A Large Sieve Zero Density Estimate for Maass Cusp Forms

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ABSTRACT

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The large sieve method has been used extensively, beginning with Bombieri in 1965, to provide bounds on the number of zeros of Dirichlet $L$-functions near the line $\sigma = 1$. Using the Kuznetsov trace formula and the work of Deshouillers and Iwaniec on Kloosterman sums, it is possible to derive large sieve inequalities for the Fourier coefficients of Maass cusp forms, which may then similarly be used to study the corresponding Hecke-Maass $L$-functions. Following an approach developed by Gallagher for Dirichlet $L$-functions, this thesis shows how the large sieve method may be used to prove a zero density estimate, averaged over the Laplace eigenvalues, for Maass cusp forms of weight zero for the congruence subgroup $\Gamma_0(q)$ for any positive integer $q$. 
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To all my loved ones
Introduction

The large sieve, first developed by Linnik [23] to study the distribution of quadratic nonresidues, has become an indispensable tool in analytic number theory. At its core, the large sieve is an analytic tool used to show cancellation in exponential sums. A simple formulation, following Davenport and Halberstam [5], is as follows: let $a_n$ be an arbitrary sequence of complex numbers, and, for real $\alpha$, let

$$S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha),$$

where $e(\theta) = e^{2\pi i \theta}$. Note that $S(\alpha)$ is periodic with period 1, so the values depend only on the equivalence class of $\alpha$ in $\mathbb{R}/\mathbb{Z}$. Let $\alpha_1, \ldots, \alpha_R$ be a set of real numbers which are “$\delta$-separated” in $\mathbb{R}/\mathbb{Z}$ in the sense that

$$||\alpha_r - \alpha_s|| \geq \delta$$

when $r \neq s$, and $|| \cdot ||$ denotes the distance to the nearest integer. Then the large sieve inequality of Davenport and Halberstam shows that

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \ll \left( N + \frac{1}{\delta} \right) \sum_{n=M+1}^{M+N} |a_n|^2.$$
In fact, as proved by Montgomery and Vaughan [26], the implied constant above can be taken to be 1, so the \( \ll \) sign may be replaced with a \( \leq \) sign.

This form of the large sieve found an application in a famous theorem proved independently by Bombieri [3] and Vinogradov [37, 38]:

**Theorem 0.1 (Bombieri-Vinogradov).** Let

\[
\psi(x; q, a) = \sum_{n \leq x \atop n \equiv a \mod q} \Lambda(n),
\]

where \( \Lambda(n) \) is the Von Mangoldt function:

\[
\Lambda(n) = \begin{cases} 
\log p & n = p^k, p \text{ prime, } k \geq 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Then for any positive constant \( A \), there exists a positive constant \( B \) such that if \( Q \leq \frac{x^{3/2}}{(\log x)^B} \),

\[
\sum_{q \leq Q} \max_{(a,q)=1} \max_{y \leq x} \left| \psi(y; q, a) - \frac{y}{\phi(q)} \right| \ll x(\log x)^{-A}.
\]

This may be viewed as an averaged form of the Generalized Riemann Hypothesis (GRH) for Dirichlet \( L \)-functions, as GRH would imply

\[
\max_{(a,q)=1} \max_{y \leq x} \left| \psi(y; q, a) - \frac{y}{\phi(q)} \right| \ll x^{1/2} \log^2 x,
\]

giving the theorem, but nothing stronger. That the theorem is as good as GRH in this averaged context shows its power and utility.
The Bombieri-Vinogradov theorem is of particular interest recently as it played a key part in the
work of Zhang [42] on bounded gaps between primes. In particular, the work of Goldston, Pintz,
and Yildirim [14] shows that the Bombieri-Vinogradov theorem is, to quote the authors, “within a
hair’s breadth” of establishing bounded gaps: if instead of requiring $Q \leq x^{1/2}(\log x)^{-B}$ we could
instead take $Q = x^{\vartheta-\epsilon}$ for some $\vartheta > 1/2$ and for all $\epsilon > 0$, it would follow that

$$\liminf_{n \to \infty}(p_{n+1} - p_n) < \infty,$$

where $p_n$ denotes the $n$th prime. The Elliot-Halberstam conjecture [8] states that it is possible to
take $\vartheta = 1$ above. With this assumption, Maynard [25] was able to prove conditionally that

$$\liminf_{n \to \infty}(p_{n+1} - p_n) \leq 12.$$

Zhang, while unable to break through the barrier of $\vartheta = 1/2$ in full generality, does so in a
limited way by restricting to moduli $q$ which do not have large prime divisors, and this is enough
to derive his celebrated result

$$\liminf_{n \to \infty}(p_{n+1} - p_n) < 7 \times 10^7.$$

All of these successes stem from the application of the large sieve method to bound sums over
Dirichlet characters. A typical example is the following bound ([10], Lemma 3):

$$\sum_{\chi \mod q} \sum_{n=M+1}^{M+N} a_n \chi(n)^2 \ll (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$
Here the asterisk means that the sum is taken over only primitive characters. A heuristic interpretation of this type of large sieve inequality is that, on average, the Dirichlet characters behave like random points on the unit circle. Indeed, a trivial application of the Cauchy-Schwarz inequality to the above would result in an \( N^2 \) term on the right-hand side above, and the improvement from \( N^2 \) to \( N \) is exactly what would be expected if the \( \chi(n) \) were replaced by uniform random variables. This aligns closely with the philosophy of GRH, which can be viewed as asserting that, for each individual \( \chi \), the sequence \( \mu(n)\chi(n) \), where \( \mu \) is the Möbius function, behaves like a random walk in a similar sense. Thus, it is intuitive that the large sieve inequality above could stand in for GRH in an averaged setting.

The large sieve for Dirichlet characters leads not only to the Bombieri-Vinogradov theorem, but also to results on the distribution of zeros of Dirichlet \( L \)-functions, providing more evidence that large sieve methods can be helpful when bounds towards GRH are needed. In fact, Bombieri’s proof of the theorem proceeds by way of the following zero density estimate:

**Theorem 0.2** (Bombieri [3], Theorem 5). Let \( Q \) be a finite set of positive integers, and define $M = \max_{q \in Q} q$ and $D = \max_{q \in Q} d(q)$, where \( d(q) \) is the number of divisors of \( q \). Let \( N(\alpha, T; \chi) \) denote the number of zeros \( s = \sigma + it \) of \( L(s, \chi) \) in the rectangle \( \alpha \leq \sigma \leq 1, |t| < T \), and let \( \tau(\chi) \) be the Gauss sum \( \sum_{a \mod q} \chi(a)e(a/q) \). Then

$$
\sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \mod q} |\tau(\chi)|^2 N(\alpha, T; \chi) \ll DT(M^2 + MT)^{\frac{4(1-\alpha)}{3(1-2\alpha)}} \log^{10}(M + T),
$$

uniformly with respect to \( Q \), for \( \frac{1}{2} \leq \alpha \leq 1, T \geq 2 \).

Though subsequent proofs of the Bombieri-Vinogradov theorem by Gallagher [9] and Vaughan
were able to dispense with the explicit step of bounding zeros, the large sieve continued to prove helpful in deriving even stronger zero density estimates than Bombieri’s. One of the first such estimates is the following result of Jutila, which improves on Bombieri’s result by a factor of $T$:

**Theorem 0.3** (Jutila, [17]). *There exists a constant $c$ such that*

$$\sum_{q \leq Q} \sum_{\chi \mod q}^{*} N(\alpha, T; \chi) \ll (Q^7 T^4)^{1 - \alpha} \log^{c}(Q + T).$$

Jutila applied this result in [17] to prove a version of the Bombieri-Vinogradov theorem for short intervals.

Another such result is the following, which is the classical inspiration for the main theorem of this thesis:

**Theorem 0.4** (Gallagher, [10]). *Let $N(\alpha, T; \chi)$ denote the number of zeros $s = \sigma + it$ of $L(s, \chi)$ in the rectangle $\alpha \leq \sigma \leq 1, |t| < T$. Then there exists a positive constant $c$ such that*

$$\sum_{q \leq T} \sum_{\chi \mod q}^{*} N(\alpha, T; \chi) \ll T^{c(1 - \alpha)}.$$  

This theorem found a striking application when Montgomery and Vaughan [27] applied it in their proof that the number of even numbers $\leq X$ which cannot be expressed as a sum of two primes is $< X^{1 - \delta}$ for an effective positive constant $\delta$.

From the point of view of the Langlands program, the Dirichlet $L$-functions are the $L$-functions attached to automorphic representations for the group $GL(1, \mathbb{A}_{\mathbb{Q}})$, and thus the simplest examples of a rich theory of automorphic $L$-functions for $GL(n)$. Thus, given the effectiveness of the large
sieve in studying Dirichlet $L$-functions, it is natural to inquire as to whether similar methods could also be used to study other families of $L$-functions. This thesis investigates the applications of the large sieve method to the study of the simplest $L$-functions beyond the Dirichlet $L$-functions; namely, those associated to cusp forms for $GL(2)$.

To extend the large sieve method to $GL(2)$, the Hecke eigenvalues of cusp forms, or relatedly, their Fourier coefficients, must play the role of the Dirichlet characters above, and one would hope that the large sieve method can show these numbers are pseudo-randomly distributed in a similar way. This introduces an additional difficulty: the values taken by Dirichlet characters are well-spaced on the unit circle, so the analytic large sieve inequality above can be used to study sums involving Dirichlet characters quite directly. This is not exactly the case, however, for Fourier coefficients of cusp forms. Thus, an additional step is needed to translate from sums involving Fourier coefficients to more directly understandable exponential sums. One important tool for this step in the case of $GL(2)$ is the trace formula developed by Bruggeman [4] and Kuznetsov [22], which relates the Fourier coefficients of cusp forms to Kloosterman sums. This process of applying the trace formula to derive large sieve inequalities was carried out extensively by Deshouillers and Iwaniec [6], whose work is a significant inspiration for the results below.

