Equivariant Gromov-Witten Theory of GKM Orbifolds

Zhengyu Zong

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ABSTRACT

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In this paper, we study the all genus Gromov-Witten theory for any GKM orbifold X. We generalize the Givental formula which is studied in the smooth case in [41] [42] [43] to the orbifold case. Specifically, we recover the higher genus Gromov-Witten invariants of a GKM orbifold X by its genus zero data. When X is toric, the genus zero Gromov-Witten invariants of X can be explicitly computed by the mirror theorem studied in [22] and our main theorem gives a closed formula for the all genus Gromov-Witten invariants of X. When X is a toric Calabi-Yau 3-orbifold, our formula leads to a proof of the remodeling conjecture in [38]. The remodeling conjecture can be viewed as an all genus mirror symmetry for toric Calabi-Yau 3-orbifolds. In this case, we apply our formula to the A-model higher genus potential and prove the remodeling conjecture by matching it to the B-model higher genus potential.

Table of Contents

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Chapter 1

Introduction

1.1 background and motivation

Let X be an algebraic GKM manifold, which means that there exists an algebraic torus T acting on X such that there are only finitely many fixed points and finitely many 1-dimensional orbit. In the sequence of papers [41] [42] [43], Givental studies the all genus equivariant Gromov-Witten theory of a GKM manifold X. He obtains a formula for the full descendent potential of X and conjectures that the same formula is true for any X with semisimple Frobenius manifold. This formula is often referred to Givental formula. It expresses the higher genus Gromov-Witten invariants of X in terms of the genus 0 data. The key point is that the genus 0 data in that formula can be expressed in abstract terms of semisimple Frobenius structures and vice versa. So one can reconstruct the higher genus Gromov-Witten theory of X by its Frobenius structure.

When X is toric, one can build the mirror symmetry between X and its Landau-Ginzburg mirror. The mirror theorem [40], [57, 58, 59, 60] for smooth toric varieties gives an isomorphism between the quantum cohomology ring of X and the Jacobi ring of its Landau-Ginzburg mirror. So one can identify the Frobenius structures of these two rings. In particular, the quantum differential equations in A and B-model can be identified. Under this identification, the genus 0 data in Givental formula can be described explicitly on B-model side. For example, the data coming from the fundamental solution of the quantum differential equation can be give by oscillatory integrals over the Lefschetz thimbles on B-model side, the norms of the canonical basis is related to the Hessians at the critical points of the super potential and so on. Of course, the above procedure can also be generalized to other homogeneous spaces such as Grassmannians.

It is natural to ask whether the Givental formula can be generalized to the orbifold case. On the Gromov-Witten theory side, one wants to study the higher genus equivariant orbifold Gromov-Witten theory for GKM orbifolds such as toric orbifolds and other homogeneous orbifold spaces. So we want to obtain an orbifold Givental formula so that we can recover the higher genus data by the Frobenius structures of the target X. When X is toric, the orbifold mirror theorem is proved in [22]. Just as the smooth case, one can identify the quantum cohomology ring of X to the Jacobi ring of its mirror. So we can give nice explicit explanations of the Frobenius structure of X in terms of corresponding structures in B-model. Therefore if we can obtain an orbifold version of the Givental formula, the higher genus Gromov-Witten theory of X can be recovered effectively by concrete data. This idea also applies to many other GIT quotients where similar mirror symmetry phenomenon arises [19].

Perhaps, the most interesting case is when X is a toric Calabi-Yau 3-orbifold. In this case, the *remodeling conjecture* is established in [65] [8] and [9]. The remodeling conjecture gives us a higher genus B-model which itself arises naturally in the matrix-model theory. The remodeling conjecture claims that the B-model higher genus potential can be identified with the A-model higher genus potential, which can be viewed as a all genus mirror symmetry statement. The higher genus B-model potential is obtained by applying the Eynard-Orantin recursion to the mirror curve of X. This potential has a graph sum formula which has exactly the same form of the corresponding graph sum of Givental formula on A-model side . The identification of the higher genus potentials is then reduced to the identification of Frobenius structures which can be deduced from the genus 0 mirror theorem [22]. So the proof of the orbifold Givental formula plays a crucial role in proving the remodeling conjecture (see [38]). Besides, one needs to study the Frobenius structures that appear in the Givental formula carefully in order to match them to the corresponding structures on the B-model side.

To prove the remodeling conjecture [38] is one of the main reasons for the author to study the orbifold Givental formula.

For higher dimensional toric orbifolds, one can try to generalize the Eynard-Orantin recursion to higher dimensional varieties. This may generalize the remodeling conjecture to higher dimensional toric orbifolds. This may be a further application of the orbifold Givental formula.

1.2 Plan of the paper

In Chapter 2, we first study the geometry of an GKM orbifold X. Then we review the definition of the Chen-Ruan orbifold cohomology of smooth Deligne-Mumford stacks. After that we move on to the equivariant Chen-Ruan cohomology of an GKM orbifold X. It turns out that the equivariant Chen-Ruan cohomology ring is a semisimple Frobenius algebra.

In Chapter 3, we first review several definitions about the orbifold Gromov-Witten theory of smooth Deligne-Mumford stacks. Then we apply these definitions to the equivariant Gromov-Witten theory of an GKM orbifold X. Then we will deduce the semisimplicity of the Frobenius manifold of X from the semisimplicity of its classical equivariant Chen-Ruan cohomology. After that we will consider the quantum differential equation and its fundamental solutions. One of them is given explicitly by the 1-primary 1-descendent genus 0 potential. This solution will play a special role in the later.

In Chapter 4, we move on to the higher genus structures. We will first review the quantization procedure which will be used in the later sections. The we obtain the orbifold Givental formula by applying Teleman's result [66] of classification of 2D semisimple field theories to our case. Since the Frobenius manifold of X is semisimple, the cohomological field theory of X coming from the Gromov-Witten theory lies in the same orbit of the trivial field theory of the same Frobenius manifold under certain group actions. The Givental formula can be derived using this group action point of view. Then we give a more explicit graph sum formula for the higher genus potential of X. This graph sum formula is crucial in the proof of the remodeling conjecture [38]. We will discuss the remodeling conjecture in Chapter 5. We also need to fix the ambiguity of the $R-$operator in Givental formula. Since we are working with equivariant Gromov-Witten theory, our Frobenius structure is not conformal. Therefore the ambiguity with the $R-$operator in Givental formula cannot be fixed by the usual method using the Euler vector field. Instead, we use the structure of solution space to the quantum differential equation (Theorem 5.1). Then we know that the ambiguity is a constant matrix. So we study the case when the degree is 0 and when there is no primary insertion and compare the Givental formula in this case with the orbifold quantum Riemann-Roch theorem in [67]. In the end, we obtain the reconstruction theorem by expressing the $R-$operator in terms of the quantum multiplication law and the explicit constant matrix given by orbifold quantum Riemann-Roch theorem.

In Chapter 5, we apply the Givental quantization formula to the case when X is a toric Calabi-Yau 3-orbifold. In this case, one can consider the remodeling conjecture which is an all genus mirror symmetry for toric Calabi-Yau 3-orbifolds. In this conjecture, the A-model higher genus potential is the open Gromov-Witten potential of X with respect to one or several Aganagic-Vafa branes on X. The higher genus B-model $\omega_{g,n}$ is obtained by applying the Eynard-Orantin [32] topological recursion to the mirror curve C of X. It is a symmetric n-form on C. When $\omega_{g,n}$ is expanded around certain points on C, the coefficients will give us the open Gromov-Witten invariants of X. On the other hand, we also have the Landau-Ginzburg mirror of X, which contains a super potential $W^T : (\mathbb{C}^*)^3 \to \mathbb{C}$. The mirror curve is related to the Landau-Ginzburg mirror by the dimensional reduction. By the genus 0 mirror theorem [22], the quantum cohomology ring of X is isomorphic to the Jacobi ring of W^T. Therefore we can identify the Frobenius structure of X with the genus 0 data coming from the Eynard-Orantin topological recursion on the mirror curve. The bridge relating the A-model and B-model higher genus potentials is the graph sum formula, which is equivalent to both the quantization formula and the recursive formula. From this point of view, the remodeling conjecture is proved by realizing both A-model and B-model higher genus potentials as quantizations on two isomorphic semi-simple Frobenius manifolds.

Chapter 2

Equivariant Chen-Ruan cohomology of GKM orbifolds

In this chapter, we discuss the geometry and basic properties of any GKM orbifold X. We will study the classical equivariant Chen-Ruan cohomology ring of X and construct its canonical basis.

The concept of a GKM manifold is first established in [44] by Goresky-Kottwitz-MacPherson. An algebraic GKM manifold is a smooth algebraic variety with an algebraic action of a torus $T = (\mathbb{C}^*)^m$, such that there are finitely many torus fixed points and finitely many one-dimensional orbits. Examples of algebraic GKM manifolds include toric manifolds, Grassmanians, flag manifolds and so on. The advantage of a GKM manifold X is that one can study the classical equivariant cohomology of X and the Atiyah-Bott localization via the combinatorics tool called GKM graphs. This might be the original motivation for people to study GKM manifolds. By generalizing the classical Atiyah-Bott localization to the virtual localization, one can compute the equivariant Gromov-Witten invariants of an algebraic GKM manifold in terms of summing over GKM graphs (see [63] for more details). The localization procedure in this case is completely similar to that in the toric case.

In this chapter, we study the classical equivariant Chen-Ruan cohomology of any GKM orbifold X, which is a generalization of the GKM manifold. The discussion of equivariant Gromov-Witten theory of X is in the next chapter.

2.1 GKM orbifolds

Let X be an $r-$dimensional smooth proper Deligne-Mumford stack with a quasi-projective coarse moduli space. Let $T = (\mathbb{C}^*)^m$ be an algebraic torus acting on X.

Definition 2.1.1. *We say that X is a* GKM orbifold *if*

1. *There are finitely many $T-$fixed points.*

2. *There are finitely many one-dimensional orbits.*

In our definition, X can be noncompact and can have nontrivial generic stabilizers.

Let $p_1, \cdots, p_n$ be the T−fixed points of X. These points may be stacky and locally around each p_σ, the tangent space $T_{p_\sigma}X$ is isomorphic to $[\mathbb{C}^r/G_\sigma]$ with G_σ a finite group and with the r axes the corresponding r one dimensional orbits containing $p_\sigma, \sigma = 1, \cdots, n$. The action of G_σ on the tangent space $T_{p_\sigma}X$ is a representation $\rho_\sigma : G_\sigma \to GL(r, \mathbb{C})$. Since T acts on X, we know that the action of T commutes with the action of G_σ. So the image of the representation $\rho_\sigma : G_\sigma \to GL(r, \mathbb{C})$ must be contained in the maximal torus (note that there are only finitely many one-dimensional orbits and hence the T−characters along any two axes of $[\mathbb{C}^r/G_\sigma]$ are linearly independent). Therefore, ρ_σ splits into r one-dimensional representations $\chi_{\sigma j} : G_\sigma \to \mathbb{C}^*,\ j = 1, \cdots, r$. Let $\boldsymbol{\mu}_{l_{\sigma j}} = \chi_{\sigma j}(G_\sigma) = \{z \in \mathbb{C}^* | z^{l_{\sigma j}} = 1\} \subseteq \mathbb{C}^*$, $\sigma = 1, \cdots, n, j = 1, \cdots, r$.

Let $N = \mathrm{Hom}(\mathbb{C}^*, T)$ be the lattice of 1-parameter subgroups of T and $M = \mathrm{Hom}(T, \mathbb{C}^*)$ the lattice of irreducible characters of T. Then M is the dual lattice of N and we have a canonical identification $M \cong H_T^2(\mathrm{pt}, \mathbb{Z})$. Here $H_T^*(\mathrm{pt}, \mathbb{Z})$ denotes the T-equivariant cohomology of a point. Let $\mathbf{w}_{\sigma j}$ be the character of the T−action along the j−th tangent direction at p_i. Then $\mathbf{w}_{\sigma j}$ lies in $H_T^2(\mathrm{pt}, \mathbb{Q}) \cong M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$. The rational coefficient here is used to adapt the orbifold structure at p_σ.

2.2 Chen-Ruan orbifold cohomology

In this section, we review the definition and basic properties of the Chen-Ruan orbifold cohomology [18] of a smooth Deligne-Mumford stack X.

2.2.1 The inertia stack

Definition 2.2.1. *Let X be a Deligne-Mumford stack. The inertia stack $\mathcal{I}X$ associated to X is defined to be the fiber product*

$$\mathcal{I}X = X \times_{\Delta, X\times X, \Delta} X,$$

where $\Delta : X \to X \times X$ is the diagonal morphism.

The objects in the category $\mathcal{I}X$ can be described as

$$\mathrm{Ob}(\mathcal{I}X) = \{(x,g) | x \in \mathrm{Ob}(X), g \in \mathrm{Aut}_X(x)\}.$$

The morphisms between two objects in the category $\mathcal{I}X$ are

$$\mathrm{Hom}_{\mathcal{I}X}((x_1,g_1),(x_2,g_2)) = \{h \in \mathrm{Hom}_X(x_1,x_2) | hg_1 = g_2 h\}.$$

In particular,

$$\mathrm{Aut}_{\mathcal{I}X}(x,g) = \{h \in \mathrm{Aut}_X(x) | hg = gh\}.$$

There is a natural projection $q : \mathcal{I}X \to X$ which, on objects level, sends (x,g) to x. There is also an involution map $\iota : \mathcal{I}X \to \mathcal{I}X$ which sends (x,g)

to (x, g^{-1}). The inertial stack $\mathcal{I}X$ is in general not connected even if X is connected. Suppose X is connected and let

$$\mathcal{I}X = \bigsqcup_{i \in I} X_i$$

be the disjoint union of connected components. There is a distinguished component

$$X_0 = \{(x, \mathrm{id}_x) | x \in \mathrm{Ob}(X)\}$$

which is isomorphic to X. The restriction of the involution map ι to X_i gives an isomorphism between X_i and another connected component of $\mathcal{I}X$. In particular, the restriction of ι to X_0 gives an identity map.

Example 2.2.2. *Let G be a finite group. Consider the quotient stack $\mathcal{B}G :=$ $[\mathrm{pt}/G]$ which is called the classifying space of G. There is only one object x in $\mathcal{B}G$ and $\mathrm{Hom}(x, x) = G$. By definition, the objects of $\mathcal{I}\mathcal{B}G$ are*

$$\mathrm{Ob}(\mathcal{I}\mathcal{B}G) = \{(x, g) | g \in G\}.$$

The morphisms between two objects $(x_1, g_1), (x_2, g_2)$ are

$$\mathrm{Hom}((x_1, g_1), (x_2, g_2)) = \{g \in G | gg_1 = g_2 g\} = \{g \in G | g_1 = g^{-1} g_2 g\}.$$

So we have

$$\mathcal{I}\mathcal{B}G \cong [G/G]$$

where G acts on G by conjugation. Therefore

$$\mathcal{I}\mathcal{B}G = \bigsqcup_{(h)\in \mathrm{Conj}(G)} (\mathcal{B}G)_{(h)} = \bigsqcup_{(h)\in \mathrm{Conj}(G)} [\mathrm{pt}/C(h)]$$

where (h) is the conjugacy class of $h \in G$ and $C(h)$ is the centralizer of h in G.

