delta}{2}\tau_1$ and

$$\langle \pi e\rangle^{\perp} = \langle \sigma_1 - \frac{\pi\delta}{2}\tau_1, \sigma_2, \tau_2, \dots, \sigma_m, \tau_m\rangle_{O_{\breve{F}}}.$$

By construction (4.2) and (4.3), $\sigma_1 = e_1 + (b_1 - \frac{\pi\delta}{2})f_1 + a_2e_2 + b_2f_2 + \cdots + a_me_m + b_mf_m$. As $b_1, a_2, b_2, \dots, a_m, b_m \in (\pi)$, we have $a_2e_2, b_2f_2, \dots, a_me_m, b_mf_m \in V\mathbb{M}$. Note that

$$\delta\pi^2 = 2\pi b_1 - \sum_{i=2}^{m} (a_i\bar{b}_i - \bar{a}_i b_i)\pi$$

where $\sum_{i=2}^{m} (a_i\bar{b}_i - \bar{a}_i b_i)\pi \in \pi^4 O_{\breve{F}}$. Hence $b_1 - \frac{\pi\delta}{2} \in \pi^3 O_{\breve{F}}$ and $(b_1 - \frac{\pi\delta}{2})f_1 \in (\pi_0)f_1 \in V\mathbb{M}$. Thus $\sigma_1 \in V\mathbb{M}$ and again by Lemma 5.10 we have

$$V\mathbb{M} = \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}}.$$

Similarly as in Case 1, using $V(\sigma_1 + \frac{\pi\delta}{2}\tau_1) = -\pi(\sigma_1 + \frac{\pi\delta}{2}\tau_1)$ we can show that if γ is one of $\sigma_1, \pi\tau_1, \sigma_2, \tau_2, \dots, \sigma_m, \tau_m$ then

$$V\gamma \in \langle \pi\sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}}.$$

Let $N = O_{\breve{F}}e \oplus \langle \pi e\rangle^\perp \subset^1 \mathbb{M}$, then $N = \langle \sigma_1, \pi\tau_1, \sigma_2, \tau_2, \dots, \sigma_m, \tau_m\rangle_{O_{\breve{F}}}$. We have shown that

$$VN \subset \langle \pi\sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}} \subset^1 V\mathbb{M}$$

and $VN \subset^1 V\mathbb{M}$. Hence

$$VN = \langle \pi\sigma_1, \pi_0\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}} = \pi N. \tag{5.23}$$

By construction (5.17) we have

$$\mathbb{L} = \pi(\langle \pi e\rangle^\perp)^* = \langle \pi^{-1}\sigma_1 - \frac{\delta}{2}\tau_1, \sigma_2, \tau_2, \dots, \sigma_m, \tau_m\rangle_{O_{\breve{F}}}. \tag{5.24}$$

From (5.23) we get

$$V\mathbb{L} = \langle \sigma_1 - \frac{\pi\delta}{2}\tau_1, \pi\sigma_2, \pi\tau_2, \dots, \pi\sigma_m, \pi\tau_m\rangle_{O_{\breve{F}}}. \tag{5.25}$$

It is clear that $V\mathbb{L} = \pi\mathbb{L}$ and $V\mathbb{L} \subset^0 V\mathbb{L} + \pi\mathbb{L}$.

Combine the two cases we see we always have $V\mathbb{L} \subset^{\leqslant 1} V\mathbb{L} + \pi\mathbb{L}$ and thus we have shown that δ^+ is surjective. $\square$

The last step is to show that δ^+ is formally étale.

Proof of Theorem 5.5. By Corollary 5.12, we see that $\mathcal{Z}_n(u)^+$ is formally smooth of relative dimension $n-2$ over $\operatorname{Spf} O_{\breve{F}}$. For simplicity we shall write $\mathcal{N}$ for $\mathcal{N}_{n-1}$ and $\mathcal{Z}$ for $\mathcal{Z}_n(u)^+$.

Also we write $\mathcal{O}$ for $\mathrm{Spf}\, O_{\breve{F}}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "\delta^+"] \arrow[dr, "f"'] & & \mathcal{Z} \arrow[dl, "g"] \\
& \mathcal{O} = \mathrm{Spf}(O_{\breve{F}}) &
\end{tikzcd}$$

where both f and g are morphisms between locally Noetherian formal schemes, which are formally locally of finite type, separated, formally smooth of relative dimension $n-2$. Then locally both f and g are pseudo-finite in the sense of [TLR05, Definition 1.3.1]. As δ^+ is a closed embedding, we have a short exact sequence of coherent $\mathcal{O}_{\mathcal{N}}$-modules ([TLR05, Proposition 2.2.8, Proposition 2.3.4])

$$\mathcal{C}_{\mathcal{N}/\mathcal{Z}} \longrightarrow (\delta^+)^*\widehat{\Omega}^1_{\mathcal{Z}/\mathcal{O}} \longrightarrow \widehat{\Omega}^1_{\mathcal{N}/\mathcal{O}} \longrightarrow 0$$

where both $(\delta^+)^*\widehat{\Omega}^1_{\mathcal{Z}/\mathcal{O}}$ and $\widehat{\Omega}^1_{\mathcal{N}/\mathcal{O}}$ are locally free $\mathcal{O}_{\mathcal{N}}$-modules of the same rank as f and g are smooth of same relative dimension ([TLR05, Proposition 2.5.5]). Then the surjection

$$(\delta^+)^*\widehat{\Omega}^1_{\mathcal{Z}/\mathcal{O}} \longrightarrow \widehat{\Omega}^1_{\mathcal{N}/\mathcal{O}}$$

must be an isomorphism. Hence δ^+ is formally étale ([TLR05, Corollary 2.5.10]). Combine with Proposition 5.14 we conclude that the closed embedding δ^+ is an isomorphism. □