One of the early applications of this method was actually a bound for the Riemann zeta function, demonstrating the connection between the study of higher-degree $L$-functions and more elementary applications:

**Theorem 0.5** (Iwaniec [16]). Let $T \geq 2$, $T_0 = T^{2/3}$ and $\epsilon > 0$. Then

$$
\int_T^{T+T_0} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \, dt \ll T_0 T^\epsilon.
$$
The main result of this thesis synthesizes the large sieve results first developed by Deshouillers and Iwaniec with the classical techniques of Gallagher used to prove Theorem 0.4. Combining these two approaches leads to a new type of zero density estimate for $GL(2)$ $L$-functions which is similar to, though weaker than, the result of Gallagher quoted above, thus demonstrating the potential strength of the large sieve method of Bombieri, Gallagher, and others, when combined with more modern techniques, to provide new insights into the distribution of zeros of more general $L$-functions.

Before stating the theorem, we require some notation, all of which is explained in detail in Chapter 1. Fix an integer $q \geq 1$. Let $\{f_j\}_{j \geq 1}$ be an orthonormal basis (with respect to the Petersson inner product) for the space of Maass cusp newforms for $\Gamma_0(q)$ such that $\Delta f_j = \left( \frac{1}{4} + t_j^2 \right) f_j$, where $\Delta$ is the Laplacian, and assume each $f_j$ is an eigenfunction of all the Hecke operators $T_n$. Write $f_j(z) = \sum_{n \neq 0} a_j(n) \sqrt{y} K_{it_j} (2\pi |n| y) e(nx)$. Then the main theorem is as follows:

**Theorem 0.6.** Let $N_j(\alpha, T)$ be the number of zeros $s = \sigma + it$ in the region $\alpha \leq \sigma \leq 1, |t| \leq T$ of the function $L(s, f_j)$. Then there exists a positive constant $c$ such that for all $\epsilon > 0$ and $\alpha$ sufficiently close to 1,

$$\sum_{|t_j| \leq T} \frac{|a_j(1)|^2}{\cosh \pi t_j} N_j(\alpha, T) \ll T^{c(1-\alpha)+\epsilon}.$$

Thus, in comparison to the averaging over the conductor $q$ which takes place in the Bombieri-Vinogradov theorem and the original result of Gallagher, the averaging in this setting is over the Laplace eigenvalues $\frac{1}{4} + t^2$ of Maass forms, with the conductor fixed. This averaging is natural from the point of view of the Kuznetsov trace formula, which exploits the spectral decomposition of the Laplace operator on $L^2(\Gamma_0(q) \backslash \mathfrak{H})$ for fixed $q$.

To put this estimate in context, the best known bound for $N(\alpha, T)$ for a single Maass form is
of the form $T^{c(1-\alpha)}$ for large $c$ (Lemke Oliver and Thorner [28]). See also Tang [32] for the bound $T^{2(1-\alpha)/\alpha}(\log T)^C$, giving an improved exponent of $T$ at the cost of a power of $\log T$.

The approach used here is not the only way of applying the large sieve concept to study higher-degree $L$-functions, and a variety of alternative methods of averaging have been successfully carried out as well. Duke and Kowalski [7] developed a large sieve inequality for automorphic forms for $GL(n)$ averaging over the conductor, leading to the following analogue of Linnik’s theorem on the least quadratic nonresidue for elliptic curves (via the famous modularity theorem of Wiles [40]):

**Theorem 0.7** (Duke-Kowalski [7]). Let $M(Q, \alpha)$ be the maximal number of isogeny classes of semistable elliptic curves over $\mathbb{Q}$ with conductor $\leq Q$ which for every prime $p \leq (\log Q)^\alpha$ have a fixed number of points (mod $p$). Then for any $\epsilon > 0$,

$$M(Q, \alpha) \ll Q^{8+\epsilon}.$$  

One possibility for generalizing the approach in this thesis is to consider the analogous problem for $GL(3)$. This has become more possible recently thanks to the work of Goldfeld and Kontorovich [13] and Blomer [2] on the $GL(3)$ Kuznetsov trace formula. In particular, Young [41] used Blomer’s version of the trace formula, along with a study of the corresponding generalized Kloosterman sums, to derive a large sieve inequality for the Hecke eigenvalues of Maass cusp forms for $SL(3, \mathbb{Z})$. Unfortunately, Young’s inequality has a term of the form $N^{3/2+\epsilon}$ in place of the $N^{1+\epsilon}$ in the $GL(2)$ case, rendering the methods below ineffective. Nevertheless, it may still be possible to obtain nontrivial zero density estimates from this work, and this provides an interesting opportunity for future research.
The exposition below begins with a summary of the basic theory of Maass forms for congruence subgroups $\Gamma_0(q) \subseteq SL(2, \mathbb{Z})$, including all the definitions, notations and main theorems which will be needed in the subsequent parts. Next is a summary and adaptation of the work of Deshouillers and Iwaniec [6], including the relevant facts on Kloosterman sums, the Kuznetsov trace formula, and the derivation of the large sieve inequality which will be used to prove the main theorem. Finally, the results of the previous chapters come together with the methodology of Gallagher [10] in the final chapter to complete the proof of the main theorem.
Chapter 1

Preliminaries on Maass forms for $\Gamma_0(q)$

1.1 Automorphic forms for $GL(2)$

We begin with a presentation of the basic theory of automorphic forms for $GL(2)$, which will be the main objects of study throughout this thesis. Most of the definitions and notation are taken from Goldfeld and Hundley [12], though there are some differences.

Definition 1.1 (Upper half plane). Let $\mathcal{H}$ denote the upper half plane $\{x + iy \mid x \in \mathbb{R}, y > 0\}$.

The group $SL(2, \mathbb{R})$ acts on $\mathcal{H}$ by Möbius transformations:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.
\]

Definition 1.2 (Extended upper half plane). Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \infty$. 

The action of the subgroup $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ can be extended to $\mathfrak{h}^*$ by defining

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \begin{cases} \frac{a}{c} & \text{if } c \neq 0, \\ \infty & \text{if } c = 0, \end{cases}
\]

and

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{m}{n} = \begin{cases} \infty & \text{if } cm + dn = 0, \\ \frac{am + bn}{cm + dn} & \text{otherwise}. \end{cases}
\]

For $q$ a positive integer, we define:

**Definition 1.3** (Congruence subgroup $\Gamma_0(q)$).

\[
\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c \equiv 0 \mod q \right\}.
\]

**Definition 1.4** (Cusp). A *cusp* for $\Gamma_0(q)$ is an equivalence class in $\mathbb{Q} \cup \infty$ with respect to the action of $\Gamma_0(q) \subseteq SL(2,\mathbb{Z})$.

**Definition 1.5** (Moderate growth). Let $f : \mathfrak{h} \to \mathbb{C}$ be a smooth function, and $a \in \mathbb{Q} \cup \infty$. Then $f$ has *moderate growth* at $a$ if, for any $\sigma_a \in SL(2,\mathbb{R})$ such that $\sigma_a\infty = a$, $f(\sigma_a(x + iy)) = O(y^N)$ for some $N$ as $y \to \infty$.

This allows for a basic definition of automorphic functions:

**Definition 1.6** (Automorphic Function). Let $k \geq 0$ be an integer. An *automorphic function* of weight $k$ for $\Gamma_0(q)$ is a smooth function $f : \mathfrak{h} \to \mathbb{C}$ such that:

- If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$, then $f(\gamma z) = (cz + d)^k f(z)$ for all $z \in \mathfrak{h}$;
- $f$ has moderate growth at all cusps for $\Gamma_0(q)$.
The space of automorphic functions is equipped with the following inner product:

**Definition 1.7** (Petersson inner product). Let \( f \) and \( g \) be automorphic functions of weight \( k \) for \( \Gamma_0(q) \). Then the Petersson inner product of \( f \) and \( g \) is

\[
\langle f, g \rangle_k = \int_{\Gamma_0(q) \backslash \mathfrak{h}} f(z) \overline{g(z)} y^{k} \frac{dxdy}{y^2},
\]

where the integral is taken over a fundamental domain for the action of \( \Gamma_0(q) \) on \( \mathfrak{h} \). The subscript \( k \) will be suppressed when it is clear from context.

While this integral will not always converge in general, we will often impose the condition \( \langle f, f \rangle < \infty \) on the functions studied below, ensuring that this is a true inner product in the relevant setting. The Hilbert space generated by all such functions is denoted \( L^2(\Gamma_0(q) \backslash \mathfrak{h}, k) \). The study of the spectral theory of this space goes back to Selberg [30], who decomposed it as direct sum of a discrete spectrum (cusp forms), a continuous spectrum (Eisenstein series), and a residual spectrum (residues of Eisenstein series).

The following operators act on the space of automorphic functions:

**Definition 1.8** (Hecke operators). Let \( n \) be a positive integer. Then for \( f \) an automorphic function for \( \Gamma_0(q) \), define the Hecke operator \( T_n \) by

\[
T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,N)=1}} \sum_{b=1}^{d} f \left( \frac{az+b}{d} \right).
\]

Also, define

\[
T_{-1} f(z) = f \left( \frac{-1}{Nz} \right).
\]
It can be verified that
\[ T_m T_n = \sum_{d \mid (m,n) \atop (d,N)=1} T_{\frac{mn}{d^2}}, \]
so the Hecke operators \( T_n \) commute with each other.

For \((n, N) = 1\), the operators \( T_n \) preserve the space of automorphic functions and are normal operators with respect to the Petersson inner product (for details, see Goldfeld [11]). Thus, any subspace of \( L^2(\Gamma_0(q)\backslash \mathfrak{h}, k) \) which is invariant under the action of the \( T_n \) for \((n, N) = 1\) has an orthonormal basis consisting of simultaneous eigenfunctions of all such \( T_n \).

### 1.2 Maass Forms of Weight Zero

**Definition 1.9** (Non-Euclidean Laplacian). Define

\[ \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]

The operator \( \Delta \) is invariant under the action of \( SL(2, \mathbb{R}) \), and hence acts on the space of \( \Gamma_0(q) \)-invariant functions. The Maass forms are the automorphic functions which are eigenfunctions of this operator.

**Definition 1.10.** A *Maass form of weight zero* for \( \Gamma_0(q) \) is a smooth function \( f : \mathfrak{h} \to \mathbb{C} \) such that:

- \( f \) is an automorphic function of weight 0 for \( \Gamma_0(q) \) in the sense of Definition 1.6,
- \( \Delta f = (\frac{1}{4} + t^2)f. \)

The notation \( \frac{1}{4} + t^2 \) for the Laplace eigenvalue will be used throughout. Note that the eigenvalues of \( \Delta \) are real and positive (Goldfeld [11], Proposition 3.3.2). Conjecturally, the \( t \) are real:
Conjecture 1.1 (Selberg [31]). Let $f$ be a non-constant function on $\Gamma_0(q) \backslash \mathfrak{H}$ with $\Delta f = \lambda f$. Then $\lambda \geq 1/4$.

