Self-duality and singularities in the Yang-Mills flow

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ABSTRACT

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We investigate the long-time behavior and smooth convergence properties of the Yang-Mills flow in dimension four. Two chapters are devoted to equivariant solutions and their precise blowup asymptotics at infinite time. The last chapter contains general results. We show that a singularity of pure $+$ or $-$ charge cannot form within finite time, in contrast to the analogous situation of harmonic maps between Riemann surfaces. This implies long-time existence given low initial self-dual energy. In this case we study convergence of the flow at infinite time: if a global weak Uhlenbeck limit is anti-self-dual and has vanishing self-dual second cohomology, then the limit exists smoothly and exponential convergence holds. We also recover the classical grafting theorem, and derive asymptotic stability of this class of instantons in the appropriate sense.

CONTENTS

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1. INTRODUCTION

The Yang-Mills flow

$$\frac{\partial A}{\partial t} = -D_A^* F_A$$

evolves a connection A on a vector or principal bundle by the L^2 gradient of the Yang-Mills functional

$$\mathrm{YM}(A) = \frac{1}{2}\int |F_A|^2 dV.$$

Over compact base manifolds of dimension two or three, it was shown by G. Daskalopoulos [7] and Rade [23] that the Yang-Mills flow exists for all time and converges. Finite-time blowup is known to occur in dimension five or higher [21], and explicit examples of Type-I shrinking solitons were produced on $\mathbb{R}^n$, $5 \leq n \leq 9$, by Weinkove [39]. Hong and Tian [18] showed that the singular set has codimension at least four, and gave a complex-analytic description in the compact Kahler case (where an application of the maximum principle shows that singularities can only form at infinite time, see [31], Ch. 1). In complex dimension two, Donaldson's early results [11] for the flow on stable holomorphic bundles have recently been generalized by Daskalopoulos and Wentworth ([8], [9]).

The behavior of the Yang-Mills flow on Riemannian manifolds of dimension four, however, has not been understood well. The foundational work of Struwe [32] gives a global weak solution with finitely many point singularities, by analogy with harmonic map flow in dimension two [33]. To date, outside of the Kahler setting, long-time existence and convergence have only been fully established in specific cases, by appealing to energy restrictions on blowup limits [28] or by imposing a symmetric Ansatz [29]. Moreover, finite-time singularities have long been known as a characteristic feature of critical harmonic map flow

[5].

In addition to a detailed study of rotationally equivariant cases, this thesis provides a number of general theorems concerning long-time existence and smooth convergence of the Yang-Mills flow in dimension four. The present section gives an overview of subsequent chapters, and of the insights they contain for this naturally intriguing problem of geometric analysis.

Preliminaries

We review the elements of differential geometry needed to understand the problem. We then introduce the Yang-Mills formalism and derive the key identities, in particular the split Bochner-Weitzenbock formula. An outline of the short-time existence theory [32] is also provided.

Equivariant examples

A connection is called *equivariant* if left unchanged by a certain group of transformations or symmetries. Due to its intrinsic nature, the Yang-Mills flow will in general preserve this property. Imposing equivariance is thus a convenient way to decrease the complexity of the system or render it more transparent.

Schlatter, Struwe and Tahvildar-Zadeh [29] studied connections over the unit ball of $\mathbb{R}^4$ equivariant under the full group of rotations $SO(4)$. In this case the flow reduces to a scalar heat equation

$$u_t = u_{rr} + \frac{1}{r}u_r - \frac{2}{r^2}u(u-1)(u-2) \tag{YM}$$

for which long-time existence was established under appropriate boundary conditions. Apart from the Kahler theory, this appears to have been the only significant class of connections in dimension four for which long-time existence of the Yang-Mills flow was previously known.

We study the slightly larger class of connections which are equivariant only under the subgroup $SU(2) \subset SO(4)$. This family is more flexible, and contains several interesting new reaction-diffusion systems with two or three parameters (§3.3). For reasons which will become clear in the course of this thesis, these new systems did not yield an example of finite-time blowup.

Asymptotics at infinite time

This chapter is concerned with blowup behavior and asymptotics for a general semi-linear heat equation

$$\partial_t u(r,t) = \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{f(u)}{r^2} \tag{1.0.1}$$

on the unit interval with Dirichlet boundary conditions

$$u(0) = 0, u(1) = \alpha. \tag{1.0.2}$$

The nonlinear term is assumed to be of the form

$$f(u) = k^2 g \cdot g'(u) \tag{1.0.3}$$

where g is a smooth function with

$$g'(u) = \pm 1 \text{ for all } u \text{ such that } g(u) = 0. \tag{1.0.4}$$

Our main result (Theorem 4.0.2) is a verification, assuming $k = 2$ in (1.0.3), of the precise blowup asymptotics for (1.0.1) predicted by Van den Berg, Hulshof and King [3]. This case takes on a very strong significance for our investigation, as we explain.

Note that (4.0.1) is the gradient flow of the natural scaling-invariant energy functional

$$E(u) = \frac{1}{2}\int \left(|\partial_r u|^2 + \frac{(kg(u))^2}{r^2} \right) r\, dr.$$

Choosing $g(u) = \sin(u)$, we obtain the Dirichlet energy $\frac{1}{2}\int |\nabla \vec{u}|^2 dA$ of k-equivariant ("co-rotational") maps $D^2 \to S^2$, originally studied by Chang, Ding, and Ye [5] (and subsequently by [1], [3], [4], [24], and [25]). The integer k corresponds to the *rotation (winding) number* about the pole of the symmetric ansatz, and (4.0.1) takes on the zeroth-order term

$$f(u) = \frac{k^2 \sin(2u)}{2}. \tag{HM}$$

For any smooth function $g(u)$ that is odd-symmetric about its zeroes, satisfying (1.0.4) and $|g'(u)| \leq 1$, (1.0.1) represents harmonic map flow for k-equivariant maps into the surface of rotation in $\mathbb{R}^3$ corresponding to g (parametrized by arc-length u along a longitudinal ray).

On the other hand, choosing $g(u) = \frac{u(2-u)}{2}$ and $k = 2$, we obtain exactly (YM), the evolution of an $SO(4)$-equivariant connection. The energy $E(u)$ now coincides with the Yang-Mills action. The analogy between harmonic maps in dimension two and Yang-Mills in dimension four thus becomes explicit in the rotationally symmetric setting, with Yang-Mills corresponding to *twice-wound* harmonic maps into a certain "surface of rotation."

It was observed by Grotowski and Shatah [14] that the difference in winding number k, rather than the choice of $g(u)$, accounts for the contrasting results of [5] and [29]. Namely, that while finite-time blowup occurs readily for (HM) with $k = 1$, it *does not* occur for (HM) with $k = 2$, or for (YM) (the result of Schlatter et. al. [29] discussed above). In fact, according to the earlier matched asymptotics of Van den Berg, Hulshof, and King [3] for (HM), the rotation number k (which can be taken positive real), as well as the initial and boundary conditions, are expected to determine a variety of blowup behaviors. For all $0 < k < 2$, one has finite-time blowup generically (and inevitably if $\alpha > \pi$ in (4.0.2)). For $k = 1$, the original case of harmonic map flow blowup [5], the generic blowup rate

$$\lambda(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2},$$

was predicted, much different than the type-I rate $\sqrt{T-t}$, as well as degenerate blowup with quantized rates for certain initial data.

In the recent work of Raphael and Schweyer ([24], [25]), $k = 1$ solutions are shown to exist with both the generic and degenerate blowup rates. The authors construct a family of model approximate solutions, and prove that there is an open set (or stable manifold) of initial data for which the flow remains trapped near a model solution. A Lyapunov functional at the H^4 level is used, but no maximum principle. This represents remarkable progress towards understanding critical blowup behavior in semilinear evolution equations.

For $k > 2$, Van den Berg et. al. [3] predict infinite-time blowup for (HM) at polynomial rate. The rotation number $k = 2$, which includes Yang-Mills, therefore marks the border between finite and infinite-time blowup. Infinite-time blowup is again expected here, but with the following subtle dependence on the boundary conditions.[1]

$$\lambda(t) \sim \begin{cases} \kappa_1 e^{-\frac{\kappa}{e_0}t} & \text{for } \alpha \in (\pi, 2\pi) \\ \kappa_2 e^{-\kappa_1 e^{\frac{\kappa}{e_0}t}} & \text{for } \alpha \in (2\pi, 3\pi) \\ \vdots & \\ \kappa_n e^{-\kappa_{n-1} e^{\cdot^{\cdot^{\cdot^{\kappa_1 e^{\frac{\kappa}{e_0}t}}}}}} & \text{for } \alpha \in (n\pi, (n+1)\pi) \end{cases} \tag{1.0.5}$$

Blowup, always Type-II in this context, arises due to the parabolic scaling symmetry

$$u(r,t) \to u(r/\lambda, t/\lambda^2).$$

The asymptotics (1.0.5) are modeled on a tree of several static harmonic maps (a.k.a. bubbles) scaling inwards, with $\lambda(t)$ the scale of the innermost bubble. Because this scaling preserves the energy, (1.0.1) is referred to as "energy-critical," as are harmonic maps in dimension two and Yang-Mills in dimension four; the coefficient $k = 2$ is also critical in the sense of finite-versus-infinite-time blowup.

In Chapter 4, we verify the exponential blowup asymptotics in the case $k = 2$, *a fortiori* confirming that blowup occurs only at infinite time. Our method (see §4.0.3) is an adaptation and simplification of that of Raphael and Schweyer [24] to the scenario of infinite-time blowup

[1] See Section 4.1 for the definitions of κ and e_0. The undetermined constants κ_i, which depend on the initial data, appear only in infinite-time blowup only for $k = 2$.

(as well as a refinement of Schlatter et. al. [29]), which allows for an exploratory attempt at using the Euler-Lagrange structure (1.0.3) to gain estimates (§4.3.3). We expect that the iterated exponential blowups in (1.0.5) are only notationally more difficult.

Self-duality and singularities

This chapter contains the main results. As with the classical Theorems of Taubes [34] and Donaldson [10], ours will rely on the splitting of two-forms into self-dual and anti-self-dual parts, as well as a number of useful observations in the parabolic setting.

5.1. (Anti)-self-dual singularities (p. 60). We give a simple yet generic criterion for long-time existence, namely, that either of F^+ or F^- does not concentrate in L^2. The proof relies on a borderline Moser iteration (Proposition 5.1.1), together with a manipulation of the local energy inequality with a logarithmic cutoff (Theorem 5.1.4). We note that this criterion is not sufficient to rule out singularity formation at infinite time. Moreover, the two results hold simultaneously only in dimension four (see Remark 5.1.8).

We draw several conclusions: first, that a singularity of pure positive or negative charge, hence modeled on an instanton, cannot occur at finite time. This suggests that finite-time singularities are very unlikely to form on low-rank bundles, and should be unstable if they do. Second, if the global self-dual energy is less than δ, a computable constant, then the flow exists for all time and blows up at most exponentially. Third, yet another, geometric proof of long-time existence in the $SO(4)$-equivariant case follows from Theorem 5.1.4 (see §3.3.2).

We note that finite-time blowup of equivariant harmonic map flow $S^2 \to S^2$, i.e. the case $k = 1$ of (HM), occurs even with low holomorphic energy [5], hence lacks this additional level of "energy quantization." In this sense, Theorem 5.1.4[2] draws a geometric contrast between the dynamics of the two flows, previously seen only at the level of the "rotation number" k. The interaction between F^+ and F^- also invites a comparison with Topping's

[2] in particular the scaling of certain Sobolev norms applied to the cutoff

repulsion estimates [36] for holomorphic and anti-holomorphic bubbles in almost-harmonic maps, which (though of a very different nature) lead to results comparable to those of the next section (§5.2).

The proof of Theorem 5.1.4 also yields a characterization of finite-time blowup in terms of the stress-energy tensor for Yang-Mills (see Remark 5.1.5). This will be a direction for future work.

5.2. Convergence at infinite time (p. 67). Next, assuming low initial self-dual energy, we give a characterization of infinite-time singularities along classical gauge-theoretic lines. If the self-dual second cohomology H^{2+} of an anti-self-dual Uhlenbeck limit is zero, e. g. if it is irreducible of charge one, then a Poincaré inequality holds on self-dual two-forms. The estimate is inherited by connections along the flow, implying the exponential decay of $||F^+||^2$. This results in smooth convergence, once one is sufficiently close to the limit modulo gauge on an open set (Theorem 5.2.8). The set of bubbling points is therefore empty and the limit unique, in this case.

We conclude that an anti-self-dual limit must have $H^{2+} \neq 0$, if bubbling occurs at infinite time. Since this need not be the case either for a general weakly convergent sequence of instantons, or a priori within Taubes's framework [35], Theorem 5.2.8 may yield additional information about the topology of the instanton moduli spaces.

5.3. Further results (p. 76). Using the precise statement of Theorem 5.2.8, we deduce further properties of the flow at low self-dual energy. We recover the grafting theorem for pointlike instantons [34], which requires a brief new gauge-fixing argument at short time for non-simply-connected M. We also obtain the following (Corollary 5.3.3).

Assume the bundle E has structure group $SU(2)$ with $c_2(E) = 1$, and the base manifold M is simply-connected with $H^{2+}(M) = 0$. If $||F^+||_{L^2} < \delta_1$ initially, then the flow exists for all time and has a smooth subsequential limit. If the limit is anti-self-dual and irreducible then it is unique, and the flow converges exponentially.

Note that on certain manifolds with $H^{2+}(M) \neq 0$, e. g. $\mathbb{CP}^2$, $SU(2)$-instantons of charge one do not exist, and therefore the flow cannot have a smooth limit. This is also the simplest demonstration that Atiyah-Bott's description of Morse theory [2] does not generalize naively to dimension four.

In the case that the ground state of a certain physical system is not locally unique, the natural question is that of "asymptotic" stability under small perturbations. This has been studied chiefly in the hyperbolic setting, but also by Gustafson, Nakanishi, and Tsai [16] for (HM) with $k \geq 2$ on $\mathbb{R}^2$ (as well as the more general Landau-Lifshitz system). In the Yang-Mills context we observe Theorem 5.3.4, which gives a general H^1 asymptotic stability result in the parabolic sense for the instantons with $H^{2+} = 0$.

2. PRELIMINARIES

2.1 Differential geometry

Vector bundles and gauge transformations

Let $\pi : E \to M$ be a vector bundle over a smooth, compact, orientable base manifold.

A *section* of E over an open set $U \subset M$ is a smooth map $s : U \to E$ such that

$$\pi \circ s = Id_U.$$

By definition, there exists a system of coordinate charts $\{U_a\}$ for M, together with a *local frame* of sections $\{e^a_\alpha\}^n_{\alpha=1}$ over U_a for each a, such that any section can be written (with no sum on a)

$$s|_{U_a \cap U} = (s^a)^\alpha e^a_\alpha. \tag{2.1.1}$$

The *transition functions* $(u^{ab})^\alpha{}_\beta$ may thus be defined over $U_a \cap U_b$ by writing

$$e^a_\beta\big|_{U_a \cap U_b} = (u^{ab})^\alpha{}_\beta \, e^b_\alpha.$$

This yields the familiar transformation law

$$\left(s^b\right)^\alpha = \left(u^{ab}\right)^\alpha{}_\beta \left(s^a\right)^\beta \tag{2.1.2}$$

for the *local components* of an arbitrary section s, defined by (2.1.1). The transition functions (invertible matrices) satisfy the *cocycle conditions*

$$u^{bc} \cdot u^{ab} = u^{ac}$$

on $U_a \cap U_b \cap U_c$. Conversely, these data are sufficient to reconstruct the bundle E.