2.2.2 Chen-Ruan orbifold cohomology

As vector spaces, the Chen-Ruan orbifold cohomology [18] of a smooth Deligne-Mumford stack X is the same as the usual cohomology of the inertia stack $\mathcal{I}X$. The difference is the definition of the degree. Let us first discuss the definition of *age* which determines the degree of the Chen-Ruan orbifold cohomology.

Given an object $(x,g) \in \mathrm{Ob}(\mathcal{I}X)$, we have a linear map $g : T_xX \to T_xX$ such that $g^l = \mathrm{id}$, where l is the order of g. Let $\zeta = e^{2\pi i/l}$ and then the eigenvalues of g are given by $\zeta^{c_1}, \cdots, \zeta^{c_r}$, where $c_i \in \{0, \cdots, l-1\}, r = \dim X$. Define

$$\mathrm{age}(x,g) = \frac{c_1 + \cdots + c_r}{l}.$$

The function $\mathrm{age} : \mathcal{I}X \to \mathbb{Q}$ is constant on each connected component X_i of $\mathcal{I}X$ and we define $\mathrm{age}(X_i)$ to be $\mathrm{age}(x,g)$ for any object (x,g) in X_i.

Definition 2.2.3. *Let X be a smooth Deligne-Mumford stack. The Chen-*

Ruan orbifold cohomology group of X is defined to be

$$H^*_{\mathrm{CR}}(X) := \bigoplus_{a\in\mathbb{Q}} H^a_{\mathrm{CR}}(X)$$

where

$$H^a_{\mathrm{CR}}(X) = \bigoplus_{i\in I} H^{a-2\mathrm{age}(X_i)}(X_i).$$

If X is proper, the orbifold Poincare pairing is defined to be

$$\langle \alpha, \beta \rangle_X = \begin{cases} \int_{X_i} \alpha \cup \iota_i^*(\beta), & X_j = \iota(X_i), \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha \in H^*(X_i)$ and $\beta \in H^*(X_j)$.

The product structure for the Chen-Ruan orbifold cohomology is more subtle.

Definition 2.2.4. *For any $\alpha, \beta \in H^*_{\mathrm{CR}}(X)$, their orbifold product $\alpha \star_X \beta$ is defined as follows: For any $\gamma \in H^*_{\mathrm{CR}}(X)$,*

$$\langle \alpha \star_X \beta, \gamma \rangle_X := \langle \alpha, \beta, \gamma \rangle^X_{0,3,0},$$

where the right hand side will be defined in Section 3.1.

2.3 Equivariant Chen-Ruan cohomology of GKM orbifolds

In this section, we focus ourselves to the case when X is a GKM orbifold. We will show that the equivariant Chen-Ruan cohomology of X is semisimple by constructing its canonical basis explicitly.

2.3.1 The simplest case: $X = [\mathbb{C}^r/G]$

Let $\mathrm{Conj}(G)$ denote the set of conjugacy classes in G. Then the inertia stack $\mathcal{I}X$ can be described as

$$\mathcal{I}X = \bigsqcup_{(h)\in \mathrm{Conj}(G)} [(\mathbb{C}^r)^h/C(h)] = \bigsqcup_{(h)\in \mathrm{Conj}(G)} X_{(h)}$$

where (h) is the conjugacy class of $h \in G$ and $C(h)$ is the centralizer of h in G.

For any $i \in \{1, \cdots, r\}$ and any $h \in G$, define $c_i(h) \in [0,1)$ and $\mathrm{age}(h)$ by

$$e^{2\pi\sqrt{-1}c_i(h)} = \chi_i(h), \tag{2.1}$$

$$\mathrm{age}(h) = \sum_{j=1}^{r} c_j(h). \tag{2.2}$$

As a graded vector space over $\mathbb{C}$, the Chen-Ruan cohomology $H^*_{\mathrm{CR}}(X;\mathbb{C})$

of X can be decomposed as

$$H^*_{\mathrm{CR}}(X;\mathbb{C}) = \bigoplus_{(h)\in\mathrm{Conj}(G)} H^*(X_{(h)};\mathbb{C})[2\mathrm{age}(h)] = \bigoplus_{(h)\in\mathrm{Conj}(G)} \mathbb{C}\mathbf{1}_{(h)},$$

where $\deg(\mathbf{1}_{(h)}) = 2\mathrm{age}(h)$ which is independent of the choice of h in its conjugacy class.

Let $\mathcal{R} = H^*(\mathcal{B}T;\mathbb{C}) = \mathbb{C}[\mathsf{u}_1,\cdots,\mathsf{u}_m]$, where $\mathsf{u}_1,\cdots,\mathsf{u}_m$ are the first Chern classes of the universal line bundles over $\mathcal{B}T$. The T-equivariant Chen-Ruan orbifold cohomology $H^*_{\mathrm{CR},T}(X;\mathbb{C}) = \oplus_{(h)\in\mathrm{Conj}(G)} H^*_T(X_{(h)};\mathbb{C})[2\mathrm{age}(h)]$ is an $\mathcal{R}$-module. Given $h\in G$, define

$$\mathbf{e}_h := \prod_{i=1}^r \mathsf{w}_i^{\delta_{c_i(h),0}} \in \mathcal{R}.$$

Here, $\mathbf{w}_i$ is the character of the $T-$action along the $i-$th direction. In particular,

$$\mathbf{e}_1 = \prod_{i=1}^r \mathsf{w}_i.$$

Then the T-equivariant Euler class of $\mathbf{0}_{(h)} := [0/C(h)]$ in $X_{(h)} = [(\mathbb{C}^r)^h/C(h)]$ is

$$e_T(T_{\mathbf{0}_{(h)}}X_{(h)}) = \mathbf{e}_h\mathbf{1}_{(h)} \in H^*_{\mathrm{CR},T}(X_{(h)};\mathbb{C}) = \mathcal{R}\mathbf{1}_{(h)}.$$

Let $\chi_1,\cdots,\chi_r : G\to\mathbb{C}^*$ and $l_1,\cdots,l_r\in\mathbb{Z}$ be defined as in the previous section. Define

$$\mathcal{R}' = \mathbb{C}[\mathsf{u}_1,\cdots,\mathsf{u}_m,\mathsf{w}_1^{\frac{1}{2l_1}},\cdots,\mathsf{w}_r^{\frac{1}{2l_r}}]$$

which is a finite extension of $\mathcal{R}$. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be the fractional fields of $\mathcal{R}$, $\mathcal{R}'$, respectively. We will consider the $T-$equivariant Chen-Ruan cohomology ring $H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$.

The $T-$equivariant Poincaré pairing of $H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$ is given by

$$\langle \mathbf{1}_{(h)}, \mathbf{1}_{(h')} \rangle_X = \frac{1}{|C(h)|} \cdot \frac{\delta_{(h^{-1}),(h')}}{\mathbf{e}_h} \in \mathcal{Q}'.$$

The T-equivariant orbifold cup product of $H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$ is given by

$$\mathbf{1}_{(h)} \star_X \mathbf{1}_{(h')} = \sum_{f_1 \in (h_1), f_2 \in (h_2)} \frac{|C(f_1 f_2)|}{|G|} \Big(\prod_{i=1}^{r} \mathsf{w}_i^{c_i(h)+c_i(h')-c_i(ff')} \Big) \mathbf{1}_{(ff')}.$$

Define

$$\bar{\mathbf{1}}_{(h)} := \frac{\mathbf{1}_{(h)}}{\prod_{i=1}^r \mathsf{w}_i^{c_i(h)}} \in H^*_{\mathrm{CR},T}(X;\mathcal{Q}'). \tag{2.3}$$

Then

$$\langle \bar{\mathbf{1}}_{(h)}, \bar{\mathbf{1}}_{(h')} \rangle_X = \frac{1}{|C(h)|} \cdot \frac{\delta_{(h^{-1}),(h')}}{\prod_{i=1}^r \mathsf{w}_i} \in \mathcal{Q}'$$

and

$$\bar{\mathbf{1}}_{(h)} \star_X \bar{\mathbf{1}}_{(h')} = \sum_{f_1 \in (h_1), f_2 \in (h_2)} \frac{|C(f_1 f_2)|}{|G|} \bar{\mathbf{1}}_{(ff')}.$$

We now define a canonical basis for the semisimple algebra $H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$. Let $\{V_\alpha\}_{\alpha=1}^{|\mathrm{Conj}(G)|}$ be the set of irreducible representations of G and let χ_α be the character of V_α. For any α, define

$$\bar{\phi}_\alpha = \frac{\dim V_\alpha}{|G|} \sum_{(h) \in \mathrm{Conj}(G)} \chi_\alpha(h^{-1}) \bar{\mathbf{1}}_{(h)}.$$

Let $\nu_\alpha = \left(\frac{\dim V_\alpha}{|G|}\right)^2, \alpha = 1, \cdots, |\mathrm{Conj}(G)|$. Then we have

$$\langle \bar{\phi}_\alpha, \bar{\phi}_{\alpha'} \rangle_X = \delta_{\alpha,\alpha'} \frac{\nu_\alpha}{\prod_{i=1}^r \mathsf{w}_i}$$

and

$$\bar{\phi}_\alpha \star_X \bar{\phi}_{\alpha'} = \delta_{\alpha,\alpha'} \bar{\phi}_\alpha.$$

Therefore, $\{\bar{\phi}_\alpha\}_{\alpha=1}^{|\mathrm{Conj}(G)|}$ is a canonical basis of the semisimple algebra $H^*_{\mathrm{CR},T}(X; \mathcal{Q}')$.

2.3.2 The general case

In general, we apply the above construction to the local geometry around each fixed point p_σ to get a set of cohomology classes $\{\bar{\phi}_{\sigma\alpha}\}_{\alpha=1}^{|\mathrm{Conj}(G_\sigma)|}, \sigma = 1, \cdots, n$. It is easy to show that

$$\langle \bar{\phi}_{\sigma\alpha}, \bar{\phi}_{\sigma'\alpha'} \rangle_X = \delta_{\sigma,\sigma'} \delta_{\alpha,\alpha'} \frac{\nu_{\sigma\alpha}}{\prod_{j=1}^r \mathsf{w}_{\sigma j}}$$

and

$$\bar{\phi}_{\sigma\alpha} \star_X \bar{\phi}_{\sigma'\alpha'} = \delta_{\sigma,\sigma'} \delta_{\alpha,\alpha'} \bar{\phi}_{\sigma\alpha}.$$

Therefore, the algebra $H^*_{\mathrm{CR},T}(X; \mathcal{Q}')$ is still semisimple and $\{\bar{\phi}_{\sigma\alpha} | \sigma = 1, \cdots, n, \alpha = 1, \cdots, |\mathrm{Conj}(G_\sigma)|\}$ is a canonical basis of $H^*_{\mathrm{CR},T}(X; \mathcal{Q}')$.

Sometimes, we also consider the normalized canonical basis $\{\tilde{\phi}_{\sigma\alpha} | \sigma = 1, \cdots, n, \alpha = 1, \cdots, |\mathrm{Conj}(G_\sigma)|\}$, where $\tilde{\phi}_{\sigma\alpha}$ is defined to be

$$\tilde{\phi}_{\sigma\alpha} = \sqrt{\frac{\prod_{j=1}^r \mathsf{w}_{\sigma j}}{\nu_{\sigma\alpha}}} \bar{\phi}_{\sigma\alpha}.$$

With this definition, we have

$$\langle \tilde{\phi}_{\sigma\alpha}, \tilde{\phi}_{\sigma'\alpha'} \rangle_X = \delta_{\sigma,\sigma'}\delta_{\alpha,\alpha'}$$

i.e. $\tilde{\phi}_{\sigma\alpha}$ has unit length.

Chapter 3

Gromov-Witten theory, quantum cohomology and Frobenius manifolds

In this chapter, we first recall some basic definitions of Gromov-Witten theory of a smooth Deligne-Mumford stack X. The foundation of Gromov-Witten theory of smooth Deligne-Mumford stacks is developed in [3]. The symplectic counterpart is developed in [17]. After that, we specialized ourselves to the equivariant Gromov-Witten theory of a GKM orbifold X. Then we will focus on the genus zero case which determines the Frobenius structure on $H^*_{\mathrm{CR},T}(X;\mathcal{Q}') \otimes_{\mathbb{C}} N(X)$ with $N(X)$ the Novikov ring. In particular, it gives us the structure of the quantum cohomology ring of X and hence determines the quantum differential equation. We will discuss the semisimplicity of the Frobenius structure and the solutions to the quantum differential equation.

3.1 Gromov-Witten theory of smooth Deligne-Mumford stacks

Let X be a smooth Deligne-Mumford stack. Let $E \subseteq H_2(X,\mathbb{Z})$ be the semigroup of effective curve classes. Define the Novikov ring $N(X)$ to be the completion of $\mathbb{C}[E]$:

$$N(X) = \widehat{\mathbb{C}[E]} = \{\sum_{\beta \in E} c_\beta Q^\beta | c_\beta \in \mathbb{C}\}.$$

Denote by $\overline{\mathcal{M}}_{g,k}(X,\beta)$ the moduli space of degree β stable maps to X from genus g curves with k marked points. The difference from the smooth case is that the marked points and nodes of the domain curve are allowed to have orbifold structures with finite cyclic isotropy groups. There are k orbifold line bundles $\mathbb{L}_1, \cdots, \mathbb{L}_k$ on $\overline{\mathcal{M}}_{g,k}(X,\beta)$. The fiber of $\mathbb{L}_i$ at a point $[f:(C,x_1,\cdots,x_k) \to X] \in \overline{\mathcal{M}}_{g,k}(X,\beta)$ is the cotangent space at x_i. Let X' be the coarse moduli space of X and let $\overline{\mathcal{M}}_{g,k}(X',\beta)$ be the usual moduli space of stable maps to X'. We still have the corresponding line bundles $\mathbb{L}'_1, \cdots, \mathbb{L}'_k$ on $\overline{\mathcal{M}}_{g,k}(X',\beta)$. Let $\pi : \overline{\mathcal{M}}_{g,k}(X,\beta) \to \overline{\mathcal{M}}_{g,k}(X',\beta)$ be the natural map forgetting the orbifold structures. We define $\psi_i = \pi^*(c_1(\mathbb{L}'_i)), i = 1, \cdots, k$. There are evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,k}(X,\beta) \to \mathcal{I}X, i = 1, \cdots, k$. Since the target of the evaluation maps is the inertia stack $\mathcal{I}X$, the Chen-Ruan orbifold cohomology $H^*_{\mathrm{CR}}(X)$ plays the role of the state space. Let $\gamma_1, \cdots, \gamma_k \in H^*_{\mathrm{CR}}(X)$ and $a_1, \cdots, a_k \in \mathbb{Z}_{\geq 0}$, define the orbifold Gromov-Witten invariants by the following correlator

$$\langle \tau_{a_1}\gamma_1, \cdots, \tau_{a_k}\gamma_k \rangle^X_{g,k,\beta} = \int_{[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{k} ((\mathrm{ev}_i^*\gamma_i)\psi_i^{a_i}).$$

where $[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is the virtual fundamental class as in the smooth case.