At present, the best bound, due to Kim and Sarnak [18] is $\lambda \geq \frac{975}{4096} \approx 0.238 \ldots$, so it is possible that $t$ may be purely imaginary of absolute value at most $\frac{7}{64}$.

The Fourier expansion of such a Maass form about each cusp may be defined as follows. Let $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ be the stabilizer of $\infty$ in $\Gamma_0(q)$. For any cusp $a$, we can choose $\sigma_a \in SL(2, \mathbb{R})$ such that $\sigma_a \infty = a$ and $\sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_\infty$, where $\Gamma_a$ is the stabilizer of $a$ in $\Gamma_0(q)$ (a detailed construction of $\sigma_a$ is given on p. 87 of [12]). Thus, there exists $g_a \in \Gamma_a \subset \Gamma_0(q)$ such that $\sigma_a^{-1} g_a \sigma_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence, for any Maass form $f$ of weight zero for $\Gamma_0(q)$, we have

$$f(\sigma_a(z + 1)) = f\left(\sigma_a \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(g_a \sigma_a z) = f(\sigma_a z),$$

so the function $z \mapsto f(\sigma_a z)$ is periodic in $x$ with period one. Thus, we may write

$$f(\sigma_a z) = \sum_{n \in \mathbb{Z}} c_{a_0}(y)e(nx),$$

where again $e(\theta) = e^{2\pi i \theta}$. This facilitates the following definition:

Definition 1.1 (Maass cusp form). A Maass cusp form of weight zero for $\Gamma_0(q)$ is a Maass form $f$ of weight zero for $\Gamma_0(q)$ such that

- $\langle f, f \rangle < \infty$,
- The Fourier coefficient $c_{a_0}(y)$ defined above vanishes for all cusps $a$.

The condition $\Delta f = \left(\frac{1}{4} + t^2\right)f$, when applied to the Fourier expansion, shows that the Fourier
coefficients $c_{an}(y)$ of a cusp form, $n \neq 0$, satisfy the differential equation

$$y^2 c''_{an}(y) = (4\pi^2 n^2 y^2 - 1 - t^2) c_{an}(y).$$

The only solutions to this equation which are compatible with the moderate growth condition are of the form

$$c_{an}(y) = a(n) \sqrt{y} K_{it}(2\pi |n| y),$$

where $K$ denotes the Bessel function

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})/2} t^s dt,$$

and thus any cusp form $f$ has the following Fourier-Whittaker expansion at the cusp $\infty$:

$$f(z) = \sum_{n \neq 0} a_f(n) \sqrt{y} K_{it}(2\pi |n| y) e(nz).$$

This is Proposition 3.5.1 in Goldfeld [11], which contains more details.

These $a_f(n)$ may be used to define a Dirichlet series, but in some cases, the resulting function will not have the typical properties of an $L$-function, such as an Euler product. In particular, if $f$ is a Maass cusp form of weight zero for the full modular group $SL(2, \mathbb{Z})$, then the function $z \mapsto f(Nz)$ will be a Maass cusp form of weight zero for $\Gamma_0(q)$. However, this function has first Fourier coefficient 0, so it is impossible to normalize in such a way that the na"ively defined $L$-function has an Euler product.

The remedy to this situation is the concept of newforms developed by Atkin and Lehner [1].
**Definition 1.12.** An *oldform* is a cusp form $f$ such that $f(z) = g(dz)$ for some form $g$ on $\Gamma_0(M)$, where $M$ is a proper divisor of $N$ and $d$ is a positive divisor of $\frac{N}{M}$. A *newform* is a cusp form which is orthogonal to all oldforms with respect to the Petersson inner product.

The Hecke operators $T_n$ (for all $n$, not just $(n, N) = 1$), preserve the space of newforms. Thus, the space of newforms has an orthonormal basis consisting of simultaneous eigenfunctions for all the $T_n$. Such a form has a well-behaved $L$-function:

**Definition 1.13 ($L$-function).** Let

$$f(z) = \sum_{n \in \mathbb{Z}} a_f(n) \sqrt{y} K_{it}(2\pi |n|y)e(nx)$$

be a newform of weight zero, and let

$$\lambda_f(n) = \frac{a_f(n)}{a_f(1)}.$$

Normalizing such that $\lambda_f(1) = 1$ has the result that $\lambda_f(n)$ is the eigenvalue of $T_n$ corresponding to $f$. Then the $L$-function associated to $f$ is defined for $\Re(s) > 1$ by

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}.$$ 

The relation

$$T_m T_n = \sum_{d | (m,n)} \sum_{(d,N)=1} T_{\frac{mn}{d^2}}$$

on the Hecke operators implies that that the Hecke eigenvalues are multiplicative in the sense that
\(\lambda_f(mn) = \lambda_f(m)\lambda_f(n)\) whenever \((m, n) = 1\). Thus, the \(L\)-function defined above has an Euler product:

**Proposition 1.1** (Euler product).

\[
L(s, f) = \prod_{\substack{p|q \leq q \leq y \leq x \leq q + y}} \left(1 - \lambda_f(p)p^{-s}\right)^{-1} \prod_{\substack{p|q \leq q \leq x \leq q + y}} \left(1 - \lambda_f(p)p^{-s} + p^{-2s}\right)^{-1}
\]

\[
= \prod_{\substack{p|q \leq q \leq y \leq x \leq q + y}} \left(1 - \lambda_f(p)p^{-s}\right)^{-1} \prod_{\substack{p|q \leq q \leq x \leq q + y}} \left(1 - \alpha_{1,f}(p)p^{-s}\right)^{-1} \left(1 - \alpha_{2,f}(p)p^{-s}\right)^{-1},
\]

where the numbers \(\alpha_{i,f}(p)\) satisfy \(\alpha_{1,f}(p) + \alpha_{2,f}(p) = \lambda_f(p)\) and \(\alpha_{1,f}(p)\alpha_{2,f}(p) = 1\).

The order of growth of \(\alpha_{i,f}(p)\), and by extension the Hecke eigenvalues \(\lambda_f(n)\), is the subject of the Ramanujan-Petersson conjecture. This conjecture holds that \(|\alpha_{i,f}(p)| = 1\), but remains an open problem. The best result to date is the following:

**Proposition 1.2** (Kim-Sarnak [18]).

\[
|\alpha_{i,f}(p)| \leq p^{7/64}.
\]

Just as with the Riemann zeta function and the Dirichlet \(L\)-functions, this \(L\)-function also has analytic continuation to the whole complex plane and satisfies a functional equation:

**Proposition 1.3** (Functional equation). Let

\[
\Lambda(s, f) = q^{s/2} \pi^{-s} \Gamma \left( \frac{s + \epsilon + it}{2} \right) \Gamma \left( \frac{s + \epsilon + it}{2} \right) L(s, f),
\]
where $\frac{1}{4} + t^2$ is the Laplace eigenvalue of $f$, and

$$
\epsilon = \begin{cases} 
0 & \text{if } T_{-1} f = f, \\
1 & \text{if } T_{-1} f = -f.
\end{cases}
$$

Then

$$
\Lambda(s, f) = (-1)^{s} \Lambda(1 - s, f).
$$

The Generalized Riemann Hypothesis in this context asserts that all zeros of $L(\sigma + it, f)$ in the critical strip $0 < \sigma < 1$ lie on the line $\sigma = 1/2$.

In the following sections, we will need a standard basis for the space of cusp forms (all such forms, not only newforms). Using the material above, we may choose this basis as follows:

**Definition 1.14 (Basis for space of cusp forms).** Let $\{u_j\}_{j \geq 1}$ be a basis for the space of Maass cusp forms of weight zero for $\Gamma_0(q)$ such that:

- The basis $\{u_j\}$ is orthonormal with respect to the Petersson inner product;
- Each $u_j$ is an eigenfunction of all the Hecke operators $T_n$ when $(n, N) = 1$;
- The basis $\{u_j\}$ contains a subset which is a basis for the space of newforms, and each $u_j$ which is a newform is an eigenfunction of the Hecke operators $T_n$ for all $n$;
- The Laplace eigenvalues $\lambda_j$ (such that $\Delta u_j = \lambda_j u_j$) are nondecreasing: $\lambda_1 \leq \lambda_2 \leq \cdots$.

Further define $u_0$ to be the constant function with Petersson norm 1. While the constant function is not a cusp form, including $u_0$ in this basis will simplify the notation below.
1.3 Eisenstein Series and the Selberg Spectral Decomposition

In addition to the Maass cusp forms of weight zero, the space \(L^2(\Gamma_0(q) \backslash \mathfrak{h})\) contains a continuous spectrum which may be expressed in terms of Eisenstein series. The Eisenstein series may be defined as follows:

**Definition 1.15 (Eisenstein Series).** Let \(a\) be a cusp for \(\Gamma_0(q)\). For \(\Re(s) > 1\) and \(z \in \mathfrak{h}\), let

\[
E_a(z, s) = \sum_{\gamma \in \Gamma_a \backslash \Gamma_0(q)} \Im(\sigma_a \gamma z)^s,
\]

where, as before \(\Gamma_a\) is the stabilizer of \(a\) and \(\sigma_a \in SL(2, \mathbb{R})\) is chosen such that \(\sigma_a \infty = a\) and \(\sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_\infty\).

The series \(E_a(z, s)\) has the following Fourier expansion about each cusp \(b\), as given in Kubota [21]:

\[
E_a(\sigma_b z, s) = \delta_{ab} y^s + \pi^{1/2} \left( \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right) \varphi_{a0}(s) y^{1-s} + \frac{2y^{1/2} \pi^s}{\Gamma(s)} \sum_{n \neq 0} |n|^{s-\frac{1}{2}} \varphi_{abn}(s) K_{s-\frac{1}{2}}(2\pi |n| y) e(nx),
\]

where

\[
\varphi_{abn}(s) = \sum_{c > 0} c^{-2s} \sum_{d} e \left( \frac{nd}{c} \right),
\]

the inner sum being over the congruence classes \(d \pmod{c}\) such that there exists \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_a^{-1} \Gamma_0(q) \sigma_b\). This expansion shows that \(E_a(z, s)\) has an analytic continuation as a function of \(s\) to the entire complex plane, and that the vector whose components are the Eisenstein series \(E_a(z, s)\) for all cusps \(a\) has a functional equation for \(s \mapsto 1 - s\).

