Towards a definition of Shimura curves in positive characteristics

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Abstract

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In the thesis, we present some answers to the question

What is an appropriate definition of Shimura curves in positive characteristics ?

The answer is obvious for Shimura curves of PEL type due to the moduli interpretation. Thus what is more interesting is the answer on Shimura curves of Hodge type.

Inspired by an example constructed by David Mumford, we find conditions on a proper smooth curve over a field of positive characteristic which guarantee that it lifts to a Shimura curve of Hodge type over the complex numbers. These conditions are in terms of geometry mod p, such as Barsotti-Tate groups, Dieudonne isocrystals, crystalline Hodge cycles and l-adic monodromy. Thus one can take them as definitions of Shimura curves in positive characteristics. More generally, We define "weak" Shimura curves in characteristic p.

Along the way, we prove if a Barsotti-Tate group is versally deformed over a proper curve over an algebraically closed field of positive characteristic, then it admits a unique deformation to the corresponding Witt ring. This deformation result serves as one of the key ingredients in the proofs.

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Chapter 1

Introduction

One of the central questions in number theory is to solve Diophantine equations, i.e., to study the common solutions of polynomials in the field of rational numbers, or more generally, in number fields. From the perspective of algebraic geometry, the zero locus of a set of polynomial equations defines an *algebraic variety* and a solution is just a point on the variety. Therefore it is important to study algebraic varieties over number fields. Among them is a special class of algebraic varieties— the Shimura varieties.

While classical Shimura varieties are defined over number fields, there is no definition of Shimura varieties in positive characteristics. In this thesis, we concentrate on Shimura curves and study the geometry of their reductions in positive characteristics. Specifically, we find conditions on a proper smooth curve over a field of positive characteristic which guarantee that it lifts to a Shimura curve of Hodge type over the complex field. These conditions are in terms of geometry mod p and thus they shed light on a (geometrical) definition of Shimura curves in positive characteristics.

Shimura varieties

The theory of Shimura varieties was established by Shimura in 1950s, later developed by Langlands, and has now become a central part of arithmetic geometry. It also plays a central role in the theory of automorphic forms and Galois representations. For instance, a Shimura variety is attached to certain linearly reductive group G over $\mathbb{Q}$, and the geometry of the Shimura variety is closely related to the theory of automorphic representations on G. Meanwhile, by studying the l-adic cohomology of the Shimura variety, one obtains Galois representations which relates to the previously mentioned automorphic representation in the manner predicted by Langlands conjecture. Harris and Taylor [19] use this philosophy to confirm the local Langlands conjecture for $GL(n)$.

We briefly state the definition of Shimura varieties. Let $\mathbb{S}$ be the Weil restriction of scalars $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. A Shimura datum (G, S) consists of a reductive group G defined over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugate class S of a cocharacter $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ such that

1. For any h in S, the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{R}}$, viewed as a conjugation representation of $\mathbb{S}$ via h, has the type $(1, -1), (0, 0)$ and $(-1, 1)$.

2. The adjoint action of $h(i)$ induces a Cartan involution on the adjoint group of $G_{\mathbb{R}}$.

3. The adjoint group of G does not have a factor H defined over $\mathbb{Q}$ such that the projection of h on H is trivial.

For a compact open subgroup K of $G(\mathbb{A}_f)$,

$$Sh_K(G, X) = G(\mathbb{Q})\backslash S \times G(\mathbb{A}_f)/K$$

is a complex algebraic variety. Take the inverse limit over K and the inverse system is called a *Shimura variety.*

A large class of Shimura varieties admits an interpretation as moduli of certain polarized abelian varieties. For instance, *Shimura varieties of PEL type* parametrize polarized abelian varieties with a prescribed endomorphism ring and polarization. For PEL type, we can take the moduli description as an equivalent definition of Shimura varieties.

Shimura varieties in positive characteristics

Though Shimura varieties in positive characteristics provide test cases for many conjectural relations among Galois representations, automorphic forms and others, there is no definition of Shimura varieties in positive characteristics. To study them, the main tool is the theory of integral models of Shimura varieties which was developed by Milne [34] [35], Kottwitz [27], Adrian Vasiu [46], Mark Kisin [26] and many others.

Such a definition can be given for Shimura varieties of PEL type. See for example the work of Kottwitz [27] and Milne [34]. Shimura varieties of PEL type have a natural moduli interpretation: polarized abelian varieties with certain endomorphism algebras. We can state the moduli problem over a field of positive characteristic as well and take the solution to be the PEL type Shimura varieties in characteristic p.

However, for Shimura varieties of Hodge type, we can not use the same strategy. Though it also has a natural moduli interpretation, there is no Hodge theory in positive characteristics and thus the moduli interpretation has no natural translation to positive characteristics. Via the work of Kisin [26], we are able to define the integral canonical model of Shimura varieties of Hodge type and also a universal family over the integral model. So one could work with the reduction of an integral canonical model at a prime $\mathfrak{p}$. The disadvantage is that it is not intrinsic in characteristic p. Our goal is to find a characterization in terms of geometry mod p of such a reduction. We are motivated by a remarkable example constructed by Mumford.

Shimura curves of Mumford type

Let K be a totally real field of degree $m+1$ and D be a quaternion division algebra over K which splits only at one real place, i.e. $D\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{H}\times\cdots\times\mathbb{H}\times M_2(\mathbb{R})$. Note there is a corestriction map $\mathrm{Cor}:\mathrm{Br}(K)\longrightarrow\mathrm{Br}(\mathbb{Q})$. Further we assume that $\mathrm{Cor}_{K/\mathbb{Q}}(D)=M_{2^{m+1}}(\mathbb{Q})$.

Let $^-$ be the standard involution of D, and let $Q=\{x\in D^*|x\bar{x}=1\}$. Then Q is a simple algebraic group over $\mathbb{Q}$ which is the $\mathbb{Q}-$form of the real algebraic group $SU(2)^{\times m}\times SL(2,\mathbb{R})$. Since $\mathrm{Cor_{K/\mathbb{Q}}(D)=M_{2^{m+1}}(\mathbb{Q})}$, Q admits a natural 2^{m+1} dimensional rational representation V whose real form is

$$\rho: SU(2)^{\times m}\times SL(2)\longrightarrow SO(2^m)\times SL(2) \text{ acting on } \mathbb{R}^{2^{m+1}}.$$

Note $Q_{\mathbb{C}}=SL(2,\mathbb{C})^{\times m+1}$. Then $V_{\mathbb{C}}$ is the tensor of $m+1$ copies of standard representation $\mathbb{C}^2$ of $SL(2,\mathbb{C})$:

$$V_C=\mathbb{C}^2\otimes\mathbb{C}^2\cdots\otimes\mathbb{C}^2\quad(m+1\text{ factors})\tag{1.1}$$

Let

$$\begin{aligned}h:\mathbb{S}_m(\mathbb{R})\longrightarrow&Q(\mathbb{R})\\ e^{i\theta}\mapsto&I_{2^m}\otimes\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}.\end{aligned}$$

Then (Q, h) defines a Shimura datum. So is $(\rho(Q), \rho \circ h)$. Generically $\rho(Q)$ is the Hodge group of V.

Let $\Gamma \subset Q_{\mathbb{R}}$ be an arithmetic subgroup such that Γ acts freely and properly discontinuous on $\mathfrak{h}$. Note $\ker \rho \subset Z(Q)$ and then it fixes h, $\Gamma \hookrightarrow \rho(Q(\mathbb{R}))$.

We call the Shimura curves corresponding to the datum (Q, h) *the Shimura curves of Mumford type* (or *Mumford curves*, for simplicity). If we don't specify level structures, Mumford curves mean a tower of Shimura curves, and any two such curves have a common finite étale covering.

Induced from the representation V, a Mumford curve M admits a family of principally polarized abelian varieties of dimension 2^m. We denote the family as $A \longrightarrow M$.

By [47, Theorem 0.5], we know any family of polarized abelian varieties over a smooth proper curve with trivial endomorphisms and maximal Higgs field is a Mumford curve, up to taking powers and isogeny.

Main results

Let k be an algebraically closed field of characteristic $p > 2$ and C be a proper smooth curve over k with genus $g(C) > 1$. Let $\Pi : \mathscr{X} \longrightarrow C$ be a nonisotrivial family of 2^m dimensional principally polarized abelian varieties over C. Let $\mathscr{E}$ be the Dieudonne crystal $R^1\Pi_{\mathrm{cris},*}(\mathcal{O}_{\mathscr{X}})$ in $\mathrm{Cris}(C/W(k))$ with Frobenius F and Verschiebung V. We use $\mathcal{A}_{g,1,n}$ to denote the moduli scheme of principally polarized abelian varieties of dimension g and level structure n.

Definition 1.0.1. (a) A curve in $\mathcal{A}_{2^m,1,n} \otimes \mathbb{C}$ is called a *special Mumford curve* if it is the image of a Mumford curve in $\mathcal{A}_{2^m,1,n} \otimes \mathbb{C}$ induced by the universal family.

(b) The family $\mathscr{X} \longrightarrow C$ is a *weak Mumford curve over* k if the image of $C \longrightarrow \mathcal{A}_{2^m,1,n}$ (induced by the family $\mathscr{X}/C$) is an irreducible component of a reduction of a special Mumford curve in $\mathcal{A}_{2^m,1,n} \otimes \mathbb{C}$.

The "weakness" of the weak Mumford curve, compared to good reductions is reflected in two aspects: first, the image of $C \longrightarrow \mathcal{A}_{2^m,1,n}$ might have singularities. Second, the reduction of a special Mumford curve at k might be reducible and the image is just one of the irreducible components.

Theorem 1.0.2. *If*

1. *$\mathscr{X}_c$ is ordinary for some closed point $c \in C$,*

2. *$\mathscr{E}$ is irreducible and isomorphic to $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_{m+1}$ as isocrystals,*

then $\mathscr{X} \longrightarrow C$ is a weak Mumford curve over k.

If we further assume the Higgs field associated to $\mathscr{X} \longrightarrow C$ is maximal, then there exists another family of polarized abelian varieties $\mathscr{Y} \longrightarrow C$ such that *(a) there exists an isogeny $\mathscr{Y} \longrightarrow \mathscr{X}$ over C,*

(b) $\mathscr{Y} \longrightarrow C$ is a good reduction of a Mumford curve.

If the family is defined over $\bar{\mathbb{F}}_p$, then we can replace condition (2) by

2'. *$\mathscr{E}$ is irreducible and isomorphic to $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \mathcal{V}_{m+1}$ as F^f-isocrystals for some positive integer f.*

Remark 6.1.4 gives more details.

Based on Theorem 1.0.2, we can give other two variants of Theorem 1.0.2, in terms of crystalline Hodge cycles and l-adic monodromy.

The first one is in terms of crystalline Hodge cycles. Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, C be a proper smooth curve over $\bar{\mathbb{F}}_p$ with absolute Frobenius σ and $g(C) > 1$. Let $\pi : X \longrightarrow C$ be a family of four dimensional principally polarized abelian varieties. Since π is of finite type, it can be defined over a finite field $\mathbb{F}_{p^{f_0}} \subset \bar{\mathbb{F}}_p$.

Let $\mathcal{E}$ be the rank 8 Dieudonne crystal $R^1\pi_{\mathrm{cris},*}(\mathcal{O}_X)$ with Frobenius F and Verschiebung V.

Theorem 1.0.3. *With notation as above, assume there exists an integer f such that f is a multiple of f_0 and*

1. *X_c is ordinary for some closed point $c \in C$,*

2. $\dim_{\mathbb{Q}_{p^f}} \Gamma((C/W(k))_{\mathrm{cris}}, \wedge^4\mathcal{E})^{F^f - p^{2f}} \otimes \mathbb{Q}_{p^f} = 1$,

3. $\Gamma((C/W(k))_{\mathrm{cris}}, \mathscr{E}nd(\wedge^2\mathcal{E}))^{F^f} \otimes \mathbb{Q}_{p^f} \cong \mathbb{Q}_{p^f}^{\times 4}$ *as algebras.*

Then $X \longrightarrow C$ is a weak Mumford curve over k.

If we further assume the Higgs field of $\mathcal{E}$ is maximal, then there exists a family of abelian fourfolds $Y \longrightarrow C'$ such that

(a) $C' \longrightarrow C$ is a finite étale covering,

(b) $Y \longrightarrow X'$ is an isogeny over C', between Y and the pullback family of X,

(c) $Y \longrightarrow C'$ is a good reduction of a Mumford curve.

It should be mentioned that this result is motivated by the classification of Hodge classes in complex abelian fourfolds, given by Ben Moonen and Yuri Zahrin [48]. From their classification, we notice that the abelian fourfolds of Mumford type have distinct exceptional Hodge classes from others. So we translate this phenomenon in terms of crystalline Hodge cycles and it also gives a characterization in the generically ordinary case. Meanwhile, due to the lack of such a characterization for higher dimensional abelian varieties, we can not generalized this result at the moment.

The second variant is given by l-adic monodromy. Let $\mathcal{E}_l$ be the lisse étale l-adic sheaf $R^1\pi_*(\mathbb{Q}_l)$ over C. Choose a geometric point $\bar{\xi}$ and a closed point c in C , and then $\mathcal{E}_l$ induces a monodromy:

$$\rho : \pi_1(C, \bar{\xi}) \longrightarrow \mathrm{Aut}(\mathcal{E}_{l,c}) \cong GL(8, \mathbb{Q}_l).$$

Let G_l be the Zariski closure of $\rho(\pi_1(C, \bar{\xi}))$ in $GL(8, \mathbb{Q}_l)$ and G_l^{geom} be that of $\rho(\pi_1^{\mathrm{geom}}(C, \bar{\xi}))$.

$$1 \longrightarrow \pi_1^{\mathrm{geom}}(C, \bar{\xi}) \longrightarrow \pi_1(C, \bar{\xi}) \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$

Then G_l^{geom} is a normal subgroup of G_l.

We can associate to every rational point $c \in C$ a unique (up to conjugation) Frobenius element F_c in $\pi_1(C, \bar{\xi})$. Its image in $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the integer $\deg(\kappa(c) : \mathbb{F}_q)$.

For notational simplicity, let us define the representation $\rho_0 : SL(2, \mathbb{Q}_l)^{\times 3} \longrightarrow GL_8(\mathbb{Q}_l)$ as the tensor product of three copies of the standard representation of $SL(2, \mathbb{Q}_l)$.

Fix an embedding $\mathbb{Q}_l \longrightarrow \mathbb{C}$ once for all. Our theorem is as follows.

Theorem 1.0.4. *Suppose there exists a closed point $c \in C$ such that*

1. $G_l^{\mathrm{geom},o} \otimes_{\mathbb{Q}_l} \mathbb{C} \cong \mathrm{im}\, \rho_0 \otimes \mathbb{C}$,

2. *X_c is an ordinary abelian variety,*

3. *In G_l, $\rho(F_c)$ generates a maximal torus which is unramified over $\mathbb{Q}_p$.*

Then $X \longrightarrow C$ is a weak Mumford curve.

If we further assume the Higgs field of $X \longrightarrow C$ is maximal, then there exists a family of polarized abelian fourfolds $Y \longrightarrow C'$ such that

(a) $C' \longrightarrow C$ is a finite étale covering,

(b) $Y \longrightarrow X'$ is an isogeny between Y and the pullback family of X over C',

(c) $Y \longrightarrow C'$ is a good reduction of a Mumford curve.

By 7.6.1 and the Chebotarev density theorem, the Frobenius element over sufficiently many points in C generates the maximal torus in G_l. From [28], we know that the torus generated by the Frobenius F_c

is defined over $\mathbb{Q}$. So it makes sense to require the torus to be unramified over $\mathbb{Q}_p$, which is equivalent to saying that the eigenvalues of the Frobenius generate a unramified extension over $\mathbb{Q}_p$.

We wonder if (3) can be replaced by a weaker condition, especially under the presence of (1) and (2). More is explained in Remark 7.6.2.

So far we only deal with the generically ordinary case. In the next theorem, we present a result in the case of general Newton slopes.

Theorem 1.0.5. *Assume*

1. *$\mathscr{X}/C$ has maximal Higgs field,*

2. *$\mathscr{E}$ is an irreducible isocrystal and isomorphic to $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_n$ as crystals where every $\mathcal{V}_i$ is an rank 2 crystal.*

Then $\mathscr{X}/C$ is a good reduction of a Shimura curve of Mumford type.

Different from Theorem 1.0.2, Theorem 1.0.5 requires $\mathscr{E} \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_r$ as *crystals*, not just isocrystals. Once we have the stronger isomorphism. we no longer need the generic ordinary condition and the proof is much shorter than Theorem 1.0.2.

In the proofs of both Theorem 1.0.2 and Theorem 1.0.5, we need the following deformation result which itself is an interesting theorem: Assume G is a Barsotti-Tate group over C.

Theorem 1.0.6. *If G is versally deformed over C, then there exists a unique curve C' over W which is a lifting of C and admits a lifting G' of G, and G' is unique up to unique isomorphism.*

Outline

In Chapter 4, we show reductions of Shimura curves of Mumford type satisfy all the conditions in the above five theorems. Then we prove the five theorems in the other four chapters.

In Chapter 3, we prove Theorem 1.0.6. In fact, we prove a general result on the deformation of a Barsotti-Tate group over a base scheme Y of any dimension, not merely curves. The proof has four steps.

I Compute the obstruction class of the lifting to W_2.

II First order deformation: a choice of a lifting of Y to W_2 can kill the obstruction class.

III Higher order deformation: recursively apply steps I and II to W_n for any n to obtain a formal scheme over $\mathrm{Spf}(W)$.

IV Algebraization: apply Grothendieck's existence theorem to algebraize the formal scheme.

We prove Theorem 1.0.5 in Chapter 5. The irreducibility of all isocrystals $\mathcal{V}_i$ implies a tensor decomposition of the Frobenius F of $\mathscr{E}$. Via an analysis of the decomposition, we show $\mathscr{E}$ is the tensor product of a Dieudonne crystal and a unit root crystal. Then it directly follows from Theorem 1.0.6 that $\mathscr{X} \longrightarrow C$ can be lifted to a formal family over the Witt ring $W(k)$. For the polarization, we show the polarization also decomposes as an isomorphism on the Dieudonne crystal and a polarization on the unit root one. Then the polarization lifts as well and then the family can be algebraized.

The proof of Theorem 1.0.2 occupies most of Chapter 6. The tensor decomposition and the generically ordinary property imply that the isomorphism in (2) of Theorem 1.0.2 is an isogeny between Dieudonne crystals: $\psi : \mathscr{E} \longrightarrow \mathcal{V}_1 \otimes (\mathcal{V}_2 \otimes \cdots \mathcal{V}_{m+1})$.

By [9, Main Theorem 1], the isogeny ψ induces an isogeny $\mathscr{Y} \longrightarrow \mathscr{X}$ between abelian schemes over C, where $\mathcal{V}_1 \otimes (\mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_m)$ is the Dieudonne crystal associated to Y. Let G be the Barsotti-Tate group corresponding to $\mathcal{V}_1$. Then in Section 6.2, we prove the second part of Theorem 1.0.3, i.e. the case with maximal Higgs field. By Theorem 3.4.1 and changing of coordinates, we can show $\mathscr{Y} \longrightarrow C$ has the maximal Higgs field. Then by Theorem 6.2.15, $\mathscr{Y} \longrightarrow C$ is a good reduction of a Mumford curve. We prove the last part, i.e. the non-maximal Higgs field case in Section 6.3. The family $\mathscr{Y} \longrightarrow C$ induces a morphism $\phi : C \longrightarrow \mathcal{A}_{2^m,d,n} \otimes k$. Then Theorem 6.3.1 shows that $\operatorname{im} \phi$ is a curve over which the universal family has maximal Higgs field. Thus by Theorem 6.2.15 $\operatorname{im} \phi$ is a reduction of a special curve in $\mathcal{A}_{2^m,d,n} \otimes \mathbb{C}$. Then the first part of Theorem 1.0.3 follows from Theorem 6.3.3.

Theorem 1.0.3 and Theorem 1.0.4 are treated in Chapter 7. The proof of Theorem 1.0.3 consists of Section 7.1, 7.2 and 7.3. In Section 7.1, we mainly prove Theorem 7.1.4, that over a proper smooth curve, a simple family of abelian varieties corresponds to a reductive group and an irreducible representation in the Tannakian formalism. Section 7.2 and 7.3 are mainly devoted to showing how to choose the finite étale covering to kill the obstruction and obtain the tensor decomposition. Then it is reduced to Theorem 1.0.2

Theorem 1.0.4 is proved in Section 7.4, 7.5 and 7.6. In Section 7.4, we study the dimensions of $H^0_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \wedge^4\mathcal{E}_l)$ and $H^0_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathscr{E}nd(\wedge^2\mathcal{E}_l))$. As a result,

$$\begin{aligned} \dim H^0_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \wedge^4\mathcal{E}_l)^{F-q^2} \otimes \mathbb{Q}_q &= 1 \\ \dim H^0_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathscr{E}nd(\wedge^2\mathcal{E}_l))^F \otimes \mathbb{Q}_q &= 4. \end{aligned} \tag{1.2}$$

In Section 7.5, we translate the above result to p-adic case via the comparison of Lefschetz trace formulas. Therefore the Frobenius eigenspaces on crystalline cohomology spaces have the correct dimensions. Further, we conclude in Section 7.6 that conditions (2) and (3) in Theorem 1.0.4 imply

$$\Gamma((C/W(k))_{\mathrm{cris}}, \mathscr{E}nd(\wedge^2\mathcal{E}))^F \otimes \mathbb{Q}_q$$

is just $\mathbb{Q}_q^{\times 4}$ as an algebra. Then it is reduced to Theorem 1.0.3.

Notation and conventions

- Let k be an algebraically closed field of characteristic $p > 2$.
- We abbreviate Barsotti-Tate groups as BT and truncated Barsotti-Tate group of level n as BT_n.
- Let $W(k)$ be the Witt ring with residue field k and $B(k)$ be the fractional field of $W(k)$.
- We use $\mathcal{A}_{g,1,n}$ to denote the moduli scheme of principally polarized abelian varieties of dimension g and level structure n.
- We let C be a proper smooth curve over k with genus greater than 1.
- We use the curling letters $\mathscr{X}, \mathscr{Y}$ to denote a family of abelian varieties of dimension 2^m over C, $\mathscr{E}$ the Dieudonne crystal of $\mathscr{X}$. The regular letters X and Y denote families of abelian fourfolds over C and $\mathcal{E}$ the Dieudonne crystal of X.
- We use $\mathbb{F}_{p^{f_0}}$ to denote the field of definition of $X \longrightarrow C$.
- Let f be an integer greater than f_0. Then the category of F^f-isocrystals is a (neutral) Tannakian category. We use (G_E, E) to denote the representation corresponding to $\mathcal{E}$. Here G_E is an algebraic group over $\mathbb{Q}_{p^f}$ and E is a eight dimensional G_E-representation over $\mathbb{Q}_{p^f}$.

Chapter 2

Preliminaries

We explain the concepts and state some results on crystals, Barsotti-Tate groups and Tannakian categories which we will use later.

2.1 Crystal

The curve C/k has a natural crystalline site $\text{cris}(C/W(k))$, induced from $(k, W(k), p)$. The higher direct image $\mathscr{E} = R^1\Pi_{\text{cris},*}(\mathcal{O}_X)$ of the abelian scheme $\pi : \mathscr{X} \longrightarrow C$ is a crystal in locally free sheaves.

Definition 2.1.1. A *Dieudonne crystal* over $\text{cris}(C/W(k))$ is a triple $(\mathscr{E}, F, V)$ where

1. $\mathscr{E}$ is a crystal in locally free sheaves,

2. $F : \mathscr{E}^\sigma \longrightarrow \mathscr{E}$ and $V : \mathscr{E} \longrightarrow \mathscr{E}^\sigma$ are homomorphisms between crystals such that $F \circ V = p.\text{Id}_{\mathcal{F}}$, $V \circ F = p.\text{Id}_{\mathscr{E}^\sigma}$.

In particular, $\mathscr{E}$ is a Dieudonne crystal.

Choose an arbitrary lifting $\tilde{C}$ of C to $W(k)$. The category of crystals in locally free sheaves in $\text{cris}(C/W(k))$ is equivalent to the category of vector bundles with an integrable connection on $\tilde{C}$. In particular, choosing an open affine subset $U \subset C$ and a lifting $\tilde{U}$ of U, we have a lifting of Frobenius $\tilde{\sigma}$ over $\tilde{U}$.

In the rest of the paper, by crystal, we mean a crystal in locally free sheaves. Therefore an F-isocrystal on $\text{cris}(U/W(k))$ corresponds to a triple (M, ∇, F), a sheaf of module on $\tilde{U}$ with an integrable connection and Frobenius $F : M^{\tilde{\sigma}} \longrightarrow M$.

2.2 Barsotti-Tate group

A Barsotti-Tate (BT) group G over C is a p-divisible, p-torsion and the p-kernel of G is a finite locally free group scheme. Each p^i-kernel $G[p^i]$ is a truncated BT group. By [2], the crystalline Dieudonne functor $\mathbb{D}(G) = \mathscr{E}xt^1(\underline{G}, \mathcal{O}_C)$ associates a Dieudonne crystal over $\text{cris}(C/W(k))$ to G. And $\mathbb{D}(G[p]) = \mathbb{D}(G)_C$ admits a filtration $0 \longrightarrow \omega_G \longrightarrow \mathbb{D}(G)_C \longrightarrow \alpha_G \longrightarrow 0$.

In the context of the Theorem 1.0.2, $\mathscr{E} = \mathbb{D}(\mathscr{X}[p^\infty])$ and the filtration on $\mathscr{E}_C$ is just the Hodge filtration of $\mathscr{X}/C$: $\omega = \pi_*\Omega_{\mathscr{X}/C}, \alpha = R^1\pi_*(\mathcal{O}_{\mathscr{X}})$. In particular, $\mathscr{E}_C$ has the Gauss-Manin connection and it induces a $\mathcal{O}_C$-linear map, called Higgs field:

$$\theta : \omega \longrightarrow \alpha \otimes \Omega^1_C$$

which is related to Kodaira-Spencer map. The Higgs field can be defined in alternative ways: one can use

$$\begin{tikzcd} \omega \arrow[r] \arrow[drr, dashed] & \mathscr{E}_C \arrow[r] \arrow[d, "\nabla"] & \alpha \\ & \mathscr{E}_C \otimes \Omega^1_C \arrow[r] & \alpha \otimes \Omega^1_C. \end{tikzcd}$$

The other is from the long exact sequence of $0 \longrightarrow \pi^*\Omega_C \longrightarrow \Omega_{\mathscr{X}} \longrightarrow \Omega_{\mathscr{X}/C} \longrightarrow 0$ and the boundary map $\cdots \longrightarrow \pi_*\Omega_{\mathscr{X}/C} \xrightarrow{\partial} R^1\pi_*\mathcal{O}_{\mathscr{X}} \otimes \Omega^1_C \longrightarrow \cdots$ gives the Higgs field. *Maximal Higgs field* just means the map θ is isomorphic.

Theorem 2.2.1. *([9, Main Theorem 1]) The category of Dieudonne crystals over* $\mathrm{cris}(C/W(k))$ *is anti-equivalent to the category of BT groups over* C.

2.3 Tannakian category

Definition 2.3.1. Let L be a field of characteristic 0. A Tannakian category T (over L) is a L-linear neutral rigid tensor abelian category with an exact fibre functor $\omega : T \longrightarrow \mathrm{Vect_L}$.

Theorem 2.3.2. *([11, Theorem 2.11]) For any Tannakian category T, there exists an L-affine group scheme G such that T is equivalent to* $\mathrm{Rep}_L(G)$ *as tensor categories.*

Example 2.3.3. Let M be the Mumford curve. Choose a point $c \in M$, the category of all modules with integral connections (MIC) on M with fibre functor $\mathcal{F} \longrightarrow \mathcal{F}_c$ form a Tannakian category. By 2.3.2, it corresponds to $\mathrm{Rep}_{\mathbb{C}}(G_{\mathrm{univ}})$.

By Riemann-Hilbert correspondence, the category of MIC on M is equivalent to $\mathrm{Rep}(\pi_1(M))$. The algebraic group G_{univ} can be constructed from $\pi_1(M)$ by $G_{\mathrm{univ}} = \lim H$ where H lists the Zariski closure of image of $\pi_1(M)$ in $GL(W)$ for all complex representations W. Note the system of H is projective. So the image of $G_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(W)$ is exactly the Zariski closure of the image of $\pi_1(M)$ in $GL(W)$.

Let $B(k)$ be the fraction field of $W(k)$.

Example 2.3.4. ([43, VI 3.1.1, 3.2.1]) Inverting p in the category $\mathrm{Cris}(C/W(k))$, we obtain the category of isocrystals $\mathrm{Isocris}(C/W(k))$. Similar to 2.3.3, the category $\mathrm{Isocris}(C/W(k))$ forms a Tannakian category over $B(k)$, with fibre functor associated to a k-point of C. So there exists a $B(k)-$affine group scheme P_{univ} such that the following two categories are equivalent.

$$\{\text{finite locally free isocrystals on } C/W(k)\} \longleftrightarrow \mathrm{Rep}_{B(k)}(P_{\mathrm{univ}}).$$

An object $\mathcal{F}'$ in $\mathrm{Isocris}(C/W(k))$ is called effective if it is from an object $\mathcal{F}$ in $\mathrm{Cris}(C/W(k))$,i.e. $\mathcal{F}' = \mathcal{F} \otimes B(k)$. For any morphism $f : \mathcal{F} \otimes B(k) \longrightarrow \mathcal{G} \otimes B(k)$ between effective objects in $\mathrm{Isocris}(C/W(k))$, there exists $m \in \mathbb{Z}$ such that $p^m f : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism in $\mathrm{Cris}(C/W(k))$.

Note different from [43, VI 3.1.1, 3.2.1], $\mathrm{Isocris}(C/W(k))$ denotes just the isocrystals, not the $F-$isocrystals. So P_{univ} is an affine group scheme over $B(k)$, note over $\mathbb{Q}_p$.

Proposition 2.3.5. *For any Tannakian category T and $W, V \in T$, let $< W >$ denote the Tannakian subcategory generated by W, with Tannakian group G_W. Similarly, $< V >= \mathrm{Rep}_L(G_V)$, $< W, V >= \mathrm{Rep}_L(K)$. Then there exists a natural injection $K \hookrightarrow G_W \times G_V$.*

Proof. Since $W, V \in \mathrm{Rep}(K)$, by [11, 2.21], K admits surjections onto G_W and G_V. Then K admits a map $K \longrightarrow G_W \times G_V$. The induced morphism $\mathrm{Rep}(G_W \times G_V) \longrightarrow \mathrm{Rep}(K)$ satisfies [11, 2.21(2)]. So the map is injective. □

If C is defined over a finite field, then for any finite étale covering C' of C and large enough integer f, F^f-isocrystals over C' also form a neutral Tannakian category, equivalent to $\mathrm{Rep}_{\mathbb{Q}_{p^f}}(G'_{\mathrm{univ}})$. Let $\mathcal{C}$(resp. $\mathcal{C}'$) denote the category of F^f-isocrystals over C(resp. C'). Then the pullback induces $\mathcal{C} \longrightarrow \mathcal{C}'$.

Let $\Gamma = \mathrm{Aut}(C'/C)$ be the finite automorphism group. Then the action of Γ on C' induces

$$\Gamma \longrightarrow \mathrm{Aut}^{\otimes}(\mathcal{C}'), \gamma \mapsto \gamma^*.$$

Obviously $1 \in \Gamma$ induces the identity on $\mathcal{C}'$.

The covering $C' \longrightarrow C$ satisfies the descent for F^f-isocrystals. In other words,

Lemma 2.3.6. *$\mathcal{C}$ can be identified as the category of*

$$\left\{(X', \{\varphi_\gamma\}_{\gamma\in\Gamma}) | X' \in \mathrm{ob}\,\mathcal{C}', \varphi_\gamma : \gamma^* X' \longrightarrow X', \varphi_{\gamma\gamma'} = \varphi_{\gamma'} \circ \varphi_\gamma\right\}.$$

Proof. The proof is an easy corollary of a more general theorem [41, Theorem 4.5]. Or we can directly compute as follows: it suffices to show any such object $(X', \{\varphi_\gamma\}_{\gamma\in\Gamma})$ can descend to C. F-isocrystal corresponds to (W, ∇_W) on C with a local Frobenius on $W(\mathbb{Q}_{p^f})$. Obviously φ_γ gives a descent datum for vector bundles on $C'_{B(k_0)}$. Note a flat connection is equivalent to the descent data on the deRham space. Thus the étale covering $C' \longrightarrow C$ also satisfies the descent for connections. Thereby we have a bundle with connection $(V', \nabla_{V'})$ on $C_{\mathbb{Q}_{p^f}}$. Similarly, locally the Frobenius can be descent to V'. Taking the intersection $V' \cap W$ gives a coherent sheaf over C with connection. And taking a double dual gives a locally free sheaf. □

Then $\mathcal{C} \longrightarrow \mathcal{C}'$ is just the forgetful functor $\{(X', \{\varphi_\gamma\}_{\gamma\in\Gamma})\} \mapsto X'$.

Proposition 2.3.7. *Notations as above, $1 \longrightarrow G'_{\mathrm{univ}} \longrightarrow G_{\mathrm{univ}} \longrightarrow \Gamma$ is a exact sequence.*

Proof. For any object $\mathcal{E}' \in \mathcal{C}'$, $\oplus_{\gamma\in\Gamma}\gamma^*\mathcal{E}'$ is invariant under Γ and thus can be descent to C. So any object in F^f-isco(C') is a subquotient of some object in F^f-isoc(C). By [11, 2.21 (b)], G'_{univ} is a subgroup of G_{univ}.

For the other side $G_{\mathrm{univ}} \longrightarrow \Gamma$, consider the kernel of $\mathcal{C} \longrightarrow \mathcal{C}'$, i.e. the full subcategory $\mathcal{K}$ in $\mathcal{C}$ consisted of objects with trivial image in $\mathcal{C}'$. Then $\mathcal{K}$ is also Tannakian and by 2.3.6 it consists of objects $(\mathcal{O}_{C'}^{\oplus n}, \{\varphi_\gamma\}_{\gamma\in\Gamma})$

where $\mathcal{O}_{C'}$ denotes the trivial isocrystal over C'. For any such object, φ induces a representation of Γ on k^n. Therefore $\mathcal{K}$ is equivalent to $\mathrm{Rep}(\Gamma)$. The inclusion $\mathcal{K} \longrightarrow \mathcal{C}$ gives a morphism between groups $G_{\mathrm{univ}} \longrightarrow \Gamma$.

Now it suffices to show the exactness of $1 \longrightarrow G'_{\mathrm{univ}} \longrightarrow G_{\mathrm{univ}} \longrightarrow \Gamma$ in the middle. Let K be the kernel of $G_{\mathrm{univ}} \longrightarrow \Gamma$. Since under $G_{\mathrm{univ}} \longrightarrow \Gamma$, G'_{univ} has trivial image, $G'_{\mathrm{univ}} \subset K$, i.e. we have

$$\mathrm{Rep}(K) \longrightarrow \mathrm{Rep}(G'_{\mathrm{univ}}).$$

For any $g \in G_{\mathrm{univ}}$ such that g is in the kernel K, consider the object $(\oplus_{\gamma\in\Gamma}\gamma^*\mathcal{E}', \{\varphi_\gamma\})$ in $\mathcal{C}$ and $\mathcal{E}' \in \mathcal{C}'$. The element g fixes each direct summand $\gamma^*\mathcal{E}$. In particular, let $\gamma = \mathrm{id}$ and g acts on $\mathcal{E}'$. Thus every $\mathcal{E}' \in \mathrm{Rep}(G'_{\mathrm{univ}})$ is a natural representation of K. So $K = G'_{\mathrm{univ}}$. $\square$

Corollary 2.3.8. $\dim G'_{\mathrm{univ}} = \dim G_{\mathrm{univ}}$.

Chapter 3

Deformation of a Barsotti-Tate group

We will prove 1.0.6 in this chapter. This deformation result is a main ingredient in lifting a curve in characteristic p to a Shimura curve.

Illusie, along with Grothendieck and Raynaud, developed the deformation theory of Barsotti-Tate (BT) groups (see [21]). One result in [21] is that, roughly speaking, BT groups are unobstructed over an affine base scheme. In this chapter, rather than an affine base, we consider the deformation of a pair of a complete curve and a BT group over this curve. We show that there is no obstruction to lift the pair (C, G) from characteristic p to characteristic 0 when G is versally deformed over C.

It should be mentioned that Shinichi Mochizuki has also considered a similar problem. In [36], he develop the theory of the indigenous bundle, which is closely related to a height 2 BT group. But our method is comparably much easier.

3.1 Main result

In this chapter, let k be an algebraically closed field of characteristic p, W Witt ring $W(k)$, W_n the truncated ring $W(k)/p^{n+1}$ and Y a projective smooth variety over k. For the definition of BT or truncated BT groups, we refer to [33, Chapter 1].

Let G be a BT group over Y. Then [21, A.2.3.6] indicates that the following two conditions are equivalent.

- For any closed point $y \in Y$, let $\hat{Y}_y$ be the completion of Y over y. The restriction of G over $\hat{Y}_y$ is the versal deformation of G_y in the category of local artinnian $k-$algebras with residue field k.
- the Kodaira-Spencer map:

$$\mathrm{ks} : T_Y \to t_G \otimes t_{G^*} \tag{3.1}$$

  is an isomorphism.

Here t_G is the degree 0 cohomology of the dual cotangent complex of G, i.e. $t_G = H^0(\check{l}_G)$. If the pair (Y, G) satisfies either one of the conditions, then we say that G is *versally deformed* of Y.