5.4 Pullback $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles

Now we want to relate intersection of special cycles on $\mathcal{N}_{n-1}$ to that on $\mathcal{N}_n$. We study this by pullbacking $\mathcal{Z}$-cycles (resp. $\mathcal{Y}$-cycles) from $\mathcal{N}_n$ to $\mathcal{N}_{n-1}$ and identify them with $\mathcal{Z}$-cycles (resp. $\mathcal{Y}$-cycles) on $\mathcal{N}_{n-1}$. More precisely, let $x \in \mathbb{V}_{n-1}$ be a nonzero element. Then x gives special cycles $\mathcal{Z}_{n-1}(x)$ and $\mathcal{Y}_{n-1}(x)$ on $\mathcal{N}_{n-1}$. Via the hermitian embedding (5.7), we can also think of $x \in \mathbb{V}_n$, which gives special cycles $\mathcal{Z}_n(x)$ and $\mathcal{Y}_n(x)$ on $\mathcal{N}_n$. We can pull back $\mathcal{Z}_n(x)$ (resp. $\mathcal{Y}_n(x)$) along the closed embedding δ^+, which is $\mathcal{Z}_n(x) \cap \mathcal{Z}_n(u)^+ \cong \mathcal{Z}_n(x) \cap \mathcal{N}_{n-1}$ as shown in the diagram below

$$\begin{tikzcd}
\mathcal{Z}_n(x) \cap \mathcal{N}_{n-1} \arrow[r, "\sim"] \arrow[d] & \mathcal{Z}_n(x) \cap \mathcal{Z}_n(u)^+ \arrow[r] \arrow[d] & \mathcal{Z}_n(x) \arrow[d] \\
\mathcal{N}_{n-1} \arrow[r, "\delta^+", "\simeq"'] & \mathcal{Z}_n(u)^+ \arrow[r] & \mathcal{N}_n.
\end{tikzcd} \tag{5.26}$$

Pullback $\mathcal{Z}$-cycle: For a $\mathrm{Spf}\, O_{\breve{F}}$-scheme S, we have

$$\mathcal{Z}_n(x) \cap \mathcal{N}_{n-1}(S) = \left\{ \begin{array}{r} (X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S) \text{ such that if its image} \\ \text{under } \delta^+ \text{ is } (X, \iota, \lambda, \rho) \in \mathcal{N}_n^+(S) \text{ then there is} \\ \text{a homomorphism } \overline{\mathcal{E}}_S \to X \text{ lifting } \rho^{-1}x \end{array} \right\},$$

$$\mathcal{Z}_{n-1}(x)(S) = \left\{ \begin{array}{r} (X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S) \text{ such that there is} \\ \text{a homomorphism } \overline{\mathcal{E}}_S \to X' \text{ lifting } (\rho')^{-1}x \end{array} \right\}.$$

Then we have a morphism denoted by δ_x^+:

$$\begin{aligned} \mathcal{Z}_n(x) \cap \mathcal{N}_{n-1} &\longrightarrow \mathcal{Z}_{n-1}(x) \\ ((X', \iota', \lambda', \rho'), \tilde{x} \colon \overline{\mathcal{E}}_S \to X) &\longmapsto ((X', \iota', \lambda', \rho'), \overline{\mathcal{E}}_S \xrightarrow{\tilde{x}} X \xrightarrow{\phi} X' \times \overline{\mathcal{E}}_S \xrightarrow{p_1} X') \end{aligned}$$

where $\phi\colon X \to X' \times \overline{\mathcal{E}}_S$ is the O_F-linear isogeny lifting ϕ_0. The morphism is well-defined because we have the following commutative diagram over $\overline{S}$

$$\begin{tikzcd}
\overline{\mathbb{E}}_{\overline{S}} \arrow[r, "x"] \arrow[dr, "{(x,0)}"'] & \mathbb{X}_{n,\overline{S}} \arrow[r, "\rho^{-1}"] \arrow[d, "\phi_0"] & X_{\overline{S}} \arrow[d, "\bar{\phi}"] \\
& \mathbb{X}_{n-1,\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \arrow[r, "(\rho')^{-1}\times \mathrm{id}"] & X'_{\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}}.
\end{tikzcd}$$

Naively we would like the morphism δ_x^+ to be an isomorphism for all $x \in \mathbb{V}_{n-1}$. However this is not possible. On one hand, $\mathcal{Z}_n(x) = \emptyset$ if $h(x,x) \notin \pi_0 O_{F_0}$. On the other hand, we have the following Proposition.

Proposition 5.15. *If $x \in \mathbb{V}_{n-1}$ has hermitian norm $h(x,x) = 1$ then we have natural isomorphism*

$$\mathcal{N}_{n-2} \xrightarrow{\sim} \mathcal{Z}_{n-1}(x).$$

Proof. By (2.21), after applying an action of $U(\mathbb{V}_{n-1})$, we only need to prove this for

$$x_0 = (0_{n-2}, \mathrm{id}_{\overline{\mathbb{E}}}) \colon \overline{\mathbb{E}} \to \mathbb{X}_{n-1} = \mathbb{X}_{n-2} \times \overline{\mathbb{E}}.$$

We have a closed embedding of formal schemes

$$\begin{array}{rccc} \delta_{\mathcal{N}}\colon & \mathcal{N}_{n-2} & \longrightarrow & \mathcal{N}_{n-1} \\ & (X, \iota, \lambda, \rho) & \longmapsto & \left(X \times \overline{\mathcal{E}}, \iota \times \iota_{\overline{\mathcal{E}}}, \lambda \times \lambda_{\overline{\mathcal{E}}}, \rho \times \rho_{\overline{\mathcal{E}}}\right) \end{array}.$$

It is direct to see that $\delta_{\mathcal{N}}$ factors through $\mathcal{Z}_{n-1}(x_0)$. By the same argument as in [Ter13, Lemma 2], we see that $\delta_{\mathcal{N}}$ induces an isomorphism $\mathcal{N}_{n-2} \simeq \mathcal{Z}_{n-1}(x_0)$. □

If we choose $x \in \mathbb{V}_{n-1}$ of hermitian norm 1 then $\mathcal{Z}_n(x) \cap \mathcal{N}_{n-1}$ is empty while $\mathcal{Z}_{n-1}(x) \simeq \mathcal{N}_{n-2}$ is not. The morphism δ_x^+ can not be an isomorphism in this case.

