TROPICAL GEOMETRY OF CURVES WITH LARGE THETA CHARACTERISTICS

by

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ABSTRACT

Tropical geometry of curves with large theta characteristics

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In this dissertation we study tropicalization curves which have a theta characteristic with large rank. This fits in the more general framework of studying the limit linear series on a curve which degenerates to a singular curve. We explore this when the singular curve is not of compact type. In particular we investigate the case when dual graph of the degenerate curve is a chain of $g$-loops.

The fundamental object under consideration is a family of curves over a complete discrete valuation ring. In the first half of the dissertation we study geometry of such a family. In the third chapter we study metric graphs and divisors on them. This could be a thought of as the theory of limit linear series on a curve of non-compact type. In the fourth chapter we make this connection via tropicalization.

We consider a family of curves with smooth generic fiber $\mathcal{X}_\eta$ of genus $g$ such that the dual graph of the special fiber is a chain of $g$ loops. The main theorem we prove is that if $\mathcal{X}_\eta$ has a theta characteristic of rank $r$ then there are at least $r$ linear relations on the edge lengths of the dual graph.
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CHAPTER 1

Introduction

1.1. Background

The study of degenerate curves plays a crucial role in our understanding of a general smooth curve. This is mainly because it is hard to get a grasp on a general curve while degenerate curves are often simpler to understand. One of the first success of this idea was the theory limit linear series developed by Griffiths and Harris \[1\] which they used to prove the Brill-Noether theorem. Their method involved analyzing the limit of a line bundle to a degenerate curve. The degenerate curves that they considered in their theory were reducible curves whose Jacobians are compact. In other words, the components of the degenerate curve do not intersect to form a loop.

In 2012 Cools, Draisma, Payne, and Robeva gave an entirely different proof of the Brill-Noether theorem using degenerate curves of non-compact type \[2\]. The degenerate curves that they consider are of non-compact type. The theory of limit linear series on such curves takes the shape of metric graphs and divisors on them \[3\]. This has been one of the cornerstones of tropical geometry. A particular curve for which this theory seems to work effectively is one whose dual graph is a chain of $g$ loops.

Chain of $g$-loops
1.2. Smoothenings of chain of \( g \) loops

Consider the following locus in \( \mathcal{M}_g \).

\[
\mathcal{M}_g^r = \{ C | C \text{ has a theta characteristic } \mathcal{L} \text{ such that } \dim H^0(C, \mathcal{L}) \geq r + 1 \}
\]

In [4] Harris proves that the codimension of \( \mathcal{M}_g^r \) in \( \mathcal{M}_g \) is at most \( r(r+1)/2 \). Further, he conjectures that when \( g >> r \) the codimension of \( \mathcal{M}_g^r \) should exactly be \( \frac{r(r+1)}{2} \). It is easy to see that the totally degenerate curve considered by Cools, Draisma, Payne, and Robeva in [2] lies in the closure of \( \mathcal{M}_g^r \).

Consider a family of curves \( \mathcal{X} \) over a complete discrete valuation ring with the special fiber \( \mathcal{X}^0 \) whose dual graph is a chain of \( g \) loops. Suppose the generic fiber \( \mathcal{X}_\eta \) is a smooth curve that lies in \( \mathcal{M}_g^r \). The dual graph of \( \mathcal{X}^0 \) has edge lengths \( l_i, m_i \) as shown above. Since the expected codimension of \( \mathcal{M}_g^r \) is \( \frac{r(r+1)}{2} \), we can hope that that in such a smoothening of \( \mathcal{X}^0 \) there are \( \frac{r(r+1)}{2} \) relation among the edge lengths. For example one can prove that if \( \mathcal{X}_\eta \) fiber is hyperelliptic then there are \( g-1 \) relations on the edge lengths which is the codimension of locus of hyperelliptic curves in \( \mathcal{M}_g \). We prove the following theorem.

**Theorem 1.2.1.** Suppose \( \mathcal{X}_\eta \) admits a theta characteristic \( \mathcal{L} \) with \( \dim H^0(\mathcal{X}, \mathcal{L}) \geq r + 1 \), then there exists a set \( J \subset \{1,2,\ldots,g\} \) of size \( r \) such that for all \( i \in J \), \( \frac{L}{m_i} \) is a ratio of two positive integers whose sum is at most equal to \( g - 1 \).

The idea behind the proof is the following. If \( X \) is a general smooth curve with a bundle \( L \), the Gieseker-Petri theorem [5] says that the multiplication map

\[
H^0(X, L) \otimes H^0(X, K_X \otimes L^{-1}) \to H^0(X, K_X)
\]

is injective. In [6] Jensen and Payne gave a tropical proof of the Gieseker-Petri theorem using generic smoothening of the chain of \( g \) loops. If in the situation of Gieseker-Petri theorem \( L \) happens to be a theta characteristic on \( X \), then the multiplication map has a non-trivial kernel whenever \( \dim H^0(X, L) \geq 2 \). Therefore not surprisingly, the ideas in the tropical proof of the Gieseker-Petri theorem play a crucial role in the proof theorem 1.2.1.
One can consider a locus analogous to \( \mathcal{M}_g \) in the moduli of tropical curves and ask if it has the expected co-dimension \( \frac{r(r+1)}{2} \). This locus would correspond to the space of metric graphs which have a theta characteristic with rank \( r \). However we construct edge lengths for the chain of \( g \) loops with only \( 2r - 1 \) linear relations such that the corresponding graph has a theta characteristic of rank at least \( r \).
CHAPTER 2

Degenerating family of curves

2.1. Background

In this section we study a family of curves over a complete discrete valuation ring $R$. We denote the fraction field of $R$ by $K$ and the residue field of $R$ by $\kappa$. We let $\pi$ denote a uniformizer of $R$ and assume that the valuation of $\pi$ is 1. We assume that the residue field $\kappa$ is algebraically closed. By a curve we mean a proper, geometrically connected scheme of dimension 1 over a field. By a family of curves $\mathcal{X} \to \text{spec}(R)$ we mean a proper, flat scheme over $\text{spec}(R)$ of relative dimension 1. We denote the generic fiber of $\mathcal{X}$ by $\mathcal{X}_\eta$ and the special fiber by $\mathcal{X}^0$. We further assume that $\mathcal{X}$ has the following properties.

- The generic fiber $\mathcal{X}_\eta$ is a smooth curve over $K$.
- The family $\mathcal{X}$ is strongly semi-stable.
  This means that the special fiber $\mathcal{X}^0$ is reduced with nodal singularities without self intersections.
- The family $\mathcal{X}$ is regular.

Let $\text{Div}(\mathcal{X})$ (resp. $\text{Div}(\mathcal{X}_\eta)$) be the group of Cartier divisors on $\mathcal{X}$ (resp. $\mathcal{X}_\eta$). Note that since $\mathcal{X}$ is regular Cartier divisors on $\mathcal{X}$ are same as Weil divisors. If $D \in \text{Div}(\mathcal{X}_\eta)$ is a divisor on the generic fiber then closure of $D$ is a divisor on $\mathcal{X}$. We call such a divisor a horizontal divisor. If a divisor on $\mathcal{X}$ is supported on $\mathcal{X}^0$ then we call it a vertical divisor. Any divisor on $\mathcal{X}$ can be written uniquely as a sum of a horizontal and a vertical divisor. We use the notation $\mathcal{X}^0_i$ to denote the components of the special fiber $\mathcal{X}^0$.

For a rational function $f \in K(\mathcal{X}_\eta)$ we denote the divisor of $f$ on $\mathcal{X}_\eta$ by $\text{div}(f)$. For a divisor $D \in \text{Div}(\mathcal{X}_\eta)$ we denote the linear series of $D$ by $H^0(\mathcal{X}_\eta, \mathcal{O}_{\mathcal{X}_\eta}(D))$. That is,
\[ H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D)) = \{ f \in K(\mathcal{X}) \mid D + \text{div}(f) \text{ is effective} \} \]

The rank of \( D \) is by definition

\[ r(D) = \dim_K(H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D))) - 1 \]

Let \( f \in K(\mathcal{X}) \) be a non-zero rational function on \( \mathcal{X} \). Suppose \( f \) vanishes to the order \( e_i \) along the component \( \mathcal{X}_i^0 \) of the special fiber. If \( \pi \) is a uniformizer of \( R \) then \( f/\pi^{e_i} \) is a unit in the local ring \( \mathcal{O}_{\mathcal{X}_i^0} \). The image of \( f/\pi^{e_i} \) in \( \mathcal{O}_{\mathcal{X}_i^0}/\mathcal{I}_{\mathcal{X}_i^0} \cong \kappa(\mathcal{X}_i^0) \) is a non-zero rational function on \( \mathcal{X}_i^0 \) which we denote by \( \tilde{f}_i \). We call this reduction of \( f \) on the component \( \mathcal{X}_i^0 \). Note that \( \tilde{f}_i \) is only well defined upto multiplication by an element in \( \kappa^* \). By convention if \( f \) is zero then we define \( \tilde{f}_i \) to be zero as well.

**Remark 2.1.1.** The assignment \( f \to \tilde{f}_i \) is multiplicative in the sense that for rational function \( f, f' \in K(\mathcal{X}) \), we have \( (ff')_i = \tilde{f}_i \cdot \tilde{f}'_i \).

**Proposition 2.1.2.** Suppose \( V \subset K(\mathcal{X}) \) is a finite dimensional vector space over \( K \). Let \( \overline{V} \) denote

\[ \overline{V} = \{ \tilde{f}_i \mid f \in V \} \subset \kappa(\mathcal{X}_i^0) \]

Then \( \overline{V} \) is a subspace of \( \kappa(\mathcal{X}_i^0) \) of the same dimension as that of \( V \).

**Proof.** It is easy to see that \( \dim_\kappa \overline{V} \leq \dim_K V \). We note that the assignment \( f \to \text{ord}_{\mathcal{X}_i^0}(f) \) is a valuation on \( K(\mathcal{X}) \). Consider the set

\[ V^+ = \{ f \in V \mid \text{ord}_{\mathcal{X}_i^0}(f) \geq 0 \} \]

Then \( V^+ \) is a finitely generated module over \( R \). Since it is torsion free we can conclude that it is free of rank equal to \( \dim_K V \). If \( \{ f_1, f_2, \ldots, f_n \} \) is a basis of \( V^+ \) as an \( R \)-module then we see that the reductions of \( \{ f_1, f_2, \ldots, f_n \} \) on \( \mathcal{X}_i^0 \) are \( \kappa \)-linearly independent. \( \square \)

**Lemma 2.1.3.** Let \( C \) be a smooth curve over \( \kappa \). Let \( V \subset \kappa(C) \) be a finite dimensional linear series. Let \( p \in C \) be a closed point on \( C \). Suppose \( \{ f_1, f_2, \ldots, f_n \} \) and
\{g_1, g_2, \ldots, g_n\} are two basis of \(V\) such that \(\{f_i\}\) have distinct orders of vanishing at \(p\).