The Selberg spectral decomposition states that the Maass cusp forms, the Eisenstein series
\( E_a(z, \frac{1}{2} + ir) \), and the residues of Eisenstein series (in this case, just the constant function \( u_0 \)) generate \( L^2(\Gamma_0(q) \setminus \mathfrak{h}) \) in the following way:

**Theorem 1.1** (Selberg spectral decomposition). Let \( f \in L^2(\Gamma_0(q) \setminus \mathfrak{h}) \), and let \( \{u_j\} \) be the basis of cusp forms from Definition 1.14. Then

\[
  f(z) = \sum_{j \geq 0} \langle f, u_j \rangle u_j(z) + \frac{1}{4\pi} \sum_a \int_{-\infty}^{\infty} \left\langle f, E_a(\cdot, \frac{1}{2} + ir) \right\rangle E_a(z, \frac{1}{2} + ir) \, dr,
\]

where \( a \) varies over the set of all cusps for \( \Gamma_0(q) \).

For more discussion of this result, see Theorem 3.16.1 in Goldfeld [11]. This immediately leads to the following:

**Proposition 1.4** (Parseval identity). Let \( f, g \in L^2(\Gamma_0(q) \setminus \mathfrak{h}) \), and let \( \{u_j\} \) be as above. Then

\[
  \langle f, g \rangle = \sum_{j \geq 0} \langle f, u_j \rangle \langle g, u_j \rangle + \frac{1}{4\pi} \sum_a \int_{-\infty}^{\infty} \left\langle f, E_a(\cdot, \frac{1}{2} + ir) \right\rangle \left\langle g, E_a(\cdot, \frac{1}{2} + ir) \right\rangle \, dr.
\]
Chapter 2

The Kuznetsov Trace Formula and Large Sieve Inequalities for Fourier Coefficients

This chapter develops the Kuznetsov trace to establish a relationship between Fourier coefficients of Maass cusp forms and Kloosterman sums, and uses this relationship to derive the large sieve inequalities which will be used in the proof of the main theorem.

2.1 Kloosterman Sums

Definition 2.1 (Kloosterman sum). Let

\[ S(m, n; c) = \sum_{d \mod c}^* e \left( \frac{md + nd}{c} \right), \]

where the asterisk means that the sum is taken over residue classes such that \((c, d) = 1\) and \(\bar{d}\) is such that \(d\bar{d} \equiv 1 \mod c\).
These sums, used by Kloosterman in [19] to provide an improvement to the Hardy-Littlewood circle method, have a number of useful properties and applications.

**Proposition 2.1.** Let \((c_1, c_2) = 1\), and choose \(c_1\) and \(c_2\) such that \(c_1c_1 \equiv 1 \mod c_2\) and \(c_2c_2 \equiv 1 \mod c_1\). Then

\[
S(m, n; c_1c_2) = S(c_2m, c_2n; c_1)c_1 S(c_1m, c_1n; c_2).
\]

**Proof.** The right hand side is

\[
\sum_{d_1 \mod c_1}^* e \left( \frac{c_2md_1 + c_2nd_1}{c_1} \right) \sum_{d_2 \mod c_2}^* e \left( \frac{c_1md_2 + c_1nd_2}{c_2} \right)
\]

\[
= \sum_{d_1 \mod c_1}^* \sum_{d_2 \mod c_2}^* e \left( \frac{c_2c_2md_1 + c_2c_2nd_1 + c_1c_1md_2 + c_1c_1nd_2}{c_1c_2} \right).
\]

The numerator is congruent to \(md_1 + nd_1 \mod c_1\) and \(md_2 + nd_2 \mod c_2\). Thus, for each \(d \mod c_1c_2\) such that \((d, c_1c_2) = 1\), if \(d_1\) and \(d_2\) denote the residue classes of \(d \mod c_1\) and \(d \mod c_2\), respectively, then the numerator will be congruent to \(md + nd \mod c_1c_2\) by the Chinese Remainder Theorem. Since, again by the Chinese Remainder Theorem, the \(d \mod c_1c_2\) such that \((d, c_1c_2) = 1\) are in bijective correspondence with the resulting pairs \((d_1, d_2)\) appearing in the above sum, it follows that the sum is just \(S(m, n; c_1c_2)\).

This multiplicativity relation reduces the problem of estimating Kloosterman sums in general to that of estimating Kloosterman sums where \(c\) is a prime power. In this case, the work of Weil [39] allows the sums to be bounded by means of the Riemann hypothesis for function fields. The result is:
Proposition 2.2 (Weil bound for Kloosterman sums).

\[ |S(m, n; c)| \leq \tau(c)(m, n, c)^{1/2}c^{1/2}, \]

where \( \tau(c) \) is the number of positive divisors of \( c \) and \((m, n, c)\) is the greatest common divisor of \( m, n, \) and \( c. \)

The following useful identity may be proved (as in Kuznetsov [22], Theorem 4) as an application of the Kuznetsov trace formula and the identity

\[ T_m T_n = \sum_{d \mid (m, n)} T_{\frac{mn}{d^2}} \]

for the Hecke operators for \( SL(2, \mathbb{Z}) \backslash \mathfrak{h}. \) An elementary proof is also possible, as shown by Matthes [24].

Proposition 2.3.

\[ S(m, n; c) = \sum_{d \mid (m, n, c)} d \cdot S\left(1, \frac{mn}{d^2}, \frac{c}{d}\right). \]

The Kloosterman sums may be used to define the Kloosterman zeta function

\[ Z_{m,n}(s) = \sum_{c=1}^{\infty} S(m, n; c)c^{-2s}. \]

The existence of an analytic continuation for the Kloosterman zeta function relates to the question of Conjecture 1.1 on the eigenvalues of the Laplace operator. Selberg [31] showed that, for real \( \sigma > 1/2, \) the Kloosterman zeta function \( Z_{m,n}(s) \) has analytic continuation to \( \Re(s) > \sigma \) if and
only if the least positive eigenvalue of $\Delta$ is at least $\sigma(1 - \sigma)$. The bound of Proposition 2.2 may be used to show continuation to $\Re(s) > 3/4$, which gives a lower bound of $3/16$ for the eigenvalues.

The application of Kloosterman sums to large sieve inequalities for Fourier coefficients requires the development of large sieve type inequalities for bilinear forms involving Kloosterman sums. The technique for this is due to Iwaniec [16], with some modifications for the purposes of this thesis.

The proof will require the use of the analytic large sieve. Recall the large sieve of Montgomery and Vaughan mentioned in the introduction:

**Theorem 2.1** (Montgomery-Vaughan [26]). Let $\alpha_1, \ldots, \alpha_R$ be a set of real numbers such that

$$||\alpha_r - \alpha_s|| \geq \delta$$

for $r \neq s$, where $|| \cdot ||$ denotes the distance to the nearest integer, and let

$$S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha).$$

Then

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq (N + \frac{1}{\delta}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Taking $\alpha_r$ to be the fractions $r/c$ for all $r \mod c$ then yields, using $\delta = 1/c$ in the above:

**Proposition 2.4.**

$$\sum_{r \mod c} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{n}{c} - \frac{r}{c}\right) \right|^2 \leq (N + c) \sum_{n=M+1}^{M+N} |a_n|^2.$$

This may be used to prove the following inequality for Kloosterman sums:
Proposition 2.5. \textit{Let}

\[ B(\theta, c, M, N) = \sum_{m=M+1}^{M+N} \sum_{n=M+1}^{M+N} b_m b_n S(m, n, c) e \left( \frac{2\sqrt{mn}}{c} \theta \right), \]

\textit{where} \( N \leq M \) \textit{are positive integers and} \( \theta > 0 \) \textit{is a real number. Suppose} \( c > M^{1-t} \). \textit{Then}

\[ |B(\theta, c, M, N)| \ll M^\epsilon (c + N) \sum_{n=M+1}^{M+N} |b_n|^2. \]

\textit{Proof.} Let \( \alpha = \frac{N}{M} \leq 1 \). Then \( 1 \leq \frac{\sqrt{mn}}{M} \leq 1 + \alpha \) \textit{for all} \( m, n \) \textit{appearing in the sum}. \textit{Thus, attaching a smooth weight function} \( \eta(\frac{\sqrt{mn}}{M}) \) \textit{such that}

\[ \eta(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 1 + \alpha, \\ 0 & \text{if } x \leq 1 - \frac{\alpha}{2} \text{ or } x \geq 1 + 2\alpha. \end{cases} \]

\textit{will not affect the sum}. \textit{By the Mellin inversion formula,}

\[ \eta(x)e(2\theta c^{-1} Mx) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} R(s)x^{-s}ds, \]

\textit{where}

\[ R(s) = \int_0^\infty \eta(x)e(2\theta c^{-1} Mx)x^{s-1}dx. \]

\textit{For} \( s = 1 + it \), \textit{this is bounded for all} \( t \) \textit{and by} \( t^{-2} \) \textit{for} \( |t| > 16\pi\theta c^{-1} M \). \textit{Applying this Mellin}
inversion to the sum gives

\[ B(\theta, c, M, N) = \sum_{m=M+1}^{M+N} \sum_{n=M+1}^{M+N} b_m \overline{b}_n S(m, n, c) \eta \left( \frac{\sqrt{mn}}{M} \right) e \left( \frac{2\sqrt{mn}}{c} \right) \]

\[ = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \sum_{m=M+1}^{M+N} \sum_{n=M+1}^{M+N} b_m \overline{b}_n S(m, n, c) R(s) \left( \frac{\sqrt{mn}}{M} \right)^{-s} ds \]

\[ \ll \int_{1-i\infty}^{1+i\infty} \sum_{d \mod c} \left| \sum_{m=M+1}^{M+N} b_m m^{-s/2} e \left( \frac{md}{c} \right) \right| \left| \sum_{n=M+1}^{M+N} \overline{b}_n n^{-s/2} e \left( \frac{nd}{c} \right) \right| M^s R(s) ds. \]

By Proposition 2.4,

\[ \sum_{d \mod c} \left| \sum_{m=M+1}^{M+N} b_m m^{-s/2} e \left( \frac{md}{c} \right) \right| \left| \sum_{n=M+1}^{M+N} \overline{b}_n n^{-s/2} e \left( \frac{nd}{c} \right) \right| \leq M^{-1} (N + c) \sum_{n=M+1}^{M+N} |b_n|^2. \]

Thus

\[ |B(\theta, c, M, N)| \ll \int_{1-i\infty}^{1+i\infty} R(s) ds \left( N + c \right) \sum_{n=M+1}^{M+N} |b_n|^2, \]

and the result follows from the bounds on \( R \) and the assumption \( c > M^{1-\epsilon} \).