Proposition 5.16. *The morphism $\delta_x^+ \colon \mathcal{Z}_n(x) \cap \mathcal{N}_{n-1} \to \mathcal{Z}_{n-1}(x)$ is an isomorphism if $h(x,x) \in \pi_0 O_{F_0}$.*

Proof. Write $n = 2m$. From the diagram (5.26) we see that δ_x^+ is *a priori* a closed embedding, so again we need to show it is surjective and formally smooth.

Follow Lemma 5.8, we write $O_{\breve{F}} e$ for the Dieudonné module of $\overline{\mathbb{E}}$. From Dieudonné theory the

$\bar{k}$-points are described as follows:

$$\mathcal{Z}_n(x)\cap\mathcal{N}_{n-1}(\bar{k})=\left\{\begin{array}{c}\text{almost } \pi\text{-modular } \mathbb{L}\subset\mathbb{V}_{n-1} \text{ satisfying condition (5.9)},\\ \pi\text{-modular } \mathbb{M}\subset\mathbb{V}_n \text{ satisfying condition (5.8) and (5.16)},\\ \text{and there exists a morphism } O_{\breve{F}}e\to\mathbb{M} \text{ induced by } x\end{array}\right\},$$

$$\mathcal{Z}_{n-1}(x)(\bar{k})=\left\{\begin{array}{c}\text{almost } \pi\text{-modular } \mathbb{L}\subset\mathbb{V}_{n-1} \text{ satisfying condition (5.9)},\\ \text{and there exists a morphism } O_{\breve{F}}e\to\mathbb{L} \text{ induced by } x\end{array}\right\}.$$

Then the morphism δ_x^+ just sends the pair $(\mathbb{L},\mathbb{M})$ to $\mathbb{L}$ with the Dieudonné morphism

$$O_{\breve{F}}e\xrightarrow{x}\mathbb{M}\subset^1\mathbb{L}\oplus O_{\breve{F}}e\xrightarrow{p_1}\mathbb{L}.$$

To show surjectivity, take $\mathbb{L}$ with the Dieudonné morphism x, then the morphism δ^+ provides us with the lattice $\mathbb{M}$ satisfying conditions (5.8) and (5.16). All we need to show is that the morphism $O_{\breve{F}}e\xrightarrow{(x,0)}\mathbb{L}\oplus O_{\breve{F}}e$ factors through $\mathbb{M}\subset^1\mathbb{L}\oplus O_{\breve{F}}e$.

As in (5.21) or (5.24), we can find a standard normal basis σ_i,τ_i of $\mathbb{M}$ such that $\pi e=\sigma_1+\frac{\pi\delta}{2}\tau_1$ and $\mathbb{L}$ has normal basis $\pi^{-1}(\sigma_1-\frac{\pi\delta}{2}\tau_1),\sigma_i,\tau_i$ for $i\geqslant 2$. Let the image of $O_{\breve{F}}e\xrightarrow{x}\mathbb{L}$ be $a\pi^{-1}(\sigma_1-\frac{\pi\delta}{2}\tau_1)+a_2\sigma_2+b_2\tau_2+\cdots+a_m\sigma_m+b_m\tau_m$, then its hermitian norm is

$$\delta a\bar{a}-\sum_{i=2}^{m}(a_i\bar{b}_i-\bar{a}_ib_i)\pi.$$

On the other hand, by (5.10) the hermitian norm should be $\delta\cdot h(x,x)\in\pi^2O_{\breve{F}_0}$ as we assume $h(x,x)\in\pi_0O_{F_0}$, so $a\in\pi O_{\breve{F}}$ and the image lies in $\mathbb{M}$ as desired.

To show it is formally smooth, we consider a Noetherian local $O_{\breve{F}}$-algebra R on which π is nilpotent, and a square zero ideal $I\subset R$. Let $S=R/I$ so that $R\to S$ is a thickening, equipped with the trivial nilpotent divided power structure on I. Suppose we have a commutative diagram consisting of solid arrows

$$\begin{tikzcd}
\operatorname{Spec} S \arrow[r] \arrow[d] & \mathcal{Z}_n(x) \cap \mathcal{N}_{n-1} \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \mathcal{Z}_{n-1}(x)
\end{tikzcd}$$

and we want to find a dotted arrow making the diagram commutative. In other words, we are given $(X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S)$ such that if we write $\delta^+(X', \iota', \lambda', \rho') = (X, \iota, \lambda, \rho)$ then there exists a lift $x\colon \overline{\mathcal{E}}_S \to X$ of $\overline{\mathbb{E}}_{\overline{S}} \xrightarrow{x} \mathbb{X}_{n,\overline{S}} \xrightarrow{\rho^{-1}} X_{\overline{S}}$, together with a lift $(\widetilde{X}', \widetilde{\iota}', \widetilde{\lambda}', \widetilde{\rho}') \in \mathcal{Z}_{n-1}(x)(R)$ of $(X', \iota', \lambda', \rho')$.

Let $\delta^+(\widetilde{X}', \widetilde{\iota}', \widetilde{\lambda}', \widetilde{\rho}') = (\widetilde{X}, \widetilde{\iota}, \widetilde{\lambda}, \widetilde{\rho})$. Then it suffices to show that the morphism $\overline{\mathcal{E}}_S \to X$ lifts to $\overline{\mathcal{E}}_R \to \widetilde{X}$. Let $\phi\colon X \to X' \times \overline{\mathcal{E}}_S$ be the O_F-isogeny lifting ϕ_0. Over S we have a commutative diagram from Dieudonné crystals:

$$\begin{tikzcd}
\mathrm{Fil}^1(\overline{\mathcal{E}}_S) \arrow[r] \arrow[d] & \mathbb{D}(\overline{\mathcal{E}}_S) \arrow[d, "\mathbb{D}(x)"] \\
\mathrm{Fil}^1(X) \arrow[r] \arrow[d] & \mathbb{D}(X) \arrow[d, "\mathbb{D}(\phi)"] \\
\mathrm{Fil}^1(X') \oplus \mathrm{Fil}^1(\overline{\mathcal{E}}_S) \arrow[r] & \mathbb{D}(X') \oplus \mathbb{D}(\overline{\mathcal{E}}_S)
\end{tikzcd}$$

and it can be extended to a commutative diagram over R:

$$\begin{tikzcd}
\mathrm{Fil}^1(\overline{\mathcal{E}}_R) \arrow[r] \arrow[dd, bend right] & \mathbb{D}(\overline{\mathcal{E}}_R) \arrow[d, "\mathbb{D}(x)_R"] & \\
\mathrm{Fil}^1(\widetilde{X}) \arrow[r] \arrow[d] & \mathbb{D}(X)_R \arrow[r] \arrow[d, "\mathbb{D}(\phi)_R"] & \mathrm{Lie}(\widetilde{X}) \arrow[d, "\mathrm{Lie}(\phi)_R"] \\
\mathrm{Fil}^1(\widetilde{X}') \oplus \mathrm{Fil}^1(\overline{\mathcal{E}}_R) \arrow[r] & \mathbb{D}(X')_R \oplus \mathbb{D}(\overline{\mathcal{E}}_R) \arrow[r] & \mathrm{Lie}(\widetilde{X}') \oplus \mathrm{Lie}(\overline{\mathcal{E}}_R)
\end{tikzcd}$$

By Grothendieck–Messing theory, it suffices to show $\mathbb{D}(x)_R$ preserves the Hodge filtration, i.e.

$$\mathbb{D}(x)_R(\mathrm{Fil}^1(\overline{\mathcal{E}}_R)) \subset \mathrm{Fil}^1(\widetilde{X}).$$

From [LL22, Lemma 2.38] we know that there is a locally direct summand R-submodule of $\mathrm{Lie}(\widetilde{X})$ of rank 1, denoted by $L_{\widetilde{X}}$, and by the same argument as [How19, Proposition 4.1] we see that the image of the induced morphism

$$\mathbb{D}(x)_R\colon\ \mathrm{Fil}^1(\overline{\mathcal{E}}_R) \longrightarrow \mathrm{Lie}(\widetilde{X})$$

lies in $L_{\widetilde{X}}$. Suppose f is an R-basis of $\mathrm{Fil}^1(\overline{\mathcal{E}}_R)$ and e' is an R-basis of $L_{\widetilde{X}}$. Write $\mathbb{D}(x)_R(f) = r\cdot e'$ for some $r \in R$. Pick any R-basis $e_1, \dots, e_n$ of $\mathrm{Lie}(\widetilde{X}') \oplus \mathrm{Lie}(\overline{\mathcal{E}}_R)$, and suppose the image of e' under $\mathrm{Lie}(\phi)_R$ in $\mathrm{Lie}(\widetilde{X}') \oplus \mathrm{Lie}(\overline{\mathcal{E}}_R)$ is given by $a_1e_1 + \cdots + a_ne_n$ for $a_1, \dots, a_n \in R$. Then as $(\widetilde{X}', \widetilde{\iota}', \widetilde{\lambda}', \widetilde{\rho}') \in \mathcal{Z}_{n-1}(x)(R)$, the image of $\mathrm{Fil}^1(\overline{\mathcal{E}}_R)$ in $\mathrm{Lie}(\widetilde{X}') \oplus \mathrm{Lie}(\overline{\mathcal{E}}_R)$ vanishes. Thus $r \cdot a_1 = \cdots = r \cdot a_n = 0$.

If one of $a_1, \dots, a_n$ is a unit, then we have $r = 0$ so that $\mathbb{D}(x)_R$ preserves the Hodge filtration and we are done. Otherwise, let $\mathfrak{m}$ be the maximal ideal of R, and assume $a_1, \dots, a_n \in \mathfrak{m}$. We shall prove by contradiction that this is impossible.

We make the base change to the residue field $\bar{k}$, then the image of $L_{\bar{k}} := (L_{\widetilde{X}})_{\bar{k}}$ in $\mathrm{Lie}(\widetilde{X}')_{\bar{k}} \oplus \mathrm{Lie}(\overline{\mathcal{E}}_R)_{\bar{k}}$ vanishes as we assume $a_1, \dots, a_n \in \mathfrak{m}$.

For any standard normal basis $e_1, f_1, \dots, e_m, f_m$ of the Dieudonné module $\mathbb{M}$ of $X_{\bar{k}}$ such that $e_1 \notin V\mathbb{M}$, by Lemma 5.10 we have

(1) $\mathrm{Fil}^1(X_{\bar{k}}) = \langle e_1, \pi e_1, \pi e_2, \pi f_2, \dots, \pi e_m, \pi f_m\rangle_{\bar{k}}$,

(2) $\mathrm{Lie}(X_{\bar{k}}) = \langle f_1, \pi f_1, e_2, f_2, \dots, e_m, f_m\rangle_{\bar{k}}$.

By [LL22, Lemma 2.42], we see that $L_{\bar{k}} = \langle \pi f_1\rangle_{\bar{k}}$.

Now let $\mathbb{L}$ be the Dieudonné module of $X'_{\bar{k}} \times \overline{\mathbb{E}}_{\bar{k}}$. Recall we have the relation (5.16)

$$N \subset^1 \mathbb{M} \subset^1 \mathbb{L}$$

where $N = \pi\mathbb{L}^*$. Then there are two cases, corresponding to the two cases in the proof of Proposition 5.14. In particular as in the proof, we can find a standard normal basis $\sigma_1, \tau_1, \dots, \sigma_m, \tau_m$

of $\mathbb{M}$ with $\sigma_1 \notin V\mathbb{M}$ (thus $L_{\bar{k}} = \langle \pi\tau_1 \rangle_{\bar{k}}$) and such that

- In the first case

$$N = \langle \sigma_1, \tau_1, \ldots, \sigma_{m-1}, \tau_{m-1}, \sigma_m, \pi\tau_m \rangle_{O_{\breve{F}}} \text{ and } \mathbb{L} = \langle \sigma_1, \tau_1, \ldots, \sigma_{m-1}, \tau_{m-1}, \pi^{-1}\sigma_m, \tau_m \rangle_{O_{\breve{F}}}.$$

  Then the inclusion $\mathbb{M} \subset^1 \mathbb{L}$ sends σ_m to $\pi \cdot \pi^{-1}\sigma_m$, $\pi\sigma_m$ to $\pi_0 \cdot \pi^{-1}\sigma_m$ and all other bases to themselves. In particular the image of $\pi\tau_1$ is again $\pi\tau_1$, which does not vanish after base change to $\bar{k}$ (the only term that vanishes is $\pi\sigma_m$);

- In the second case

$$N = \langle \sigma_1, \pi\tau_1, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m \rangle_{O_{\breve{F}}} \text{ and } \mathbb{L} = \langle \pi^{-1}\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m \rangle_{O_{\breve{F}}}.$$

  Then the inclusion $\mathbb{M} \subset^1 \mathbb{L}$ sends σ_1 to $\pi \cdot \pi^{-1}\sigma_1$ and $\pi\sigma_1$ to $\pi_0 \cdot \pi^{-1}\sigma_1$, and all other bases to themselves. In particular $\pi\tau_1$ maps to $\pi\tau_1$, which will not vanish after base change to $\bar{k}$ (the only term that vanishes is $\pi\sigma_1$).