When $g = 0, \beta = 0, k = 3$, the above correlator determines the orbifold multiplication structure of $H^*_{\mathrm{CR}}(X)$ as in Definition 2.2.4.

3.2 Gromov-Witten theory of GKM orbifolds

Let X be a GKM orbifold. The $T-$action on X naturally induces a $T-$action on $\overline{\mathcal{M}}_{g,k}(X,\beta)$. For any nonnegative integer a, we consider the cohomology class $t_a = \sum_{\sigma,\alpha} t_a^{\sigma\alpha}\tilde{\phi}_{\sigma\alpha} \in H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$. For convenience, we combine the two indices σ, α into a single index μ so that μ runs over the set $\Sigma_X := \{(\sigma,\alpha) | \sigma = 1, \cdots, n, \alpha = 1, \cdots, |\mathrm{Conj}(G_\sigma)|\}$. So we can write t_a as $t_a = \sum_{\mu\in\Sigma_X} t_a^{\mu}\tilde{\phi}_\mu \in H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$. Define the genus g correlator to be

$$\langle \mathbf{t}(\psi_1), \cdots, \mathbf{t}(\psi_k) \rangle^{X,T}_{g,k,\beta} = \int_{[\overline{\mathcal{M}}_{g,k}(X,\beta)^T]^{\mathrm{vir}}} \frac{\prod_{j=1}^{k}(\sum_{a=0}^{\infty}(\mathrm{ev}_j^* t_a)\psi_j^a)}{e_T(N^{\mathrm{vir}})}.$$

Here ψ_j is the 1-st Chern class of the universal cotangent line bundle over $\overline{\mathcal{M}}_{g,k}(X,\beta)$ corresponding to the $j-$th marked point and ev_j is the $j-$th evaluation map. The insertion $\mathbf{t}(\psi_j) := t_0 + t_1\psi_j + t_2\psi_j^2 + \cdots$ is viewed as a formal power series in ψ_i with coefficients in $H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$.

Let $t = \sum_{\mu\in\Sigma_X} t^\mu \tilde{\phi}_\mu \in H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$. We will be interested in the following descendent potential with primary insertions:

$$F^{X,T}_{g,k}(\mathbf{t},t) = \sum_{s=0}^{\infty}\sum_{\beta\in E}\frac{Q^\beta}{s!}\langle \mathbf{t}(\psi_1),\cdots,\mathbf{t}(\psi_k),t,\cdots,t\rangle^{X,T}_{g,k+s,\beta}.$$

Sometimes we also denote $F^{X,T}_{g,k}(\mathbf{t},t)$ by the double bracket:

$$\langle\!\langle \mathbf{t}(\psi_1),\cdots,\mathbf{t}(\psi_k)\rangle\!\rangle^{X,T}_{g,k} := F^{X,T}_{g,k}(\mathbf{t},t).$$

We define the full descendent potential $\mathcal{D}_X$ of X to be

$$\mathcal{D}_X := \exp\Big(\sum_{k\geq 0}\sum_{g\geq 0}\frac{\hbar^{g-1}}{k!}F^{X,T}_{g,k}(\mathbf{t},0)\Big).$$

Let $\overline{\mathcal{M}}_{g,k}$ denote the moduli space of genus g nodal curves with k marked points. Consider the map $\pi:\overline{\mathcal{M}}_{g,k+s}(X,\beta)\to\overline{\mathcal{M}}_{g,k}$ which forgets the map to the target and the last s marked points. Let $\bar{\psi}_i := \pi^*(\psi_i)$ be the pull-backs of the classes $\psi_i, i = 1,\cdots,k$, from $\overline{\mathcal{M}}_{g,k}$. Then similarly we can define the ancestor potential with primary insertions to be

$$\bar{F}^{X,T}_{g,k}(\mathbf{t},t) = \sum_{s=0}^{\infty}\sum_{\beta\in E}\frac{Q^\beta}{s!}\langle \mathbf{t}(\bar{\psi}_1),\cdots,\mathbf{t}(\bar{\psi}_k),t,\cdots,t\rangle^{X,T}_{g,k+s,\beta}.$$

Similar to the descendent potential, we also denote $\bar{F}^{X,T}_{g,k}(\mathbf{t},t)$ by the double bracket:

$$\langle\langle \mathbf{t}(\bar{\psi}_1), \cdots, \mathbf{t}(\bar{\psi}_k) \rangle\rangle_{g,k}^{X,T} := \bar{F}_{g,k}^{X,T}(\mathbf{t}, t).$$

We define the total ancestor potential $\mathcal{A}_X(t)$ of X to be

$$\mathcal{A}_X(t) := \exp\Big(\sum_{k \geq 0} \sum_{g \geq 0} \frac{\hbar^{g-1}}{k!} \bar{F}_{g,k}^{X,T}(\mathbf{t}, t) \Big).$$

3.3 Frobenius manifolds and semisimplicity

In this section, we focus ourselves to the genus zero case. The genus zero data of X determines a Frobenius structure on $H^*_{\mathrm{CR},T}(X; \mathcal{Q}') \otimes_{\mathbb{C}} N(X) = H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ which is going to be proved to be semisimple. For a general introduction to Frobenius manifolds and quantum cohomology, the reader is refered to [64] and [55].

Definition 3.3.1. *Given a point $t \in H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ and any two cohomology classes $a, b \in H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$, the* quantum product $\star_t$ *of a and b at t is defined to be*

$$\langle a \star_t b, c \rangle_X = \langle\langle a, b, c \rangle\rangle_{0,3}^{X,T},$$

where $c \in H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ is any cohomology class.

The quantum product $\star_t$ gives us a product structure on the tangent space $T_t H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ which depends smoothly on t. The quantum product is associative due to the WDVV equation.

So far, $H^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$ is equipped with the following structures

1. A flat pseudo-Riemannian metric $\langle\rangle_X$ which is given by the Poincaré pairing on $H^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$.

2. An associative commutative multiplication $\star_t$ satisfying $\langle a\star_t b,c\rangle_X = \langle a,b\star_t c\rangle_X$, on the tangent space $T_tH^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$ which depends smoothly on t.

3. A vector field $\mathbf{1}$ which is flat under the metric $\langle\rangle_X$ and is the unit for the product structure.

With the above structures, $H^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$ is called a *Frobenius manifold* (see [64] and [55] for more details).

Notice that at the origin $t=0, Q=0$, the quantum product $\star_0$ is just the classical equivariant orbifold product $\star$ introduced in Definition 2.2.4. The classical equivariant Chen-Ruan cohomology $H^*_{\mathrm{CR},T}(X;\mathcal{Q}')$ is semisimple as we proved in Section 2.3. Therefore by the criterion of semisimplicity (see Lemma 18 and Lemma 23 in [55]), we know that the Frobenius manifold $H^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$ is also semisimple. So there exists a system of *canonical* coordinates $\{u^i(t)\}_{i=1}^N$ on $H^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$, where $N = \dim H^*_{\mathrm{CR},T}(X;\mathcal{Q}') = \sum_\sigma |\mathrm{Conj}(G_\sigma)|$, characterized by the property that the corresponding vector fields $\{\partial/\partial u^i\}_{i=1}^N$ form a canonical basis of the quantum product on $\star_t$ $T_tH^*_{\mathrm{CR},T}(X;\mathcal{Q}'\otimes_{\mathbb{C}} N(X))$. This characterization determines the canonical coordinates $\{u^i(t)\}_{i=1}^N$ uniquely up to reordering and additive constants. We choose the canonical coordinates $\{u^i(t)\}_{i=1}^N$ such that they vanish when $t = 0, Q = 0$.

Let $\Delta_i := \frac{1}{\langle \partial/\partial u^i, \partial/\partial u^i \rangle_X}$. Denote by Ψ the transition matrix between flat and normalized canonical basis: $\Delta_i^{-\frac{1}{2}} du^i = \sum_{\mu \in \Sigma_X} \Psi_\mu{}^i dt^\mu$. Here we use the convention that the left index of a matrix is for the rows and the right index is for the columns.

3.4 Solutions to the quantum differential equations

Let $H = H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$. We consider the Dubrovin connection ∇ on the tangent bundle TH:

$$\nabla_\mu = \frac{\partial}{\partial t^\mu} + \frac{1}{z} \tilde{\phi}_\mu \star_t$$

for any $\mu \in \Sigma_X$. Here z is a formal variable. The equation $\nabla \tau = 0$ for a section τ of TH is called the *quantum differential equation*. Consider the operator S_t defined as follows: for any $a, b \in H$,

$$\langle a, S_t b \rangle_X = \langle\langle a, \frac{b}{z - \psi} \rangle\rangle_{0,2}^{X,T}.$$

The operator S_t satisfies the following nice property: For any $a \in H^*_{\mathrm{CR},T}(X; \mathcal{Q}')$, the section $S_t a$ satisfies the quantum differential equation i.e. we have

$$\nabla S_t a = 0.$$

Such an operator is called a *fundamental solution* to the quantum differential equation. The proof for S_t being a fundamental solution can be found in [23] for the smooth case and in [49] for the orbifold case which is a direct generalization of the smooth case.

The operator $S = \mathbf{1} + S_1/z + S_2/z^2 + \cdots$ is a formal power series in $1/z$ with operator-valued coefficients.

Chapter 4

Higher genus Gromov-Witten potential

4.1 quantization of quadratic Hamiltonians

In this section, we review the basic concepts of the quantization of quadratic Hamiltonians (see [42] for more details). The quantization procedure provides a way to recover the higher genus theory from the genus zero data which we will use in the next section.

4.1.1 Symplectic space formalism

So far, we have been working on the space $H = H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ which provides us the Frobenius structure and state space of the corresponding Gromov-Witten theory. When we consider the descendent theory of X, however, additional parameters are needed. As we have seen in section

3.1, the insertion $\mathbf{t}(\psi) = t_0 + t_1\psi + t_2\psi^2 + \cdots$ is a formal power series in ψ with an integer index that keeps track in the power of ψ. Similarly, the $S-$operator studied in the previous section is a formal power series in $1/z$. These phenomena lead to the study of the symplectic space formalism.

Let z be a formal variable. We consider the space $\mathcal{H}$ which is the space of Laurent polynomials in one variable z with coefficients in H. We define the symplectic form Ω on $\mathcal{H}$ by

$$\Omega(f,g) = \mathrm{Res}_{z=0}\langle f(-z), g(z)\rangle\rangle_X dz$$

for any $f, g \in \mathcal{H}$. Note that we have $\Omega(f,g) = -\Omega(g,f)$. There is a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ corresponding to the decomposition $f(z,z^{-1}) = f_+(z) + f_-(z^{-1})z^{-1}$ of laurent polynomials into polynomial and polar parts. It is easy to see that $\mathcal{H}_+$ and $\mathcal{H}_-$ are both Lagrangian subspaces of $\mathcal{H}$ with respect to Ω.

Introduce a Darboux coordinate system $\{p_a^\mu, q_b^\nu\}$ on $\mathcal{H}$ with respect to the above polarization. This means that we write a general element $f \in \mathcal{H}$ in the form

$$\sum_{a\geq 0,\mu\in\Sigma_X} p_a^\mu \tilde{\phi}^\mu(-z)^{-a-1} + \sum_{b\geq 0,\nu\in\Sigma_X} q_b^\nu \tilde{\phi}_\nu z^b,$$

where $\{\tilde{\phi}^\mu\}$ is the dual basis of $\{\tilde{\phi}_\mu\}$. Denote

$$\begin{aligned}
\mathbf{p}(z) : &= p_0(-z)^{-1} + p_1(-z)^{-2} + \cdots \\
\mathbf{q}(z) : &= q_0 z + q_1 z^2 + \cdots,
\end{aligned}$$

where $p_a = \sum_\mu p_a^\mu \tilde{\phi}^\mu$ and $q_b = \sum_\mu q_b^\nu \tilde{\phi}_\nu$.

Recall that when we discussed the Gromov-Witten theory of X, we introduced the formal power series $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots$. With z replaced by ψ, $\mathbf{t}$ appears as the insertion in the genus g correlator. We relate $\mathbf{t}(z)$ to the Darboux coordinates by introducing the *dilaton shift*: $\mathbf{q}(z) = \mathbf{t}(z) - \mathbf{1}z$. The dilaton shift appears naturally in the quantization procedure. We will explain this phenomenon as a group action on Cohomological field theories in the next section.

4.1.2 Quantization of quadratic Hamiltonians

Let $A : \mathcal{H} \to$ ch be a linear infinitesimally symplectic transformation, i.e. $\Omega(Af, g) + \Omega(f, Ag) = 0$ for any $f, g \in \mathcal{H}$. Under the Darboux coordinates, the quadratic Hamiltonian

$$f \to \frac{1}{2}\Omega(Af, f)$$

is a series of homogeneous degree two monomials in $\{p_a^\mu, q_b^\nu\}$. Let $\hbar$ be a formal variable and define the quantization of quadratic monomials as

$$\widehat{q_a^\mu q_b^\nu} = \frac{q_a^\mu q_b^\nu}{\hbar}, \widehat{q_a^\mu p_b^\nu} = q_a^\mu \frac{\partial}{\partial q_b^\nu}, \widehat{p_a^\mu p_b^\nu} = \hbar \frac{\partial}{\partial q_a^\mu} \frac{\partial}{\partial q_b^\nu}.$$

We define the quantization $\widehat{A}$ by extending the above equalities linearly. The differential operators $\widehat{q_a^\mu q_b^\nu}, \widehat{q_a^\mu p_b^\nu}, \widehat{p_a^\mu p_b^\nu}$ act on the so called Fock space *Fock* which is the space of formal functions in $\mathbf{t}(z) \in \mathcal{H}_+$. For example, the descendent potential and ancestor potential are regarded as elements in *Fock*. The quantization operator $\widehat{A}$ does not act on *Fock* in general since it may contain infinitely many monomials. However, the actions of quantization operators in our paper are well-defined. The quantization of a symplectic transform of the form $\exp(A)$, with A infinitesimally symplectic, is defined to be $\exp(\widehat{A}) = \sum_{n\geq 0} \frac{\widehat{A}^n}{n!}$.