In the case of smaller \( c \), we have the following:

**Proposition 2.6.** Let \( B(\theta, c, M, N) \) be as in Proposition 2.5, where again \( N \leq M \), and now assume \( c \leq M^{1-\epsilon} \). Assume further that \( 0 < \theta < 2 \). Then

\[ |B(\theta, c, M, N)| \ll \theta^{-\frac{1}{2}} c^{\frac{1}{2}} N^{\frac{1}{2}} M^{\frac{1}{2}+\epsilon} \sum_{n=M+1}^{M+N} |b_n|^2. \]
Proof. By Proposition 2.3,

\[ B(\theta, c, M, N) = \sum_{d|c} d \cdot X \left( \theta, \frac{c}{d}, \frac{M}{d}, \frac{N}{d} \right), \]

where for \( x_n = b_{dn}, q = c/d, K = M/d, L = N/d, \)

\[ X(\theta, q, K, L) = \sum_{K < m, n \leq K + L} \overline{x_m} x_n S(1, mn; q) e \left( \frac{2\sqrt{mn}}{q} \right). \]

By Cauchy-Schwartz,

\[
|X(\theta, q, K, L)|^2 \leq \left( \sum_{K < n \leq K + L} |x_n|^2 \right) \sum_{n \in \mathbb{Z}} \eta \left( \frac{n}{K} \right) \left| \sum_{K < m \leq K + L} \overline{x_m} S(1, mn; q) e \left( \frac{2\sqrt{mn}}{q} \right) \right|^2 \\
= \left( \sum_{K < n \leq K + L} |x_n|^2 \right) \sum_{K < m_1, m_2 \leq K + L} x_{m_1} x_{m_2} \sum_{h_1, h_2 \mod q}^{*} e \left( \frac{h_1 - h_2}{q} \right) \sum_{n \in \mathbb{Z}} f(n),
\]

where \( \eta \) is the smooth weight function from the previous proposition, the asterisk as usual means a sum over the classes where \( (h_1, h_2, q) = 1 \), the residue classes \( \overline{h_i} \mod q \) are defined such that \( h_i \overline{h_i} \equiv 1 \mod q \), and

\[ f(n) = \eta \left( \frac{n}{K} \right) e \left( \frac{h_1 m_1 - h_2 m_2}{q} n + \frac{2(\sqrt{m_1} - \sqrt{m_2})\theta}{q} \sqrt{n} \right). \]

For brevity, define \( A \) and \( B \) so that \( f(n) = \eta \left( \frac{n}{K} \right) e \left( An + B\sqrt{n} \right) \). By the Poisson summation formula,

\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{u \in \mathbb{Z}} \hat{f}(u), \]
where

\[ \hat{f}(u) = \int_{-\infty}^{\infty} \eta \left( \frac{t}{K} \right) e \left( (A - u)t + B \sqrt{t} \right) dt. \]

Suppose \( u \neq A. \) Then \( |A - u| \geq \frac{1}{q}. \) Also, for all \( t \) such that \( \eta \left( \frac{t}{K} \right) \neq 0 \) (and thus \( t > K/2 \)),

\[ \frac{|B|}{2\sqrt{t}} < \frac{\frac{\sqrt{K+L} - \sqrt{K}}{\sqrt{K/2}}}{q} \ll \frac{1}{q} \]

Integrating by parts \( p \) times gives:

\[
\hat{f}(u) = \left( -\frac{1}{2\pi i} \right)^p \int_{-\infty}^{\infty} \left( \cdots \left( \left( \eta \left( \frac{t}{K} \right) \frac{1}{A - u + \frac{B}{2\sqrt{t}}} \right)' \frac{1}{A - u + \frac{B}{2\sqrt{t}}} \right)' \cdots \right) \\
\cdot e \left( (A - u)t + B \sqrt{t} \right) dt,
\]

so

\[ \hat{f}(u) \ll K(|A - u|K)^{-p}. \]

Thus

\[ \sum_{u \neq A} \hat{f}(u) \ll K \left( \frac{K}{q} \right)^{-p} = \frac{M}{d} \left( \frac{M}{c} \right)^{-p} \ll \frac{M}{d} M^{-\epsilon p} \ll K^{-1} \]

for \( p \) sufficiently large. The remaining case is \( u = A, \) which from the definition of \( A \) implies

\[ h_1m_1 \equiv h_2m_2 \bmod q. \]

In this case,

\[ \sum_{h_1, h_2 \bmod q}^* e \left( \frac{h_1 - h_2}{q} \right) \ll (m_1 - m_2, q), \]

where the sum is taken over residue classes with \( h_1m_1 \equiv h_2m_2 \bmod q. \)

If \( m_1 = m_2 \) then trivially \( \hat{f}(A) \ll L. \) Otherwise, integration by parts gives

\[ \hat{f}(A) = \int_{-\infty}^{\infty} \eta \left( \frac{t}{K} \right) e \left( B \sqrt{t} \right) dt \ll \frac{\sqrt{L}}{|B|} \ll \frac{q\sqrt{KL}}{\theta|m_1 - m_2|}. \]
Combining everything above gives

\[ \sum_{K<m_1,m_2 \leq K+L} x_{m_1} x_{m_2} \sum_{h_1, h_2 \mod q} * e \left( \frac{h_1 - h_2}{q} \right) \sum_{n \in \mathbb{Z}} f(n) \]

\[ \ll q(q + L) \sum_{K<m \leq K+L} |x_m|^2 + \theta^{-1} q \sqrt{KL} \sum_{K<m_1,m_2 \leq K+L \atop m_1 \neq m_2} \frac{(m_1 - m_2, q)}{|m_1 - m_2|} |x_{m_1} x_{m_2}| \]

\[ \ll \theta^{-1} q K^\varepsilon \sqrt{KL} \sum_{K<m \leq K+L} |x_m|^2. \]

and thus

\[ |X(\theta, q, K, L)|^2 \ll \theta^{-1} q K^\varepsilon \sqrt{KL} \left( \sum_{K<m \leq K+L} |x_m|^2 \right)^2. \]

Taking square roots and summing over \( d | c \) gives the result.

2.2 Kuznetsov Trace Formula

To relate Kloosterman sums to Fourier coefficients of cusp forms, we use the trace formula developed by Kuznetsov [22]. The key to this formula is the non-holomorphic Poincaré series, which is defined as follows (see Selberg [31]):

**Definition 2.2.** The non-holomorphic Poincaré series (at the cusp 1) for \( \Gamma_0(q) \), for integer \( m \geq 0 \), is

\[ P_m(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \text{Im}(\gamma z)^s e(m \gamma z). \]

This is a generalization of the Eisenstein series, which results from setting \( m = 0 \). The Poincaré series converges absolutely for \( \Re(s) > 1 \) and satisfies \( P_m(\gamma z, s) = P_m(z, s) \) by construction. Furthermore, it is square integrable, so \( P_m(\cdot, s) \in L^2(\Gamma_0(q) \backslash \mathbb{H}) \).
The Kuznetsov trace formula is derived by computing the inner product $\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2}) \rangle$ in two ways. First, it may be computed directly by means of the Fourier expansion of $P_m(z, s)$, which exists because $P_m(z, s)$ is automorphic. The expansion is as follows (details can be found in Kuznetsov [22]), and accounts for the appearance of Kloosterman sums in the trace formula:

**Proposition 2.7.** Suppose $\Re(s) > 1$ and $m \geq 0$. Then

$$P_m(z, s) = \sum_{n \in \mathbb{Z}} B_n(m, y, s)e(nx),$$

where

$$B_n(m, y, s) = \delta_{mn}y^s e^{-2\pi ny} + y^s \sum_{c > 0, c \equiv 0 \mod q} c^{-2s} S(m, n; c) \int_{-\infty}^{\infty} (x^2 + y^2)^{-s} e \left( -nx - \frac{m}{c^2(x + iy)} \right) dx.$$

Based on the fact that, by the Weil bound (Proposition 2.2), the Kloosterman zeta function $Z_m,n(s) = \sum_{c=1}^{\infty} S(m, n; c)c^{-2s}$ converges for $\Re(s) > 3/4$, this formula may be used to give an analytic continuation of the Poincaré series $P_m(z, s)$ to $\Re(s) > 3/4$ as well. This expansion gives the following formula for the inner product of two Poincaré series:

**Proposition 2.8.** Let $m, n \geq 1$. Then

$$\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2}) \rangle = \int_{0}^{\infty} B_n(m, y, s_1)y^{s_2-2}e^{-2\pi ny} dy.$$
Proof. Let \( \mathcal{D} \) be a fundamental domain for the action of \( \Gamma_0(q) \) on \( \mathfrak{h} \). Then

\[
\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2}) \rangle = \int \int_{\mathcal{D}} P_m(z, s_1) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \text{Im}(\gamma z)^{s_2} e(nz) \frac{dxdy}{y^2}
\]

\[
= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \int \int_{\mathcal{D}} P_m(\gamma^{-1}z, s_1) y^{s_2} e(nz) \frac{dxdy}{y^2}
\]

\[
= \int \int_{E} P_m(z, s_1) e(-nx) y^{s_2 - 2} e^{-2\pi my} dxdy,
\]

where \( E \) is a fundamental domain for the action of \( \Gamma_\infty \). Such a fundamental domain is given by the vertical strip \( 0 \leq x < 1, y > 0 \). Replacing \( P_m(z, s_1) \) with its Fourier expansion then gives

\[
\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2}) \rangle = \int_{0}^{\infty} \int_{0}^{1} \sum_{k \in \mathbb{Z}} B_k(m, y, s_1) e(kx) e(-nx) y^{s_2 - 2} e^{-2\pi my} dxdy.
\]

The integral over \( x \) isolates the case \( k = n \), finishing the proof.