Proposition 3.1.1. *If G is a versally deformed over Y and $H^2(Y, T_Y) = \Gamma(Y, T_Y) = 0$, then there exists a unique lifting Y' of Y to W which admits a lifting G' of G,*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G' \arrow[d] \\ Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ k \arrow[r] & W. \end{tikzcd}$$

Further, G' is unique up to a unique isomorphism.

Theorem 1.0.6 is an immediate corollary of 3.1.1.

Remark 3.1.2. We also have a similar result for truncated BT groups but the uniqueness is not guaranteed. For details, see 3.3.9.

Though more general than 1.0.6, we have not found any example of such Y as in 3.1.1 other than curves. Such an example over a curve can be obtained from the well-known fake elliptic curve.

In Section 3.2, we compute the obstruction class of the lifting to W_2 and deal with the first deformation. In Section 3.3, we solve the higher order deformation and algebraization problem. In Section 3.4, we consider another extremal case, that is, a BT group with trivial Kodaira-Spencer map.

3.2 First order deformation

3.2.1 Step I

First recall the following theorems by Grothendieck and Illusie:

Proposition 3.2.2. *([21, Theorem 4.4]) Let $S_0 \longrightarrow S \xrightarrow{i} S'$ be a closed immersion defined by ideal sheaf $J \subset K$ with $JK = 0$ and $p^N \mathcal{O}_{S_0} = 0$. Let G be a BT_n group over S with $n \geq N$ or a BT group. Assume S' is affine and p is nilpotent on S'. Then*

1. *there exists a BT_n group G' on S' as a deformation of G,*
2. *the set of deformations up to isomorphism is a torsor under the group $(t_G \otimes t_{G^*} \otimes J)(S_0)$*
3. *if $n \geq k \geq N$, then the morphism*

$$\mathrm{Def}(G, i) \longrightarrow \mathrm{Def}(G[p^k], i)$$

is bijective.

Proposition 3.2.3. *([21, Corollary 4.7]) Assumption as 3.2.2. Let H be a BT group on S.*

1. *If $n \geq N$, then the map*

$$\mathrm{Def}(H, i) \longrightarrow \mathrm{Def}(H[p^n], i)$$

is bijective.

2. *The automorphism group of the deformation H' of H to S' is trivial.*

Corollary 3.2.4. *Let $S_0 \longrightarrow S \xrightarrow{i} S'$ be a closed immersion defined by ideal sheaf $J \subset K$ with $JK = 0$. Let G be a BT_n group over S with $n \geq 1$ or a BT group. Assume p is nilpotent on S' and S_0 is a proper smooth curve over an algebraically closed field k of characteristic p. If $G[p]$ can be lifted to S', then G can also be lifted to S'. So in this case, the obstruction to lift G is the same as that of $G[p]$.*

Proof. By 3.2.3 (1), we know that over any affine open $U \subset S$, G_U can be uniquely lifted to $U' \subset S'$ which is compatible with the lifting of $G[p]_U$. Over $U \cap V$, $G_{U'}$ and $G_{V'}$ both induce the same lifting of $G[p]_{U \cap V}$. Therefore due to the bijection in 3.2.3(1), $G_{U'} \cong G_{V'}$. Since S_0 is a curve, there is no need to check the coboundry condition. Thus $\{G_{U'}\}$ glue to a global BT group over S.

By 3.2.2 (3), the same argument works for the truncated case. ∎

Remark 3.2.5. From 3.2.4, if $G[p]$ is unobstructed, then we only know G is liftable but it may not be globally compatible with the lifting of $G[p]$.

Corollary 3.2.6. *Assumptions as 3.2.4 and further assume G is a BT group. The deformation space* $\mathrm{Def}(G,i)$ *is a torsor under* $H^0(S_0, t_G \otimes t_{G^*} \otimes J)$.

Proof. Let $\{U_i\}$ be an affine open cover of S. Let $\{U_i'\}$ be the lifting of $\{U_i\}$ to S' and $\{U_{i0}\}$ be the corresponding affine open cover on S_0.

By [21, 3.2(b)], the deformation space $\mathrm{Def}_{S'}(G)$ (up to isomorphism) is a torsor under some $k-$vector space V. By 3.2.2, for any $v \in V$ and any i, there exists $s_i \in (t_G \otimes t_{G^*} \otimes J)(U_{i0})$ such that $v_{|U_i'} = s_i$. Over each U_{ij},

$$s_{i|U_{ij0}} = v_{|U_{ij}'} = s_{j|U_{ij0}}.$$

So $\{s_i\}$ patch to a global section s of $t_G \otimes t_{G^*} \otimes J$. Therefore we have the linear transformation

$$\begin{aligned} V &\longrightarrow H^0(S_0, t_G \otimes t_{G^*} \otimes J) \\ v &\mapsto s \end{aligned} \tag{3.2}$$

It is easy to show this transformation is surjective.

The kernel of morphism (3.2) consists of the elements which induce locally isomorphisms between any two deformations of G. Since G is a BT group, by 3.2.3 (2), over each U_i, the isomorphism between two deformations G_1 and G_2 which induces the identity on $G_{|U_i}$ is unique. So the local isomorphisms over each U_i can glue to a global isomorphism. Therefore the kernel is trivial. The space $\mathrm{Def}(G,i)$ is a torsor under $H^0(S_0, t_G \otimes t_{G^*} \otimes J)$ ∎

Theorem 3.2.7. *Assumptions as 3.1.1, for any deformation Y_2 of Y over W_2,*

1. *the obstruction class* $\mathrm{ob}_{Y_2}(G)$ *of the deformation of G to Y_2 is in* $H^1(Y, T_Y)$,
2. *if the obstruction class vanishes, then the deformation space is a torsor under* $H^0(Y, T_Y)$.

Proof. (1) By 3.2.3 (2) and 3.2.4, we only need to consider the truncated BT group G. Choose any open affine cover $\{U_i\}$ of Y and the deformation U_i' of U_i to W_2 is unique up to isomorphisms. By 3.2.2, G is locally liftable and the set of deformations of G_{U_i} (up to isomorphisms) is a torsor under the group $(t_G \otimes t_{G^*} \otimes J)(U_i)$. Since the thickening just has the first order, $J \cong k$ and the set of lifting is a torsor under $(t_G \otimes t_{G*})(U_i)$. On $U_{ij} = U_i \cap U_j$, the restriction from $G_{U_i'}$ and $G_{U_j'}$ give two deformations of $G_{U_{ij}}$. We denote the two deformations as $[G_{U_{ij}'}]$ and $[G_{U_{ji}'}]$ respectively.

By 3.2.2, we can choose s_{ij} the element in $(t_G \otimes t_{G^*})(U_{ij})$ sending $[G_{U'_{ij}}]$ to $[G_{U'_{ij}}]$, i.e.

$$s_{ij} = [G_{U'_{ij}}] - [G_{U'_{ji}}] \in (t_G \otimes t_{G^*})(U_{ij}).$$

Over U_{ijk}, $s_{ijk} = s_{ij} + s_{jk} - s_{ik} = 0$. So $\{s_{ij}\}$ is a Cech cocycle in $H^1(Y, T_Y)$. The vanishing of $\{s_{ij}\}$ gives a deformation of G. Note by 3.2.4, we can adjust the deformation on U_i by elements in $(t_G \otimes t_{G^*})(U_i)$. Therefore the obstruction is in $H^1(Y, t_G \otimes t_{G^*})$.

Since G is a versally deformed over Y, the Kodaira-Spencer map gives

$$t_G \otimes t_{G^*} \cong T_Y.$$

(2) It directly follows from 3.2.6. □

Remark 3.2.8. From the proof, we know 3.2.7 (1) also holds for truncated BT groups.

Remark 3.2.9. As in 1.0.6, if the genus $g(C) \geq 2$, then $H^0(C, T_C) = 0$ and hence for the BT group G, there is a unique (up to a unique isomorphism, 3.2.3(2)) deformation G' to C', if exists. And according to [31, Corollary 5.4], if a curve along with a family of abelian varieties is liftable, preserving the maximal Higgs field, then the genus of the curve is necessarily greater than 2.

3.2.10 Step II

Recall that the deformation space of Y to W_2 is $H^1(Y, T_C)$ and Kodaira-Spencer map 3.1 induces an isomorphism $H^1(Y, T_Y) \longrightarrow H^1(Y, t_G \otimes t_{G^*})$. Let U, V be two affine open sets of Y, with the unique deformations U' and V' to W_2.

Fix the embedding $(U \cap V)' \longrightarrow V'$ and alternating the embedding $(U \cap V)' \longrightarrow U'$ gives a different lifting of Y. And all the deformations of Y can be obtained in this way.

$$\begin{tikzcd} U' & (U \cap V)' \arrow[l, Leftarrow] \arrow[r] & V' \\ U \arrow[u] & U \cap V \arrow[l] \arrow[r] \arrow[u] & V \arrow[u] \end{tikzcd} \tag{3.3}$$

Since all the schemes involved in (3.3) are affine, we can interpret the diagram in the level of commutative algebras:

$$\begin{tikzcd} B' \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d, "/p"'] & A' \arrow[d, "/p"'] & C' \arrow[l] \arrow[d, "/p"'] \\ B \arrow[r] & A & C \arrow[l] \end{tikzcd}$$

where $\psi_1 - \psi_2 = p\delta$ where $\delta \in \mathrm{Der(A)}$. Here $/p$ means quotient of p.

Let $G_{U'}$ be any deformation of G_U. Then the map ψ_1 and ψ_2 induce two deformations $G_{U'} \otimes_{\psi_1} A'$ and $G_{U'} \otimes_{\psi_2} A'$ of $G_{U\cap V}$. By 3.2.2 (2), we can denote the element in $t_G \otimes t_{G^*}(U \cap V)$ taking $G_{U'} \otimes_{\psi_2} A'$ to $G_{U'} \otimes_{\psi_1} A'$ as $[G_{U'} \otimes_{\psi_1} A'] - [G_{U'} \otimes_{\psi_2} A']$.

Theorem 3.2.11. *Restricting the* ks *to* $U \cap V$ *(for the definition of* ks*, see the morphism (3.1)), we have*

$$ks(\delta) = [G_{U'} \otimes_{\psi_1} A'] - [G_{U'} \otimes_{\psi_2} A'].$$

Proof. By [21, 4.8.1], we have

$$\mathrm{ks}(\delta|_x) = [f^*G] - [G_{|x} \otimes_k k[\epsilon]] \tag{3.4}$$

where $\epsilon^2 = 0$ and $f : \mathrm{Spec}\ k[\epsilon] \longrightarrow Y$ is induced from the tangent vector $\delta_{|x}$. Then f induces a ring morphism $f^* : A \longrightarrow k[\epsilon]$. Note A_x is regular and then $\mathfrak{m}_x$ is generated by $s_1, s_2, \cdots, s_h$ where h is the dimension of Y. We can assume that $f^*(s_1)$ is nonzero.

For any $x \in U \cap V$, by Hensel's lemma, there exists a section x' of $Y' \longrightarrow W_2$ lifting the point x. We can choose the ideal $I_{x'}$ such that the x' is given by $A' \longrightarrow A'/I_{x'}$ locally on $U \cap V$.

Based on (3.4) it suffices to show

$$[G_{U'} \otimes_{\psi_1} A'/I_{x'}] - [G_{U'} \otimes_{\psi_2} A'/I_{x'}] = [f^*G] - [G_{|x} \otimes_k k[\epsilon]].$$

We can prove the result by local computation.

$$\begin{tikzcd} B \arrow[r, dashed] \arrow[d, ".p"] & A \arrow[r, "/\mathfrak{m}_x"] \arrow[d, ".p"] & k \arrow[d, ".p"] \\ B' \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d] & A' \arrow[r, "/I_{x'}"] \arrow[d] & W_2 \arrow[d] \\ B \arrow[r] & A \arrow[r] & k \end{tikzcd} \tag{3.5}$$

where $\psi_2 - \psi_1 = p\delta$. It is easy to see that the top arrow $A \longrightarrow k$ is the quotient by the maximal ideal $\mathfrak{m}_x$ of x.

Let $I_1 = (\psi_1)^{-1}(I_{x'})$ and $I_2 = (\psi_2)^{-1}(I_{x'}) = (\psi_1 + p\delta)^{-1}(I_{x'})$. Note $U' = \text{Spec } B'$ and then

$$G_1 := G_{U'} \otimes B'/I_1, G_2 := G_{U'} \otimes B'/I_2$$

are both deformations of $G_U \otimes k$ to W_2 with

$$[G_{U'} \otimes_{\psi_1} A'/I_{x'}] - [G_{U'} \otimes_{\psi_2} A'/I_{x'}] = [G_2] - [G_1].$$

Since $s_1 \in I_1/p = \mathfrak{m}_x$, we can choose a lifting $\tilde{s}_i$ of s_i to B' such that $\tilde{s}_i \in I_1$. Then $\psi_2(\tilde{s}_1 - p) = (\psi_1 + p\delta)(\tilde{s}_1 - p) \in I_{x'}$, i.e. $\tilde{s} - p \in I_2$. Since $\mathfrak{m}_x = (s)$, we have

$$I_1 = (\tilde{s}_1, \tilde{s}_2, \cdots,), I_2 = (\tilde{s}_1 - p, \tilde{s}_2, \cdots).$$

Let $I = (\tilde{s}_1^2, \tilde{s}_2, \cdots, ps_1, \cdots, ps_n)$. Note $I \subset I_1 \cap I_2$ and since U' is an affine flat scheme over W_2, there exists an injection

$$i : W_2 \hookrightarrow B'/I. \tag{3.6}$$

Note $B'/(\tilde{s}_1, p) = k$ and there exists a section $k \longrightarrow B'/(\tilde{s}_1^2, p)$. Thus $B'/(\tilde{s}_1^2, p) = k[\epsilon]/\epsilon^2$. We have the following diagram:

$$\begin{tikzcd}
& k[\epsilon] \arrow[dl] & \\
k & & B'/I \arrow[ll, "J"'] \arrow[ul, "\phi_3"'] \arrow[dl, shift right, "\phi_1"'] \arrow[dl, shift left, "\phi_2"] \\
& W_2 \arrow[ul] &
\end{tikzcd} \tag{3.7}$$

where $\phi_1(\tilde{s}_1) = 0, \phi_2(\tilde{s}_1) = p, \phi_3(p) = 0$ and $(B'/I)/J = k$. The ideal J is generated by $(\tilde{s}, p)$.

We consider the triple

$$\{G_{U'} \otimes B'/I, G_1 \otimes_{W_2} B'/I, G_2 \otimes_{W_2} B'/I\}$$

of deformations of $G_U \otimes k$ on B'/I where $G_i \otimes_{W_2} B'/I$ is through i (see (3.6)). Let

$$\begin{aligned}
\xi_1 &= [G_{U'} \otimes B'/I] - [G_1 \otimes B'/I] \\
\xi_2 &= [G_{U'} \otimes B'/I] - [G_2 \otimes B'/I] \\
\xi_3 &= [G_2 \otimes B'/I] - [G_1 \otimes B'/I]
\end{aligned}$$

where $\xi_i \in t_{G_x} \otimes t_{G_x^*} \otimes_k J($ 3.2.2).

Then ϕ_i induces

$$\phi_i^* : t_G \otimes t_{G^*} \otimes_k J \longrightarrow t_G \otimes t_{G^*} \otimes_k pW_2, i \in \{1,2\}.$$

$$\phi_3^* : t_G \otimes t_{G^*} \otimes_k J \longrightarrow t_G \otimes t_{G^*} \otimes_k k[\epsilon].$$

Choose the canonical basis $(\epsilon, p), (p), (\epsilon)$ for $J, pW_2, \epsilon k[\epsilon]$, respectively. Then ϕ_1, ϕ_2, ϕ_3 are induced by

$$J \longrightarrow pW_2, a\epsilon + bp \mapsto b,$$

$$J \longrightarrow pW_2, a\epsilon + bp \mapsto a + b,$$

$$J \longrightarrow k[\epsilon], a\epsilon + bp \mapsto a$$

respectively. In particular,

$$\phi_2^* = \phi_1^* + \phi_3^*. \tag{3.8}$$

Then we have the following relations:

1. $\phi_1^*(\xi_1) = 0$, $\xi_1 = \xi_2 + \xi_3$,
2. $\phi_2^*(\xi_2) = 0$, $\phi_3^*(\xi_2) = [G_U \otimes_f k[\epsilon]] - [G_U \otimes_k k[\epsilon]]$,
3. $\phi_1^*(\xi_3) = \phi_2^*(\xi_3) = [G_2] - [G_1]$.

For (1), the pull back of the pair $\{G_{U'} \otimes B'/I, G_1 \otimes B'/I\}$ through ϕ_1 is identically G_1. For (2), $\phi_3^*(B'/I) = B/(s^2)$ and then $\phi_3^*(G_{U'} \otimes B'/I) = G_U \otimes_f k[\epsilon]$. For (3), it follows from that $\phi_1 \circ i = \phi_2 \circ i = \mathrm{Id}$.

Similarly, one can verify the other formulas.

So

$$0 = \phi_1^*(\xi_1) = (\phi_2^* - \phi_3^*)(\xi_2 + \xi_3) = \phi_2^*(\xi_3) - \phi_3^*(\xi_2) \tag{3.9}$$

which implies

$$[G_1] - [G_2] = [G_U \otimes_k k[\epsilon]] - [G_U \otimes_f k[\epsilon]]$$

as an element in $(t_G \otimes t_{G^*})(k)$.

Therefore, the difference of the two classes $[G \otimes_{\psi_1} A'_{|x'}] - [G \otimes_{\psi_2} A'_{|x'}]$ is the same as the difference between the trivial deformation to $k[\epsilon]$ and the deformation given by s, which is exactly $\mathrm{ks}(\delta_{|x})$. $\square$

Fix a lifting $Y \longrightarrow Y_2$ of Y. By 3.2.7, $\mathrm{ob}_{Y_2}(G) = (s_{ij}) \in H^1(Y, t_G \otimes t_{G^*})$ where $s_{ij} = [G_{U'_{ij}}] - [G_{U'_{ji}}]$. Since Y is liftable to W_2 and $H^1(Y, T_Y) \cong H^1(Y, t_G \otimes t_{G^*})$, $\{s_{ij}\}$ takes Y_2 to another lifting of Y. Further, by 3.2.11, the Kodaira-Spencer map induces a bijection

$$\begin{aligned} \mathrm{Def}_{W_2}(Y) \longrightarrow & H^1(Y, t_G \otimes t_{G^*}) \ni \ \mathrm{ob}_{Y_2}(G) \\ Y_2' \mapsto & \mathrm{ks}([Y_2'] - [Y_2]). \end{aligned} \tag{3.10}$$

Then there is a lifting of Y to W_2 which admits a deformation of G. Explicitly, we change the gluing $U_{ij}' \longrightarrow U_i'$ by the derivation $\delta_{ij} = \mathrm{ks}^{-1}(-s_{ij})$ and then the new $G_{U_i'}$ is isomorphic to $G_{U_j'}$ and U_i' still patch to a scheme which lifts Y.

That is how we adjust the lifting of Y to kill the obstruction. It is clear that such a lifting is unique. Therefore we obtain a unique first-order deformation of Y such that G can be deformed.

We have finished the first order thickening.

3.3 Higher order deformation

3.3.1 Step III

Suppose we have solved the deformation problem up to order n, i.e. there exists a unique deformation $G_n \longrightarrow Y_n$ of $G \longrightarrow Y$. Since $H^2(Y, T_Y) = 0$, the deformation of Y_n to W_{n+1} is unobstructed.

We would like to complete the following diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G_n \arrow[r, dashed] \arrow[d] & ? \arrow[d, dashed] \\ Y \arrow[r] & Y_n \arrow[r] & Y_{n+1}. \end{tikzcd}$$

It is easy to check that 3.2.7(1) is still true for the higher order deformation, and thus the obstruction class $\mathrm{ob}_{Y_{n+1}}(G_n)$ is still in $H^1(Y, T_Y)$.

Proposition 3.3.2. *Theorem 3.2.11 is also true for higher order thickening.*

Proof. We prove it by induction.

Based on the induction hypothesis that the proposition is true for $n-$th order thickening, we just mimic the proof of 3.2.11. Similar to the diagram 3.3, we can adjust the lifting $Y_n \longrightarrow Y_{n+1}$ by

$$\begin{tikzcd} U_{n+1} & (U_{n+1} \cap V_{n+1}) \arrow[l, Leftarrow] \arrow[r] & V_{n+1} \\ U_n \arrow[u] & U_n \cap V_n \arrow[l] \arrow[r] \arrow[u] & V_n \arrow[u] \end{tikzcd}$$

In terms of algebra, correspondingly, the diagram (3.5) becomes :

$$\begin{tikzcd}
B \arrow[r, dashed, "\delta"] \arrow[d, ".p^n"] & A \arrow[r, "/\mathfrak{m}_x"] \arrow[d, ".p^n"] & k \arrow[d, ".p^n"] \\
B_{n+1} \arrow[r, shift left, "\psi+p^n\delta"] \arrow[r, shift right, "\psi"'] \arrow[d] & A_{n+1} \arrow[r, "/I_{x_{n+1}}"] \arrow[d] & W_{n+1}(k) \arrow[d] \\
B_n \arrow[r] & A_n \arrow[r, "/I_{x_n}"] & W_n
\end{tikzcd}$$

where $\delta_{|x}$ is given by $f : k[\epsilon] \longrightarrow U \cap V, \epsilon \mapsto s \in \mathfrak{m}_x/\mathfrak{m}_x^2$.

Let $G_{U_{n+1}}$ be any deformation of $(G_n)_{U_n}$ to U_{n+1}. Let

$$I_1 := \psi^{-1}(I_{x_{n+1}}), I_2 := (\psi + p\delta)^{-1}(I_{x_{n+1}}).$$

Then $B_{n+1}/I_1 \cong B_{n+1}/I_2 \cong W_{n+1}(k)$. We use $\tilde{s}$ to denote the lifting of s to B_{n+1} such that $\tilde{s} \in I_1$. Then

$$I_2 = (\tilde{s} - p^n, \cdots), I_1 = (\tilde{s}, \cdots).$$

Let

$$G_1 := G_{U_{n+1}} \otimes B_{n+1}/I_1, G_2 := G_{U_{n+1}} \otimes B_{n+1}/I_2.$$

Choose $I = (\tilde{s}^2, \cdots, p\tilde{s})$ and $J = (\tilde{s}, \cdots, p^n)$. Similar to diagram 3.7, we have

$$\begin{tikzcd}
 & & B_{n+1}/I_1 \arrow[dl] & \\
k & W_n \arrow[l] & B_{n+1}/I_2 \arrow[l] & B_{n+1}/I \arrow[l, "\phi_2"'] \arrow[ul, "\phi_1"'] \arrow[dl, "\phi_3"] \\
 & & W_n[\epsilon]/(p\epsilon) \arrow[ul] &
\end{tikzcd}$$

where $\phi_1(\tilde{s}) = 0, \phi_2(\tilde{s}) = p^n, \phi_3(p^n) = 0$ and $(B_{n+1}/I)/J \cong W_n$.

As in the proof of 3.2.11, let

$$\xi_1 = [G_{U_{n+1}} \otimes B_{n+1}/I] - [G_1 \otimes B_{n+1}/I]$$

$$\xi_2 = [G_{U_{n+1}} \otimes B_{n+1}/I] - [G_2 \otimes B_{n+1}/I]$$

$$\xi_3 = [G_2 \otimes B_{n+1}/I] - [G_1 \otimes B_{n+1}/I]$$

where $\xi_i \in t_{G_x} \otimes t_{G_x^*} \otimes J$.

Similar to Equation 3.8, we have

$$\phi_2^* = \phi_1^* + \phi_3^*.$$

Therefore the same result as Equation 3.9 holds for the higher order deformation.

We obtain that

$$[G_{U_{n+1}} \otimes B_{n+1}/I_1] - [G_{U_{n+1}} \otimes B_{n+1}/I_2] = [G_n \otimes B_n/I] - [G_n \otimes_{B_n/I_{x_n}} W_n(\epsilon)].$$

The right hand side is the deformation of $G_{|U_n} \otimes W_n$ and mod p, it is just

$$[G \otimes_f k[\epsilon]] - [G \otimes_k k[\epsilon]] = \mathrm{ks}(\delta).$$

Therefore by functorality, the right hand side is

$$[G_n \otimes B_n/I] - [G_n \otimes_{W_n} W_n(\epsilon)] = \mathrm{ks}(\delta).$$

Therefore the result of 3.2.11 is also true for the higher order deformation. □

Let $\iota : \mathrm{Spec}\ W_n \longrightarrow \mathrm{Spec}\ W_{n+1}$. Proposition 3.3.2 implies (3.10) holds in the higher order case:

$$\mathrm{Def}(Y_n, \iota) \longrightarrow H^1(Y, t_G \otimes t_{G^*})$$

is bijective. Then we can always adjust the lifting Y_{n+1} so that it admits a lifting G_{n+1} as BT or BT_n groups.

In the following, we only consider the BT group case. By 3.2.9, for a BT group G, the lifting of G to Y_2 is strongly unique. Since 3.2.3(2) and 3.2.6 also hold for higher deformation case, the lifting of BT group G is strongly unique for the deformation problem $Y \hookrightarrow Y_{n-1} \hookrightarrow Y_n$. In this case, we have a canonical choice of deformations $\{Y_n\}$ of Y such that each Y_n admits the unique BT group G_n.

Proposition 3.3.3. *For a BT group G, the sequence of deformations $\{Y_n\}$ of Y which admits a deformation of G is unique.*

3.3.4 Step IV

In this step, we always assume G is a BT group. By 3.3.3, we have a formal scheme

$$\hat{G} = \lim G_n \longrightarrow \hat{Y} = \lim Y_n \longrightarrow \mathrm{Spf(W)}.$$

A natural question is whether there exists actual schemes

$$G' \longrightarrow Y' \longrightarrow W$$

that base change to W_n, it is $G_n \longrightarrow Y_n$.

Recall the Grothendieck existence theorem: let A be a noetherian ring, I an ideal of A and $W = \mathrm{Spec}\, A$, $W_n = \mathrm{Spec}\, A/I^{n+1}$, $\hat{W} = \mathrm{colim}_n W_n = \mathrm{Spf}(A)$.

Theorem 3.3.5. *([17, 5.1.4]) Let X be a noetherian scheme, separated and of finite type over Y, and let $\hat{X}$ be its $I-$adic completion. Then the functor $F \mapsto \hat{F}$ from the category of coherent sheaves on X whose support is proper over Y to the category of coherent sheaves on $\hat{X}$ whose support is proper over $\hat{Y}$ is an equivalence.*

Theorem 3.3.6. *([17, 5.4.5]) Let $\mathscr{X} \longrightarrow \mathrm{Spf}(A)$ be a formal scheme and $\mathscr{L}$ an ample line bundle on over $\mathscr{X}$, then $(\mathscr{X}, \mathscr{L})$ are algebraizable.*

Thus the formal projective variety $\{Y_n\}$ is algebraizable. Denote the algebraic variety as Y'.

Corollary 3.3.7. *Let X/W be as the 3.3.5. Then*

1. *$Z \longrightarrow \hat{Z}$ is a bijection from the set of closed subschemes of X which are proper over W to the set of closed formal subschemes of $\hat{X}$ which are proper over $\hat{W}$.*

2. *$Z \longrightarrow \hat{Z}$ is an equivalence from the category of finite $X-$schemes which are proper over W to the category of finite $\hat{X}-$formal schemes which are proper over $\hat{W}$.*

3. *If both of X and Z are proper, then $(f : X \longrightarrow Z) \mapsto (\hat{f} : \hat{X} \longrightarrow \hat{Z})$ is a bijection from the set of morphisms between X and Z to the set of formal morphisms between $\hat{X}$ and $\hat{Z}$. In particular, any finite locally free group schemes over X_n can be algebraized to a finite locally free group scheme over X.*

Proof. The first and second assertions are [23, 4.5, 4.6]. For any $f_n : X_n \longrightarrow Z_n$ morphism between formal schemes, the graph of morphism: $(X_n \cong)\Gamma_n \subset X_n \times Z_n$ can be algebraized to $\Gamma \subset X \times Z$ which is still isomorphic to X and then the morphism can also be uniquely algebraized. □

Proposition 3.3.8. *The formal scheme*

$$\hat{G} \longrightarrow \hat{Y} \longrightarrow \mathrm{Spf}(W)$$

can be algebraized.

Proof. First consider the truncated case, we have an algebraization $G' \longrightarrow Y'$ of $G_n \longrightarrow Y_n$ where G' is a finite locally free group scheme over Y'. It remains to show G' is BT_n. It suffices to show it satisfies the exact sequence for any $l \leq k$:

$$0 \longrightarrow G'[p^l] \longrightarrow G' \longrightarrow G'[p^{k-l}] \longrightarrow 0.$$

Since the algebraization of closed subschemes and morphisms are unique up to isomorphism, the algebraization of $G_n[p^l] \longrightarrow G_n$ gives a p^l-torsion subgroup $G'[p^l] \longrightarrow G'$ and the composition $G'[p^l] \longrightarrow G' \longrightarrow G'[p^{k-l}]$ is trivial. Furthermore, the morphism $G_n/G_n[p^l] \cong G_n[p^{k-l}]$ can also be algebraized.

For the BT groups, any BT group G_n can be represented as a colimit $\mathrm{colim}_k G_n[p^k]$ and $G_n[p^k]$ can be algebraized to $G'[p^k]$ for each k. Since the algebraization preserves the morphism, $G'[p^k]$ is still an inductive system and $\mathrm{colim}_k G'[p^k]$ is a BT group. Therefore the BT group G_n can be algebraized too. $\square$

We have just finished the proof of 3.1.1 and 1.0.6.

3.3.9 The truncated case

For a BT_n group G versally deformed over Y/k, by 3.3.2, G is unobstructed. But since 3.2.7(2) may not be true for BT_n, the lifting of G may not be unique and consequently the choice of Y_m may not be unique if m is large enough. But we still have the following weaker result:

Theorem 3.3.10. *Let Y be a smooth projective variety. If a BT_n group G is versally deformed over Y, liftable to W and $\Gamma(Y, T_Y) = 0$, then there exists a unique variety Y_n/W_n and BT_n(may not be unique) G_n/Y_n such that the following diagram*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G_n \arrow[d] \\ Y \arrow[r] \arrow[d] & Y_n \arrow[d] \\ k \arrow[r] & W_n \end{tikzcd}$$

is a fibre product.

Proof. We have known the existence from Step I, II and III. So it suffices to show the uniqueness of Y_n. We prove it by induction.

From 3.2.11, Y_2 is unique for any $n \geq 1$. By [21, Theorem 4.4(d)], we know for any $G' \in \mathrm{Def}_{Y_2}(G)$, the morphism

$$\mathrm{Aut}(G') \longrightarrow \mathrm{Aut}(G'[p^{n-1}]) \tag{3.11}$$

is trivial. Note we have $H^0(Y, t_G \otimes t_{G^*}) \cong H^0(Y, T_Y) = 0$ and the morphism (3.2) is null . So for any $G', G'' \in \mathrm{Def}_{Y_2}(G)$, G' and G'' are locally isomorphic. Restricting to $G'[p^{n-1}]$ and $G''[p^{n-1}]$, these local isomorphism are unique and then compatible on the overlapping. Thus we have

$$G'[p^{n-1}] \cong G''[p^{n-1}]$$

globally. By 3.2.4, all the deformation to Y_2 has the same obstruction class to the second order deformation. Then by 3.3.2, there exits a unique Y_3 admits the deformation.

Assume that Y_{k-1} is unique with $k-2 < n$ and all the deformations in $\mathrm{Def}_{Y_{k-1}}(G)$ share the same $p^{n-k+2}-$kernel, i.e. for any $G', G'' \in \mathrm{Def}_{Y_{k-1}}(G)$, globally

$$G'[p^{n-k+2}] \cong G''[p^{n-k+2}].$$

Then by 3.2.4 and 3.3.2, the obstruction to deform $G'[p^{n-k+2}]$ from Y_{k-1} gives a unique lifting $Y_{k-1} \longrightarrow Y_k$. And by the same argument in the Y_2 case, the different lifting of $G'[p^{n-k+2}]$ share the same $p-$ kernel, i.e. for any $K', K'' \in \mathrm{Def}_{Y_k}(G)$,

$$K'[p^{n-k+1}] \cong K''[p^{n-k+1}].$$

So we can keep induction until $k = n$. □

For higher order deformations, there may not exist a canonical choice of Y_m. However, since the lifting of BT_n is always unobstructed, using the result in Step IV, we have

Theorem 3.3.11. *Assumption as 3.3.10. There exists a smooth projective variety Y' over $W(k)$ and a BT_n*

group $G' \longrightarrow Y'$ *such that the following diagrams*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G' \arrow[d] \\ Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ k \arrow[r] & W \end{tikzcd}$$

are both fibre products.

3.4 A BT group with trivial Kodaira-Spencer map

Another extremal case is G with trivial Kodaira-Spencer map over Y, or equivalently, a Dieudonne crystal over $\mathrm{cris}(Y/W)$ with trivial Higgs field. Let $\mathcal{V}$ be the Dieudonne crystal $\mathbb{D}(G)$ associated to G with the Hodge filtration

$$0 \longrightarrow \omega \longrightarrow (\mathcal{V})_Y \longrightarrow t \longrightarrow 0$$

and trivial Higgs field $\theta : \omega \longrightarrow t \otimes \Omega^1_Y$. Let F and V be the Frobenius and Verschiebung for $\mathcal{V}$. Then over Y, $\ker F = \operatorname{im} V = \omega^\sigma$.

Theorem 3.4.1. *If a BT group* G *has trivial Kodaira-Spencer map over* Y*, then there exists a BT group* H *such that* $H^{(p)} = G$*.*

Proof. Since $\theta = 0$, the connection preserves the subbundle ω. Choose a lifting Y' of Y to W. Then the crystal $\mathcal{V}$ corresponds to a bundle with an integral connection. For notational simplicity, we still denote it as $(\mathcal{V}, \nabla)$. Let $L = \pi^{-1}(\omega) \subset \mathcal{V}$ be the inverse image of ω under $\pi : \mathcal{V} \longrightarrow \mathcal{V}_Y$. Since ∇ preserves ω, ∇ preserves L, $\nabla : L \longrightarrow L \otimes \Omega^1_{Y'}$.

For any affine open subset $U \subset Y'$, choose a lifting of the absolute Frobenius σ to U. Then $L^\sigma \subset \mathcal{V}^\sigma$. Since $\pi(L^\sigma) = \pi(V(\mathcal{V}))$, we have $L^\sigma = V(\mathcal{V}) \cong \mathcal{V}$. In particular, L^σ is isomorphic to a vector bundle over U. It implies that L itself can be viewed as a vector bundle over Y'.

It remains to define the Frobenius F_L and Verschiebung V_L for L locally. Over any U, $V : \mathcal{V}^\sigma \longrightarrow \mathcal{V}$ induces just the inclusion $L \longrightarrow L^\sigma \cong \mathcal{V}$, which can be chosen to be V_L. And $F(L^\sigma) = F(V(\mathcal{V})) = p.\mathcal{V} \subset L$. So F induces a morphism $F_L : L^\sigma \longrightarrow L$. Since both of F_L and V_L are restriction of F and V, we have $F_L \circ V_L = V_L \circ F_L = p.\mathrm{Id}$.

Therefore (L, ∇, F_L, V_L) corresponds to a Dieudonne crystal whose Frobenius pullback is isomorphic to $\mathcal{V}$. By [9, Main Theorem 1], such a crystal corresponds to a BT group H and $H^{(p)} = G$. $\square$

Chapter 4

Examples

In this chapter, we prove that some reduction of Mumford curves satisfies all the assumptions in the theorems. We state the main result in the first section. In Section 4.2, we introduce the basic definitions. By Lefschetz principle 4.2.4, a Mumford curve with the universal family can descend to a Witt ring whose special fibre X/C is smooth. The definition of Mumford curves implies the Dieudonne crystal $\mathbb{D}(X/C)$ of the abelian scheme X is a tensor product of $m+1$ rank 2 crystals:

$$\mathbb{D}(X) \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.$$

In Section 2.3, we set up some notation and basic facts of Tannakian categories. In Section 4.3, we prove two important lemmas (4.3.2, 4.3.3) in context of abstract Tannakian categories. They are key ingredients in determining the Tannakian groups of the rank 2 isocrystals and their Frobenius pullback.

In Section 4.4, we describe the structure of $\mathcal{V}_i$ in the terminology of Tannakian categories. It is shown in 4.4.2 that the Tannakian group of each isocrystals $\mathcal{V}_i$ is $SL(2)$. Furthermore, we investigate the tensor decomposition of the Frobenius morphism F. Firstly, it is shown in 4.4.3 that F can be decomposed to the tensor product of ϕ_i which are morphisms between rank 2 crystals. Secondly, imposing the generically ordinary assumption permits a refinement, i.e. the permutation s in 4.4.3 fixes an index, say 1. Lastly, we adjust ϕ_1 to be an actual Frobenius morphism of a rank 2 crystal. That requires us to prove that $\sigma^* - \mathrm{Id}$ on Pic $(C/W(k)_{\mathrm{cris}})$ is surjective. This step is in Section 4.5.

Summarizing the above results in Section 4.6, we construct a rank 2 Dieudonne crystal $\mathcal{V}$ and a rank 2^m unit root crystal $\mathcal{T}$, such that $\mathbb{D}(\tilde{X}/\tilde{C}) \cong \mathcal{V} \otimes \mathcal{T}$. We conclude the proof of 4.1.2 in Section 4.7 by studying the BT groups corresponding to $\mathcal{V}$ and $\mathcal{T}$.