Thus it is a contradiction that the image of $L_{\bar{k}}$ will vanish. $\square$

Pullback $\mathcal{Y}$-cycles: For a $\operatorname{Spf} O_{\breve{F}}$-scheme S, we have

$$\mathcal{Y}_{n-1}(x)(S) = \left\{ \begin{array}{c} (X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S) \text{ such that there exists a morphism} \\ \overline{\mathcal{E}}_S \to (X')^\vee \text{ lifting } \overline{\mathbb{E}}_{\overline{S}} \xrightarrow{x} \mathbb{X}_{n-1,\overline{S}} \xrightarrow{(\rho')^{-1}} X'_{\overline{S}} \xrightarrow{\lambda'} (X')^\vee_{\overline{S}}. \end{array} \right\},$$

$$\mathcal{N}_{n-1} \cap \mathcal{Y}_n(x)(S) = \left\{ \begin{array}{c} (X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S) \text{ such that if its image in } \mathcal{N}_n^+(S) \\ \text{is } (X, \iota, \lambda, \rho) \text{ then there exists a morphism } \overline{\mathcal{E}}_S \to X^\vee \text{ lifting} \\ \overline{\mathbb{E}}_{\overline{S}} \xrightarrow{x} \mathbb{X}_{n,\overline{S}} \xrightarrow{\rho^{-1}} X_{\overline{S}} \xrightarrow{\lambda} X^\vee_{\overline{S}}. \end{array} \right\}.$$

Then we have a morphism denoted by γ_x^+:

$$\begin{gathered} \mathcal{Y}_{n-1}(x) \longrightarrow \mathcal{N}_{n-1} \cap \mathcal{Y}_n(x) \\ ((X', \iota', \lambda', \rho'), \widetilde{x} \colon \overline{\mathcal{E}}_S \to (X')^\vee) \longmapsto ((X', \iota', \lambda', \rho'), \overline{\mathcal{E}}_S \xrightarrow{(\widetilde{x}, 0)} (X')^\vee \times \overline{\mathcal{E}}_S^\vee \xrightarrow{\phi^\vee} X^\vee) \end{gathered}$$

where $\phi\colon X \to X' \times \overline{\mathcal{E}}_S$ is the O_F-linear isogney lifting ϕ_0. The morphism is well-defined because we have the following commutative diagram over $\overline{S}$

$$\begin{tikzcd}
\overline{\mathbb{E}}_{\overline{S}} \arrow[r, "{(x,0)}"] \arrow[dr, "x"] & \mathbb{X}_{n-1,\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \arrow[r, "(\rho')^{-1}\times\mathrm{id}"] & X'_{\overline{S}} \times \overline{\mathbb{E}}_{\overline{S}} \arrow[r, "\lambda'\times\lambda_{\overline{\mathbb{E}}}"] & (X'_{\overline{S}})^{\vee} \times \overline{\mathbb{E}}^{\vee}_{\overline{S}} \arrow[d, "\phi^{\vee}_{\overline{S}}"] \\
& \mathbb{X}_{n,\overline{S}} \arrow[u, "\phi_0"'] \arrow[r, "\rho^{-1}"] & X_{\overline{S}} \arrow[u, "\phi_{\overline{S}}"] \arrow[r, "\lambda"] & X^{\vee}_{\overline{S}}.
\end{tikzcd}$$

Proposition 5.17. *The morphism γ_x^+ is an isomorphism for any nonzero $x \in \mathbb{V}_{n-1}$.*

Proof. The morphism γ_x^+ completes the closed embedding diagram

$$\begin{tikzcd}
\mathcal{Y}_{n-1}(x) \arrow[rr, "\gamma_x^+"] \arrow[dr] & & \mathcal{N}_{n-1} \cap \mathcal{Y}_n(x) \arrow[dl] \\
& \mathcal{N}_{n-1} &
\end{tikzcd}$$

so it is also a closed embedding. It remains to show it is surjective and formally smooth.

We firstly show it is surjective. We shall consider the $\bar{k}$-points and the argument for arbitrary algebraically closed field is the same. Let $O_{\breve{F}}e$ denote the Dieudonné module of $\overline{\mathbb{E}}$ as in Lemma 5.8. We have the following description in terms of Dieudonné module:

$$\mathcal{N}_{n-1} \cap \mathcal{Y}_n(x)(\bar{k}) = \left\{ \begin{array}{c} \text{almost } \pi\text{-modular } \mathbb{L} \subset \mathbb{N}_{n-1} \text{ satisfying condition (5.9)}, \\ \pi\text{-modular } \mathbb{M} \subset \mathbb{N}_n \text{ satisfying condition (5.8) and (5.16)}, \\ \text{and there exists a morphism } O_{\breve{F}}e \to \mathbb{M}^* \text{ induced by } x. \end{array} \right\},$$

$$\mathcal{Y}_{n-1}(x)(\bar{k}) = \left\{ \begin{array}{c} \text{almost } \pi\text{-modular } \mathbb{L} \subset \mathbb{N}_{n-1} \text{ satisfying condition (5.9)}, \\ \text{and there exists a morphism } O_{\breve{F}}e \to \mathbb{L}^* \text{ induced by } x. \end{array} \right\}.$$

The morphism γ_x^+ just sends $(\mathbb{L}, x\colon O_{\breve{F}}e \to \mathbb{L}^*)$ to $(\mathbb{L}, \mathbb{M}, O_{\breve{F}}e \stackrel{(x,0)}{\to} \mathbb{L}^* \oplus O_{\breve{F}}e \subset^1 \pi^{-1}\mathbb{M} = \mathbb{M}^*)$ where $\mathbb{M} = \delta^+(\mathbb{L})$. To show it is surjective, it suffices to show given a morphism $O_{\breve{F}}e \to \mathbb{M}^*$ induced by x, it actually factors through $O_{\breve{F}}e \to \mathbb{L}^*$.