Substituting the expression from Proposition 2.7 for \( B_k(m, y, s_1) \) allows this inner product to be expressed in terms of Kloosterman sums. Again using the Weil bound for Kloosterman sums, this allows for the analytic continuation of \( \langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2}) \rangle \) for \( \Re(s_1) > 3/4, \Re(s_2) > 3/4, \) and, hence, it makes sense to consider \( s_1 = 1 + it, s_2 = 1 - it \). In this case, the explicit calculations were carried out by Kuznetsov [22], resulting in the following:

**Proposition 2.9.** Let \( m, n \geq 1 \). Then

\[
\langle P_m(\cdot, 1 + it), P_n(\cdot, 1 + it) \rangle = \frac{\delta_{mn}}{4\pi n} - 2i \left( \frac{n}{m} \right)^{it} \sum_{c > 0 \mod q} \frac{S(m, n; c)}{c^2} \int_{-i}^{i} K_{2it} \left( \frac{4\pi \sqrt{mn}}{c} v \right) \frac{dv}{v}.
\]

The inner product \( \langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2}) \rangle \) may also be computed using the Selberg spectral
decomposition (Theorem 1.1), taking advantage of the fact that \( P_m(\cdot, s) \in L^2(\Gamma_0(q) \setminus \mathfrak{h}) \). This version of the computation can be expressed in terms of the Fourier coefficients of cusp forms, since the inner product of a Poincaré series \( P_m \) and a cusp form relates to the \( m \)-th Fourier coefficient as follows:

**Proposition 2.10.** Let \( f(z) = \sum_{n \neq 0} a_f(n) \sqrt{y} K_{it}(2\pi |n| y) e(nz) \) be a Maass cusp form of weight zero for \( \Gamma_0(q) \) with Laplace eigenvalue \( \frac{1}{4} + t^2 \). Then, for \( m \geq 1 \) and \( \Re(s) > 1 \),

\[
\langle P_m(\cdot, s), f \rangle = (4\pi m)^{\frac{1}{2}-s} \pi^{\frac{1}{4}+t^2} a_f(m) \frac{\Gamma(s - \frac{1}{2} + it)\Gamma(s - \frac{1}{2} - it)}{\Gamma(s)}.
\]

**Proof.** The proof is similar to the proof of Proposition 2.8, and the same notation will be used.

\[
\langle P_m(\cdot, s), f \rangle = \int \int \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_0(q)} \text{Im}(\gamma z)^s e(m \gamma z) \overline{f(\gamma z)} \frac{dxdy}{y^2} \]

\[
= \int \int \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_0(q)} \text{Im}(z)^s e(mz) \overline{f(\gamma^{-1} z)} \frac{dxdy}{y^2} \]

\[
= \int_E \int f(z) e(mx) y^{s-\frac{1}{2}} e^{-2\pi my} dxdy \]

\[
= \int_0^{\infty} \int_0^1 \sum_{n \neq 0} a_f(n) K_{it}(2\pi |n| y) e(-nz) e(mx) y^{s-\frac{3}{2}} e^{-2\pi my} dxdy \]

\[
= a_f(m) \int_0^{\infty} K_{it}(2\pi my) y^{s-\frac{3}{2}} e^{-2\pi my} dy.
\]

The computation

\[
\int_0^{\infty} K_{it}(y) y^{s-\frac{3}{2}} e^{-y} dy = \pi^{\frac{1}{2}+s} \frac{\Gamma(s - \frac{1}{2} + it)\Gamma(s - \frac{1}{2} - it)}{\Gamma(s)}
\]
A similar argument applies for the Eisenstein series, giving:

**Proposition 2.11.** Let \( m \geq 1 \) and \( \Re(s) > 1 \), and let \( a \) be a cusp for \( \Gamma_0(q) \). Then

\[
\left\langle P_m(\cdot, s), E_a\left(\cdot, \frac{1}{2} + ir\right)\right\rangle = 2^{2-2s}m^{\frac{1}{2}-s-ir} \pi^{\frac{1}{2}-s-ir} \varphi_{\infty m} \left(\frac{1}{2} + ir\right) \frac{\Gamma(s - \frac{1}{2} + ir) \Gamma(s - \frac{1}{2} - ir)}{\Gamma(s) \Gamma(\frac{1}{2} - ir)}.
\]

The Parseval identity (Proposition 1.4) and the above computations immediately give:

**Proposition 2.12.** Let \( u_j(z) = \sum_{n \neq 0} a_j(n) \sqrt{y} \, K_{it}(2\pi|n|y)e(nx) \) be as in Definition 1.14, where the Laplace eigenvalue of \( u_j \) is \( \frac{1}{4} + t_j^2 \), and let \( m, n \geq 1 \). Then

\[
\langle P_m(\cdot, s_1), P_n(\cdot, s_2) \rangle = \frac{\pi m^{\frac{1}{2}-s_1} n^{\frac{1}{2}-s_2}}{(4\pi)^{s_1+s_2-1} \Gamma(s_1) \Gamma(s_2)} \left( \sum_{j \geq 1} a_j(m) a_j(n) \Lambda(s_1, s_2; t_j) \right)
+ \sum_a \int_{-\infty}^{\infty} \left( \frac{n}{m} \right)^{ir} \varphi_{\infty m} \left(\frac{1}{2} + ir\right) \varphi_{\infty n} \left(\frac{1}{2} + ir\right) \frac{\Lambda(s_1, s_2; r)}{|\Gamma(\frac{1}{2} + ir)|^2} dr,
\]

where

\[
\Lambda(s_1, s_2; r) = \Gamma\left(s_1 - \frac{1}{2} + ir\right) \Gamma\left(s_1 - \frac{1}{2} - ir\right) \Gamma\left(s_2 - \frac{1}{2} + ir\right) \Gamma\left(s_2 - \frac{1}{2} - ir\right).
\]

Specializing as above to the case \( s_1 = 1 + it, s_2 = 1 - it \) gives:
Proposition 2.13. Keeping the notation as above,

\[ \langle P_m(\cdot, 1 + it), P_n(\cdot, 1 + it) \rangle = \frac{1}{4\sqrt{mn}} \left( \frac{n}{m} \right)^{it} \frac{\sinh \pi t}{t} \left( \prod_{j \geq 1} \cfrac{a_j(m)a_j(n)}{\cosh \pi(t_j - t) \cosh \pi(t_j + t)} + \sum_{a} \int_{-\infty}^{\infty} \left( \frac{n}{m} \right)^{it} \frac{\varphi_{\alpha\alpha\alpha\alpha}(\frac{1}{2} + ir)\varphi_{\alpha\alpha\alpha\alpha}(\frac{1}{2} + ir)}{\cosh \pi(r - t) \cosh \pi(r + t)} \cosh \pi \right) dr \right). \]

The following version of the Kuznetsov trace formula (Deshouillers-Iwaniec [6], Lemma 4.7) then follows by equating the expressions in Propositions 2.9 and 2.13 and dividing both sides by \(\frac{1}{4\sqrt{mn}} \left( \frac{n}{m} \right)^{it} \frac{\sinh \pi t}{t} \).

Theorem 2.2 (Kuznetsov trace formula). Let \( \{u_j\} \) be a basis as in Definition 1.14 for the space of Maass cusp forms of weight zero for \( \Gamma_0(q) \), with Laplace eigenvalues \( \frac{1}{4} + t_j^2 \) and Fourier expansions \( u_j(z) = \sum_{n \neq 0} a_j(n) \sqrt{y} K_{it}(2\pi|n|y)e(nx) \). Let \( m, n \geq 1 \) be integers, and let \( t \in \mathbb{R} \). Then

\[
\frac{-2it}{\sinh \pi t} \sum_{c \equiv 0 \mod q} \frac{4\pi \sqrt{mn}}{c^2} S(m, n; c) \int_{-i}^{i} K_{2it} \left( \frac{4\pi \sqrt{mn}}{c} v \right) \frac{dv}{v} + \delta_{mn} \frac{t}{\pi \sinh \pi t}
\]

\[= \pi \sum_{j \geq 1} \frac{a_j(m)a_j(n)}{\cosh \pi(t_j - t) \cosh \pi(t_j + t)} + \sum_{a} \int_{-\infty}^{\infty} \left( \frac{n}{m} \right)^{it} \frac{\varphi_{\alpha\alpha\alpha\alpha}(\frac{1}{2} + ir)\varphi_{\alpha\alpha\alpha\alpha}(\frac{1}{2} + ir)}{\cosh \pi(r - t) \cosh \pi(r + t)} \cosh \pi \right) dr. \]

Here as usual \( a \) varies over the cusps of \( \Gamma_0(q) \).

2.3 Large Sieve Inequality for Maass Forms

Theorem 2.2 allows for sums involving Fourier coefficients of the Maass cusp forms \( \{u_j\} \) of weight zero (as in Definition 1.14) to be studied by means of the Kloosterman sums appearing on the opposite side of the trace formula. The result is the following large sieve inequality:
Theorem 2.3. With the notation from Theorem 2.2, let \( \{b_n\} \) be any sequence of complex numbers, and let \( M \geq N \) be integers. Then

\[
\sum_{t_j \leq K} \frac{1}{\cosh \pi t_j} \left| \sum_{n=M+1}^{M+N} b_n a_j(n) \right|^2 \ll (K^2 + M \epsilon N) \sum_{n=M+1}^{M+N} |b_n|^2.
\]

Proof. Multiplying both sides of Theorem 2.2 by \( t \sinh t e^{-\left(t/\sqrt{K^2 + M \epsilon N}\right)^2} b_m b_n \), and integrating over \( t \) from \(-\infty \) to \( \infty \), we have ([6], (5.5)):

\[
\sum_{j \geq 1} \frac{t_j + 1}{\cosh \pi t_j} e^{-\left(t_j/\sqrt{K^2 + M \epsilon N}\right)^2} \left| \sum_{n=M+1}^{M+N} b_n a_j(n) \right|^2
\]

\[
+ \sum_{c > 0} \int_{-\infty}^{\infty} \left( |r| + 1 \right) e^{-\left(r/\sqrt{K^2 + M \epsilon N}\right)^2} \left| \sum_{n=M+1}^{M+N} b_n n^{ir} \varphi_{a\infty} \left( \frac{1}{2} + ir \right) \right|^2 dr
\]

\[
\ll K^3 \sum_{n=M+1}^{M+N} |b_n|^2 + \sum_{c > 0} \sum_{c \equiv 0 \mod q} \sum_{m=M+1}^{M+N} \sum_{n=M+1}^{M+N} b_m b_n \frac{4\pi \sqrt{mn}}{c} S(m, n; c) \Phi \left( \frac{4\pi \sqrt{mn}}{c} \right),
\]

where

\[
\Phi (x) = \int_{-\infty}^{\infty} t^2 e^{-\left(t/\sqrt{K^2 + M \epsilon N}\right)^2} \int_{-i}^{i} K_{2i\pi}(xv) \frac{dv}{v}.
\]