4.1 Main result

Definition 4.1.1. For any prime number p and integer m, we say the pair $(\tilde{X} \longrightarrow \tilde{C}, k)$ satisfies $(*_{p,m})$ if it satisfies the following properties:

1. k is an algebraically closed field of characteristic p,
2. $\tilde{C}$ is a proper smooth curve over $W(k)$ and $\tilde{X} \longrightarrow \tilde{C}$ is a family of abelian varieties of dimension 2^m over $\tilde{C}$,
3. there exists a versally deformed height 2 Barsotti-Tate (BT) group $\tilde{G}$ and a height 2^m etale BT group $\tilde{H}$ over $\tilde{C}$ such that $\tilde{X}[p^\infty] \cong \tilde{G} \otimes \tilde{H} := \mathrm{colim}_n(\tilde{G}[p^n] \otimes \tilde{H}[p^n])$,
4. the reduction of $\tilde{X} \longrightarrow \tilde{C}$ at k is generically ordinary.

A height 2 BT group $\tilde{G}$ over $\tilde{C}$ is versally deformed if the Kodaira-Spencer map $T_{\tilde{C}} \to t_{\tilde{G}} \otimes t_{\tilde{G}^*}$ is an isomorphism, or equivalently the Higgs field $\theta_{\tilde{G}}$ (see Section 4.7) is an isomorphism.

Theorem 4.1.2. *Let $A \longrightarrow M$ be a Mumford curve, parameterizing principally polarized abelian varieties of dimension 2^m. For infinitely many primes p, there exists a pair $(\tilde{X} \longrightarrow \tilde{C}, k)$ satisfying $(*_{p,m})$ (see Definition 4.1.1) and*

$$(\tilde{X} \longrightarrow \tilde{C}) \otimes_{W(k)} \mathbb{C} = (A \longrightarrow M).$$

Remark 4.1.3. To choose p, it suffices to require that

1. $p > 2$, see 4.5.7
2. M admits a good reduction at the place p, see 4.2.4
3. the reflex field of M is splitting over p, see 4.4.6.

Then such primes have positive density among all primes.

Let H be the reduction of $\tilde{H}$ over C.During the proof of 4.1.2, we have that the crystal $\mathbb{D}(H)$ is isomorphic to $\mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$ (4.3.3). The Frobenius F is a morphism permuting $\mathcal{V}_i$.

4.1.4 For 1.0.2

The decomposition $\mathscr{E} \cong \mathcal{V} \otimes \mathcal{V}_2 \cdots \otimes \mathcal{V}_{m+1}$ as isocrystals follows directly from 4.1.2 and above remarks. So the reduction $\mathscr{X} \longrightarrow C$ of $\tilde{\mathscr{X}} \longrightarrow \tilde{C}$ satisfies (1) and (2) in 1.0.2.

By [37, 0.9], the universal family over a Shimura curve of Hodge type has maximal Higgs field, and hence the Higgs field of the special fibre $\mathscr{X}/C$ is also maximal.

4.1.5 For 1.0.3

We use $\tilde{X} \xrightarrow{\tilde{f}} \tilde{C}$ to denote the Mumford curve over the Witt ring $W(k)$, parametrizing abelian fourfolds. Then $\mathcal{E} := \mathbb{D}(X/C)$ has the decomposition $\mathcal{E} \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$ as F^2-isocrystals. We can identify $\mathcal{E}$ as its corresponding vector bundle with a connection $R^1\tilde{\pi}_*(\Omega^{\cdot}_{\tilde{X}/\tilde{C}})$.

For Condition (3) in 1.0.3, note $\mathbb{Q}_{p^{f_0}}$ is the field of definition. The self product of the polarization gives

$$\dim_{\mathbb{Q}_{p^{f_0}}} \Gamma(\tilde{C}, R^4\tilde{\pi}_*(\Omega^{\cdot}_{\tilde{X}/\tilde{C}})(2f))^{F^f} \otimes \mathbb{Q}_{p^{f_0}} \geq 1$$

for any integer f.

Base change to algebraically closed field $\bar{\mathbb{Q}}_p$ and $\mathcal{E}$ becomes a local system, with algebraic monodromy $G = SL(2)^{\times 3}$. Hence $\mathcal{E}$ corresponds to a tensor product of three standard representation of $SL(2, \bar{\mathbb{Q}}_p)$. Direct computation of the invariants gives $\Gamma(\tilde{C}, \wedge^4 R^1\tilde{\pi}_*(\Omega^{\cdot}_{\tilde{X}/\tilde{C}})) \otimes \bar{\mathbb{Q}}_{p^{f_0}} = 1$. We have

$$\dim_{\mathbb{Q}_{p^{f_0}}} \Gamma(\tilde{C}, R^4\tilde{\pi}_*(\Omega^{\cdot}_{\tilde{X}/\tilde{C}}))^{F^f - q^{2f}} \otimes \mathbb{Q}_{p^{f_0}} \leq 1.$$

Therefore (3) in 1.0.3 is satisfied.

For Condition (2) in 1.0.3, since $\mathcal{E} \cong \mathcal{V} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$ as F^{2f}-isocrystals for any natural number f and $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ are irreducible as isocrystals,

$$\wedge^2\mathcal{E} = S^2\mathcal{V}_2 \otimes S^2\mathcal{V}_3 \oplus S^2\mathcal{V} \otimes S^2\mathcal{V}_2 \oplus S^2\mathcal{V} \otimes S^2\mathcal{V}_3 \oplus \mathcal{O}_C$$

as direct sum of simple isocrystals. Thereby the algebra $\mathrm{End}(\wedge^2\mathcal{E})^{F^f}$ is isomorphic to $\mathbb{Q}_{p^f}^{\times 4}$ for some f.

For Condition (1) in 1.0.3, it follows from (2) in Theorem 4.1.2.

For the maximal Higgs field, since the universal family over the Shimura curve $\tilde{C}$ has maximal Higgs field (see [37, Theorem 0.9]), the Higgs field of the special fibre X/C is also maximal.

Therefore the special reduction of Mumford curve at k satisfies all the conditions of Theorem 1.0.3.

4.1.6 For 1.0.4

Let $A \longrightarrow M$ be the universal family over a Mumford curve defined over $\mathbb{C}$. The curve M is defined over the reflex field K. For any p which K splits over, $A \longrightarrow M$ admits a smooth and generically ordinary reduction $X \longrightarrow C$ over p ([32]). By the definition of Mumford curve, the image of

$$\rho_{\mathbb{C}} : \pi_1(M) \longrightarrow \mathrm{Aut}(R^1\pi_*(\mathbb{C}))$$

is $SL(2,\mathbb{C})^{\times 3}$. By Grothendieck specialization of algebraic monodromy, $\rho_{\mathbb{C}}$ factors through

$$\rho : \pi_1(C, \bar{\xi}) \longrightarrow \mathrm{Aut}(R^1\pi_*(\mathbb{C}))$$

with a surjection $\pi_1(M) \longrightarrow \pi_1(\bar{C}, \bar{\xi})$. By the comparison of the l-adic cohomology and de Rham cohomology $R^1 f_*(\mathbb{C}) \cong R^1 f_*(\mathbb{Q}_l) \otimes \mathbb{C}$ for any $l \neq p$, we know condition (1) and (2) holds for such C.

For condition (3), we can choose c to be a CM point on M. Since A_c is simple, there is a field $L \subset \mathrm{End}^o(A_c)$ with $[L : \mathbb{Q}] = 8$. We can choose a prime p such that L is unramified over p. Then look at the reduction $\bar{c}$ and $\mathbb{Q}[F_{\bar{c}}]$ is the center of $\mathrm{End}(X_{\bar{c}})$. Since $L \subset \mathrm{End}(X_{\bar{c}})$ is the maximal commutative subalgebra, $F \in L$ and in particular, F is unramified over p.

So with a careful choice of p, the resultant reduction of a Mumford curve satisfies all the conditions in 1.0.4.

4.2 The reduction of Mumford curves

4.2.1 Monodromy

In Chapter 1, we review the construction of a Mumford curve $M = \Gamma\backslash\mathfrak{h}$, from the representation of reductive group $\rho : G \longrightarrow \mathrm{Aut}(V)$ over $\mathbb{Q}$. Since $\mathfrak{h}$ is simply connected, $\pi_1(M) = \Gamma$. The local system V induces a monodromy $\Gamma \longrightarrow \mathrm{Aut}(V_{\mathbb{C}})$. Further, the tensor components $\mathbb{C}^2$ of $V_{\mathbb{C}}$ also admit representations of Γ and hence also monodromy. Since $\Gamma_{\mathbb{C}} \subset Q_{\mathbb{C}} \cong SL(2)^{\times 3}$, $\wedge^2\mathbb{C}^2$ is a trivial representation of Γ.

Definition 4.2.2. For any monodromy $\Gamma \longrightarrow GL(n)$, the *algebraic monodromy group* is defined to be the Zariski closure of the image of the monodromy. The *connected algebraic monodromy group* is the connected component of the identity of the algebraic monodromy.

Proposition 4.2.3. *The algebraic monodromy group induced by $\mathbb{C}^2$ in (1.1) is $SL(2,\mathbb{C})$ and that of $V_{\mathbb{C}}$ is the image of $SL(2,\mathbb{C})^{\times m+1}$ in $\mathrm{Aut}(V_{\mathbb{C}})$.*

Proof. From above, the monodromy induced by the representation of Q, is tensor of $m+1$ copies of monodromies $\mathbb{C}^2$. Let $K_i, 1 \leq i \leq n+1$ be the corresponding algebraic monodromy groups. Since $\wedge^2\mathbb{C}^2$ is a trivial representation of $\Gamma_{\mathbb{C}}$, $K_i \subset SL(2,\mathbb{C})$.

By [1], the connected algebraic monodromy on $V_{\mathbb{Q}}$ is a normal subgroup in the Hodge group $\rho(Q)$. Since Q is simple, $\rho(Q)$ is also simple over $\mathbb{Q}$. Thus the connected algebraic monodromy is $\rho(Q)$. Since $\rho(Q)_{\mathbb{C}} = \mathrm{im}\,(SL(2,\mathbb{C})^{\times m+1} \longrightarrow \mathrm{Aut}(V_{\mathbb{C}}))$ is connected, the connected complex algebraic monodromy of $V_{\mathbb{C}}$ is $\mathrm{im}\,(SL(2,\mathbb{C})^{\times m+1} \longrightarrow \mathrm{Aut}(V_{\mathbb{C}}))$.

Note the complex algebraic monodromy of $V_{\mathbb{C}}$ is im $(\prod_i K_i \longrightarrow \mathrm{Aut}(V_{\mathbb{C}}))$. Therefore

$$\mathrm{im}\,(\prod_i K_i \longrightarrow \mathrm{Aut}(V_{\mathbb{C}}))^o = \mathrm{im}\,(SL(2,\mathbb{C})^{\times m+1} \longrightarrow \mathrm{Aut}(V_{\mathbb{C}})).$$

Then necessarily, $K_i = SL(2,\mathbb{C})$ for each i. □

4.2.4 Lefschetz Principle

By Lefschetz Principle (see [30]), we mean the process that all the coefficients of polynomials, defining a variety of finite type over a field, generate a subring R of finite type over $\mathbb{Z}$, such that the variety can be defined over R. Note this process can be easily generalized to morphisms of finite type or vector bundles of finite rank.

Apply Lefschetz Principle to $A \longrightarrow M$ and the flat vector bundles induced by $\mathbb{C}^2$. We obtain these data can descend from K to a ring R finite type over $\mathbb{Z}$. Throwing away finite places, we can assume R is smooth over $\mathbb{Z}$. Let k be a residue field of R with characteristic> 2 such that M admits a good reduction over k. By smoothness of R, we have the a lifting from Spec $W_n(k)$ to Spec R:

$$\begin{tikzcd} \mathrm{Spec}\ k \arrow[r] \arrow[d] & \mathrm{Spec}\ R \arrow[d] \\ \mathrm{Spec}\ W_n(k) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\ \mathbb{Z} \end{tikzcd}$$

Therefore we find a morphism Spec $W(k) \longrightarrow$ Spec R.

Let $\tilde{X} \xrightarrow{\tilde{\pi}} \tilde{C}$ be the base change $A \longrightarrow M$ from Spec R to Spec $W(k)$ and $\mathcal{V}_i$ be the descent of the rank 2 flat bundle induced by $\mathbb{C}^2$ to $W(k)$. Let X, C be the special fibre of $\tilde{X}, \tilde{C}$. Let $\mathcal{E}$ be the Hodge bundle $\mathcal{E} = R^1\tilde{\pi}_*(\Omega^{\cdot}_{\tilde{X}/\tilde{C}})$ and $\mathcal{E}$ admits the Gauss-Manin connection. By [4, Theorem 6.6], the category of crystals on C is equivalent to the category of modules with an integrable connection (MIC). In particular, the Hodge bundle $\mathcal{E}$ corresponds to the Dieudonne crystal $R^1\pi_{*,\mathrm{cris}}(\mathcal{O}_X)$. Let us denote the crystal still as $\mathcal{E}$. The vector bundles $\mathcal{V}_i$ also correspond to crystals and denote the corresponding crystals as $\mathcal{V}_i$ as well. Then as crystals

$$\mathcal{E} \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.$$

4.3 Some lemmas on Tannakian categories

In this section, we prove some lemmas about general Tannakian categories. These lemmas will be applied to proving 4.1.2 in the next section.

Lemma 4.3.1. *For any $g \in GL(2)$, the centralizer $Z(g)$ of g has dimension ≥ 2 as a variety.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The centralizer of g

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

implies

$$bz = cy, (a-d)y = b(x-w).$$

Note $\dim GL(2) = 4$. As a subvariety of $GL(2)$, $Z(g)$ has dimension at least 2. $\square$

The Tannakian category of isocrystals on $C/W(k)$ is equivalent to $\mathrm{Rep}(P_{\mathrm{univ}})$.

Lemma 4.3.2. *Let $W_i, V_i \in \mathrm{Rep}(P_{\mathrm{univ}})$ be representations over $B(k)$, $1 \leq i \leq n$. Let $E \cong \otimes_i V_i$, $P_i = \mathrm{im}\,(P_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(V_i))$ and $Q_i = \mathrm{im}\,(P_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(W_i))$. Suppose we have that*

$$F : W_1 \otimes \cdots \otimes W_{m+1} \longrightarrow V_1 \otimes \cdots \otimes V_{m+1}$$

is an isomorphism between representations and $P_i = SL(2)$ for each i. Then

$$Q_i = GL(2) \text{ or } SL(2) \times \mu_k \text{ for some } k.$$

Let Q' be the image of $P_{\mathrm{univ}} \longrightarrow \prod Q_i$, and then the projections $Q' \longrightarrow Q_i$ are surjective for each i.

Proof. Let P' be $\mathrm{im}\,(Q' \longrightarrow \mathrm{Aut}(E) \cong GL(2^{m+1}))$. Then we have the following commutative diagram:

$$\begin{tikzcd}
Q_1 \times Q_2 \times Q_3 \cdots \times Q_{m+1} \arrow[d, hook] & Q' \arrow[l, hook'] \arrow[d] & SL(2)^{\times m+1} \arrow[dl, "\text{twisted by } F"] \\
GL(2)^{\times m+1} \arrow[r] & GL(2^{m+1}) &
\end{tikzcd}$$

Note $SL(2)^{\times m+1} \longrightarrow GL(2^{m+1})$ is twisted by F. The right triangle can be specified as

$$\begin{tikzcd}
Q' \arrow[d, two heads] & SL(2)^{\times m+1} \arrow[dl, two heads] \\
P' \arrow[d] & \\
GL(2^{m+1}) &
\end{tikzcd}$$

where P' is the common image.

Since $SL(2)$ is semisimple, so is $P'/Z(P')$. Since $\ker(Q' \longrightarrow GL(2^{m+1})) \subset \ker(GL(2)^{\times m+1} \longrightarrow GL(2^{m+1}))$, the kernel of $Q' \longrightarrow P'$ consists of just central elements. The group Q' is an extension of central elements and a semisimple group. Therefore Q' is reductive and P' is the adjoint group of Q'. Further, $Q' \longrightarrow P'$ induces a morphism from the derived group $[Q', Q']$ to P' which further induces a surjection to $P'/Z(P')$.

$$[Q', Q'] \longrightarrow Q' \longrightarrow P' \longrightarrow P'/Z(P').$$

If the projection of $[Q', Q']$ to some factor $GL(2)$ has dimension less than 3, then one of the projections must have dimension 4 because of $\dim P' = 3 \times 2^m$. So one of the projections would be $GL(2)$. Since the kernel of $Q' \longrightarrow P'$ is finite, P', as the image of $SL(2)^{m+1} \longrightarrow GL(2^{m+1})$ would have infinitely many centers, contradiction.

Now we have the projections of $[Q', Q']$ to each factor have precisely dimension 3. Therefore each projection has the form $SL(2) \times \mu_k$. By comparing the dimensions, $SL(2)^{\times m+1} \subset \mathrm{im}\,(Q' \longrightarrow GL(2)^{\times m+1})$. Then we have a lifting

$$SL(2)^{\times m+1} \longrightarrow [Q', Q'] \subset Q'$$

such that the right triangle is commutative

$$\begin{tikzcd}
Q_1 \times Q_2 \times Q_3 \cdots \times Q_{m+1} \arrow[d, hook] & Q' \arrow[l, hook'] \arrow[d] & SL(2)^{\times m+1} \arrow[l, dashed] \arrow[dl, "\text{twisted by } F"] \\
GL(2)^{\times m+1} \arrow[r] & GL(2^{m+1}) &
\end{tikzcd}.$$

Now we classify the elements with finite kernel in $\mathrm{Hom}(SL(2)^{\times m+1}, GL(2)^{\times m+1})$.

First, recall that all automorphisms of $SL(2)$ are inner and hence $\mathrm{Hom}(SL(2), GL(2))$ consists of the trivial morphism and the conjugation by some element in $GL(2)$. For any morphism $f \in \mathrm{Hom}(SL(2)^{\times m+1}, GL(2)^{\times m+1})$, restricting to each factor of $SL(2)$ gives $m+1$ inclusions $SL(2) \hookrightarrow GL(2)$. Explicitly,

$$(g_1, 1, 1, \cdots) \mapsto (\psi_{11}(g_1), \psi_{12}(g_1), \psi_{13}(g_1), \cdots)$$
$$(1, g_2, 1, \cdots) \mapsto (\psi_{21}(g_2), \psi_{22}(g_2), \psi_{23}(g_2), \cdots)$$
$$(1, 1, g_3, \cdots) \mapsto (\psi_{31}(g_3), \psi_{32}(g_3), \psi_{33}(g_3), \cdots).$$

Then $\psi_{11}(g_1)$ and $\psi_{21}(g_2)$ commute for any $g_i \in SL(2)$. Note all the automorphisms of $SL(2)$ are inner. So if neither of ψ_{11} and ψ_{12} is an identity, then there exists $h, k \in GL(2)$ such that ψ_{11}, ψ_{21} are conjugation by h and k, respectively. Then for any $g_i \in SL(2)$,

$$hg_1h^{-1}kg_2k^{-1} = kg_2k^{-1}hg_1h^{-1} \tag{4.1}$$

$$k^{-1}hg_1h^{-1}kg_2k^{-1}h = g_2k^{-1}hg_1, \tag{4.2}$$

Since $Z(k^{-1}g)$ in $GL(2)$ has dimension at least 2 by 4.3.1, we can choose $g_2 \in SL(2)$ such that $g_2 \neq \pm I$ and $g_2 \in Z(k^{-1}g)$. But then from (2), g_2 has to commute with g_1, i.e. $g_2 \in Z(SL(2)) = \pm 1$, contradiction. Therefore at least one of ψ_{11} and ψ_{21} is identity. Further, each column ψ_{*i} has at least m identities. So each factor $SL(2)$ is embedded into exactly one of the $m+1$ copies of $GL(2)$ and trivially to the others.

Then $\dim Q_i = 3$ or 4. Since $Q_i \subset GL(2)$, $Q_i = GL(2)$ or $SL(2) \times \mu_k$ for some integer k. $\square$

Lemma 4.3.3. *Assumptions as 4.3.2, there exist a permutation $s \in S_{m+1}$, dimension 1 representation L_i with $\otimes_i L_i$ trivial and isomorphisms*

$$\phi_i : W_i \longrightarrow V_{s(i)} \otimes L_i$$

such that

$$F = \otimes_i \phi_i.$$

4.3.4 Proof of 4.3.3

From the proof of 4.3.2, for each i, there exists a unique $P_j = SL(2)$ such that $P_j \hookrightarrow Q_i$. This inclusion is an isomorphism between P_j and Q_i^0, which is a conjugation by some $l \in GL(2)$. Without loss of generality, assume $i = 1$ and $j = 2$.

Note we have the following diagram:

$$\begin{tikzcd}
& P_{\text{univ}} \arrow[dl, two heads] \arrow[dr, two heads] & \\
Q_1 \arrow[dr, two heads, "f_1"] & & P_2 \arrow[dl, two heads, "f_2"'] \\
& PGL(2) &
\end{tikzcd}$$

The morphism f_1 is the usual quotient by the center $Q_i \longrightarrow PGL(2)$. The morphism f_2 is twisted by the conjugation by l.

Claim 1. this diagram is commutative.

Proof. We have

$$\begin{tikzcd}
& P_{\text{univ}} \arrow[d, two heads] \arrow[dr] & \\
Q_1 \times Q_2 \times Q_3 \cdots \times Q_{m+1} \arrow[d, hook] & Q' \arrow[l, hook'] \arrow[d] & \prod P_i \arrow[l, dashed] \arrow[dl, "\text{twisted by } F"] \\
GL(2)^{\times m+1} \arrow[r] & GL(2^{m+1}) &
\end{tikzcd}$$

For any $h \in P_{\text{univ}}$, let $(h_1, h_2, h_3, \cdots, h_{m+1})$ be the image of h in $\prod P_i$ and $(g_1, g_2, g_3, \cdots, g_{m+1})$ image in Q'. Then $\prod P_i \dashrightarrow Q'$ permutes the factors and sends $(h_1, h_2, h_3, \cdots)$ to $(lh_2l^{-1}, \cdots)$. Then $(lh_2l^{-1}, \cdots)$ and $(g_1, g_2, g_3, \cdots)$ have the same image under $GL(2)^{\times m+1} \longrightarrow GL(2^{m+1})$. Therefore $C_l(h_2) = tg_1$ for some scalar $t \in B(k)$ where C_l is the adjoint action by l. In particular, $f_2(h_2) = f_1(g_1)$.The claim is true. $\square$

Then $P_{\text{univ}} \longrightarrow Q_1 \times P_2$ factors through the limit of

$$\begin{tikzcd}
Q_1 \arrow[dr, two heads, "f_1"] & & P_2 \arrow[dl, two heads, "f_2"'] \\
& PGL(2) &
\end{tikzcd} \quad .$$

Claim 2. the limit of the above diagram is $P_2 \times Z(Q_1) = SL(2) \times \mu_n$ or $SL(2) \times \mathbb{G}_m$ with

$$P_2 \times Z(Q_1) \longrightarrow P_2 \qquad\qquad P_2 \times Z(Q_1) \longrightarrow Q_1$$

$$(h, k) \mapsto h \qquad\qquad (h, k) \mapsto (klhl^{-1}).$$

Proof. We can prove it directly: for any K' fitting in the diagram

$$\begin{tikzcd}
& K' \arrow[dl, two heads, "s_1"'] \arrow[dr, two heads, "s_2"] & \\
Q_1 \arrow[dr, two heads, "f_1"'] & & P_2 \arrow[ll, dashed, "C_l"] \arrow[dl, two heads, "f_2"] \\
& PGL(2) &
\end{tikzcd},$$

we construct the map

$$K' \longrightarrow Z(Q_1) \times SL(2)$$

$$k \mapsto (s_1(k)C_l(s_2(k))^{-1}, s_2(k)).$$

Since the lower triangle is commutative, the map is well defined and obviously it is unique. □

Consider the Tannakian category generated by $\{W_1, V_2\}$. Then it is isomorphic to $\mathrm{Rep}(K_{12})$ for some algebraic group K_{12}. By 2.3.5,

$$K_{12} = \mathrm{im}\,(P_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(W_1) \times \mathrm{Aut}(V_2)) \subset Q_1 \times P_2.$$

Therefore by Claim 2, $K_{12} \subset P_2 \times Z(Q_1) = SL(2) \times Z(Q_1)$.

If $Q_1 = GL(2)$, then $\dim K_{12} = 4$ and by $GL(2)$ connected, $K_{12} = SL(2) \times \mathbb{G}_m$.

If $Q_1^0 = SL(2)$ and $Z(Q_1) = \mu_n$, then $\dim K_{12}^0 = 3$ and hence $K_{12}^0 = SL(2)$. It suffices to determine the number of the connected components of K_{12}. Let ζ be a generator of μ_n. Then ζ and $-\zeta$ are in the same component of Q_1.

1. If $n \equiv 0 \pmod 4$, then $-\zeta$ is also a generator of μ_n. Therefore K_{12} has to be $SL(2) \times \mu_n$ to cover the whole Q_1.

2. If $n \equiv 2 \pmod 4$, then $\mu_n = \pm I \times \mu_{\frac{n}{2}}$ and hence $Q_1 \cong SL(2) \times \mu_{\frac{n}{2}}$. So besides $SL(2) \times \mu_n$, K_{12} also can be Q_1.

In summary, $K_{12} = SL(2) \times \mathbb{G}_m$ or $SL(2) \times \mu_k$ for some k.

Therefore as an irreducible $K-$representation, W_i is tensor of a $SL(2)-$representation and an irreducible μ_k or $\mathbb{G}_m$ representation, i.e. $W_i = V_{\sigma(i)} \otimes L_i$.

This is the end of the proof of 4.3.3.

4.4 Tensor decomposition of the Frobenius

Now we come back to the context of 4.1.2. The Dieudonne crystal $\mathcal{E} = R^1\pi_{\mathrm{cris}\,*}(\mathcal{O}_X)$ admits the Frobenius map:

$$\mathcal{E}^\sigma \xrightarrow{F} \mathcal{E}.$$

Then we have

$$F : \mathcal{V}_1^\sigma \otimes \mathcal{V}_2^\sigma \otimes \mathcal{V}_3^\sigma \cdots \otimes \mathcal{V}_{m+1}^\sigma \otimes B(k) \xrightarrow{\cong} \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \otimes B(k). \tag{4.3}$$

where $B(k)$ is the fractional field of $W(k)$. By 2.3.4, the category of isocrystals over C is Tannakian.

Proposition 4.4.1. *For each i, $P_i \cong SL(2, B(k))$ and $P_{\mathrm{univ}} \longrightarrow \prod P_i$ is surjective.*

Proof. Since by [4, Theorem 6.6] the crystals on $C/W(k)_{\mathrm{cris}}$ are exactly vector bundles with a connection over $\tilde{C}$, $\mathrm{Rep}_{\mathbb{C}}(P_{\mathrm{univ}} \otimes \mathbb{C})$ is a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{C}}(G_{\mathrm{univ}})$. By functorality, $G_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(E \otimes \mathbb{C})$ factors through $P_{\mathrm{univ}} \otimes \mathbb{C}$. By 4.2.3 and 2.3.3 ,

$$\begin{aligned} P \otimes \mathbb{C} =& \mathrm{im}\,(G_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(V)) = \mathrm{im}\,(SL(2,\mathbb{C})^{\times m+1} \longrightarrow \mathrm{Aut}(\mathbb{C}^{2^{\otimes m+1}})) \\ P_i \otimes \mathbb{C} =& \mathrm{im}\,(G_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(\mathbb{C}^2)) = SL(2,\mathbb{C}). \end{aligned}$$

The group P_i is a $B(k)-$form of $SL(2)$ and admits a faithful two dimensional representation. Therefore $P_i \cong SL(2, B(k))$.

Therefore $P = \mathrm{im}\,(P_{\mathrm{univ}} \longrightarrow \prod_i P_i \longrightarrow \mathrm{Aut}(E))$ is the same as $\prod_i P_i \longrightarrow \mathrm{Aut}(E)$, after tensoring with $\mathbb{C}$. Since it is faithfully flat, it is also true over $B(k)$ and $P = \mathrm{im}\,(\prod_i P_i \xrightarrow{\otimes} \mathrm{Aut}(E))$. Further, since the kernel of $(\prod P_i \longrightarrow \mathrm{Aut}(E))$ is finite, $\mathrm{im}\,(P_{\mathrm{univ}} \longrightarrow \prod_i P_i)$ is an algebraic subgroup of $\prod_i P_i$ with the same dimension. Since $\prod_i P_i = SL(2, B(k))^{\times m+1}$ are connected,

$$\mathrm{im}\,(P_{\mathrm{univ}} \longrightarrow \prod_i P_i) = \prod_i P_i,$$

i.e. $P_{\mathrm{univ}} \longrightarrow \prod_i P_i$ is surjective. □

Now we can interpret isomorphism (4.3) as follows. We already have a rank 2^{m+1} isocrystal admitting a tensor decomposition to $m+1$ rank 2 isocrystals, each corresponding to a standard representation of $SL(2)$. Then for another tensor decomposition to $m+1$ rank 2 isocrystals, just as left hand side of (4.3), we expect that each component also corresponds to a $SL(2)-$representation which is a corollary of 4.3.2.

Proposition 4.4.2. *For each i, $Q_i \cong SL(2, B(k))$.*

Proof. By 4.4.1, $\mathcal{V}_i$, $\mathcal{V}_i^\sigma$ and the isomorphism (4.3) satisfy the conditions of 4.3.2. Therefore the Tannakian group Q_i corresponds to $\mathcal{V}_i^\sigma$ is either $GL(2)$ or $SL(2) \times \mu_k$.

Furthermore, note $\mathcal{V}_i$ comes from $\mathbb{C}^2$ in (1.1). Since the local system $\mathbb{C}^2$ on M has a trivial determinant, each isocrystal $\mathcal{V}_i$ has $\wedge^2 \mathcal{V}_i = \mathcal{O}_{\tilde{C}}$. So correspondingly $\det Q_i = 1$ and thus $Q_i = SL(2)$. □

Apply 4.3.3 and note that W_1 and V_2 are the corresponding objects of $\mathcal{V}_1^\sigma \otimes B(k)$ and $\mathcal{V}_2 \otimes B(k)$ in $\mathrm{Rep}(P_{\mathrm{univ}})$, respectively. We have that there exist a permutation $s \in S_{m+1}$, rank 1 crystals $\mathcal{L}_i$ with $\otimes_i \mathcal{L}_i \cong \mathcal{O}_{\tilde{C}}$ and isomorphisms

$$\phi_i : \mathcal{V}_i^\sigma \otimes B(k) \longrightarrow \mathcal{V}_{s(i)} \otimes \mathcal{L}_i \otimes B(k)$$

such that

$$F = \otimes_i \phi_i.$$

In fact, we can refine ϕ_i to be a morphism between crystals.

Proposition 4.4.3. *There exist a permutation $s \in S_{m+1}$, rank 1 crystals $\mathcal{L}_i$ with $\otimes_i \mathcal{L}_i \cong \mathcal{O}_{\tilde{C}}$ and isomorphisms*

$$\phi_i : \mathcal{V}_i^\sigma \longrightarrow \mathcal{V}_{s(i)} \otimes \mathcal{L}_i$$

such that

$$F = \otimes_i \phi_i.$$

Proof. Since $\mathcal{E}$ is an $F-$crystal, we still have $F : \otimes \mathcal{V}_i^\sigma \longrightarrow \otimes \mathcal{V}_i$. Since each ϕ_i is a morphism between effective isocrystals, by 2.3.4, there exists an integer k_i such that $p^{k_i}\phi_i$ is a morphism in $\mathrm{Cris}(C)$. We can assume $p^{k_i}\phi_i \neq 0 \pmod p$ at the generic point. Then $p^{-k_1-k_2-\cdots-k_m}\phi_{m+1}$ is also a morphism in $\mathrm{Cris}(C)$. In fact, for any $U \subset C$ and $a_{m+1} \in \mathcal{V}_{m+1}^\sigma(U)$, we can find $a_1 \in \mathcal{V}_1^\sigma(U), a_2 \in \mathcal{V}_2^\sigma(U), \cdots$ such that

$$p^{k_i}\phi_i(a_i) \neq 0 \pmod p$$

for $1 \leq i \leq m$. Then $p^{-k_1-k_2-\cdots-k_m}\phi_{m+1}(a_{m+1}) \in \mathcal{V}_{m+1}(U)$. Otherwise,

$$F(a_1 \otimes a_2 \cdots \otimes a_{n+1}) = p^{k_1}\phi_1(a_1) \otimes_{B(k)} p^{k_2}\phi_2(a_2) \cdots \otimes_{B(k)} p^{-k_1-k_2-\cdots-k_m}\phi_{m+1}(a_{m+1})$$

is not in $\mathcal{V}_1 \otimes \mathcal{V}_2 \cdots \otimes \mathcal{V}_{m+1}(U)$.

□

A straightforward corollary of 4.4.3 is that

Corollary 4.4.4. *Viewed as a morphism between crystals, F still preserves pure tensors.*

Let η be the generic point of C and $\mathcal{V}_{i,\eta}$ denote the restriction of $\mathcal{V}_i$ to the crystalline site $\mathrm{cris}(\eta/W(k))$.

Since C parametrizes a family of polarized abelian varieties (with a level structure), it admits a map to the moduli scheme $\mathcal{A}_{2^m,d,n} \otimes k$. If the image intersects with the ordinary locus in $\mathcal{A}_{2^m,d,n} \otimes k$, we say "$C$ intersects the ordinary locus" for simplicity. Note since the ordinary locus is open in $\mathcal{A}_{2^m,d,n} \otimes k$, the statement is equivalent to the universal family over C is generically ordinary. Let

$$0 \longrightarrow \omega \longrightarrow \mathcal{E} \longrightarrow \alpha \longrightarrow 0$$

be the weight 1 Hodge filtration associated to $\tilde{X}/\tilde{C}$. Then from the definition of Mumford curves, especially the action of Hodge group Q on V, we know ω is constructed from a line bundle $\mathscr{L}$ in $\mathcal{V}_i$ for some i, say $i = 1$, then

$$\omega \cong \mathscr{L} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}. \tag{4.4}$$

Correspondingly $\alpha \cong \mathcal{V}_1/\mathscr{L} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$ and the Hodge filtration of $\mathcal{E}$ comes from a filtration $\mathscr{L} \subset \mathcal{V}_1$:

$$\mathscr{L} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \subset \mathcal{E} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.$$

Base change from $W(k)$ to k. Denote the reduction of $\tilde{C}$ over k as C and the reduction of $\mathcal{E}$ as $\mathcal{E}_C$. Then the Frobenius ${\mathcal{E}_C}^{(p)} \xrightarrow{F} \mathcal{E}_C$ factors through ${\alpha_C}^{(p)}$ and then we have the conjugate spectral sequence:

$$0 \longrightarrow {\alpha_C}^{(p)} \longrightarrow \mathcal{E}_C \longrightarrow {\omega_C}^{(p)} \longrightarrow 0. \tag{4.5}$$

Proposition 4.4.5. *If C intersects the ordinary locus, then $s(1) = 1$.*

Proof. Let c be a closed point in the intersection of ordinary locus and C. Then restricted to c, consider the composition $F' : \mathcal{E}_C^{(p)} \longrightarrow \alpha_C$ in the following diagram

$$\begin{tikzcd} & \omega_C \arrow[d] \\ \mathcal{E}_C^{(p)} \arrow[r, "F_C"] \arrow[dr, dashed, "F'"] & \mathcal{E}_C \arrow[d, "\pi"] \\ & \alpha_C \end{tikzcd} .$$

Since X_c is ordinary, F'_c is surjective.

If $s(1) \neq 1$, Without loss of generality, suppose $s(1) = 2$. Note by 4.4.3,

$$\begin{aligned} F'(\mathcal{E}_C^{(p)}) &= \pi \circ F_C(\mathcal{V}_1^{(p)} \otimes \mathcal{V}_2^{(p)} \otimes \mathcal{V}_3^{(p)} \cdots \otimes \mathcal{V}_{m+1}^{(p)})_C \\ &= \pi \circ \otimes \phi_i(\mathcal{V}_1^{(p)} \otimes \mathcal{V}_2^{(p)} \otimes \mathcal{V}_3^{(p)} \cdots \otimes \mathcal{V}_{m+1}^{(p)})_C \end{aligned}$$

From the conjugate spectral sequence, F_C factors through $\alpha_C^{(p)}$ and $\alpha = (\mathcal{V}_1/\mathscr{L}) \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{n+1}$, thus $\phi_1 : \mathcal{V}_{1\,C}^{(p)} \longrightarrow \mathcal{V}_{2\,C} \otimes \mathcal{L}_{1\,C}$ factors through $\mathcal{V}_{1\,C}^{(p)}/\mathscr{L}^{(p)}$, and the image of $\bar{\phi}_1$ has rank 1. But $\dim_k \mathcal{V}_{2|c} = 2$. So F' can not be surjective. Contradiction. $\square$

Therefore we have

$$\phi_1 : \mathcal{V}_1^{\sigma} \longrightarrow \mathcal{V}_1 \otimes \mathcal{L}_1 \tag{4.6}$$

Remark 4.4.6. The Mumford curve M is defined over the reflex field K, and let $\mathfrak{p}$ be the prime of K over p.