4.2 Higher genus structure

In this section, we recover the higher genus data of a GKM orbifold X by its Frobenius structure. Recall that in section 3.3, we have proved that the Frobenius manifold $H = H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ of any GKM orbifold X is semisimple. This means that the underlining cohomological field theory, which comes from the Gromov-Witten theory of X, is semisimple. So we can use Teleman's result [66], which classifies the 2D semisimple field theories, to express the ancestor potential of X in terms of certain group action (which is basically the action of quantization operators) on the trivial cohomological field theory with the same Frobenius manifold. The key point here is that the cohomological field theory that we are considering is not conformal since we are working with the equivariant Gromov-Witten theory. So the ambiguity with the corresponding group element, which acts on the trivial theory, cannot be fixed by the usual method using Euler vector field. Instead, we will fix the ambiguity by studying the degree 0 case of the ancestor potential and by using the structure of the solution space to the quantum differential equation. We will study this ambiguity in the next section. In the end, the descendent potential is related to the ancestor potential by the S–operator in the standard way (see [42] and [21]).

4.2.1 The quantization procedure and group actions on cohomological field theories

Recall that in section 3.3, we defined the transition matrix Ψ between flat and normalized canonical basis: $\Delta_i^{-\frac{1}{2}} du^i = \sum_{\mu \in \Sigma_X} {\Psi_\mu}^i dt^\mu$. If we view Ψ as an operator on H which sends the flat basis $\{\tilde{\phi}_\mu\}$ to the normalized canonical basis $\{\Delta_i^{\frac{1}{2}} \frac{\partial}{\partial u^i}\}$, then $({\Psi_\mu}^i)$ is the corresponding matrix expression under the basis $\{\tilde{\phi}_\mu\}$. Note that when $t = 0, Q = 0$, the normalized canonical basis $\{\Delta_i^{\frac{1}{2}} \frac{\partial}{\partial u^i}\}$ coincides with the flat basis $\{\tilde{\phi}_\mu\}$. So we have a canonical 1-1 correspondence between the two sets $\{\Delta_i^{\frac{1}{2}} \frac{\partial}{\partial u^i}\}$ and $\{\tilde{\phi}_\mu\}$. Let U denote the diagonal matrix $\mathrm{diag}(u^1, \cdots, u^N)$. Using the above correspondence, we can define the operator $e^{U/z}$ which has the matrix expression $e^{U/z}$ under the basis $\{\tilde{\phi}_\mu\}$.

Now we can state the following theorem which characterizes the solution space of the quantum differential equation. The proof of this theorem can be found in [41].

Theorem 4.2.1. *1. The quantum differential equation $\nabla \tilde{S} = 0$ in a neighborhood of a semisimple point u has a fundamental solution in the form: $\Psi_u \tilde{R}_u(z) e^{U/z}$, where $\tilde{R}_u(z) = \mathbf{1} + \tilde{R}_1 z + \tilde{R}_2 z^2 + \cdots$ is a formal matrix* *power series satisfying the unitary condition* $\tilde{R}_u^*(-z)\tilde{R}_u(z) = \mathbf{1}$*, where* $\tilde{R}_u^*$ *is the adjoint of* $\tilde{R}_u$.

2. *The series* $\tilde{R}_u(z)$ *satisfying the unitary condition in (a) is unique up to right multiplication by diagonal matrices* $\exp(a_1 z + a_3 z^3 + a_5 z^5 + \cdots)$ *where* a_{2k-1} *are constant diagonal matrices.*

3. *In the case of conformal Frobenius manifolds the series* $\tilde{R}_u(z)$ *satisfying the unitary condition in (a) is uniquely determined by the homogeneity condition* $(z\partial_z + \sum u^i \partial_{u^i})\tilde{R}_u(z) = 0$.

From this theorem, we know that there are ambiguities with the R-matrix in the fundamental solution $\tilde{S}$ if we work with non-conformal Frobenius manifolds. However, the ambiguity is a *constant* matrix and so we can fix it by studying the case when $t = 0, Q = 0$. This will be done in the next section.

Remark 4.2.2. *The fundamental solution* $\tilde{S}$ *in Theorem 4.2.1 is viewed as a matrix with entries in* $\mathcal{Q}'((z))[[Q, t^\mu]]$*. Since we choose the canonical coordinates* $\{u^i(t)\}_{i=1}^N$ *such that they vanish when* $Q = 0, t = 0$*, if we fix the powers of* Q *and* $t^\mu, \mu \in \Sigma_X$*, only finitely many terms in the expansion of* $e^{U/z}$ *contribute. So the multiplication* $\Psi_u \tilde{R}_u(z) e^{U/z}$ *is well defined and the result matrix indeed has entries in* $\mathcal{Q}'((z))[[Q, t^\mu]]$.

Remark 4.2.3. *For a general abstract semi-simple Frobenius manifold defined over a ring A, the expression* $\tilde{S} = \Psi\tilde{R}(z)e^{U/z}$ *in Theorem 4.2.1 can be understood in the following way. We consider the free module* $M = \langle e^{u^1/z}\rangle \oplus \cdots \oplus \langle e^{u^N/z}\rangle$ *over the ring* $A((z))[[t^1,\cdots,t^N]]$ *where* $t^1,\cdots,t^N$ *are the flat coordinates of the Frobenius manifold. We formally define the differential* $de^{u^i/z} = e^{u^i/z}\frac{du^i}{z}$ *and we extend the differential to* M *by the product rule. Then we have a map* $d : M \to Mdt^1 \oplus \cdots \oplus Mdt^N$*. We consider the fundamental solution* $\tilde{S} = \Psi\tilde{R}(z)e^{U/z}$ *as a matrix with entries in* M*. The meaning that* $\tilde{S}$ *satisfies the quantum differential equation is understood by the above formal differential.*

The operator $\tilde{R}$ in Theorem 4.2.1 plays a central role in the quantization procedure. Before we move on to the quantization process, let us consider the potential functions of the trivial field theory I_H. When we use the 1-1 correspondence between $\{\Delta_i^{\frac{1}{2}}\frac{\partial}{\partial u^i}\}$ and $\{\tilde{\phi}_\mu\}$ by identifying them at the origin $t = 0, Q = 0$, we can use the same index i for both of the two basis. Define the correlator $\langle\rangle_{g,k}^{I_H}$ to be

$$\langle\tau_{a_1}(\tilde{\phi}_{i_1}),\cdots,\tau_{a_k}(\tilde{\phi}_{i_k})\rangle_{g,k}^{I_H} = \begin{cases} \Delta_i^{g-1+k/2}\int_{\overline{\mathcal{M}}_{g,k}}\psi_1^{a_1}\cdots\psi_k^{a_k}, & \text{if} \quad i_1 = i_2 = \cdots = i_k = i, \\ 0, & \text{otherwise} \end{cases}$$

where $a_1,\cdots,a_k$ are nonnegative integers. Let

$$\mathcal{D}_{I_H} = \exp\Big(\sum_{g\geq 0}\sum_{k\geq 0}\sum_{a_1,\cdots,a_k\geq 0}\sum_{i_1,\cdots,i_k\in\{1,\cdots,N\}}\frac{\hbar^{g-1}}{a_1!\cdots a_k!}\langle\tau_{a_1}(\tilde{\phi}_{i_1}),\cdots,\tau_{a_k}(\tilde{\phi}_{i_k})\rangle_{g,k}^{I_H}\Big).$$

The following theorem is the result of the semisimplicity of the Frobenius manifold $H = H^*_{\mathrm{CR},T}(X; \mathcal{Q}' \otimes_{\mathbb{C}} N(X))$ and Teleman's classification of semisimple cohomological field theories (see Proposition 8.3 in [66]):

Theorem 4.2.4 (Givental formula for ancestor potentials of GKM orbifolds)**.** *There exists a fundamental solution $\tilde{S} = \Psi_u \tilde{R}_u(z) e^{U/z}$ to the quantum differential equation $\nabla \tilde{S} = 0$ with $\tilde{R}_u(z)$ satisfying the unitary condition such that*

$$\mathcal{A}_X(t) = \widehat{\Psi} \widehat{\tilde{R}} \mathcal{D}_{I_H}.$$

Here $\widehat{\Psi}$ is the operator $\mathcal{G}(\Psi^{-1}\mathbf{q}) \mapsto \mathcal{G}(\mathbf{q})$ for any element $\mathcal{G}$ in the Fock space.

Remark 4.2.5. *In [66], $\widehat{\tilde{R}}$ is explained as an element in a certain group acting on the cohomological field theories and Theorem 4.2.4 is equivalent to say that $\mathcal{A}_X(t)$ and $\mathcal{D}_{I_H}$ lie in the same orbit under this group action. The dilaton shift in the quantization process is equivalent to the conjugate action of the translation T_z in [66].*

Recall that we have defined a particular fundamental solution S_t in section 3.4 by the 1-primary, 1-descendent correlator. The quantization of the operator S_t relates the ancestor potential $\mathcal{A}_X(t)$ to the descendent potential $\mathcal{D}_X$. Specifically, we have the following theorem:

Theorem 4.2.6 (Givental formula for descendent potentials of GKM orbifolds)**.** *Let $F_1(t) = \sum_{k\geq 0} \frac{1}{k!} F^{X,T}_{1,k}(\mathbf{t}, 0)|_{t_0=1, t_1=t_2=\cdots=0}$. Then we have*

$$\mathcal{D}_X = \exp(F_1(t)) \widehat{S_t}^{-1} \mathcal{A}_X(t) = \exp(F_1(t)) \widehat{S_t}^{-1} \widehat{\Psi} \widehat{\tilde{R}} \mathcal{D}_{I_H}.$$

The proof of the relation $\mathcal{D}_X = \exp(F_1(t))\widehat{S_t}^{-1}\mathcal{A}_t$ can be found in Theorem 1.5.1 of [21] for the smooth case. The strategy is to use the comparison lemma type argument to relate ψ to $\bar{\psi}$. The proof for the orbifold case is completely similar to the smooth case.

We should notice that although $\mathcal{A}_X(t)$ depends on t, the total descendent potential $\mathcal{D}_X$ is indepentdent of t. For our purpose, we are more interested in the descendent potential with primary insertions i.e the function $F_{g,k}^{X,T}(\mathbf{t},t)$ defined in section 3.2. This potential function will eventually correspond to the B-model potential under the all genus mirror symmetry studied in [38]. The relation between $F_{g,k}^{X,T}(\mathbf{t},t)$ and the ancestor potential $\bar{F}_{g,k}^{X,T}(\mathbf{t},t)$ is even easier:

Proposition 4.2.7. *For $2g-2+k>0$, we have the following relation*

$$F_{g,k}^{X,T}(\mathbf{t},t) = \bar{F}_{g,k}^{X,T}([S_t\mathbf{t}]_+,t).$$

Here we consider $\mathbf{t}=\mathbf{t}(z)$ as element in $\mathcal{H}$ and $[S_t\mathbf{t}]_+$ is the part of $S_t\mathbf{t}$ containing nonnegative powers of z.

The proof of the proposition can also be found in the proof of Theorem 1.5.1 of [21].

4.2.2 The graph sum formula

The Givental quantization formula, which involves differential operators, can be expressed in terms of graph sums. This follows from the standard correspondence between differential operators and Feynman type graph sum formulas (see [29]). In this section, we describe a graph sum formula of $\bar{F}^{X,T}_{g,k}(\mathbf{t},t)$ which is equivalent to Theorem 4.2.4. By using Proposition 4.2.7, we obtain the graph sum formula for $F^{X,T}_{g,k}(\mathbf{t},t)$. The graph sum formula gives us a more explicit expression of the Givental formula. In [38], we prove the all genus mirror symmetry for toric Calabi-Yau 3-orbifolds by expressing both the A-model and B-model potentials as graph sums and by identifying each term in the graph sum. We will also sketch this proof in Chapter 5.

In this subsection, every matrix expression of the corresponding linear operator is under the basis $\{\tilde{\phi}_\mu\}$.

Given a connected graph Γ, we introduce the following notation.

1. $V(\Gamma)$ is the set of vertices in Γ.
2. $E(\Gamma)$ is the set of edges in Γ.
3. $H(\Gamma)$ is the set of half edges in Γ.
4. $L^o(\Gamma)$ is the set of ordinary leaves in Γ.
5. $L^1(\Gamma)$ is the set of dilaton leaves in Γ.

With the above notation, we introduce the following labels:

1. (genus) $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
2. (marking) $i : V(\Gamma) \to \{1, \cdots, N\}$. This induces $i : L(\Gamma) = L^o(\Gamma) \cup L^1(\Gamma) \to \{1, \cdots, N\}$, as follows: if $l \in L(\Gamma)$ is a leaf attached to a vertex $v \in V(\Gamma)$, define $i(l) = i(v)$.

3. (height) $a : H(\Gamma) \to \mathbb{Z}_{\geq 0}$.

Given an edge e, let $h_1(e), h_2(e)$ be the two half edges associated to e. The order of the two half edges does not affect the graph sum formula in this paper. Given a vertex $v \in V(\Gamma)$, let $H(v)$ denote the set of half edges emanating from v. The valency of the vertex v is equal to the cardinality of the set $H(v)$: $\mathrm{val}(v) = |H(v)|$. A labeled graph $\vec{\Gamma} = (\Gamma, g, i, a)$ is *stable* if

$$2g(v) - 2 + \mathrm{val}(v) > 0$$

for all $v \in V(\Gamma)$.

Let $\mathbf{\Gamma}(X)$ denote the set of all stable labeled graphs $\vec{\Gamma} = (\Gamma, g, i, a)$. The genus of a stable labeled graph $\vec{\Gamma}$ is defined to be

$$g(\vec{\Gamma}) := \sum_{v \in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1 = \sum_{v \in V(\Gamma)} (g(v) - 1) + (\sum_{e \in E(\Gamma)} 1) + 1.$$

Define

$$\mathbf{\Gamma}_{g,k}(X) = \{\vec{\Gamma} = (\Gamma, g, i, a) \in \mathbf{\Gamma}(X) : g(\vec{\Gamma}) = g, |L^o(\Gamma)| = k\}.$$

Let $\mathbf{t}(z) = \sum_\mu \mathbf{t}^\mu(z) \tilde{\phi}_\mu$.

We assign weights to leaves, edges, and vertices of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(X)$ as follows.