The function \( \Phi \) has the following representations ([6], (5.6), (5.7)):

\[
\Phi (x) = \sqrt{\pi i K^3} \int_{0}^{\infty} e^{-\left(\xi^2/\sqrt{K^2 + M \epsilon N}\right)^2} \xi \tanh \xi \sin(x \cosh \xi) d\xi \ll 1
\]

\[
x \Phi (x) = \sqrt{\pi i K^3} \int_{0}^{\infty} e^{-\left(\xi^2/\sqrt{K^2 + M \epsilon N}\right)^2} \left( 1 - \xi \tanh \xi - 2\xi^2 K^2 \right) \cos(x \cosh \xi) \frac{d\xi}{\cosh \xi} \ll x K^{-2}.
\]

The first representation will be used for \( c > K^{-2} M \) and the second for \( c \leq K^{-2} M \). Let

\[
\phi(c, K, M, N) = \sum_{m=M+1}^{M+N} \sum_{n=M+1}^{M+N} b_m b_n \frac{4\pi \sqrt{mn}}{c} S(m, n; c) \Phi \left( \frac{4\pi \sqrt{mn}}{c} \right).
\]
For $c > M^2$, the Weil bound (Proposition 2.2) gives

$$\Phi(c, K, M, N) \ll c^{-\frac{1}{2} + \epsilon} M^2 \sum_{n=M+1}^{M+N} |b_n|^2.$$  

For $M^{1-\epsilon} < c \leq M^2$, Proposition 2.5 gives

$$\Phi(c, K, M, N) \ll M'N \sum_{n=M+1}^{M+N} |b_n|^2.$$  

For $K^{-2}M < c \leq M^{1-\epsilon}$, the first integral representation above, combined with Propositions 2.5 (for $\xi \leq 1$) and 2.6 (for $\xi \geq 1$), gives

$$\Phi(c, K, M, N) \ll M'(Ke^{-K^2}c^{-1}MN + K^2c^{\frac{1}{2}}M^{\frac{2}{3}}N^{\frac{1}{3}}) \sum_{n=M+1}^{M+N} |b_n|^2.$$  

Finally, for $c \ll K^{-2}M$, the second integral above gives

$$\Phi(c, K, M, N) \ll M'(Ke^{-K^2}c^{-1}MN + c^{\frac{3}{2}}M^{\frac{1}{2}}N^{\frac{1}{2}}) \sum_{n=M+1}^{M+N} |b_n|^2.$$  

Summing all the above estimates gives

$$\sum_{c>0} \frac{1}{c} |\phi(c, K, M, N)| \ll KN(M' + MKe^{-K^2}) \sum_{n=M+1}^{M+N} |b_n|^2,$$

and thus, substituting into the trace formula expression above and discarding the Eisenstein series
term gives

\[ \sum_{t_j \leq K} \frac{t_j + 1}{\cosh \pi t_j} \left| \sum_{n=M+1}^{M+N} b_n a_j(n) \right|^2 \ll K \left( K^2 + M^\epsilon N + M N K e^{-K^2} \right) \sum_{n=M+1}^{M+N} |b_n|^2. \]

Since the left-hand side is nondecreasing as a function of $K$, we may replace $K$ by $K + M^\epsilon$ on the right hand side, reducing the expression to

\[ \sum_{t_j \leq K} \frac{t_j + 1}{\cosh \pi t_j} \left| \sum_{n=M+1}^{M+N} b_n a_j(n) \right|^2 \ll K \left( K^2 + M^\epsilon N \right) \sum_{n=M+1}^{M+N} |b_n|^2, \]

and then by partial summation

\[ \sum_{t_j \leq K} \frac{1}{\cosh \pi t_j} \left| \sum_{n=M+1}^{M+N} b_n a_j(n) \right|^2 \ll (K^2 + M^\epsilon N) \sum_{n=M+1}^{M+N} |b_n|^2. \]

\[ \square \]
Chapter 3

Zero Density Estimate for $L$-functions

The aim of this chapter is to apply the large sieve inequality in Theorem 2.3 to derive a zero density estimate for the corresponding $L$-functions. The technique is derived from Gallagher’s proof in [10] of the analogous estimate given in Theorem 0.4 for Dirichlet $L$-functions. The proof combines a bound derived from the large sieve inequality with the zero-counting method developed by Turán [33, 35, 34].

3.1 Turán’s Method

The following theorem will be used to link the zeros of $L$-functions to sums over Fourier coefficients. It is essentially Theorem 5 in Gallagher [10] adapted from the case of Dirichlet $L$-functions to the case of $L$-functions for $GL(2)$ automorphic forms. A few facts will be needed which carry over directly from the theory of Dirichlet $L$-functions, beginning with the following formula for the logarithmic derivative (as in Prachar [29], Satz 4.1, p. 225), which follows from the functional equation (Proposition 1.3):
Proposition 3.1. Let $L(s, f)$ be the $L$-function as in Definition 1.13 of a newform for $\Gamma_0(q)$, and let $w = 1 + iv$. Then

$$\frac{L'(s, f)}{L(s, f)} = \sum_\rho \frac{1}{s - \rho} + O(\log q(|v| + 2)),$$

where the sum is over zeros $\rho$ of $L(s, f)$ in the disk $|\rho - w| \leq 1$.

This leads to the following bound on the density of zeros in small disks near the critical line, which is due to Linnik in the case of Dirichlet $L$-functions (Prachar [29], p. 331, Lemma 2.1, see also Kowalski and Michel [20], Lemma 21):

Proposition 3.2. Let $L(s, f)$ be as above, and let $Q(f, v, r)$ be the number of zeros of $L(s, f)$ in the disc $|s - (1 + iv)| \leq r$. Then

$$Q(f, v, r) \ll r \log q(|v| + 2) + 1.$$

The following theorem on power sums is due to Turán [35]:

Proposition 3.3. Let $m \geq 0$ be an integer, $\{b_j\}$ be an arbitrary sequence of complex numbers, and

$$|z_1| \geq |z_2| \geq \cdots \geq |z_n|.$$

Then there exists an integer $\nu$ with $m + 1 \leq \nu \leq m + n$ and

$$|b_1 z_1^\nu + b_2 z_2^\nu + \cdots + b_n z_n^\nu| \geq |z_1|^{\nu} \left( \frac{n}{24e^2(m + 2n)} \right) \min_{1 \leq j \leq n} |b_1 + \cdots + b_j|.$$

The main theorem of this section is as follows:
**Theorem 3.1.** Let $L(s, f)$ be as above, and let $w = 1 + iv$ with $|v| \leq T$. Then there exist positive constants $r_0, A, B, C$ such that if $L(s, f)$ has a zero in the disc $|s - w| \leq r$, where $\frac{1}{\log T} \leq r \leq r_0$, then for all $x \geq T^A$,

$$
\int_x^{x^B} \left| \sum_{x < p \leq y} \frac{a_f(p) \log p}{a_f(1)p^w} \right| \frac{dy}{y} \gg x^{-Cr} \log^2 x,
$$

where the sum is taken over primes $p$.

**Proof.** This proof follows Gallagher [10], with slight modifications for the present setting. By Proposition 3.1, for $|s - w| \leq 1/2$,

$$
\frac{L'}{L}(s, f) = \sum_{|\rho - w| \leq 1} \frac{1}{s - \rho} + O(\log T).
$$

Then, by Cauchy’s inequality for Taylor series coefficients, for $|s - w| \leq 1/4$

$$
\frac{1}{k!} \frac{d^k}{ds^k} \frac{L'}{L}(s, f) = (-1)^k \sum_{|\rho - w| \leq 1} \frac{1}{(s - \rho)^{k+1}} + O(4^k \log T).
$$

Take $s = w + r$ in this formula (as we may assume $r < 1/4$). Let $\lambda$ be such that $r \leq \lambda \leq 1/4$. Then by Proposition 3.2, the number of $\rho$ in the above sum with $2^j \lambda < |\rho - w| \leq 2^{j+1} \lambda$ is $\ll 2^j \lambda \log T$, and for such $\rho$, $\frac{1}{(s - \rho)^{k+1}} \ll (2^j \lambda)^{-(k+1)}$. Thus the contribution of all terms with $|\rho - w| > \lambda$ is

$$
\ll \sum_{j=0}^{\infty} (2^j \lambda)^{-k} \log T \ll \lambda^{-k} \log T.
$$

Thus, since $\lambda \leq 1/4$, we may eliminate the terms with $|\rho - w| > \lambda$ from the sum and write

$$
\frac{1}{k!} \frac{d^k}{ds^k} \frac{L'}{L}(s, f) = (-1)^k \sum_{|\rho - w| \leq \lambda} \frac{1}{(s - \rho)^{k+1}} + O(\lambda^{-k} \log T).
$$
Again using Proposition 3.2, the number of remaining zeros is \( \leq A_1 \lambda \log T \) for some constant \( A_1 \).

If there is a zero in the disc \( |s - w| \leq r \), then \( \min_\rho |s - \rho| \leq 2r \). Now apply proposition 3.3, with the \( z_j \) equal to \( \frac{1}{s_\rho} \), the \( b_j = 1 \), and \( |z_1| \geq \frac{1}{2r} \). Taking \( m \geq A_1 \lambda \log T \), this implies that there exists \( k \) with \( m \leq k \leq 2m \) and

\[
\left| \sum_{\rho \in \mathcal{R}, |\rho - w| \leq \Lambda} \frac{1}{(s - \rho)^{k+1}} \right| \geq (Dr)^{-k+1}
\]

for positive constant \( D \). Choosing \( \lambda = A_2 Dr \) (setting \( r_0 \) as necessary to assure that \( \lambda \leq 1/4 \)) for sufficiently large \( A_2 \), the above sum dominates the remainder term \( O(\lambda^{-k} \log T) \) (using the fact that \( r \geq \frac{1}{\log T} \)), and thus

\[
\frac{1}{k!} \frac{d^k}{ds^k} \frac{L'}{L}(s, f) \gg (Dr)^{-k+1},
\]

where still \( m \leq k \leq 2m \) and \( m \geq A_1 \lambda \log T = Er \log T \) where \( E = A_1 A_2 D \).