Let $r = [K_{\mathfrak{p}} : \mathbb{Q}_p]$. Then by [32, Theorem 1.2], there are two Newton polynomials in C/k, it is either $\{2^{m+1+\epsilon(D)} \times \frac{1}{2}\}$ or $\{2^{m+1-r+\epsilon(D)} \times 0, 2^{m+1-r+\epsilon(D)}.\binom{r}{i} \times \frac{i}{r} \cdots, 2^{m+1-r+\epsilon(D)} \times 1\}$. So C intersects with ordinary locus if and only if $r = 1$.

So there are infinitely many prime p over which the reduction of Mumford curve at p is generically ordinary.

4.5 The surjectivity of $(\sigma^* - \mathrm{Id})$ on the Picard group

Our purpose is to construct a rank 2 Dieudonne crystal in the tensor decomposition of $\mathcal{E}$. We already have

$$\phi_1 : \mathcal{V}_1 \longrightarrow \mathcal{V}_1 \otimes \mathcal{L}_1.$$

So it only remains to "eliminate" $\mathcal{L}_1$. We can achieve this goal in next section and the key ingredient is 4.5.1 which we will prove in this section.

Let σ be the absolute Frobenius of C/k and Pic $(C/W(k)_{\mathrm{cris}})$ denote the group of the rank 1 crystals on C. The following general principle guarantees that there exists a rank 1 crystal $\mathscr{L}'$ such that $\mathcal{L}_1 = (\sigma^* - \mathrm{Id})(\mathscr{L}')$.

Proposition 4.5.1. *The group endomorphism $(\sigma^* - \mathrm{Id})$ of Pic $(C/W(k)_{\mathrm{cris}})$ is surjective for any positive integer f.*

4.5.2 The proof

Lemma 4.5.3. $(\sigma^* - \mathrm{Id})$ *acts on* $W(k)$ *surjectively.*

Proof. Since k is algebraically closed, $(\sigma^* - \mathrm{Id})$ acts on k surjectively. Then for any $b \in W(k)$, we can find

$$(\sigma^* - \mathrm{Id})(a_0) = b - pb_1$$

and there exists $a_1, a_2, a_3, \cdots$ such that

$$(\sigma^* - \mathrm{Id})(a_1) = b_1 - pb_2,$$

$$(\sigma^* - \mathrm{Id})(a_2) = b_2 - pb_3,$$

$$(\sigma^* - \mathrm{Id})(a_3) = b_3 - pb_4 \cdots$$

Then $(\sigma^* - \mathrm{Id})(\sum_i p^i a_i) = b$. In fact, since $W(k)$ is p-adically complete, $\sum p^i a_i \in W(k)$ and $(\sigma^* - \mathrm{Id})(\sum p^i a_i) - b$ is contained in $p^n W(k)$ for any n. Therefore $(\sigma^* - \mathrm{Id})(a + \sum p^i a_i) - b = 0$. $\square$

Now we recall the definition of Atiyah class. For a more detailed explanation, we refer the reader to [20, 10.1].

Let $\mathscr{I}$ be the ideal sheaf of the diagonal set of $\tilde{X} \times \tilde{X}$ and $\mathcal{O}_{2\Delta} = \mathcal{O}_{\tilde{X} \times \tilde{X}} / \mathscr{I}^2$.

Definition 4.5.4. For any smooth proper variety $\tilde{X}$ and vector bundle V over $\tilde{X}$, the Atiyah class is the extension class of

$$0 \longrightarrow V \otimes \Omega^1_{\tilde{X}} \longrightarrow p_{1*}(p_2^* V \otimes \mathcal{O}_{2\Delta}) \longrightarrow V \longrightarrow 0.$$

Atiyah class is the unique obstruction to the existence of a connection on V.

By [42, Remark 3.7], the Atiyah class of any line bundle coincides with its first Chern class. So line bundles with a connection over a curve are exactly those of degree 0.

Lemma 4.5.5. *The restriction of* $(\sigma^* - \mathrm{Id})$ *to* $\mathrm{Pic}^0(C/k_{\mathrm{cris}})$ *is surjective.*

Proof. Note the rank 1 crystal on the site C/k_{cris} is equivalent to a line bundle on C/k with connection. For any $\mathscr{L} \in \mathrm{Pic}\ (C/k_{\mathrm{cris}})$, $\sigma^*(\mathscr{L}) = \mathscr{L}^p$. So it suffices to show that for any degree 0 line bundle with connection $(\mathscr{L}, \nabla)$, there exists a line bundle with connection (L, ∇_L) such that

$$(L, \nabla_L)^{p-1} \cong (\mathscr{L}, \nabla).$$

Since k is algebraically closed, the Jacobian $\mathrm{Jac}(C/k)$ is a divisible group. Therefore we always can find a line bundle $L \in \mathrm{Jac}(C/k)$ such that $L^{p-1} \cong \mathscr{L}$.

Note the set of connections of $\mathscr{L}$ is a torsor under

$$\mathrm{Hom}(\mathscr{L}, \mathscr{L} \otimes \omega_C) = \Gamma(\omega_C) = k^g.$$

The same for L. For any two connections ∇_L, ∇'_L on L, let $\nabla_L - \nabla'_L = h \in \Gamma(\omega_C)$.

Thus to find the connection ∇_L, it suffices to show the $(p-1)-$th power is an injection from the connections on L to the connections on $\mathscr{L}$. Then for any local section $\otimes_i s_i$,

$$\begin{aligned}
&((\nabla_L + h)^{p^f-1} - \nabla_L^{p^f-1})(\otimes_{i=1}^{p^f-1} s_i) \\
=&\sum_{i=1}^{p^f-1} \cdots \otimes h.s_i \otimes \cdots \\
=&(p^f-1)(\prod_i s_i)h.(1 \otimes 1 \otimes \cdots \otimes 1) \neq 0.
\end{aligned}$$

Therefore for any connection $\nabla_{\mathscr{L}}$, if $g = \nabla_{\mathscr{L}} - \nabla_L^{p-1}$, then $(\nabla_L + \frac{g}{p^f-1})^{p-1} = \nabla_{\mathscr{L}}$. So $(\sigma^* - \mathrm{Id})$ acts on $\mathrm{Pic}^{\,0}(C/k_{\mathrm{cris}})$ surjectively. ☐

Lemma 4.5.6. $(\sigma^* - \mathrm{Id})$ *maps* $H^1(C/W(k)_{\mathrm{cris}}, \mathcal{O}_{\tilde{C}})$ *to itself surjectively.*

Proof. By comparison theorem,

$$H^1(C/W(k)_{\mathrm{cris}}, \mathcal{O}_{\tilde{C}}) \cong \mathbb{H}^1(\tilde{C}, \Omega^\cdot_{\tilde{C}}) \cong W(k)^{2g}.$$

Let N denote the free $W(k)-$module with σ^* action. Then $V := N/pN$ is a $k-$vector space with $p-$linear action. By a result in [39, Page 143],

$$V = V_s \oplus V_n$$

where V_s is the semisimple part and V_n the nilpotent part. On V_n, since σ^* acts nilpotently, $(\sigma^* - \mathrm{Id})$ is invertible and hence surjective. On V_s, by 4.5.3, we can find λ such that $(\sigma^* - \mathrm{Id})(\lambda) = 1$. Then for each k, $(\sigma^* - \mathrm{Id})(\lambda x_k) = x_k$. Therefore $(\sigma^* - \mathrm{Id})$ acts on V surjectively.

Back to N, for any $b \in N$, we can choose a_0 such that

$$(\sigma^* - \mathrm{Id})(a_0) = b - pb_1.$$

Then choose a_1 such that

$$(\sigma^* - \mathrm{Id})(a_1) = b_1 - pb_2.$$

Following this way, we can find $a_2, a_3, \cdots$. Similar to the proof of 4.5.3, we have

$$(\sigma^* - \mathrm{Id})(a_0 + pa_1 + p^2 a_2 + \cdots + p^n a_n + \cdots) = b.$$

□

Now we can prove 4.5.1:

Proof. Note Pic $(C/W(k)_{\mathrm{cris}}) \cong H^1(C/W(k)_{\mathrm{cris}}, \mathcal{O}_C^*)$. We have the sequence

$$0 \longrightarrow (1 + p\mathcal{O}_C)^* \longrightarrow \mathcal{O}_C^* \longrightarrow (\mathcal{O}_C/p)^* \longrightarrow 0$$

and $(\sigma^* - \mathrm{Id})$ acts on the long exact sequence. Since char $k > 2$, the exponential and logarithm maps converge and thus give an isomorphism between abelian groups

$$\mathcal{O}_C \cong (1 + p\mathcal{O}_C)^*.$$

So the cohomology groups are isomorphic:

$$H^1(C/W(k)_{\mathrm{cris}}, \mathcal{O}_C) \cong H^1(C/W(k)_{\mathrm{cris}}, (1 + p\mathcal{O}_C)^*).$$

We have the long exact sequence

$$\begin{aligned} &H^1(C/W(k)_{\mathrm{cris}}, \mathcal{O}_C) \longrightarrow H^1(C/W(k)_{\mathrm{cris}}, \mathcal{O}_C^*) \xrightarrow{g} \\ &H^1(C/W(k)_{\mathrm{cris}}, (\mathcal{O}_C/p)^*) \cong \mathrm{Pic}\ (C/k_{\mathrm{cris}}) \longrightarrow H^2(C/W(k)_{\mathrm{cris}}, \mathcal{O}_C) \end{aligned} \quad .$$

By [4, Theorem 6.6] , the category of crystals on C is equivalent to the category of vector bundles with a connection on $\tilde{C}$. Therefore Pic $(C/W(k)_{\mathrm{cris}})$ is isomorphic to the group of line bundles with a connection on $\tilde{C}$ and g is the pull back of such line bundle from $\tilde{C}$ to C. Therefore im $g \subset$ Pic ${}^0(C/k_{\mathrm{cris}})$.

Since the obstruction to deform the line bundle from C to $\tilde{C}$ vanishes and the deformation preserves the degree, Pic ${}^0(\tilde{C}) \longrightarrow$ Pic ${}^0(C)$ is surjective. In fact, for any degree 0 line bundle $\mathscr{L}$ on $\tilde{C}$, it corresponds to a divisor $\sum_i n_i p_i$ with each p_i a $k-$point. Then by Hensel's lemma, each p_i lifts to a $W(k)-$point $\tilde{p}_i$(though not uniquely). Let $\sum_i n_i \tilde{p}_i = \tilde{\mathscr{L}} \in$ Pic ${}^0(\tilde{C})$ and then $\tilde{\mathscr{L}}$ reduces to $\mathscr{L}$.

For the connection, for any $(\mathscr{L}, \nabla) \in \text{Pic}\ ^0(C/k_{\text{cris}})$, choose a lifting $\tilde{\mathscr{L}} \in \text{Pic}\ ^0(\tilde{C})$ of $\mathscr{L}$ and a connection $\tilde{\nabla}$ on $\tilde{\mathscr{L}}$. Let ∇' be the reduction of $\tilde{\nabla}$, then $\nabla' - \nabla = f \in \Gamma(\omega_C)$. Choose $\tilde{f} \in \Gamma(\omega_{\tilde{C}})$ such that $\tilde{f}$ reduces to f. Then $\tilde{\nabla} - \tilde{f}$ reduces to ∇. And $g(\tilde{\mathscr{L}}, \tilde{\nabla} - \tilde{f}) = (\mathscr{L}, \nabla)$.

Therefore $\text{im}\, g = \text{Pic}\ ^0(C/k_{\text{cris}})$. So we have the following sequence:

$$H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \longrightarrow H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^*) \longrightarrow \text{Pic}\ ^0(C/k_{\text{cris}}) \longrightarrow 0$$

By 4.5.5 and 4.5.6, we know that $(\sigma^* - \text{Id})$ induces surjective endomorphisms on $H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C)$ and $\text{Pic}\ ^0(C/k_{\text{cris}})$. Therefore $(\sigma^* - \text{Id})$ maps $H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^*)$ surjectively on itself. □

Remark 4.5.7. In the proof of 4.5.1, we use the convergence of exponential and logarithm, which are true if and only if the characteristic $p > 2$.

4.6 The Dieudonne crystal $\mathcal{V}$ and the unit crystal $\mathcal{T}$

Now by 4.5.1 we can choose $\mathscr{L}' \in \text{Pic}\ (C/W(k)_{\text{cris}})$ such that $(\sigma^* - \text{Id})(\mathscr{L}') = \mathcal{L}_1^{-1}$(for $\mathcal{L}_1$ see 4.6). Then ϕ_1 induces an isomorphism

$$\gamma : \mathcal{V}_1^\sigma \otimes \mathscr{L}'^\sigma \otimes B(k) \longrightarrow \mathcal{V}_1 \otimes \mathscr{L}' \otimes B(k). \tag{4.7}$$

Let $\mathcal{V} = \mathcal{V}_1 \otimes \mathscr{L}$. Similarly, we have the isomorphism

$$\beta : \mathcal{V}_2^\sigma \otimes \mathcal{V}_3^\sigma \cdots \otimes \mathcal{V}_{m+1}^\sigma \otimes (\mathscr{L}'^{-1})^\sigma \otimes B(k) \longrightarrow \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \otimes \mathscr{L}'^{-1} \otimes B(k).$$

Denote $\mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \otimes \mathscr{L}'^{-1}$ as $\mathcal{T}$. Therefore as crystals,

$$\mathcal{E} \cong \mathcal{V} \otimes \mathcal{T}$$

and as a morphism between crystals $F = \gamma \otimes \beta$. Then $V = pF^{-1} = p\gamma^{-1} \otimes \beta^{-1}$.

Lemma 4.6.1. *The morphism $\beta : \mathcal{T}^\sigma \longrightarrow \mathcal{T}$ is an isomorphism between crystals.*

Proof. We have known that $\gamma \neq 0 \pmod p$. Over C, 4.5shows the Frobenius

$$F_C = \gamma_C \otimes \beta_C : \mathcal{E}_C{}^\sigma \longrightarrow \mathcal{E}_C$$

induces an injection

$$\alpha_C^{(p)} = (\mathcal{V}_1/\mathscr{L} \otimes \bar{\mathscr{L}}')_C^{(p)} \otimes \mathcal{T}_C^{(p)} \longrightarrow \mathcal{E}_C \cong \mathcal{V}_C \otimes \mathcal{T}_C.$$

Therefore β_C is an isomorphism between $\mathcal{T}_C^\sigma$ and $\mathcal{T}_C$.

Note the fact that for any $W(k)-$algebra R and any $r \in R$, if the image $\bar{r} \in \bar{R}$ over k is a unit, then r is a unit in R. So β is an isomorphism between crystals $\mathcal{T}^\sigma$ and $\mathcal{T}$. $\square$

Then β^{-1} is also a morphism between crystals. Since $V = pF^{-1} = p\gamma^{-1} \otimes \beta^{-1}$, so is $p\gamma^{-1}$. Therefore $F_{\mathcal{V}} := \gamma$ and $V_{\mathcal{V}} := \gamma^{-1}$ can serve as Frobenius and Verschiebung of $\mathcal{V}$, which makes $\mathcal{V}$ a Dieudonne crystal. The fact that $p^{-k'}\beta^{-1}$ and $p^{-k}\beta$ are isomorphisms implies $\mathcal{T}$ is a unit root crystal. We have the following summary.

Corollary 4.6.2.

$$(\mathcal{V}, F_{\mathcal{V}} = p^k\gamma, V_{\mathcal{V}} = p^{k'+1}\gamma^{-1})$$

is a Dieudonne crystal,

$$(\mathcal{T}, F_{\mathcal{T}} = p^{-k}\beta)$$

is a unit root crystal and

$$(\mathcal{E}, F) \cong (\mathcal{V}, F_{\mathcal{V}}) \otimes (\mathcal{T}, F_{\mathcal{T}}).$$

The Hodge filtration of $\mathcal{E}$ comes from a sub line bundle $\mathscr{L} \otimes \mathscr{L}'$ of $\mathcal{V}$.

Let the filtration $\mathrm{Fil}_{\mathcal{V}}$ be $\mathscr{L} \otimes \mathscr{L}' \subset \mathcal{V}$ and $\mathrm{Fil}_{\mathcal{T}}$ be the trivial filtration.

Now we switch to BT groups.

4.7 The Barsotti-Tate groups corresponding to $\mathcal{V}$, $\mathcal{T}$ and $\mathcal{E}$

From [9, Main Theorem 1], we know that over a smooth curve C/k, the category of finite locally free Dieudonne crystals on $\mathrm{cris}(C/W(k))$ is equivalent to the category of BT groups on C. Obviously $(\mathcal{E}, F, V)$ corresponds to $X[p^\infty]$. Let G be the Barsotti-Tate (BT) group over C corresponding to $(\mathcal{V}, F_{\mathcal{V}}, V_{\mathcal{V}})$. From [9], we know the BT group $\bar{G}$ induces a filtration of $\mathbb{D}(\bar{G})_C = \mathcal{V}_C$:

$$0 \longrightarrow \omega_G \longrightarrow \mathcal{V}_C \longrightarrow t_{G^*} \longrightarrow 0. \tag{4.8}$$

Lemma 4.7.1. *The above filtration 4.8 coincides with the filtration* $\mathrm{Fil}_{\mathcal{V}} \pmod p$.

Proof. From 4.6.2, $\ker F_{\mathcal{V}\,C} = (\mathscr{L} \otimes \mathscr{L}')_C^{(p)}$. By [9, Theorem 2.5.2 and Remark 2.5.5], the subbundle of $\mathcal{V}$ satisfying this condition is unique and $\omega_G \cong \mathscr{L} \otimes \mathscr{L}'_C$. $\square$

Then the filtration $\mathrm{Fil}_{\mathcal{V}}$ is just

$$0 \longrightarrow \omega_G \longrightarrow \mathcal{V}_C \longrightarrow t_{\bar{G}^*} \longrightarrow 0.$$

Note $\mathcal{V}_C$ admits a connection $\nabla : \mathcal{V}_C \longrightarrow \mathcal{V}_C \otimes \Omega_C$. The connection and the filtration induce the Higgs field: $\theta_G : \omega_G \longrightarrow t_G \otimes \Omega_C$.

$$\begin{tikzcd} 0 \arrow[r] & \omega_G \arrow[r] \arrow[drr, bend right=10] & \mathcal{V}_C \arrow[d, "\bar{\nabla}"] & & \\ & & \mathcal{V}_C \otimes \Omega^1_C \arrow[r] & t_{\bar{G}^*} \otimes \Omega^1_C \arrow[r] & 0. \end{tikzcd}$$

Since $\mathcal{T}$ is a unit root crystal, by [3, 2.4.10], $\mathcal{T}$ comes from an etale BT group $\bar{H}$ over C. In particular, $\mathbb{D}(\bar{H}[p^n]) \cong \mathcal{T}/p^n$ and each truncated $\mathcal{T}/p^n$ comes from a local system [8, Theorem 2.2]

$$\rho_n : \pi_1(C, c) \longrightarrow GL(4, \mathbb{Z}/p^n).$$

Then there exists a finite etale covering $f_n : C' \longrightarrow C$ such that $\pi_1(C', c) \cong \ker \rho_n$. Therefore we have $f_n^*(\mathcal{T}/p^n) \cong \mathcal{O}_{C/W_n}^{\oplus m}$ as unit root $F-$crystals. By [3, 2.4.1], over smooth curve C, the category of finite locally free etale group schemes is equivalent to the category of $p-$torsion unit crystals. Thus

$$f_n^*(\bar{H}[p^n]) \cong (\mathbb{Z}/p^n)^{\oplus m}. \tag{4.9}$$

Definition 4.7.2. Define the binary operation between two BT groups:

$$G \otimes H := \mathrm{colim}_n(G[p^n] \otimes_{\mathbb{Z}} H[p^n]).$$

Remark 4.7.3. Note in general $G \otimes H$ is just an abelian sheaf rather than a group scheme. But in our case, H is etale and $G \otimes H$ is indeed a BT group and $(G \otimes H)[p^n] = G[p^n] \otimes_{\mathbb{Z}} H[p^n]$.

Proposition 4.7.4.

$$X[p^n] \cong \bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n].$$

Proof. We will show that $\mathbb{D}(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n]) = \mathcal{V} \otimes \mathcal{T}/p^n = \mathcal{E}/p^n($ 4.6.2$)$ as Dieudonne crystals. Over C',

$$\mathbb{D}_{C'}(f_n^*(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n])) \cong f_n^*(\mathcal{V}/p^n)^{\oplus m} \cong f_n^*\mathcal{V}/p^n \otimes_{\mathcal{O}_{\tilde{C}}} f^*\mathcal{T}/p^n \tag{*}$$

as Dieudonne crystals. Both sides have effective descent datum with respect to $C' \longrightarrow C$. For any $g \in$

Aut(C'/C), g^* acts on both of $f^*(\bar{G}[n]\otimes_{\mathbb{Z}}\bar{H}[n])$ and $f_n^*\mathcal{V}/p^n\otimes_{\mathcal{O}_{\tilde{C}}} f^*\mathcal{T}/p^n$ which is compatible with the functor $\mathbb{D}_{C'}$:

$$\begin{tikzcd} f^*(\bar{G}[n]\otimes_{\mathbb{Z}}\bar{H}[n]) \arrow[r, "g^*"] \arrow[d, "\mathbb{D}_{C'}"] & f^*(\bar{G}[n]\otimes_{\mathbb{Z}}\bar{H}[n]) \arrow[d, "\mathbb{D}_{C'}"] \\ f^*(\mathcal{V}\otimes\mathcal{T}/p^n) \arrow[r, "g^*"] & f^*(\mathcal{V}\otimes\mathcal{T}/p^n) \end{tikzcd}$$

is commutative(we leave the details leave to the reader). Therefore the isomorphism (*) between effective descent datum also descends to C.

Then we have

$$\mathbb{D}_C(\bar{G}[n]\otimes_{\mathbb{Z}}\bar{H}[n]) = (\mathcal{V}\otimes\mathcal{T}/p^n, F_{\mathcal{V}}\otimes F_{\mathcal{T}}, V_{\mathcal{V}}\otimes F_{\mathcal{T}}^{-1}).$$

Since C is smooth over an algebraically closed field k, it has locally p-basis. Therefore we can apply [3, 4.1.1], the Dieudonne functor is fully faithful, so

$$\bar{G}[n]\otimes_{\mathbb{Z}}\bar{H}[n]\cong X[p^n].$$

∎

Corollary 4.7.5. $\bar{G}\otimes H = X[p^\infty]$.

To complete the proof of 4.1.2, it remains to show the isomorphism in 4.7.4 lifts to $\tilde{C}$.

Proposition 4.7.6. *The BT group G is versally deformed over the curve C.*

Proof. By [37, Theorem 0.9], any Shimura curve of Hodge type admits the maximal Higgs field. So $\tilde{C}$ and thus C has maximal Higgs field:

$$\begin{tikzcd} 0 \arrow[r] & \omega_C \arrow[r] \arrow[drr, bend right] & \mathcal{E}_C \arrow[d, "\bar{\nabla}"] & & \\ & & \mathcal{E}_C\otimes\Omega_C^1 \arrow[r] & \alpha_C\otimes\Omega_C^1 \arrow[r] & 0 \end{tikzcd}$$

the induce map $\theta:\omega_C\longrightarrow\alpha_C\otimes\Omega_C^1$ is an isomorphism.

By 4.6.2, the filtration of $\mathcal{E}_C$ comes from that of $\mathcal{V}_C$. So $\theta=\theta_G\otimes\mathrm{Id}_{\mathcal{T}}$. Therefore the Higgs field of $\mathcal{V}_C/C$ is maximal and combining with 7.1.9, it implies $\omega_G\cong t_{G^*}\otimes\Omega_C^1$. By [21, A.2.3.6], the BT group G is versally deformed over C. ∎

From 1.0.6, such a curve C admits a lifting to $W(k)$ over which $\bar{G}$ admits a lifting as a BT group.

Proposition 4.7.7. *The lifting of C coincides with $\tilde{C}$.*

Proof. From [33, V, Theorem 1.6], it is known that the lifting of the BT group G is equivalent to lifting the filtration $\omega_G \hookrightarrow \mathcal{V}_C$. By 7.1.9, it is equivalent to lifting $\mathscr{L} \otimes \mathscr{L}' \hookrightarrow \mathcal{V}_C$ and hence the curve $\tilde{C}/W(k)$ admits a lifting of G. From 1.0.6, we know the lifting of G is unique. $\square$

Let $\tilde{G}$ be the lifting of G on $\tilde{C}$ and $\tilde{H}$ be the lifting of H on $\tilde{C}$. Since H is etale, $\tilde{H}$ is etale and unique up to isomorphism. Since $\tilde{H}$ is étale, $\tilde{G} \otimes \tilde{H}$ is a BT group.

Proposition 4.7.8.

$$\tilde{X}[p^\infty] \cong \tilde{G} \otimes \tilde{H}.$$

Proof. In 4.7.4, we have shown that $X[p^\infty] \cong G \otimes H$ as BT groups over C. Both sides are liftable to $\tilde{C}$(4.7.7), induced by the same filtration $\omega_{\tilde{G}} \otimes \mathcal{T} \hookrightarrow \mathcal{E}$(7.1.9). Again by [33, V Theorem 1.6], $\tilde{X}[p^\infty] \cong \tilde{G} \otimes \tilde{H}$. $\square$

Now the proof of 4.1.2 is complete.

Chapter 5

Tensor decomposition of crystals

We prove 1.0.5 in this chapter. The proof is technical yet the strategy is fairly simple. Firstly 1.0.6 implies that the family $\mathscr{X} \longrightarrow C$ lifts to a formal family of abelian varieties over W. Then we prove any polarization lifts as well, which justifies that the formal family is algebraizable. As before, C is a proper smooth curve with $g(C) > 1$ and $\mathscr{X} \longrightarrow C$ is a family of abelian varieties satisfying the assumption of 1.0.5, that is, $\mathscr{X}/C$ has maximal Higgs field and $\mathscr{E} := \mathbb{D}(\mathscr{X}) \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ as $F-$crystals where $\mathscr{E}$ is an irreducible isocrystal and each $\mathcal{V}_i$ is a rank 2 crystal.

5.1 Lifting of X/C

Recall that we define the binary operation between two BT groups as

$$G \otimes H := \mathrm{colim}_n (G[p^n] \otimes_{\mathbb{Z}_p} H[p^n]).$$

If H is étale, then one can show easily that $G \otimes H$ is indeed a BT group by descent theory.

Lemma 5.1.1. *The BT group $X[p^\infty]$ is isomorphic to $G \otimes H$ where H is an étale BT group.*

Proof. Let Φ be the Frobenius of $\mathscr{E}$. Since each $\mathcal{V}_i$ is an irreducible isocrystal, it corresponds to a $SL(2)$-representation in the Tannakian formalism. Then by 4.3.3, there exists a permutation s and $\phi_i : \mathcal{V}_i^\sigma \longrightarrow \mathcal{V}_{s(i)}$ between isocrystals such that $\Phi \cong \otimes \phi_i$. We can adjust each ϕ_i by some power of p so that it is a morphism between crystals and $\phi_i \neq 0 \pmod p$. And we still have $\Phi \cong \otimes\phi_i$ since Φ is a morphism between crystals. Decompose $s = (s_1) \cdots (s_k)$ by cycles and then Φ can be decomposed as $\Phi = \otimes F_k$ where $F_k : \otimes_{j\in s_k} \mathcal{V}_j^\sigma \longrightarrow \otimes_{j\in s_i} \mathcal{V}_j$ is the Frobenius of $\otimes_{j \in s_k} \mathcal{V}_j$. Similarly we have Verschiebungs $V_k : \otimes_{j\in s_k} \mathcal{V}_j \longrightarrow \otimes_{j\in s_k} \mathcal{V}_j^\sigma$. So $\otimes(F_k \circ V_k) = p$. Then one of $F_k \circ V_k$ is p.Id and the others are identity maps. Without loss of generality, assume $F_1 \circ V_1 = p$. Then $\otimes_{j\in s_1} \mathcal{V}_j$ is a Dieudonne crystal and hence it corresponds to a BT group G by [9, Main Theorem 1]. The other part corresponds to an étale BT group H.

It suffices to show $\mathbb{D}(G \otimes H) \cong \mathscr{E}$. Let $\mathcal{T}$ be the unit crystal $\mathbb{D}(H)$ and $\mathcal{V}$ be the Dieudonne crystal $\mathbb{D}(G)$. Since H is etale, for any n, there exists a finite etale morphism $f_n : T_n \longrightarrow C$ such that $f_n^*(H[p^n])$ is trivial.

Over T_n,

$$\mathbb{D}_{T_n}(f_n^*(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n])) \cong f_n^*(\mathcal{V}/p^n)^{\oplus \mathrm{ht}(H)} \cong f_n^*\mathcal{V}/p^n \otimes_{\mathcal{O}_C} f_n^*\mathcal{T}/p^n \tag{*}$$

as Dieudonne crystals. Both sides are effective descent datum with respect to $T_n \longrightarrow C$. For any $g \in \mathrm{Aut}(T_n/C)$, g^* acts on both of $f_n^*(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n])$ and $f_n^*\mathcal{V}/p^n \otimes_{\mathcal{O}_C} f_n^*\mathcal{T}/p^n$ which is compatible with the functor $\mathbb{D}_{T_n}$, i.e. the diagram

$$\begin{tikzcd}
f^*(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n]) \arrow[r, "g^*"] \arrow[d, "\mathbb{D}_{C'}"] & f^*(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n]) \arrow[d, "\mathbb{D}_{C'}"] \\
f^*(\mathcal{V} \otimes \mathcal{T}/p^n) \arrow[r, "g^*"] & f^*(\mathcal{V} \otimes \mathcal{T}/p^n)
\end{tikzcd}$$

is commutative. Thus the isomorphism (*) between effective descent datum also descends to C. Then we have for any n, $\mathbb{D}(\bar{G}[n] \otimes_{\mathbb{Z}} \bar{H}[n]) = (\mathcal{V} \otimes \mathcal{T}/p^n, F_{\mathcal{V}} \otimes F_{\mathcal{T}}, V_{\mathcal{V}} \otimes F_{\mathcal{T}}^{-1})$. So $\mathbb{D}(G \otimes H) = \mathcal{V} \otimes \mathcal{T}$. □

We can assume $\mathbb{D}(G) = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_r$ and the corresponding Frobenius $F = \otimes_i \phi_i$ where

$$\phi_i : \mathcal{V}_i^\sigma \longrightarrow \mathcal{V}_{i+1}, i < r \text{ and } \phi_r : \mathcal{V}_r^\sigma \longrightarrow \mathcal{V}_1.$$

Proposition 5.1.2. *The family $\mathscr{X} \longrightarrow C$ can be lifted to a formal abelian scheme $\{\mathscr{X}_n \longrightarrow C_n\}$ over W.*

Note if $r = 1$, then G is versally deformed over C and hence 5.1.2 directly follows from 1.0.4.

Proof. Since $\ker F$ has only rank 2^{r-1}, all $\phi_i (1 \le i \le r)$ are isomorphisms except one with a rank 1 kernel. By rearranging the indices, we can assume ϕ_r is the one with the nontrivial kernel. Then $\mathcal{V}_i \cong \mathcal{V}_1^{\sigma^{i-1}}$ for $1 < i \le r$.

Let ${\mathcal{L}_1}^\sigma$ be the kernel of $\phi_{r\,C} : \mathcal{V}_{r\,C}^\sigma \longrightarrow \mathcal{V}_{1\,C}$. Recall that $\omega_G \hookrightarrow \mathcal{V}_C$ is the Hodge filtration associated to G. Since ω_G^σ is the kernel of F, $\omega_G \cong \otimes_{i=1}^{r-1} \mathcal{V}_i \otimes \mathcal{L}_1$ and the Hodge filtration of $\mathcal{V}$ inherits from the inclusion $\mathcal{L}_1 \hookrightarrow \mathcal{V}_{r\,C}$. Then it suffices to deform the filtration $\mathcal{L}_1 \hookrightarrow \mathcal{V}_{r\,C}$.

Let $\mathcal{L}_2$ be the quotient $\mathcal{V}_{r\,C}/\mathcal{L}_1$. Since $\mathscr{X}/C$ has maximal Higgs field, $\theta : \omega_G \longrightarrow \alpha_G \otimes \Omega_C^1$ is an isomorphism. Note $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_r$ as crystals. Let ∇_i be the connection of $\mathcal{V}_{i\,C}$. The connection ∇_C on $\mathcal{V}_C$ has the form of $\nabla_1 \otimes \mathrm{Id} \otimes \mathrm{Id} \cdots \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla_2 \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} + \cdots\cdots + \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \nabla_r$. Correspondingly, the Higgs field is a tensor product of an isomorphism $\mathcal{L}_1 \longrightarrow \mathcal{L}_2 \otimes \Omega_C^1$ and identity on $\mathcal{V}_{2\,C} \otimes \cdots \otimes \mathcal{V}_{r\,C}$. In particular, $T_C \cong {\mathcal{L}_1}^{-1} \otimes \mathcal{L}_2$. Over each open subset U of C, the deformation of the filtration is unobstructed with deformation space

$$\mathrm{Ext}^0(\mathcal{L}_{1|U}, \mathcal{L}_{2|U}) \cong T_C(U).$$

So for any deformation C_2 of C to $W_2(k)$, the obstruction class of the deformation of the filtration to C_2 is in $H^1(C, T_C)$. Then replacing the BT groups by the filtration, 3.2.11 shows that the maximal Higgs field induces a bijection $\mathrm{Def}_{W_2(k)}(C) \longrightarrow H^1(C, {\mathcal{L}_1}^{-1} \otimes \mathcal{L}_2)$ and there exists a unique lifting C_2 of C over which the deformation of the filtration is unobstructed. Repeating this process for higher order deformation, as in Section 3.3, we have a lifting of the Hodge filtration. By Grothendiek-Messing theory [33, Chapter V, Theorem 1.6, Theorem 2.3], we have a formal family $\{\mathscr{X}_n \longrightarrow C_n\}$ over W. □

5.2 Lifting of polarization

Choose any polarization $\lambda : \mathscr{X} \longrightarrow \mathscr{X}^t$ over C and fix it throughout the rest of the chapter. Once we can lift the polarization λ, the formal scheme in 5.1.2 can be algebraized.

Note we can always assume $\lambda_1 : \mathscr{X}[p] \longrightarrow \mathscr{X}^t[p]$ induced by λ is not zero. Otherwise, $\mathscr{X}[p] \subset \ker \lambda$ and since $\mathscr{X}[p]$ always lifts, we can replace λ by $\frac{1}{p}\lambda$. Since $\mathscr{X}[p^\infty] \cong G \otimes H$,

$$\mathscr{X}^t[p^\infty] \cong G^t \otimes H^t$$

where H^t corresponds to the dual Galois representation of H.

We aim to prove λ can be decomposed into a tensor product of an isomorphism between G and G^t and a morphism between H and H^t, after base changing to a finite étale covering of C.

Let η be the generic point of C and h be the height of H. Let $\mathbb{D}(G) = \mathcal{V}$ and $\mathbb{D}(H) = \mathcal{T}$. Then $\mathcal{V} \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_r$ as crystals. As in the proof of 5.1.2, we can assume the Hodge filtration of $\mathcal{V}$ inherits from the filtration

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{V}_{r\,C} \longrightarrow \mathcal{L}_2 \longrightarrow 0. \tag{5.1}$$

Lemma 5.2.1. *Any nonzero morphism $\gamma_r : \mathcal{V}_r^\vee \longrightarrow \mathcal{V}_r$ between crystals which preserves the filtration 5.1 is an isomorphism.*

Proof. It suffices to prove the restriction $\gamma_{r\,C}$ is isomorphic. Since $\gamma_{r\,C}$ preserves the Hodge filtration and the Higgs field,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {\mathcal{L}_2}^\vee & \longrightarrow & \mathcal{V}_{r\,C}^\vee & \longrightarrow & {\mathcal{L}_1}^\vee & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \gamma_{r\,C}} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{V}_{r\,C} & \longrightarrow & \mathcal{L}_2 & \longrightarrow & 0 \end{array}$$

the restrictions of $\gamma_{r\,C}$ to ${\mathcal{L}_2}^\vee$ and ${\mathcal{L}_1}^\vee$ are both nonzero or both zero. If they are zero, then $\gamma_{r\,C}$ factors through ${\mathcal{L}_1}^\vee \longrightarrow \mathcal{L}_1$. But since the Higgs field is maximal, $\mathcal{L}_1$ is not a horizontal sub line bundle of $\mathcal{V}_{r\,C}$ which is a contradiction to the fact that γ_r preserves the connection. Thus γ_r is an isomorphism. $\square$

Lemma 5.2.2. *The endomorphism ring of $\mathcal{V}_{r\,C}$ in the category of filtered crystals is isomorphic to k.*

Proof. For any such nonzero endomorphism ρ, it induces a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{V}_{r\,C} & \longrightarrow & \mathcal{L}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{V}_{r\,C} & \longrightarrow & \mathcal{L}_2 & \longrightarrow & 0. \end{array}$$

Since ρ preserves the Higgs field, the morphism ρ induces the same scalar multiplication on $\mathcal{L}_1$ and $\mathcal{L}_2$. Then the subtraction of the scalar multiplication from ρ induces a morphism $\mathcal{V}_{r\,C} \longrightarrow \mathcal{L}_1$ which preserves the connection. So the subtraction has to be zero. $\square$

Lemma 5.2.3. *Let ψ be a nonzero endomorphism of $\otimes_{i=1}^{r}\mathcal{V}_{i\,C}$ in the category of crystals. If ψ preserves the Hodge filtration, then there exists an endomorphism T in $\mathrm{End}(\mathcal{V}_{1\,C}\otimes\mathcal{V}_{3\,C}\cdots\otimes\mathcal{V}_{r-1\,C})$ such that $\psi = Id_{\mathcal{V}_{r\,C}}\otimes T$.*

Proof. Note the morphism ψ preserves the Hodge filtration $\mathcal{L}_1\hookrightarrow\mathcal{V}_{r\,C}$ and the Higgs field.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_1\otimes\mathcal{V}_{1\,C}\cdots\otimes\mathcal{V}_{r-1\,C} \arrow[r] \arrow[d] & \otimes_{i=1}^{r}\mathcal{V}_{i\,C} \arrow[r] \arrow[d, "\psi"] & \mathcal{L}_2\otimes\mathcal{V}_{1\,C}\cdots\otimes\mathcal{V}_{r-1\,C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L}_1\otimes\mathcal{V}_{1\,C}\cdots\otimes\mathcal{V}_{r-1\,C} \arrow[r] & \otimes_{i=1}^{r}\mathcal{V}_{i\,C} \arrow[r] & \mathcal{L}_2\otimes\mathcal{V}_{1\,C}\cdots\otimes\mathcal{V}_{r-1\,C} \arrow[r] & 0.
\end{tikzcd}$$

Let T be the first vertical arrow. Then $T\in\mathrm{End}(\mathcal{V}_{1\,C}\otimes\mathcal{V}_{3\,C}\cdots\otimes\mathcal{V}_{r-1\,C})$. Then $\psi-\mathrm{Id}\otimes T$ induces a horizontal morphism $\otimes_{i=1}^{r}\mathcal{V}_{i\,C}\longrightarrow\mathcal{L}_1\otimes\mathcal{V}_{2\,C}\cdots\otimes\mathcal{V}_{r-1\,C}$. Due to the maximal Higgs field, it has to be zero and $\psi=\mathrm{Id}_{\mathcal{V}_{r\,C}}\otimes T$. ☐

Corollary 5.2.4. *The endomorphism ring of $\otimes_{i=1}^{r}\mathcal{V}_{i\,C}$ in the category of filtered F-crystals is isomorphic to k.*

Proof. Let ψ be a nonzero endomorphism of $\otimes_{i=1}^{r}\mathcal{V}_{i\,C}$. Then $\psi=\mathrm{Id}\otimes T$.