Recall any $x \in \mathbb{V}_n$ induces a morphism of rational Dieudonné modules $x\colon \breve{F}e \to \mathbb{N}_n$, and we have the following relation from (5.10)

$$h(xe, ye) = h(e, e)h(x, y) = \delta h(x, y).$$

Consider the inclusion $\mathbb{L}^* \oplus O_{\breve{F}}e \subset^1 \mathbb{M}^*$. The element $u \in \mathbb{V}_n$ induces a morphism $\breve{F}e \to \mathbb{N}_n$ sending e to $u(e) = \pi e$. As we are taking $x \in \mathbb{V}_{n-1}$, we have $h(x, u) = 0$. This implies $h(x(e), u(e)) = 0$ so $x(e)$ lies in the orthogonal complement of $e \in \mathbb{M}^*$, which is just $\mathbb{L}^*$. Thus the morphism $x \colon O_{\breve{F}}e \to \mathbb{M}^*$ naturally factors through $\mathbb{L}^*$.

Next we show the morphism γ_x^+ is formally smooth. Consider a Noetherian local $O_{\breve{F}}$-algebra R on which π is nilpotent, and a square zero ideal $I \subset R$. Let $S = R/I$ so that $R \to S$ is a thickening, equipped with the trivial nilpotent divided power structure on I. Suppose we have a commutative diagram consisting of solid arrows

$$\begin{tikzcd}
\operatorname{Spec} S \arrow[r] \arrow[d] & \mathcal{Y}_{n-1}(x) \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \mathcal{N}_{n-1} \cap \mathcal{Y}_n(x)
\end{tikzcd}$$

and we want to find a dotted arrow making the diagram commutative.

Equivalently, we have $(X', \iota', \lambda', \rho') \in \mathcal{Y}_{n-1}(x)(S)$ and a lift $(\widetilde{X'}, \widetilde{\iota'}, \widetilde{\lambda'}, \widetilde{\rho'}) \in \mathcal{N}_{n-1}(R)$ such that under δ^+ its image $(\widetilde{X}, \widetilde{\iota}, \widetilde{\lambda}, \widetilde{\rho}) \in \mathcal{Y}_n(x)(R)$. We want to show that $(\widetilde{X'}, \widetilde{\iota'}, \widetilde{\lambda'}, \widetilde{\rho'}) \in \mathcal{Y}_{n-1}(x)(R)$.

Write $\delta^+(X', \iota', \lambda', \rho') = (X, \iota, \lambda, \rho)$. Let $\phi \colon X \to X' \times \overline{\mathcal{E}}_S$ and $\widetilde{\phi} \colon \widetilde{X} \to \widetilde{X'} \times \overline{\mathcal{E}}_R$ be the O_F-linear isogenies lifting ϕ_0. Over S we have a commutative diagram from Dieudonné crystals:

$$\begin{tikzcd}
\operatorname{Fil}^1(\overline{\mathcal{E}}_S) \arrow[r] \arrow[d] & \mathbb{D}(\overline{\mathcal{E}}_S) \arrow[d, "\mathbb{D}(x)\oplus 0"] \\
\operatorname{Fil}^1((X')^\vee) \oplus \operatorname{Fil}^1(\overline{\mathcal{E}}_S) \arrow[r] \arrow[d] & \mathbb{D}((X')^\vee) \oplus \mathbb{D}(\overline{\mathcal{E}}_S) \arrow[d, "\mathbb{D}(\phi^\vee)"] \\
\operatorname{Fil}^1(X^\vee) \arrow[r] & \mathbb{D}(X^\vee)
\end{tikzcd}$$

and it can be extended to a commutative diagram over R:

$$\begin{tikzcd}
\mathrm{Fil}^1(\overline{\mathcal{E}}_R) \arrow[r] \arrow[dd, bend right] & \mathbb{D}(\overline{\mathcal{E}}_R) \arrow[d, "\mathbb{D}(x)_R\oplus 0"] & \\
\mathrm{Fil}^1(\widetilde{X'}^\vee)\oplus\mathrm{Fil}^1(\overline{\mathcal{E}}_R) \arrow[r] \arrow[d] & \mathbb{D}((X')^\vee)_R\oplus\mathbb{D}(\overline{\mathcal{E}}_R) \arrow[r] \arrow[d, "\mathbb{D}(\widetilde{\phi}^\vee)"] & \mathrm{Lie}(\widetilde{X'}^\vee)\oplus\mathrm{Lie}(\overline{\mathcal{E}}_R) \arrow[d, "\mathrm{Lie}(\widetilde{\phi}^\vee)"] \\
\mathrm{Fil}^1(\widetilde{X}^\vee) \arrow[r] & \mathbb{D}(X^\vee)_R \arrow[r] & \mathrm{Lie}(\widetilde{X}^\vee).
\end{tikzcd}$$

Suppose f is an R-basis of $\mathrm{Fil}^1(\overline{\mathcal{E}}_R)$ and consider its image f' under $\mathbb{D}(x)_R$ in $\mathrm{Lie}(\widetilde{X'}^\vee)$. By Grothendieck–Messing theory, it suffices to show the image is zero, as then we will have a lift of the morphism $\overline{\mathcal{E}}_S \to (X')^\vee$ over S to $\overline{\mathcal{E}}_R \to \widetilde{X'}^\vee$ over R.