Using the Euler product in Proposition 1.1, we have

\[
\frac{L'}{L}(s, f) = \sum_{n=1}^{\infty} \Lambda_f(n)n^{-s},
\]

where

\[
\Lambda_f(p^j) = \begin{cases} 
\lambda_f(p^j) \log p & \text{if } p|q, \\
(\alpha_{f,1}(p^j) + \alpha_{f,2}(p^j)) \log p & \text{if } p \nmid q
\end{cases}
\]

and \( \Lambda_f(n) = 0 \) for \( n \) not a prime power. In particular, \( \Lambda_f(p) = \lambda_f(p) \log p \) and \( \Lambda(f)(p^j) \ll p^{\frac{1}{2}j+\epsilon} \) by Proposition 1.2. Thus the above bound may be written as

\[
\sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^w} p_k(r \log n) \gg \frac{D^{-k}}{r},
\]

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where

\[ p_k(u) = \frac{e^{-u} u^k}{k!}. \]

Using Stirling’s formula, it can be shown that there exist constants \( B_1, B_2 \), such that the function \( p_k \) satisfies \( p_k(u) \leq (2D)^{-k} \) for \( u \leq B_1 k \) and \( p_k(u) \leq (2D)^{-k} e^{-u/2} \) for \( u \geq B_1 k \). Set \( A = B_1 E \). For \( x \geq T^A \), choose \( m = B_1^{-1} r \log x \), so that \( m \geq Er \log T \) as required and there exists an appropriate \( k \) with \( m \leq k \leq 2m \). Let \( B = 2B_2 / B_1 \). Then it follows that:

\[
\sum_{n \leq x} \frac{\Lambda_f(n)}{n^{1+\epsilon}} p_k(r \log n) \ll (2D)^{-k} \sum_{n \leq x} \frac{\Lambda_f(n)}{n} \ll (2D)^{-k} \frac{k}{r},
\]

\[
\sum_{n > x^B} \frac{\Lambda_f(n)}{n^{1+\epsilon}} p_k(r \log n) \ll (2D)^{-k} \sum_{n > x^B} \frac{\Lambda_f(n)}{n^{1+\frac{1}{2}}} \ll (2D)^{-k} \frac{k}{r}.
\]

The second step in each line above requires the prime number theorem for the function \( L(s, f) \) in the form \( \sum_{n \leq x} \Lambda_f(n) \ll x \), which follows in particular from the zero-free region established by Hoffstein and Ramakrishnan [15]. Thus the terms for \( n \leq x \) and \( n > x^B \) may be discarded, leaving

\[
\sum_{x < n \leq x^B} \frac{\Lambda_f(n)}{n^{1+\epsilon}} p_k(r \log n) \gg \frac{D^{-k}}{r}.
\]

Finally, the bound \( \Lambda(f)(p^j) \ll p^{\frac{7}{64} j + \epsilon} \) means the terms for \( p^j \) with \( j \geq 2 \) may be safely discarded as well. Set

\[
S(y) = \sum_{x < p \leq y} \frac{\Lambda_f(p)}{p^{1+\epsilon}} = \sum_{x < p \leq y} \frac{\Lambda_f(p) \log p}{p^{1+\epsilon}} = \sum_{x < p \leq y} \frac{a_f(p) \log p}{a_f(1) p^{1+\epsilon}}.
\]

Then the left-hand side of the above inequality is

\[
\int_x^{x^B} p_k(r \log y) dS(y) = p_k(r \log x^B) S(x^B) - \int_x^{x^B} S(y) p'_k(r \log y) r \frac{dy}{y}.
\]
As before, the first term on the right is \( \ll \frac{(2D)^{-k}}{r} \), and then the bound \( p'_k \ll 1 \) gives

\[
\int_x^{x^B} |S(y)| \frac{dy}{y} \gg D^{-k} \gg x^{-C r \log^2 x}.
\]

\[\square\]

### 3.2 Exponential Sums with Fourier Coefficients

The next step in the proof of the main theorem is a bound on certain exponential sums involving the Fourier coefficients of cusp forms (this is an analogue of Gallagher’s Theorem 2 in [10]).

The bound will use the following theorem from Gallagher [10] on exponential sums:

**Theorem 3.2** (Gallagher [10], Theorem 1). Let \( S(t) = \sum_{n=1}^{\infty} a_n n^it \) be an absolutely convergent Dirichlet series. Then

\[
\int_{-T}^{T} |S(t)|^2 dt \ll T^2 \int_{0}^{\infty} \left| \sum_{y < n \leq y e^{1/T}} a_n \right|^2 \frac{dy}{y}.
\]

Applying this to sums over Fourier coefficients gives the following:

**Theorem 3.3.** Let \( \{u_j\} \) be a basis as in Definition 1.14 for the space of Maass cusp forms of weight zero for \( \Gamma_0(q) \), such that \( u_j \) has Laplace eigenvalue \( \frac{1}{4} + t_j^2 \) and Fourier expansion \( u_j(z) = \sum_{n \neq 0} a_j(n) \sqrt{j} K_{it} (2\pi |n| y) e(nx) \). Let \( \{b_n\} \) be any sequence such that the series \( \sum_{n=1}^{\infty} b_n a_j(n) \) converges absolutely. Then

\[
\sum_{|t_j| \leq K} \frac{1}{\cosh \pi t_j} \int_{-T}^{T} \left| \sum_{n=1}^{\infty} b_n a_j(n) n^it \right|^2 dt \ll \sum_{n=1}^{\infty} (K^2 T + n^{1+\epsilon}) |b_n|^2.
\]
Proof. By Theorem 3.2, the left-hand side is

\[ \ll T^2 \int_0^\infty \sum_{|t_j| \leq K} \frac{1}{\cosh \pi t_j} \left| \sum_y y e^{1/T} b_n a_j(n) \right|^2 \frac{dy}{y}. \]  

(3.1)

Applying Theorem 2.3 then gives

\[ \sum_{|t_j| \leq K} \frac{1}{\cosh \pi t_j} \left| \sum_y y e^{1/T} b_n a_j(n) \right|^2 \ll \left( K^2 + y^r (e^{1/T} - 1) \right) \sum_y |b_n|^2. \]

Substituting this in (3.1), we see that the coefficient of $|b_n|^2$ is

\[ \ll T^2 K^2 \int_{ne^{-1/T}}^n \frac{dy}{y} + T^2 \left( e^{1/T} - 1 \right) \int_{ne^{-1/T}}^n y^r dy \]

\[ \ll K^2 T + T^2 \left( e^{1/T} - 1 \right) \left( 1 - e^{-1/T} \right)^{1+\epsilon} n^{1+\epsilon} \]

\[ \ll K^2 T + n^{1+\epsilon} \]

for $T$ sufficiently large.

\[ \square \]

3.3 Proof of the Main Theorem

The results of the two previous sections may now be combined just as in Gallagher [10] to prove Theorem 0.6.

Proof. It suffices to consider the case where $\frac{1}{\log T} \leq 2(1 - \alpha) \leq r_0$, since for $1 - \alpha \ll \frac{1}{\log T}$, the left-hand side is a decreasing function of $\alpha$ while the right-hand side is essentially constant. Thus we may take $r = 2(1 - \alpha)$ in Theorem 3.1. Choose $x = T^{\max(A,3)}$. Then if $L(s, f_j)$ has a zero in
the disc $|s - w| \leq r$, we have

$$x^{Cr} \log^{-2} x \int_{x}^{x^B} \left| \sum_{x \leq p \leq y} \frac{a_j(p) \log p}{a_j(1)p^w} \right| \frac{dy}{y} \gg 1,$$

where the sum is over primes $p$. Applying Cauchy-Schwarz, and letting $c = 4C \max(A, 3)$, we obtain

$$T^{c(1-\alpha)} \log^{-3} T \int_{x}^{x^B} \left| \sum_{x \leq p \leq y} \frac{a_j(p) \log p}{a_j(1)p^w} \right|^2 \frac{dy}{y} \gg 1.$$

Each disc $|s - w| \leq 2(1 - \alpha)$ contains $\ll (1 - \alpha) \log T$ zeros, and each zero with real part at least $1 - \alpha$ will be contained in the disc as $w$ ranges over an interval of length $\gg 1 - \alpha$ along the line $w = 1 + iv$. Thus we obtain

$$N_j(\alpha, T) \ll T^{c(1-\alpha)} \log^{-2} T \int_{x}^{x^B} \int_{-T}^{T} \left| \sum_{x \leq p \leq y} \frac{a_j(p) \log p}{a_j(1)p^{1+iv}} \right|^2 dv \frac{dy}{y}.$$

The basis $\{f_j\}$ may be assumed to be a subset of a basis $\{u_j\}$ as in Definition 1.14. Thus,

$$\sum_{|t_j| \leq T} \frac{|a_j(1)|^2}{\cosh \pi t_j} N_j(\alpha, T) \ll \sum_{|t_j| \leq T} \frac{1}{\cosh \pi t_j} T^{c(1-\alpha)} \log^{-2} T \int_{x}^{x^B} \int_{-T}^{T} \left| \sum_{x \leq p \leq y} \frac{a_j(p) \log p}{p^{1+iv}} \right|^2 dv \frac{dy}{y},$$

where the sum on the left hand side is over the basis $\{f_j\}$ of newforms and the sum on the right hand side is over the basis $\{u_j\}$ of all cusp forms. Now, applying Theorem 3.3 to the inner integral with $b_p = \frac{\log p}{p}$ and $b_n = 0$ otherwise, we obtain

$$\sum_{|t_j| \leq T} \frac{|a_j(1)|^2}{\cosh \pi t_j} N_j(\alpha, T) \ll T^{c(1-\alpha)} \log^{-2} T \int_{x}^{x^B} \sum_{x \leq p \leq y} (T^3 + p^{1+\epsilon}) \left( \frac{\log p}{p} \right)^2 \frac{dy}{y}.$$
Recalling that $x \geq T^a$, we see that the integral is

$$
\ll \log T \sum_{x \leq p \leq x^B} p^a \log^2 p \ll T^c,
$$

giving the theorem. □
Bibliography