Choosing the local frames to be orthonormal

$$\langle e^a_\alpha, e^a_\beta \rangle = \delta_{\alpha\beta}$$

ensures that the u^{ab} lie inside the orthogonal group $O(n)$. Should these lie within a subgroup $G \subset O(n)$, we say that E has *structure group* G. Since any compact Lie group G embeds into $O(n)$ for some n, studying vector rather than principal bundles with compact structure group entails no loss of generality.[1]

Henceforth we will suppress the chart label and local frame, writing s^α for a section of E in local components, with Greek index, and s_α for a section of E^*. A Latin index v^i corresponds to the section $v^i \frac{\partial}{\partial x^i}$ of the tangent bundle TM, and v_i to a section $v_i dx^i$ of the cotangent bundle T^*M. We will use the bracket $\langle \cdot, \cdot \rangle$ also to denote the full induced pointwise inner product on tensor bundles formed from these.

The set of *gauge transformations* $\mathfrak{G}_E \subset \text{End}E$ consists of the orthogonal matrices at each point (or elements of the structure group G), and a (smooth) section u of $\mathfrak{G}_E|_U$ gives a local metric-preserving automorphism of E (c. f. 3.1.2). The vector bundle of *infinitesimal gauge transformations* $\mathfrak{g}_E \subset \text{End}E$ consists of skew-symmetric matrices (or elements of $\mathfrak{g}$). Sections of $\mathfrak{g}_E|_U$ correspond to the Lie algebra of $\mathfrak{G}_E|_U$ via exponentiation within End$E$. We denote the induced action of u on any tensor by $u(\cdot)$, which on $\mathfrak{g}_E$ coincides with the adjoint action.

We write $\Omega^k(E)$ for the bundle of *E-valued k-forms*, or alternating elements of $(T^*M)^{\otimes k} \otimes$

[1] It will be clear that if the connection takes values in the Lie algebra $\mathfrak{g}$ of the group G, then this property will be preserved as long as we deal with smooth connections and gauge transformations, and in fact more generally (see [12]).

E, with inner-product $g(\cdot,\cdot)$ induced from the standard orthonormal basis of wedge elements. The components of a two-form, for instance, are defined by

$$\sum_{i<j} \omega_{ij} dx^i \wedge dx^j = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j.$$

Write $\Omega^k(\mathfrak{g}_E) \subset \Omega^k(\mathrm{End}E)$ for the *Lie-algebra valued k-forms.* For $\omega, \eta \in \Omega^2(\mathrm{End}E)$, and similarly for forms of any degree, we define the *wedge product*

$$(\omega \wedge \eta)^{\alpha}{}_{\beta} = \frac{1}{4}\omega_{ij}{}^{\alpha}{}_{\gamma}\, \eta_{k\ell}{}^{\gamma}{}_{\beta} \left(dx^i \wedge dx^j \wedge dx^k \wedge dx^{\ell} \right).$$

Defining the operator $* : \Omega^k(\mathfrak{g}_E) \to \Omega^{4-k}(\mathfrak{g}_E)$ as the linear extension of the ordinary Hodge star on differential forms, we obtain the relation

$$-Tr\, \omega \wedge *\eta = g(\omega, \eta)\, dV \tag{2.1.3}$$

for $\omega, \eta \in \Omega^k(\mathfrak{g}_E)$.

In dimension four, we have

$$*^2 = (-1)^{k(4-k)} = (-1)^k$$

on Ω^k. For this reason, the two-forms (valued in any bundle) split into orthogonal positive and negative eigenspaces

$$\Omega^2 = \Omega^{2+} \oplus \Omega^{2-}.$$

A form $\omega \in \Omega^{2\pm}$ which satisfies $*\omega = \pm\omega$ is called *self-dual* or *anti-self-dual*, respectively. In normal coordinates at a point, this amounts to the three relations

$$\omega_{12} = \pm\omega_{34} \qquad \omega_{13} = \mp\omega_{24} \qquad \omega_{14} = \pm\omega_{23}. \tag{2.1.4}$$

Connections and covariant derivatives

A *connection* A is a metric-preserving rule for transporting fiber elements of E, which is linear in the tangent directions of M.

Formally, a connection is equivalent to a *covariant derivative*, or an $\mathbb{R}$-linear map

$$s \mapsto \nabla_A s$$

from sections of E to sections of $T^*M \otimes E$, satisfying

$$\begin{aligned} \nabla_A(f \cdot s) &= df \otimes s + f\nabla_A s \\ d\langle s,t\rangle &= \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle. \end{aligned}$$

In local coordinates, writing $(\nabla_A s)(\partial_i) = \nabla_i s$, we may define the components

$$A^\alpha_{i\beta} = \langle e_\alpha, (\nabla_i e_\beta)\rangle$$

in order to obtain the well-known formula

$$\nabla_i s^\alpha := (\nabla_A s)_i{}^\alpha = \partial_i s^\alpha + A^\alpha_{i\beta} s^\beta.$$

The connection A functions independently as follows. Given a path $\gamma(t)^i$ in a local chart of M and a section s along γ, the rule

$$\frac{d\gamma^i}{dt}\nabla_i s = 0$$

defines *parallel transport* via the connection A, written explicitly

$$\frac{ds^\alpha(\gamma(t))}{dt} = -A^\alpha_{i\beta}\frac{d\gamma(t)^i}{dt}s^\beta.$$

By this linear ordinary differential equation, any local smooth $\mathfrak{g}$-valued functions A_i define an identification by elements of G of the fibers along γ. On the other hand, for a given connection A, if the local frame e_α is chosen via parallel transport along geodesics from a point $x \in M$, one can achieve the identical vanishing of the radial component of A, hence of all components at the single point x ("*radial gauge*").

Under a gauge transformation or change-of-frame u, the components of A transform according to the requirement

$$u(\nabla_A s) = \nabla_{u(A)}(u(s)) \tag{2.1.5}$$

or in matrix notation

$$u(A) = u \cdot A \cdot u^{-1} - du \cdot u^{-1}. \tag{2.1.6}$$

From this transformation law, it is evident that the difference of any two connections defines a genuine section of $\Omega^1(\mathfrak{g}_E)$, as does the derivative $\dot{A}$ of a smooth family of connections.

In order to be compatible with traces, A is defined to act on E^* by

$$\nabla_i s_\beta = \partial_i s_\beta - A^\alpha_{i\beta} s_\alpha.$$

Using the Levi-Civita connection Γ^j_{ik} on TM and T^*M, we may uniquely extend the connection ∇_A to all tensor bundles via the requirements

$$\nabla_i(s \otimes t) = \nabla_i s \otimes t + s \otimes \nabla_i t$$
$$\partial_i \langle s, t \rangle = \langle \nabla_i s, t \rangle + \langle s, \nabla_i t \rangle.$$

We also define the *covariant differential* on sections $\Omega^k(E) \to \Omega^{k+1}(E)$ by the rule

$$D_A(s^\alpha dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \nabla_i s^\alpha dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

By abuse of notation, we may consider A locally as a $\mathfrak{g}$-valued "*connection 1-form,*" $A^\alpha_{i\beta} dx^i$, and rewrite D_A in terms of the wedge product as follows. For $\alpha \in \Omega^k(E)$, we have

$$D_A\alpha = d\alpha + A \wedge \alpha$$

and for $\omega \in \Omega^k(\mathrm{End}E)$

$$D_A\omega = d\omega + A \wedge \omega + (-1)^{k+1}\omega \wedge A. \tag{2.1.7}$$

Define the L^2-*adjoint*

$$(\nabla_A^*\omega)_{i_1\cdots i_k} = -g^{\ell j}\nabla_\ell \omega_{j i_1\cdots i_k} = -\nabla^j \omega_{j i_1 \cdots i_k}$$

which agrees on form components with the adjoint of the covariant differential, namely

$$D_A^* = - * D_A * .$$

Curvature and Bianchi identities

The *curvature* F_A of the connection A is defined as the operator on sections of E

$$\begin{aligned}(D_A)^2 s &= D_A(ds + A \cdot s)\\ &= d^2 s + dA \cdot s - A \wedge ds + A \wedge ds + A \wedge A \cdot s\\ &= (dA + A \wedge A)s.\end{aligned}$$

This operator is evidently C^∞-linear, and therefore defines a section $\frac{1}{2}F_{ij}dx^i \wedge dx^j \in \Omega^2(\mathfrak{g}_E)$ with components

$$F_{ij}{}^\alpha{}_\beta = \partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{i\gamma}A^\gamma_{j\beta} - A^\alpha_{j\gamma}A^\gamma_{i\beta}.$$

Writing $R_{ij}{}^k{}_\ell$ for the curvature of Γ on TM, we obtain the commutation formula

$$\begin{aligned}[\nabla_i, \nabla_j]\, t^k{}_\ell{}^\alpha{}_\beta &= R_{ij}{}^k{}_m t^m{}_\ell{}^\alpha{}_\beta - R_{ij}{}^n{}_\ell t^k{}_n{}^\alpha{}_\beta\\ &\quad + F_{ij}{}^\alpha{}_\gamma t^k{}_j{}^\gamma{}_\beta - F_{ij}{}^\gamma{}_\beta t^k{}_j{}^\alpha{}_\gamma\end{aligned} \tag{2.1.8}$$

and similar formulae in general. We may derive for F_A the so-called *first Bianchi identity*

$$\begin{aligned}(D_A^*)^2 F_A &= \nabla^i\nabla^j F_{ij} = \frac{1}{2}(\nabla^i\nabla^j - \nabla^j\nabla^i)F_{ij}\\ &= \frac{1}{2}\left(-R^{ijn}{}_i F_{nj} - R^{ijn}{}_j F_{in} + \left[F^{ij}, F_{ij}\right]\right)\\ &= 0.\end{aligned}$$

Using (2.1.7), we derive the *second Bianchi identity*

$$\begin{aligned}D_A F_A &= d(dA + A\wedge A) + A\wedge dA - dA\wedge A + A\wedge(A\wedge A) - (A\wedge A)\wedge A\\ &= dA\wedge A - A\wedge dA + A\wedge dA - dA\wedge A\\ &= 0.\end{aligned}$$

This is equivalent to the familiar identity on component matrices

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0.$$

2.2 Yang-Mills theory

2.2.1 Yang-Mills functional and instantons

Writing $|F_A|^2$ for the pointwise norm of the curvature form in the fixed metric g, the Yang-Mills energy is defined as above. We may compute its gradient using the formula

$$\begin{aligned}F_{A+a} &= F_A + da + A\wedge a + a\wedge A + a\wedge a\\ &= D_A a + a\wedge a\end{aligned} \tag{2.2.1}$$

in order to obtain

$$\begin{aligned}\frac{d}{dt}YM(A+ta) &= \frac{1}{2}\frac{d}{dt}\left(\int\left(|F_A|^2 + 2t\left\langle F_A, D_A a\right\rangle\right)dV + O(t^2)\right)\\ &= \int\langle a, D_A^* F_A\rangle dV.\end{aligned}$$

We conclude that a critical point, or *Yang-Mills connection*, satisfies

$$D_A^* F_A = 0.$$

Moreover the Yang-Mills flow is given in local components

$$\frac{\partial}{\partial t} A_{j\beta}^{\alpha} = \nabla^i F_{ij}{}^{\alpha}{}_{\beta}.$$

By definition, we have the *energy inequality*

$$\mathrm{YM}(A(0)) - \mathrm{YM}(A(T)) = \int_0^T ||D_A^* F_A||^2 dt$$

as long as the connection is sufficiently smooth. Therefore, if the flow exists for all time, we expect a weak limit which, if not an absolute minimum of YM, is at least a Yang-Mills connection. Note that we will often abbreviate

$$|| \cdot ||_{L^2(M)} = || \cdot ||.$$

We will write

$$F^{\pm} = \frac{1}{2}(F \pm *F)$$

for the self-dual and anti-self-dual parts of the curvature form, respectively. In normal coordinates, these satisfy the relations

$$F_{12}^{\pm} = \pm F_{34}^{\pm} \qquad F_{13}^{\pm} = \mp F_{24}^{\pm} \qquad F_{14}^{\pm} = \pm F_{23}^{\pm}. \tag{2.2.2}$$

From the second Bianchi identity, remark that

$$\begin{aligned} 2D_A^* F^{\pm} &= - * (D * F \pm D *^2 F) \\ &= D_A^* F. \end{aligned} \tag{2.2.3}$$

Therefore, if a connection is anti-self-dual ($F^+ = 0$) or self-dual ($F^- = 0$), then it is a critical point of YM. These special critical points are called *instantons*.

Recall from Chern-Weil theory that the integer

$$\kappa(E) = \frac{1}{8\pi^2} \int TrF_A \wedge F_A$$

is a topological invariant which does not depend on the connection A (for complex bundles, this coincides with the second Chern character). From the definition of the Hodge star operator, we compute

$$\begin{aligned} \int TrF_A \wedge F_A &= - \int g\left(F^+ + F^-, F^+ - F^-\right) dV \\ &= ||F^-||^2 - ||F^+||^2 \end{aligned}$$

but by orthogonality, also

$$||F||^2 = ||F^+||^2 + ||F^-||^2.$$

Changing the orientation of M if necessary, we may assume that κ is nonnegative. We obtain the formula

$$||F||^2 = 8\pi^2\kappa + 2||F^+||^2. \tag{2.2.4}$$

Thus a connection is anti-self-dual if and only if it attains the energy $8\pi^2\kappa$, which then must be the absolute minimum for connections on E.

2.2.2 Evolution of curvature and Weitzenbock formulae

From (2.2.1), we compute the evolution

$$\frac{\partial}{\partial t} F_A = D_A(-D_A^* F_A).$$

In view of the second Bianchi identity $D_A F_A = 0$, we may rewrite this as the tensorial heat equation

$$\left(\frac{\partial}{\partial t} + \Delta_A \right) F_A = 0$$

where $\Delta_A = DD^* + D^*D$ is the Hodge Laplacian with respect to the evolving connection.

We compute, for $\omega \in \Omega^k(\mathfrak{g}_E)$

$$\begin{aligned}(D^*D + DD^*)\omega_{i_1 \cdots i_k} = &-\nabla^j \left(\nabla_j \omega_{i_1 \cdots i_k} - \nabla_{i_1} \omega_{j i_2 \cdots i_k} - \cdots - \nabla_{i_k} \omega_{i_1 \cdots i_{k-1} j} \right) \\ &- \nabla_{i_1} \nabla^j \omega_{j i_2 \cdots i_k} + \nabla_{i_2} \nabla^j \omega_{j i_1 i_3 \cdots i_k} + \cdots + \nabla_{i_k} \nabla^j \omega_{j i_2 \cdots i_{k-1} i_1}.\end{aligned}$$

Permuting j and i_1 in the positive terms of the second line, we may group all but the very first term into commutators. We obtain the *Weitzenbock formula*

$$(D^*D + DD^*)\omega_{i_1 \cdots i_k} = \nabla^* \nabla \omega_{i_1 \cdots i_k} + Rm \# \omega - \left[F_{i_1}{}^j, \omega_{j i_2 \cdots i_k} \right] - \cdots - \left[F_{i_k}{}^j, \omega_{i_1 \cdots i_{k-1} j} \right]$$

In particular, for a two-form, we have

$$\begin{aligned}-\Delta_A \omega_{ij} = &\nabla^k \nabla_k \omega_{ij} + \left[F_i{}^k, \omega_{kj} \right] - \left[F_j{}^k, \omega_{ki} \right] \\ &- R_i{}^{k\ell}{}_k \omega_{\ell j} - R_i{}^{k\ell}{}_j \omega_{k\ell} + R_j{}^{k\ell}{}_k \omega_{\ell i} + R_j{}^{k\ell}{}_i \omega_{k\ell}\end{aligned} \tag{2.2.5}$$

We now make a simple observation about the zeroth-order terms (see [19], appendix). Assume we are in geodesic coordinates at a point, so (anti)-self-duality is defined as in (2.2.2). For $\omega \in \Omega^{2+}$ and $\eta \in \Omega^{2-}$, we may write

$$\begin{aligned}\omega_{1k}\eta_{k2} - \omega_{2k}\eta_{k1} &= \omega_{13}\eta_{32} - \omega_{23}\eta_{31} + \omega_{14}\eta_{42} - \omega_{24}\eta_{41} \\ &= (-\omega_{24})(-\eta_{41}) - \omega_{14}\eta_{42} + \omega_{14}\eta_{42} - \omega_{24}\eta_{41} \\ &= 0\end{aligned} \tag{2.2.6}$$

and similarly for any choice of indices. A similar calculation shows that for ω, ω' self-dual,

$\omega_{1k}\omega'_{k2} - \omega_{2k}\omega'_{k1}$ is again self-dual. These facts amount to the splitting of Lie algebras

$$so(4) = so(3) \oplus so(3).$$

For the Rm terms, one notes that the first and third are skew in i, j, as are the second and fourth, and that these are each self-dual if the same is true of ω (as explained in [13], appendix). We conclude that the extra terms of the Weitzenbock formula (2.2.5) in fact split into self-dual and anti-self-dual parts. Note also that $\Delta_A * = *\Delta_A$, and the trace Laplacian clearly preserves the identites (2.2.2) in an orthonormal frame.