1. *Ordinary leaves.* To each ordinary leaf $l \in L^o(\Gamma)$ with $i(l) = i \in$

$\{1,\cdots,N\}$ and $a(l)=a\in\mathbb{Z}_{\geq 0}$, we assign:

$$(\mathcal{L}^{\mathbf{t}})^i_a(l)=[z^a](\sum_{\mu,j=1,\cdots,N}\mathbf{t}^{\mu}(z)\Psi_{\mu}{}^{j}\tilde{R}_j{}^{i}(-z)).$$

2. *Dilaton leaves.* To each dilaton leaf $l\in L^1(\Gamma)$ with $i(l)=i\in\{1,\cdots,N\}$ and $2\leq a(l)=a\in\mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^1)^i_a(l)=[z^{a-1}](-\sum_{j=1,\cdots,N}\frac{1}{\sqrt{\Delta^j}}\tilde{R}_j{}^{i}(-z)).$$

3. *Edges.* To an edge connected a vertex marked by $i\in\{1,\cdots,N\}$ to a vertex marked by $j\in\{1,\cdots,N\}$ and with heights a and b at the corresponding half-edges, we assign

$$\mathcal{E}^{i,j}_{a,b}(e)=[z^aw^b]\Big(\frac{1}{z+w}(\delta_{i,j}-\sum_{p=1,\cdots,N}\tilde{R}_p{}^{i}(-z)\tilde{R}_p{}^{j}(-w)\Big).$$

4. *Vertices.* To a vertex v with genus $g(v)=g\in\mathbb{Z}_{\geq 0}$ and with marking $i(v)=i$, with k_1 ordinary leaves and half-edges attached to it with heights $a_1,...,a_n\in\mathbb{Z}_{\geq 0}$ and k_2 more dilaton leaves with heights $a_{k_1+1},\ldots,a_{k_1+k_2}\in\mathbb{Z}_{\geq 0}$, we assign

$$\int_{\overline{\mathcal{M}}_{g,k_1+k_2}}\psi_1^{a_1}\cdots\psi_{k_1+k_2}^{a_{n+m}}.$$

We define the weight of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(\mathbb{P}^1)$ to be

$$\begin{aligned} w(\vec{\Gamma}) &= \prod_{v \in V(\Gamma)} (\sqrt{\Delta^{i(v)}})^{2g(v)-2+\mathrm{val}(v)} \langle \prod_{h \in H(v)} \tau_{a(h)} \rangle_{g(v)} \prod_{e \in E(\Gamma)} \mathcal{E}^{i(v_1(e)), i(v_2(e))}_{a(h_1(e)), a(h_2(e))}(e) \\ &\cdot \prod_{l \in L^o(\Gamma)} (\mathcal{L}^{\mathbf{t}})^{i(l)}_{a(l)}(l) \prod_{l \in L^1(\Gamma)} (\mathcal{L}^1)^{i(l)}_{a(l)}(l). \end{aligned}$$

Then

$$\log(\mathcal{A}_X(t)) = \sum_{\vec{\Gamma} \in \mathbf{\Gamma}(X)} \frac{\hbar^{g(\vec{\Gamma})-1} w(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|} = \sum_{g \geq 0} \hbar^{g-1} \sum_{k \geq 0} \sum_{\vec{\Gamma} \in \mathbf{\Gamma}_{g,k}(X)} \frac{w(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

By Proposition 4.2.7, $F^{X,T}_{g,k}(\mathbf{t}, t)$ can be obtained by $\bar{F}^{X,T}_{g,k}(\mathbf{t}, t)$ by change of variables defined by the operator $S_t(z)$. So in order to get the graph sum formula for $F^{X,T}_{g,k}(\mathbf{t}, t)$, we only need to modify the ordinary leaves:

(1)' *Ordinary leaves.* To each ordinary leaf $l \in L^o(\Gamma)$ with $i(l) = i \in \{1, \cdots, N\}$ and $a(l) = a \in \mathbb{Z}_{\geq 0}$, we assign:

$$(\mathring{\mathcal{L}}^{\mathbf{t}})^i_a(l) = [z^a](\sum_{\mu, \nu, j = 1, \cdots, N} \mathbf{t}^{\mu}(z) S_t(z)^{\nu}{}_{\mu} \Psi_{\nu}{}^{j} \tilde{R}_j{}^{i}(-z)).$$

We define a new weight of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(X)$ to be

$$\begin{aligned} \mathring{w}(\vec{\Gamma}) &= \prod_{v \in V(\Gamma)} (\sqrt{\Delta^{i(v)}})^{2g(v)-2+\mathrm{val}(v)} \langle \prod_{h \in H(v)} \tau_{a(h)} \rangle_{g(v)} \prod_{e \in E(\Gamma)} \mathcal{E}^{i(v_1(e)), i(v_2(e))}_{a(h_1(e)), a(h_2(e))}(e) \\ &\cdot \prod_{l \in L^o(\Gamma)} (\mathring{\mathcal{L}}^{\mathbf{t}})^{i(l)}_{a(l)}(l) \prod_{l \in L^1(\Gamma)} (\mathcal{L}^1)^{i(l)}_{a(l)}(l). \end{aligned}$$

Then

$$\sum_{g\geq 0}\hbar^{g-1}\sum_{k\geq 0}\frac{1}{k!}F^{X,T}_{g,k}(\mathbf{t},t)=\sum_{\vec{\Gamma}\in\mathbf{\Gamma}(X)}\frac{\hbar^{g(\vec{\Gamma})-1}\mathring{w}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}=\sum_{g\geq 0}\hbar^{g-1}\sum_{k\geq 0}\sum_{\vec{\Gamma}\in\mathbf{\Gamma}_{g,k}(X)}\frac{\mathring{w}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

4.3 Reconstruction from genus zero data

As mentioned earlier, the operator $\tilde{R}$ is not uniquely determined since we are working with non-conformal Frobenius manifold. However, by Theorem 4.2.1 the ambiguity is a constant matrix which allows us to fix the ambiguity by passing to the case when $t = 0, Q = 0$. In this case, the domain curve is contracted to one of the torus fixed points $p_1, \cdots, p_n$ of X and there is no primary insertions. So we can reduce the problem to the case when $X = [\mathbb{C}^r/G]$. In this case, we can study the Gromov-Witten theory of X by orbifold quantum Riemann-Roch theorem in [67]

4.3.1 The case $X = [\mathbb{C}^r/G]$ and orbifold quantum Riemann-Roch theorem

In this section, we apply the orbifold quantum Riemann-Roch theorem to $X = [\mathbb{C}^r/G]$ to get a formula for $\mathcal{D}_X$. Then we can compare this formula with the Givental formula in the previous section to fix the ambiguity with the operator $\tilde{R}$.

Recall that the Bernoulli polynomials $B_m(x)$ are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{m \geq 0} \frac{B_m(x)t^m}{m!}.$$

The Bernoulli numbers are given by $B_m := B_m(0)$.

Let $\mathcal{B}G$ be the classifying stack of the finite group G. Then by Example 2.2.2,

$$\mathcal{I}\mathcal{B}G = \bigsqcup_{(h) \in \mathrm{Conj}(G)} [pt/C(h)]$$

and

$$H^*(\mathcal{I}\mathcal{B}G; \mathbb{C}) = \bigoplus_{(h) \in \mathrm{Conj}(G)} \mathbb{C}\mathbf{1}_{(h)}.$$

Now we consider the Chen-Ruan cohomology $H^*_{\mathrm{CR}}(\mathcal{B}G; \mathcal{Q}')$. For each irreducible representation V_α of G, let

$$\phi_\alpha = \frac{\dim V_\alpha}{|G|} \sum_{(h) \in \mathrm{Conj}(G)} \chi_\alpha(h^{-1}) \mathbf{1}_{(h)}.$$

Then by [50] (or by specializing the computation in Section 2.3 to the case when $r = 0$), we have

$$\phi_\alpha \star \phi_{\alpha'} = \delta_{\alpha,\alpha'} \frac{|G|}{\dim V_\alpha} \phi_\alpha$$

and

$$\langle \phi_\alpha, \phi_{\alpha'} \rangle_{\mathcal{B}G} = \delta_{\alpha,\alpha'}.$$

So $\{\phi_\alpha\}$ is a normalized canonical basis for $H^*_{\mathrm{CR}}(\mathcal{B}G;\mathcal{Q}')$.

For each integer $m \geq 0$ and $i = 1,\cdots,r$, define an linear operator A^i_m : $H^*(\mathcal{IB}G) \to H^*(\mathcal{IB}G)$ by

$$A^i_m(\mathbf{1}_{(h)}) := B_m(c_i(h))\mathbf{1}_{(h)}.$$

Then we define the symplectic operator $P(z)$ to be

$$P(z) := \prod_{i=1}^{r} \exp\Big(\sum_{m\geq 1} \frac{(-1)^m}{m(m+1)} \sum_{i=1}^{r} A^i_{m+1}(\frac{z}{\mathsf{w}_i})^m\Big),$$

where w_i is the torus character in the i−th tangent direction.

Let $\tilde{H} := H^*_{\mathrm{CR}}(\mathcal{B}G;\mathcal{Q}')$ and we define the cohomological field theory $I_{\tilde{H}}$ as follows. Define the genus g correlator $\langle\rangle^{I_{\tilde{H}}}_{g,k}$ to be

$$\langle\tau_{a_1}(\phi_{i_1}),\cdots,\tau_{a_k}(\phi_{i_k})\rangle^{I_{\tilde{H}}}_{g,k} = \begin{cases} (\frac{|G|\sqrt{\prod_j \mathsf{w}_j}}{\dim V_i})^{2g-2+k}\int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{a_1}\cdots\psi_k^{a_k}, & \text{if} \quad i_1 = i_2 = \cdots = i_k = i, \\ 0, & \text{otherwise} \end{cases}$$

where $a_1,\cdots,a_k$ are nonnegative integers. Let

$$\mathcal{D}_{I_{\tilde{H}}} = \exp\Big(\sum_{g\geq 0}\sum_{k\geq 0}\sum_{a_1,\cdots,a_k\geq 0}\sum_{i_1,\cdots,i_k\in\{1,\cdots,|\mathrm{Conj}(G)|\}} \frac{\hbar^{g-1}}{a_1!\cdots a_k!}\langle\tau_{a_1}(\tilde{\phi}_{i_1}),\cdots,\tau_{a_k}(\tilde{\phi}_{i_k})\rangle^{I_{\tilde{H}}}_{g,k}\Big).$$

Then we have the following orbifold quantum Riemann-Roch theorem [67]:

Theorem 4.3.1 (orbifold quantum Riemann-Roch theorem for X).

$$\mathcal{D}_X = \widehat{P}\mathcal{D}_{I_{\tilde{H}}}.$$

Now we make the following key observation. When $t = 0, Q = 0$, $\Delta_i = \frac{|G|^2 \prod_j \mathsf{w}_j}{(\dim V_i)^2}$. This means that the Frobenius algebra $H|_{t=0,Q=0}$ is isomorphic to $\tilde{H}$ and the isomorphism is given by

$$\tilde{\phi}_i \mapsto \phi_i$$

for $i = 1, \cdots, |\mathrm{Conj}(G)|$. On the other hand, when $t = 0, Q = 0$ we have $\mathcal{D}_X = \mathcal{A}_X$ and Ψ is trivial. So by comparing theorem 4.2.4 and theorem 4.3.1, we know that

$$\tilde{R}_j{}^i|_{t=0,Q=0} = P_j{}^i,$$

where $(P_j{}^i)$ is the matrix expression of P under the basis $\{\phi_\alpha\}$. Therefore if we let α_i, α_j be the corresponding irreducible representation, we have

$$\tilde{R}_j{}^i|_{t=0,Q=0} = \frac{1}{|G|} \sum_{(h) \in \mathrm{Conj}(G)} \chi_{\alpha_j}(h) \chi_{\alpha_i}(h^{-1}) \prod_{k=1}^{r} \exp\Big(\sum_{m=1}^{\infty} \frac{(-1)^m}{m(m+1)} B_{m+1}(c_k(h)) (\frac{z}{\mathsf{w}_k})^m \Big). \tag{4.1}$$

4.3.2 The general case

In general, when $t = 0, Q = 0$, the domain curve is contracted to one of the torus fixed points $p_1, \cdots, p_n$. So we apply the quantum Riemann-Roch theorem to each $[\mathbb{C}^r/G_\sigma]$ for $\sigma = 1, \cdots, n$ and we obtain n operators $P_1, \cdots, P_n$. So we have the following theorem which fixes the ambiguity with $\tilde{R}$.

Theorem 4.3.2. *The operator $\tilde{R}$ in theorem 4.2.4 is uniquely determined by the property that*

$$(\tilde{R}_j{}^i)|_{t=0,Q=0} = \mathrm{diag}\Big(((P_\sigma)_j{}^i)\Big)$$

where the block $((P_\sigma)_j{}^i)$ is given by

$$(P_\sigma)_j{}^i = \frac{1}{|G_\sigma|}\sum_{(h)\in\mathrm{Conj}(G_\sigma)}\chi_{\alpha_j}(h)\chi_{\alpha_i}(h^{-1})\prod_{k=1}^{r}\exp\Big(\sum_{m=1}^{\infty}\frac{(-1)^m}{m(m+1)}B_{m+1}(c_{\sigma k}(h))(\frac{z}{\mathsf{w}_{\sigma k}})^m\Big).$$

Theorem 4.2.1 is proved by substituting $\Psi_u\tilde{R}_u(z)e^{U/z}$ into the quantum differential equation and solve each $(\tilde{R}_u)_k$ inductively (see [41]). In this process, the constant terms in the diagonal entries of each $(\tilde{R}_u)_{2k-1}$ are ambiguous and this ambiguity is fixed by Theorem 4.3.2. So the matrix $\tilde{R}_u(z)$ can be uniquely reconstructed from the quantum differential equation and Theorem 4.3.2. Recall that the quantum differential equation is determined by the quantum product $\star_t$ which is given by the genus zero three points function $\langle\!\langle\cdots\rangle\!\rangle_{0,3}^{X,T}$. So by combining Theorem 4.2.4 and Proposition 4.2.7, we have the following reconstruction theorem from genus zero data:

Theorem 4.3.3 (Reconstruction from the Frobenius structure)**.** *The descendent potential $F_{g,k}^{X,T}(\mathbf{t}, t)$ of a GKM orbifold X can be uniquely reconstructed from the operator $\tilde{R}$ which is uniquely determined by the quantum* *multiplication law and the property*

$$(\tilde{R}_j{}^i)|_{t=0,Q=0} = \mathrm{diag}\big(((P_\sigma)_j{}^i)\big)$$

where $\mathrm{diag}\big(((P_\sigma)_j{}^i)\big)$ *is a constant matrix which is explicitly given by Theorem 4.3.2.*

Chapter 5

Application: all genus mirror symmetry for toric Calabi-Yau 3-orbifolds

5.1 Introduction

In this chapter, we consider the case when X is a toric Calabi-Yau 3-orbifold. Then X contains an algebraic torus $(\mathbb{C}^*)^3$ as Zariski dense open subset. We choose the torus T to be the two dimensional sub-torus of $(\mathbb{C}^*)^3$ which acts trivially on $\Lambda^3 TX$. In this case, there exists a particularly nice duality. This story comes from the *remodeling conjecture*, which is established in [65] [8] and [9]. The remodeling conjecture can be viewed as an all genus mirror symmetry for toric Calabi-Yau 3-orbifolds. In this conjecture, the

A-model higher genus potential is the open Gromov-Witten potential of X with respect to one or several Aganagic-Vafa branes on X. The higher genus B-model $\omega_{g,n}$ is obtained by applying the Eynard-Orantin [32] topological recursion to the mirror curve C of X. It is a symmetric n-form on C. When $\omega_{g,n}$ is expanded around certain points on C, the coefficients will give us the open Gromov-Witten invariants of X. More than one Aganagic-Vafa branes can be put on X depending on around which points we expend $\omega_{g,n}$. The interesting thing here is that the Eynard-Orantin topological recursion itself is motivated by matrix model theory, which was not expected to be related to mirror symmetry at the beginning. The Eynard-Orantin theory itself also has its own interests and can be applied to much more general targets such as quantum curves, higher dimensional manifolds and so on.