Then the order of vanishing of \(\prod_i^n f_i\) at \(p\) is greater or equal to that of \(\prod_i^n g_i\).

**Proof.** Suppose the basis elements are numbered such that \(\{f_i\}\) have strictly increasing orders of vanishing at \(p\) and \(\{g_i\}\) have non-decreasing orders of vanishing at \(p\).

We will prove that \(\text{ord}_p(f_j) \geq \text{ord}_p(g_j)\). Clearly this is sufficient to prove the lemma.

Suppose \(g_j = \sum_i^n a_i^j \cdot f_i\). Since \(f_i\) have have distinct orders of vanishing

\[\text{ord}_p(g_j) = \text{ord}_p(f_i')\]

where \(i'\) is the smallest index such that \(a_i^j\) is non-zero. Therefore it suffices to argue

that the smallest non-zero entry in the vector \(\{a_i^j\}\) occurs at some \(i \leq j\). If this is false then \(g_j\) lies in the span of \(\{f_{j+1}, f_{j+2}, \ldots, f_n\}\). We proceed to derive a contradiction.

Since \(\text{ord}_p(g_{j+1}) \geq \text{ord}_p(g_j)\) by writing \(g_{j+1}\) in the basis \(\{f_i\}\), we can see that \(g_{j+1}\) lies in the span of \(\{f_{j+1}, f_{j+2}, \ldots, f_n\}\) as well. Inductively we can argue that \(g_k\) lies in the span of \(\{f_{j+1}, f_{j+2}, \ldots, f_n\}\) for all \(k \geq j\). Therefore the elements \(g_j, g_{j+1}, \ldots, g_n\) lie in the subspace of spanned by \(\{f_{j+1}, f_{j+2}, \ldots, f_n\}\). But this contradicts the linear independence of \(g_j, g_{j+1}, \ldots, g_n\). \(\square\)

**Lemma 2.1.4.** Let \(\mathcal{X}_1^0, \mathcal{X}_2^0\) be two components of \(\mathcal{X}^0\) and \(p \in \mathcal{X}_1^0\) be any point on \(\mathcal{X}_1^0\). Let \(V \subset K(\mathcal{X}_1^0)\) be a finite dimensional linear series. Then there exists a basis \(\{f_1, f_2, \ldots, f_n\}\) of \(V\) such that

1. the reductions of \(\{f_i\}\) in \(\kappa(\mathcal{X}_1^0)\) have distinct orders of vanishing at \(p\) and
2. the reductions of \(\{f_i\}\) in \(\kappa(\mathcal{X}_2^0)\) are linearly independent.

**Proof.** This is proved in [6]. We give a proof here for the sake of completeness. Suppose \(\{g_1, g_2, \ldots, g_n\}\) is a basis of \(V\) such that the reductions of \(\{g_i\}\) on \(\mathcal{X}_1^0\) are \(\kappa\)-linearly independent. By taking linear combinations with coefficients in \(R^\times\) we may assume that the reductions of \(\{g_i\}\) have distinct orders of vanishing at \(p\). Now we multiply each \(g_i\) by a suitable power of \(\pi\) and relabel the indices so that

1. For each \(i\) the order of vanishing of \(g_i\) along \(\mathcal{X}_2^0\) is zero.
2. The reductions of \(\{g_i\}\) on \(\mathcal{X}_1^0\) have strictly decreasing orders of vanishing at \(p\).
As in proof of Proposition 2.1.2 we consider the $R$ module

$$V_2^+ = \{ f \in V \mid \text{ord}_{\mathcal{X}_0^0}(f) \geq 0 \}$$

The elements $\{g_i\}_i \subset V_2^+$ are $R$-linearly independent. To obtain a basis over $R$ we can replace $g_i$ by

$$g'_i = \pi^{-e} \sum_{j \leq i} a_j \cdot g_j \in V_2^+$$

where $e$ is a non-negative integer and $a_j$ are units in $R$. Then the reduction of $g'_i$ on $\mathcal{X}_1^0$ has the same order of vanishing at $p$ as that of $g_i$. Also the reductions of $g'_i$ on $\mathcal{X}_2^0$ are $\kappa$-linearly independent. □

2.2. Local geometry

Let $p$ be a closed point on the special fiber $\mathcal{X}^0$. We denote the local ring of $\mathcal{X}$ at $p$ by $\mathcal{O}_{\mathcal{X}, p}$ and its completion with respect to the maximal ideal by $\hat{\mathcal{O}}_{\mathcal{X}, p}$. We denote the fraction field of $\hat{\mathcal{O}}_{\mathcal{X}, p}$ by $\hat{K}(\mathcal{X})_p$. Thus we have the following diagram of schemes over Spec($R$).

$$\begin{array}{ccc}
\text{Spec}(\hat{\mathcal{O}}_{\mathcal{X}, p}) & \longrightarrow & \text{Spec}(\mathcal{O}_{\mathcal{X}, p}) \longrightarrow \mathcal{X} \\
\mathcal{X} & \phi_p & \\
\end{array}$$

In this section we study the geometry of $\text{Spec}(\hat{\mathcal{O}}_{\mathcal{X}, p}) \rightarrow \text{Spec}(R)$.

**Lemma 2.2.1.** Let $q \in \mathcal{X}_q$ be a closed point. Then the closure of $q$ in $\mathcal{X}$ intersects the special fiber $\mathcal{X}^0$ at a unique point.

**Proof.** Let $K'$ be the residue field $q$. Since $K$ is complete the valuation on $K$ extends uniquely to $K'$. Let $R' \subset K'$ denote the valuation ring of $K'$. Since $\mathcal{X}$ is proper and separated over Spec($R$), the map Spec($K'$) $\rightarrow \mathcal{X}$ can be uniquely lifted to Spec($R'$). Thus we have the following commutative diagram.
The image \( \text{Spec}(R') \) in \( \mathcal{X} \) is the closure of \( q \). Since \( \text{Spec}(R') \) has a unique closed point the closure of \( q \) intersects \( \mathcal{X}^0 \) at a unique point. \( \square \)

**Definition 2.2.2.** Let \( q \in \mathcal{X}_\eta \) be a closed point whose closure intersects the special fiber at \( q_0 \). Then we say \( q \) specializes to \( q_0 \).

**Lemma 2.2.3.** The morphism \( \text{Spec}(\mathcal{O}_{X,p}) \to \text{Spec}(\hat{\mathcal{O}}_{X,p}) \) is bijective on points.

**Proof.** Since \( \mathcal{O}_{X,p} \to \hat{\mathcal{O}}_{X,p} \) is faithfully flat \( \text{Spec}(\mathcal{O}_{X,p}) \to \text{Spec}(\hat{\mathcal{O}}_{X,p}) \) is surjective on points. Since \( \text{Spec}(\mathcal{O}_{X,p}) \) and \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \) have dimension two we have to argue that \( \mathcal{O}_{X,p} \to \hat{\mathcal{O}}_{X,p} \) induces bijection on the height one primes. The height one primes in \( \mathcal{O}_{X,p} \) correspond to the irreducible divisors in \( \mathcal{X} \) which contain \( p \). If \( p \) is a prime corresponding to a horizontal divisor in \( \mathcal{X} \) then \( \mathcal{O}_{X,p}/p \) is a finite extension of \( R \) and therefore is complete. Hence the map \( \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/p \) uniquely lifts to \( \hat{\mathcal{O}}_{X,p} \). Therefore \( p\hat{\mathcal{O}}_{X,p} \) is a prime ideal in \( \hat{\mathcal{O}}_{X,p} \).

Suppose \( p \subset \mathcal{O}_{X,p} \) corresponds to a vertical divisor of \( \mathcal{X} \). If \( p \) lies on the smooth locus of \( \mathcal{X}^0 \) then \( p\hat{\mathcal{O}}_{X,p} \) is prime in \( \hat{\mathcal{O}}_{X,p} \). Since \( \mathcal{X} \) is strongly semi-stable \( p\hat{\mathcal{O}}_{X,p} \) is still prime in \( \hat{\mathcal{O}}_{X,p} \) when \( p \) is a node. \( \square \)

Since \( \hat{\mathcal{O}}_{X,p} \) is regular, Weil divisors are same as Cartier divisors on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \). In view of the Lemma 2.2.3 we can describe the divisors on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \) as follows. If \( q \) is a point on the generic fiber that specializes to \( p \) then the closure of \( q \) is a divisor on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \). More generally if \( E \) is a divisor on \( \mathcal{X}_\eta \) which is supported on points which specialize to \( p \) then the closure of \( E \) is a divisor on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \). We call such a divisor a horizontal divisor. The special fiber of \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \) is by definition the fiber over the closed point of \( \text{Spec}(R) \). If a divisor is supported on the special fiber then we call it a vertical divisor. The irreducible vertical divisors on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \) are in one to one correspondence with the generic points of the components of \( \mathcal{X}^0 \) which contain \( p \).
A divisor on \( \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X},p}) \) can be uniquely written as a sum of a vertical and a horizontal divisor.

**Definition 2.2.4.** Let \( D \) be a horizontal divisor on \( \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X},p}) \). Then \( D \) is the closure of a divisor \( D' \in \text{Div}(\mathcal{X}_\eta) \) which is supported on points which specialize to \( p \). We define degree of \( D \) to be the degree of \( D' \) on \( \mathcal{X}_\eta \).

**Lemma 2.2.5.** All divisors on \( \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X},p}) \) are principal.

**Proof.** The ring \( \hat{\mathcal{O}}_{\mathcal{X},p} \) is a regular local ring and therefore it is a unique factorization domain. Hence all divisors are principal.

**Lemma 2.2.6.** Let \( \phi_p \) denote the morphism \( \phi_p : \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X},p}) \rightarrow \mathcal{X} \) and let \( \pi \) be a uniformizer of \( R \).