We obtain, finally, for ω self-dual

$$-\Delta_A \omega_{ij} = \nabla^k \nabla_k \omega_{ij} + \left[F_i^{+k}, \omega_{kj}\right] - \left[F_j^{+k}, \omega_{ki}\right] + Rm\#\omega \tag{2.2.7}$$

as well as a similar formula for anti-self-dual forms. Applied to the self-dual curvature F^+, this yields the key evolution equation

$$\frac{\partial}{\partial t} F^+_{ij} = \nabla^k \nabla_k F^+_{ij} + 2\left[F_i^{+k}, F^+_{kj}\right] + Rm\#F^+. \tag{2.2.8}$$

2.2.3 Sobolev spaces

Any connection can be uniquely written $A_{ref} + A$, with $A \in \Omega^1(\mathfrak{g}_E)$, and any norms applied to a connection will be applied to the global one-form A.

We define the Sobolev norms

$$||\omega||_{H^k} = \left(\sum_{\ell=0}^{k} ||\nabla^{\ell}_{ref}\omega||^2_{L^2}\right)^{\frac{1}{2}}$$

as well as the corresponding spaces of forms and connections over any open set $\Omega \subset M$. A different reference connection defines uniformly equivalent norms. Our proofs will not deal directly with Sobolev spaces of gauge transformations and connections, as we are able to cite the highly developed regularity theory.

For any $\Omega' \subset\subset \Omega$, there is a local Sobolev inequality

$$||\omega||^2_{L^4(\Omega')} < C_{\Omega',\Omega}||\omega||^2_{H^1(\Omega)}$$

for the norms defined with respect to A_{ref}. The difficulty with Yang-Mills in dimension four and above is that due to the zeroth-order terms of the Weitzenbock formula, the Sobolev constant for $D_A \oplus D^*_A$ blows up as the curvature of A concentrates.

2.3 Short-time existence

We review Struwe's construction [32] of a solution $D(t) = D_{ref} + A(t)$ with initial connection D_0. In subsequent chapters, by a solution of the Yang-Mills flow, we will always mean a solution of this form, although $A(t)$ may not be unique if it is reducible. We will also assume that all initial data is smooth, as justified by the construction. However, the H^1 local existence statement of Theorem 2.3.1 is required for the proof of Theorem 5.3.4.

Although the flow is not strictly parabolic, short-time existence is guaranteed for smooth data by a De Turck-type trick (due to Donaldson). Let $D(t)$ be a family of connections depending smoothly on time, $u_t \in \mathfrak{G}_E$ a family of gauge transformations with $u_{t_0} = Id$, and define $\bar{D}(t) = u_t(D(t))$. The transformation law (2.1.6) and Leibniz rule give

$$\frac{d}{dt}\bar{D}(t_0) = \frac{d}{dt}D - Ds \tag{2.3.1}$$

where $s = \frac{d}{dt}u_t\Big|_{t=t_0} \in \mathfrak{g}_E$. In general, consider the gauge transformation $u_t \cdot u_{t_0}^{-1}$ in (2.3.1), and apply u_{t_0} to the both sides. This yields

$$\frac{d}{dt}\bar{D}(t_0) = u_{t_0}\left(\frac{d}{dt}D(t_0)\right) - \bar{D}s(t_0) \tag{2.3.2}$$

where $s = u_t^{-1}\frac{d}{dt}u_t \in \mathfrak{g}_E$. Since t_0 was arbitrary, (2.3.2) holds for all time.

Now, to solve the Yang-Mills flow, write $\bar{D} = D_0 + a$ and consider the alternative equation

$$\frac{d}{dt}\bar{D} = \frac{d}{dt}a = -\bar{D}^*\bar{F} + \bar{D}\left(-\bar{D}^*a\right), \quad a(0) = 0. \tag{2.3.3}$$

Recall that

$$\bar{F} = F(D_0) + D_0 a + a\#a = F(D_0) + \bar{D}a + a\#a$$

so we may rewrite (2.3.3)

$$\frac{d}{dt}a = -\left(\bar{D}\bar{D}^* + \bar{D}^*\bar{D}\right)a - \bar{D}^*\left(F(D_0) + a\#a\right).$$

This is a smoothly perturbed heat equation, hence by standard parabolic theory, a unique smooth solution exists for a small time $0 \le t \le \tau$. Moreover, we may define a gauge-transformation $u = u_t$ by the pointwise ODE

$$s = u^{-1}\frac{d}{dt}u = \bar{D}^*a, \quad u_0 = Id. \tag{2.3.4}$$

By (2.3.3), $D = u^{-1}(\bar{D})$ is a smooth solution of the Yang-Mills flow.

For initial data $D_0 \in H^1$, Struwe writes

$$D(t) = D_1 + A_{bg}(t) + a(t)$$

where D_1 is a smooth connection near D_0, and A_{bg} solves the ordinary heat equation with respect to D_1 with

$$A_{bg}(0) = A_0 = D_0 - D_1.$$

The remaining piece $a(t)$ is determined by a fixed-point argument. The result can be summarized as follows.

Theorem 2.3.1. *([32] §4.2-4.3) Given a smooth connection D_1, there exist C and $\epsilon > 0$ (depending only on the bundle E) and τ (depending on D_1) as follows. For any $A_0 \in H^1$*

with $||A_0||_{H^1} < \epsilon$, *there exists a smooth solution* $\bar{D}(t) = D_1 + \bar{A}(t)$ *to (2.3.3) for* $0 < t \leq \tau$, *with*

$$||\bar{A}(t)||_{H^1} \leq C||A_0||_{H^1}$$

and $\bar{A}(t) \to A_0$ *strongly in* H^1 *as* $t \to 0$.

Fixing any time $0 < t_0 < \tau$ (or $t_0 = 0$ if D_0 is smooth), let $\hat{D}(t)$ be the solution of the Yang-Mills flow with $\hat{D}(t_0) = \bar{D}(t_0)$, obtained by solving (2.3.4) backwards and forwards in time, which is smooth for $0 < t < \tau$. For any sequence of times $t_i \to 0$, by definition there exist smooth gauge transformations u_i such that $u_i(\hat{D}(t_i)) = \bar{D}(t_i)$. By Theorem 2.3.1, $u_i(\hat{D}(t_i)) \to D_0$ in H^1. Struwe also finds an H^1 limit $u_i \to u_0$, and defines $D(t) = u_0(\hat{D})$ as the desired weak solution of the flow with $D(t) \to D_0$ in L^2.

The solution $D(t) = D_{ref} + A(t)$ is therefore smooth for $0 < t < \tau$, modulo the constant gauge transformation u_0 (if D_0 is singular). Struwe then gives the following long-time existence result, using arguments similar to those of Section 5.2 below. For a certain $\epsilon_0 > 0$, we say that the curvature $F(t) = F_{A(t)}$ *concentrates* in L^2 at $x \in M$ if

$$\inf_{R>0} \limsup_{t \to T} \int_{B_R(x)} |F(t)|^2 dV \geq \epsilon_0.$$

Theorem 2.3.2. (Struwe [32], Theorem 2.3) *The maximal smooth existence time* T *of* $A(t)$ *is characterized by concentration of the curvature* $F(t)$ *at some* $x \in M$ *as* $t \to T$.

It remains to study the concentration of curvature along the Yang-Mills flow.

3. EQUIVARIANT CASES

This chapter consists of an extended example, discussed somewhat informally at times.

While not intended for publication, its inclusion serves several purposes: to review the work of Schlatter, Struwe, and Tahvildar-Zadeh [29] in the light of this thesis; to demonstrate the complexity of the flow, even in the simplest concrete cases; and, finally, to record the specific motivation for the general results that follow.

3.1 Equivariant connections

$SU(2)$ and quaternions.

Let

$$SU(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \;\middle|\; z, w \in \mathbb{C},\;\; |\,z|^2 + |\,w|^2 = 1 \right\}$$

be the group of unitary matrices acting on $\mathbb{C}^2$ with determinant one.

These also act as orthogonal matrices on $\mathbb{R}^4$, containing the four basic elements

$$q^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad q^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$q^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad q^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

As matrix coefficients, these are

$$(q^\mu)^\alpha{}_\beta = \epsilon_{\mu\alpha\beta} + \delta_{\mu\alpha}\delta_{\beta 0} - \delta_{\mu\beta}\delta_{\alpha 0} + \delta_{\mu 0}\delta_{\alpha\beta}.$$

Here ϵ is the completely antisymmetric tensor with $\epsilon_{123} = 1$ and $\epsilon_{0\mu\nu} = 0$.

The q^μ in turn span the four-dimensional algebra of *quaternions*

$$\begin{aligned} x &= x_0 q^0 + x_1 q^1 + x_2 q^2 + x_3 q^3 \\ &= x_\mu q^\mu \end{aligned}$$

in which $SU(2)$ forms the unit sphere S^3. The coordinates x_i of a quaternion x can be read off from the first column of the corresponding matrix; hence the matrix form of x corresponds to the linear map given by left-multiplication $(x \cdot -)$ in the quaternion algebra.

The multiplication law reads[1]

$$q^\mu \cdot q^\nu = -\epsilon_{\mu\nu k} q^k - \delta_{\mu\nu} q^0 + \delta_{0\mu} q^\nu + \delta_{0\nu} q^\mu.$$

We identify the tangent space $su(2)$ to the identity in $SU(2)$ with the purely imaginary quaternions spanned by q^i, $i > 0$, having the commutation rule

$$\left[q^i, q^j\right] = -2\epsilon_{ijk} q^k.$$

Throughout this chapter, Latin indices will be limited to the values $1-3$ (the Lie-algebra indices) while Greek indices may take the values $0-3$.

Define the quaternion conjugate

$$x^* = x_0 q^0 - x_1 q^1 - x_2 q^2 - x_3 q^3$$

which agrees both with the transpose as a real matrix and the hermitian conjugate as a complex matrix. Also define

$$\mathrm{Re}\,(x) = \frac{x+x^*}{2} \qquad \mathrm{Im}\,(x) = \frac{x-x^*}{2} \qquad \langle x, y\rangle = \mathrm{Re}\,(x \cdot y^*).$$

$SU(2)$ and $SO(4)$.

For any tranformation $\Lambda \in SO(4)$ acting on $x \in \mathbb{R}^4$, there exist $p, q \in SU(2)$ such that

$$\Lambda x = q \cdot x \cdot p^* \;\; \forall \;\; x \in \mathbb{R}^4.$$

[1] Notice that these quaternion generators differ from Hamilton's by a sign

$$(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) = (q^0, -q^1, -q^2, -q^3)$$

and also that the imaginary unit $\sqrt{-1}$ is different both from $\mathbf{i}$ and from q^1. The Lie bracket used by Schlatter et. al. [29] has generators

$$\sigma_i = -q^i/2.$$

As the pair (q, p) corresponding to Λ is unique up to sign, we have a double-cover

$$SU(2) \times SU(2) \to SO(4).$$

Because left- and right-multiplication commute, this is a direct product, and any connection with structure group $so(4)$ decouples. It therefore entails no loss of generality to assume that all connections have structure group $su(2)$ in the left factor, as above.

Transformation rule.

Consider a diffeomorphism σ covered by a bundle map u.

$$\begin{array}{ccc} E & \xrightarrow{u} & F \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & Y \end{array}$$

Given a connection A on E, we obtain a connection A^σ on F as follows (c. f. (2.1.5)). For a section $s(x)$ and tangent vector v at x, define

$$u\left(\nabla_A(s)(x, v)\right) = \nabla_{A^\sigma}(u(s))\left(\sigma(x), \sigma_* v\right)$$

or in coordinates with $y = \sigma(x)$

$$A^\sigma_\nu(y) = \frac{\partial x^\mu}{\partial y^\nu}\left(u \cdot A_\mu(x) \cdot u^{-1} - \partial_\mu u \cdot u^{-1}\right). \tag{3.1.1}$$

Definition 3.1.1. Let G be a group of transformations acting as above, with $X = Y$ and $E = F$. We say that a connection A is *equivariant* under G if $A^\sigma = A$ for all $\sigma \in G$.

Consider the simple case $\sigma = u = \Lambda \in SO(4)$, acting by a constant matrix on both points and tangent vectors of $\mathbb{R}^4$. Since $d\Lambda = 0$, we have

$$A^{\Lambda^{-1}}(x, v) = \Lambda^{-1} A(\Lambda x, \Lambda v)\Lambda.$$

Write $\Lambda = \mathcal{L}_q \circ \mathcal{R}_{p^*}$, for (p, q) unique up to sign, and also $\Lambda^{-1} = \mathcal{L}_{q^*} \circ \mathcal{R}_p$. Since $A(x, v)$ lies inside $su(2)$ by assumption, it is clear that the corresponding transformation law is

$$A^{\Lambda^{-1}}(x, v) = q^* A(qxp^*, qvp^*)q. \tag{3.1.2}$$

Explicitly, one can observe that the connection matrix acts by left-multiplication in the quaternions (as remarked at the beginning), left- and right-multiplication commute, and $\mathcal{R}_p \mathcal{R}_{p^*} = \mathrm{Id}$.

$SO(4)$-Equivariant connections.

Following [29], we derive the form of an $su(2)$-connection equivariant under the whole of $SU(2) \times SU(2)$ based on (3.1.2). Write

$$\tilde{x} = x/|x|.$$

Let $q = q^0$ and $p = \tilde{x}$ in (3.1.2), to obtain

$$A(x, v) = A(|x|q^0, v\tilde{x}^*)$$

by which it suffices to determine the values of A for $x = |x|q^0$. The diagonal (with $p = q$) covers the subgroup $SO(3) \subset SO(4)$ which fixes the q^0-axis. Hence for all $q \in SU(2)$ and $v \in \mathbb{R}^4$

$$A(|x|q^0, v) = q^* A(|x|q^0, qvq^*)q.$$

Choosing $v = q^0$ gives that $A(|x|q^0, q^0) \in su(2)$ commutes with every $q \in SU(2)$, hence $A(|x|q^0, q^0) = 0$.

Taking $v \in su(2)$, consider the stabilizer $S^1 = \{q \in SU(2) \mid qvq^* = v\}$. By (3.1.2), the matrix $A(|x|q^0, v) \in su(2)$ must be fixed by this S^1, and is collinear with v. We conclude

$$A(|x|q^0, q^j) = \tilde{a}(|x|)q^j.$$

Letting $r = |x|$, we find that an $SO(4)$-equivariant $su(2)$-connection has the unique[2] form

$$\begin{aligned} A(x, v) = A(|x|q^0, vx^*) &= \tilde{a}(r)\mathrm{Im}\,(vx^*) \\ &= \tilde{a}(r)\left[vx^* - \langle v, \tilde{x}^*\rangle q^0\right] \end{aligned} \tag{3.1.3}$$

or in coordinates

$$A_\mu(x) = a(|x|)(\delta_{\lambda\mu}x_0 - \delta_{0\mu}x_\lambda - \epsilon_{\mu\gamma\lambda}x_\gamma)q^\lambda$$

with $a(r) = \tilde{a}(r)/r$.