Consider the reduction to residue field. Then there are two cases corresponding to the two cases in the proof of Proposition 5.14:

Case 1: As in the proof, we can find a standard normal basis $\sigma_1, \tau_1, \dots, \sigma_m, \tau_m$ of the Dieudonné module M^* of $X_{\bar{k}}^\vee$ with $\sigma_1 \notin VM^*$ such that element $e \in M^*$ is given by $\sigma_m + \frac{\pi\delta}{2}\tau_m$. Let $\varphi = \sigma_m - \frac{\pi\delta}{2}\tau_m$. By Lemma 5.10 we have

$$\mathrm{Lie}(X_{\bar{k}}^\vee) = \langle \tau_1, \pi\tau_1, \sigma_2, \tau_2, \dots, \sigma_m, \tau_m\rangle_{\bar{k}}.$$

Denote the Dieudonné module of $(X')_{\bar{k}}^\vee$ by L^*. Then by (5.17) we have

$$L^* = \langle e\rangle^\perp = \langle \sigma_1, \tau_1, \dots, \sigma_{m-1}, \tau_{m-1}, \varphi\rangle_{O_{\breve{F}}},$$

by (5.7) and (5.22) we have

$$VL^* = \pi_0(VL)^* = \langle \sigma_1, \pi_0\tau_1, \pi\sigma_2, \dots, \pi\tau_{m-1}, \pi\varphi\rangle_{O_{\breve{F}}},$$
$$\mathrm{Lie}((X')_{\bar{k}}^\vee) = L^*/VL^* = \langle \tau_1, \pi\tau_1, \sigma_2, \dots, \tau_{m-1}, \varphi\rangle_{\bar{k}}.$$

The map $\mathbb{D}(\phi^\vee)_{\bar{k}} \colon \mathbb{D}((X')^\vee_{\bar{k}}) \oplus \mathbb{D}(\overline{\mathbb{E}}) \to \mathbb{D}(X^\vee_{\bar{k}})$ then can be described as

$$\begin{aligned} \sigma_i &\mapsto \sigma_i, & \tau_i &\mapsto \tau_i \text{ for } 1 \leqslant i \leqslant m-1, \\ \varphi &\mapsto \sigma_m - \frac{\pi\delta}{2}\tau_m, & e &\mapsto \sigma_m + \frac{\pi\delta}{2}\tau_m. \end{aligned} \tag{5.27}$$

We may lift the $\bar{k}$-basis $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m, \varphi, e$ to an R-basis, denoted by $\widetilde{\sigma}_1, \widetilde{\tau}_1, \ldots, \widetilde{\sigma}_m, \widetilde{\tau}_m, \widetilde{\varphi}, \widetilde{e}$ such that the map $\mathbb{D}(\widetilde{\phi}^\vee)$ has the same description as (5.27) with every element replaced by its R-lift, and

$$\mathrm{Lie}(\widetilde{X'}^\vee) = \langle \widetilde{\tau}_1, \pi\widetilde{\tau}_1, \widetilde{\sigma}_2, \ldots, \widetilde{\tau}_{m-1}, \widetilde{\varphi} \rangle_R.$$

Write f' in terms of the basis as

$$f' = a\widetilde{\tau}_1 + a'\pi\widetilde{\tau}_1 + a_2\widetilde{\sigma}_2 + b_2\widetilde{\tau}_2 + \cdots + a_{m-1}\widetilde{\sigma}_{m-1} + b_{m-1}\widetilde{\tau}_{m-1} + \alpha\widetilde{\varphi}.$$

Then under the map $\mathrm{Lie}(\widetilde{\phi}^\vee)$, it maps to

$$a\widetilde{\tau}_1 + a'\pi\widetilde{\tau}_1 + a_2\widetilde{\sigma}_2 + b_2\widetilde{\tau}_2 + \cdots + a_{m-1}\widetilde{\sigma}_{m-1} + b_{m-1}\widetilde{\tau}_{m-1} + \alpha\widetilde{\sigma}_m - \alpha\frac{\pi\delta}{2}\widetilde{\tau}_m$$

which should be zero as we have the morphism $\overline{\mathcal{E}}_R \to \widetilde{X}^\vee$. This implies that $a = a' = a_2 = b_2 = \cdots = a_{m-1} = b_{m-1} = \alpha = 0$ and $f' = 0$.

Case 2: Again we can find a standard normal basis $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m$ of M^* with $\sigma_1 \notin VM^*$ such that $e \in M^*$ is given by $\sigma_1 + \frac{\pi\delta}{2}\tau_1$. Let $\varphi = \sigma_1 - \frac{\pi\delta}{2}\tau_1$. Denote the Dieudonné module of $(X')^\vee_{\bar{k}}$ by L^*. Then

$$\begin{aligned} L^* &= \langle \varphi, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m \rangle_{O_{\breve{F}}}, \\ VL^* = \pi L^* &= \langle \pi\varphi, \pi\sigma_2, \pi\tau_2, \ldots, \pi\sigma_m, \pi\tau_m \rangle_{O_{\breve{F}}}, \\ \mathrm{Lie}((X')^\vee_{\bar{k}}) &= \langle \varphi, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m \rangle_{\bar{k}}, \\ \mathrm{Lie}(X^\vee_{\bar{k}}) &= \langle \tau_1, \pi\tau_1, \sigma_2, \tau_2, \ldots, \sigma_m, \tau_m \rangle_{\bar{k}}. \end{aligned}$$

The map $\mathbb{D}(\phi^\vee)_{\bar{k}} \colon \mathbb{D}((X')^\vee_{\bar{k}}) \oplus \mathbb{D}(\overline{\mathbb{E}}) \to \mathbb{D}(X^\vee_{\bar{k}})$ then can be described as

$$\begin{aligned} &\varphi \mapsto \sigma_1 - \frac{\pi\delta}{2}\tau_1, \quad && e \mapsto \sigma_1 + \frac{\pi\delta}{2}\tau_1, \\ &\sigma_i \mapsto \sigma_i, && \tau_i \mapsto \tau_i \text{ for } 2 \leqslant i \leqslant m. \end{aligned} \tag{5.28}$$

We may lift the $\bar{k}$-basis to an R-basis denoted by $\widetilde{\sigma}_1, \widetilde{\tau}_1, \ldots, \widetilde{\sigma}_m, \widetilde{\tau}_m, \widetilde{\varphi}, \widetilde{e}$ such that the map $\mathbb{D}(\widetilde{\phi}^\vee)$ has the same description as (5.28) with every element replaced by its R-lift. Write f' in terms of the basis as

$$f' = \alpha\widetilde{\varphi} + a_2\widetilde{\sigma}_2 + b_2\widetilde{\tau}_2 + \cdots + a_m\widetilde{\sigma}_m + b_m\widetilde{\tau}_m.$$