1. Suppose \( p \) is a smooth point of \( \mathcal{X}_0 \) lying on the component \( \mathcal{X}_k \). Then
   \[
   \hat{\mathcal{O}}_{\mathcal{X},p}/(\pi) \cong \hat{\mathcal{O}}_{\mathcal{X}_k,p}
   \]

2. Suppose \( p \) is a node of \( \mathcal{X}_0 \) lying on the intersection of \( \mathcal{X}_i \) and \( \mathcal{X}_j \). Suppose \( \phi_p^{-1}(\mathcal{X}_i) = \text{div}(x) \) and \( \phi_p^{-1}(\mathcal{X}_j) = \text{div}(y) \) for some \( x, y \in \hat{\mathcal{O}}_{\mathcal{X},p} \). Then
   \[
   \hat{\mathcal{O}}_{\mathcal{X},p}/(x) \cong \hat{\mathcal{O}}_{\mathcal{X}_i,p}, \quad \hat{\mathcal{O}}_{\mathcal{X},p}/(y) \cong \hat{\mathcal{O}}_{\mathcal{X}_j,p}
   \]

**Proof.** This follows from the fact in algebra that taking completion of a local ring commutes with taking quotient by an ideal.

**Proposition 2.2.7.** Let \( E \) be a horizontal divisor on \( \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X},p}) \). Suppose \( E = \text{div}(a/b) \) for some \( a, b \in \hat{\mathcal{O}}_{\mathcal{X},p} \). Then

\[
\deg(E) = \dim_n \hat{\mathcal{O}}_{\mathcal{X},p}/(\pi, a) - \dim_n \hat{\mathcal{O}}_{\mathcal{X},p}/(\pi, b)
\]

**Proof.** Since both sides are additive it suffices to prove this when \( E \) is an irreducible horizontal divisor. So we may assume that \( E \) is the closure of a point \( q \) on \( \mathcal{X}_\eta \) that specializes to \( p \).
Let $K'$ be the residue field of $q$. Suppose $f : \hat{O}_{X,p} \to R'$ denote the ring map associated to the morphism $\text{Spec}(R') \to \text{Spec}(\hat{O}_{X,p})$. Then $\ker f \subset \hat{O}_{X,p}$ is a principal ideal. If $a \in \hat{O}_{X,p}$ generates $\ker f$ then $E = \text{div}(a)$. Thus we need to prove that
\[
\dim_k \frac{\hat{O}_{X,p}}{(\pi, a)} = [K' : K]
\]
If $R''$ denotes the image of $f$ then $\hat{O}_{X,p} \simeq R''$. Since $R''$ is a finitely generated torsion free module over $R$ it is free. The fraction field of $R''$ is $K'$. Therefore rank of $R''$ over $R$ is $[K' : K]$. Hence
\[
\dim_k \frac{\hat{O}_{X,p}}{(\pi, a)} = \dim_k \frac{R''}{(\pi)} = [K' : K]
\]
\[\square\]

**Proposition 2.2.8.** Let $p$ be a smooth point on $X^0$ which lies on the component $X^0_i$. Let $f \in K(X^0_{\eta})$ be a rational function. We write
\[\text{div}(f) = E + D\]
where $E$ and $D$ are divisors on $X_{\eta}$ such that $E$ is supported on points which specialize to $p$ and the closure of support of $D$ is disjoint from $p$. Then the order of vanishing of $\tilde{f}_i$ at $p$ equals $\deg(E)$.

**Proof.** Let $\tilde{f}_p$ denote the germ of $f$ at $\tilde{K}((X)_p)$. Let $E'$ denote the closure of $E$ in $\text{Spec}(\hat{O}_{X,p})$. We can write
\[\tilde{f}_p = \pi^l \cdot a/b\]
where $a, b \in \hat{O}_{X,p}$ are not divisible by $\pi$. By the hypothesis the horizontal component in $\text{div}(\tilde{f}_p)$ is $E'$ which means $\text{div}(a/b) = E'$. By Lemma 2.2.6 the order of vanishing of $\tilde{f}_i$ at $p$ is
\[
\dim_k \frac{\hat{O}_{X,p}}{(\pi, a)} - \dim_k \frac{\hat{O}_{X,p}}{(\pi, b)}
\]
By Proposition 2.2.7 this equals the degree of $E$.

\[\square\]
CHAPTER 3

Metric Graphs

3.1. Definitions and Terminology

By a graph we mean a connected graph with finite number of vertices and edges where
the edges are undirected. We allow multiple edges between vertices but do not allow an
edge from a vertex to itself. A metric graph is a pair \((G, \Lambda)\) where \(G\) is a graph and \(\Lambda\) is
a metric space obtained by gluing vertices and line segments according to the incidence
correspondence of the graph \(G\). A metric on this space is equivalent to assigning a length
to every edge. We will use the notation \(V(G)\) and \(E(G)\) to denote the vertices of \(G\) and
the edges of \(G\) respectively. We will often refer to a metric graph \((G, \Lambda)\) simply as \(\Lambda\)
when there is no ambiguity or when the structure of \(G\) is immaterial. By slight abuse
of notation we sometimes use vertices in \(V(G)\) to refer to corresponding points on \(\Lambda\). If
\(p \in \Lambda\) is point that does not correspond to a vertex of \(G\) then it lies in the interior of an
edge. In that case we define \(\deg(p)\) to be two as there are two directions coming towards
\(p\).

**Definition 3.1.1.** A divisor on a metric graph \(\Lambda\) is a finite formal sum
\(D = \sum_i a_i \cdot p_i\) where \(a_i\) are integers and \(p_i\) are points on \(\Lambda\). The degree of \(D\) is defined as
\(\sum_i a_i\).

The set of all divisors on \(\Lambda\) form a free abelian group which we denote by \(\text{Div}(\Lambda)\). The
divisors of degree zero form a subgroup of \(\text{Div}(\Lambda)\) which we write as \(\text{Div}^0(\Lambda)\). A divisor
\(D\) is said to be effective whenever all the coefficients in \(D\) are non-negative. We write
this as \(D \geq 0\). The genus of \(\Lambda\) is \(|V(G)| - |E(G)| + 1\) which also equals the topological
genus of \(\Lambda\). We denote the genus by \(g\). The canonical divisor \(K_\Lambda\) is defined as

\[
K_\Lambda = \sum_{p \in \Lambda} \deg(p) - 2
\]
The degree of $K_\Lambda$ is $2g - 2$.

**Definition 3.1.2.** A piece-wise linear function on $\Lambda$ is a function $f : \Lambda \to \mathbb{R}$ which is continuous and piece-wise affine with integer slopes. We denote the set of all piece-wise linear functions on $\Lambda$ by $PL(\Lambda)$.

**Definition 3.1.3.** The divisor of a piece-wise linear function $f$ is defined as

$$\text{div}(f) = \sum_{p \in \Lambda} a_f(p)p$$

where $a_f(p)$ is the sum of the slopes of $f$ at $p$ along all directions coming towards $p$. We call such a divisor a principal divisor.

**Proposition 3.1.4.** Let $f, g \in PL(\Lambda)$.

1. The degree of $\text{div}(f)$ is zero.
2. $\text{div}(f + g) = \text{div}(f) + \text{div}(g)$. 

**Proof.** We refer the reader to [2].

The set all principal divisors on $\Lambda$ form a subgroup of $\text{Div}^0(\Lambda)$ which we denote by $\text{Prin}(\Lambda)$. If $f_1, f_2 \in PL(\Lambda)$ we define the minimum of $f_1, f_2$ by setting

$$\min(f_1, f_2)(p) = \min(f_1(p), f_2(p))$$

It is easy to see that this defines a piece-wise linear function on $\Lambda$.

**Definition 3.1.5.** Let $D, D'$ be divisors on $\Lambda$. We say they are linearly equivalent and write $D \sim D'$ if $D - D'$ is a principal divisor.

**Definition 3.1.6.** Let $D$ be a divisor on $\Lambda$. The tropical module which we denote by $R(D)$ is defined as

$$R(D) = \{ f \in PL(\Lambda) \mid D + \text{div}(f) \text{ is effective} \}$$

**Proposition 3.1.7.** Let $D \in \text{Div}(\Lambda)$ be a divisor. Then the tropical module $R(D)$ is closed under taking minimums.
Proof. Let \( p \) be a point on \( \Lambda \) and suppose \( f, h \in R(D) \) are piece-wise linear functions. If the values of \( f \) and \( h \) at \( p \) are the same then the slope of \( \min(f, h) \) towards \( p \) along any direction is greater or equal to the corresponding slopes of \( f \) and \( h \). If \( f \) and \( h \) have different values at \( p \) then \( \min(f, h) \) equals the smaller function in a small neighborhood. \( \square \)

**Definition 3.1.8.** The rank of divisor \( D \) is the largest number \( r \) such that for every effective divisor \( T \) of degree \( r \) there exists a piece-wise linear function \( f \) in \( R(D) \) such that \( D - T + \text{div}(f) \) is effective.

Let \( D \in \text{Div}(\Lambda) \) be a divisor. Traditionally we refer to \( D \) as chip configuration on \( \Lambda \). For example if \( D = \sum_i a_i p_i \) we say \( D \) has \( a_i \) chips at \( p_i \). If \( D' \) is linearly equivalent to \( D \) then we say we can move chips from the configuration of \( D \) to that of \( D' \). For example see [2] for chip firing.

Let \( S \) be a finite set of points on \( \Lambda \). We say \( S \) is a rank determining set if for any divisor \( D \) the rank of \( D \) can be computed by effective divisors supported on \( S \). That is, \( D \) has rank \( r \) if and only if for every effective divisor \( T \) of degree \( r \) which is supported on \( S \) the divisor \( D - T \) is linearly equivalent to an effective divisor.

**Theorem 3.1.9 (Luo’s theorem).** Let \( S \subset \Lambda \) be a finite set. If the closure of every connected component of \( \Lambda - S \) is contractible then \( S \) is a rank determining set.

Proof. We refer to [7]. Thus in particular the set of vertices of \( G \) is a rank determining set. \( \square \)

**Theorem 3.1.10 (Tropical Riemann Roch).** For any divisor \( D \in \text{Div}(\Lambda) \) we have

\[
r(D) - r(K_\Lambda - D) = \deg(D) + 1 - g
\]

Proof. We refer to [8] or [9] for a proof. \( \square \)

**Lemma 3.1.11.** Let \( D \in \text{Div}(\Lambda) \) be a divisor. Suppose \( f_1, f_2, \ldots, f_n \in R(D) \) are piece-wise linear functions. Let \( p \) be any point on \( \Lambda \) and \( Y \) be a connected set in \( \Lambda \). Set

\[
\theta = \min\{f_1, f_2, \ldots, f_n\}
\]
(1) If $D + \text{div}(f_i)$ has a chip at $p$ for all $i$ then $D + \text{div}(\theta)$ has a chip at $p$ as well.

(2) If $D + \text{div}(f_i)$ has a chip in $Y$ for all $i$ then so does $D + \text{div}(\theta)$.