We note that the more obvious equivariant connection

$$\begin{aligned} a(r)\tilde{x}\,\mathrm{Im}\,(\tilde{x}^*v)\,\tilde{x}^* &= a(r)\,\tilde{x}\,(\tilde{x}^*v - \mathrm{Re}\,(\tilde{x}^*v))\,\tilde{x}^* \\ &= a(r)\,\left(v\tilde{x}^* - \mathrm{Re}\,(v\tilde{x}^*)\right) \\ &= a(r)\,\mathrm{Im}\,(v\tilde{x}^*) \end{aligned} \tag{3.1.4}$$

agrees with (3.1.3).

Example 3.1.2. The *standard instanton* ('t Hooft solution) is the equivariant anti-self-dual[3] connection

$$A(x, v) = \frac{\mathrm{Im}\,(vx^*)}{1 + r^2}.$$

The instanton with center x_0 and scale λ is given by

$$A_{x_0,\lambda}(x, v) = A\left(\frac{x - x_0}{\lambda}, \frac{v}{\lambda}\right) = \frac{\mathrm{Im}\ (v(x - x_0)^*)}{\lambda^2 + |x - x_0|^2}.$$

[2] We could equally well have considered connections with values in the other $SU(2)$-factor. The general $SO(4)$-equivariant $so(4)$-connection is a linear combination of these two uncoupled components at each point. For instance, the Ansatz in Weinkove's paper [39] and several others is the sum of these two. This point (unique to dimension four) was omitted in O. Dumitrescu's 1982 paper.

[3] With the generators used by [29], the standard instanton is self-dual (and carries a factor of 2). Also, as shown by 3.1.4, A and A^* are exchanged by the singular gauge change from (3.1.5) to (3.1.6).

This 5-dimensional family contains all instantons of charge one on $\mathbb{R}^4$. The conjugate

$$A^*(x,v) = \frac{\mathrm{Im}\,(xv^*)}{1+r^2}$$

is self-dual (Remark 3.3.1 below).

$SU(2)$-equivariant connections.

To expand on this example, we consider $su(2)$-connections which are equivariant only for $SU(2)\times \mathrm{Id}$. This group acts on S^3 freely and transitively, hence the four components $A(|x|\,q^0, q^\mu)$ can be taken to be arbitrary traceless skew-Hermitian matrices depending smoothly on $r>0$. These determine an $SU(2)$-equivariant connection by the formula

$$A(x,v) = \tilde{x}A(|x|q^0, \tilde{x}^* v)\tilde{x}^*. \tag{3.1.5}$$

Connections with this broken symmetry could be described as "homogeneous but not isotropic" on each S^3_r. This form is also compatible with different choices of base for which $SU(2)$ acts, for instance $\mathbb{CP}^2$.[4] While a general $SU(2)$-equivariant connection has 12 free parameters, only 9 remain after choosing the radial gauge $A_0 = 0$ (which commutes with rotations).

Choosing the action on the bundle to be trivial, an equivariant connection takes the form

$$A(x)^\alpha_{\mu\beta} = \left(\frac{x^*}{r}\right)^\nu{}_\mu A(|x|q^0)^\alpha_{\nu\beta}. \tag{3.1.6}$$

This Ansatz is related to (3.1.5) by the singular gauge transformation $\tilde{x} = (x/r)$.[5]

[4] In fact, the example of $\overline{\mathbb{CP}^2}$ with this action led Donaldson to realize the general structure of the ASD moduli spaces!

[5] This gauge change has the effect (according to (3.1.1))

$$A_i^{3.1.5}(rq^0) = A_i^{3.1.6}(rq^0) - \frac{q^i}{r}$$

so smooth connections in one gauge have a pole in the other.

3.2 Evolution of $SU(2)$-equivariant connections

We derive the Yang-Mills flow equation for the $SU(2)$-equivariant connection (3.1.6). The relevant computations now hinge on the matrix coefficient

$$D^{\nu}{}_{\mu} := \left(\frac{x^*}{r}\right)^{\nu}{}_{\mu} = \frac{-\epsilon_{\nu\mu k}x_k + \delta_{\nu 0}x_{\mu} - \delta_{\mu 0}x_{\nu} + \delta_{\mu\nu}x_0}{r}$$

and its derivatives

$$\begin{aligned} D^{\nu}{}_{\mu}|_{rq^0} &= \delta_{\mu\nu} & \partial_{\alpha}D^{\nu}{}_{\mu}|_{rq^0} &= \frac{1}{r}\left(\epsilon_{\alpha\mu\nu} + \delta_{\nu 0}\delta_{\alpha\mu} - \delta_{\mu 0}\delta_{\alpha\nu}\right) \\ \partial_j\partial_k D^{\nu}{}_{\mu}|_{rq^0} &= \frac{-\delta_{jk}\delta_{\mu\nu}}{r^2} & \partial_0\partial_{\alpha}D^{\nu}{}_{\mu}|_{rq^0} &= \frac{-1}{r^2}\left(\epsilon_{\alpha\mu\nu} + \delta_{\nu 0}\delta_{\alpha\mu} - \delta_{\mu 0}\delta_{\alpha\nu}\right). \end{aligned}$$

We remind the reader that Latin indices are reserved for the values $1, 2, 3$, while Greek indices take values $0, 1, 2, 3$. Recall also that for any radial function

$$\partial_{\alpha}\partial_{\beta}f(r)|_{rq^0} = f''(r)\delta_{0\alpha}\delta_{0\beta} + \frac{f'(r)}{r}\left(\delta_{\alpha\beta} - \delta_{0\alpha}\delta_{0\beta}\right).$$

The flow equation[6] is derived by repeat application of the previous formulae and Leibniz rule to (3.1.6).

$$\begin{aligned} \partial_t A_i|_{rq^0} &= \nabla_0^2 A_i(r) + \frac{3}{r}\nabla_0 A_i(r) - \frac{3}{r^2}A_i + \frac{3\epsilon_{ijk}}{r}\left[A_k, A_j\right] + \left[A_{\ell}\left[A_{\ell}, A_i\right]\right] \\ &= A_i''(r) + \frac{3}{r}A_i'(r) - \frac{3}{r^2}A_i + \frac{3\epsilon_{ijk}}{r}\left[A_k, A_j\right] + \left[A_{\ell}\left[A_{\ell}, A_i\right]\right] \\ &\quad + \left[A_0'(r), A_i\right] + 2\left[A_0, A_i'(r)\right] + \frac{3}{r}\left[A_0, A_i\right] + \left[A_0\left[A_0, A_i\right]\right] \qquad (3.2.1) \\ \partial_t A_0|_{rq^0} &= \left[\nabla_0 A_i, A_i\right] \\ &= \left[A_i'(r), A_i\right] + \left[A_i, \left[A_i, A_0\right]\right]. \end{aligned}$$

[6] A different choice of metric on the base (e. g. S^4 or $\mathbb{CP}^2$) does not affect the bundle curvature F_A, but produces some extra terms from the adjoint D_A^*. These scale away when blowing up around a point, however, and would appear to be of secondary importance for finite-time dynamics.

3.2.1 Evolution of ASD curvature

The evolution of $|F^-|^2$ is obtained from (2.2.8)

$$\frac{\partial}{\partial t}|F^-|^2 = \Delta|F^-|^2 - 2|\nabla F^-|^2 + 4\langle F^-_{\mu\nu}, \left[F^-_{\mu\lambda}, F^-_{\lambda\nu}\right]\rangle. \tag{3.2.2}$$

To apply these in the equivariant case, we compute

$$\begin{aligned}\partial_k F^{\pm}_{\mu\nu}\big|_{rq^0} &= \partial_k \left(D^\eta{}_\mu D^\lambda{}_\nu F^{\pm}_{\eta\lambda}(r)\right) \\ &= \frac{1}{r}\left(\epsilon_{k\mu\eta} + \delta_{\eta 0}\delta_{k\mu} - \delta_{\mu 0}\delta_{k\eta}\right) F^{\pm}_{\eta\nu} + \frac{1}{r}\left(\epsilon_{k\nu\lambda} + \delta_{\lambda 0}\delta_{k\nu} - \delta_{\nu 0}\delta_{k\lambda}\right) F^{\pm}_{\mu\lambda}\end{aligned}$$

which gives

$$\nabla_k F^{\pm}_{0j}\big|_{rq^0} = \frac{1}{r}\left(\epsilon_{kj\lambda}F^{\pm}_{0\lambda} - F^{\pm}_{kj}\right) + \left[A_k, F^{\pm}_{0j}\right].$$

In particular we have the divergent results

$$\begin{aligned}\nabla_k F^{+}_{0j}\big|_{rq^0} &= \left[A_k, F^{+}_{0j}(r)\right] \\ \nabla_k F^{-}_{0j}\big|_{rq^0} &= \frac{2\epsilon_{kj\ell}}{r}F^-_{0\ell} + \left[A_k, F^-_{0j}\right].\end{aligned} \tag{3.2.3}$$

For equivariant connections, we obtain

$$(\partial_t - \Delta)\,|F^-|^2\ \big|_{rq^0} = -2|\nabla_0 F^-|^2 - 8\sum_{k,j}\left|\frac{2\epsilon_{kj\ell}}{r}F^-_{0\ell} + \left[A_k, F^-_{0j}\right]\right|^2 + 16\epsilon_{jk\ell}\langle F^-_{0j}, \left[F^-_{0k}, F^-_{0\ell}\right]\rangle. \tag{3.2.4}$$

Note that for (3.2.4) to hold for a smooth solution as $r \to 0$, we must have $F^-(0,t) = 0$ for all time. We have also the following.

Theorem 3.2.1. *Any smooth $SU(2)$-equivariant Yang-Mills connection is self-dual.*[7]

Proof. Rewrite (3.2.3)

$$\epsilon_{kj\ell}\nabla_k F^-_{0j} = \frac{4}{r}F^-_{0\ell} + \epsilon_{kj\ell}\left[A_k, F^-_{0j}\right].$$

[7] Here we have chosen the action on the bundle to be trivial, hence the connection takes the form (3.1.6) as opposed to (3.1.5), in which case the result would be anti-self dual. Further arguments show that the result must be a standard instanton (3.1.2), up to the scale λ and a constant gauge transformation.

The Yang-Mills equation $D^*F = 0$ reads

$$\begin{aligned}\nabla_0 F^-_{0j} &= -\nabla_i F^-_{ij} \\ &= \nabla_i F^-_{ji} \\ &= \nabla_i \left(-\epsilon_{kji} F^-_{0k}\right) \\ &= -\epsilon_{ikj} \nabla_i F^-_{0k} \\ &= -\frac{4}{r} F^-_{0j} - \epsilon_{ikj} \left[A_i, F_{0k}\right].\end{aligned} \tag{3.2.5}$$

Contracting with F^-_{0j} and writing $u = |F^-|^2$ gives the ODE

$$\frac{du}{dr} + \frac{8}{r} u = F^- \# F^-$$

with # a smooth bilinear function (depending on A). Therefore

$$\left|\frac{d\left(r^8 u\right)}{dr}\right| \leq C r^8 u$$

and $u \equiv 0$. [8] □

3.3 Systems with several parameters

The additional complexity of the $SU(2)$-equivariant case over the $SO(4)$ case is twofold.

(A) The connection has independent components A_μ for the base directions.

(B) These components each have three parameters (the coefficients of $q^i, i > 0$).

It is possible, however, to add on these complications independently. Taking $A_0 = 0$ and

$$A_i = \frac{f_i(r,t)}{r} q^i$$

[8] For the system (C), this was shown by Parker [22]; although in the same paper he bypasses the Theorem by adding a rotationally symmetric perturbation to the metric. Evidently the perturbed metric is not diagonal away from the origin (in normal coordinates based at the origin), and the radial derivative cannot be isolated and controlled as in (3.2.5).

for $i = 1, 2, 3$, the system becomes

$$\dot{f}_i = f_i''(r) + \frac{1}{r} f_i'(r) - \frac{4}{r^2} f_i + \frac{12}{r^2} f_j f_k - \frac{4}{r^2}(f_j^2 + f_k^2) f_i \tag{A}$$

where $i = 1, 2, 3$ and i, j, k are distinct; moreover, the Ansatz is preserved by the flow.[9]

On the other hand, the Ansatz

$$\begin{aligned} A_0(r) &= w(r)(q^1 + q^2 + q^3) \\ rA_1(r) &= x(r)q^1 + y(r)q^2 + z(r)q^3 \\ rA_2(r) &= z(r)q^1 + x(r)q^2 + y(r)q^3 \\ rA_3(r) &= y(r)q^1 + z(r)q^2 + x(r)q^3 \end{aligned}$$

is also preserved. One can see this from invariance of the system of equations under cyclic permutation of the indices, or by observing that these are exactly the equivariant connections for $SU(2) \times \mathbb{Z}_3$. The system becomes

$$\begin{aligned} \dot{x} &= x''(r) + \frac{1}{r} x'(r) + \frac{1}{r^2}(-4x((2x-1)(x-1) + (y-z)^2) + 4yz(z+y-3)) \\ &+ \; 2\left(\left(w'(r) + \frac{1}{r} w(r)\right)(y-z) + 2w\,(y'(r) - z'(r))\right) + 4w^2(y+z-2x) \\ \dot{y}, \dot{z} &= \text{cyclic} \\ \dot{w} &= \frac{2}{r}\left((x-z)(-2(x-z)w + y') + \text{cyclic}\right). \end{aligned} \tag{B}$$

Observe that the conditions

$$w(r) = 0 \qquad y(r) = z(r) =: \frac{s(r)}{\sqrt{2}}$$

[9] Attempts to find a compact Ansatz along the lines of (3.1.3) for the system (A) yielded the expression

$$A_1(x) = \frac{1}{r^4}\left(-r^2 \vec{f} \cdot \text{Im}\,(q^1 x^*) + x_2 x_3 (f_2 - f_3) \text{Im}\, x^* + x_0 x_3 (f_1 - f_2) \text{Im}\,(q^2 x^*) - x_0 x_2 (f_1 - f_3) \text{Im}\,(q^3 x^*)\right)$$

where we put $(1 - f_i)/2$ in place of f_i to conform with (3.1.5) and [29]. This shows that for a smooth connection of the form (A) we must have $f_i - f_j = O(r^4)$ as $r \to 0$. For (B) we should likewise have $y, z = O(r^4)$ under the boundary condition $x(0) = 1$.

are preserved by (B). Writing $X = x$, the system becomes

$$\begin{aligned}\dot{X} &= X'' + \frac{1}{r}X' - \frac{1}{r^2}\left(4X(2X-1)(X-1) + 2s^2(\sqrt{2}s - 3)\right) \\ \dot{s} &= s'' + \frac{1}{r}s' - \frac{4}{r^2}s\left(\frac{3}{2}(s-\sqrt{2})(s-\sqrt{2}X) + 1\right).\end{aligned} \tag{C}$$

There is a way to simplify (B) by exploiting the continuous gauge symmetry of the family. The Ansatz, although not the individual connections, is preserved under the gauge transformation

$$u^\theta = \cos(\theta)q^0 + \sin(\theta)(q^1 + q^2 + q^3)/\sqrt{3}.$$

This has the effect of rotating (x, y, z) counterclockwise around the axis $(1,1,1)/\sqrt{3}$ by the angle θ, which commutes with cyclic permutation $u^{2\pi/3}$. Let

$$\begin{aligned}(x,y,z) &= u^\theta(X, \sqrt{2}s, \sqrt{2}s) \\ d &= \sqrt{2}X - s \\ h &= X + \sqrt{2}s \\ q &= \theta'(r) + 2\sqrt{3}w(r).\end{aligned}$$