Then under the map $\mathrm{Lie}(\widetilde{\phi}^\vee)$, it maps to

$$\alpha\widetilde{\sigma}_1 - \alpha\frac{\pi\delta}{2}\widetilde{\tau}_1 + a_2\widetilde{\sigma}_2 + b_2\widetilde{\tau}_2 + \cdots + a_m\widetilde{\sigma}_m + b_m\widetilde{\tau}_m$$

which should be zero as we have the morphism $\overline{\mathcal{E}}_R \to \widetilde{X}^\vee$. This implies that $\alpha = a_2 = b_2 = \cdots = a_m = b_m = 0$ and $f' = 0$.

This finishes the proof that γ_x^+ is formally smooth and is an isomorphism. $\square$

Proposition 5.18. *Let $L \subset \mathbb{V}_{n-1}$ be an O_F-lattice. View L as an O_F-lattice (of rank $n-1$) in $\mathbb{V}_n$ via the hermitian embedding $\mathbb{V}_{n-1} \to \mathbb{V}_n$ and set $L^\# = L \ominus \langle f \rangle_{O_F}$. Then*

$$\mathrm{Int}_{n-1,\mathcal{Y}}(L) = \frac{1}{2}\mathrm{Int}_{n,\mathcal{Y}}(L^\#).$$

In particular, the intersection number $\mathrm{Int}_{n-1,\mathcal{Y}}(L)$ is independent of a choice of basis of L.

Proof. From Lemma 3.11 and Theorem 5.5 we have

$$\mathcal{Y}_n(f) \simeq \mathcal{Z}_n(\pi f) = \mathcal{Z}_n(u)\,, \mathcal{Y}_n(f) \cap \mathcal{N}_n^+ \simeq \mathcal{Z}_n(u)^+ \simeq \mathcal{N}_{n-1}.$$

From Proposition 5.17 we have

$$\mathcal{N}_n^+ \cap \mathcal{Y}_n(f) \cap \mathcal{Y}_n(x) \simeq \mathcal{Y}_{n-1}(x)$$

for any nonzero $x \in \mathbb{V}_{n-1}$. The same is true for $\mathcal{N}_n^-$. The results then follow from definition. $\square$

Consider nonzero $x \in \mathbb{V}_{n-1}$ and assume $h(x, x) \in \pi_0 O_{F_0}$. By Lemma 3.11 and Proposition 5.17 we have

$$\mathcal{Y}_{n-1}(x) \simeq \mathcal{N}_{n-1} \cap \mathcal{Y}_n(x) \simeq \mathcal{N}_{n-1} \cap \mathcal{Z}_n(\pi x).$$

By Proposition 5.16 we have $\mathcal{N}_{n-1} \cap \mathcal{Z}_n(\pi x) \simeq \mathcal{Z}_{n-1}(\pi x)$. Combine the results we get

Corollary 5.19. *For any nonzero $x \in \mathbb{V}_{n-1}$ such that $h(x, x) \in \pi_0 O_{F_0}$, we have identification*

$$\mathcal{Y}_{n-1}(x) \simeq \mathcal{Z}_{n-1}(\pi x).$$

Explicitly the isomorphism is given by

$$(X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S), x' \colon \overline{\mathcal{E}}_S \to (X')^\vee \longmapsto (X', \iota', \lambda', \rho') \in \mathcal{N}_{n-1}(S), \overline{\mathcal{E}}_S \xrightarrow{x'} (X')^\vee \xrightarrow{r} X'$$

where $r \colon (X')^\vee \to X'$ is the unique morphism such that $r \circ \lambda' = \iota'(\pi)$.

Chapter 6: Proof of main Theorem

In this section we collect results in previous sections and prove Theorem 3.8.

- When n is even, the Theorem is proved in Corollary 3.12.
- When $n = 1$, the condition (2.13) is redundant and $\mathcal{N}_1 \simeq \operatorname{Spf} O_{\breve{F}}$. As the polarization is principal, $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles are the same. Let $L = \langle \ell \rangle_{O_F} \subset \mathbb{V}_1$ be a hermitian O_F-lattice with $\operatorname{val}_{\pi_0} h(\ell, \ell) = a$. Set $L^\# = L \obot I_1^{-1}$. Then on analytic side, by Proposition 4.5, we have $\partial\mathrm{Den}(L) = \frac{1}{2}\partial\mathrm{Den}(L^\#)$ and by [LL22, Lemma 2.43], $\partial\mathrm{Den}(L^\#) = 2(a+1)$ and so $\partial\mathrm{Den}(L) = a+1$. On the geometric side, by the theory of canonical lifting ([Gro86]) we have $\mathrm{Int}_{\mathcal{Y}}(L) = a+1$. Hence $\mathrm{Int}_{\mathcal{Y}}(L) = \partial\mathrm{Den}(L)$.
- Now assume $n \geqslant 3$ is odd. Let $L \subset \mathbb{V}_n$ be any O_F-lattice. View it as in $\mathbb{V}_{n+1} = \mathbb{V}_n \obot \langle f \rangle_F$ via (5.7) and set $L^\# = L \obot \langle f \rangle_{O_F}$.

  On the analytic side, by Proposition 4.5 we have

  $$\partial\mathrm{Den}(L) = \frac{1}{2}\partial\mathrm{Den}(L^\#).$$

  On the geometric side, by Proposition 5.18 we have

  $$\mathrm{Int}_{n,\mathcal{Y}}(L) = \frac{1}{2}\mathrm{Int}_{n+1,\mathcal{Y}}(L^\#).$$

  By Corollary 3.12 we have $\partial\mathrm{Den}(L^\#) = \mathrm{Int}_{n+1,\mathcal{Y}}(L^\#)$. Thus we get $\partial\mathrm{Den}(L) = \mathrm{Int}_{n,\mathcal{Y}}(L)$. The Theorem is fully proved.

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