**Proof.** If $D + \text{div}(f_i)$ has a chip at $p$ then $f_i \in R(D - p)$. Since tropical modules are closed under taking minimums we can conclude that $\theta \in R(D - p)$. This proves part 1.

The second part is a generalization of part 1. We refer to [1] for a proof. □

### 3.2. Picard group

The Picard group of $\Lambda$ which we denote by $\text{Pic}(\Lambda)$ is defined as the quotient $\text{Div}(\Lambda)/\text{Prin}(\Lambda)$.

**Remark 3.2.1.** It is easy to see that the rank of $D$ only depends on the linear class of $D$ in $\text{Pic}(\Lambda)$.

Since principal divisors on $\Lambda$ have degree zero we have the following short exact sequence.

$$
0 \longrightarrow \text{Pic}^0(\Lambda) \longrightarrow \text{Pic}(\Lambda) \xrightarrow{\text{deg}} \mathbb{Z} \longrightarrow 0
$$

The group $\text{Pic}^0(\Lambda)$ is $\frac{\text{Div}^0(\Lambda)}{\text{Prin}(\Lambda)}$. It turns out that $\text{Pic}^0(\Lambda)$ is a real torus of dimension $g$.

In this section we sketch a proof of this fact. We refer the reader to [10] for details.

**Definition 3.2.2.** A 1-chain in $\Lambda$ is a formal finite sum $c = \sum a_i \cdot [p_i, q_i]$ where $a_i \in \mathbb{Z}$ and $[p_i, q_i]$ are (directed) line segments in $\Lambda$ with end points $p_i, q_i$. We denote the set 1-chains in $\Lambda$ by $C_1(\Lambda, \mathbb{Z})$. The boundary of $c$ denoted by $\partial c$ is the divisor

$$
\partial c = \sum a_i \cdot (p_i - q_i)
$$

**Definition 3.2.3.** A 1-cycle is a 1-chain whose boundary is zero.

**Definition 3.2.4.** We say a 1-chain is homologous to zero if it can be written as a boundary of a 2-chain.

A 2-chain could be defined as maps from a triangle to $\Lambda$. For example if $[p_1, p_2]$ and $[p_2, p_3]$ are small enough line segments in $\Lambda$ then $[p_1, p_2] + [p_2, p_3] - [p_1, p_3]$ is homologous
to zero as we can define a map from a 2-cell to \( \Lambda \) such that the boundary of the 2-cell traces out the segments \([p_1, p_2], [p_2, p_3]\) and \([p_3, p_1]\). In our case the group consisting of boundaries of 2-chains is generated by \([p_1, p_2] + [p_2, p_3] - [p_1, p_3]\) where \([p_i, p_j]\) are small enough segments. Note that if 1-chain is homologous to zero then it is a 1-cycle. We say two 1-chains \(c_1\) and \(c_2\) are homologous if the difference \(c_1 - c_2\) is homologous to zero. The set of 1-cycles modulo this equivalence relation can be identified with the first homology group \(H_1(\Lambda, \mathbb{Z}) \cong \mathbb{Z}^g\).

**Definition 3.2.5.** A 1-form on \( \Lambda \) is a homomorphism \(C_1(\Lambda, \mathbb{Z}) \to \mathbb{R}\).

**Definition 3.2.6.** Let \(f : \Lambda \to \mathbb{R}\) be a real valued function on \(\Lambda\). The differential of \(f\) denoted by \(\delta f\) is a 1-form defined by

\[
\delta f \left( \sum_i a_i[p_i, q_i] \right) = \sum_i a_i \cdot (f(p_i) - f(q_i))
\]

If we quotient the 1-forms by the differentials we get the first cohomology group \(H^1(\Lambda, \mathbb{R}) \cong \mathbb{R}^g\). We have the integration pairing

\[
H^1(\Lambda, \mathbb{R}) \times H_1(\Lambda, \mathbb{Z}) \to \mathbb{R}
\]

By this pairing we can identify \(H_1(\Lambda, \mathbb{Z})\) as a lattice in \(H^1(\Lambda, \mathbb{R})^*\).

**Definition 3.2.7.** We say a 1-form is harmonic if locally it is differential of a piece-wise linear function. In other words, there exists an open cover \(\{U_\alpha\}_\alpha\) of \(\Lambda\) such that the restriction of the 1-form to each \(U_\alpha\) is differential of a piece-wise linear function on \(U_\alpha\).

If \(dx\) is a harmonic 1-form and \(c \in C_1(\Lambda, \mathbb{Z})\) is a 1-chain then we can integrate \(dx\) along \(c\) to get the integral \(\int_c dx\) which only depends on the homology class of \(c\). It is a fact that we can find a basis of \(H^1(\Lambda, \mathbb{R})\) which consists of harmonic 1-forms. Therefore by assigning \(c \to \int_c\) we get a map

\[
C_1(\Lambda, \mathbb{Z}) \to H^1(\Lambda, \mathbb{R})^*
\]

which is compatible with the integration pairing defined earlier.
**Proposition 3.2.8.** The group \( \text{Pic}^0(\Lambda) \) is isomorphic to \( \frac{H^1(\Lambda, \mathbb{R})^*}{H_1(\Lambda, \mathbb{Z})} \).

**Proof (Sketch).** Let \( D \) be a degree zero divisor. Since \( \Lambda \) is connected we can write \( D \) as boundary of a 1-chain \( c \in C_1(\Lambda, \mathbb{Z}) \). The image of \( \int_c \in \frac{H^1(\Lambda, \mathbb{R})^*}{H_1(\Lambda, \mathbb{Z})} \) does not depend on choice of \( c \). Therefore we get a homomorphism \( \text{Div}^0(\Lambda) \to \frac{H^1(\Lambda, \mathbb{R})^*}{H_1(\Lambda, \mathbb{Z})} \). The kernel of this map is precisely \( \text{Prin}(\Lambda) \). \( \Box \)

**Example 3.2.9.** Suppose \( \Lambda \) consists of a single loop as shown below. The vertices of this graph are \( v \) and \( w \) which are joined by two edges of lengths \( l \) and \( m \). The graph \( \Lambda \) has a harmonic form \( dx \) whose integral along the loop is \( l + m \). Consider the divisor \( D = aw - av \). We can write \( D \) as a boundary of \( a \cdot e \) where \( e \) goes from \( v \) to \( w \) along the edge of length \( l \). Then \( \int_{a\cdot e} dx = a \cdot l \). Therefore \( D \) is a principal divisor if and only if

\[
a \cdot l \equiv 0 \pmod{l + m}
\]

This happens precisely when \( l/m \) is a ratio of two positive integers whose sum equals \( a \). In other words we can move \( a \) chips from \( v \) to \( w \) if and only if this condition is met.

Suppose we have \( a \) chips at \( v \) and one chip at some point \( p \). Then without any restrictions on the edge lengths we can move \( a \) chips to \( w \). This is because we can find a suitable point \( q \) such that

\[
a \cdot l + \int_{[p, q]} dx \equiv 0 \pmod{l + m}
\]

Then \( a \cdot v + p \) is linearly equivalent to \( a \cdot w + q \).

### 3.3. Chain of \( g \)-loops

Here we prove some geometric properties of a specific metric graph which is a chain of \( g \)-loops.
In this metric graph, \( v_1, v_2, \ldots, v_{g+1}, w_0, w_2, \ldots, w_g \) are the vertices. The edges which form the \( g \) loops have edge lengths \( \{l_i, m_i\}_{1 \leq i \leq g} \) as shown in the figure. We denote this metric graph by \( \Gamma \). Let \( \gamma_i \) denote the \( i \)th loop minus the vertex \( w_i \).

**Proposition 3.3.1.** Let \( E \) an effective divisor on \( \Gamma \) linearly equivalent to \( K_\Gamma \) then \( \gamma_i \) contains no point in at least on of the sets \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_g \).

**Proof.** Suppose \( E \) contains a point \( p_i \) on \( \gamma_i \) for each \( i \). Then \( E - p_1 - p_2 - \ldots - p_g \) is effective and therefore has rank at least 0. By tropical Riemann Roch 3.1.10 the rank of \( p_1 + p_2 + \ldots + p_g \) is at least one. However it can be seen that the rank of \( p_1 + p_2 + \ldots + p_g \) is 0 (See Dhar’s burning algorithm and reduced divisors in [2]).

**Proposition 3.3.2.** Let \( E \) be an effective divisor on \( \Gamma \). Suppose we have two piecewise linear functions \( f_0, f_1 \in R(E) \) with distinct slopes at \( v_i \) from the left. Assume that neither \( E + \text{div}(f_0) \) nor \( E + \text{div}(f_1) \) contain a point on \( \gamma_i \). Then \( \frac{1}{m_i} = a/b \) for some positive integers \( a \) and \( b \) whose sum is at most equal to the degree of \( E \).

**Proof.** This is essentially proved in [6]. We reproduce the proof for the sake of completeness. Let \( v \) be a point on bridge left of \( v_i \) which is close enough to \( v \) so that

1. The functions \( f_1, f_2 \) have constant slope on \([v, v_i]\).
2. The divisor \( E \) does not contain a chip on \([v, v_i]\).

Suppose the slopes of \( f_1, f_2 \) on \([v, v_i]\) are \( s_1, s_2 \) respectively. Let \( \Gamma' \) denote the graph from \( v \) to \( w_i \) including the endpoints. Let \( E' \) be the restriction of \( E \) to \( \Gamma' \). Let \( f_1', f_2' \)
be restrictions of $f_1, f_2$ on $\Gamma'$. Then then the divisor $E' + \text{div}(f_1')$ and $E' + \text{div}(f_2')$ are supported on $\{v, w_i\}$. By taking their difference we see that $(s_1 - s_2)v - (s_2 - s_1)w_i$ is a principal divisor on $\Gamma'$. As in Example 3.2.9 this allows us to conclude that $l_i/m_i$ is ratio of two positive integers whose sum is $|s_1 - s_2|$. Since $E$ is effective the slopes $s_i$ are bounded by the number of chips of $E$ which are to the left of $v$. Therefore the result follows. □
CHAPTER 4

Tropicalization

4.1. Metric graph associated to a family of curves

Let $\mathcal{X} \to \text{Spec}(R)$ be a family of curves as in Section 2.1. The metric graph associated to $\mathcal{X}$ is obtained as follows. The vertices of the graph are in one to one correspondence with the components of $\mathcal{X}^0$. For a component $\mathcal{X}_i^0$ of the special fiber, we denote the corresponding vertex by $v_i$. For every intersection of $\mathcal{X}_i^0$ and $\mathcal{X}_j^0$ there is an edge between $v_i$ and $v_j$. We denote this graph by $G$. We define a metric on the graph by declaring every edge to have length 1. We denote this metric graph by $\Lambda$.