Here d and h are the distance from and height along the axis (up to scale), and the parameter q appears naturally in the calculations. The system in the variables X, s, w, θ can in fact be re-expressed

$$\begin{aligned}\dot{d} &= d'' + \frac{1}{r}d' + \frac{2}{r^2}d\left(-2 + 6h - (2h^2 + d^2)\right) - q^2 d \\ \dot{h} &= h'' + \frac{1}{r}h' + \frac{2}{r^2}\left(-2h + d^2(3-2h)\right) \\ \dot{q} &= q'' + \left(\frac{1}{r} + \frac{2d'}{d}\right)q' + \left(\frac{2d''}{d} - 2\frac{(d')^2}{d^2} - 4\frac{d^2}{r^2} - \frac{1}{r^2}\right)q.\end{aligned} \tag{B'}$$

The variables w and θ can be re-obtained from q by the formulae

$$w(r,T) = -\int_0^T \frac{2\sqrt{3}}{3r^2}(\sqrt{2}X - s)^2 q \; dt$$

$$\theta(r,T) = \int_0^T \left(q' + \left(\frac{1}{r} + 2\frac{\sqrt{2}X' - s'}{\sqrt{2}X - s} \right) q \right) \, dt.$$

Upon setting $q = 0$ one obtains the 2-parameter system

$$\begin{aligned} \dot{d} &= d'' + \frac{1}{r}d' + \frac{2}{r^2}\, d\left(-2 + 6h - (2h^2 + d^2)\right) \\ \dot{h} &= h'' + \frac{1}{r}h' + \frac{2}{r^2}\left(-2h(1+d^2) + 3d^2\right). \end{aligned} \tag{C'}$$

Remark 3.3.1. As a partial check on these derivations, one can place

$$\frac{f(r)}{2} = \begin{cases} f_i(r)\,\forall\, i & \text{(A)} \\ X(r), s = 0 & \text{(B)} \\ h(r) = d(r)/\sqrt{2} & \text{(C')} \end{cases}$$

in order to recover the scalar equation of the $SO(4)$-equivariant case

$$\dot{f} = f''(r) + \frac{1}{r}f'(r) - \frac{2}{r^2}f(f-1)(f-2). \tag{YM}$$

The standard instanton (3.1.2) corresponds to

$$f(r) = Q(r) = \frac{2r^2}{1+r^2}.$$

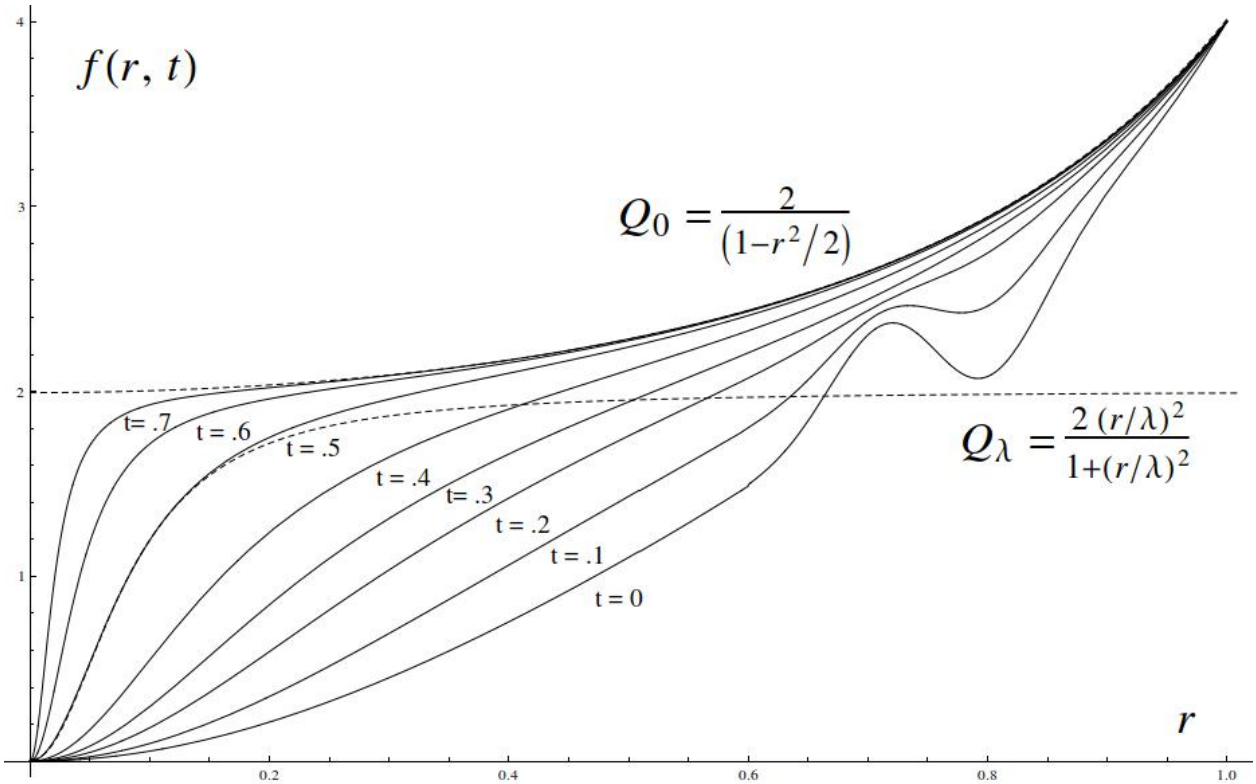
It follows from the formula

$$F_{0j}^{\pm} = \frac{1}{2r}\left(f'(r) \pm \frac{2f(2-f)}{r} \right) q^j \tag{3.3.1}$$

that this is self-dual.

3.3.1 Qualitative description

The following is a schematic picture of blowup for (YM). The asymptotics as $t \to \infty$ will be rigorously justified in the next chapter, for initial data of the type shown.



The solution searches for a critical point of the Yang-Mills functional; but none exist with the given boundary conditions $f(0) = 0, f(1) = 4$. Instead, $f(r, t)$ attempts to interpolate between the two instantons Q_λ and Q_0 (which are self-dual and anti-self-dual, respectively, according to (3.3.1)). Provided that $\lambda \to 0$, this is possible: $f(r, t)$ approaches Q_0 on the original scale, and Q_λ on the scale λ. Thus a singularity (or "bubble") forms in a highly controlled fashion at the origin, allowing the connection to change between the topological classes (3.1.6) and (3.1.5) as $t \to \infty$.

It is not known whether singularities of the Yang-Mills flow are always of this mild type. The behavior is difficult to see even in the $SU(2)$-equivariant case, where, though each scarcely removed from (YM), the two systems (A) and (C') would appear quite different. In both these cases, however, calculations directly comparing with (YM), as well as computer simulations, did not appear to indicate finite-time blowup.[10] This can be explained within the framework of the thesis.

The general blowup behavior of Yang-Mills flow in dimension four is as follows (largely

[10] Computer simulations were conducted using a Crank-Nicolson algorithm programmed in Mathematica.

justified by [27], [32]). Whether at finite or at infinite time T, an interior singularity must consume at least ϵ_0 of energy. If one rescales on a sequence of times $t_i \to T$ such that this energy is attained on a unit ball, then, from standard compactness theory, a Yang-Mills connection must appear as a subsequential limit modulo gauge. In fact this limit can be extended to all of $\mathbb{R}^4$. After removing this innermost Yang-Mills bubble, one can obtain additional limits on a further subsequence of times. Ultimately, one would expect to see a full "bubble tree," accounting for all of the energy concentrating at the singular point.

In the $SU(2)$-equivariant case, it is still clear that a singularity can only occur at the origin, and hence that the blowup limit should again be equivariant. But Theorem 3.2.1 implies that it must be self-dual (in fact a standard instanton, as in the above picture). And the results of Chapter 5, in particular Theorem 5.1.4, imply that bubbles of pure positive (or negative) charge cannot form in finite time.

There remains the possibility of a tree of self-dual and anti-self-dual bubbles concentrating at the origin at the same time. In such a case, however, there should always be an outermost bubble, which is again an equivariant connection (say) of pure negative charge. On the scale of this bubble, an L^∞ bound on the positive curvature should hold, sufficient (by Theorem 5.1.4) to imply that the full curvature does not concentrate in finite time. We expect that this argument could be made rigorous, ruling out finite-time singularities in the $SU(2)$-equivariant case.[11]

[11] This description is supported by (C'), also studied by Parker [22] (not known to the author at the time of writing) in his construction of non-self-dual Yang-Mills connections on S^4 with perturbed metric. With the boundary conditions

$$d(0) = \sqrt{2} \qquad d(1) = -\sqrt{2}$$
$$h(0) = h(1) = 1$$

a nontrivial bubble-tree must form; however, the outermost bubble still moves at most exponentially. The boundary condition $d(\infty) = -\sqrt{2}$ is also valid on S^4, corresponding to a nontrivial family of connections on the trivial bundle. This provides a compact example where an infinite-time singularity must form, despite the existence of a (trivial) minimizer on the same bundle.

3.3.2 Proof in $SO(4)$-equivariant case

We sketch a proof, using the main result of Chapter 5, of the exponential blowup result of Schlatter, Struwe, and Tahvildar-Zadeh [29] for (YM).

Let $f(r,t)$ be a solution of (YM) with smooth initial data and Dirichlet boundary conditions $f(0)=0$ and $f(1)=\alpha$. Write $A_j(r) = \frac{f(r)}{2r} q^j$ and

$$F^-_{0j}(r) = \frac{1}{2r}\left(f'(r) - 2\frac{f(2-f)}{r}\right) q^j =: F(r) q^j.$$

Equation (3.2.4) reads

$$(\partial_t - \Delta) F^2 = -2|\partial_r F|^2 - C_1 \left(\frac{f-2}{r} F\right)^2 - C_2 F^3. \tag{3.3.2}$$

Assume first that $F(r,0) \geq 0$. By the maximum principle, $F(r,t) \geq 0$ for $t \geq 0$. But then the right-hand-side of (3.3.2) is non-positive, and we conclude that $F(r,t) \leq \sup_r F(r,0)$. Therefore $|F^-|$ is uniformly bounded for all time. By Theorem 5.1.4 (an interior estimate), blowup occurs at most exponentially.

If $F(r,0)$ has mixed sign, there nonetheless exists a smooth function $\bar{f}_0(r)$ with nonnegative F such that $\bar{f}_0(r) \geq f(r,0)$ (see e. g. Proposition 4.2.2 of the next chapter). The flow $\bar{f}(r,t)$, with initial data $\bar{f}(r,0) = \bar{f}_0(r)$, forms an upper barrier. For fixed $\mu > 0$ sufficiently small, the static solution $\underline{f}(r,t) = -\frac{2(r/\mu)^2}{1-(r/\mu)^2}$ forms a lower barrier. By the maximum principle applied directly to (YM), we conclude

$$|f(r,t)| \leq C e^{kt} r^2.$$

By Lemma 4.1.1 of the next chapter, the solution exists smoothly for all time.

4. ASYMPTOTICS OF INFINITE-TIME BLOWUP

This chapter is concerned with blowup asymptotics for the semi-linear heat equation

$$\partial_t u(r,t) = \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{f(u)}{r^2} \tag{4.0.1}$$

on the unit interval with Dirichlet boundary conditions

$$u(0) = 0 \qquad u(1) = \alpha. \tag{4.0.2}$$

We assume that the nonlinear term is of the form

$$f(u) = 4g \cdot g'(u) \tag{4.0.3}$$

for a smooth function $g(u)$. For simplicity, in the present treatment we let g have only the two zeroes[1] $g(0) = g(2) = 0$, with

$$g'(0) = 1 \qquad g'(2) = -1. \tag{4.0.4}$$

We also take the boundary condition

$$u(1) = \alpha > 2. \tag{4.0.5}$$

For a discussion of the origins of the problem, the reader is referred to the Introduction.

Our main result is the following.

[1] In the harmonic map case $g(u) = \sin u$, our proofs apply for $\pi < \alpha < 2\pi$.

Theorem 4.0.2. *Let $u_0 \in C^2([0,1])$, satisfying*

$$u_0 = O(r^2) \quad (r \to 0) \qquad u_0(1) = \alpha \qquad 0 \leq u_0 \leq Q_0. \tag{4.0.6}$$

With the above assumptions (4.0.2) - (4.0.5), there exists a unique global solution $u(r,t)$ to (4.0.1) with $u(r,0) = u_0(r)$, which blows up at the origin as $t \to \infty$ with the rate

$$ce^{2\frac{\kappa}{e_0}t} \leq \sup_{0<r\leq 1} \frac{u(r,t)}{r^2} \leq Ce^{2\frac{\kappa}{e_0}t}.$$

Here $\kappa, e_0 > 0$ are the constants defined in Lemma 4.1.4 and (4.1.5), respectively, and depend only on $g(u)$ and α. The constants c and C depend on u_0.

4.0.3 Outline of argument

We introduce a version of the scheme of Raphael and Schweyer [24] which is sufficient for studying infinite-time blowup, via the maximum principle. A very useful picture is given in §3.3.1 above.

Let $Q(r)$ be a static solution of (4.0.1), and $Q_\lambda = Q(r/\lambda)$. In the geometric cases we have

$$Q(r) = \begin{cases} 2\arctan(r^2) & \text{(HM)} \\ \dfrac{2r^2}{1+r^2} & \text{(YM).} \end{cases}$$

The former corresponds to the harmonic map $\mathbb{R}^2 \to S^2$ obtained by composing the holomorphic map $z \to z^2$ with the inverse stereographic projection, and the latter to the standard instanton on $\mathbb{R}^4$ (Example 3.1.2). Denote the scaling operator

$$\Lambda u = r \cdot \partial_r u = - \left. \partial_\lambda u\,(r/\lambda)\right|_{\lambda=1}.$$

In Section 4.2, we construct approximate solutions. This is achieved by solving the nonlinear ODE

$$\frac{d^2u_\lambda}{dr^2} + \frac{1}{r}\frac{du_\lambda}{dr} - \frac{f(u_\lambda)}{r^2} = b_\lambda \Lambda Q_\lambda$$
$$u_\lambda \sim Q_\lambda \quad (r \to 0) \tag{$*$}$$
$$u_\lambda(1) = \alpha$$

by a fixed-point method (Proposition 4.2.1), yielding a family of solutions

$$(u_\lambda(r), b_\lambda)$$

parametrized by λ. Here the coefficient b_λ is chosen in order to satisfy the boundary condition at $r = 1$, and by an elementary matching with Q_0, must tend to the constant κ/e_0 as $\lambda \to 0$ (Proposition 4.2.2).

In Section 4.3, we show that this family is smooth and well-behaved for $0 < \lambda < \lambda^*$. In lieu of the matched asymptotics of [3] or the slowly modulated series and cutoffs of [24], our estimates make direct use of a nonlinear conservation-law-type reduction of $(*)$.

We then obtain sub- and super-solutions to (4.0.1) of the form

$$u_\pm(r,t) = u_{\lambda(t)}(\mu(t)r) \pm \epsilon(t) T_0(\mu(t)r)$$

Here $\lambda = \lambda_\pm(t)$ is chosen with $\lambda(0) = \lambda^* << 1$ and

$$\frac{d}{dt}\log\lambda = -b_\lambda + O(\lambda^a).$$

The additional term ϵT_0 is needed to obtain a sub/supersolution on the scale $\lambda^{1/2} < r \leq 1$, but requires the extra factor $\mu(t) \approx 1$ in order to satisfy the boundary condition $u_\pm(1) = \alpha$.

4.1 Preliminaries

Lemma 4.1.1. *Let $u_0 \in C^2([0,1])$ with $u_0 = O(r^2)$ as $r \to 0$ and $u_0(1) = \alpha$. There exists a unique short-time solution to (4.0.1) with $|u(r,t)| \leq Cr^2$ and $u(r,0) = u_0(r)$. The solution* *exists and remains smooth as long as such a C exists.*

Proof. By Taylor's Theorem, we may write $u_0 = r^2 v_0$, with $v_0 \in C^0([0,1])$.