Before we discuss the all genus mirror symmetry for X, we should notice that one can build the genus 0 mirror symmetry for toric orbifolds of any dimensions by using their Landau-Ginzburg mirrors, which contains a super potential $W^T : (\mathbb{C}^*)^r \to \mathbb{C}$. Here T is the torus acting on the $r-$dimensional toric orbifold. In the general toric case, the genus 0 mirror symmetry has been studied in [40] for smooth toric varieties and in [22] for toric Deligne-Mumford (DM) stacks. The genus 0 mirror theorem gives us an isomorphism between the quantum cohomology ring of a toric orbifold and the Jacobi ring of its Landau-Ginzburg mirror. So one can identify the Frobenius structures of these two rings. In particular, the quantum differential equations in A and B-model can be identified.

So far, we have two kinds of mirrors associated to a toric Calabi-Yau 3-orbifold X: the mirror curve of and the Landau-Ginzburg mirror of X. It is natural to ask whether these two B-models are related. Let the mirror curve C be defined by an equation H:

$$C = \{(X,Y)|H(X,Y) = 0\} \subset (\mathbb{C}^*)^2.$$

On the Landau-Ginzburg B-model side, we have a super potential

$$W^T : (\mathbb{C}^*)^3 \to \mathbb{C}$$

with

$$W^T = H(X,Y)Z - u_1 \log X - u_2 \log Y,$$

where $H^*(\mathcal{B}T;\mathbb{C}) = \mathbb{C}[u_1,u_2]$. This explicit expression of W^T gives us the relation between the mirror curve of and the Landau-Ginzburg mirror of X. If we view $X : C \to \mathbb{C}^*$ as a super potential on C, then the structure of the mirror curve gives us a singularity theory on C. The above relation can be viewed as the dimensional reduction from the singularity theory on $(\mathbb{C}^*)^3$ to the singularity theory on C.

The remodeling conjecture, which relates the formal generating functions of higher genus Gromov-Witten invariants of X to the analytic symmetric forms on C, seems to be mysterious at the beginning. The structure behind this conjecture is that both A-model and B-model higher genus potentials can be realized as quantizations on on two isomorphic semisimple Frobenius manifolds. With this structure in mind, the remodeling conjecture becomes natural to understand. The genus 0 mirror theorem identifies the Frobenius structure of the quantum cohomology of X and the Frobenius structure of the Jacobi ring of W^T. By the dimensional reduction, the Frobenius structure on the Landau-Ginzburg B-model reduces to the structure on the mirror curve C. In particular, we can consider the fundamental solution to the B-model quantum differential equation and the B-model $R-$matrix. The A-model and B-model $R-$matrices are then identified by comparing the ambiguities in Theorem 4.2.1. The A-model higher genus potential is obtained by Givental quantization formula developed in the previous chapters. The B-model higher genus potential is obtained by the Eynard-Orantin topological recursion on C. The bridge relating these two potentials is the graph sum formula, which is equivalent to both the quantization formula and the recursive formula. Finally, the remodeling conjecture is proved by identifying the A-model and B-model graph sums, which is a direct result of the identification of A-model and B-model $R-$matrices.

The proof of the remodeling conjecture for general toric Calabi-Yau 3-orbifolds is in [38]. In the following sections, we illustrating the strategy of the proof and ideas behind the proof. In particular, we will see that the Givental quantization formula developed in the previous chapters gives the key to relate the A-model and B-model higher genus potentials.

5.2 A-model geometry

In this section, we discuss the basis properties of toric Calabi-Yau 3-orbifolds. Then we apply the Givental quantization formula to this special case to study the higher genus Gromov-Witten potentials of toric Calabi-Yau 3-orbifolds.

5.2.1 Toric Calabi-Yau 3-orbifolds

Let X be a semi-projective toric Calabi-Yau 3-orbifold defined by a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$ [7]. Then X contains the algebraic torus $\mathbb{T} = (\mathbb{C}^*)^3$ as Zariski dense open subset, and $N = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}) \cong \mathbb{Z}^3$. We assume that there is at least one $\mathbb{T}-$fixed point on X. Let $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) = N^\vee$ be the group of irreducible characters of $\mathbb{T}$. The course moduli space of X is the simplicial toric variety X'_Σ defined by the simplicial fan Σ, and the canonical divisor $K_{X'_\Sigma}$ of X'_Σ is trivial. Let $X'_0 \coloneqq \mathrm{Spec} H^0(X', \mathcal{O}_{X'})$ be the affinization of X'. Then X'_0 is an affine Gorenstein toric variety defined by a cone $\sigma_0 \subset N_\mathbb{R} \coloneqq N \otimes_\mathbb{R} \mathbb{R}$.

In this case, we choose our torus T to be the 2-dimensional sub-torus of $\mathbb{T}$ such that T acts trivially on $\Lambda^3 TX$. Then $\mathbb{T} \cong T \times \mathbb{C}^*$ and $N \cong N' \times \mathbb{Z}$, where $N' = \mathrm{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^2$. Then σ_0 is the cone over $P \times \{1\} \subset N'_\mathbb{R} \times \mathbb{R}$, where P is a convex polytope in $N'_\mathbb{R}$ with vertices in N'. There is a one-to-one correspondence between the T-fixed points $p_1, \cdots, p_n$ in X and the 3-dimensional cones in Σ. Let $\sigma \in \Sigma(3)$ be a 3-dimensional cone and let N_σ

be the sublattice spanned by σ. Then $G_\sigma = N/N_\sigma$ is a finite abelian group and there is a one-to-one correspondence between elements in $\mathrm{Box}(\sigma) \subset N$ and elements in G_σ:

$$b \in \mathrm{Box}(\sigma) \mapsto b + N_\sigma \in N/N_\sigma = G.$$

The cone σ defines an affine toric Calabi-Yau 3-fold $X_\sigma \cong [\mathbb{C}^3/G_\sigma] \subset X$ which is a local neighborhood of p_σ.

5.2.2 A-model topological string

Recall that in Section 3.2, we have defined the double correlator

$$\langle\!\langle \mathbf{t}(\psi_1), \cdots, \mathbf{t}(\psi_k) \rangle\!\rangle_{g,k}^{X,T} := \sum_{s=0}^{\infty} \sum_{\beta \in E} \frac{Q^\beta}{s!} \langle \mathbf{t}(\psi_1), \cdots, \mathbf{t}(\psi_k), t, \cdots, t \rangle_{g,k+s,\beta}^{X,T}.$$

Then we studied the quantization formula for this descendent potential. In order to apply this general formula to the remodeling conjecture, we need to slightly modify this potential. In this subsection, we describe these modifications which are all straightforward.

First, we need to restrict the primary insertion t to the subspace $H^2_{\mathrm{CR}}(X;\mathbb{C})$, where the corresponding coordinates $\{t_i\}$ are called *kahler parameters*. Second, note that $H^2(X;\mathbb{Z})$ is dual to $H_2(X;\mathbb{Z})$, the numner Novikov variables Q_i are equal to the rank of $H^2(X;\mathbb{Z})$. When we deal with the quantum cohomology of X, we can use the divisor equation to pull out the $H^2(X;\mathbb{C})$ part of t to get the factor e^{t_i} in front of the correlator. The Novikov variable

Q_i just rescales e^{t_i}. This point of view will be used to match the Frobenius structures on A-model and B-model.

The third step is to modify the formal variable $\mathbf{t}$ in the double correlator $\langle\langle \mathbf{t}(\psi_1), \cdots, \mathbf{t}(\psi_k) \rangle\rangle_{g,k}^{X,T}$ by k *ordered* variables $\mathbf{t}_1, \cdots, \mathbf{t}_k$ with $\mathbf{t}_i(z) = \sum_{a \geq 0} \sum_{\sigma,\alpha} (t_i)_a^{\sigma\alpha} \tilde{\phi}_{\sigma\alpha} z^a$. In this case, the only change in the graph sum formula for $\langle\langle \mathbf{t}_1(\psi_1), \cdots, \mathbf{t}_k(\psi_k) \rangle\rangle_{g,k}^{X,T}$ is that we have k *ordered* ordinary leaves in the graph.

The fourth step is to modify this descendent potential to the open Gromov-Witten potential with respect to an Aganagic-Vafa brane $\mathcal{L}$ on X. This means that we consider maps from Riemann surfaces with boundaries to X such that the boundary circles are mapped to $\mathcal{L}$. Since X is toric, the open Gromov-Witten potential is related to the descendent potential in a simple way. Let $\mathcal{L} \subset \mathcal{X}$ be an Aganagic-Vafa brane. Then there is a unque 1-dimensional orbit closure $\mathfrak{l}_\tau$, where $\tau \in \Sigma(2)$, such that $\mathcal{L} \cap \mathfrak{l}_\tau$ is non-empty. The coarse moduli space ℓ_τ of $\mathfrak{l}_\tau$ is either the complex projective line $\mathbb{P}^1$ or the complex affine line $\mathbb{C}$. For simplicity, we assume that $\mathcal{L}$ is outer, i.e. $\ell_\tau \cong \mathbb{C}$. Then there is a unique $\sigma \in \Sigma(3)$ such that $\tau \subset \sigma$. Let the affine chart around p_σ be $[\mathbb{C}^3/G_\sigma]$. Since X does not have generic stabilizer, it is easy to see that there exists a short exact sequence:

$$1 \to \boldsymbol{\mu}_m \to G_\sigma \xrightarrow{\chi_1} \boldsymbol{\mu}_r \to 1.$$

where χ_1 is the representation of G_σ along the first axis of $[\mathbb{C}^3/G_\sigma]$. By localization, in the fixed loci the domain curve degenerates into a closed curve with several disks attaching to it. The disks are mapped to $\mathfrak{l}_\tau$ and the maps are of the form $z \to z^d$ for some $d \in \mathbb{Z}_{>0}$ called the *winding number*. The closed part of the curve contributes the descendent Gromov-Witten invariant with descendent markings the nodes connecting the disks. These descendent markings are of course mapped to p_σ. So we only need to replace the formal variables $(t_i)_a^{\sigma\alpha}$ in $\langle\!\langle \mathbf{t}_1(\psi_1), \cdots, \mathbf{t}_k(\psi_k) \rangle\!\rangle_{g,k}^{X,T}$ by the *disk functions* $\widetilde{\xi}_a^{\sigma,\alpha}(X_i)$, where X_i is a formal variable and $\widetilde{\xi}_a^{\sigma,\alpha}(X_i)$ is a power series of X_i. The disk function $\widetilde{\xi}_a^{\sigma,\alpha}(X_i)$ is the contribution of the $i-$th disk and the power of X_i records the winding number of the $i-$th disk. The function $\widetilde{\xi}_a^{\sigma,\alpha}(X_i)$ is an explicit function and the reader is referred to [36] for the expression of $\widetilde{\xi}_a^{\sigma,\alpha}(X_i)$. One should notice that if $m > 1$, $\mathfrak{l}_\tau$ is gerby. So in the definition of open Gromov-Witten invariants with respective to $\mathcal{L}$, we need to input the additional data of monodromies at infinity of the disks besides the data of winding numbers. Therefore, the disk function $\widetilde{\xi}_a^{\sigma,\alpha}(X_i)$ is in fact $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m)$-valued.

The last step is to introduce the framing. Consider the affine chart $[\mathbb{C}^3/G_\sigma]$ in the fourth step. Recall that there exists a short exact sequence:

$$1 \to \boldsymbol{\mu}_m \to G_\sigma \xrightarrow{\chi_1} \boldsymbol{\mu}_r \to 1.$$

where χ_1 is the representation of G_σ along the first axis of $[\mathbb{C}^3/G_\sigma]$. Let $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3$ be the three torus weights along the three axes of $[\mathbb{C}^3/G_\sigma]$ respectively. Then we do the following specialization:

$$\mathsf{w}_1 \mapsto \frac{1}{r}\mathsf{v} = w_1\mathsf{v}, \quad \mathsf{w}_2 \mapsto \frac{s+rf}{rm}\mathsf{v} = w_2\mathsf{v}, \quad \mathsf{w}_3 \mapsto \frac{-s-rf-m}{rm}\mathsf{v} = w_3\mathsf{v}.$$

with f an integer. Introducing the framing f is equivalent to the following. Consider the subtorus

$$T_f \hookrightarrow T$$

which corresponds to the character v above. Then we project the open Gromov-Witten potential to the fractional field of $H^*_{\mathrm{tw}}(\mathcal{B}T_f;\mathbb{C})$. Then as priori, our open Gromov-Witten potential is defined over $\mathbb{C}(\mathsf{v})$. But since our primary insertions are restricted to $H^2_{\mathrm{tw}}(X;\mathbb{C})$, the open Gromov-Witten potential is in fact independent of v.

After the above five steps, we obtain an open Gromov-Witten potential $F^{\mathcal{X},(\mathcal{L},f)}_{g,n}(t;X_1,\ldots,X_n)$ from the descendent potential $\langle\!\langle \mathbf{t}(\psi_1),\cdots,\mathbf{t}(\psi_k)\rangle\!\rangle^{X,T}_{g,k}$. We still have the graph sum formula for $F^{\mathcal{X},(\mathcal{L},f)}_{g,n}(t;X_1,\ldots,X_n)$ as in Section 4.2.2. The difference is that we have k ordered ordinary leaves and the formal variables $(t_i)^{\sigma\alpha}_a$ are replaced by the *disk functions* $\widetilde{\xi}^{\sigma,\alpha}_a(X_i)$. This replacement only changes the ordinary leaves in the graph sum formula.