Since ψ is compatible with the Frobenius F and the conjugate filtration, we have

$$\begin{tikzcd}
0 \arrow[r] & {\mathcal{L}_2}^{p}\otimes_{i=2}^{r}\mathcal{V}_{i\,C} \arrow[r] \arrow[d, "\mathrm{Id}\otimes T^{\sigma}"] & \otimes_{i=1}^{r}\mathcal{V}_{i\,C} \arrow[r] \arrow[d, "\psi"] & {\mathcal{L}_1}^{p}\otimes\otimes_{i=2}^{r}\mathcal{V}_{i\,C} \arrow[r] \arrow[d, "\mathrm{Id}\otimes T^{\sigma}"] & 0 \\
0 \arrow[r] & {\mathcal{L}_2}^{p}\otimes_{i=2}^{r}\mathcal{V}_{i\,C} \arrow[r] & \otimes_{i=1}^{r}\mathcal{V}_{i\,C} \arrow[r] & {\mathcal{L}_1}^{p}\otimes\otimes_{i=2}^{r}\mathcal{V}_{i\,C} \arrow[r] & 0.
\end{tikzcd}$$

Note ${\mathcal{L}_2}^{p}\otimes_{i=2}^{r-1}\mathcal{V}_{i\,C}\otimes\mathcal{L}_1\longrightarrow{\mathcal{L}_2}^{p}\otimes_{i=2}^{r}\mathcal{V}_{i\,C}$ is a sub-filtration of the Hodge filtration of $\otimes_{i=1}^{r}\mathcal{V}_{i\,C}$. Thus $T^{\sigma}:\otimes_{i=2}^{r}\mathcal{V}_{i\,C}\longrightarrow\otimes_{i=2}^{r}\mathcal{V}_{i\,C}$ is a morphism between crystals which preserves the Hodge filtration. Apply 5.2.3 to T^{σ} and we have $T^{\sigma}=\mathrm{Id}_{\mathcal{V}_{r\,C}}\otimes T'$. Then T' fits in the following diagram where the top horizontal row is induced by $F\circ F$:

$$\begin{tikzcd}
{\mathcal{L}_2}^{p}\otimes{\mathcal{L}_2}^{p^2}\otimes_{i=3}^{r}\mathcal{V}_{i\,C} \arrow[r, "F^2"'] \arrow[d, "\mathrm{Id}\otimes T'^{\sigma}"] & \otimes_{i=1}^{r}\mathcal{V}_{i\,C} \arrow[d, "\psi"] \\
{\mathcal{L}_2}^{p}\otimes{\mathcal{L}_2}^{p^2}\otimes_{i=3}^{r}\mathcal{V}_{i\,C} \arrow[r] & \otimes_{i=1}^{r}\mathcal{V}_{i\,C}.
\end{tikzcd}$$

So T'^{σ} again preserves the Hodge filtration and the connection. We can reapply 5.2.3 to T'^{σ}. Repeat this process and we can conclude that $T^{\sigma^{r-1}}$ is just $t.\mathrm{Id}$ for some $t\in k^{*}$. Thus T itself is a scalar multiplication and so is ψ. ☐

Lemma 5.2.5. *There exists a pair $\{\tilde{C},\gamma\}$ such that*

1. *$\tilde{C} \longrightarrow C$ is a finite étale covering ;*

2. *Let $\otimes_{i=1}^r \tilde{\mathcal{V}}_i$ denote the pull back Dieudonne crystal of $\otimes_{i=1}^r \mathcal{V}_i$ from C to $\tilde{C}$. Then $\gamma : \otimes_{i=1}^r \tilde{\mathcal{V}}_i^\vee \longrightarrow \otimes_{i=1}^r \tilde{\mathcal{V}}_i$ is a nonzero morphism.*

Proof. First we consider $\wedge^{2^r}\mathcal{V}$. It is an F-crystal over C and hence corresponds to a triple $(\mathscr{L}, \nabla, F)$, an invertible sheaf over C' with integrable connection and Frobenius. Obviously as an F-isocrystal, $\mathscr{L}$ is isoclinical of slope p. Replace F by $p^{-1}F$ and then $\mathscr{L}$ is a unit root. Therefore, it is equivalent to a one dimensional representation

$$\pi_1(C, \bar{c}) \longrightarrow \mathbb{Z}_p^*.$$

Note $\pi_1^{\mathrm{ab}}(C, \bar{c})$ is an extension of $\hat{\mathbb{Z}}$ with a finite group. So the image of the monodromy is just a finite subgroup of $\mathbb{Z}_p^*$. And then there exists a finite étale covering $\tilde{C} \longrightarrow C$ such that the pullback of the monodromy to $\tilde{C}$ is trivial.

Let $\tilde{\mathcal{V}}$ denote the pull back of $\mathcal{V}$ to $\tilde{C}$. Then $\wedge^{2^r}\tilde{\mathcal{V}}$ is trivial as an $F-$crystal. Consider the following morphism

$$\otimes\tilde{\mathcal{V}}_i \leftarrow \otimes(\tilde{\mathcal{V}}_i \otimes \tilde{\mathcal{V}}_i \otimes \tilde{\mathcal{V}}_i^\vee) \leftarrow \otimes(\wedge^2\tilde{\mathcal{V}}_i \otimes \tilde{\mathcal{V}}_i^\vee) \cong \otimes(\wedge^2\tilde{\mathcal{V}}_i) \otimes \tilde{\mathcal{V}}_i \cong \otimes\tilde{\mathcal{V}}^\vee.$$

Obviously it is nonzero and we denote it as γ. □

Taking a finite étale covering if necessary, we can assume $\wedge^2\mathcal{V}_{r\,\tilde{C}}$ is trivial.

Lemma 5.2.6. *There exists $\alpha : \mathcal{V}_{r\,\tilde{C}}^\vee \longrightarrow \mathcal{V}_{r\,\tilde{C}}$ in the category of crystals which preserves 5.1.*

Proof. Since $\wedge^2\mathcal{V}_{r\,\tilde{C}}$ is a trivial crystal, we have a nonzero morphism $\alpha : \mathcal{V}_{r\,\tilde{C}}^\vee \longrightarrow \mathcal{V}_{r\,\tilde{C}}$ in the category of crystals. It suffices to show α preserves 5.1. If not, then α induces a nonzero morphism from $\tilde{\mathcal{L}_1} \longrightarrow \tilde{\mathcal{L}_2}$. Since $\deg\tilde{\mathcal{L}_1} > \deg\tilde{\mathcal{L}_2}$, we know such a morphism does not exist. □

Let γ_1 be the restriction of γ to $\tilde{\mathcal{V}}_{\tilde{C}}^\vee$.

Lemma 5.2.7. *There exists $\alpha_1 \in \mathrm{Hom}(\tilde{\mathcal{V}}_{1\,\tilde{C}}^\vee, \tilde{\mathcal{V}}_{1\,\tilde{C}})$ such that $\gamma_1 = \alpha_1 \otimes \alpha_1^\sigma \cdots \otimes \alpha_1^{\sigma^{r-1}}$ as morphisms in the category of F-crystals over* $\mathrm{cris}(\tilde{C}/k)$.

Proof. Since γ_1 preserves the Frobenius, it preserves the Hodge filtration $\tilde{\mathcal{L}_1} \hookrightarrow \tilde{\mathcal{V}}_{r\,\tilde{C}}$.

$$\begin{tikzcd}
0 \arrow[r] & (\tilde{\mathcal{L}_2})^\vee \otimes \tilde{\mathcal{V}}_{1,\tilde{C}}^\vee \cdots \otimes \tilde{\mathcal{V}}_{r-1\,\tilde{C}}^\vee \arrow[r] \arrow[d] & \tilde{\mathcal{V}}_{1\,\tilde{C}}^\vee \otimes \tilde{\mathcal{V}}_{2,\tilde{C}}^\vee \cdots \otimes \tilde{\mathcal{V}}_{r\,\tilde{C}}^\vee \arrow[r] \arrow[d] & (\tilde{\mathcal{L}_1})^\vee \otimes \tilde{\mathcal{V}}_{1,\tilde{C}}^\vee \cdots \otimes \tilde{\mathcal{V}}_{r-1\,\tilde{C}}^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\mathcal{L}_1} \otimes \tilde{\mathcal{V}}_{1,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}_{r-1\,\tilde{C}} \arrow[r] & \tilde{\mathcal{V}}_{1\,\tilde{C}} \otimes \tilde{\mathcal{V}}_{2,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}_{r\,\tilde{C}} \arrow[r] & (\tilde{\mathcal{L}_2}) \otimes \tilde{\mathcal{V}}_{2,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}_{r\,\tilde{C}} \arrow[r] & 0
\end{tikzcd}$$

Since $\wedge^2 \mathcal{V}_{r\,\tilde{C}}$ is trivial, $(\mathcal{L}_1)^\vee \cong \mathcal{L}_2$. Since γ_1 preserves the Higgs fields, the first and the third vertical arrows are identical, viewed as a morphisms between $\tilde{\mathcal{V}}^\vee_{1,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}^\vee_{r-1\,\tilde{C}}$ and $\tilde{\mathcal{V}}_{1,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}_{r-1\,\tilde{C}}$. Denote it as α'. Then $\gamma_1 - \alpha \otimes \alpha'$ induces a horizontal morphism $\tilde{\mathcal{V}}^\vee_{1\,\tilde{C}} \otimes \tilde{\mathcal{V}}^\vee_{2,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}^\vee_{r\,\tilde{C}} \longrightarrow \tilde{\mathcal{L}}_1 \otimes \tilde{\mathcal{V}}_{1,\tilde{C}} \cdots \otimes \tilde{\mathcal{V}}_{r-1\,\tilde{C}}$. So it has to be zero and $\gamma_1 = \alpha \otimes \alpha'$.

Since the Frobenius acts as simply permuting the indices and γ_1 is compatible with any power of Frobenius, as in the proof of 5.2.4, there exists $\alpha_1 \in \mathrm{Hom}(\tilde{\mathcal{V}}^\vee_{1\,\tilde{C}})$ such that $\gamma_1 = \alpha_1 \otimes \alpha_1^\sigma \cdots \otimes \alpha_1^{\sigma^{r-1}}$ and by 5.2.1, $\alpha_1^{\sigma^{r-1}} = \alpha$ is an isomorphism. □

Corollary 5.2.8. *The morphism γ is an isomorphism.*

Let $\mathcal{T}^\vee$ be the unit root crystal $\mathbb{D}(H^\vee)$ corresponding to the dual $H^\vee$ and $\tilde{H}$ (resp. $\tilde{\mathcal{T}}$) be the pull back of H (resp. $\mathcal{T}$) to $\tilde{C}$.

Then the polarization $\lambda_1 : \mathscr{X}[p] \longrightarrow \mathscr{X}^t[p]$ induces $\tilde{\lambda}_1^* : \tilde{\mathcal{V}}^\vee_{\tilde{C}} \otimes \tilde{\mathcal{T}}^\vee_{\tilde{C}} \longrightarrow \tilde{\mathcal{V}}_{\tilde{C}} \otimes \tilde{\mathcal{T}}_{\tilde{C}}$.

Lemma 5.2.9. *The morphism $\tilde{\lambda}_1^*$ has a tensor decomposition*

$$\tilde{\lambda}_1^* = \gamma_1 \otimes \beta_1$$

where β_1 is a morphism between $\tilde{\mathcal{T}}^\vee_{\tilde{C}}$ and $\tilde{\mathcal{T}}_{\tilde{C}}$.

Proof. Note $\tilde{\lambda}_1^*$ is compatible with the Frobenius and the Higgs field. Further, the Hodge filtration of $\mathcal{E}$ is inherited from that of $\tilde{\mathcal{V}}_{\tilde{C}}$ and the Frobenius $F_{\mathscr{E}} \cong F_{\mathcal{V}} \otimes F_{\mathcal{T}}$ with $F_{\mathcal{T}}$ an isomorphism. So we can repeat the arguments in 5.2.7 to $\tilde{\lambda}_1^* : \tilde{\mathcal{V}}^\vee_{\tilde{C}} \otimes \tilde{\mathcal{T}}^\vee_{\tilde{C}} \longrightarrow \tilde{\mathcal{V}}_{\tilde{C}} \otimes \tilde{\mathcal{T}}_{\tilde{C}}$ and there exists $\gamma' \in \mathrm{Hom}(\tilde{\mathcal{V}}^\vee_{\tilde{C}}, \tilde{\mathcal{V}}_{\tilde{C}})$ and $\beta' \in \mathrm{Hom}(\tilde{\mathcal{T}}^\vee_{\tilde{C}}, \tilde{\mathcal{T}}_{\tilde{C}})$ such that $\tilde{\lambda}_1^* = \gamma' \otimes \beta'$ and γ' is an isomorphism. By 5.2.4, we can adjust by a scalar such that $\gamma' = \gamma_1$. □

In the proof of 5.2.9, the only special property of the polarization $\tilde{\lambda}$ we use is that it preserves the Hodge filtration, Frobenius and the Higgs field. So we obtain the following corollary.

Corollary 5.2.10. *In 5.2.9, we can replace $\tilde{\lambda}_1^*$ by any nonzero morphism between $\tilde{\mathcal{V}}_{\tilde{C}} \otimes \tilde{\mathcal{T}}$ and $\tilde{\mathcal{V}}^\vee_{\tilde{C}} \otimes \tilde{\mathcal{T}}^\vee$ in the category of $F-$crystals over* $\mathrm{cris}(\tilde{C}/k)$ *and the result still holds.*

Since $\tilde{H}$ is an étale BT group, for each n there exists a finite étale covering T_n of $\tilde{C}$ such that the pull back of $\tilde{H}[p^n]$ to T_n is trivial. Then $T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow \tilde{C}$ is a tower of finite étale coverings of $\tilde{C}$.

Proposition 5.2.11. *As a morphism between Dieudonne crystals, $\tilde{\lambda}^* = \gamma \otimes \beta$ where γ is as in 5.2.5 and β is a morphism between $\mathcal{T}$ and $\mathcal{T}^\vee$.*

Proof. We will prove the tensor decomposition by induction on the group order. Let λ_n^* be the polarization

$$\lambda_n^* : \mathbb{D}(G^t[p^n]) \otimes \mathbb{D}(H^\vee[p^n]) \longrightarrow \mathbb{D}(G[p^n]) \otimes \mathbb{D}(H[p^n])$$

It is true for $n = 1$. If it is true for $n - 1$, then for n,

$$\begin{tikzcd}
\mathbb{D}(\tilde{G}[p]) \otimes \mathbb{D}(\tilde{H}[p]) & \mathbb{D}(\tilde{G}^t[p]) \otimes \mathbb{D}(\tilde{H}[p])^t \arrow[l, "\gamma_1 \otimes \beta_1"'] \\
\mathbb{D}(\tilde{G}[p^n]) \otimes \mathbb{D}(\tilde{H}[p^n]) \arrow[u, "p^{n-1}"] & \mathbb{D}(\tilde{G}^t[p^n]) \otimes \mathbb{D}(\tilde{H}[p^n])^t \arrow[l, "\tilde{\lambda}_n"'] \arrow[u, "p^{n-1}"'] \\
\mathbb{D}(\tilde{G}[p^{n-1}]) \otimes \mathbb{D}(\tilde{H}[p^{n-1}]) \arrow[u] & \mathbb{D}(\tilde{G}^t[p^{n-1}]) \otimes \mathbb{D}(\tilde{H}[p^{n-1}])^t. \arrow[l, "\gamma_{n-1} \otimes \beta_{n-1}"'] \arrow[u]
\end{tikzcd}$$

After base change to T_n, since $H[p^n]$ is trivial on T_n, we find

$$\beta_{T_n} : \mathbb{D}(H_{T_n}[p^n])^t \longrightarrow \mathbb{D}(H_{T_n}[p^n])$$

so that the restriction of β_{T_n} to $\mathbb{D}(H_{T_{n-1}}[p^{n-1}])^t$ yields the base change of β_{n-1} to T_{n-1}.

Therefore $(\lambda_n^*)_{T_n} - \gamma_{T_n} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \beta_{T_n}$ factors through

$$\alpha : \mathbb{D}(G_{T_n}[p])^\vee \otimes \mathbb{D}(H_{T_n}[p])^t \longrightarrow \mathbb{D}(G_{T_n}[p^{n-1}]) \otimes \mathbb{D}(H_{T_n}[p^{n-1}]).$$

Definitely $\mathrm{im}\,\alpha \subset \mathbb{D}(G_{T_n}[p]) \otimes \mathbb{D}(H_{T_n}[p])$. If α is nonzero, by 5.2.10, $\alpha = \gamma_1 \otimes \beta'$ where $\beta' : \mathbb{D}(H_{T_n}[p])^t \longrightarrow \mathbb{D}(H_{T_n}[p^{n-1}])$.

Note $\gamma_1 = \gamma_{T_n}|_{p^{n-1}\mathbb{D}(G_{T_n}[p^n])^\vee}$, we have that α can be written as the following composition

$$\mathbb{D}(G^t_{T_n}[p^n]) \otimes_{\mathbb{Z}/p^n} \mathbb{D}(H_{T_n}[p^n])^t \xrightarrow{\mathrm{Id} \otimes p^{n-1}} \mathbb{D}(G^t_{T_n}[p^n]) \otimes_{\mathbb{Z}/p^n} \mathbb{D}(H_{T_n}[p])^t$$
$$\xrightarrow{\gamma_{T_n} \otimes \beta'} \mathbb{D}(G_{T_n}[p^{n-1}]) \otimes_{\mathbb{Z}/p^n} \mathbb{D}(H_{T_n}[p^{n-1}]).$$

Thus $(\lambda_n^*)_{T_n} - \gamma_{T_n} \otimes_{\mathbb{Z}/p^n} \beta_{T_n} = \gamma_{T_n} \otimes_{\mathbb{Z}/p^n} p^{n-1}\beta'$. So $(\lambda_n^*)_{T_n} = \gamma_{T_n} \otimes (\bar{\beta}_{T_n} + p^{n-1}\beta')$.

Since both λ_n^* and γ_{T_n} can be descended to $\tilde{C}$, $\bar{\beta}_{T_n} + p^{n-1}\beta'$ can be descended to $\tilde{C}$ as well. Therefore, as a morphism between Dieudonne crystal over $\tilde{C}$, $\tilde{\lambda}^* = \gamma \otimes \beta$. ∎

By the equivalence between Dieudonne crystals and BT groups over $\tilde{C}$([9, Main Theorem 1]), we know $\tilde{\lambda}$ can be decomposed into a tensor product of an isomorphism between $\tilde{G}$ and $\tilde{G}^t$ and a morphism between

H and H^t. Applying 1.0.4 to $(\tilde{C}, \tilde{G})$, there exists a strongly unique pair $(\tilde{C}', \tilde{G}')$ over W which is a lifting of $(\tilde{C}, \tilde{G})$. In particular, the isomorphism between $\tilde{G}$ and $\tilde{G}^t$ lifts to an isomorphism $\tilde{G}'$ and $\tilde{G}'^t$. Since H is étale over C, any morphism between $\tilde{H}$ and $\tilde{H}^t$ naturally lifts. Therefore the polarization $\tilde{\lambda}$ lifts.

For each n, let $\tilde{\lambda}_n$ denote the lifting of λ to $\tilde{\mathscr{X}}_n/\tilde{C}_n$. Since $\mathrm{Aut}(\tilde{C}_n/C_n) = \mathrm{Aut}(\tilde{C}/C)$, $\tilde{\lambda}_n$ is a descent datum as $\tilde{\lambda}$ and hence descends to a polarization λ_n on $\mathscr{X}_n/C_n$.

To infer that $\{\lambda_n\}$ is a compatible system, we need to show such a lifting λ_n is unique. We shall appeal to a general fact: if $\mathscr{X} \longrightarrow \mathscr{C}$ is any abelian scheme over $\mathscr{C}$ and l is a prime different from p, then the collection of closed subschemes $\mathscr{X}[l^m]$ for all $m \geq 1$ is universally schematically dominant in $\mathscr{X}$ with respect to $\mathscr{C}$ in the sense of [18, 11.10.8]. Therefore λ_n is uniquely determined by its restriction on $\mathscr{X}_n[l^m]$, which is further uniquely determined by λ, due to the equivalence between the étale sites over $\mathscr{X}$ and $\mathscr{X}_n$. So such a lifting λ_n is unique. So we have a compatible family of polarization $\{\lambda_n\}$ and hence the formal family $\{\mathscr{X}_n \longrightarrow C_n\}$ can be algebraized to a family of abelian varieties $\mathscr{X}' \longrightarrow C'$ over W.

Remark 5.2.12. If we know a priori that $\mathscr{X}/C$ is a family of principally polarized abelian varieties, i.e. λ is isomorphic, then λ naturally lifts to any $\mathscr{X}_n$ since the lifting $\mathscr{X}_n$ of $\mathscr{X}$ is strongly unique.

5.2.13 Lift to Shimura curves

In the following proposition, we show $\mathscr{X}' \longrightarrow C'$ is a Shimura curve of Mumford type.

Proposition 5.2.14. *The generic fibre $(\mathscr{X}'/C') \otimes \mathbb{C}$ is a Shimura curve of Mumford type.*

Proof. Obviously the family $\mathscr{X}' \longrightarrow C'$ has maximal Higgs field. By Theorem 0.5 in [47], we directly conclude that $(\mathscr{X}'/C') \otimes \mathbb{C}$ is a Shimura curve of Mumford type. Note in that theorem, a family that reaches the Arakelov bound is equivalent to a family with the maximal Higgs field. So we can apply that theorem in our situation. ◻

Chapter 6

Tensor decomposition of isocrystals

We devote this chapter to prove 1.0.2. As before, k is an algebraically closed field of characteristic $p > 2$ and C is a proper smooth curve over k. The family of principally polarized abelian varieties $\Pi : \mathscr{X} \longrightarrow C$ satisfies the assumptions of 1.0.2, that is, the family is generically ordinary and $\mathscr{E} := \mathbb{D}(\mathscr{X}/C)$ irreducible with isomorphism $\mathscr{E} \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{m+1}$ as isocrystals. After analysing the decomposition of the Frobenius F of $\mathscr{E}$, we firstly prove the second part, i.e. assuming maximal Higgs field, in Section 6.2. Then the first part is proved in Section 6.3.

6.1 The structure of $\mathscr{E}$ as crystals

In this section, we choose an F-crystal model of $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$.

Note $\mathscr{E}$ is an F-isocrystal. So

$$F : \mathcal{V}_1^\sigma \otimes \mathcal{V}_2^\sigma \otimes \mathcal{V}_3^\sigma \cdots \otimes \mathcal{V}_{m+1}^\sigma \longrightarrow \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$$

is a morphism in the category of isocrystals.

By condition (2) in 1.0.2, each $\mathcal{V}_i$ is irreducible as an isocrystal. Let $B(k)$ be the fractional field of the Witt ring and H_i be the Tannakian group associated to $\mathcal{V}_i$ in $\mathrm{Rep}(G_{\mathrm{univ}})$.

Lemma 6.1.1. *H_i has the form $SL(2) \times \mu_m$ for some integer m.*

Proof. Since $\mathcal{V}_i$ are all irreducible isocrystals, each of the group H_i admits a faithful irreducible representation of dimension 2. Thereby H_i is a reductive group over $B(k)$. Let $\tilde{C}$ be a lifting of C to the Witt ring $W(k)$. If we view $\mathcal{V}_i$ as a vector bundle with a connection over $\tilde{C}$, then we know $\mathcal{V}_i$ is also irreducible over $\tilde{C}_{\mathbb{C}}$ and thus the representation of H_i is geometrically irreducible. Thus base change to $\bar{B}(k)$, $H_i \otimes \bar{B}(k)$ must be a central extension of $SL(2, \bar{B}(k))$. Since $H_i \hookrightarrow GL(2, B(k))$ and the field extension is faithfully flat, H_i is $SL(2, B(k)) \times \mu_m$ or $GL(2, B(k))$.

Since $\mathscr{X} \longrightarrow C$ is polarized, we have $\mathscr{E} \cong \mathscr{E}^\vee$ as isocrystals, up to some twist. Thereby as representations, $\wedge^8 E \cong \wedge^8 E^\vee$ and hence $\wedge^2 V_1 \otimes \cdots \wedge^2 V_{m+1}$ is 2-torsion. Therefore $\det H_i$ can not be $\mathbb{G}_m$, i.e. H_i can not be $GL(2)$. □

As an irreducible representation of $SL(2) \times \mu_m$, $V_i \cong V_i' \otimes L_i$ where V_i' the standard representation of $SL(2)$ and L_i an irreducible representation of μ_m. Thus we can adjust each V_i such that $H_i = SL(2)$ for $1 \le i \le m$ and only $H_{m+1} = SL(2) \times \mu_m$.

Since $\mathscr{E}$ is irreducible, $\otimes \mathcal{V}_i$ corresponds to an irreducible $SL(2)^{\times m+1} \times \mu_m$-representation. Then we can modify the proof of 4.3.3 in this case and obtain a similar result: there exist

$$\phi_i : \mathcal{V}_i^\sigma \longrightarrow \mathcal{V}_{s(i)} \otimes \mathcal{L}_i$$

such that $F = \otimes \phi_i$. We can find an f such that $s^f = \mathrm{Id}$. Then F^f can be decomposed to the tensor product of

$$\phi_i' : \mathcal{V}_i^{\sigma^f} \longrightarrow \mathcal{V}_i \otimes \mathscr{L}_i$$

for some $\mathscr{L}_i$.

By 4.5.1, we can find $\mathcal{L}'_i$ such that $\mathscr{L}_i^{-1} = ((\sigma^f)^* - \mathrm{Id})(\mathcal{L}'_i)$, then ϕ'_i induces a morphism

$$\mathcal{V}_i^{\sigma^f} \otimes \mathcal{L}_i'^{\sigma^f} \longrightarrow \mathcal{V}_i \otimes \mathcal{L}'_i.$$

Replace $\mathcal{V}_i$ by $\mathcal{V}_i \otimes \mathcal{L}'_i$ and let us still denote it as $\mathcal{V}_i$. Therefore

$$\phi'_i : \mathcal{V}_i^{\sigma^f} \longrightarrow \mathcal{V}_i.$$

So each $\mathcal{V}_i$ is an F^f-isocrystal.

By condition (1) of 1.0.2, as an F^f-isocrystal, $\mathscr{E}$ has generic slopes $\{0 \times 2^m, f \times 2^m\}$.

Lemma 6.1.2. *Generically, the slopes of $\mathcal{V}_i$ are all $\{0,0\}$ except one $\{0,f\}$.*

Proof. Use $\mathcal{V}_{i,\eta}$ to denote the restriction of $\mathcal{V}_i$ to the crystalline site $\mathrm{cris}(W(\eta^-)/\eta^-)$ where η^- is the geometric generic point. Then slopes of $\mathcal{V}_{1,\eta} \otimes \mathcal{V}_{2,\eta} \otimes \mathcal{V}_{3,\eta} \cdots \otimes \mathcal{V}_{m+1,\eta}$ are $\{0 \times 2^m, f \times 2^m\}$. Now we can compute directly the slopes of each $\mathcal{V}_i$. For notational simplicity, here we just show the case $m = 2$.

Assume $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$ are slopes of $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$, respectively. Further, we could assume the slopes of each $\mathcal{V}_i$ are non-negative. Then the slopes of $\mathscr{E}$ are

$$a_i + b_j + c_k, 1 \le i, j, k \le 2.$$

If $a_1 \le a_2$ and $b_1 \le b_2$, then we can adjust that $a_1 = b_1 = 0$. Then $\{c_1, c_2\} = \{0, f\}$ or $\{0, 0\}$. The former case forces all a_i, b_j to be zero, while the latter case implies $\{a_2, b_2\} = \{0, f\}$. □

Proposition 6.1.3. *$s(j) = j$ for some $j \in \{1, 2, 3 \cdots, m+1\}$.*

Proof. From 6.1.2, we can assume the slopes of $\mathcal{V}_1$, restricting to $\mathrm{cris}(W(\eta^-)/\eta^-)$, are $\{0, f\}$. If $s(1) \neq 1$, say $s(2) = 1$, then over $W(\eta^-)$, $\mathrm{im}\, \phi_2 = \mathcal{V}_{1,\eta}$ has only one-dimension subspace of slope 0. Iterate the Frobenius f times and note $(\otimes \phi_i)^f = F^f = F_1 \otimes F_2 \otimes F_3 \cdots \otimes F_{m+1}$. Thus $\mathcal{V}_{2,\eta} = \mathrm{im}\, F_2$ has at most one dimensional subspace of slope 0, contradicting to $\mathcal{V}_2$ with slopes $\{0, 0\}$. So $s(1) = 1$. □

Without loss of generality, assume $s(1) = 1$. Then $\phi_1 : \mathcal{V}_1^\sigma \longrightarrow \mathcal{V}_1 \otimes \mathcal{L}_1$.

By 4.5.1, we can find $\mathcal{L}'$ such that $\mathcal{L}_1^{-1} = (\sigma^* - \mathrm{Id})(\mathcal{L}')$, then ϕ_1 induces a morphism

$$\mathcal{V}_1^\sigma \otimes \mathcal{L}'^\sigma \longrightarrow \mathcal{V}_1 \otimes \mathcal{L}'.$$

Replace $\mathcal{V}_1$ by $\mathcal{V}_1 \otimes \mathcal{L}'$ and let us still denote it as $\mathcal{V}_1$. Therefore

$$\phi_1 : \mathcal{V}_1^{\sigma} \longrightarrow \mathcal{V}_1.$$

Then $\mathcal{V}_1$ and $\mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$ are actually F-isocrystals. And $\mathscr{E} \cong \mathcal{V}_1 \otimes (\mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1})$ as F-isocrystals. Since $\mathscr{E}$ is further a Dieudonne crystal, the Verschiebung V also can be decomposed to $V_1 \otimes V_2$ where both of $V_1 : \mathcal{V}_1 \longrightarrow \mathcal{V}_1^{\sigma}, V_2 : \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \longrightarrow \mathcal{V}_2^{\sigma} \otimes \mathcal{V}_3^{\sigma} \cdots \otimes \mathcal{V}_{m+1}^{\sigma}$ are isomorphisms.

Since all the slopes of $\mathcal{V}_i$ are nonnegative, we can choose an F^f-crystal model for each $\mathcal{V}_i$ (see Appendix A.2). Since as an F-isocrystal $\mathcal{V}_1$ has slopes $\{0, 1\}$, there exists $V_1 : \mathcal{V}_1 \longrightarrow \mathcal{V}_1^{\sigma}$ such that $F_1 \circ V_1 = V_1 \circ F_1 = p$. Thereby we can choose the descent of Frobenius and Verschiebung so that $\mathcal{V}_1$ is a Dieudonne crystal. For simplicity, we still denote the corresponding crystal as $\mathcal{V}_i$. Though the isomorphism in 1.0.2 (2) may not be true in the level of crystals, we still can choose

$$\psi : \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \longrightarrow \mathscr{E}$$

injectively between F-crystals and $\psi[\frac{1}{p}]$ is the isomorphism. Multiplying some power of p if necessary, we can assume $\psi \mod p$ is not zero and $\mathscr{E}/\operatorname{im}\psi$ is p-torsion. By 6.1.2, we have $\mathcal{V}_1$ is generically ordinary and $\mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$ is a unit root F-crystal. For simplicity, we use $\mathscr{T}$ to denote the F-crystal $\mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$ in the rest of the chapter.

Remark 6.1.4. We observe that if we change the condition (2) in 1.0.2 to the following

$$\mathscr{E} \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{m+1} \text{ as } F^f\text{-isocrystals for some integers } f \text{ where } \mathcal{V}_i \text{ are all irreducible of rank 2,}$$

and if $\mathscr{X} \longrightarrow C$ is defined over a finite field, then we can consider the (neutral) Tannakian category of F^f-isocrystals instead of isocrystals. Mimic the above arguments and we can have the same result that $\mathscr{E}$ is isogenous to a tensor decomposition of $\mathcal{V} \otimes \mathscr{T}$.

6.2 Proof of 1.0.2: with maximal Higgs field

In this section, we prove the second part of 1.0.2, the case that $\mathscr{X} \longrightarrow C$ has maximal Higgs field.

By [9, Main Theorem 1], $\mathcal{V}_1$ corresponds to a BT group over C. In particular, it admits a Hodge filtration: $0 \longrightarrow \mathscr{L}_1 \longrightarrow \mathcal{V}_{1|C} \longrightarrow \mathscr{L}_2 \longrightarrow 0$ with the Higgs field $\theta_1 : \mathscr{L}_1 \longrightarrow \mathscr{L}_2 \otimes \Omega_C^1$. Since $\mathscr{T}$ is a unit root crystal, the Hodge filtration of $\mathscr{T}$ is trivial. Therefore the Hodge filtration of the Dieudonne crystal $\mathcal{V}_1 \otimes \mathscr{T}$ is

$$0 \longrightarrow \mathscr{L}_1 \otimes \mathscr{T}_C \longrightarrow (\mathcal{V}_1 \otimes \mathscr{T})_C \longrightarrow \mathscr{L}_2 \otimes \mathscr{T}_C \longrightarrow 0.$$

The associated Higgs field θ' is $\theta_1 \otimes \mathrm{id}_{\mathscr{T}}$ where θ_1 is the Higgs field of $\mathcal{V}_1$.

6.2.1 Hodge and conjugate filtrations

For any Dieudonne crystal $\mathcal{F}$ over $\mathrm{cris}(C/W(k))$, $\mathcal{F}_C$ admits two filtrations: Hodge filtration and conjugation filtration. The relation between the two filtrations is shown in [9, Proposition 2.5.2]: conjugation filtration is given by the kernel of F_C: $\omega^{(p)} \subset \mathcal{F}_C^{(p)}$, which the Frobenius pullback of the Hodge filtration $\omega \subset \mathcal{F}_C$. Since the isogeny $\psi : \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{m+1} \longrightarrow \mathscr{E}$ is compatible with the Frobenius, ψ_C induces actually a morphism between the Hodge filtrations and conjugate filtration. Since ψ preserves the connections, it induces a morphism between the Higgs fields.

Lemma 6.2.2. *The Hodge and conjugate filtrations of $\mathscr{E}$ induce a morphism $\alpha^{(p)} \longrightarrow \alpha$ which is generically surjective over C.*

Proof. Combine the two filtrations in one diagram:

$$\begin{tikzcd}
 & \alpha^{(p)} \arrow[d] \arrow[dr, dashed, "h"] & \\
\omega \arrow[r] & \mathscr{E}_C \arrow[r] \arrow[d] & \alpha \\
 & \omega^{(p)} &
\end{tikzcd}$$

For any $x \in C$, let the stalk $\alpha_x^{(p)}$ generate by $\{a_1 \otimes 1, \cdots, a_{2^m} \otimes 1\}$ where $a_i \in \alpha_x$. Since $\nabla(a_i \otimes 1) = 0$ and the Higgs field is maximal, $l(a_i \otimes 1)$ are linearly independent in α_x. Otherwise, some section $a \otimes 1$ is a local section of ω and due to maximal Higgs field, any local section of ω is not horizontal, contradiction. Therefore $a_i \otimes 1$ generate the stalk α_x generically. Therefore l is generically surjective. ☐

6.2.3 The Higgs field is nonzero

Lemma 6.2.4. *If the Higgs field θ_1 of $\mathcal{V}_1$ is zero, then* $\deg \mathscr{L}_2 \neq 0$.

Since $\deg \mathcal{V}_{1\,C} = \deg \mathscr{L}_1 + \deg \mathscr{L}_2 = \deg \mathscr{L}_1^p + \deg \mathscr{L}_2^p$, $\deg \mathcal{V}_{1\,C} = 0$. So we have $\deg \mathscr{L}_1 \neq 0$ as well.