Letting $u = r^2 v$, an elementary computation (c. f. [29]) shows that (4.0.1) is equivalent to

$$\partial_t v = \partial_r^2 v + \frac{5}{r}\partial_r v + v^2 \tilde{f}\left(r^2 v\right) \tag{4.1.1}$$

for a smooth function $\tilde{f}(u)$. If $v(r)$ is viewed as a radially symmetric function on the unit ball of $\mathbb{R}^6$, then (4.1.1) is simply a heat equation with smooth zeroth-order nonlinearity. Hence by standard parabolic theory, a unique solution $v(r,t)$ with $v(r,0) = v_0(r)$ exists, is smooth for $t > 0$, and continues as long as $||v(\cdot,t)||_{L^\infty} \le C$. This amounts to the desired statements for the solution $u(r,t) = r^2 v(r,t)$ of (4.0.1). □

Remark 4.1.2. Because the nonlinearity of (4.1.1) is bounded as long as the solutions exist, a comparison principle holds, and so too for the solutions $u = r^2 v$ of (4.0.1).

Definition 4.1.3. Let $\Lambda = r\partial_r$ be the scaling operator, as above. We will write

$$f(r) = g(r) + O_\Lambda(h(r)) \quad (r \to 0)$$

if

$$\Lambda^i f(r) = \Lambda^i g(r) + O(\Lambda^i h(r)) \quad (r \to 0)$$

for $0 \le i < \infty$, and similarly as $r \to \infty$. Observe that for any $a, b \in \mathbb{R}$, there holds

$$\Lambda\left(r^a (\log r)^b\right) = O_\Lambda\left(r^a (\log r)^b\right)$$

both as $r \to 0$ and $r \to \infty$.

Lemma 4.1.4. *There is a unique static, smooth, increasing solution $Q(r)$ to (4.0.1) with* *the asymptotics*

$$Q(r) = \begin{cases} cr^2 + O_\Lambda(r^3) & (r \to 0) \\ 2 - 2r^{-2} + O_\Lambda(r^{-3}) & (r \to \infty). \end{cases}$$

There also exists a unique solution Q_0 *with*

$$Q_0 \sim 2 + \kappa r^2 \quad (r \to 0)$$

$$Q_0(1) = \alpha > 2$$

where $\kappa > 0$ *is a constant.*

Proof. Set $\partial_t u = 0$, and multiply (4.0.1) by $r\Lambda u = r^2 \partial_r u$ to obtain

$$\begin{aligned} 0 &= (\Lambda u)\,(\partial_r \Lambda u) - \partial_r u f(u) \\ &= \frac{1}{2}\,\partial_r((\Lambda u)^2 - 4g(u)^2) \\ A &= (\Lambda u)^2 - 4g(u)^2. \end{aligned} \tag{4.1.2}$$

Since g is positive on $(0, 2)$, any solution $\Lambda Q = 2g(Q)$ with $Q(1) \in (0, 2)$ must satisfy $\lim_{r\to 0} Q(r) = 0$ and $\lim_{r\to\infty} Q(r) = 2$ (see [24] for the asymptotics).

Let Q_0 be the unique solution of $\Lambda Q_0 = -2g(Q_0)$ with $Q_0(1) = \alpha > 2$. □

Consider the linearized Hamiltonian at Q (see [24] for its factorization properties)

$$H = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{f'(Q)}{r^2}.$$

By scale-invariance, ΛQ is in the kernel. The equation $Hu = 0$ admits also the singular solution

$$\Gamma(r) = \Lambda Q \int_1^r \frac{dx}{x \Lambda Q^2} = \begin{cases} -\dfrac{1}{16r^2} + O_\Lambda(r^2 \log r) & (r \to 0) \\ \dfrac{r^2}{16} + O_\Lambda\left(\dfrac{\log r}{r^2}\right) & (r \to \infty) \end{cases}$$

The equation $Hu = -f$ can be solved at the origin by the formula

$$u = \Gamma(r) \int_0^r f \Lambda Q \, x dx - \Lambda Q(r) \int_0^r f \Gamma \, x dx. \tag{4.1.3}$$

Lemma 4.1.5. *Let $k > m \geq 0$, $\ell \geq 0$ with $k - \ell \neq 0, -4$, and assume that*

$$f \in C^m([0, M]) \cap C^n((0, M])$$

satisfies

$$|\Lambda^i f(r)| \leq K \frac{r^k}{1 + r^\ell} \quad (i \leq n)$$

for some constant K. Then u given by (4.1.3) solves $Hu = -f$ uniquely with $u = o(r^2)$ as $r \to 0$. Moreover

$$u \in C^{m+2}([0, M]) \cap C^{n+2}((0, M])$$

with

$$|\Lambda^i u(r)| \leq C_i K \frac{r^{k+2}}{1 + r^{\min(k,\ell)}} \quad (i \leq n + 2).$$

Here C_i depends only on k, ℓ.

Proof. The first statements are standard, in view of the above asymptotics for ΛQ and Γ. Next, let $r \leq 1$ and note that

$$\begin{aligned} |u(r)| &\leq K \left(|\Gamma| \int_0^r x^k \Lambda Q \, x dx + \Lambda Q \int_0^r x^k |\Gamma| \, x dx \right) \\ &\leq C K r^{k+2} \end{aligned}$$

since $k > 0$ in the second integral (recall also that $\Lambda Q \geq 0$). For $r \geq 1$, we have

$$\begin{aligned} |u(r)| &\leq CK \left(r^2 \left(1 + \int_1^r x^{k-\ell-1} \, dx \right) + \frac{1}{r^2} \left(1 + \int_0^r x^{k-\ell+3} \, dx \right) \right) \\ &\leq CK \left(r^2 + r^{k-\ell+2} \right) \end{aligned}$$

given the assumptions on k and ℓ. The estimates for $i > 0$ follow similarly after applying

$\Lambda = r\partial_r$. □

In view of Lemma 4.1.5, we will write $H^{-1}f = -u$. Letting

$$T_1 = -H^{-1}\Lambda Q \tag{4.1.4}$$

we have

$$T_1 = \begin{cases} c_0 r^4 + O_\Lambda(r^6) & r \to 0 \\ e_0 r^2 - 1 + O_\Lambda(\log(r)/r^2) & r \to \infty \end{cases} \tag{4.1.5}$$

with $c_0, e_0 > 0$.

4.2 Construction of approximate solutions

We now aim to solve $(*)$. We first solve near the fixed solution Q with arbitrary b, calling these solutions u_b. Then we rescale and relabel to obtain the desired approximate solutions u_λ satisfying the boundary condition $u_\lambda(1) = \alpha$.

Recall that $Q(r)$ and $T_1(r)$ are defined in Lemma 4.1.4 and (4.1.4), respectively.

Proposition 4.2.1. *There exists $\epsilon^* > 0$ such that for any $b, M > 0$ with*

$$bM^2 = \epsilon \leq \epsilon^*$$

a unique solution u_b exists on $[0, M]$ to the ODE

$$\begin{gathered} \frac{d^2 u_b}{dr^2} + \frac{1}{r}\frac{du_b}{dr} - \frac{f(u_b)}{r^2} = b\Lambda Q \\ u_b \sim Q \text{ as } r \to 0. \end{gathered} \tag{4.2.1}$$

Writing $u_b = Q + \tilde{T}_b$, for $i \in \mathbb{N}$ there holds

$$\left|\Lambda^i\left(\tilde{T}_b - bT_1\right)\right| + b\left|\frac{d\Lambda^i\tilde{T}_b}{db} - \Lambda^i T_1\right| \leq C_i\, b^2 \frac{r^8}{1+r^4} \leq C_i\,\epsilon^2. \tag{4.2.2}$$

Proof. Let

$$R(z) = f(Q+z) - f(Q) - f'(Q)z.$$

By Taylor's Theorem, since $0 \le Q \le 2$, we have

$$|R(z)| \le \frac{1}{2}\left(\sup_{-1\le u\le 3} |f''(u)|\right) z^2 \qquad (|z| \le 1)\,.$$

To solve (4.2.1), write further

$$u_b = Q + \tilde{T}_b = Q + bT_1 + T_2$$

where $T_1 = -H^{-1}\Lambda Q$ is defined by (4.1.4). Equation (4.2.1) becomes

$$\begin{aligned} -H\tilde{T}_b + \frac{R(\tilde{T}_b)}{r^2} &= b\Lambda Q \\ HT_2 &= \frac{R(\tilde{T}_b)}{r^2} \\ T_2 &= H^{-1}\left(\frac{R(bT_1+T_2)}{r^2}\right) =: \Pi_b(T_2). \end{aligned} \tag{4.2.3}$$

Note that $|T_1| \le C\frac{r^4}{1+r^2}$ for all M. For $bM^2 \le (2C)^{-1}$, consider

$$B_{M,b} = \left\{ T \,\middle|\, |T(r)| \le b\frac{r^4}{1+r^2} \right\}.$$

For $T_2 \in B_{M,b}$ we have

$$\left|\frac{R(\tilde{T}_b)}{r^2}\right| \le Cb^2\frac{r^6}{1+r^4}.$$

Lemma 4.1.5 and (4.2.3) imply

$$|\Pi_b(T_2)| \le Cb^2\frac{r^8}{1+r^4} \le \left(CbM^2\right) b\frac{r^4}{1+r^2}$$

Next, note that

$$|R'(z)| \le C|z| \text{ for } |z| \le 1$$

so for $T, S \in B_{M,b}$ with $T - S \in B_{M,d}$, we likewise have

$$|\Pi_b(T_2) - \Pi_b(S_2)| \leq Cbd\frac{r^8}{1+r^4} \leq (CbM^2)d\frac{r^4}{1+r^2} \tag{4.2.4}$$

Hence for

$$bM^2 < C^{-1}/2 =: \epsilon^*$$

the map Π_b is a contraction on $B_{M,b}$ for the weighted norm

$$\left|\left|\frac{1+r^2}{r^4}T(r)\right|\right|_{C^0[0,M]}$$

and we have the desired solution $u_b = Q + bT_1 + T_2$ to (4.2.1). The stated estimates on T_2 follow from the equation $\Pi_b(T_2) = T_2$ and Lemma 4.1.5, and the derivative estimates follow in similar fashion.[2] □

Proposition 4.2.2. *There exists $\lambda^* > 0$ and a family $(u_\lambda(r), b_\lambda)$ of solutions to $(*)$, for $0 < \lambda < \lambda^*$, with*

$$\Lambda u_\lambda > 0 \quad (r > 0)$$

$$u_\lambda < Q_0 \quad (0 \leq r < 1)$$

$$\lim_{\lambda \to 0} b_\lambda = \frac{\kappa}{e_0}.$$

Proof. Applying Proposition 4.2.1 with $b = \tilde{b}\lambda^2$ and rescaling, we obtain a solution $u = u_{\tilde{b},\lambda}(r)$ to $(*)$ near the origin, smoothly varying in $\tilde{b}, \lambda > 0$. It remains to choose $\tilde{b} = b_\lambda$ to satisfy the boundary condition $u_\lambda(1) = \alpha$.

[2] The bounds can be improved if $f''(0) = 0 \Rightarrow R(z) = O(z^3)$ or higher, as in the case of the round sphere. We also note that the r^8 bound at the origin is not important: in fact one does not fully need the "rotation number" assumption (4.0.4) at $u = 0$ but only at $u = 2$, which in effect drives the blowup (and to which finite- versus infinite-time blowup was attributed by [14]).

As in the proof of Lemma 4.1.4, multiplying $(*)$ by $r\Lambda u = r^2\partial_r u$ yields

$$\begin{aligned}\frac{1}{2}\partial_r((\Lambda u)^2 - 4g(u)^2) &= \tilde{b}\Lambda Q_\lambda \Lambda u \cdot r \\ (\Lambda u)^2 - 4g(u)^2 &= 2\int_0^r \tilde{b}\Lambda Q_\lambda \Lambda u\, x\, dx =: A(r) \qquad (4.2.5) \\ \Lambda u(r) &= \sqrt{4g(u(r))^2 + A(r)}.\end{aligned}$$

Since $\Lambda Q, \Lambda u$, and $A(r)$ are positive for small $r > 0$, in view of this expression they are in fact positive for all $r > 0$, with $A(r)$ increasing. Therefore for each $\lambda < \lambda^*(\alpha)$ there exists a unique minimal $\tilde{b} > 0$ such that $u(1) = \alpha$, and we define $b_\lambda := \tilde{b}, u_\lambda := u$. As $\Lambda Q_\lambda = 2|g(Q_\lambda)|$ by construction, it is also clear from (4.2.5) that u_λ crosses each Q_λ at most once, and crosses Q_0 only at $r = 1$.

Next, change variables and apply Lemma 4.1.4 to obtain

$$A(r) = 2\tilde{b}\lambda^2\left(\int_0^M (\Lambda Q(x))(\Lambda u_\lambda(\lambda x))\, xdx + \int_M^{r/\lambda}(4/x^2)\left(1 + O(M^{-1})\right)(\Lambda u_\lambda(\lambda x))\; xdx\right)$$

In view of Proposition 4.2.1 and the fact that u_λ cannot cross Q_0 before $r = 1$, the first term tends to $||\Lambda Q||^2$ as $b = \tilde{b}\lambda^2 \to 0$ and $M \to \infty$. For the second term, note that $\frac{4}{r^2}r\Lambda u = 4\partial_r u \geq 0$. Hence for any i and $\epsilon > 0$ there exist $\lambda^*, M > 0$ such that

$$\left|\frac{\Lambda^i A(r)}{\tilde{b}\lambda^2} - 2\Lambda^i\left(||\Lambda Q||_{L^2}^2 + 4u(r)\right)\right| < \epsilon$$

for $0 < \lambda < \lambda^*, \lambda M < r \leq 1$.

In view of the fact that $A(r) \to 0$ in (4.2.5), by standard ODE we must have

$$\lim_{\lambda\to 0}\Lambda^i u_\lambda = \Lambda^i Q_0 \qquad (4.2.6)$$

uniformly on $[r_0, 1]$ for any $r_0 > 0$. Hence by Proposition 4.2.1, for any $r_0 > 0$ such that

$Q_0(r_0) < 2 + 2\kappa r_0^2$ and $2\kappa r_0^2 < \epsilon^*$, we must have

$$\lim_{\lambda \to 0} |b_\lambda \lambda^2 e_0(r_0/\lambda)^2 - \kappa r_0^2| < C(\kappa r_0^2)^2$$
$$\lim_{\lambda \to 0} |b_\lambda e_0 - \kappa| < C r_0^2.$$

Therefore $\lim_{\lambda \to 0} b_\lambda = \kappa / e_0$. □

4.3 Estimates on (*)

Lemma 4.3.1. *Let u_λ be the solution to (*) on $[0,1]$ constructed in Proposition 4.2.2. Then*

$$Q_0(r) - u_\lambda(r) \le C\lambda^2 \frac{1 - r^4}{r^2}$$

for $C\lambda^{1/2} \le r \le 1$ and $0 < \lambda < \lambda^*$.

Proof. Choose κ_0 such that $2 + \kappa_0 r^2 < Q_0(r)$ for $0 < r \le 1$, and let r_1 such that

$$2 + (\kappa_0/2) r^2 \le u_\lambda(r) \le Q_0$$

for $r_1 \le r \le 1$. Then for any u with $u_\lambda(r) \le u \le Q_0(r)$, we have also

$$C^{-1} r^2 \le |g(u)| \le C r^2$$

$$|g'(u) - 1| \le C r^2$$

for $r_1 \le r \le 1$ and C independent of r_1.