5.3 B-model geometry

5.3.1 Mirror curve and dimensional reduction of the Landau-Ginzburg model

Recall that in Section 5.2.1, we have a polytope P in $N'_{\mathbb{R}}$ with vertices in N'. If we choose an isomorphism $N' \cong \mathbb{Z}^2$, then the mirror curve is defined by

$$\{(X,Y) \in (\mathbb{C}^*)^2 : H(X,Y) = 0\}$$

where

$$H(X,Y) = X^r Y^{-s-rf} + Y^m + 1 + \sum_{a=1}^{p} q_a X^{m_a} Y^{n_a - m_a f}$$

$$P \cap \mathbb{Z}^2 = \{(r,-s),(0,m),(0,0)\} \cup \{(m_a,n_a) : a = 1,\dots,p\}, \quad p = \dim_{\mathbb{C}} H^2_{\mathrm{CR}}(\mathcal{X}).$$

Here $q_1,\cdots,q_p$ are complex parameters on B-model. If we let $t_1,\cdots,t_p$ be coordinates of $H^2_{\mathrm{CR}}(\mathcal{X})$, which means that $t_1,\cdots,t_p$ are Kahler parameters on A-model. Then $q_1,\cdots,q_p$ and $t_1,\cdots,t_p$ are related by closed mirror maps [35] [38].

On the other hand, The T-equivariant mirror of X is a Landau-Gizburg model $((\mathbb{C}^*)^3, W^T)$, where $W^T : (\mathbb{C}^*)^3 \to \mathbb{C}$ is the T-equivariant superpotential

$$W^T = H(X,Y)Z - \log X$$

which is multi-valued. The differential

$$dW^T = \frac{\partial W^T}{\partial X} dX + \frac{\partial W^T}{\partial Y} dY + \frac{\partial W^T}{\partial Z} dZ = Z dH + H dZ - \frac{dX}{X}$$

is a well-defined holomorphic 1-form on $(\mathbb{C}^*)^3$.

We have

$$\begin{aligned} \frac{\partial W^T}{\partial X}(X,Y,Z) &= Z\frac{\partial H}{\partial X}(X,Y) - \frac{1}{X} \\ \frac{\partial W^T}{\partial Y}(X,Y,Z) &= Z\frac{\partial H}{\partial Y}(X,Y) \\ \frac{\partial W^T}{\partial Z}(X,Y,Z) &= H(X,Y) \end{aligned}$$

Therefore,

$$\frac{\partial W^T}{\partial X} = 0, \quad \frac{\partial W^T}{\partial Y} = 0, \quad \frac{\partial W^T}{\partial Z} = 0$$

are equivalent to

$$H(X,Y) = 0, \quad \frac{\partial H}{\partial X}(X,Y) = -\frac{1}{Z}\frac{\partial x}{\partial X}, \quad \frac{\partial H}{\partial Y}(X,Y) = 0$$

where $x = -\ln X$. Therefore, the critical points of $W^T(X,Y,Z)$, which are zeros of the holomorphic differential dW^T on $(\mathbb{C}^*)^3$, can be identified with the critical points of p, which are zeros of the holomorphic differential

$$dx = -\frac{dX}{X}$$

on the mirror curve

$$\Sigma_q = \{(X,Y) \in (\mathbb{C}^*)^2 : H(X,Y) = 0\}.$$

Define $I_\Sigma := \{(\sigma,\alpha) : \sigma \in \Sigma(3), \alpha \in G_\sigma^*\}$. Then there is a bijection between the zeros of dW^T and the set I_Σ. Let $P_{\sigma,\alpha} \in (\mathbb{C}^*)^3$ be the zero of dW^T associated to $(\sigma,\alpha) \in I_\Sigma$ and let $p_{\sigma,\alpha} \in \Sigma_q$ be the corresponding critical points of x on Σ_q.

The Jacobian ring of W^T is

$$\mathrm{Jac}(W^T) := \frac{\mathbb{C}[X,X^{-1},Y,Y^{-1},Z,Z^{-1}]}{\langle \frac{\partial W^T}{\partial X}, \frac{\partial W^T}{\partial Y}, \frac{\partial W^T}{\partial Z}\rangle} \cong H_B := \frac{\mathbb{C}[X,X^{-1},Y,Y^{-1}]}{\langle H(X,Y), -Y\frac{\partial H}{\partial Y}(X,Y)\rangle}$$

There is pairing on $\mathrm{Jac}(W^T)$:

$$(f,g) := \frac{-1}{(2\pi\sqrt{-1})^3}\int_{|dW^T|=\epsilon} \frac{fg dx \wedge dy \wedge dz}{\frac{\partial W^T}{\partial x}\frac{\partial W^{\mathbb{T}'}}{\partial y}\frac{\partial W^T}{\partial z}}$$

where $x = -\ln X, y = -\ln Y, z = -\ln Z$.

5.3.2 The Liouville form

Let

$$\lambda = y dx.$$

be the Liouville form on $\mathbb{C}^2$. Then $d\lambda = -dx \wedge dy$. Let

$$\Phi := \lambda|_{\Sigma_q},$$

5.3.3 Lefshetz thimbles

Around critical points $p_{\sigma,\alpha}$ of $x:\Sigma_q\to\mathbb{C}$, we have have

$$
\begin{aligned}
x &= \check{u}^{\sigma,\alpha}+\zeta^2\\
y &= \check{v}^{\sigma,\alpha}+\sum_{d=1}^{\infty}h_d^{\sigma,\alpha}\zeta^d
\end{aligned}
$$

where

$$h_1^{\sigma,\alpha}=\sqrt{\frac{2}{\frac{d^2x}{dy^2}(\check{v}^{\sigma,\alpha})}}$$

Let $\Gamma_{\sigma,\alpha}$ be the Lefshetz thimble of the superpotential $x:\Sigma_q\to\mathbb{C}$ such that $x(\Gamma_{\sigma,\alpha})=\check{u}^{\sigma,\alpha}+\mathbb{R}_{\geq 0}$. Then $\{\Gamma_{\sigma,\alpha}:(\sigma,\alpha)\in I_\Sigma\}$ is a basis of the relative homology group $H_1(\Sigma_q,\{\hat{x}\gg 0\})$.

5.3.4 Differentials of the second kind

Let $B(p_1,p_2)$ be the fundamental normalized differential of the second kind on $\bar{\Sigma}_q$ (see e.g. [33]), where $\bar{\Sigma}_q$ is the compactification of Σ_q. It is also called the Bergman kernel in [32].

Remark 5.3.1. *The compactification of Σ_q is a little bit subtle. Recall that we have a polytope P which appears in the definition of X and the equation $H(X,Y)$ of the mirror curve. The polytope P determines a toric surface S together with an ample line bundle L on S. The function $H(X,Y)$ extends uniquely to a section of L and the zero locus of this section in S gives us the compactification of Σ_q (see [38] for more details).*

Following [31], given $\alpha = 1, 2$ and $d \in \mathbb{Z}_{\geq 0}$, define

$$d\xi_{\sigma,\alpha,d}(p) := (2d-1)!!2^{-d}\mathrm{Res}_{p' \to p_{\sigma,\alpha}} B(p,p')\zeta_{\sigma,\alpha}^{-2d-1}.$$

Then $d\xi_{\sigma,\alpha,d}$ satisfies the following properties.

1. $d\xi_{\sigma,\alpha,d}$ is a meromorphic 1-form on $\bar{\Sigma}_q$ with a single pole of order $2d+2$ at $p_{\sigma,\alpha}$.

2. In local coordinate $\zeta = \sqrt{x - \check{u}^{\sigma,\alpha}}$ near $p_{\sigma,\alpha}$,

$$d\xi_{\sigma,\alpha,d} = \Big(\frac{-(2d+1)!!}{2^d \zeta^{2d+2}} + f(\zeta)\Big)d\zeta,$$

   where $f(\zeta)$ is analytic around $p_{\sigma,\alpha}$. The residue of $d\xi_{\sigma,\alpha,d}$ at $p_{\sigma,\alpha}$ is zero, so $d\xi_{\sigma,\alpha,d}$ is a differential of the second kind.

The meromorphic 1-form $d\xi_{\sigma,\alpha,d}$ is characterized by the above properties; $d\xi_{\sigma,\alpha,d}$ can be viewed as a section in $H^0(\bar{\Sigma}_q, \omega_{\bar{\Sigma}_q}((2d+2)p_{\sigma,\alpha}))$.

Define

$$\hat{\xi}_{\sigma,\alpha,k} := (-1)^k(\frac{d}{dx})^{k-1}(\frac{d\xi_{\sigma,\alpha,0}}{dx}) = (X\frac{d}{dX})^{k-1}(X\frac{d\xi_{\sigma,\alpha,0}}{dX}), \tag{5.1}$$

which is a meromorphic function on $\bar{\Sigma}_q$. We also define $d\hat{\xi}_{\sigma,\alpha,0} := d\xi_{\sigma,\alpha,0}$. Then we have the following lemma (see [36])

Lemma 5.3.2.

$$d\xi_{\sigma,\alpha,k} = d\hat{\xi}_{\sigma,\alpha,k} - \sum_{i=0}^{k-1} \sum_{(\rho,\beta)\in I_\Sigma} \check{B}^{\sigma,\alpha,\rho,\beta}_{k-1-i,0} d\hat{\xi}_{\rho,\beta,i}.$$

where $\check{B}^{\sigma,\alpha,\beta}_{k-1-i,0}$ *is defined in Section 5.3.7.*

5.3.5 Oscillating integrals and the B-model S-matrix

Given $(\sigma,\alpha) \in I_\Sigma$, there exists $\widetilde{\Gamma}_{\sigma,\alpha} \in H^3((\mathbb{C}^*)^3, \{\Re(\frac{W^T}{z} \ll 0\}; \mathbb{Z})$ such that

$$I_{\sigma,\alpha} := \int_{\widetilde{\Gamma}_{\sigma,\alpha}} e^{\frac{W^T}{z}} \frac{dX}{X}\frac{dY}{Y}\frac{dZ}{Z} = 2\pi\sqrt{-1}z \int_{\Gamma_{\sigma,\alpha}} e^{-x/z}\Phi$$

where $\Phi = ydx$. There exists a differential operator $D_{\sigma,\alpha}$ on the closed string moduli such that the flat basis $\tilde{\phi}_{\sigma,\alpha}$ corresponds to $D_{\sigma,\alpha}W^T$ under the isomoprhism $H^*_{\mathrm{CR},T}(X) \cong \mathrm{Jac}(W^T)$. Then for any $\Gamma \in H_1(\Sigma_q, \{x \geq 0\})$,

$$\int_\Gamma e^{-x/z} D_{\sigma',\alpha'}\Phi = \sum_{(\sigma,\alpha)\in I_\Sigma} \Psi_{\sigma',\alpha'}{}^{\sigma,\alpha} \int_\Gamma e^{-x/z} \frac{d\xi_{\sigma,\alpha,0}}{\sqrt{-2}}$$

Define

$$\check{S}_{\sigma',\alpha'}{}^{\sigma,\alpha} = \int_{\Gamma_{\sigma,\alpha}} e^{-x/z} D_{\sigma,\alpha}\Phi$$

The matrix $\check{S}$ plays the role of the fundamental solution of the B-model quantum differential equation.

5.3.6 The f-matrix and the B-model R-matrix

Let

$$f_{\rho,\delta}{}^{\sigma,\alpha}(u) = \frac{e^{u\check{u}^{\sigma,\alpha}}}{2\sqrt{\pi u}} \int_{\Gamma_{\sigma,\alpha}} e^{-ux} d\xi_{\rho,\delta,0}$$

and let

$$\check{R}_{\rho,\delta}{}^{\sigma,\alpha}(z) = f_{\rho,\delta}{}^{\sigma,\alpha}(\frac{-1}{z})$$

Then by [38, 39], we have the following asymptotic expansion

$$\check{S}_{\sigma',\alpha'}{}^{\sigma,\alpha} = \frac{\sqrt{-2\pi}}{z}(\Psi\check{R})_{\sigma',\alpha'}{}^{\sigma,\alpha} e^{-\check{u}^{\alpha,\sigma}/z}.$$

5.3.7 The Eynard-Orantin topological recursion and the B-model graph sum

Let $\omega_{g,n}$ be defined recursively by the Eynard-Orantin topological recursion [32]:

$$\omega_{0,1} = 0, \quad \omega_{0,2} = B(p_1, p_2).$$

When $2g - 2 + n > 0$,

$$\begin{aligned}\omega_{g,n}(p_1,\dots,p_n) &= \sum_{(\sigma,\alpha)\in I_\Sigma} \mathrm{Res}_{p\to p_{\sigma,\alpha}} \frac{\int_{\xi=p}^{\bar{p}} B(p_n,\xi)}{2(\Phi(p)-\Phi(\bar{p}))}\Big(\omega_{g-1,n+1}(p,\bar{p},p_1,\dots,p_{n-1}) \\ &\quad + \sum_{g_1+g_2=g} \sum_{\substack{I\cup J=\{1,\dots,n-1\}\\ I\cap J=\emptyset}} \omega_{g_1,|I|+1}(p,p_I)\omega_{g_2,|J|+1}(\bar{p},p_J)\end{aligned}$$

Following [30], the B-model invariants $\omega_{g,n}$ are expressed in terms of graph sums. Given a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(\mathcal{X})$ with $L^o(\Gamma) = \{l_1,\dots,l_n\}$, we define its weight to be

$$\begin{aligned} w_B(\vec{\Gamma}) &= (-1)^{g(\vec{\Gamma})-1+n} \prod_{v\in V(\Gamma)} \Big(\frac{h_1^{i(v)}}{\sqrt{-2}}\Big)^{2-2g-\mathrm{val}(v)} \langle \prod_{h\in H(v)} \tau_{k(h)}\rangle_{g(v)} \prod_{e\in E(\Gamma)} \check{B}^{i(v_1(e)),i(v_2(e))}_{k(e),l(e)} \\ &\cdot \prod_{j=1}^{n} \frac{1}{\sqrt{-2}} d\xi^{i(l_j)}_{k(l_j)}(Y_j) \prod_{l\in\mathcal{L}^1(\Gamma)} (-\frac{1}{\sqrt{-2}}) \check{h}^{i(l)}_{k(l)}. \end{aligned}$$

Here, for any $i, j \in I_\Sigma$,

$$B(p_1,p_2) = \Big(\frac{\delta_{i,j}}{(\zeta_i-\zeta_j)^2} + \sum_{k,l\in\mathbb{Z}_{\geq 0}} B^{i,j}_{k,l}\zeta_i^k\zeta_j^l\Big) d\zeta_i d\zeta_j.$$

We define

$$\check{B}^{i,j}_{k,l} = \frac{(2k-1)!!(2l-1)!!}{2^{k+l+1}} B^{i,j}_{2k,2l}. \tag{5.2}$$

And we define

$$\check{h}^i_k = 2(2k-1)!! h^i_{2k-1}.$$

We have the following property for $\check{B}^{i,j}_{k,l}$:

$$\begin{aligned} \check{B}^{i,j}_{k,l} &= [u^{-k}v^{-l}]\left(\frac{uv}{u+v}(\delta_{i,j} - \sum_{\gamma\in I_\Sigma} f^i_\gamma(u) f^j_\gamma(v))\right) \\ &= [z^k w^l]\left(\frac{1}{z+w}(\delta_{i,j} - \sum_{\gamma\in I_\Sigma} f^i_\gamma(\frac{1}{z}) f^j_\gamma(\frac{1}{w}))\right). \end{aligned}$$

In our notation [30, Theorem 3.7] is equivalent to:

Theorem 5.3.3 (Dunin-Barkowski–Orantin–Shadrin–Spitz [30])**.** *For $2g - 2 + n > 0$,*

$$\omega_{g,n} = \sum_{\Gamma\in\mathbf{\Gamma}_{g,n}(\mathcal{X})} \frac{w_B(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

5.4 The remodeling conjecture: all genus open-closed mirror symmetry

In this section, we sketch the proof of the remodeling conjecture for toric Calabi-Yau 3-orbifolds. The detail of the proof is contained in [38].