Proof. We prove it by contradiction. Consider the Hodge and conjugate filtration of $(\mathcal{V}_1)_C$:

$$\begin{tikzcd} & \mathscr{L}_2^p \arrow[d, "F"] \arrow[dr, "l"] & \\ \mathscr{L}_1 \arrow[r] & (\mathcal{V}_1)_C \arrow[r] \arrow[d, "V"] & \mathscr{L}_2 \\ & \mathscr{L}_1^p. & \end{tikzcd}$$

Choose any open affine subset $U \subset C$ over which $\mathscr{L}_1$ is free. Let t be a generator of $\mathscr{L}_1$ over U. Then if $l(t) = 0$, then $F(t) \in \mathscr{L}_1(U)$. Since $\psi_1 = 0$, then for any section $s \in \mathscr{T}(U)$, $\psi(t \otimes s) = 0$. However, $\psi(t\otimes s) = \psi_2^{(p)}(t\otimes s)$ and hence ψ_2 is zero over U. Since $\operatorname{im}\psi_2$ is torsion-free, $\psi_2 = 0$ globally, contradiction. Thereby l is injective and it induces an injection $\mathcal{O}_C \hookrightarrow \mathscr{L}_2^{1-p}$. In particular, $\mathscr{L}_2^{1-p}$ is effective. If $\deg \mathscr{L}_2 = 0$, then $\mathscr{L}_2^{1-p} \cong \mathcal{O}_C$ and hence l is an isomorphism.

Note $\mathscr{T}$ is a unit root crystal. The isomorphism l induces isomorphism $\mathscr{L}_2^p \otimes \mathscr{T}^{(p)} \longrightarrow \mathscr{L}_2 \otimes \mathscr{T}$. It implies $\mathscr{X}$ is everywhere ordinary over C. However, C is proper and so it can not be contained in the ordinary locus of $\mathcal{A}_{2^m,1,k}$. □

Proposition 6.2.5. *If $\theta_1 = 0$, then there exists a rank 2 Dieudonne crystal $\mathcal{V}$ such that $\mathcal{V}_1 = \mathcal{V}^\sigma$ and the $\mathcal{V}_1 \otimes \mathscr{T} \longrightarrow \mathscr{E}$ factors through the Frobenius map $F_\mathcal{V} : \mathcal{V}_1 \otimes \mathscr{T} \longrightarrow \mathcal{V} \otimes \mathscr{T}$ of $\mathcal{V} \otimes \mathscr{T}$.*

Proof. The Higgs field of $\mathcal{V}_1$ is trivial and then $\psi_1 = 0$. By 3.4.1, $\mathcal{V}_1$ is the Frobenius pullback of some Dieudonne crystal $\mathcal{V}$. Let $F_\mathcal{V}, V_\mathcal{V}$ denote the Froebnius and Verschiebung of $\mathcal{V}$. Since $\psi_1 = 0$, the image of $\psi \circ V_\mathcal{V}$ is contained in $p\mathscr{E}$. Thereby $\frac{\psi\circ V_\mathcal{V}}{p} : \mathcal{V} \otimes \mathscr{T} \longrightarrow \mathscr{E}$ is a well defined morphism. Since $\frac{\psi\circ V_\mathcal{V}}{p} \circ F_\mathcal{V} = \psi$, ψ factors through $F_\mathcal{V}$:

$$\begin{tikzcd} & \mathcal{V} \otimes \mathscr{T} \arrow[dr] & \\ \mathcal{V}_1 \otimes \mathscr{T} \arrow[ur, "F_\mathcal{V}"] \arrow[rr, "\psi"] & & \mathscr{E}. \end{tikzcd}$$

□

Remark 6.2.6. If the Higgs field of $\mathcal{V}$ is still trivial, then we repeat the process of 6.2.5. Note the Hodge filtration $\mathscr{L} \subset \mathcal{V}_C$ of $\mathcal{V}$ satisfies $\mathscr{L}^p = \mathscr{L}_1$. Since by 6.2.4 $\deg \mathscr{L}_1 \neq 0$, $|\deg \mathscr{L}| < |\deg \mathscr{L}_1|$. After a finite steps, the degree of the sub line bundle from the Hodge filtration is no longer divisible by p and hence the Gauss Manin connection no longer preserves the Hodge filtration. The Higgs field is therefore nonzero. Then we can assume $\mathcal{V} \otimes \mathscr{T}$ with nontrivial Higgs field $\theta = \theta_\mathcal{V} \otimes \mathrm{id}_\mathscr{T}$.

6.2.7 The Higgs field is maximal

In the following, we will show the Higgs field θ of $\mathcal{V}\otimes\mathscr{T}$ is an isomorphism.

Let $0\longrightarrow\mathscr{L}\longrightarrow\mathcal{V}_C\longrightarrow\mathscr{L}'\longrightarrow 0$ be the Hodge filtration of $\mathcal{V}_C$. By [9, Main Theorem 1], the Dieudonne crystal $\mathcal{V}\otimes\mathscr{T}$ corresponds to a BT group $G\otimes U$ over C where G is a height 2, generically ordinary BT group with nontrivial Kodaira Spencer map and U is a height 2^m etale BT group. The BT groups associated to $\mathcal{V}\otimes\mathscr{T}$ and $\mathscr{E}$ are isogenous and therefore $G\otimes U$ also comes from an abelian scheme $\mathscr{Y}$ over C which is isogenous to $\mathscr{X}$. Let

$$g:\mathscr{Y}\longrightarrow\mathscr{X}$$

be the isogeny.

Note $\mathscr{Y}[p^\infty]\cong G\otimes U$. Let

$$K=\ker g.$$

Then K is a finite flat group scheme. Let $G[p^n]\otimes U[p^n]$ be the smallest truncated BT group containing K.

Then we can choose a finite etale covering of C such that the pullback of $U[p^n]$ is trivial. Then the pullback of $\mathscr{Y}[p^n]$ is $G[p^n]^{\times 2^m}$ and $\mathscr{Y}$ has maximal Higgs field if and only if the pull back of $\mathscr{Y}$ does. For simplicity, we still denote the etale covering as C.

Lemma 6.2.8. *For any n, any nontrivial subgroup scheme of $G_{n,\eta}$ contains μ_p.*

Proof. Let H be a nontrivial subgroup scheme of $G_{n,\eta}$. We can reduce n so that H is not contained in G_{n-1}.

$$\begin{tikzcd} & H \arrow[d, hook] \arrow[dr] & \\ G_{n-1,\eta} \arrow[r] & G_{n,\eta} \arrow[r, "p^{n-1}"] & G_{1,\eta}. \end{tikzcd}$$

Then H has a nontrivial image in $G_{1,\eta}$ which is flat (over the field). Since $G_{1,\eta}$ has maximal Higgs field, any subgroup scheme of $G_{1,\eta}$ contains μ_p. So this image contains μ_p. Note p^{n-1} is an endomorphism of H. Thus $\mu_p\subset H$. ☐

Now we prove that $K\cap(1^{\times 2^m-1}\times G_n)=1$. The key observation is that since $\dim\operatorname{Lie}K\leq 2^m-1$(see the proof of Theorem 6.2.11), by changing of coordinates, we can make K_η be contained in the product of first three factors $G_{n,\eta}^{\times 2^m-1}$ of $G_{n,\eta}^{\times 2^m}$.

Since G is generically ordinary, the filtration

$$0\longrightarrow G_{n,\eta,\mathrm{mult}}\longrightarrow G_{n,\eta}\longrightarrow G_{n,\eta,\mathrm{et}}\longrightarrow 0.$$

Since the group schemes are defined over a field, we may assume the multiplicative part is isomorphic to μ_{p^n}, the etale part isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Since over a field, all the group scheme involved are automatically flat. We start with the following proposition

Proposition 6.2.9. *Over k, for any n, if H is a subgroup of $\mu_{p^n}^{\times 2^m}$ with $\dim Lie\ H \leq 2^m - 1$, then via changing of coordinates, H is contained in the product of first $2^m - 1$ factors $\mu_{p^n}^{\times 2^m - 1}$.*

Proof. We consider the case $\dim \mathrm{Lie}\ H = 2^m - 1$ and we prove it by induction.

If $n = 1$, then $H \subset \mu_p^{\times 2^m}$ such that H has height $2^m - 1$. Since μ_p is simple, $H \cong \mu_p^{\times 2^m - 1}$ and one of projections of H to the product of any $2^m - 1$ factors $H \longrightarrow \mu_p^{\times 2^m - 1}$ is injective. Assume the projection to the first $2^m - 1$ factors is isomorphic. Then by change of coordinates, we can put $H = \mu_p^{\times 2^m - 1} \subset \mu_p^{\times 2^m}$.

If the proposition is true for $k - 1$, then for $n = k$, after changing of coordinates, we have $H \cap \mu_{p^{k-1}}^{\times 2^m} \subset \mu_{p^{k-1}}^{\times 2^m - 1}$. Let pr_{2^m} be the projection of H to the 2^m-th factor and $K_{2^m} = \ker \mathrm{pr}_{2^m}$

$$K_{2^m} \hookrightarrow H \xrightarrow{\mathrm{pr}_{2^m}} \mu_{p^k}.$$

We may assume pr_{2^m} is nontrivial. Obviously $K_{2^m} \subset \mu_{p^k}^{\times 2^m - 1}$. Since $pH \subset H \cap \mu_{p^{k-1}}^{\times 2^m} \subset \mu_{p^{k-1}}^{\times 2^m - 1}$, $\mathrm{im}\, \mathrm{pr}_{2^m} \subset \mu_p$. And $p^{k-1} K_{2^m} \subset \mu_p^{\times 2^m - 1}$ is a flat sub group scheme with height h. Since pr_{2^m} is nontrivial, $h \leq 2^m - 2$. Note the reduction

$$\mathrm{Aut}(\mu_{p^k}^{\times 2^m - 1}) \cong GL_{2^m - 1}(\mathbb{Z}/p^k\mathbb{Z}) \longrightarrow \mathrm{Aut}(\mu_p^{\times 2^m - 1}) \cong GL_{2^m - 1}(\mathbb{F}_p)$$

is surjective. Thus we can apply some automorphism of $\mu_{p^k}^{\times 2^m - 1}$ to implement the coordinate change so that we can make $p^{k-1} K_{2^m}$ be exactly the product of first h factors $\mu_p^{\times h}$.

Consider the following diagram

$$\begin{tikzcd} & H \arrow[r] \arrow[d, hook] & H/H \cap \mu_{p^{k-1}}^{\times 2^m} \arrow[d, hook] \\ \mu_{p^{k-1}}^{\times 2^m} \arrow[r] & \mu_{p^k}^{\times 2^m} \arrow[r, "p^{k-1}"] & \mu_p^{\times 2^m}. \end{tikzcd}$$

Since $H \cap \mu_{p^{k-1}}^{\times 2^m} = K_{2^m} \cap \mu_{p^{k-1}}^{\times 2^m - 1}$, the morphism pr_{2^m} factors through $H/H \cap \mu_{p^{k-1}}^{\times 2^m}$ and hence $\mu_p^{\times 2^m} \supset H/H \cap \mu_{p^{k-1}}^{\times 2^m} \longrightarrow \mu_p$ surjectively, induced by pr_{2^m}. The kernel of this surjective map, denoting as pr'_{2^m}, is exactly $K_{2^m}/K_{2^m} \cap \mu_{p^{k-1}}^{\times 2^m - 1} \cong \mu_p^{\times h}$. Therefore the graph of pr'_{2^m} is a height 1 subgroup scheme in $\mu_p^{\times (2^m - h)}$ and pr'_{2^m} is actually the last factor projection of this subgroup scheme. There exists an element in $\mathrm{Aut}(\mu_{p^k}^{\times 2^m})$ whose reduction mod p transforms the height 1 subgroup scheme to be the first factor in $\mu_p^{\times (2^m - h)}$, i.e. the image of pr_{2^m} is trivial. Note the automorphisms involved here are of the form $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^{k-1}a & p^{k-1}b & p^{k-1}c & 1 \end{pmatrix}$ which fixes the first 2^m-1 factors $\mu_{p^k}^{\times 2^m-1}$. Thereby H is contained in $\mu_{p^k}^{\times 2^m-1}$.

For the easier cases $\dim \mathrm{Lie}\ H < 2^m-1$, we can adjust above argument accordingly. ☐

Now we come back to the original case. Firstly we need the following lemma which relates the 6.2.9 and our target:

Lemma 6.2.10. *The restriction* $\mathrm{Aut}(G[p^n]_\eta) \longrightarrow \mathrm{Aut}(\mu_{p^n})$ *is surjective.*

Proof. It follows from $\mathrm{Aut}(\mu_{p^n}) = (\mathbb{Z}/p^n\mathbb{Z})^*$ and $(\mathbb{Z}/p^n\mathbb{Z})^*$ naturally acts on any p^n torsion group scheme, in particular, $G_{n,\eta}$. ☐

Proposition 6.2.11. *By changing coordinates,* K_η *is contained in the product of the first* 2^m-1 *factors* $G_{n,\eta}^{\times 2^m-1}$.

Proof. Since the Higgs field θ_1 is nonzero by 6.2.6, $f: \mathscr{Y} \longrightarrow \mathscr{X}$ induces a nonzero map on the tangent bundles. Thereby the dimension of Lie K is at most 2^m-1. By 6.2.9 and 6.2.10, we can assume $K_\eta \cap \mu_{p^n}^{\times 2^m} \subset \mu_{p^n}^{\times 2^m-1}$. Let pr_{2^m} be the projection $G_{n,\eta}^{\times 2^m} \longrightarrow G_{n,\eta}$ to the 2^m factor and K_{2^m} be the kernel of the restriction of $\mathrm{pr}_{2^m}: K_\eta \longrightarrow G[p^n]_\eta$.

$$\begin{array}{ccccc} & & K_\eta & & \\ & & \big\downarrow & & \\ \mu_{p^n}^{\times 2^m} & \longrightarrow & G_{n,\eta}^{\times 2^m} & \longrightarrow & \mathbb{Z}/p^n\mathbb{Z}^{\times 2^m}. \end{array}$$

Obviously $K_\eta \cap \mu_{p^n}^{\times 2^m} \supset K_{2^m} \cap \mu_{p^n}^{\times 2^m-1}$. Since $K_\eta \cap \mu_{p^n}^{\times 2^m} \subset \mu_{p^n}^{\times 2^m-1}$, $\mathrm{pr}_{2^m}(K_\eta \cap \mu_{p^n}^{\times 2^m}) = 1$ and hence $K_\eta \cap \mu_{p^n}^{\times 2^m} \subset K_{2^m} \cap \mu_{p^n}^{\times 2^m-1}$. So $K_\eta \cap \mu_{p^n}^{\times 2^m} = K_{2^m} \cap \mu_{p^n}^{\times 2^m-1}$

Thereby on one hand, K_η/K_{2^m} is a quotient of an étale group scheme $K_\eta/(K_\eta \cap \mu_{p^n}^{\times 2^m}) \subset (\mathbb{Z}/p^n\mathbb{Z})^{\times 2^m}$ by the subgroup $K_{2^m}/(K_{2^m} \cap \mu_{p^n}^{\times 2^m-1})$. On the other hand, it is also a subgroup of $G[p^n]_\eta$. By 6.2.8, if it is nontrivial, $\mu_p \subset K_\eta/K_{2^m}$, contradiction to etale. Thus $K_\eta = K_{2^m}$ and $\mathrm{im}\,\mathrm{pr}_{2^m}$ is trivial. ☐

Proposition 6.2.12. *As subgroup schemes of* $G^{\times 2^m}$, *we can change the coordinates so that*

$$K \cap (1^{\times 2^m-1} \times G_n) = 1.$$

Proof. By 6.2.11, since $K_\eta \subset G_{n,\eta}^{\times 2^m-1} \subset G_{n,\eta}^{\times 2^m}$ and K, G_n are flat, taking the closure in $G^{\times 2^m}$ gives $K \subset G_n^{\times 2^m-1}$. Thereby globally $K \cap (1^{\times 2^m-1} \times G_n) = 1$. ☐

Theorem 6.2.13. *The Higgs field θ of $\mathcal{V} \otimes \mathscr{T}$ is maximal.*

Proof. Note $f : \mathscr{Y} \longrightarrow \mathscr{X}$ induces a diagram

$$\begin{tikzcd} \omega \arrow[r] \arrow[d, "\cong"] & \mathscr{L}_1 \otimes \mathscr{T} \arrow[d, "\theta \cong \theta_1 \otimes \mathrm{Id}"] \\ \alpha \otimes \Omega^1_C \arrow[r, "f^*"] & \mathscr{L}_2 \otimes \mathscr{T} \otimes \Omega^1_C. \end{tikzcd}$$

We have known that θ is not zero. Since $\theta_1 : \mathscr{L}_1 \longrightarrow \mathscr{L}_2 \otimes \Omega^1_C$, over each fibre of C, θ is either zero or an isomorphism. It suffices to show f^* is nonzero over each fibre.

Consider the composition

$$G_n \hookrightarrow G \otimes H \cong \mathscr{Y}[p^\infty] \longrightarrow \mathscr{X}[p^\infty]$$

over C where the first arrow is the embedding to the fourth factor. By 6.2.12, this composition is injective. Therefore im f contains a generically ordinary height 2 BT_n subgroup of $\mathscr{X}[p^\infty]$ over C. Such G_n provides nonzero elements in im f^* over each fibre. $\square$

Now $\mathscr{Y} \longrightarrow C$ satisfies

1. $\mathscr{Y}/C$ has the maximal Higgs field,

2. $\mathbb{D}(\mathscr{Y}[p^\infty]) \cong \mathcal{V} \otimes \mathscr{T}$ where $\mathcal{V}$ is a Dieudonne crystal and $\mathscr{T}$ is a rank 2^m unit root crystal.

The last step is based on Serre-Tate theory. For a given ring R and an ideal $I \subset R$, let $R_0 = R/I$. Write $AS(R)$ for the category of abelian schemes over R and write $BT - \mathrm{Def}(R_0, R)$ for the category of triples (A_0, G, α) where A_0 is an abelian scheme over R_0, G is a BT group over R and α is an isomorphism $\alpha : G_0 = G \otimes R_0 \cong A_0[p^\infty]$.

Theorem 6.2.14. *[33, Chapter 5, 1.6] Let R be a ring in which the prime p is nilpotent, let $I \subset R$ be a nilpotent ideal and write $R_0 = R/I$. Then the functor $AS(R) \longrightarrow BT - \mathrm{Def}(R, R_0)$ obtained by sending A to the triplet*

$$(A_0 = A \otimes R_0, A[p^\infty], \textit{the natural isomorphism } \alpha)$$

is an equivalence.

Theorem 6.2.15. *The family of polarized abelian varieties $\mathscr{Y} \longrightarrow C$ as above has a unique lifting to $W(k)$ which is a family of abelian fourfolds with maximal Higgs field.*

Proof. By [9, Main Theorem 1], the rank 2 Dieudonne crystal $\mathcal{V}$ corresponds to a height 2 BT group G and the unit root crystal $\mathscr{T}$ corresponds to a height 2^m etale BT group. By 1.0.6, the height 2 BT group gives a lifting $\tilde{C}$ of C to $W(k)$. The lifting of G gives a lifting of $\mathscr{Y}[p^\infty]$. By 6.2.14, $\mathscr{Y} \longrightarrow C$ lifts to a formal aeblian scheme over $\tilde{C}$ over $W(k)$. Again by 1.0.5, the polarization lifts as well and hence the family $\mathscr{Y} \longrightarrow C$ lifts to a polarized abelian scheme $\tilde{\mathscr{Y}} \longrightarrow \tilde{C}$ over $W(k)$.

Since the special fibre has maximal Higgs field, so is $\tilde{\mathscr{Y}} \longrightarrow \tilde{C}$. □

Remark 6.2.16. From the proof of 6.2.15, the weight 1 variation of Hodge structure $R^1\tilde{\pi}_*(\Omega^\cdot_{\tilde{\mathscr{Y}}/\tilde{C}}) \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{m+1}$ has the Hodge filtration coming from $\mathscr{L} \hookrightarrow \mathcal{V}_1$.

To show the $\tilde{C}$ is a Mumford curve, we use Theorem 0.5 in [47]. The family in our case is smooth so that there is no unitary part. Maximal Higgs field implies the family reaches the Arakelov bound. Apply [47, Theorem 0.5] and we have $\tilde{\mathscr{Y}} \longrightarrow \tilde{C}$ is isogenous to a Mumford curve. In particular, the generic fibre $(\tilde{\mathscr{Y}} \longrightarrow \tilde{C}) \otimes \mathbb{C}$ is a Shimura curve (with a universal family). From 6.2.16, there can be more than one copy of Mumford curve Z appearing in the decomposition of $\tilde{\mathscr{Y}}$. Therefore $(\tilde{\mathscr{Y}} \longrightarrow \tilde{C}) \otimes \mathbb{C}$ is a Mumford curve.

It finishes the proof of the case with maximal Higgs field.

6.3 Proof of 1.0.2: without maximal Higgs field

In this part, we will prove without maximal Higgs field, $\mathscr{X} \longrightarrow C$ is a weak Shimura curve over k. Note the arguments before 6.2.7 still hold for the non-maximal Higgs field case. So $\mathscr{E}$ is isogenous to the tensor product $\mathcal{V} \otimes \mathscr{T}$ with $\mathcal{V}$ a rank 2 Dieudonne crystal and $\mathscr{T}$ a rank 2^m unit root crystal. Equivalently, there exists a family of abelian varieties $\mathscr{Y} \longrightarrow C$ such that

$$\mathscr{Y}[p^\infty] = G \otimes H$$

where G is a height 2 BT group and H is a height 2^m etale BT group. And the family $\mathscr{Y} \longrightarrow C$ induces a morphism $\phi : C \longrightarrow \mathcal{A}_{2^m,d}$

Let us introduce the categories $\mathcal{C}$ and $\hat{\mathcal{C}}$, following the terminology of [44]. The objects of $\mathcal{C}$ are the artinian local algebras R such that $R/\mathfrak{m}_R \cong k$. The morphisms in $\mathcal{C}$ are the homomorphisms of algebras. Then $\hat{\mathcal{C}}$ is defined as the category of complete noetherian local algebras $\mathcal{R}$ such that $\mathcal{R}/\mathfrak{m}^i_{\mathcal{R}}$ is in $\mathcal{C}$ for all i. Again the morphisms are just the homomorphisms of algebras. Notice that $\mathcal{C}$ is a full subcategory in $\hat{\mathcal{C}}$. We consider the $\mathcal{R}$ in $\hat{\mathcal{C}}$ with their $\mathfrak{m}_{\mathcal{R}}$-adic topology; for R in $\mathcal{C}$ this is just the discrete topology.

For any scheme Z_0 over k, we define a formal deformation functor $\mathrm{Def}_{Z_0} : \mathcal{C} \longrightarrow \mathrm{Sets}$, given by

$$\mathrm{Def}_{Z_0}(R) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (Z, \psi), \text{ where } Z \text{ is an scheme} \\ \text{over Spec } R \text{ and } \psi \text{ is an isomorphism } \psi : Z \otimes k \cong Z_0 \end{array} \right\}$$

In particular, we can define deformation functor $\mathrm{Def}_{X_0,\lambda}$ for an abelian scheme X_0 and its polarization λ. And we can extend these deformation functors to the category $\hat{\mathcal{C}}$ by defining $\mathrm{Def}_*(\mathcal{R})$ as the projective limit of the $\mathrm{Def}_*(\mathcal{R}/\mathfrak{m}_{\mathcal{R}}^i)$.

Note the functor Def_{X_0} is isomorphic to $\mathrm{Hom}(T_pX_0 \otimes T_pX_0^t, \hat{\mathbb{G}}_m)$ and the functor $\mathrm{Def}_{(X_0,\lambda)}$ is represented by the formal completion of $\mathcal{A}_{g,d,n}$ at X_0 which is a subformal torus of $\mathrm{Hom}(T_pX_0 \otimes T_pX_0^t, \hat{\mathbb{G}}_m)$.

Proposition 6.3.1. *Under the above assumption, if $\mathscr{Y}$ is not isotrivial, then the universal family over* $\mathrm{im}\,\phi$ *has maximal Higgs field.*

Proof. Since $\mathscr{Y} \longrightarrow C$ is not isotrivial, the morphism ϕ is not a point. Then the Higgs field of $\mathrm{im}\,\phi$ is not zero.

For any closed point $c \in C$, let G_c be the restriction of the height 2 BT group G to the point c. Since $\mathscr{Y}$ is not isotrivial, $\mathrm{im}\,(\mathrm{Spec}\ \hat{\mathcal{O}}_{C,c} \longrightarrow \mathrm{Def}_{G_c})$ has dimension ≥ 1. By [21], the local formal deformation space Def_{G_c} over k is one-dimensional and isomorphic to $k[[t]]$. We know $\mathrm{Spec}\ \hat{\mathcal{O}}_{C,c} \longrightarrow \mathrm{Def}_{G_c}$ is a surjection.

Let λ be a polarization on $\mathscr{Y}_c$. Then $\mathscr{Y}_c$ has two deformation spaces $\mathrm{Def}_{\mathscr{Y}_c}$ and $\mathrm{Def}_{\mathscr{Y}_c,\lambda}$. And there exists a natural embedding $\mathrm{Def}_{(\mathscr{Y}_c,\lambda)} \longrightarrow \mathrm{Def}_{\mathscr{Y}_c}$.

Since $\mathscr{Y}_c[p^\infty] \cong G_c \otimes H_c$, by Serre-Tate theory(6.2.14), a deformation of G_c induces deformation of $\mathscr{Y}_c$. Thereby we have a morphism between local deformation space $\mathrm{Def}(G_c) \longrightarrow \mathrm{Def}(\mathscr{Y}_c)$ which is a local embedding.

Consider

$$\begin{tikzcd} & \mathrm{Spec}\ \hat{\mathcal{O}}_{C,c} \arrow[dl, two heads] \arrow[dr] & \\ \mathrm{Def}(G_c) \arrow[rr, dashed] \arrow[dr] & & \mathrm{Def}(\mathscr{Y}_c, \lambda) \arrow[dl, hook] \\ & \mathrm{Def}(\mathscr{Y}_c) & \end{tikzcd}.$$

Since $\mathrm{im}\,(\mathrm{Spec}\ \hat{\mathcal{O}}_{C,c} \longrightarrow \mathrm{Def}(\mathscr{Y}_c)) \cong \mathrm{Def}(G_c)$ and $\mathrm{Def}_{(\mathscr{Y}_c,\lambda)} \longrightarrow \mathrm{Def}_{\mathscr{Y}_c}$ is an embedding, we have a morphism $\mathrm{Def}(G_c) \longrightarrow \mathrm{Def}(\mathscr{Y}_c, \lambda)$. Thereby $\mathrm{im}\,(\mathrm{Spec}\ \hat{\mathcal{O}}_{C,c} \longrightarrow \mathrm{Def}(\mathscr{Y}_c, \lambda)) \cong \mathrm{Def}(G_c) \cong k[[t]]$. Thus $\mathcal{O}_{\mathrm{im}\,\phi,c}$ maps surjectively to $\mathrm{Def}(G_c)$ which implies the universal family over $\mathrm{im}\,\hat{\phi}$ has maximal Higgs field. $\square$

Remark 6.3.2. Now denote $\mathrm{im}\,\phi$ as C'.

1. From the proof of 6.3.1, C' has at worst ordinary singularity.

2. We can apply the 6.2.15 to the normalization of C' and then obtain $\mathscr{Y} \longrightarrow C'$ is a semistable reduction of a special Mumford curve in $\mathcal{A}_{2^m,d,n} \otimes \mathbb{C}$.

Then the following proposition justifies the non-maximal Higgs field case.

Proposition 6.3.3. *Notations as 1.0.2. If $\mathscr{Y} \longrightarrow C$ is another family of polarized abelian varieties over C such that*

1. *$\mathscr{Y}_c$ is ordinary for some closed point $c \in C$,*

2. *the image of $C \longrightarrow \mathcal{A}_{2^m,d,n} \otimes k$ induced by $\mathscr{Y}$ is a reduction of a special Mumford curve,*

3. *$f : \mathscr{Y} \longrightarrow \mathscr{X}$ is an isogeny compatible with polarization.*

Then the image of $C \longrightarrow \mathcal{A}_{2^m,1,n}$ induced by $\mathscr{X}/C$ is an irreducible component of a reduction of a special Mumford curve.

The proof of 6.3.3 takes the rest of the chapter. First let us review the isogeny scheme, referring to [14].

6.3.4 The isogeny scheme

Let Isog_g be the moduli stack of isogenies between polarized abelian schemes of relative dimension g, so that for an arbitrary base S, $\mathrm{Isog}_g(S)$ is the category in groupoids whose objects are the isogenies $\phi : A_1 \longrightarrow A_2$ over S between polarized abelian schemes $(A_1 \longrightarrow S, \lambda_1)$ and $(A_2 \longrightarrow S, \lambda_2)$ of relative dimension g such that $\phi^*\lambda_2 = p^e.\lambda_1$ for some $e \in \mathbb{N}$. Here we do not require A_i to be principally polarized.

Assigning ϕ to its source(resp. target) defines a morphism $\mathrm{pr}_1 : \mathrm{Isog}_g \longrightarrow \mathcal{A}_{g,d}$(resp. $\mathrm{pr}_2 : \mathrm{Isog}_g \longrightarrow \mathcal{A}_{g,d'}$). Bounding the degree of the isogeny gives a substack of Isog_g. We write $\mathrm{Isog}(p^l)$ for the stack of p-isogenies of degree less than or equal to p^l which is of finite type over $\mathcal{A}_{g,d}$.

As a variant, we can take level structure into account. Choose an integer $n > 3$ and consider $\mathcal{A}_{g,d,n}$. Making isogenies compatible with level structures, we obtain a scheme $\mathrm{Isog}(p^l)$ over $\mathcal{A}_{g,d,n} \times \mathcal{A}_{g,d',n}$. To keep notations easy, we omit the subscripts n.

Let $\mathrm{Isog}^o(p^l)_g$ (resp. $\mathcal{A}^o_{g,d}$) denote the ordinary locus of $\mathrm{Isog}_g(p^l)$ (resp. $\mathcal{A}_{g,d}$) over any base, i.e., $\mathcal{A}_{g,d}$ denote the open stack of $\mathcal{A}_{g,d}$ whose fibre in characteristics different from p coincide with $\mathcal{A}_{g,d}$, while in the fibre over $\mathbb{F}_p$, the stack $\mathcal{A}^o_{d,g}$ denote the ordinary locus.

We need the following result to prove 6.3.3. To ease the notations, we do not write the bound of the degree p^l.

Proposition 6.3.5. *The projections $pr_i : Isog_g^o \longrightarrow \mathcal{A}_{g,1}^o$ are finite and flat.*

Proof. This proposition is essentially the same as the Proposition 4.1(i) in the Chapter VII of [14]. Here we give a proof.

It is known that the morphism $\mathrm{Isog}_g \longrightarrow \mathcal{A}_{g,1}$ is proper. Let Y be any ordinary abelian variety over any algebraically closed field. Note the degree of the isogeny is bounded and there is only finitely many subgroup in $Y[p^n] \cong Y^{\mathrm{et}}[p^n] \times Y^{\mathrm{mult}}[p^n] \cong \mu_{p^n} \times \mathbb{Z}/p^n$. Therefore the fibre of projection $\mathrm{Isog}_g \longrightarrow \mathcal{A}_{g,1}$ over X is finite which implies that $\mathrm{pr}_i : \mathrm{Isog}_g^o \longrightarrow \mathcal{A}_{g,1}^o$ is quasi-finite. So pr_i is a finite morphism.

Let η (resp. ξ) be a closed point in $\mathcal{A}_{g,1}$(resp. Isog_g) with stalk A_η(resp. B_ξ) and assume $\mathrm{pr}(\xi) = \eta$. Since both A_η and B_ξ are local rings, by local criterion of flatness, it suffices to show the image of A_η in B_ξ contains no zerodivisor. Then we can assume A_η and B_ξ are complete with respect to the maximal ideals.

Let t be any element in the image of A_η and $\mathfrak{q}$ be a minimal point in the zero set $V(t) \subset \mathrm{Spec}\ B_\xi$. Then $\mathrm{ht}\mathfrak{q} \leq 1$. Let $\mathfrak{p} = \mathfrak{q} \cap A_\eta$.

On the one hand, t is a zerodivisor if and only if $\mathrm{ht}\,\mathfrak{q} = 0$.

On the other hand, for the principally polarized abelian variety X defined over $\mathrm{Spec}\ (A_\eta)_\mathfrak{p}$ and an isogeny $g : X \otimes k \longrightarrow Y$ over the special fibre of $\mathrm{Spec}\ (A_\eta)_\mathfrak{p}$. Let $K = \ker g$ and thus $K \subset X_k[p^e] \cong X_k[p^e]^{\mathrm{mult}} \times X_k[p^e]^{\mathrm{et}}$. Then correspondingly, $K = K^{\mathrm{mult}} \times K^{\mathrm{et}}$. Since A_η is complete, the splitting $X_k[p^e] \cong X_k[p^e]^{\mathrm{mult}} \times X_k[p^e]^{\mathrm{et}}$ can be extended to $\mathrm{Spec}\ (A_\eta)_\mathfrak{p}$. Since K^{et} is étale, it extends to a constant group scheme over $\mathrm{Spec}\ (A_\eta)_\mathfrak{p}$. The group K^{mult} lifts uniquely to a subgroup scheme of $X_k[p^e]^{\mathrm{mult}}$ by the rigidity of groups of multiplicative type. Then K can be lifted to $\mathrm{Spec}\ (A_\eta)_\mathfrak{p}$ as a subgroup scheme. Thereby we have a deformation of the BT group $Y[p^\infty]$. By Serre-Tate theory [33, Chapter 5, 1.6], the abelian variety Y can be lifted to a formal abelian scheme over $\mathrm{Spf}(A_\eta)_\mathfrak{p}$. Since Y is principally polarized, any polarization on Y can be lifted due to the uniqueness of the lifting of Y. Thus the isogeny ψ can be lifted.

Therefore the image of $\mathrm{Spec}\ (B_\xi)_\mathfrak{q} \longrightarrow \mathrm{Spec}\ (A_\eta)_\mathfrak{p}$ is not just the special fibre, i.e. $\mathrm{ht}(\mathfrak{q}) > 0$. So pr is flat over the point Y. ☐

Remark 6.3.6. In the proof of 6.3.5, the principal polarization is only used in the last step, i.e. to lift the polarization. Therefore, if X is principally polarized and the polarization λ_Y satisfies

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\lambda_Y"] & X \arrow[d, "\lambda_X"] \\ Y^t & X^t \arrow[l] \end{tikzcd}$$

is commutative, then the proof still works. So we have $\mathrm{Isog}_g^o \longrightarrow \mathcal{A}_{g,\deg\lambda_Y}^o$ is also flat at the point corresponding to Y.

6.3.7 Proof of 6.3.3

Let K be the kernel of $X \longrightarrow Y$. In our case, we mainly focus on the Isog_{2^m} associated to $\mathcal{A}_{2^m,1}$ and $\mathcal{A}_{2^m,d}$ where $d = 2|K|$.

$$\begin{tikzcd}
& \mathrm{Isog}_{2^m} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
\mathcal{A}_{2^m,1} & & \mathcal{A}_{2^m,d}
\end{tikzcd}$$

Let us denote the lifting of the image of $C \longrightarrow \mathcal{A}_{2^m,d,n}$ as $\tilde{C}$ and $\mathcal{O}_{\tilde{C},\eta}$ be the local ring of $\tilde{C}$ at η. The isogeny $\mathscr{Y}_\eta \longrightarrow \mathscr{X}_\eta$ gives a point $\xi \in \mathrm{Isog}^o_{2^m}$ such that $\mathrm{pr}_2(\xi) = \eta$. By 6.3.6, pr_2 is finite and flat over ξ. Therefore there exists a local ring R such that subscheme Spec $R \subset \mathrm{Isog}_{2^m}$ of dimension 1 with special fibre ξ. Further, $\mathrm{pr}_2(\mathrm{Spec}\ R) = \mathrm{Spec}\ \mathcal{O}_{\tilde{C},\eta}$.

$$\begin{tikzcd}
\mathrm{Spec}\ R \arrow[r] \arrow[d] & \mathrm{Isog}_{2^m} \times \tilde{C} \arrow[r] \arrow[d] & \mathrm{Isog}_{2^m} \arrow[d] \\
\mathrm{Spec}\ \mathcal{O}_{\tilde{C},\eta} \arrow[r] & \tilde{C} \arrow[r] & \mathcal{A}_{2^m,d}
\end{tikzcd}$$

Let $\tilde{C}' = (\mathrm{Spec}\ R)^-$ be the closure of Spec R in $\mathrm{Isog}_{2^m} \times \tilde{C}$ Then $\tilde{C}'$ admits a finite surjective morphism to $\tilde{C}$. Then the morphism $\tilde{C}' \longrightarrow \mathrm{Isog}_{2^m}$ induces a lifting

$$\tilde{\mathscr{Y}}' \longrightarrow \tilde{\mathscr{X}}'$$

of the isogeny $\mathscr{Y} \longrightarrow \mathscr{X}$ to $\tilde{C}'$.