Now subtract $\Lambda Q_0 = 2|g(Q_0)|$ from (4.2.5) to obtain

$$\begin{aligned} \Lambda(u_\lambda - Q_0) &= \sqrt{4g(u_\lambda)^2 + A(r)} - 2|g(Q_0)| \\ |\Lambda(u_\lambda - Q_0) - 2\left(|g(u_\lambda)| - |g(Q_0)|\right)| &\le \frac{A(r)}{4|g(u_\lambda)|} \\ |\Lambda(u_\lambda - Q_0) - 2\left(1 + \epsilon(r)\right)(u_\lambda - Q_0)| &\le \frac{CA(r)}{r^2} \le \frac{Cb_\lambda \lambda^2}{r^2} \end{aligned} \tag{4.3.1}$$

with $|\epsilon(r)| \leq Cr^2$ for $r_1 \leq r \leq 1$.

Write $s = \log(r), \partial_s = \Lambda$, and define the integrating factor $I(r) = e^{-2\int_0^s (1+\epsilon(r))ds'}$. Because

$$\left| \int_0^s |\epsilon(r)| ds' \right| \leq \left| \int_0^s Cr^2 ds' \right| \leq C$$

it follows that

$$\frac{C^{-1}}{r^2} \leq I(r) \leq \frac{C}{r^2}.$$

The differential inequality (4.3.1) becomes

$$\left| \frac{d}{ds} \left(I \cdot (u_\lambda - Q_0) \right) \right| \leq CI(r) \frac{b_\lambda \lambda^2}{r^2} \leq C \frac{\lambda^2}{r^4}.$$

Since

$$u_\lambda(1) = Q_0(1) = \alpha$$

by definition, integration from $r = 1$ gives

$$\begin{aligned} I(r) \left(Q_0(r) - u_\lambda(r) \right) &\leq C\lambda^2 \left(\frac{1}{r^4} - 1 \right) \\ \left(Q_0(r) - u_\lambda(r) \right) &\leq C\lambda^2 \left(\frac{1}{r^2} - r^2 \right) \end{aligned} \tag{4.3.2}$$

for $r_1 \leq r \leq 1$. We may therefore choose r_1 such that

$$C \frac{\lambda^2}{r_1^2} \leq \frac{\kappa_0}{2} r_1^2$$

or $r_1 = C\lambda^{1/2}$. □

Proposition 4.3.2. *For $C\lambda^{1/2} \leq r \leq 1$ and $0 < \mu \leq \lambda < \lambda^*$, there holds*

$$|u_\lambda - u_\mu| \leq C\lambda^2 \frac{1 - r^4}{r^2} \left(|b_\lambda - b_\mu| + \frac{\lambda - \mu}{\mu} \right).$$

Proof. We will assume $C\lambda^{1/2} \leq r$ as in the previous Lemma. From (4.2.5) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dr}\left(\left(\Lambda u_\lambda + \Lambda u_\mu\right)\Lambda(u_\lambda - u_\mu) - 4\left(g(u_\lambda) + g(u_\mu)\right)\left(g(u_\lambda) - g(u_\mu)\right)\right)\\ &\quad = b_\lambda \Lambda Q_\lambda \Lambda u_\lambda r - b_\mu \Lambda Q_\mu \Lambda u_\mu r\\ &\quad = b_\lambda \Lambda Q_\lambda \Lambda(u_\lambda - u_\mu) r + (b_\lambda - b_\mu)\Lambda Q_\mu \Lambda u_\mu r + b_\lambda\left(\Lambda Q_\lambda - \Lambda Q_\mu\right)\Lambda u_\mu r.\end{aligned}$$

Integrating from $r = 1$, we obtain

$$\Lambda(u_\lambda - u_\mu) - 4\frac{g(u_\lambda) + g(u_\mu)}{\Lambda u_\lambda + \Lambda u_\mu}\left(g(u_\lambda) - g(u_\mu)\right) = \frac{2}{\Lambda u_\lambda + \Lambda u_\mu} B(r). \tag{4.3.3}$$

Let $\kappa_1 > 0$, and $r_2 \geq r_1$ such that

$$|\Lambda(u_\lambda - u_\mu)| \leq \kappa_1 r^2 \tag{4.3.4}$$

for $r_2 \leq r \leq 1$. Then

$$\begin{aligned}|B(r)| &\leq C\left(b_\lambda \lambda^2 \kappa_1 + |b_\lambda - b_\mu|\mu^2 + b_\lambda \frac{(\lambda - \mu)}{\mu}\lambda^2\right)\int_r^1 \frac{1}{x^2}\cdot x^2 \cdot x dx\\ &\leq C\left(|b_\lambda - b_\mu|\mu^2 + \lambda^2\left(\kappa_1 + \frac{\lambda - \mu}{\mu}\right)\right).\end{aligned}$$

Hence for $r_2 \leq r \leq 1$, (4.3.3) reduces to

$$|\Lambda(u_\lambda - u_\mu) - 2\left(1 + \epsilon(r)\right)(u_\lambda - u_\mu)| \leq \frac{C}{r^2}\left(|b_\lambda - b_\mu|\mu^2 + \lambda^2\left(\kappa_1 + \frac{\lambda - \mu}{\mu}\right)\right).$$

We integrate as in the previous Lemma to conclude

$$|u_\lambda - u_\mu| \leq C\lambda^2\left(\frac{1}{r^2} - r^2\right)\left(|b_\lambda - b_\mu| + \kappa_1 + \frac{\lambda - \mu}{\mu}\right).$$

Substituting back to (4.3.3) we have also

$$|\Lambda(u_\lambda - u_\mu)| \leq C\frac{\lambda^2}{r^2}\left(|b_\lambda - b_\mu| + \kappa_1 + \frac{\lambda - \mu}{\mu}\right).$$

Therefore our assumption (4.3.4) is justified for

$$C\lambda^2\left(|b_\lambda - b_\mu| + \kappa_1 + \frac{\lambda-\mu}{\mu}\right) \leq \kappa_1 r_2^4 \leq \kappa_1 r^4$$

which holds for $\kappa_1 = \max\left(|b_\lambda - b_\mu|, \frac{\lambda-\mu}{\mu}\right)$ and $r_2 = C\lambda^{1/2}$. □

Proposition 4.3.3. *The family* (u_λ, b_λ) *is smooth for* $0 \leq r \leq 1$ *and* $0 < \lambda < \lambda^*$, *with*

$$\left|\frac{db_\lambda}{d\lambda}\right| \leq C\lambda$$

$$\left|\frac{\partial T_\lambda}{\partial \lambda}\right|(r) \leq C\lambda \min\left(\left(\frac{r}{\lambda}\right)^4, 1\right)$$

where $u_\lambda = Q_\lambda + T_\lambda$.

Proof. Recalling that $T_\lambda = \tilde{T}_b(r/\lambda)$, with $b = b_\lambda\lambda^2$, for

$$b(r/\lambda)^2 = b_\lambda r^2 \leq \epsilon^* \tag{4.3.5}$$

we write

$$\begin{aligned} T_\lambda - T_\mu &= \left(\tilde{T}_{b_\lambda\lambda^2}(r/\lambda) - \tilde{T}_{b_\mu\mu^2}(r/\lambda)\right) + \left(\tilde{T}_{b_\mu\mu^2}(r/\lambda) - \tilde{T}_{b_\mu\mu^2}(r/\mu)\right) \\ &= \left(b_\lambda\lambda^2 - b_\mu\mu^2\right)\left(e_0\left(r/\lambda\right)^2 + \eta(r)\right) - b_\mu\mu^2\frac{\lambda-\mu}{\mu}\left(2e_0\left(r/\lambda\right)^2 + \eta(r)\right) \end{aligned} \tag{4.3.6}$$

where the $\eta(r)$ represent generic bounded functions, by Proposition 4.2.1. Writing

$$b_\lambda\lambda^2 - b_\mu\mu^2 = b_\mu(\lambda^2 - \mu^2) + (b_\lambda - b_\mu)\lambda^2$$

gives the factorization

$$\begin{aligned} T_\lambda - T_\mu &= b_\mu(\lambda-\mu)(\lambda+\mu)e_0\left(r/\lambda\right)^2 - b_\mu(\lambda-\mu)\mu\left(2e_0\left(r/\lambda\right)^2\right) \\ &\quad + \left(b_\lambda - b_\mu\right)e_0\lambda^2\left(r/\lambda\right)^2 + \eta(r)\left(b_\mu(\lambda-\mu)(\lambda+\mu) + (b_\lambda-b_\mu)\lambda^2 + b_\mu\left(\lambda-\mu\right)\mu\right) \\ &= b_\mu\frac{(\lambda-\mu)^2}{\lambda^2}e_0r^2 + \left(b_\lambda-b_\mu\right)\left(e_0r^2+\eta(r)\lambda^2\right) + \eta(r)b_\mu(\lambda-\mu)\lambda. \end{aligned} \tag{4.3.7}$$

On the other hand, by the previous Proposition, for

$$C\lambda^{1/2} \leq r \leq 1 \tag{4.3.8}$$

we have

$$|T_\lambda - T_\mu| \leq |Q_\lambda - Q_\mu| + C\frac{\lambda^2}{r^2}\left(|b_\lambda - b_\mu| + \frac{\lambda-\mu}{\mu}\right)$$

and so

$$\left|b_\mu\frac{(\lambda-\mu)^2}{\lambda^2}e_0r^2 + \left(b_\lambda-b_\mu\right)\left(e_0r^2+\eta(r)\lambda^2\right) + \eta(r)b_\mu(\lambda-\mu)\lambda\right| \leq C\frac{\lambda^2}{r^2}\left(\frac{\lambda-\mu}{\mu} + |b_\lambda - b_\mu|\right).$$

Dividing by $\lambda - \mu$ yields

$$\frac{|b_\lambda - b_\mu|}{\lambda-\mu}\left(e_0r^2 + \eta(r)(\lambda/r)^2\right) \leq C\left(\frac{\lambda-\mu}{\lambda^2}r^2 + \frac{\lambda}{r^2}\right). \tag{4.3.9}$$

For $0 < \lambda < \lambda^*$, there exists a fixed r_0 with

$$C\lambda \leq b_\lambda r_0^2 < \epsilon^*$$

hence satisfying both (4.3.5) and (4.3.8), as well as

$$\eta(r_0)\lambda^2 < e_0r_0^4/2$$

for η in (4.3.9). Taking the limit $\mu \to \lambda$ gives $|\dot{b_\lambda}| \leq C\lambda$ in the Lipschitz sense. In the context of Lemma 4.2.2 (i. e. varying $\tilde{b} = b_\lambda$ independently of λ), (4.3.7) and Proposition 4.3.2 with $\mu \to \lambda$ clearly imply that b_λ is smooth, via the implicit function theorem.

The estimate on $\dot{T}_\lambda$ follows from (4.3.6) on $[0, \lambda]$, (4.3.7) on $[\lambda, r_0]$ and Proposition 4.3.2 on $[r_0, 1]$. □

4.4 Proof of Theorem 4.0.2

Proof. We construct sub- and super-solutions $u_\pm(r)$ of (4.0.1) by choosing a time-dependent λ (given by (4.4.5)) in the family u_λ of solutions to ($*$) constructed above, after adding a small modification on the unit scale.

Let T_0 be the solution of

$$\left(-\Delta + \frac{f'(Q_0)}{r^2}\right) T_0 = \Lambda Q_0$$

given by (4.1.3), with $T_0 = O(r^4)$ at the origin. For $\lambda(t) < \lambda^*$ and $\epsilon(t) < 1$ to be determined (4.4.5), write

$$\tilde{u}_\lambda = u_\lambda + \epsilon T_0 = Q_\lambda + T_\lambda + \epsilon T_0.$$

We compute

$$\begin{aligned}
(\partial_t - \Delta)\,\tilde{u}_\lambda + \frac{f(\tilde{u}_\lambda)}{r^2} &= \left[(\partial_t - \Delta)\,u_\lambda + \frac{f(u_\lambda)}{r^2}\right] + \frac{f(u_\lambda + \epsilon T_0) - f(u_\lambda)}{r^2} + (\partial_t - \Delta)\,\epsilon T_0 \\
&= \partial_t u_\lambda - b_\lambda \Lambda Q_\lambda + (\partial_t - \Delta)\,\epsilon T_0 + \frac{f'(u_\lambda)\epsilon T_0}{r^2} \\
&\quad + \frac{f(u_\lambda + \epsilon T_0) - f(u_\lambda) - f'(u_\lambda)\epsilon T_0}{r^2} \\
&= \left(-\frac{\lambda_t}{\lambda} - b_\lambda\right)\Lambda Q_\lambda + \lambda_t \partial_\lambda T_\lambda + \left(\partial_t - \Delta + \frac{f'(Q_0)}{r^2}\right)\epsilon T_0 \\
&\quad + \frac{f'(u_\lambda) - f'(Q_0)}{r^2}\epsilon T_0 + \frac{f(u_\lambda + \epsilon T_0) - f(u_\lambda) - f'(u_\lambda)\epsilon T_0}{r^2} \\
&= \left(-\frac{\lambda_t}{\lambda} - b_\lambda\right)\Lambda Q_\lambda + \lambda_t \partial_\lambda T_\lambda + \epsilon_t T_0 + \epsilon \Lambda Q_0 + R
\end{aligned} \tag{4.4.1}$$

where R comprises the two error terms. We estimate

$$\begin{aligned}|R(r)| &\leq \sup|f''|\left(\frac{|u_\lambda - Q_0||\epsilon T_0|}{r^2} + \frac{\epsilon^2 T_0^2}{2r^2}\right)\\ &\leq C\epsilon r^2\left(|u_\lambda - Q_0| + \epsilon r^4\right)\end{aligned}$$

and so

$$|R(r)| \leq C\epsilon \begin{cases}(\lambda^2 + \epsilon r^6) & C\lambda^{1/2} \leq r \leq 1\\ \lambda & \lambda \leq r \leq C\lambda^{1/2}\\ \lambda^2\left(\frac{r}{\lambda}\right)^2 & 0 \leq r \leq \lambda\end{cases} \tag{4.4.2}$$

where the first line is from Lemma 4.3.1.