The goal of this chapter is to define tropicalizations of rational functions and divisors on $\mathcal{X}_\eta$. We denote these maps by

$$\text{trop} : K(\mathcal{X}_\eta) \to \text{PL}(\Lambda)$$

$$\text{Trop} : \text{Div}(\mathcal{X}_\eta) \to \text{Div}(\Lambda)$$

This would facilitate us to use the divisor theory on the metric graph $\Lambda$ to study linear systems on $\mathcal{X}_\eta$.

**Remark 4.1.1.** If $\mathcal{X}$ is not regular then the edge lengths of $\Lambda$ are defined as follows. If the local equation of a node is given by

$$\kappa[[x,y]]/xy - \pi^e$$

then the corresponding edge has length $e$.

4.2. Local tropicalization

Let $\mathcal{X}_i^0$ and $\mathcal{X}_j^0$ be two components of $\mathcal{X}^0$ which intersect at a node $p$. Let $l$ denote the edge between $v_i$ and $v_j$ corresponding to this node. The generic fiber of $\text{Spec}(\hat{O}_{\mathcal{X},p}) \to \text{Spec}(R)$ by is $\text{Spec}(\hat{O}_{\mathcal{X},p}[1/\pi])$. We denote it by $\mathcal{X}_\eta^p$. Since $\hat{O}_{\mathcal{X},p}$ is
a unique factorization domain so is \( \hat{\mathcal{O}}_{X,p}[1/\pi] \). The divisors on \( \mathcal{X}_n^p \) are precisely the horizontal divisors on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \). If \( f \in \hat{K}(\mathcal{X})_p \) is a rational function on \( \mathcal{X}_n^p \) then the divisor of \( f \) on \( \mathcal{X}_n^p \) is the horizontal component of the divisor of \( f \) on \( \text{Spec}(\hat{\mathcal{O}}_{X,p}) \).

We now define tropicalizations of the divisors and the rational functions on \( \mathcal{X}_n^p \). We refer to these as the local tropicalization maps. The tropicalization of a divisors would be a divisor on the edge \( l \). Similarly the tropicalization of a rational function would be a piece-wise linear on \( l \). The divisors and the piece-wise linear functions on \( l \) are as defined in \[3.1\]. If \( f \) is a piece-wise linear function \( l \), then \( \text{div}(f) \) is as defined in \[3.1.3\]. We denote the local tropicalization maps by

\[
\text{trop}_p : \hat{K}(\mathcal{X})_p \to \text{PL}(l)
\]

\[
\text{Trop}_p : \text{Div}(\mathcal{X}_n^p) \to \text{Div}(l)
\]

We can realize the complete local ring at \( p \) as

\[
\hat{\mathcal{O}}_{X,p} \cong \frac{R[[x, y]]}{xy - \pi}
\]

where \( \pi \) is a uniformizer of \( R \) and \( x = 0 \) (resp. \( y = 0 \)) locally define \( \mathcal{X}_i^0 \) (resp. \( \mathcal{X}_j^0 \)) (See \[2.2.6\]). For \( t \in [0, 1] \) we can extend the valuation on \( K \) to \( \hat{K}(\mathcal{X})_p \) by setting

\[
\text{val}_t(x) = 1 - t, \text{val}_t(y) = t
\]

We interpret \( t \) as a parameter parameterizing the points of \( l \) such that \( t = 0 \) corresponds to \( v_i \) and \( t = 1 \) corresponds to \( v_j \). For \( h \in \hat{K}(\mathcal{X})_p \) we define \( \text{trop}_p(h) \) by

\[
\text{trop}_p(h)(t) = \text{val}_t(h)
\]

**Lemma 4.2.1.** Let \( f, g \in \hat{K}(\mathcal{X})_p \).

1. \( \text{trop}_p(fg) = \text{trop}_p(f) + \text{trop}_p(g) \)
2. The function \( \text{trop}_p(f) \) is a piece-wise linear function on \( l \).  
3. If \( f \in \hat{\mathcal{O}}_{X,p} \) then the function \( \text{trop}_p(f) \) is concave downward.  
4. If \( f \in \hat{\mathcal{O}}_{X,p} \) is a unit in \( \hat{\mathcal{O}}_{X,p} \) then \( \text{trop}_p(f) \) is a constant zero function.
(5) If $f \in K$ then $\text{trop}_p(K)$ is a constant function with value equal to the valuation of $f$ in $K$.

(6) The value of $\text{trop}_p(h)$ at $v_i$ (resp. $v_j$) is the order of vanishing of $h$ along $\mathcal{X}_i^0$ (resp. $\mathcal{X}_j^0$).

**Proof.** Since $\text{val}_i$ is a valuation on $\hat{K}(\mathcal{X})_p$ we have $1$. It suffices to prove $2$ for $f \in R[[x,y]]/(xy - \pi)$. If $f$ is a monomial in then $\text{trop}_p(f)$ is a linear function without any breakpoints in the interior. For a general $f$ the graph $\text{trop}_p(f)$ is obtained by taking the minimum of graphs of all monomials in $f$. Therefore we see that $\text{trop}_p(f)$ is piecewise linear. This also proves $3, 4, 5$. The order of vanishing of $f$ along $\mathcal{X}_i^0$ is the power of $x$ that appears in the factorization of $h$ in $\hat{O}_{\mathcal{X},p}$. We can write $f = x^e a/b$ where $a, b \in \hat{O}_{\mathcal{X},p}$ are not divisible by $x$. Therefore if $x$ has valuation $1$ and $y$ has valuation $0$ then the valuation of $f$ is $e$. $\Box$

**Example 4.2.2.** Suppose $h \in \frac{R[[x,y]]}{xy - \pi}$ is given by $h = x^2 + y^6 + \pi$. Then $\text{trop}_p(h)$ looks as shown below.

![Diagram](image)

$$\text{trop}_p(x^2 + y^6 + \pi)$$

**Proposition 4.2.3.** Let $f \in \hat{K}(\mathcal{X})_p$ We can write $f = x^e \cdot a/b$ where $a, b \in \hat{O}_{\mathcal{X},p}$ are not divisible by $x$. Then the slope of $\text{trop}_p(f)$ in the direction away from $v_i$ is

$$-e + \dim_{\kappa} \hat{O}_{\mathcal{X},p}(x, a) - \dim_{\kappa} \hat{O}_{\mathcal{X},p}(x, b)$$

**Proof.** The function $\text{trop}_p(x^e)$ is a linear function on $l$ with slope $-e$ without any breakpoints in the interior of $l$. Since $\text{trop}_p(f) = \text{trop}_p(x^e) + \text{trop}_p(a) - \text{trop}_p(b)$ it suffices to prove that the slope of $\text{trop}_p(a)$ at $v_i$ equals $\dim_{\kappa} \hat{O}_{\mathcal{X},p}(x, a)$. We can write $a \in \frac{R[[x,y]]}{xy - \pi}$ as

$$a = x^m u + y^n v + \pi^r h$$
where \( m, n, r \) are non-negative integers and \( u, v, h \) are units in \( R[[x]], R[[y]] \), and \( R \) respectively. Since \( u, v, h \) are units their valuation is zero regardless of the parameter \( t \). Recall that \( \text{val}_t(x) = 1 - t \) and \( \text{val}_t(y) = t \). Therefore we see that for small enough values of \( t \) the valuation of \( a \) is given by \( nt \). Thus the slope of \( \text{trop}_p(a) \) at \( v_i \) equals \( n \). The quotient \( \hat{O}_{\mathcal{X}_p}/(x,a) \) is isomorphic to \( R[[x,y]]/(y^n,x) \). Therefore we see that the dimension of \( \hat{O}_{\mathcal{X}_p}(x,a) \) over \( \kappa \) is also \( n \). □

Our next step is to define tropicalization of divisors on \( \mathcal{X}_\eta^p \). We recall that the divisors on \( \mathcal{X}_\eta^p \) are precisely the horizontal divisors on \( \text{Spec}(\hat{O}_{\mathcal{X}_p}) \). Suppose \( E = \text{div}(g) \) is a horizontal divisor on \( \text{Spec}(\hat{O}_{\mathcal{X}_p}) \) for some \( g \in \hat{K}(\mathcal{X}_p) \). Then we define \( \text{Trop}_p(E) \) as

\[
\text{Trop}_p(E) = \text{div}(\text{trop}_p(g)) \bigg|_{\text{interior of } l}
\]

Since \( \text{trop}_p \) of a unit in \( \hat{O}_{\mathcal{X}_p} \) is a constant zero function the above definition does not depend on the choice of \( g \).

**Lemma 4.2.4.** Let \( f \in \hat{K}(\mathcal{X})_p \) be a rational function on \( \text{Spec}(\hat{O}_{\mathcal{X}_p}) \). Let \( E \) be the horizontal component of the divisor of \( f \) on \( \text{Spec}(\hat{O}_{\mathcal{X}_p}) \) Then the divisors \( \text{div}(\text{trop}_p(f)) \) and \( \text{Trop}_p(E) \) agree on the interior of \( l \).

**Proof.** We can write \( f = x^iy^j \cdot a/b \) where \( a, b \in \hat{O}_{\mathcal{X}_p} \) are not divisible by \( x \) or \( y \). Then the horizontal component in divisor of \( f \) on \( \text{Spec}(\hat{O}_{\mathcal{X}_p}) \) is \( \text{div}(a/b) \). Therefore \( \text{Trop}_p(E) \) is

\[
\text{Trop}_p(E) = \text{div}(\text{trop}_p(a/b)) \bigg|_{\text{interior of } l}
\]

We know that \( \text{trop}_p(f) = \text{trop}_p(x^iy^j) + \text{trop}_p(a/b) \). It is easy to see that \( \text{trop}_p(x^iy^j) \) is a linear function on \( l \) without any break points in the interior of \( l \). Therefore,

\[
\text{div}(\text{trop}_p(f)) = \text{div}(\text{trop}_p(x^iy^j)) + \text{div}(\text{trop}_p(a/b))
\]

which agrees with \( \text{div}(\text{trop}_p(a/b)) \) on the interior of \( l \). □

**Proposition 4.2.5.** Let \( E \) be a horizontal divisor on \( \text{Spec}(\hat{O}_{\mathcal{X}_p}) \).

\( 1 \) If \( E \) is effective then so is \( \text{Trop}_p(E) \).
(2) The degree of $E$ equals the degree of $\text{Trop}_p(E)$.