By 6.3.2, the family $\tilde{\mathscr{Y}} \longrightarrow \tilde{C}$ over $W(k)$ whose generic fibre is a special Mumford curve in $\mathcal{A}_{2^m,d,n} \otimes \mathbb{C}$. Note $\mathrm{pr}_2(\tilde{C}') = \tilde{C}$ and hence $\mathrm{pr}_1(\tilde{C}')$ also has a special Mumford curve as the generic fibre. Note $\tilde{\mathscr{X}}' \longrightarrow \mathrm{pr}_1(\tilde{C}')$ is a lifting of the image of $C \longrightarrow \mathcal{A}_{2^m,1,n,k}$. Since the reduction of $\tilde{C}'$ may be reducible, we can at best have that the image of $C \longrightarrow \mathcal{A}_{2^m,1,n,k}$ is an irreducible component of the reduction. So $\mathscr{X} \longrightarrow C$ is a weak Shimura curve over k.

Now we complete the proof of 1.0.2.

Chapter 7

Crystalline Hodge cycles and l-adic monodromy

In this chapter, we prove 1.0.3 and 1.0.4 by reducing them to 1.0.2. In the course of proof, we obtain an equivalence 7.1.4 between simplicity of an abelian scheme and its corresponding F-isocrystal .

Let C be a smooth proper curve over $\bar{\mathbb{F}}_p$ with genus > 1 and $X \longrightarrow C$ is a family of principally polarized abelian fourfolds. Let $\mathcal{E}$ be the rank 8 Dieudonne crystal $\mathbb{D}(X/C)$. By comparing between l-adic and crystalline Lefschetz Trace formulas, we can reduce 1.0.4 to 1.0.3. Since the family $X \longrightarrow C$ is defined over a finite field, for a large integer f, the category of F^f-isocrystals over C forms a neutral Tannakian category. We can induce from conditions (1) and (2) in 1.0.3 that $\mathcal{E}$ corresponds to a $SL(2)^{\times 3}$-representation. After taking a finite étale covering, $\mathcal{E}$ has the decomposition as assumed in 1.0.2.

7.1 The structure of G_E

In Section 7.1 and 7.2, we prove Theorem 1.0.3.

Note in the Tannakian formalism, the F^f-crystal $\mathcal{E}$ gives rise to a linear algebraic group G_E acting on an 8-dimensional vector space E. We will prove G_E is reductive, which follows from a general result 7.1.4. Then conditions (2) and (3) can be translated to

$$\dim_{B(k_0)}(\wedge^4 E)^{G_E} = 1, \mathrm{End}(\wedge^2 E)^{G_E} \cong \mathbb{Q}_{p^f}^{\times 4}.$$

Using the classification of representations of simple Lie algebras, we can show G_E geometrically has the shape of $SL(2)^{\times 3}$ and E geometrically corresponds to a tensor product of three copies of standard representations of $SL(2)$.

The obstruction to descend the decomposition from $\mathbb{C}$ to the base field $\mathbb{Q}_{p^f}$ is a certain cohomology class (see 7.2.1,7.2.3 and 7.2.4). It is where we have to choose a finite étale cover of C.

7.1.1 Simplicity of X/C

Proposition 7.1.2. *Under the assumption of 1.0.3, there is no proper abelian subvariety $Z \hookrightarrow X$ over C.*

Proof. If there exists a proper abelian subvariety $Z \subset X$, then by Poincare irreducibility theorem, we have $Z' \subset X$ and an isogeny $g : X \longrightarrow Z \times Z'$. The isogeny induces a morphism $g^* : H^4_{\mathrm{cris}}(Z \times Z', \mathcal{O}) \longrightarrow H^4_{\mathrm{cris}}(X, \mathcal{O})$. By Kunneth formula for crystalline cohomology,

$$H^4_{\mathrm{cris}}(Z \times Z', \mathcal{O}) = H^4_{\mathrm{cris}}(Z) \oplus H^4_{\mathrm{cris}}(Z') \oplus H^2_{\mathrm{cris}}(Z) \otimes H^2_{\mathrm{cris}}(Z') \oplus \cdots.$$

The self product of polarizations on Z and Z' gives the nontrivial elements in the second cohomology groups which are all Frobenius eigen-elements. If $\dim Z \geq 2$, then $H^4_{\mathrm{cris}}(Z)^{F-p^2}$ contains the self product of the polarization. Thereby $H^4_{\mathrm{cris}}(Z \times Z', \mathcal{O})^{F-p^2}$ contains at least two linearly independent idempotents.

The Leray spectral sequence $H^4_{\mathrm{cris}}(Z \times Z', \mathcal{O}) \longrightarrow H^0((C/W)_{\mathrm{cris}}, R^4\pi_*(\mathcal{O}_{Z\times Z'}))$ induces diagram

$$\begin{tikzcd}
H^4_{\mathrm{cris}}(Z \times Z', \mathcal{O})^{F^f} \arrow[r, "g^*"] \arrow[d, "\mathrm{pr}"] & H^4_{\mathrm{cris}}(X, \mathcal{O})^{F^f} \arrow[d] \\
H^0((C/W)_{\mathrm{cris}}, R^4\pi_*(\mathcal{O}_{Z\times Z'}))^{F^f} \arrow[r] & H^0((C/W)_{\mathrm{cris}}, R^4\pi_*(\mathcal{O}_X))^{F^f}.
\end{tikzcd}$$

The idempotents in $H^4_{\mathrm{cris}}(Z \times Z', \mathcal{O})$ gives linearly independent elements in

$$\Gamma((C/W)_{\mathrm{cris}}, R^4\pi_*(\mathcal{O}_X))^{F^f}.$$

Then $\dim_{\mathbb{Q}_p} \Gamma((C/W)_{\mathrm{cris}}, R^4\pi_*(\mathcal{O}_X))^{F^f} \geq 2$ for any f, contradicting to condition (3) in 1.0.3. $\square$

7.1.3 G_E reductive

We will show that G_E is reductive. The idea is to show E is a faithful irreducible representation of G_E. Firstly we show the following general fact.

Theorem 7.1.4. *The abelian scheme X/C is simple if and only if $\mathbb{D}(X/C)$ is an irreducible F-isocrystal over C.*

7.1.5 The proof of 7.1.4

If $\mathcal{E}$ is not irreducible, then there exists a F-isocrystal $\mathcal{G}$ such that $\mathcal{E} \longrightarrow \mathcal{G}$ is surjective. The slopes of $\mathcal{G}$ are between 0 and 1. Hence $\mathcal{G}$ has a model of F-crystal over $C/W(k)$ (see Appendix A.2), which we still denote as $\mathcal{G}$. It is easy to show the Verschiebung V also descends to $W(k)$ and hence $\mathcal{G}$ is a Dieudonne crystal.

However, the morphism between Dieudonne crystals

$$\gamma : \mathcal{E} \longrightarrow \mathcal{G}$$

may not be surjective. We only know that $\operatorname{im}\gamma \supset p^k\mathcal{G}$ for some integer k.

By [9, Main Theorem 1], γ corresponds to a morphism between BT groups:

$$\rho : B \longrightarrow X[p^\infty].$$

Since $p^k\mathcal{G} \subset \operatorname{im}\gamma$, the kernel of ρ is a subgroup scheme of $B[p^k]$. In particular, $\ker\rho$ is finite. Let $\eta \in C$ be the generic point. Though $\operatorname{im}\rho$ is merely an fppf abelian sheaf, its generic fibre $\operatorname{im}\rho_\eta \subset X[p^\infty]_\eta$ is a BT group. Further, due to the following lemma, we can further assume ρ_η is injective.

Lemma 7.1.6. *The morphism ρ factors through a BT group B' such that $B' \longrightarrow X[p^\infty]$ is generically injective.*

Proof. Let $K = (\ker\rho_\eta)^-$ be the closure of $\ker\rho_\eta$ in B. Then K is a finite flat group scheme over C. Further, $\rho(K) = 0$ since $\rho(K)_\eta = 0$ and K is flat. Therefore $K \subset \ker\rho$. Let $B' = B/K$. Then B' is a BT group and ρ factors through B'. $\square$

Since ρ_η is injective, $\rho_\eta(B_\eta[p^n])^- = \rho_\eta(B_\eta)[p^n]^-$. We denote it as K_n. Then K_n is a finite flat group scheme over C. Therefore $X[p^\infty]/K_n$ is a BT group and applying the crystalline Dieudonne functor gives that $\mathcal{T}_n := \mathbb{D}(X/K_n)$ is a finite locally free subsheaf of $\mathcal{E}$. And they form a filtration $\mathcal{T}_0 = \mathcal{E} \supset \cdots \mathcal{T}_{n-1} \supset \mathcal{T}_n \supset \mathcal{T}_{n+1} \cdots$ with subquotient $\mathcal{T}_{n-1}/\mathcal{T}_n = \mathbb{D}(K_n/K_{n-1})$. Another filtration on $\mathcal{E}$ is that $\{\mathcal{I}_n = \gamma^{-1}(p^n\mathcal{G})\}$.

Lemma 7.1.7. $\mathcal{T}_n \subset \mathcal{I}_n$.

Proof. On the side of BT groups, we have the following diagram:

$$\begin{tikzcd}
B[p^n] \arrow[r, hook] \arrow[d, hook] & K_n \arrow[d, hook] \\
B \arrow[r, "\rho"] \arrow[d, two heads] & X[p^\infty] \arrow[d, two heads] \\
B \arrow[r] & X[p^\infty]/K_n.
\end{tikzcd}$$

Dually, on the side of Dieudonne crystals,

$$\begin{tikzcd}
\mathbb{D}(B[p^n]) & \mathbb{D}(K_n) \arrow[l, two heads] \\
\mathcal{G} \arrow[u, two heads] & \mathcal{E} \arrow[l, "\gamma"'] \arrow[u, two heads] \\
\mathcal{G} \arrow[u, hook, "p^n"'] & \mathcal{T}_n. \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

The above diagram gives that the image of $\mathcal{T}_n$ in $\mathbb{D}(B)$, composing the upper and right arrows, is contained in $p^n\mathcal{G}$. Therefore $\mathcal{T}_n \subset \mathcal{I}_n$. $\square$

On one hand, we restrict the two diagrams above at the generic point η. Note γ_η is surjective.

$$\begin{tikzcd}
\mathbb{D}(B_\eta[p^n]) & \mathbb{D}(K_{n,\eta}) \arrow[l, "\cong"'] \\
\mathcal{G}_\eta \arrow[u, two heads] & \mathcal{E}_\eta \arrow[l, two heads, "\gamma_\eta"'] \arrow[u, two heads] \\
\mathcal{G}_\eta \arrow[u, hook] & \mathcal{T}_{n,\eta} \arrow[l, two heads] \arrow[u, hook]
\end{tikzcd}$$

Since $B_\eta[p^n] \cong K_{n,\eta}$, we have $\mathcal{E}_\eta/\mathcal{T}_{n,\eta} \cong \mathbb{D}(B_\eta[p^n]) \cong \mathcal{G}_\eta/p^n\mathcal{G}_\eta \cong \mathcal{E}_\eta/\mathcal{I}_{n,\eta}$. We already have $\mathcal{T}_n \subset \mathcal{I}_n$. Hence $\mathcal{T}_{n,\eta} = \mathcal{I}_{n,\eta}$.

On the other hand, since $\mathbb{D}(K_n)$ is p^n-torsion, $p^n\mathcal{E} \subset \mathcal{T}_n$. Therefore $\mathcal{T}_n \otimes \mathcal{O}_{\tilde{C}}[\frac{1}{p}] \cong \mathcal{E} \otimes \mathcal{O}_{\tilde{C}}[\frac{1}{p}]$. In particular, $\mathcal{T}_n \otimes \mathcal{O}_{\tilde{C}}[\frac{1}{p}] \cong \mathcal{I}_n \otimes \mathcal{O}_{\tilde{C}}[\frac{1}{p}]$.

Therefore, $\mathcal{T}_n \subset \mathcal{I}_n$ induces isomorphisms over generical fibre of $\tilde{C}$ and generic point of C. In particular, it is isomorphic on every height 1 points in $\tilde{C}$.

Lemma 7.1.8. *Let $(D, \mathfrak{m})$ be a regular local domain of dimension 2 and*

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of D-modules with M finite free, N torsion free and $\operatorname{supp}(Q) \subset \{\mathfrak{m}\}$, then $Q = 0$.

Proof. Since D is regular of dimension 2, we have the reflexive module $N^{\vee\vee}$ of N is free([15, Chapter 2, Proposition 25]). The new short exact sequence $0 \longrightarrow M \xrightarrow{T} N^{\vee\vee} \longrightarrow Q' \longrightarrow 0$ still satisfies that $\operatorname{supp}(Q') \subset \{\mathfrak{m}\}$. Since N is torsion free, still by [15, Chapter 2, Corollary 21], we have $N \subset N^{\vee\vee}$. Thereby $\operatorname{supp}(Q) \subset \operatorname{supp}(Q')$. Note both of M and $N^{\vee\vee}$ are free of the same rank and hence T can be represented as a square matrix with entries in D. So the support of Q' is the zero set of $\det T$. But if $\det T$ is a nonunit, then the dimension of the zero set $Z(\det T)$ is of dimension 1. Hence $Q = 0$. ☐

Corollary 7.1.9. *$\mathcal{T}_n = \mathcal{I}_n$.*

Proof. Localize the injection $\mathcal{T}_n \hookrightarrow \mathcal{I}_n$ at each closed point of $\tilde{C}$ and apply 7.1.8. ☐

Let J_n be K_n/K_{n-1}.

Proposition 7.1.10. *$J_n \cong J_{n+1}$ if n large enough.*

Proof. By 7.1.9, we have $\gamma(\mathcal{T}_n) = p^n\mathbb{D}(B) \cap \gamma(\mathcal{E})$. By Artin-Rees lemma, there exists an integer k such that $\gamma(\mathcal{T}_n) = p^{n-k}\gamma(\mathcal{T}_k)$ for any integer $n > k$. Since via 7.1.9, $\ker\gamma \subset \mathcal{T}_n$ for each n, $\mathcal{T}_{n-1}/\mathcal{T}_n \cong \gamma(\mathcal{T}_{n-1})/\gamma(\mathcal{T}_n) \cong \gamma(\mathcal{T}_k) \otimes \mathcal{O}_{\tilde{C}}[\frac{1}{p}] \cong \gamma(\mathcal{T}_{n-2})/\gamma(\mathcal{T}_{n-1}) \cong \mathcal{T}_{n-2}/\mathcal{T}_{n-1}$. So $J_n \cong J_{n+1}$ for $n > k$. ☐

Fix the integer k in the proof of 7.1.10. Let $\rho_\eta(B_\eta)^-$ denote the union $\cup_n K_n = \cup_n \rho_\eta(B_\eta[p^n])$ and J denote $\rho_\eta(B_\eta)^-/K_k$. Then we have

Proposition 7.1.11. *J is a BT group over C.*

Proof. Obviously $\rho_\eta(B_\eta)^-/K_k$ is p-torsion. And we have $\rho_\eta(B_\eta)^-/K_k[p] = J_{k+1}$ is a finite locally free group scheme. It remains to show that J is p-divisible. From Proposition 7.1.10, $J[p^2] \xrightarrow{p} J[p]$ is surjective. Now we proceed by induction. Suppose $J[p^n] \xrightarrow{p} J[p^{n-1}]$ is surjective. The following diagram is always commutative:

$$\begin{tikzcd} J[p^n] \arrow[r, two heads, "p"] \arrow[d, hook] & J[p^{n-1}] \arrow[d, hook] \\ J[p^{n+1}] \arrow[r, "p"] & J[p^n]. \end{tikzcd}$$

Again by 7.1.10, the induced morphism on cokernels $J[p^{n+1}]/J[p^n] \longrightarrow J[p^n]/J[p^{n-1}]$ is isomorphic. Hence $J[p^{n+1}] \xrightarrow{p} J[p^n]$ is surjective. Therefore J is p-divisible. □

Note J is a subquotient BT group of $X[p^\infty]$. The following standard trick follows from the proof of Theorem 2.6 in [10].

Let $Z_k = X/K_k$. Then Z_k is an abelian scheme over C and J is a sub BT group of $Z_k[p^\infty]$. Put $Z'_n = Z_k/J[p^n]$. We have an exact sequence of truncated BT group schemes of level 1 over C as follows

$$0 \longrightarrow J[p] \longrightarrow Z_k[p] \longrightarrow Z'_n[p] \longrightarrow J[p] \longrightarrow 0.$$

And hence an exact sequence

$$0 \longrightarrow \omega_J \longrightarrow \omega_{Z'_n} \longrightarrow \omega_{Z_k} \longrightarrow \omega_J \longrightarrow 0.$$

We conclude that $\det(\omega_{Z'_n}) \cong \det(\omega_{Z_k})$ independent of n. It is known ([48]) that this implies there are only a finite number of isomorphism classes of abelian schemes among Z'_n. So we can find an abelian scheme Z'_l such that there exists infinitely many $f_n \in \mathrm{Hom}(Z_k, Z'_l)$ such that $\ker f_n = J[p^n]$. Fix a f_{n_0}. And let $g_n \in \mathrm{End}(Z_k)$ be $(f_{n_0})^{-1} \circ f_n$. Then $\ker g_n[p^\infty]$ is an extension of a fixed finite group scheme and $J[p^n]$. Let g be the limit of g_n in $\mathrm{End}(Z_k)$ and hence $\ker g[p^\infty]$ is an extension of a finite group scheme and J. Therefore $\mathrm{im}\, g \subset Z_k$ is a proper subvariety. Since Z_k and X are isogenous, g induces a morphism in $\mathrm{End}(X/C)$ which is not surjective. It contradicts to the assumption X/C is simple.

This is the end of the proof of 7.1.4.

Now let us consider the specific crystal $\mathcal{E}$ in 1.0.3. Recall that G_{univ} is the group scheme corresponding to $F^f - \mathrm{Isocris}(C)$. Let E be the representation corresponding to $\mathcal{E}$ and $G_E = \mathrm{im}\,(G_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(E))$.

Lemma 7.1.12. *Notation as above, we have* $\dim_{\mathbb{Q}_{p^f}}(\wedge^4 E)^{G_E} = 1$ *and* $\mathbb{Q}_{p^f} \otimes \mathrm{End}(\wedge^2 E)^{G_E} \cong \mathbb{Q}_{p^f}^{\times 4}$ *as algebras.*

Proof. Condition (2) is equivalent to $\dim_{\mathbb{Q}_{p^f}} \Gamma((C/W)_{\mathrm{cris}}, \wedge^4\mathcal{E}(2f))^{F^f} \otimes \mathbb{Q}_{p^f} = 1$. The space

$$\Gamma((C/W)_{\mathrm{cris}}, (\wedge^4\mathcal{E})(2f))^{F^f}$$

consists of invariant elements in $(\wedge^4\mathcal{E})(2f)$. The representation associated to $\wedge^4\mathcal{E}(2f)$ is $E \otimes \chi^{2f}$ where χ is the character of P_{univ} corresponding to Tate twist. Then $\dim_{\mathbb{Q}_{p^f}}(E \otimes \chi^{2f})^{P_{\mathrm{univ}}} = 1$. Note Tate twist only affects the weight, $\dim_{\mathbb{Q}_{p^f}}(E)^G = \dim_{\mathbb{Q}_{p^f}}(E)^{P_{\mathrm{univ}}} = \dim_{\mathbb{Q}_{p^f}}(E \otimes \chi)^{P_{\mathrm{univ}}} = 1$.

The isomorphism $\mathbb{Q}_{p^f} \otimes \mathrm{End}(\wedge^2 E)^G \cong \mathbb{Q}_{p^f}^{\times 4}$ follows directly. □

Proposition 7.1.13. *The representation E is irreducible.*

Proof. If E is not an irreducible G_E-representation, then E has a proper sub-representation. Let V be an irreducible sub-representation of E with the smallest dimension. Then V gives a proper sub object $\mathcal{V} \subset \mathcal{E}$ with minimal rank. Since $F : \mathcal{E}^\sigma \longrightarrow \mathcal{E}$ is an isomorphism between isocrystals, $F(\mathcal{V}^\sigma)$ is also irreducible of the smallest rank and hence $F(\mathcal{V}^\sigma) \cap \mathcal{V} = 0$. Consider $\sum_n F^n(\mathcal{V}^{\sigma^n})$. It is a proper sub-isocrystal of $\mathcal{E}$ and invariant under F. By 7.1.4, $\sum_n F^n(\mathcal{V}^{\sigma^n}) = \mathcal{E}$. As a quotient of the sum of irreducible elements,

$$\mathcal{E} \cong \oplus_{i \in I} \mathcal{V}^{\sigma^i}$$

for some index set I.

Therefore $E \cong \oplus_i V_i$. Since each V_i has the same rank, the number of direct summands is either 1, 2, 4 or 8. This number is not greater than 2 otherwise it violates $\mathrm{End}(\wedge^2 E)^{G_E} \cong \mathbb{Q}_{p^f}^{\times 4}$. If there are two direct summands, let

$$E \cong V_1 \oplus V_2.$$

Note E admits a symplectic form λ and $(\wedge^4 E)^{G_E}$ is generated by the self-product of λ.

If λ preserves the direct summands V_1 and V_2, then in the decomposition

$$\wedge^4 E \cong \wedge^4 V_1 \oplus \wedge^4 V_2 \oplus \cdots,$$

the self product λ^2 has nontrivial components in $\wedge^4 V_1$ and $\wedge^4 \mathcal{V}_2$. Let λ_1^2 and λ_2^2 be the two components. Then both of them are invariant under G_E, contradicting to $\dim_{\mathbb{Q}_{p^f}}(\wedge^4 E)^{G_E} = 1$.

If the polarization does not preserve the direct summands, then λ induces isomorphisms $V_1 \longrightarrow {V_2}^\vee$ and $V_2 \longrightarrow V_1^\vee$. Note $\wedge^2 E \cong \wedge^2 V_1 \oplus \wedge^2 V_2 \oplus V_1 \otimes V_2$ and there is a surjection

$$S^2(\wedge^2 E) \longrightarrow \wedge^4 E, w_1.w_2 \mapsto w_1 \wedge w_2.$$

And then λ^2 lies in

$$S^2(V_1 \otimes V_2) \cong S^2V_1 \otimes S^2V_2 \oplus \wedge^2V_1 \otimes \wedge^2V_2.$$

Let the λ_1^2 and λ_2^2 be the image λ in the two components in $S^2(V_1 \otimes V_2)$. Then they are invariant under G_E, again contradicting to $\dim_{\mathbb{Q}_{p^f}}(\wedge^4E)^{G_E} = 1$. □

Corollary 7.1.14. *The group G_E is reductive.*

Proof. Note G_E is reductive if G_E admits a faithful and completely reducible representation. □

7.1.15 G_E nonsimple

Since X/C is polarized, the F-isocrystal $\mathcal{E}$ admits a non-degenerate alternating form([2, Section 5.1]). So E also admits an alternating form which is preserved by G_E. Therefore the action of G_E factors through $Sp(8, \mathbb{Q}_{p^f})$. Then the reductive Lie algebra $\mathfrak{g}_E$ factors through $\mathfrak{sp}(8)$.

Proposition 7.1.16. *G_E is not simple and the semisimple part $(\mathfrak{g}_E)^{ss}_{\mathbb{C}} \cong \mathfrak{sl}(2)^{\times 3}$.*

Proof. If G_E is simple, then the Lie algebra $\mathfrak{g}_E$ is a simple subalgebra of $\mathfrak{sp}(8, \mathbb{Q}_{p^f})$. Then condition 7.1.12 can be stated in terms of Lie algebra:

$$\dim_{\mathbb{Q}_{p^f}}(\wedge^4E)^{\mathfrak{g}_E} = 1, \mathrm{End}(\wedge^2E)^{\mathfrak{g}_E} \cong \mathbb{Q}_{p^f}^{\times 4}.$$

Base change to $\mathbb{C}$. By Appendix A.1, there is no simple complex Lie algebras satisfying the conditions above. Therefore $G_{E\mathbb{C}}$ is not simple. Then the semisimple part $\mathfrak{g}^{ss}_{\mathbb{C}} = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ and $E_{\mathbb{C}} = E_1 \otimes E_2 \otimes E_3$ where E_i is the standard representation of $\mathfrak{sl}(2)$. Therefore

$$\begin{aligned}\wedge^2 E_{\mathbb{C}} \cong (S^2E_1 \otimes S^2E_2 \otimes \wedge^2E_3) \oplus (\wedge^2E_1 \otimes S^2E_2 \otimes S^2E_3)\oplus \\ (S^2E_1 \otimes \wedge^2E_2 \otimes S^2E_3) \oplus (\wedge^2E_1 \otimes \wedge^2E_2 \otimes \wedge^2E_3) \quad (7.1)\end{aligned}$$

as direct sum of irreducible representations. Therefore $\mathrm{End}_{G_E}(\wedge^2E)_{\mathbb{C}} \cong \mathbb{C}^{\oplus 4}$ and $\wedge^2E_1 \otimes \wedge^2E_2 \otimes \wedge^2E_3$ is the polarization.

For $\mathfrak{g}_E$, $\mathrm{End}_{\mathfrak{g}}(\wedge^2E) \cong \mathbb{Q}_{p^f}^{\times 4}$. Therefore each of the 4 components of $\wedge^2E_{\mathbb{C}}$ is defined over $\mathbb{Q}_{p^f}$. Note they are all nonfaithful nontrivial representations of $\mathfrak{g}$. Thereby $\mathfrak{g}$ and then G_E is not simple. □

So the Lie algebra $\mathfrak{g}^{ss}_E = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$. Then $E = E_1 \otimes \cdots \otimes E_n$, where $n \geq 2$, the $\mathfrak{g}_i$ are simple Lie algebras and E_i is a faithful representation of $\mathfrak{g}_i$. Clearly at least one of the E_i say E_1, has dimension 2, and this implies that $\mathfrak{g}_1 = \mathfrak{sl}(2)$. Then $E_2 \otimes \cdots \otimes E_n$ is a 4-dimensional representation of $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_n$. Since

E_1 is already symplectic, $E_2 \otimes \cdots \otimes E_n$ must be orthogonal. For simplicity, let $\mathfrak{h} = \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_n$ which acts orthogonally on $E_2 \otimes \cdots \otimes E_n$. Let us denote it as W.

Corollary 7.1.17.

$$\wedge^2 E \cong \mathbb{Q}_{p^f} \oplus W_1 \otimes W_2 \oplus W_1 \otimes W_3 \oplus W_2 \otimes W_3 \tag{7.2}$$

over $\mathbb{Q}_{p^f}$ where each W_i is a dimension 3 representation of G_E. Further W_i admits a symmetric product.

Proof. Since $E = E_1 \otimes_{\mathbb{Q}_{p^f}} W$, $\wedge^2 E \cong S^2 W \oplus S^2 E_1 \otimes \wedge^2 W$. Let W_1 be $S^2 E_1$. Since the four idemponents in $\mathrm{End}_{G_E}(\wedge^2 E)_{\mathbb{C}}$ are defined over $\mathbb{Q}_{p^f}$, comparing with the decomposition over $\mathbb{C}$, $\wedge^2 W$ is the direct sum of two rank 3 representations, say W_2, W_3 and then

$$\wedge^2 E = S^2 W \oplus W_1 \otimes W_2 \oplus W_2 \otimes W_3.$$

As a subrepresentation of $\wedge^2 E$, $W_1 \otimes W_2$ admits a symmetric product. Therefore $S^2(W_1 \otimes W_2)$ has a one-dimensional trivial direct summand. Note

$$\begin{aligned} S^2(W_1 \otimes W_2) &= S^2 W_1 \otimes S^2 W_2 \oplus \wedge^2 W_1 \otimes \wedge^2 W_2 \\ &= (\mathbb{Q}_{p^f} \oplus W_1') \otimes S^2 W_2 \oplus \wedge^2 W_1 \otimes \wedge^2 W_2 \end{aligned}$$

where W_1' is irreducible. The 1-dimensional direct summand can only come from $\mathbb{Q}_{p^f} \otimes S^2 W_2$. Therefore $S^2 W_2$ has a one-dimensional trivial direct summand and hence W_2, as well as W_3, also admits a symmetric product. $\square$

However, we don't know whether E_i are defined over $\mathbb{Q}_{p^f}$ yet. So we can not write W_i as $S^2 V_i$ for some rank 2 representation V_i. To remedy this situation, we take an étale covering of C and increase the power f if necessary.

7.2 The choice of the étale covering

In this section, we show how to choose the étale covering of C for the existence of V_i.

We consider a more general setting: if we have a rank 3 F^f-isocrystal $\mathcal{W}$ with symmetric product, when can we write $\mathcal{W}$ as $S^2\mathcal{V}$ for some rank 2 F^f-isocrystal $\mathcal{V}$ which corresponds to a $SL(2)$-representation? If not in general, we compute the obstruction.

Firstly as crystals, $\mathcal{W}$ corresponds to a bundle with an integrable connection over $\tilde{C}$.

Proposition 7.2.1. *The obstruction o_1 to the existence of a rank 2 bundle $\mathcal{V}$ over $\tilde{C}$ such that $\mathcal{W} = S^2\mathcal{V}[\frac{1}{p}]$ is in $H^2_{\mathrm{et}}(\tilde{C}, \mu_2)$.*

Proof. Note the symmetric form $(,)$ gives an element in $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(2)$. So it defines a conic bundle over $\tilde{C}$. If this conic bundle is isomorphic to $\mathbb{P}(\mathcal{V})$ for some bundle $\mathcal{V}$ on $\tilde{C}$, then $\mathcal{W} \cong S^2\mathcal{V}$ and the symmetric product $(,)$ on $\mathcal{W}$ is a scalar multiple of $S^2 <,>$ where $<,>$ is an alternating form on $\mathcal{V}$. So the obstruction is just the special Brauer class in $H^2_{\mathrm{et}}(\tilde{C}, \mu_2)$. $\square$

From the proof, if $\mathcal{W} = S^2\mathcal{V}$, we have $(,) = aS^2 <,>$ for some $a \in \mathbb{Q}^*_{p^f}$. In particular, $(x^2, x^2) = 0$ for any local section x of $\mathcal{V}$.

Remark 7.2.2. For the dimension 3 $PGL(2)$-representation W and standard representation V of $SL(2)$, $W = S^2V$ and there exists a 2-uple embedding $\mathbb{P}(V) \cong \mathbb{P}^1 \hookrightarrow \mathbb{P}(W)$.

Proposition 7.2.3. *If $o_1 = 0$, then there exists a rank 2 bundle with connection $(\mathcal{V}, \nabla_{\mathcal{V}})$ on $\tilde{C}$ such that $S^2\mathcal{V} \cong \mathcal{W}$ as modules with connection.*

Proof. If $o_1 = 0$, the let $\mathcal{V}$ be the rank 2 bundle on $\tilde{C}$ such that $S^2\mathcal{V} \cong \mathcal{W}$ over $\tilde{C}[\frac{1}{p}]$. For any affine open subset $U \subset \tilde{C}$ such that $\mathcal{V}(U)$ and $\Omega^1_{\tilde{C}}(U)$ are free, suppose $T_{\tilde{C}}(U)$ is generated by τ. Let $N = \mathcal{V}(U)$ and $M = \mathcal{W}(U)$. For any section $x \in N$, $x^2 \in M$. Since $\nabla_{\mathcal{W}}$ is compatible with the symmetric product and $(x^2, x^2) = 0$, $((\nabla_{\mathcal{W}})_\tau(x^2), x^2) = 0$. Then $(\nabla_{\mathcal{W}})_\tau(x^2) = x.v$ for some local section v of $N[\frac{1}{p}]$. Define a map $\nabla_{\mathcal{V}}$ locally as $(\nabla_{\mathcal{V}})_\tau(x) = v$. It is easy to check $\nabla_{\mathcal{V}}$ is a well-defined connection and it can be defined globally over $\tilde{C}[\frac{1}{p}]$. Further it is easy to show $S^2\nabla_{\mathcal{V}} = \nabla_{\mathcal{W}}$.

From $S^2N[\frac{1}{p}] \cong M[\frac{1}{p}]$, there is an injective morphism $S^2N \longrightarrow M$. Now locally over U, let $N' = \sum_n (\nabla_{\mathcal{V}})^{(n)}_\tau(N)$ and $(S^2N)' = \sum_n (\nabla_{\mathcal{V}})^{(n)}_\tau(S^2N)$. Then $N \subset N' \subset N[\frac{1}{p}]$. Since $(S^2N)' \subset (S^2N)[\frac{1}{p}] \cap M$, $(S^2N)'$ is noetherian and hence there exists k_0 such that $(S^2N)' \subset \frac{1}{p^{k_0}}S^2N$.

Next we prove N' is also finitely generated. Choose generators $\{x, y\}$ of N. For simplicity, we use ∇ to denote $(\nabla_{\mathcal{V}})_\tau$. Then

$$\begin{pmatrix} \nabla x \\ \nabla y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a 2×2 matrix with entries in $\mathcal{O}_U[\frac{1}{p}]$. Then roughly

$$\begin{pmatrix} \nabla^n x \\ \nabla^n y \end{pmatrix} = (\nabla^n A + \cdots + A^n) \begin{pmatrix} x \\ y \end{pmatrix}.$$

For any $z \in N'$(resp. $(S^2N)'$), let the order $\mathrm{ord}(z)$ of z be the minimal integer $-k$ such that $z \in p^{-k}N$ (resp. $p^{-k}S^2N$). Similarly, the order $\mathrm{ord}(A)$ of a matrix A is the minimal integer among the orders of its entries. Let $A = p^{\mathrm{ord}(A)} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We explore some identities of the order. Obviously $\mathrm{ord}(zz') = \mathrm{ord}(z) + \mathrm{ord}(z')$, $\mathrm{ord}(AB) \leq \mathrm{ord}(A) + \mathrm{ord}(B)$. Since ∇ commutes with p, $\mathrm{ord}(\nabla^n A) = \mathrm{ord}(A)$ for any n and $\mathrm{ord}(A\nabla A) = \mathrm{ord}(A^2)$.

If the sequence $\{\mathrm{ord}(A^n)\}$ is bounded below, then by the identities above, $\mathrm{ord}(\binom{\nabla^n x}{\nabla^n y})$ is also bounded below. Thus N' is finitely generated.

Now we assume $\{\mathrm{ord}(A^n)\}$ is not bounded below. By direct computation, it is easy to show the sequence $\{\mathrm{ord}(A^n)\}$ is strictly decreasing. Without loss of generality, we can assume $\mathrm{ord}(\nabla x) = \mathrm{ord}(A)$. Then $\mathrm{ord}((1,0)A^n) = \mathrm{ord}(A^n)$. Since

$$\nabla^n(x) = (1,0)(\nabla^n A + nA\nabla^n A + \cdots + A^n)\begin{pmatrix} x \\ y \end{pmatrix},$$

$\mathrm{ord}(\nabla^n x) = \mathrm{ord}(A^n)$. Consider $\nabla^n(x^2) = 2\sum_{k=0}^{n} \nabla^k(x)\nabla^{n-k}(x)$. The order of the product $\nabla^k(x)\nabla^{n-k}(x)$ is given by $(1,0)A^k\binom{x}{y}.(1,0)A^{n-k}\binom{x}{y}$.

Let $A^k = p^{\mathrm{ord}(A^k)} \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$. Then

$$\nabla^n(x^2) = [(\sum a_k a_{n-k})x^2 + 2(\sum a_k b_{n-k})xy + (\sum b_k b_{n-k})y^2]p^{\mathrm{ord}(A^n)} + \text{lower order terms} .$$

Note $\nabla^n(x^2) \in (S^2N)' \subset p^{k_0}S^2N$. For n large, the coefficient of $p^{\mathrm{ord}(A^n)}$ has to be zero. Since x^2, xy, y^2 are basis of S^2N, we have

$$\sum a_k a_{n-k} = \sum a_k b_{n-k} = \sum b_k b_{n-k} = 0.$$

Repeat the analysis for $\nabla^n(xy)$ and $\nabla^n(y^2)$. We have for n large, the following terms

$$\sum a_k c_{n-k}, \sum b_k d_{n-k}, \sum c_k c_{n-k}, \sum d_k d_{n-k}, \sum c_k d_{n-k}, \sum(a_k d_{n-k} + b_k c_{n-k})$$

are all zero. Note $A^n = p^{\mathrm{ord}(A^n)}\begin{pmatrix}\sum a_k a_{n-k} + b_k c_{n-k} & \sum a_k b_{n-k} + b_k d_{n-k} \\ \sum c_k a_{n-k} + d_k c_{n-k} & \sum c_k b_{n-k} + d_k d_{n-k}\end{pmatrix}$ for any $k \leq n$. Thus

$$\begin{aligned} nA^n = p^{\mathrm{ord}(A^n)} &\begin{pmatrix}\sum_k a_k a_{n-k} + b_k c_{n-k} & \sum_k a_k b_{n-k} + b_k d_{n-k} \\ \sum_k c_k a_{n-k} + d_k c_{n-k} & \sum_k c_k b_{n-k} + d_k d_{n-k}\end{pmatrix} \\ &= p^{\mathrm{ord}(A^n)}\begin{pmatrix}\sum_k b_k c_{n-k} & 0 \\ 0 & \sum_k c_k b_{n-k}\end{pmatrix}. \end{aligned} \tag{7.3}$$

In particular, for n large, A^n is always diagonal which implies A has to be diagonal, i.e. $b_k = c_k = 0$. Then A^n is further the zero matrix, contradiction to the assumption $\{\mathrm{ord}(A^n)\}$ not bounded below.

Hence N' is finitely generated.

Since $\tilde{C}$ is a regular dimension 2 scheme, similar to the proof of 7.1.8, the double dual $N'^{\vee\vee}$ is locally free which also admits a connection.

Since all the arguments above are canonical, the existence holds globally. □

Therefore there is no obstruction to the connection. Now since the étale cohomology group $H^2_{\mathrm{et}}(\tilde{C}, \mu_2)$ is killed by any 2:1 étale covering of C. So over any 2:1 étale covering C' of C, there exists a rank 2 crystal $\mathcal{V}$ such that $S^2\mathcal{V} \cong \mathcal{W}_{C'}$. Next we consider the Frobenius.