5.4.1 Identification of fundamental solutions of A-model and B-model quantum differential equations

In [22], the genus 0 mirror theorem of any semi-projective toric orbifolds is proved. It implies that the quantum cohomology ring of X is isomorphic to the Jacobi ring of W^T as Frobenius algebras. This isomorphism is of course under the closed mirror map between the Kahlar parameters $t_1, \cdots, t_p$ and complex parameters $q_1, \cdots, q_p$. Under this isomorphism $h_1^{\sigma,\alpha} = \sqrt{\frac{2}{\frac{d^2x}{dy^2}(\check{v}^{\sigma,\alpha})}}$ is identified with $\sqrt{\frac{-2}{\Delta^{\sigma,\alpha}}}$. This isomorphism implies the following theorem which is proved in [38].

Theorem 5.4.1. *Under the closed mirror map and the isomorphism between the quantum cohomology ring of X and the Jacobi ring of W^T, the matrix $(\check{S}_{\sigma',\alpha'}{}^{\sigma,\alpha})$ in Section 5.3.5 is a fundamental solution of the quantum differential equation $\nabla S = 0$ in Section 3.4.*

Since we are working with non-conformal Frobenius manifolds, the solution of the quantum differential equation is not unique. The ambiguity is fixed by the following theorem which is proved in [38, 36].

Theorem 5.4.2.

$$\tilde{S}|_{t=0,Q=0} = \check{S}|_{q=0}$$

where $\tilde{S}$ is the A-model fundamental solution.

In the proof of this theorem, the A-model fundamental solution is computed by Theorem 4.3.2.

By the above two theorems and and Theorem 4.2.1 we can identify the A-model and B-model $R-$matrices.

$$\tilde{R}(z) = \check{R}(z).$$

Here we use the asymptotic expansion

$$\check{S}_{\sigma',\alpha'}{}^{\sigma,\alpha} = \frac{\sqrt{-2\pi}}{z}(\Psi\check{R})_{\sigma',\alpha'}{}^{\sigma,\alpha}e^{-\check{u}^{\alpha,\sigma}/z}.$$

5.4.2 Identification of graph sums

In this subsection, we prove the remodeling conjecture by identifying the graph sums on A-model and B-model. This shows that the A-model and B-model higher genus potentials are quantizations on two isomorphic semi-simple Frobenius manifolds.

Define the A-model graph weight $w_A(\Gamma)$ to be the graph weight $\mathring{w}(\Gamma)$ in Section 4.2.2 after the modification described in Section 5.2.2. For $l = 1, \cdots, k$

and $i \in I_\Sigma$, let

$$\tilde{\mathbf{u}}_l^i(z) = \sum_{a\geq 0} (\tilde{u}_l)_a^i z^a := \sum_{\mu,\nu=1,\cdots,N} (\sum_{b\geq 0} \widetilde{\xi}_b^\mu(X_l) z^b) S_t(z)^\nu{}_\mu \Psi_\nu{}^i.$$

The identification $\tilde{R}(z) = \check{R}(z)$ implies the following theorem:

Theorem 5.4.3. *For any $\vec{\Gamma} \in \mathbf{\Gamma}_{g,k}(X)$,*

$$w_B(\vec{\Gamma})|_{\frac{1}{\sqrt{-2}} d\hat{\xi}_{i,a}(p_l) = (\tilde{u}_l)_a^\tau} = (-1)^{g(\vec{\Gamma})-1+k} w_A(\vec{\Gamma}),$$

under the closed mirror map.

Proof. 1. Vertex. By the discussion in Section 5.4.1, $h_1^i = \sqrt{\frac{-2}{\Delta^i}}$ for any $i \in I_\Sigma$. So in the model vertex term, $\frac{h_1^i}{\sqrt{-2}} = \sqrt{\frac{1}{\Delta^i}}$. Therefore the B-model vertex matches the A-model vertex.

2. Edge. By the property for $\check{B}_{a,b}^{i,j}$,

$$\begin{aligned} \check{B}_{a,b}^{i,j} &= [u^{-a}v^{-b}]\left(\frac{uv}{u+v}(\delta_{i,j} - \sum_{\gamma\in I_\Sigma} f_\gamma^i(u) f_\gamma^j(v))\right) \\ &= [z^a w^b]\left(\frac{1}{z+w}(\delta_{i,j} - \sum_{\gamma\in I_\Sigma} f_\gamma^i(\frac{1}{z}) f_\gamma^j(\frac{1}{w}))\right). \end{aligned}$$

Therefore, the identification $\tilde{R}(z) = \check{R}(z) = (f_j^i(-\frac{1}{z}))$ gives us

$$\check{B}_{a,b}^{i,j} = \mathcal{E}_{a,b}^{i,j}.$$

3. Ordinary leaf. By Lemma 5.3.2, we have the following expression for

$d\xi_{i,a}$:

$$d\xi_{i,a} = d\hat{\xi}_{i,a} - \sum_{c=0}^{a-1} \sum_{j \in I_\Sigma} \check{B}^{i,j}_{a-1-c,0} d\hat{\xi}_{j,c}.$$

By item 2 (Edge) above, for $a, b \in \mathbb{Z}_{\geq 0}$,

$$\check{B}^{i,j}_{a,b} = [z^a w^b] \left(\frac{1}{z+w} (\delta_{i,j} - \sum_{\gamma \in I_\Sigma} \tilde{R}_\gamma{}^i(-z) \tilde{R}_\gamma{}^j(-w)) \right).$$

We also have

$$[z^0](\tilde{R}_j{}^i(-z)) = \delta_{i,j}.$$

Therefore,

$$d\xi_{i,a} = \sum_{c=0}^{a} \sum_{j \in I_\Sigma} ([z^{a-c}](\tilde{R}_j{}^i(-z)) d\hat{\xi}_{j,c}.$$

So under the identification

$$\frac{1}{\sqrt{-2}} d\hat{\xi}_{i,a}(p_l) = (\tilde{u}_l)^\tau_a.$$

The B-model ordinary leaf matches the A-model ordinary leaf

4. Dilaton leaf. We have the following relation between $\check{h}^i_a$ and $f^i_j(u)$ (see [36])

$$\check{h}^i_a = [u^{1-k}] \sum_{j \in I_\Sigma} h^j_1 f^i_j(u).$$

By the relation

$$\tilde{R}_j{}^i(z) = f^i_j(-\frac{1}{z})$$

and the fact $h^i_1 = \sqrt{\frac{-2}{\Delta^i}}$, it is easy to see that the B-model dilaton leaf matches the A-model dilaton leaf.

□

Theorem 5.4.3 almost proves the remodeling conjecture. The only thing remaining is to expand $d\hat{\xi}_{i,a}(p_l)$ at suitable points on $\bar{\Sigma}_q$ to recover the information of $\tilde{\mathbf{u}}_l^i(z) = \sum_{a\geq 0}(\tilde{u}_l)_a^i z^a = \sum_{\mu,\nu=1,\cdots,N}(\sum_{b\geq 0}\widetilde{\xi}_b^\mu(X_l)z^b)S_t(z)^\nu{}_\mu\Psi_\nu{}^i$. We sketch this process now and the details are contained in [36, 38].

Recall that our Aganagic-Vafa brane lies at $\mathfrak{l}_\tau$ which is a $\mathbb{Z}_m$ gerbe and the disk functions $\widetilde{\xi}_a^\mu(X_l), l = 1,\cdots,k, \mu \in I_\Sigma$ are $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m)$-valued. On the B-model side, there are m punctures $\bar{p}_1,\cdots,\bar{p}_m$ of the mirror curve $\bar{\Sigma}_q$ with $X(\bar{p}_1) = \cdots = X(\bar{p}_m) = 0$. Let D^ℓ be an open neighborhood of $\bar{p}_\ell \in X^{-1}(0)$, $\ell = 0,\ldots,m-1$ such that the restriction

$$X^\ell : D^\ell \to D_\delta = \{X \in \mathbb{C} : |X| < \delta\},$$

gives an isomorphism. Define

$$\rho^{\ell_1,\ldots,\ell_n} := (X^{\ell_1})^{-1} \times \cdots \times (X^{\ell_n})^{-1} : (D_\delta)^n \to D^{\ell_1} \times \cdots \times D^{\ell_n} \subset (\bar{\Sigma}_q)^n.$$

Define

$$\psi_\ell := \frac{1}{m}\sum_{k=0}^{m-1}\omega_m^{-k\ell}\mathbf{1}_{\frac{k}{m}}, \quad \ell = 0,1,\ldots,m-1,$$

where $\omega_m = e^{2\pi\sqrt{-1}/m}$ and $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m) = \bigoplus_{k=0}^{m-1}\mathbb{C}\mathbf{1}_{\frac{k}{m}}$ is the natural decomposition with respect to the components of $\mathcal{IB}\mathbb{Z}_m$. Let

$$d\hat{\xi}_{i,a}(X'_l) = \sum_{\ell\in\mathbb{Z}_m} \Big(\int_0^{X'_l} (\rho^\ell)^*(d\hat{\xi}_{i,a}) \Big) \psi_\ell.$$

The reason why we use the notation X'_l is that there is an *open mirror map* [35, 38] $U : (X_1, \cdots, X_k) \mapsto (X'_1, \cdots, X'_k)$ where $(X_1, \cdots, X_k)$ are the formal variables on A-model recording the winding numbers. The following lemma is proved in [38]

Lemma 5.4.4. *Under the open mirror map, we have*

$$(\tilde{u}_l)^\tau_a(X_l) = \frac{1}{|G_\sigma|\sqrt{-2}} d\hat{\xi}_{i,a}(X'_l).$$

For $2g - 2 + k > 0$, define

$$W_{g,n}(q; X'_1, \ldots, X'_k) := \sum_{\ell_1,\ldots,\ell_k\in\mathbb{Z}_m} \int_0^{X'_1} \cdots \int_0^{X'_k} (\rho_q^{\ell_1,\ldots,\ell_k})^* \omega_{g,k} \cdot \psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_k},$$

which takes values in $H^*_{\mathrm{CR}}(\mathcal{B}\boldsymbol{\mu}_m; \mathbb{C})^{\otimes k}$. Combining Theorem 5.4.3 and Lemma 5.4.4, we finally obtain the following theorem

Theorem 5.4.5 (All genus open-closed mirror symmetry)**.** *Under the open and closed mirror maps,*

$$W_{g,k}(q, X'_1, \ldots, X'_k) = (-1)^{g-1+k} |G_\sigma|^k F_{g,k}^{X,(\mathcal{L},f)}(t; X_1, \ldots, X_k).$$

for $2g - 2 + k > 0$.

Remark 5.4.6. *1. For* $(g,k) = (0,1),(0,2)$*, one can still obtain the above theorem of mirror symmetry by studying the genus 0 data directly, see [35, 36, 38].*

2. One can also generalized the above theorem to the case in which we have several Aganagic-Vafa branes (outer or inner).

3. By taking the oscillating integral of $\omega_{g,k}$ *over suitable Lefschetz thimbles, one can obtain the descendent version of the above theorem: the oscillating integral of* $\omega_{g,k}$ *equals to the descendent potential of* X*, see [38].*

5.4.3 Generalization to the multi-branes case

Theorem 5.4.5 can be easily generalized to the case when several Aganagic-Vafa branes are put in X. In order to do this, the most convenient way is to consider the open Gromov-Witten potential before the introduction of framing and meanwhile consider the general Landau-Ginzburg B-model which we will introduce below.

The general T-equivariant Landau-Ginzburg mirror of X contains a super potential

$$W^T = H(X,Y)Z - u_1 \log X - u_2 \log Y$$

where u_1, u_2 span $H^2(\mathcal{B}T,\mathbb{C})$.

For the mirror curve of X, instead of considering the framed mirror curve, we consider the curve defined by

$$\{(X,Y) \in (\mathbb{C}^*)^2 : H(X,Y) = 0\},$$

where

$$H(X,Y) = X^r Y^{-s} + Y^m + 1 + \sum_{a=1}^{p} q_a X^{m_a} Y^{n_a}.$$

Then we consider the meromorphic function $\hat{x} = u_1 x + u_2 y$ instead of x. With this definition, the dimensional reduction process still works and we get a B-model higher genus potential $\omega_{g,k}$ which depends on u_1, u_2.

By specializing u_1, u_2 to some multiples of v (which is equivalent to introducing the subtorus $T_f \hookrightarrow T$), we get a particular $\hat{x}$ which depends on the framing f. The advantage of this point of view is that it is convenient to use different local coordinates on the mirror of X to expand $\omega_{g,k}$. These local coordinates correspond to the Aganagic-Vafa branes and the framing of X. So for each brane $\mathcal{L}$ and each framing f, we get a corresponding $\hat{x}$ and we use this particular $\hat{x}$ to expand those variables in $\omega_{g,k}$ which correspond to the boundary circles mapping to $\mathcal{L}$. After this expansion, we get a B-model potential $W_{g,k}(q, X_1', \ldots, X_k')$ and the we have a generalization of Theorem 5.4.5 (see [38] for more details).

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