**Proof.** Let $E = \text{div}(f)$ for some $f \in \hat{K}(\mathcal{X})_p$. If $E$ is effective then $f$ belongs to $\hat{\mathcal{O}}_{\mathcal{X},p}$. By Lemma 4.2.1 the graph of $\text{trop}_p(f)$ is concave downward. Therefore for any point $x$ on $l$ the slopes of $\text{trop}_p(f)$ towards $x$ are positive. Therefore $\text{Trop}_p(E)$ is effective.

We may assume that that $E$ is effective and irreducible to prove 2. Therefore we can say $E = \text{div}(f)$ for some $f \in \hat{\mathcal{O}}_{\mathcal{X},p}$ which is irreducible. Since the graph of $\text{trop}_p(f)$ is concave downward we can conclude that the degree of $\text{div}(\text{trop}_p(f))$ is the sum of slopes of $\text{trop}_p(f)$ at the two endpoints of $l$. Since $x$ is not a zero divisor in $\hat{\mathcal{O}}_{\mathcal{X},p}/(f)$ we have the following exact sequence.

$$0 \to \hat{\mathcal{O}}_{\mathcal{X},p}(f,x) \to \hat{\mathcal{O}}_{\mathcal{X},p}(f,xy) \to \hat{\mathcal{O}}_{\mathcal{X},p}(f,y) \to 0$$

By Proposition 4.2.3 the slopes of $\text{trop}_p(f)$ at $v_i$ and at $v_j$ are given by $\dim_{\kappa} \hat{\mathcal{O}}_{\mathcal{X},p}(f,x)$ and $\dim_{\kappa} \hat{\mathcal{O}}_{\mathcal{X},p}(f,y)$ respectively. By the above exact sequence

$$\dim_{\kappa} \hat{\mathcal{O}}_{\mathcal{X},p}(f,x) + \dim_{\kappa} \hat{\mathcal{O}}_{\mathcal{X},p}(f,y) = \dim_{\kappa} \hat{\mathcal{O}}_{\mathcal{X},p}(f,xy)$$

By Proposition 2.2.7 $\dim_{\kappa} \hat{\mathcal{O}}_{\mathcal{X},p}(f,xy)$ is the degree of $E$. \qed

### 4.3. Global tropicalization

Let $f \in K(\mathcal{X})$ be a rational function. Suppose $p$ is a node of $\mathcal{X}^0$ which lies on the intersection of $\mathcal{X}_i^0$ and $\mathcal{X}_j^0$. Let $\hat{f}_p$ denotes the germ of $f$ in $\hat{K}(\mathcal{X})_p$. We define $\text{trop}(f)$ on $l$ by

$$\text{trop}(f) = \text{trop}_p(\hat{f}_p)$$

By lemma 4.2.1 the value of $\text{trop}_p(\hat{f}_p)$ on $v_i$ (resp. $v_j$) is the order of vanishing of $f$ along $\mathcal{X}_i^0$ (resp. $\mathcal{X}_j^0$). Thus if we define $\text{trop}(f)$ on all edges of $\Lambda$ in a similar fashion we get a continuous piece-wise linear function on $\Lambda$.

**Proposition 4.3.1.** Let $f, g \in K(\mathcal{X}_\eta)$.

(1) $\text{trop}(f \cdot g) = \text{trop}(f) + \text{trop}(g)$
If \( f \in K \) then \( \text{trop}(f) \) is a constant function with value equal to the valuation of \( f \) in \( K \).

**Proof.** These properties follow directly from Lemma 4.2.1. \( \square \)

**Corollary 4.3.2.** For a point \( s \in \Lambda \) the assignment \( f \rightarrow \text{trop}(f)(s) \) is a valuation on \( K(\mathcal{X}) \) which extends the valuation on \( K \). We denote this valuation by \( \text{val}_s \).

**Proposition 4.3.3.** Suppose \( \mathcal{X}_i^0 \) and \( \mathcal{X}_j^0 \) intersect at a node \( p \) which corresponds the edge \( l \) in \( \Lambda \). Let \( f \in K(\mathcal{X}_\eta) \) be rational function. Then the slope of \( \text{trop}(f) \) along \( l \) in the direction away from \( v_i \) equals the order of vanishing of \( \tilde{f}_i \) at the point \( p \).

**Proof.** Let \( \hat{f}_p \) denote the germ of \( f \) in \( \hat{K}(\mathcal{X})_p \). We can realize \( \hat{O}_{\mathcal{X},p} \) as \( R[[x,y]]/(xy-\pi) \) where \( x = 0 \) and \( y = 0 \) locally define \( \mathcal{X}_i^0 \) and \( \mathcal{X}_j^0 \) respectively. If \( f \) vanishes to order \( e \) along \( \mathcal{X}_i^0 \) then we can write \( f = x^e \cdot a/b \) where \( a, b \in \hat{O}_{\mathcal{X},p} \) are not divisible by \( x \). Therefore \( \hat{f}_p/\pi^e = y^{-e} \cdot a/b \). Therefore the order of vanishing of \( \tilde{f}_i \) at \( p \) is

\[
-e + \dim(x,a) - \dim(x,b)
\]

By Proposition 4.2.3 this equals the slope of \( \text{trop}_p(\hat{f}_p) \) at \( v_i \). \( \square \)

Our next step is to construct \( \text{Trop} : \text{Div} (\mathcal{X}_\eta) \rightarrow \text{Div}(\Lambda) \). Let \( q \in \mathcal{X}_\eta \) be a closed point on \( \mathcal{X}_\eta \). Suppose \( q \) specializes to \( q_0 \). We distinguish cases depending on whether \( q_0 \) is a smooth point of \( \mathcal{X}^0 \) or a node.

**Case 1:** Suppose \( q_0 \) is a smooth point of \( \mathcal{X}^0 \). Then it lies on a unique component of \( \mathcal{X}^0 \), say \( \mathcal{X}_i^0 \). Then we define \( \text{Trop}(q) \) to be

\[
\text{Trop}(q) = \text{deg}(q) \cdot v_i
\]

**Case 2:** Suppose \( q_0 \) is a node of \( \mathcal{X}^0 \) lying on the intersection of \( \mathcal{X}_i^0 \) and \( \mathcal{X}_j^0 \). Then the closure of \( q \) is a horizontal divisor on \( \text{Spec}(\hat{O}_{\mathcal{X},q_0}) \). Let \( E \) denote this divisor. Then we define \( \text{Trop}(q) \) by

\[
\text{Trop}(q) = \text{Trop}_{q_0}(E)
\]

We extend \( \text{Trop} \) linearly on \( \text{Div}(\mathcal{X}_\eta) \).
**Proposition 4.3.4.** Let \( D \in \text{Div}(\mathcal{X}_\eta) \) be a divisor.

1. If \( D \) is effective then so is \( \text{Trop}(D) \).
2. \( \deg(D) = \deg(\text{Trop}(D)) \)

**Proof.** These properties follow immediately from Proposition [4.2.5] \( \square \)

**Proposition 4.3.5.** Let \( f \in K(\mathcal{X}_\eta) \) be a rational function. Then

\[
\text{trop}(\text{div}(f)) = \text{div}(\text{Trop}(f))
\]

**Proof.** We prove this is in two steps. First we prove that for any edge \( l \) in \( \Lambda \), the two divisors agree when restricted to the interior of \( l \). Then we argue that the multiplicity of both the divisors at any vertex is equal.

Let \( l \) be an edge in \( \Lambda \) corresponding to the node \( p \) in \( \mathcal{X}_0 \) which lies on the intersection of \( \mathcal{X}_i \) and \( \mathcal{X}_j \). We can write

\[
\text{div}(f) = E + D
\]

where \( E \) is supported on points of \( \mathcal{X}_\eta \) which specialize to \( p \) and the closure of the support of \( D \) is disjoint from \( p \). Let \( \hat{f}_p \) denote the image of \( f \) in \( \hat{O}_{\mathcal{X},p} \). Then \( \text{div}(\hat{f}_p) \) is a divisor on \( \text{Spec}(\hat{O}_{\mathcal{X},p}) \) whose horizontal components is precisely the closure of \( E \) in \( \text{Spec}(\hat{O}_{\mathcal{X},p}) \). The support of \( \text{trop}(D) \) is disjoint from the interior of \( l \). Therefore

\[
\text{trop}(\text{div}(f)) \big|_{\text{interior of } l} = \text{trop}_p(E)
\]

By definition \( \text{trop}(f) \) equals \( \text{trop}_p(\hat{f}_p) \) on the edge \( l \). By Lemma [4.2.4] the divisors \( \text{div}(\text{trop}_p(\hat{f}_p)) \) and \( \text{trop}_p(E) \) agree on the interior of \( l \). This completes the first step of the proof.

Let \( v_k \) be a vertex of \( \Lambda \) corresponding to the component \( \mathcal{X}_k^0 \). Suppose the multiplicities of \( v_k \) in \( \text{div}(\text{Trop}(f)) \) and \( \text{trop}(\text{div}(f)) \) are \( a_k \) and \( a'_k \) respectively. By Proposition [4.3.3] the slope of \( \text{Trop}(f) \) in the direction away from \( v_k \) along an edge equals the order of vanishing of \( \tilde{f}_k \) at the corresponding node. Therefore if \( S \) denotes the set of nodes on \( \mathcal{X}_k^0 \) then

\[
a_k = \sum_{p \in S} -1 \cdot \text{ord}_p(\tilde{f}_k)
\]
Suppose $q \in \mathcal{X}_\eta$ is a point in the support of $\text{div}(f)$ that specializes to a point $q_0$ on $\mathcal{X}_k^0$ which is not a node. By Proposition 2.2.8 we can say that

$$\text{ord}_q(f) \cdot \deg(q) = \text{ord}_{q_0}(\tilde{f}_k)$$

Therefore

$$a'_k = \sum_{p \in \mathcal{X}_k^0(\kappa) - S} \text{ord}_p(\tilde{f}_k)$$

Since $\sum_{p \in \mathcal{X}_k^0(\kappa)} \text{ord}_p(\tilde{f}_k) = 0$, we see that $a'_k = a_k$. \qed

**Corollary 4.3.6.** If $f \in H^0(\mathcal{X}_\eta, \mathcal{O}_{\mathcal{X}_\eta}(D))$ then $\text{trop}(f) \in R(\text{Trop}(D))$.