Proposition 7.2.4. *There exists an étale covering $C' \longrightarrow C$ such that over C', there is a rank 2 F^f-isocrystal $\mathcal{V}$ with $S^2\mathcal{V} = \mathcal{W}_{C'}$.*

Proof. By 7.2.1, 7.2.3, we can choose any 2:1 covering killing the obstruction to the existence of a crystal. So we assume there exists crystal $\mathcal{V}$ such that $S^2\mathcal{V} = \mathcal{W}$ over $\tilde{C}$.

Notations as in the proof of 7.2.3, we choose an open affine subset $U = \mathrm{Spec}\ A \subset \tilde{C}$ and $N = \mathcal{V}(U), M = \mathcal{W}(U)$. So $S^2N = M$. Let $\tilde{\sigma}$ be the lifting of σ to U. Note $\mathcal{W}$ is an F^f-isocrystal. Without loss of generality, we can assume $f = 1$ and then $F_M : M^{\tilde{\sigma}} \longrightarrow M$ and it is compatible with the symmetric forms $(,)$ and $(,)^{\tilde{\sigma}}$.

Shrinking U if necessary, assume further N is free and generated by (x, y). Then correspondingly, M is generated by (x^2, xy, y^2). Since $(x^2, x^2) = 0$, $(F_M(x^2), F_M(x^2)) = 0$. Thereby $F_M(x^2)$ is contained in the conic bundle. Because the obstruction o_1 is trivial, $F_M(x^2)$ comes from a square of some element in N. Thus $F_M(x^2)$ is also a square up to a scalar: $F_M(x^2) = \lambda x'^2$. Similarly, $F_M(y^2) = \mu y'^2$. Here $x'.y'$ are elements in N and $\lambda, \mu \in A[\frac{1}{p}]^*$. Since $(xy, x^2) = (xy, y^2) = 0$, $F_M(xy) = \nu x'y'$.

Note $x^2 + sxy + y^2$ is also a square in M and thus

$$F_M(x^2 + sxy + y^2) = \lambda x'^2 + 2\nu x'y' + \mu y'^2$$

is also a square up to scalar. We can rewrite it as

$$F_M(x^2) = \omega x''^2$$
$$F_M(xy) = \omega x'' y''$$
$$F_M(y^2) = \omega y''^2$$

for some $\omega \in A[\frac{1}{p}]^*$. Therefore we choose F_N such that $F_N(x) = x''$ and $F_N(y) = y''$ and

$$F_M = \omega S^2 F_N.$$

We also need the F_N to be horizontal. Note

$$\nabla_{\mathcal{W}}(F_M(x^2)) = \nabla_{\mathcal{W}}(\omega x^2) = d\omega \otimes x'^2 + \omega(2x')\nabla_{\mathcal{V}} x'$$

and

$$\nabla_{\mathcal{W}}(F_M(x^2)) = F_M(\nabla_{\mathcal{V}}(x^2)) = F_M(2x\nabla_{\mathcal{V}} x) = 2\omega F_N(x) F_N(\nabla_{\mathcal{V}} x).$$

Similarly, we can compute for $F_M(y^2)$. So F_N commutes with ∇ if $d\omega = 0$.

Write $\omega = p^v.u$ where $u \in A^*$. If v is not even, then consider F_M^2 instead of F_M in which case ω is replaced by $\tilde{\sigma}(\omega)\omega = p^{2v}u'$. So we can assume v is even.

For the unit u, there exists 2:1 étale covering $U' \xrightarrow{f} U$ (note $p > 2$), such that $f^*(u)$ is a square in A'^*. Then $f^*(\omega) = p^v f^*(u)$ is a square ω'^2. Adjust F_N such that $F_N(x) = \omega' x'', F_N(y) = \omega' y''$. Then

$$F_M = S^2 F_N$$

and F_N is compatible with the connection $\nabla_{\mathcal{V}}$ and alternating form $<,>$.

Now we consider the global case. For any affine covering $\tilde{C} = \cup_i U_i$, there exist 2:1 covering $U_i' \longrightarrow U_i$ such that we can find $F_{N,i}$ with $F_M = S^2 F_{N,i}$. Over $U_i' \times_C U_j'$, $S^2 F_{N,i} = S^2 F_{N,j}$ and thus $F_{N,i} = \tau F_{N,j}$ with $\tau^2 = 1$, i.e.

$$\tau \in \mu_2(A[\frac{1}{p}]) = \mu_2(A) = \mu_2(A/p).$$

Therefore the obstruction of the existence of $F_{\mathcal{V}}$ is in $H^1_{\text{et}}(C, \mu_2)$. This étale cohomology group can be killed by some 2:1 étale covering of C. $\square$

Each W_i in (7.2) corresponds to a rank 3 isocrystal $\mathcal{W}_i$ over C with a symmetric form. Hence we can choose a finite étale covering $C' \longrightarrow C$. The base change of $\mathcal{W}_i$ to C' are the second power symmetric product of some rank 2 isocrystals $S^2\mathcal{V}_i$.

Remark 7.2.5. In the rest of the chapter, we consider all the datum base change to C' and use $\mathcal{E}'$, E', G'_E to denote the pullback of $\mathcal{E}, E, G_E$ to C'.

7.3 Proof of 1.0.3

In this section, we will show that as an F^f-isocrystal over the étale covering C' of C, $\mathcal{E}$ has a tensor decomposition $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$.

In the Tannakian formalism, the representation V_i corresponding to $\mathcal{V}_i$ is dimension 2 with an alternating form and $S^2V_i = W_i$. Therefore P'_{univ} acting on V_i factors through $SL(2)$ and then the Tannakian group of $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$ is $\operatorname{im}(P'_{\text{univ}} \longrightarrow SL(2)^{\times 3})$. Now (7.2) transforms to

$$\wedge^2 E' \cong \wedge^2V_1 \otimes \wedge^2V_2 \otimes \wedge^2V_3 \oplus S^2V_1 \otimes S^2V_2 \otimes \wedge^2V_3 \oplus S^2V_1 \otimes \wedge^2V_2 \otimes S^2V_3 \oplus \wedge^2V_1 \otimes S^2V_2 \otimes S^2V_3. \quad (7.4)$$

Thus we have the commutative diagram:

$$\begin{tikzcd}
 & G'_E \arrow[dr] & \\
P'_{\text{univ}} \arrow[ur] \arrow[dr] & & PGL(2)^{\times 3} \\
 & SL(2)^{\times 3} \arrow[ur] &
\end{tikzcd}$$

Remark 7.3.1. For E, we have

$$\begin{tikzcd}
P'_{\text{univ}} \arrow[dd, hook] \arrow[dr] & \\
 & \operatorname{Aut}(E) \\
P_{\text{univ}} \arrow[ur] &
\end{tikzcd}$$

which induces $G'_E \hookrightarrow G_E$. Since $\dim P'_{\text{univ}} = \dim P_{\text{univ}}$, we have $\dim G'_E = \dim G_E$.

Lemma 7.3.2. *The map $G'_E \longrightarrow PGL(2)^{\times 3}$ is surjective.*

Proof. From (7.2), we know that $\ker(G'_E \longrightarrow PGL(2)^{\times 3}) = \ker(G'_E \longrightarrow \operatorname{Aut}(\wedge^2 E'))$. Since E' is a faithful representation of G'_E, the kernel is just $\pm I$.

By 7.3.1, since $\dim \mathfrak{g}_{\mathbb{C}}^{ss} = 9$, $\dim G_E'^{\mathrm{der}} = \dim G_E^{\mathrm{der}} = 9$. Hence $\mathrm{im}\,(G'_E \longrightarrow PGL(2)^{\times 3})$ has dimension 9. Since $PGL(2)$ is simple, it must be surjective. $\square$

Since $SL(2)$ is simply connected as an algebraic group and both of $SL(2)^{\times 3}$ and $G_E'^{\mathrm{der}}$ are finite covering of $PGL(2)^{\times 3}$, there is a natural surjection $SL(2)^{\times 3} \longrightarrow G_E'^{\mathrm{der}}$ with kernel isomorphic to $\mu_2 \times \mu_2$. Then

$$\mathrm{im}\,(SL(2)^{\times 3} \longrightarrow \mathrm{Aut}(V_1 \otimes V_2 \otimes V_3)) \cong G'_E.$$

Since the action of $G_E'^{der}$ on E' is absolutely irreducible, the center $Z(G'_E)$ acts as scalars on E'. So the action of G'_E on E' is the same as $SL(2)^{\times 3}$ up to a character on G'_E. Let χ be the character.

Now we have two morphisms from P'_{univ} to G'_E, one induced by $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$ and the other induced by $\mathcal{E}'$. Denote them by f_1, f_2 respectively:

$$P'_{\mathrm{univ}} \underset{f_2}{\overset{f_1}{\rightrightarrows}} G'_E.$$

For any $g \in P'_{\mathrm{univ}}$, $f_1(g) = \chi(g) f_2(g)$, i.e. the image of

$$P'_{\mathrm{univ}} \xrightarrow{f_1 \times f_2} G'_E \times G'_E$$

is contained in $\Delta \cup \Delta'$ where $\Delta' = \{(g, \chi(g)g) | g \in G'_E\}$. The image is isomorphic to a subgroup of $G'_E \times \mathbb{G}_m$.

Proposition 7.3.3. *There exists an F^f- isocrystal $\mathcal{L}$ such that*

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cong \mathcal{E}' \otimes \mathcal{L}$$

as F^f-isocrystals.

Proof. Consider the sub Tannakian category generated by $\{\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3, \mathcal{E}'\}$. The group corresponds to this sub category is given by

$$\mathrm{im}\,(P'_{\mathrm{univ}} \longrightarrow \mathrm{Aut}(V_1 \otimes V_2 \otimes V_3) \times G'_E) \hookrightarrow G'_E \times \mathbb{G}_m.$$

Hence we have

$$\begin{tikzcd} & & G'_E \\ P'_{\mathrm{univ}} \arrow[r] & G'_E \times \mathbb{G}_m \arrow[ur, "\mathrm{pr}"] \arrow[dr, "m"'] & \\ & & G'_E \end{tikzcd}$$

where pr is the first factor projection while m is the multiplication. Through m, $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cong \mathcal{E}' \otimes \mathcal{L}$ where $\mathcal{L}$ is a rank 1 F^f-isocrystal. $\square$

Replace $\mathcal{V}_3$ by $\mathcal{V}_3 \otimes \mathcal{L}^{-1}$ and we still denote it as $\mathcal{V}_3$. Summarize the results and we have the following theorem.

Theorem 7.3.4. *There exists a finite étale covering C' of C such that after base change to* $\mathrm{cris}(C'/W(k))$*, we have an isomorphism of F^f-isocrystals:*

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cong \mathcal{E}'$$

where rank $\mathcal{V}_i = 2$.

Theorem 7.3.4 follows from 1.0.2. It finishes the proof of 1.0.3.

7.4 Frobenius eigenvalues on $\wedge^4\mathcal{E}$ or $\mathcal{E}nd(\wedge^2\mathcal{E})$

We prove 1.0.4 in Section 7.4, 7.5 and 7.6. In this section, we compute the dimension of Frobenius eigenspace at the target spaces and the corresponding eigenvalues.

Recall the short exact sequence

$$1 \longrightarrow \pi_1^{\mathrm{geom}}(C, \bar{\xi}) \longrightarrow \pi_1(C, \bar{\xi}) \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$

The Zariski closure G_l^{geom}, of $\rho(\pi_1^{\mathrm{geom}}(C, \bar{\xi}))$ is a normal subgroup of G_l. Since $\mathcal{E}_l$ is pure, it follows from [12, 1.3.9, 3.4.1(iii)] that G_l^{geom} is semisimple. The connected component of identity $G_l^{\mathrm{geom},o}$ is the derived group of G_l^o.

Since $G_l^{\mathrm{geom},o} \otimes \mathbb{C} \subset GL(8, \mathbb{C})$ is $SL(2, \mathbb{C})^{\times 3}$ modulo finite central elements, $G_l \otimes \mathbb{C} \subset GL(8, \mathbb{C})$ is contained in the normalizer $\mathbb{C}^*.SL(2, \mathbb{C})^{\times 3} \rtimes S_3$. Taking some finite étale covering of C if necessary, we can assume that $G_l \otimes \mathbb{C}$ is a subgroup of $\mathbb{C}^*.SL(2)^{\times 3}$.

Remark 7.4.1. If we assume $G_l^{\mathrm{geom}} \otimes \mathbb{C}$ is entirely contained in $SL(2,\mathbb{C})^{\times 3}$, then we only need to enlarge the base field $\mathbb{F}_q$ to kill the S_3 part.

For every closed point c of C, the action of F_c on $\mathcal{E}_{l\,c} \cong H^1_{\mathrm{et}}(X_{\bar{c}}, \mathbb{Q}_l)$ is semisimple (see [6]). Therefore F_c acts on $\wedge^4\mathcal{E}_{l,c}$ and $\mathrm{End}(\wedge^2\mathcal{E}_{l,c})$ semisimply.

Firstly let us consider the space

$$H^0_{\mathrm{et}}(C_{\bar{\mathbb{F}}_q}, \mathscr{E}nd(\wedge^2\mathcal{E}_l)) \cong \mathrm{End}(\wedge^2(\mathcal{E}_{l,c}))^{\pi_1^{\mathrm{geom}}(C,\bar{c})}.$$

Since F_c acts on $\mathrm{End}(\wedge^2\mathcal{E}_{l,c})$ semisimply, we can calculate its eigenvalues over $\mathbb{C}$.

Let V be the dimension 2 standard representation of $SL(2,\mathbb{C})$. Condition (1) of Theorem 1.0.4 shows that $\mathcal{E}_{l.c} \otimes \mathbb{C} \cong V^{\otimes 3}$ is the tensor product of three standard representations of $SL(2,\mathbb{C})$. So base change to $\mathbb{C}$,

$$H^0_{\mathrm{et}}(C_{\bar{\mathbb{F}}_q}, \mathscr{E}nd(\wedge^2\mathcal{E}_l)) \otimes \mathbb{C} \cong \mathrm{End}(\wedge^2(V^{\otimes 3}))^{SL(2,\mathbb{C})^{\times 3}}.$$

As a representation of $SL(2,\mathbb{C})^{\times 3}$, $\wedge^2(V^{\otimes 3})$ decomposes into four distinct irreducible components

$$\wedge^2(V^{\otimes 3}) \cong \oplus_{i=1}^4 W_i.$$

There W_1, W_2, W_3 are all $S^2V \otimes S^2V$ and W_4 is the trivial representation of $SL(2,\mathbb{C})^{\times 3}$. Yet W_1, W_2, W_3 are pairwisely non-isomorphic $SL(2,\mathbb{C})^{\times 3}$ representations.

Let p_k (i_k, resp.) be the projection from $\wedge^2(V^{\otimes 3})$ to W_k (inclusion from W_k to $\wedge^2(V^{\otimes 3})$, resp.). We have

$$\mathrm{End}(\wedge^2(V^{\otimes 3}))^{SL(2,\mathbb{C})^{\times 3}} \cong \oplus_{k=1}^4 (i_k \circ \mathrm{id}_{W_k} \circ p_k).$$

Note each id_{W_k} is invariant under the action of $SL(2,\mathbb{C})^{\times 3}$ and the scalar multiplication on W_K. Thus id_{W_k} is further invariant under the action of G_l. In particular, the Frobenius F_c fixes each id_{V_i}. So the invariant space $H^0_{\mathrm{et}}(C_{\bar{\mathbb{F}}_q}, \mathscr{E}nd(\wedge^2\mathcal{E}_l))^F$ has dimension 4.

Secondly,

$$H^0_{\mathrm{et}}(C_{\bar{\mathbb{F}}_q}, \wedge^4\mathcal{E}_l) = (\wedge^4\mathcal{E}_{l\,c})^{\pi_1^{\mathrm{geom}}(C,\bar{\xi})}.$$

Similarly, base change to $\mathbb{C}$ and it is isomorphic to $\wedge^4(V^{\otimes 3})^{SL(2,\mathbb{C})^{\times 3}}$. One can directly compute by hand, or see the proof of Theorem 4.1 in [38] to conclude that this space only has dimension 1 which is generated by the polarization. Therefore the corresponding Frobenius eigenvalues are q^2.

In summary, we have the following results:

$$\begin{aligned} \dim H^0_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathscr{E}nd(\wedge^2\mathcal{E}_l))^F &= 4, \\ \dim H^0_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \wedge^4\mathcal{E}_l)^{F-q^2} &= 1. \end{aligned} \tag{7.5}$$

7.5 Comparison of Lefschetz Trace Formulas

In this section, we compare Lefschetz Trace Formulas to obtain a similar result to (7.5) in the case of crystalline cohomology.

We firstly consider $\mathcal{E}_p := R^1\pi_{\text{cris},*}(\mathcal{O}_X)$. Since σ is the identity on $\mathbb{F}_q$, the absolute Frobenius F acts linearly on $\mathcal{E}_{p,c}$. Since the local crystalline characteristic polynomial coincides with the l-adic one ([22, 1.3.5])

$$\det(1 - tF|_{\mathcal{E}_{p\,c}}) = \det(1 - tF|_{\mathcal{E}_{l,c}}), \tag{7.6}$$

the eigenvalues of F on $\mathcal{E}_{l,c}$ and $\mathcal{E}_{p\,c}$ are identical.

Let $\mathcal{F}_l$ be either $\wedge^4\mathcal{E}_l$ or $\mathscr{E}nd(\wedge^2\mathcal{E}_l)$. Since $\mathcal{E}_l$ comes from geometry, by Deligne's Weil II, the l-adic relative Lefschetz Trace Formula provides

$$\prod_{c\in C} \det(1 - tF|_{\mathcal{F}_{l,c}}) = \prod_i \det(1 - tF|_{H^i_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)})^{(-1)^i}.$$

In the p-adic case, we still use $\mathcal{F}_p$ to represent either $\wedge^4\mathcal{E}_p$ or $\mathscr{E}nd(\wedge^2\mathcal{E}_p)$. Since $\mathcal{E}_p$ is a Dieudonne crystal, $\mathcal{F}_p$ is automatically overconvergent. By a theorem of Etesse and le Stum ([25, 2.1.2]), we also have a Lefschetz Trace Formula within crystalline cohomology setting

$$\prod_{c\in C} \det(1 - tF|_{\mathcal{F}_{p,c}}) = \prod_i \det(1 - tF|_{H^i_{\text{cris}}(C/\mathbb{Z}_q, \mathcal{F}_p)})^{(-1)^i}.$$

Combining with equality (7.6), we have

$$\prod_i \det(1 - tF|_{H^i_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)})^{(-1)^i} = \prod_i \det(1 - tF|_{H^i_{\text{cris}}(C/\mathbb{Z}_q, \mathcal{F}_p)})^{(-1)^i}. \tag{7.7}$$

By Deligne's Weil II [12], the étale cohomology groups $H^i_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)$ is pure of weight $i + j$ where $\mathcal{F}_l$ has weight j. Since $\mathcal{F}_p$ is pointwisely pure, by [25, Theorem 5.3.2], $H^i_{\text{cris}}(C/\mathbb{Z}_q, \mathcal{F}_p)$ has the purity which implies, on each side of equality (7.7), there is no cancellation between the numerator and the denominator. All zeros or poles have the expected complex norms. Then we have the following termwise equality from (7.7).

$$\det(1 - tF|_{H^i_{\mathrm{et}}(C_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)}) = \det(1 - tF|_{H^i_{\mathrm{cris}}(C/\mathbb{Z}_q, \mathcal{F}_p)}). \tag{7.8}$$

Note the crystalline Frobenius acts semisimply on each fiber $\mathcal{F}_{p,c}$ [13, Page 372, Lemma 2.10]. So the eigenvalues of F on $H^0(C/\mathbb{Z}_q, \wedge^4\mathcal{E}_p) \otimes \mathbb{Q}_q$ and $H^0(C/\mathbb{Z}_q, \mathscr{E}nd(\wedge^2\mathcal{E}_p)) \otimes \mathbb{Q}_q$ are identical as on their l-adic counterparts. In particular,

$$\begin{aligned} \dim H^0(C/\mathbb{Z}_q, \wedge^4\mathcal{E}_p)^{F-q^2} \otimes \mathbb{Q}_q = 1, \\ \dim_{\mathbb{Q}_q} H^0(C/\mathbb{Z}_q, \mathscr{E}nd(\wedge^2\mathcal{E}_p))^F \otimes \mathbb{Q}_q = 4. \end{aligned} \tag{7.9}$$

7.6 Proof of 1.0.4

In order to apply 1.0.3, we need to prove that $\mathrm{End}^0(\wedge^2\mathcal{E}_p)^F \cong \mathbb{Q}_q^{\times 4}$ as algebras.

7.6.1 Frobenius Torus

By [45, Theorem 2], $\mathbb{Q}[F] \cong \prod K_i$ where K_i are number fields. The multiplicative group $\mathbb{Q}[F]^*$ defines a $\mathbb{Q}$-torus

$$T = \prod_i \mathrm{Res}_{\mathrm{K_i}/\mathbb{Q}}(\mathbb{G}_\mathrm{m}).$$

Viewing F as an element in G_l, T can be regarded as the $\mathbb{Q}$-model of the connected component of 1 in the Zariski closure of the set $\{\rho(F)^n | n \in \mathbb{Z}\}$ in G_l (cf. [5, Chapter II, Section 13, Proposition 3]). In particular, T is contained in a maximal torus of G_l.

By [7, Theorem 3.7] and Chebotarev density theorem for the function field, generic points c on C satisfy that F_c generates a maximal torus. For every c, the torus T is defined over $\mathbb{Q}$. We say T is *unramified* over $\mathbb{Q}_p$ if the splitting field of T is unramified over prime p, and equivalently, the eigenvalues of F_c are unramified over p.

Remark 7.6.2. Varying the prime l, we obtain a compatible system of l-adic representation as stated in [28, 6.5]. The existence of a point c satisfying (3) in 1.0.4 requires G_l to be unramified over $\mathbb{Q}_p$.

On one hand, by [28, Proposition 8.9] and [29, Proposition 1.2, Theorem 3.2], for a subset of primes l of density 1(or even l large enough), G_l is unramified over Q_l. However, we start with a fixed prime p. Most results in the two paper have involved Dirichlet density restriction and hence can not be applied directly to our case.

On the other hand, we expect that if G_l is unramified over $\mathbb{Q}_q$, then there always exists a closed point c satisfying (3) in 1.0.4.

We also think generic ordinary property of $X \longrightarrow C$ should also provide more information on Frobenius eigenvalues.

7.6.3 Eigenvalues of F_c on $\mathcal{E}_{p\,c}$

Note up to now, we have not used condition (2) and (3) in 1.0.4. Under the condition (2), by 7.6.1, we always can find c such that X_c ordinary and $\rho(F_c)$ a maximal torus. Further with the condition (3), there exists a closed point c which satisfies the following two conditions:

1. X_c is ordinary,
2. the Frobenius torus T is a maximal torus in G_l.

Now we study the eigenvalues of the Frobenius on the fibre over c. Since X_c is ordinary, $\mathcal{E}_{p\,c}$ is the product of a unit root crystal $\mathcal{U}_c$ and its dual $\mathcal{U}_c^\vee$. Let $\lambda_1, \cdots, \lambda_4$ be the eigenvalues of F_c on $\mathcal{U}_c$. Then on $\mathcal{U}_c^\vee$, the eigenvalues of F_c are $\dfrac{q}{\lambda_i}$. Since $\mathcal{U}_c$ is a unit root crystal, λ_i are all p-adic units. Since $\mathcal{E}_{p\,c}$ has pure weight 1, λ_i all have complex norm $q^{\frac{1}{2}}$.

By 7.6, the Frobenius F_c also has eigenvalues $\lambda_1, \cdots, \lambda_4, \dfrac{q}{\lambda_1}, \cdots, \dfrac{q}{\lambda_4}$ on $\mathcal{E}_{l,c}$. Since the Frobenius torus is the maximal torus, the Frobenius eigenvalues λ_i correspond to the weights in the $SL(2)^{\times 3}$ representation $V^{\otimes 3}$. Let a, b, c be the three highest weights in the three standard representation of $SL(2, \mathbb{C})$. Then the eight weights of $V^{\otimes 3}$ are of the form $\pm a \pm b \pm c$ and they have a configuration as vertices of a cube. In this cube, the four p-adic units $\lambda_1, \cdots, \lambda_4$ lie in the same face. Without loss of generality, we can assume that λ_1 corresponds to the highest weight $a + b + c$ and $\lambda_2, \lambda_3, \lambda_4$ correspond to $a + b - c, a + c - b$ and $a - b - c$. Then the only relation between $\lambda_1, \cdots, \lambda_4$ is $\lambda_1\lambda_4 = \lambda_2\lambda_3$. So we have the following lemma.

Lemma 7.6.4. *Under the above choice of c, the eigenvalues λ_i have no relations other than those generated by $\lambda_i \dfrac{q}{\lambda_i} = q$ and $\lambda_1\lambda_4 = \lambda_2\lambda_3$.*

Remark 7.6.5. Lemma 7.6.4 also follows from the arguments in [40, Section 4].

Proposition 7.6.6.

$$\mathrm{End}^0(\wedge^2 \mathcal{E}_p)^F \cong \mathbb{Q}_q^{\times 4}$$

as algebras.

Proof. From 7.1.16 or basic representation theory of $SL(2)$, we know the condition 7.9 implies

$$\mathrm{End}(\wedge^2 \mathcal{E}_p)^F \otimes_{\mathbb{Q}_q} \mathbb{C} \cong \mathbb{C}^{\times 4}$$

as algebras. In particular, the algebra $\mathrm{End}(\wedge^2\mathcal{E}_p)^F$ is commutative. Therefore $\mathrm{End}^0(\wedge^2\mathcal{E}_p)^F$ is a product of fields.

Note $\wedge^2\mathcal{E}_p$ has the polarization as a direct summand. So

$$\mathrm{End}(\wedge^2\mathcal{E}_p)^F \cong \mathbb{Q}_q^{\times 4}, \mathbb{Q}_q \times K \text{ or } \mathbb{Q}_q^{\times 2} \times L$$

where K is a degree 3 field extension of $\mathbb{Q}_q$ and L has degree 2. Comparing with the decomposition over $\mathbb{C}$, there exists $\eta_K \in K$ or $\eta_L \in L$ such that $\mathrm{im}\,\eta_K$ and $\mathrm{im}\,\eta_L$ are subcrystals in $\wedge^2\mathcal{E}_p$. Further, rank $\mathrm{im}\,\eta_K = 27$ and rank $\mathrm{im}\,\eta_L = 18$.

If K or L is unramified over $\mathbb{Q}_q$, then by enlarging f in $q = p^f$, it becomes a product of copies of $\mathbb{Q}_q$. Therefore we only need to consider the case K or L ramified over $\mathbb{Q}_q$. Since $p \neq 2$ or 3, we can assume $L \cong \mathbb{Q}_q(\sqrt{p})$ and $K \cong \mathbb{Q}_q(\sqrt[3]{p})$ and we can choose $\eta_K = \sqrt[3]{p}$, $\eta_L = \sqrt{p}$.

Note $\mathcal{E}_{p\,c} \cong \mathcal{U}_c \oplus \mathcal{U}_c^\vee$. Since the eigenvalues have distinct p-adic values, there is no F_c-invariant morphisms between $\wedge^2\mathcal{U}_c, \wedge^2\mathcal{U}_c^\vee$ and $\mathcal{U}_c \otimes \mathcal{U}_c^\vee$. Thus we have the decomposition

$$\mathrm{End}(\wedge^2\mathcal{E}_p)^F \longrightarrow \mathrm{End}(\wedge^2\mathcal{E}_{p\,c})^F \cong \mathrm{End}(\wedge^2\mathcal{U}_c) \oplus \mathrm{End}(\wedge^2\mathcal{U}_c^\vee) \oplus \mathrm{End}(\mathcal{U}_c \otimes \mathcal{U}_c^\vee).$$

The restriction of F to $\mathrm{End}(\wedge^2\mathcal{E}_{p\,c})$ is just as F_c. Then by 7.6.4, all the eigenvalues of F on $\wedge^2\mathcal{U}_c$ are $\lambda_1\lambda_2, \cdots, \lambda_3\lambda_4$ and there is no more relations between the eigenvalues of $\wedge^2\mathcal{U}_c$ other than $\lambda_4\lambda_1 = \lambda_2\lambda_3$. So each eigenspace $U_{\lambda_i\lambda_j}$ has dimension 1 except for $(1,4)$ or $(2,3)$. Thereby

$$\begin{aligned}\mathrm{End}(\wedge^2\mathcal{U}_c)^F &\cong \oplus_{(i,j)\neq(1,3),(2,4)}\mathrm{End}(U_{\lambda_i\lambda_j}) \oplus \mathrm{End}(U_{\lambda_1\lambda_3}) \\ &\cong \oplus_{(i,j)\neq(1,3),(2,4)}\mathbb{Q}_q(\lambda_i\lambda_j) \oplus M_2(\mathbb{Q}_q(\lambda_1\lambda_4)).\end{aligned} \tag{7.10}$$

Since the four eigenvalues λ_i are all unramified over $\mathbb{Q}_q$ and L or K is ramified, the image of the composition

$$L \text{ or } K \longrightarrow \mathrm{End}(\wedge^2\mathcal{E}_p)^F \longrightarrow \mathrm{End}(\wedge^2\mathcal{U}_c)^F$$

lies only in $\mathrm{End}(U_{\lambda_1\lambda_4}) \cong M_2(\mathbb{Q}_q(\lambda_1\lambda_4))$. Otherwise, it would induce an embedding L or $K \hookrightarrow \mathbb{Q}_q(\lambda_i\lambda_j)$. In particular, $\eta_{K|\wedge^2\mathcal{U}_c}$ or $\eta_{L|\wedge^2\mathcal{U}_c}$ has only rank 2.

Restricted to point c, the image of η_K has dimension at most only 20. Contradiction.

For L, we know that $\eta_{L|\mathcal{U}_c\otimes\mathcal{U}_c^\vee}$ is a surjection. Note the eigenvalues of F_c on $\mathcal{U}_c \otimes \mathcal{U}_c^\vee$ have the form $\dfrac{q\lambda_i}{\lambda_j}$.

Again by 7.6.4, among these eigenvalues, $\dfrac{q\lambda_1}{\lambda_4}$ has only multiplicity 1. Therefore

$$\mathrm{End}(\mathcal{U}_c \otimes \mathcal{U}_c^{\vee})^F \cong \mathrm{End}(\mathcal{U}_{\frac{q\lambda_1}{\lambda_4}}) \oplus \cdots \cong \mathbb{Q}_q(\frac{q\lambda_1}{\lambda_4}) \oplus \cdots$$

as algebras. Since $\mathbb{Q}_q(\frac{q\lambda_1}{\lambda_4})$ is unramified over $\mathbb{Q}_q$, the image of L in $\mathrm{End}(\mathcal{U}_c \otimes \mathcal{U}_c^{\vee})^F$ excludes $\mathrm{End}(\mathcal{U}_{\frac{q\lambda_1}{\lambda_4}})$ and hence η_L can not be a surjection. The contradiction concludes the proof. $\square$

Now we have reduced 1.0.4 to 1.0.3 and thus it finishes the proof.

Chapter 8

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Appendix A

A.1 A classification result on complex simple Lie algebra representation

We classify all complex simple Lie algebras $\mathfrak{g}$ with an irreducible faithful symplectic 8-dimensional representation. In other words, we look for an embedding of $\mathfrak{g} \longrightarrow \mathfrak{sp}(8)$ such that the standard representation of $\mathfrak{sp}(8)$ is $\mathfrak{g}$-irreducible.

Since $\mathfrak{g}$ is simple, $\dim \mathfrak{g} \geq 2$. Note $\mathrm{rank}(\mathfrak{g}) \leq \mathrm{rank}(\mathfrak{sp}(8)) = 4$.

If $\mathrm{rank}(\mathfrak{g}) = 4$, the adjusting by conjugation, we can assume the embedding $\mathfrak{g} \longrightarrow \mathfrak{sp}(8)$ maps the Cartan subalgebra of $\mathfrak{g}$ into Cartan subalgebra of $\mathfrak{sp}(8)$, with the direct sum of positive roots $\mathfrak{g}^+$ to $\mathfrak{sp}(8)^+$, $\mathfrak{g}^-$ to $\mathfrak{sp}(8)^-$. Hence each root space of $\mathfrak{g}$ maps to a root space of $\mathfrak{sp}(8)$, which induces a map between Dynkin diagrams. Comparing the Dynkin diagrams of A_4, B_4, D_4 with $\mathfrak{sp}(8)$ yields that none of them can be embedded into $\mathfrak{sp}(8)$.

Therefore the only possible Lie algebras are $A_1, A_2, A_3, B_2, B_3, C_3, C_4$.

In each of the following cases, let V always denote the standard representation of corresponding Lie algebras.

1. $A_1 = \mathfrak{sl}(2)$.

   The unique 8-dimension irreducible representation of $\mathfrak{sl}(2)$ is the symmetric power S^7V. Since $\mathfrak{sl}(2) \cong \mathfrak{sp}(1)$, S^7V is symplectic.

   Now consider $\wedge^4(S^7V)^{\mathfrak{sl}(2)}$. By [16, 11.35], $\wedge^4(S^7V) \cong S^4(S^4V)$. Counting the dimension of each weight spaces yields the decomposition

$$S^4(S^4V) \cong V_{16} \oplus V_{12} \oplus V_{10} \oplus V_8 \oplus V_6 \oplus V_4^{\oplus 2} \oplus V_0^{\oplus 3}.$$

Therefore $\dim_{\mathbb{C}} \wedge^4(S^7V)^{\mathfrak{sl}(2)} = 3$ which is too big.

2. $A_2 = \mathfrak{sl}(3)$

   By [16, 15.17], for any irreducible representation $\Gamma_{a,b}$ with highest weight $aL_1 - bL_3$, the dimension $\dim_{\mathbb{C}} \Gamma_{a,b} = (a+b+2)(a+1)(b+1)/2$. Then $\Gamma_{a,b}$ is 8-dimensional if and only if $a = b = 1$. Note $\Gamma_{1,1}$ is nothing but the adjoint representation of $\mathfrak{sl}(3)$ which is the traceless subrepresentation of $\mathrm{End}(V)$. So it has a nondegenerate symmetric, not alternating form. Therefore $\mathfrak{sl}(3)$ does not have a symplectic irreducible representation.

3. $A_3 = \mathfrak{sl}(4)$

   Still by [16, 15.17], $\dim \Gamma_{a_1,a_2,a_3} = (a_1+1)(a_2+1)(a_3+1)(a_1+a_2+2)(a_2+a_3+2)(a_1+a_2+a_3+3)/12$ where each a_i is a nonnegative integer. No such a_i makes $\dim \Gamma_{a_1,a_2,a_3} = 8$. Hence $\mathfrak{sl}(4)$ has no 8-dimensional irreducible representation.

4. $B_2 = \mathfrak{so}(5)$

   By [16, 24.30], $\dim \Gamma_{a_1,a_2} = (a_1+1)(a_1+a_2+2)(2a_1+a_2+3)(a_2+1)/6$. No a_i makes it dimension 8. So B_2 has no irreducible representation of dimension 8.

5. $B_3 = \mathfrak{so}(7)$

   Again by [16, 24.30], $\dim \Gamma_{a_1,a_2,a_3} = (a_1+1)(a_3+1)^2(a_1+a_2+2)(a_1+2a_2+a_3+4)(a_1+a_2+a_3+3)(a_2+a_3+2)(2a_1+2a_2+a_3+5)(2a_2+a_3+3)/720$. Still no a_i make it 8. Therefore B_3 has no 8-dimensional irreducible representation.

6. $C_3 = \mathfrak{sp}(6)$

   By [16, 24.20], $\dim \Gamma_{a_1,a_2,a_3} = (a_3+1)(a_2+a_3+2)(a_1+a_2+a_3+3)(a_1+1)(a_2+1)(a_1+a_2+2)(a_1+2a_2+2a_3+5)(a_1+a_2+2a_3+4)(a_2+2a_3+3)/720$. No a_i make it 8. Hence $\mathfrak{sp}(6)$ has no irreducible 8-dimensional representation.

7. $C_4 = \mathfrak{sp}(8)$

   Then the second exterior product of standard representation $\wedge^2 V$ decomposes to $\wedge^2 V \cong \mathbb{C} \oplus W$ with W irreducible $\mathfrak{sp}(8)$-representation. So $\mathrm{End}_{\mathfrak{sp}(8)}(\wedge^2 V) = 2 < 3$.

A.2 Crystal model

Assume we have a F-isocrystal $\mathcal{V}$ over C. Then it is a prior a crystal over C. Locally, it corresponds to a module with connection $(M.\nabla)$ over $\tilde{C}$ such that $M \otimes B(k)$ admits a σ-linear morphism F. The point is to descend the Frobenius F.

Though F may not descend to M, we can consider $M' = \sum_n F^{(n)}(M)$. Then $M \subset M' \subset M \otimes B(k_0)$. One can mimic the the proof of [24], Theorem 2.6.1 to show M' is finitely generated provided that all slopes are nonnegative.

Since $\tilde{C}$ is regular of dimension 2, taking the double dual $M'^{\vee\vee}$ gives a locally free sheaf over $\tilde{C}$.

If $\mathcal{V}$ is an isocrystal with slopes between 0 and 1, then we can further choose a morphism $V : \mathcal{V} \longrightarrow \mathcal{V}^\sigma$ such that $V \circ F = F \circ V = p$. Hence such an isocrystal has a model of Dieudonne crystal.