**Proof.** Since we know that $\text{trop}$ maps effective divisors to effective divisors, this follows immediately. \qed

**Example 4.3.7.** Suppose $\mathcal{X} = \text{Proj} \ R \left[ \frac{x,y,z}{xy-\pi z^2} \right]$.

The special fiber has two components $\mathcal{X}_1^0$ and $\mathcal{X}_2^0$ which intersect at one node. Consider the rational function $f = x + y - \pi$. The divisor of $f$ on $\mathcal{X}_\eta$ is $q_3 - q_1 - q_2$. The points $q_1, q_2$ are degree 1 points which specialize to smooth points on $\mathcal{X}_1^0$ and $\mathcal{X}_2^0$ respectively. The point $q_2$ has degree 2 and it specializes to the node of $\mathcal{X}^0$. The dual graph of $\mathcal{X}^0$ is simple an edge of length 1. The function $\text{trop}(f)$ looks as shown below. The incoming
slopes of \( \text{trop}(x + y + \pi) \) at \( v_1 \) and at \( v_2 \) are \(-1\). Therefore we have

\[
\text{div}(\text{trop}(f)) = \text{Trop}(\text{div}(f)) = 2m - v_1 - v_2
\]

We can conclude from Proposition 4.3.5 that \( \text{trop} \) maps principal divisors on \( \mathcal{X}_\eta \) to principal divisors on \( \Lambda \). Therefore \( \text{trop} \) descends to a homomorphism

\[
\text{trop} : \text{Pic}(\mathcal{X}_\eta) \to \text{Pic}(\Lambda).
\]

Now we describe an alternative way to realize this homomorphism.

**Lemma 4.3.8.** The restriction map \( \text{Pic}(\mathcal{X}) \to \text{Pic}(\mathcal{X}_\eta) \) is surjective.

**Proof.** Let \( \mathcal{L}_\eta \) be a line bundle on \( \mathcal{X}_\eta \). Suppose \( \mathcal{L}_\eta \cong \mathcal{O}_{\mathcal{X}_\eta}(D) \) for some divisor \( D \in \text{Div}(\mathcal{X}_\eta) \). Since \( \mathcal{X} \) is regular the closure \( \overline{D} \) of \( D \) in \( \mathcal{X} \) is a Cartier divisor on \( \mathcal{X} \). The restriction of \( \mathcal{O}_{\mathcal{X}}(\overline{D}) \) to \( \mathcal{X}_\eta \) is precisely \( \mathcal{O}_{\mathcal{X}_\eta}(D) \). \( \Box \)

**Proposition 4.3.9.** Let \( \mathcal{L}_\eta \) be a line bundle on \( \mathcal{X}_\eta \). Suppose \( \mathcal{L} \) is a line bundle on \( \mathcal{X} \) which is an extension of \( \mathcal{L}_\eta \). Let \( a_i = \deg \mathcal{L}|_{\mathcal{X}_\eta^i} \). Let \( E \) be the divisor \( \sum a_i v_i \) on \( \Lambda \).

1. The divisor class of \( E \) only depends on \( \mathcal{L}_\eta \).
2. If we assign \( \mathcal{L}_\eta \) to the divisor class of \( E \) we recover the homomorphism \( \text{trop} : \text{Pic}(\mathcal{X}_\eta) \to \text{Pic}(\Lambda) \).

**Proof.** We do a sequence of reductions to prove \( \Box \) First we observe that it suffices to prove that if \( \mathcal{L}_\eta \cong \mathcal{O}_{\mathcal{X}_\eta} \) then \( E \) is a principal divisor on \( \Lambda \). Suppose \( \mathcal{L} \cong \mathcal{O}_{\mathcal{X}}(D) \) for some divisor \( D \in \text{Div}(\mathcal{X}) \). Since \( \mathcal{L}_\eta \cong \mathcal{O}_{\mathcal{X}_\eta} \) we may take \( D \) to be a vertical divisor. We
may further reduce to the case when \( D = \mathcal{X}_k^0 \) for some component \( \mathcal{X}_k^0 \) of the special fiber. Now if \( \mathcal{X}_i^0 \) is a component different from \( \mathcal{X}_k^0 \) then \( \deg \mathcal{O}_X(\mathcal{X}_k^0)|_{\mathcal{X}_i^0} \) is the number of intersection points between \( \mathcal{X}_k^0 \) and \( \mathcal{X}_i^0 \). In another words \( a_i \) is the number of edges between \( v_k \) and \( v_i \) for all \( i \) different from \( k \). Since degree of a line bundle is constant in a flat family we can conclude that \( a_k = -\deg(v_k) \). Consider a bump function \( \phi_k \) on \( \Lambda \) which is 1 at \( v_k \) and decreases to 0 along edges incident at \( v_k \). Then \( E = \sum_i a_i v_i \) is \( \text{div}(\phi_k) \).

Let \( D \in \text{Div}(\mathcal{X}) \) be a horizontal divisor such that \( \mathcal{O}_X(D)|_{\mathcal{X}_q} \cong \mathcal{L}_q \) and let \( a_i = \deg \mathcal{O}_X(D)|_{\mathcal{X}_i^0} \). To prove \(^2\) we have to prove that \( \text{trop}(D) \) is linearly equivalent to \( E = \sum_i a_i v_i \). We can reduce to the case when \( D \) is closure of a point \( q \) on \( \mathcal{X}_q \). The trivial case is when \( q \) specializes to a smooth point of \( \mathcal{X}^0 \). In this case \( \text{trop}(D) \) equals \( E \). Suppose \( q \) specialize to a node \( q_0 \) lying on the intersection of \( \mathcal{X}_i^0 \) and \( \mathcal{X}_j^0 \). Then is supported on the vertices \( \{v_i, v_j\} \) and \( \text{trop}(D) \) is supported on the interior of the edge corresponding to \( q_0 \) between \( v_i \) and \( v_j \). Let us denote this edge by \( l \). Suppose the pullback of \( D \) on \( \text{Spec}(\hat{O}_{\mathcal{X}, q_0}) \) is \( \text{div}(f) \) for some \( f \in \text{Spec}(\hat{O}_{\mathcal{X}, q_0}) \). Since \( \text{div}(f) \) is a horizontal divisor on \( \text{Spec}(\hat{O}_{\mathcal{X}, p}) \) \( \text{trop}_{q_0}(f) \) has value zero on \( v_i \) and \( v_j \). (See Lemma 4.2.1). Therefore we can extend \( \text{trop}_{q_0}(f) \) on all of \( \Lambda \) by declaring it to be zero outside \( l \). We can write \( \hat{O}_{\mathcal{X}, q_0} \) as \( R[[x, y]]/(xy - \pi) \) where \( x \) and \( y \) define \( \mathcal{X}_i^0 \) and \( \mathcal{X}_j^0 \) respectively on \( \text{Spec}(\hat{O}_{\mathcal{X}, q_0}) \). Now the slope of \( \text{trop}_{q_0}(f) \) along \( l \) in the direction towards \( v_i \) is \( \dim_x \frac{\partial \hat{O}_{\mathcal{X}, q_0}}{\partial (x, f)} \).

(See Proposition 4.2.3). This is the multiplicity of \( D \cap \mathcal{X}_i^0 \) at \( q_0 \). Therefore

\[
\deg \mathcal{O}_X(D)|_{\mathcal{X}_i} = \dim_x \frac{\partial \hat{O}_{\mathcal{X}, q_0}}{\partial (x, f)}
\]

Thus we see that the divisor of \( \text{trop}_{q_0}(f) \) is \( E - \text{trop}(D) \). This completes the proof. \( \square \)

**Corollary 4.3.10.** Suppose every component of \( \mathcal{X}^0 \) is of genus zero. Let \( \omega_{\mathcal{X}} \) be the canonical line bundle of \( \mathcal{X}_q \). Then \( \text{trop}(\omega_{\mathcal{X}}) = K_\Lambda \).

**Proof.** In our case the relative dualizing sheaf \( \omega_{\mathcal{X}}/\text{Spec}(R) \) on \( \mathcal{X} \) is invertible. The restriction of \( \omega_{\mathcal{X}}/\text{Spec}(R) \) to a fiber is the dualizing sheaf of the fiber. If \( \mathcal{X}_i^0 \) has genus 0 and contains \( n \) nodes then the degree of \( \omega_{\mathcal{X}^0}|_{\mathcal{X}_i^0} \) is \( -2 + n \). This completes the proof. \( \square \)
Proposition 4.3.11. Let $D \in \text{Div}(\mathcal{X}_\eta)$ be a divisor of rank $r$. Let $T = \sum a_i v_i$ be an effective divisor supported on the vertices of $G$. Suppose $T$ has degree $d \leq r$. Then there exists a subspace $V \subset H^0(\mathcal{X}_\eta, O_{\mathcal{X}_\eta}(D))$ of dimension at least $r + 1 - d$ such that for any $f \in V$ the divisor $\text{Trop}(D) + \text{trop}(f) - T$ is effective.

Proof. Recall the rank of $D$ is by definition $\dim_K H^0(\mathcal{X}_\eta, O_{\mathcal{X}_\eta}(D)) - 1$. Since $R$ is Henselian with algebraically closed residue field, given any point on the smooth locus of $\mathcal{X}^0$ we can find a point $q \in \mathcal{X}_\eta(K)$ that specializes to it. Therefore we can find points $q_i$ on $\mathcal{X}_\eta$ which are defined over $K$ such that $q_i$ specializes to a smooth point on $\mathcal{X}_i^0$. By imposing vanishing conditions at $q_i$ we can find $V \subset H^0(\mathcal{X}_\eta, O_{\mathcal{X}_\eta}(D))$ such that for any $f \in V$ the divisor $D - \sum_i a_i q_i + \text{div}(f)$ is effective. It follows that $V$ has the required property and dimension. □

Corollary 4.3.12. For any divisor $D \in \text{Div}(\mathcal{X}_\eta)$ we have $r(D) \leq r(\text{trop}(D))$.

Proof. The vertices $v_i$ form a rank determining set (See Theorem 3.1.9). Therefore the corollary immediately follows. This inequality is known as specialization lemma. □

4.4. Main theorem

Let $\mathcal{X} \to \text{Spec}(R)$ be a flat and proper family of curves over $\text{Spec}(R)$ as in Section 2.1. To recall $R$ is a complete dvr with value group $\mathbb{Z}$. The family $\mathcal{X}$ has the following properties.

- The generic fiber $\mathcal{X}_\eta$ is a smooth curve over $K$.
- The family $\mathcal{X}$ is strongly semi-stable.
- The family $\mathcal{X}$ is regular.

Suppose each component of the special fiber $\mathcal{X}^0$ has genus 0, intersecting in such a way that the dual graph of $\mathcal{X}^0_0$ is a chain of $g$ loops. We prove that if $\mathcal{X}_\eta$ has a theta characteristic with large number of sections then the lengths $\{l_i, m_i\}$ are forced to satisfy some linear conditions. The precise statement is in Theorem 4.4.1.

We denote the dual graph of $\mathcal{X}^0$ by $\Gamma$. The vertices of $\Gamma$ are $v_1, v_2, \ldots, v_{g+1}, w_0, w_1, \ldots, w_g$. We denote the component corresponding to the vertex $v_i$ by $\mathcal{X}^0_{v_i}$ and the component corresponding to the vertex $w_j$ by $\mathcal{X}^0_{w_j}$. For $1 \leq i \leq g$ there are two chains of genus zero.
curves which connect $X^0_{v_i}$ to $X^0_{w_i}$ which correspond to two edges between $v_i$ and $w_i$ of lengths $l_i$ and $m_i$ respectively. The component $X^0_{w_i}$ meets $X^0_{v_{i+1}}$ at exactly one node creating bridges of length 1 between the loops.

**Theorem 4.4.1.** Suppose $X_\eta$ admits a theta characteristic $\mathcal{L}$ with $\dim_K H^0(X_\eta, \mathcal{L}) \geq r + 1$, then there exists a set $J \subset \{1, 2, \ldots, g\}$ of size $r$ such that for all $i \in J$, $\frac{l_i}{m_i}$ is a ratio of two positive integers whose sum is at most equal to $g - 1$.

**Proof.** The strategy of the proof is as follows. We prove that for any set of indices $J' \subset \{1, 2, 3, \ldots, g\}$ of size $r - 1$ there exists an index $i$ not contained in $J'$ such that $\frac{l_i}{m_i}$ is a ratio of two positive integers whose sum is at most equal to $g - 1$. This will inductively prove that there are $r$ relations as stated in the theorem.

Let $\mathcal{L} \cong \mathcal{O}_{X_\eta}(D)$ for some effective divisor $D$ on $X_\eta$. Let $E$ denote the divisor $\text{Trop}(D)$ on $\Gamma$. By Proposition 4.3.11 we can find a subspace $V \subset H^0(X_\eta, \mathcal{L})$ of dimension two such that for each $f \in V$ the divisor $E + \text{div}(\text{trop}(f))$ contains at least one chip at $v_j$ for all $j \in J'$. For $1 \leq i \leq g$ let $p_i$ denote the node on $X^0$ which is the unique intersection of $X^0_{w_{i-1}}$ and $X^0_{v_{i}}$. By Lemma 2.1.4 we can find a basis $f^i_1, f^i_2$ of $V$ such that reductions of $f^i_1, f^i_2$ on $X^0_{v_i}$ have distinct orders of vanishing at $p_i$ and reductions of $f^i_1, f^i_2$ on $X^0_{v_{i+1}}$ are linearly independent. We set

$$\theta_i = \text{trop}(f^i_1) + \text{trop}(f^i_2) + \text{val}(a_i) = \text{trop}(f^i_1 \cdot f^i_2 \cdot a_i)$$

where $a_i \in K$ are chosen such that the functions $\theta_i$ and $\theta_{i+1}$ have the same value on $v_{i+1}$. Note that since $f^i_1, f^i_2 \in V$, the divisor $2E + \text{div}(\theta_i)$ is effective and contains at least two chips on $v_i$. 

Dual graph of $\mathcal{D}^0$
We patch the functions $\theta_i$ to make a piece-wise linear function $\theta$ as follows. The function $\theta$ equals $\theta_1$ from $w_0$ to $v_2$. For $2 \leq i \leq g$ the function $\theta$ which agrees with $\theta_i$ on the part of $\Gamma$ between $v_i$ and $v_{i+1}$. Since $\theta_i(v_i) = \theta_{i+1}(v_i)$ we get a continuous piece-wise linear function. For $h \in \text{PL}(\Gamma)$ let $s_i(h)$ denote the slope of $h$ at $v_i$ from the left. We claim that

$$s_i(\theta_i) \leq s_i(\theta)$$

By Lemma 2.1.3 the order of vanishing of $f_i^{-1} \cdot f_i^{-1}$ at $p_i$ is less than or equal to the order of vanishing of $f_1 \cdot f_2$ at $p_i$. By Proposition 4.3.3 we see that $s_i(\theta_i) \leq s_i(\theta)$.

Now we claim that

1. The function $\theta$ belongs to the tropical module $R(2E)$.
2. The divisor $\text{div}(\theta) + 2E$ contains at least two chips on $v_j$ for $j \in J'$.

The function $\theta$ agrees with $\theta_i$ on open subgraph between $v_i$ and $v_{i+1}$. Therefore the divisor $2E + \text{div}(\theta)$ is effective on that open subgraph. Therefore we only need to show that the coefficient of $v_i$ in $2E + \text{div}(\theta)$ is greater than the coefficient of $v_i$ in $2E + \text{div}(\theta_i)$. But this follows since we proved $s_i(\theta_i) \leq s_i(\theta)$. This proves both of our claims.

Since $2E$ is linearly equivalent to the canonical divisor of $\Gamma$, by Proposition 3.3.1 there exists a loop $\gamma_i$ such that $2E + \text{trop}(\theta)$ does not have a chip on $\gamma_i$. Since $2E + \text{trop}(\theta)$ has at least two chips on $v_j$ for all $j \in J'$ the index $i$ lies outside the set $J'$. On $\gamma_i$ the function $\theta$ equals $\text{trop}(f_i^1) + \text{trop}(f_i^2) + \text{val}(a_i)$. Therefore the divisor $2E + \text{trop}(f_i^1) + \text{trop}(f_i^2) = 2E + \text{trop}(\theta_i)$ does not have a chip on $\gamma_i$ except possibly at $v_i$. But as we argued before the coefficient of $v_i$ in $2E + \text{trop}(\theta_i)$ is less than or equal to that in $2E + \text{trop}(\theta)$. Therefore $2E + \text{trop}(f_i^1) + \text{trop}(f_i^2)$ does not have a chip on $\gamma_i$. Now we have two piece-wise linear functions $\text{trop}(f_i^1)$ and $\text{trop}(f_i^2)$ in $R(E)$ which satisfy the hypothesis of Proposition 3.3.2. Therefore we get that $l_i/m_i$ is a ratio of two positive integers whose sum is less than or equal to $g - 1$.

$\square$
4.5. Metric graphs with large theta characteristics

In this sections we construct examples to prove that there exist metric graphs with only $2r - 1$ relations on the edge lengths that have a theta characteristic of rank $r$. To be explicit we take $g = 8$ and $r = 2$ but we indicate how the construction works for arbitrary genus and rank. The metric graph we consider is chain of $g$ loops as shown below.

![Diagram of a metric graph with $g$ loops](image)

The edge lengths of the $i$th loop are $l_i, m_i$ as before. For $i \in \{2, 3, 4\}$ we have $l_i = m_i$ and the rest of the edge lengths are arbitrary. The points $p_5, p_6, p_7$ are diametrically opposite to $w_5, w_6, w_7$ respectively. Let $E$ be the divisor:

$$E = 4w_1 + p_5 + p_6 + p_7$$

The canonical divisor $K_\Gamma$ is linearly equivalent to

$$K_\Gamma = \sum_{i=1}^{7} w_i + \sum_{i=2}^{8} v_i$$

We have can choose a basis of harmonic forms $\{dx_i\}_{1 \leq i \leq g}$ such that $dx_i$ integrates to $l_i + m_i$ along the $i$th loop and is zero along 1-chains whose support is outside the $i$th loop. (See Section 3.2). To see that $2E - K$ is a principal divisor we use the same strategy as in Example 3.2.9. We construct a 1-chain $c$ such that the boundary $c$ is $D$. Let us start from the extreme right. We join $v_8$ and $w_7$ to $p_7$; $v_7$ and $w_6$ to $p_6$; and $v_6$ and $w_5$ to $p_5$. For $1 \leq i \leq 4$ join $w_i$ and $v_{i+1}$ to $w_1$. We may assume that these segments are along the lower edges of $\Gamma$. It is not very hard to verify that for any of the $g$ harmonic forms $dx_i$ we have $\int_c dx_i \equiv 0 \mod (l_i + m_i)$. For $dx_2$, $dx_3$ and $dx_4$ we observe that there even number segments in $c$ that cross the corresponding loops. Therefore $E$ is a theta characteristic on $\Gamma$.

Since the edge between $w_1$ and $v_2$ is a bride the 4 chips on $w_1$ can be moved to $v_2$. Since $w_2$ is diametrically opposite to $v_2$ they can be moved to $w_2$ (See Example 3.2.9). By the same argument they can be moved to $v_i$ and $w_i$ for $2 \leq i \leq 4$. Now since we have
an additional chip on loops 6, 7 and 8, we can carry 4 chips to $v_i$ and $w_i$ for $5 \leq i \leq 7$ (See Example 3.2.9). In the general case the theta characteristic $E$ has $2r$ chips an $w_1$. The next $2r - 1$ loops have equal edge lengths, and the last but one loop has exactly one chip. We can carry the $2r$ chips on $w_1$ to $v_i$ and to $w_i$ for $2 \leq i \leq g - 1$.

We claim that $E$ has rank at least $r$. We consider a rank determining set $S$ (See theorem 3.1.9).

\[ S = \bigcup_{1 \leq i \leq g} \{ a_i, b_i \} \]

For $1 \leq i \leq g - 1$, the points $a_i, b_i$ are on the $i^{th}$ loop and are equidistant from $w_i$. The set $S$ contains points $a_g, b_g$ on the last loop which are equidistant from $v_g$. Suppose $T = \sum_{i}^{g} \alpha_i a_i + \beta_i b_i$ is an effective divisor of degree $r$ supported on $S$. We need to prove that we can move chips in $E$ such that there at least $\alpha_i$ chips on $a_i$ and at least $\beta_i$ chips on $b_i$. Suppose $\alpha_1$ is the maximum of $\{ \alpha_1, \beta_1 \}$. Then we can move $2\alpha_1$ chips away from $w_1$ such that we leave $\alpha_1$ chips at both $a_1$ and $b_1$. We are left with $2r - 2\alpha_1$ chips on $w_1$. Since this is an even number we can move them to $w_2$. We again pick maximum of $\{ \alpha_2, \beta_2 \}$ and leave max $\{ \alpha_2, \beta_2 \}$ chips on $a_2$ and $b_2$. Thus it follows that we can carry an even number of chips forward and leave required number of chips on $S$. Since $S$ is a rank determining set we conclude that rank of $E$ is at least $r$. 

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Bibliography