We let

$$\epsilon_\pm = \pm\lambda_\pm^a$$

for a constant $a > 0$ to be determined. To obtain sub- and super-solutions with the boundary condition $u_\pm(1) = \alpha$, we must also scale $\tilde{u}$ slightly. For ϵ sufficiently small, i.e. $\lambda < \lambda^*$, $\tilde{u}'_\lambda(r)$ is uniformly positive for r near 1 (since u_λ approaches Q_0 and T_0 is fixed). Hence we may define $\mu_\pm$ depending on $\lambda_\pm$ such that

$$\tilde{u}_{\lambda_\pm}(\mu_\pm) = \alpha \tag{4.4.3}$$

which satisfies

$$|\mu_\pm - 1| < C\epsilon_\pm. \tag{4.4.4}$$

For K_1 to be determined, we finally define $\lambda_\pm(t)$ to solve

$$-\frac{(\lambda_\pm)_t}{\lambda_\pm} - \mu_\pm^2 b_{\lambda_\pm} = \pm K_1\lambda_\pm^a \tag{4.4.5}$$

Here b_λ is the smooth function of $\lambda < \lambda^*$ defined in Proposition 4.2.2. Let

$$u_\pm(r,t) = \tilde{u}_{\lambda_\pm(t)}\left(\mu_\pm(t)\cdot r\right).$$

We estimate the time-derivatives of the parameters. By (4.4.5), abbreviating $\lambda = \lambda_\pm$ and $\epsilon = \epsilon_\pm$, we have

$$\begin{aligned} |\lambda_t| &\leq C\left(\lambda + K_1\lambda^{1+a}\right) \\ |\epsilon_t| &= a|\lambda_t|\lambda^{a-1} \leq Ca\lambda^a\left(1 + K_1\lambda^a\right) \end{aligned} \tag{4.4.6}$$

Differentiating (4.4.3) yields

$$0 = \partial_t \tilde{u}(\mu) = \lambda_t \partial_\lambda u_\lambda(\mu) + \epsilon_t T_0(\mu) + \mu_t \tilde{u}'_\lambda(\mu).$$

Solving for μ_t, by Proposition 4.3.2 we may bound the first term to obtain

$$\begin{aligned} |\mu_t| &\leq C\left(|\lambda_t|\lambda\,|\mu - 1| + |\epsilon_t|\right) \\ &\leq C\lambda(1 + K_1\lambda^a)\left(\lambda|\epsilon| + a\lambda^{a-1}\right) \\ &\leq C\lambda^a\left(1 + K_1\lambda^a\right)\left(\lambda^2 + a\right). \end{aligned} \tag{4.4.7}$$

Writing $\lambda = \lambda_\pm$, etc., and carrying the factor $\mu(t)$ through the calculation (4.4.1) yields

$$\begin{aligned} \left((\partial_t - \Delta)\,u_\pm + \frac{f(u_\pm)}{r^2}\right)(r) = \left(\pm K_1\lambda^a\Lambda Q_\lambda + \lambda_t\partial_\lambda T_\lambda + \epsilon_t T_0\right)(\mu r) + \mu^2\left(\epsilon\Lambda Q_0 + R\right)(\mu r) \\ + \frac{\mu_t}{\mu}\left(\Lambda\tilde{u}_\lambda\right)(\mu r) \end{aligned}$$

Note that $\Lambda Q_0 = O(r^2)$ and is positive for $r > 0$, whereas $|T_0| = O(r^4)$. Hence by (4.4.6), for $a > 0$ sufficiently small (independent of K_1) there holds

$$|\epsilon_t||T_0| \leq \frac{1}{3}\epsilon\Lambda Q_0,$$

for $0 < \lambda < \lambda^*$. Therefore we may write

$$\pm\left(\epsilon_t T_0 + \mu^2\epsilon\Lambda Q_0\right) \geq \frac{1}{2}\lambda^a\Lambda Q_0.$$

Hence

$$\begin{aligned}\pm\left(\left(\partial_t-\Delta\right)u_\pm+\frac{f(u_\pm)}{r^2}\right)(r)\geq\left(K_1\lambda^a\Lambda Q_\lambda+\frac{1}{2}\lambda^a\Lambda Q_0\right)(\mu r)\\ -C\lambda\left(1+K_1\lambda^a\right)\left|\partial_\lambda T_\lambda\right|-\mu^2\left|R\right|-\left|\frac{\mu_t}{\mu}\Lambda\tilde{u}_\lambda\right|\\ =: A(\mu r)-B(\mu r)\end{aligned}$$

which we claim is positive for $r>0$.

We have the bounds from below

$$A(r)\geq C^{-1}\lambda^a\begin{cases}r^2\geq\lambda & C\lambda^{1/2}\leq r\leq 1\\ K_1\left(\frac{\lambda}{r}\right)^2\geq K_1\lambda & \lambda\leq r\leq C\lambda^{1/2}\\ K_1(\frac{r}{\lambda})^2 & 0\leq r\leq\lambda.\end{cases}\tag{4.4.8}$$

Note that

$$\begin{aligned}|\Lambda\tilde{u}|&\leq|\Lambda Q_\lambda+\Lambda T_\lambda+\epsilon\Lambda T_0|\\ &\leq|\Lambda Q_\lambda|+C\left(b_\lambda r^2+\epsilon r^4\right).\end{aligned}$$

Combining this with Proposition 4.3.3 and (4.4.2), as well as (4.4.7), for $0<\lambda<\lambda^*$ sufficiently small, we have

$$|B(r)|\leq C\begin{cases}\lambda^2+\lambda^a(\lambda^2+\lambda^a r^6)+\lambda^a\left(\lambda^2+a\right)\left(\lambda+r^2\right) & C\lambda^{1/2}\leq r\leq 1\\ \lambda^2+\lambda^{1+a}+\lambda^a(\lambda^2+a)\left(\frac{\lambda}{r}\right)^2 & \lambda\leq r\leq C\lambda^{1/2}\\ \lambda^2\left(\frac{r}{\lambda}\right)^4+\lambda^2\left(\frac{r}{\lambda}\right)^2+\lambda^a(\lambda^2+a)\left(\frac{r}{\lambda}\right)^2 & 0\leq r\leq\lambda.\end{cases}\tag{4.4.9}$$

Comparing (4.4.8) and (4.4.9), we see that for K_1 sufficiently large and a,λ^* sufficiently small, $A(r)$ dominates $|B(r)|$. Therefore $u_\pm$ are sub/supersolutions, as desired.

Equation (4.4.5) can be rewritten

$$-\lambda_t/\lambda = -\left(\log\lambda\right)_t = \mu^2\kappa/e_0 \pm K_1\lambda^a + O(\lambda)$$
$$= \kappa/e_0 + O(\lambda^a)$$

by (4.4.4). Note that $\log\lambda < \log\lambda^* - (\kappa/2e_0)t$, so the remainder terms is integrable in time. We obtain

$$\lambda_\pm(t) \sim C_\pm e^{-\frac{\kappa}{e_0}t}$$

as $t \to \infty$. Assuming

$$u_-(r,0) \le u_0(r) \le u_+(r,t_0)$$

for some t_0, the comparison principle (Remark 4.1.2) implies that

$$u_-(r,t) \le u(r,t) \le u_+(r,t_0+t)$$

which implies the stated bounds on u.

For arbitrary initial data u_0 with the given properties, by the strict maximum principle applied to (4.1.1), for $\tau > 0$ there must hold

$$u(r,\tau) < Q_0(r) \quad (r < 1)$$
$$u'(1,\tau) < Q_0'(1).$$

Therefore there exists t_0 with $0 \le u(r,\tau) \le u_+(r,t_0)$. Lemma 4.1.1 implies that $u(r,t)$ exists for all time with the desired bound from above.

Fixing $\tau > 0$, again by the strict maximum principle for (4.1.1) with the positive Dirichlet boundary condition $v(1) = \alpha$, there exists c such that

$$cr^2 \le u(r,\tau)$$

for c independent of u_0. Now let $\mu_0 > 0$ be such that

$$u_-(\mu_0 r, 0) \leq cr^2.$$

By construction, there exists $\epsilon_0 > 0$ such that

$$-\Delta\left(u_-(r,0)\right) + \frac{f(u_-(r,0))}{r^2} \geq \epsilon_0 \Lambda u_-(r,0)$$

for $r \in [0,1]$. Therefore, letting $\bar{\mu}(t) = \mu_0 e^{\epsilon_0 t}$, we obtain a subsolution

$$\bar{u}_-(r,t) := u_-(\bar{\mu}(t) r)$$

for $r \in [0, \bar{\mu}(t)^{-1}]$. This interval includes $[0,1]$, for $t \leq t_0$ such that $\mu_0 e^{\epsilon_0 t_0} = 1$, and note that

$$\bar{u}_-(1, t_0) = u_-(1,0) = \alpha.$$

But then $u_-(r,0) \leq u(r,t_0)$, and for $t \geq t_0$ we have

$$u_-(r, t - t_0) \leq u(r,t)$$

which is the desired lower bound. □

Remark 4.4.1. A slightly weaker asymptotic appears in the thesis of Michelangelo Vargas Rivera [26] for (HM), by an explicit construction. For (YM), Schlatter et. al. [29] proved that blowup was at most exponential, but their method does not produce subsolutions or the precise rate.

5. SELF-DUALITY AND SINGULARITIES

This chapter provides several theorems concerning long-time existence and smooth convergence of the Yang-Mills flow in dimension four. See the Introduction for a detailed summary.

Note on dependence of constants

Several of our estimates will have constants, e. g. $C_{5.1.1}$, with a particular dependence which we state in the corresponding proposition. The letter C itself denotes a numerical constant which can be taken to be increasing throughout the paper, although it will be used similarly within individual proofs. The constant C_M also depends on the geometry of the fixed base manifold M. In Section 5.2.1 we will also define a Poincaré constant C_A, labeled by the corresponding connection.

5.1 (Anti)-self-dual singularities

Let (M, g) be a compact Riemannian manifold of dimension four, $\pi : E \to M$ a vector bundle with fiber $\mathbb{R}^n$, fiberwise inner-product $\langle \cdot, \cdot \rangle$ and smooth reference connection D_{ref}. Let $D(t) = D_{ref} + A(t)$ be a smooth solution of the Yang-Mills flow, as constructed in Section 2.3.

In order to obtain separate control of the self-dual curvature, we apply the inner-product with F^+ to (2.2.8). Letting $u = |F^+|^2$, we obtain the differential inequality

$$\left(\frac{\partial}{\partial t} + \Delta\right) u \leq -2|\nabla F^+|^2 + Au^{3/2} + Bu \tag{5.1.1}$$

where B is a multiple of $||Rm||_{L^\infty(M)}$.

Proposition 5.1.1. *Let $u(x,t) \geq 0$ be a smooth function satisfying*

$$\left(\frac{\partial}{\partial t} + \Delta\right) u \leq A u^{3/2} + Bu.$$

on $M \times [0,T)$, with M compact of dimension four. There exist $R_0 > 0$ (depending on the geometry of M) and $\delta > 0$ (depending on A, B, R_0) as follows:

Assume $R < R_0$ is such that $\int_{B_R(x_0)} u(x,t)dx < \delta^2$ for all $x_0 \in M, 0 \leq t < T$. Then

$$||u(t)||_{L^\infty(M)} \leq C_{5.1.1}\, ||u||_{L^1(M\times[t-\tau,t])} \quad (\tau \leq t < T)\,.$$

The constant depends on $||u(0)||_{L^2}, R$, and τ. If u is defined for all time, then

$$\limsup_{t\to\infty} ||u(t)||_{L^\infty(M)} \leq C_M/R^4.$$

Proof. Let $\varphi \in C_0^\infty(B_R(x))$. Multiplying by $\varphi^2 u$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int \varphi^2 u^2\right) + \int \nabla(\varphi^2 u)\cdot \nabla u \leq A\int \varphi^2 u^{5/2} + B\int \varphi^2 u^2$$
$$\frac{1}{2}\frac{d}{dt}\left(\int \varphi^2 u^2\right) + \int |\nabla(\varphi u)|^2 \leq \int |\nabla \varphi|^2 u^2 + A\int \varphi^2 u^{5/2} + B\int \varphi^2 u^2.$$

Applying the Sobolev and Hölder inequalities on B_R,

$$\frac{1}{2}\frac{d}{dt}\int \varphi^2 u^2 + \left(\frac{1}{C_S} - A\delta\right)\left(\int (\varphi u)^4\right)^{1/2} \leq ||\nabla\varphi||^2_{L^\infty}\int_{B_R} u^2 + B\int \varphi^2 u^2.$$

Assuming $R < R_0$, depending on the geometry of M, we have $Vol(B_R(x)) \leq c^2 R^4$ for all $x \in X$ as well as a uniform Sobolev constant C_S. We may also choose a cover of M by geodesic balls $B_{R/2}(x_i)$ in such a way that no more than N of the balls $B_i = B_R(x_i)$ intersect a fixed ball, with N universal in dimension four. For each i, let $\tilde{\varphi}_i$ be a standard cutoff for $B_{R/2}(x_i) \subset B_R(x_i)$ with $||\nabla\tilde{\varphi}_i||_{L^\infty} < 4/R$. Define $\varphi_i = \tilde{\varphi}_i/\sqrt{\sum_j \tilde{\varphi}_j^2}$, so that $\{\varphi_i^2\}$ is a partition of unity with $||\nabla\varphi_i||_{L^\infty} < C/R$.

We now apply the above differential inequality to φ_i and sum

$$\sum_i \left(\frac{1}{2}\frac{d}{dt}\int \varphi_i^2 u^2 + (C_S^{-1} - A\delta)\left(\int (\varphi_i u)^4\right)^{1/2}\right) \leq \sum_i \left(CR^{-2}\int_{B_i}(\sum_j \varphi_j^2)u^2 + B\int \varphi_i^2 u^2 \right)$$
$$\leq \left(\frac{CN}{R^2} + B\right)\sum_i \int \varphi_i^2 u^2.$$

Note that for $\theta > 0$, we have by Hölder's and Young's inequalities

$$\int (\varphi_i u)^2 \leq \delta\left(\int_{B_R}(\varphi_i u)^3\right)^{1/2} \leq \delta\left(\int_{B_R}\left(\theta^3 + \frac{(\varphi_i u)^4}{\theta}\right)\right)^{1/2}$$
$$\leq \delta\left((CR^4\theta^3)^{1/2} + \theta^{-1/2}\left(\int (\varphi_i u)^4\right)^{1/2}\right).$$

Taking $\theta = R^{-4}$, we obtain

$$\sum_i \left(\frac{1}{2}\frac{d}{dt}\int \varphi_i^2 u^2 + (C_S^{-1} - A\delta)\left(\int (\varphi_i u)^4\right)^{1/2}\right) \leq \delta\left(\frac{C}{R^2} + B\right)\sum_i \left(\frac{C}{R^4} + R^2\left(\int (\varphi_i u)^4\right)^{1/2}\right)$$

and subtracting the last term

$$\sum_i \left(\frac{d}{dt}\int \varphi_i^2 u^2 + \epsilon\left(\int (\varphi_i u)^4\right)^{1/2}\right) \leq \frac{C\delta\,(1 + BR^2)}{R^6}\,(\#\text{ of balls}),$$

where we now choose δ so that

$$\epsilon = 2\left(C_S^{-1} - \delta\left(A + (C + BR_0^2)\right)\right) > 0.$$

We may finally apply Hölder's inequality to the left-hand side and absorb the partition of unity

$$\sum_i \left(\frac{d}{dt}\int \varphi_i^2 u^2 + \frac{\epsilon}{cR^2}\int \varphi_i^2 u^2\right) \leq \frac{C\delta\,(1 + BR^2)}{R^6}\left(\frac{Vol(M)}{R^4}\right)$$
$$\frac{d}{dt}\int u^2 + \frac{\epsilon}{cR^2}\int u^2 \leq \frac{C\delta\,(1 + BR^2)\,Vol(M)}{R^{10}}.$$
$$\frac{d}{dt}\left(e^{\frac{\epsilon}{cR^2}t}\int u(t)^2\right) \leq e^{\frac{\epsilon}{cR^2}t}\frac{C\delta\,(1 + BR^2)\,Vol(M)}{R^{10}}.$$

Integrating, we obtain the estimate

$$\int u(t)^2 \leq e^{-\frac{\epsilon}{cR^2}t} \int u(0)^2 + \frac{C\delta\,(1+BR^2)\,Vol(M)}{\epsilon R^8}\left(1 - e^{-\frac{\epsilon}{cR^2}t}\right).$$

This gives a uniform L^2 bound on $u(t)$ for $t > 0$, hence a uniform L^4 bound on $Au^{1/2} + B$. Standard Moser iteration (see [20] Lemma 19.1) on cylinders of radius R_0 and height τ then implies the stated L^∞ bounds. □

Lemma 5.1.2. *(C. f. [12], 7.2.10) There is a constant L and for any $N \geq 2, R > 0$ a smooth function $\beta = \beta_{N,R}$ on $\mathbb{R}^4$ with $0 \leq \beta(x) \leq 1$ and*

$$\beta(x) = \begin{cases} 1 & |x| \leq R/N \\ 0 & |x| \geq R \end{cases}$$

and

$$||\nabla\beta||_{L^4}, ||\nabla^2\beta||_{L^2} < \frac{L}{\sqrt{\log N}}.$$

Assuming $R < R_0$, the same holds for $\beta(x - x_0)$ on any geodesic ball $B_R(x_0) \subset M$.

Proof. We take

$$\beta(x) = \tilde{\varphi}\left(\frac{\log \frac{N}{R}x}{\log N}\right)$$

where

$$\tilde{\varphi}(s) = \begin{cases} 1 & s \leq 0 \\ 0 & s \geq 1 \end{cases}$$

is a standard cutoff function (with respect to the cylindrical coordinate s). □

Remark 5.1.3. The construction of Lemma 5.1.2 is possible in dimension four and above due to the scaling of the L^4 norm on 1-forms (L^2 norm on 2-tensors), together with the failure of these norms to control the supremum. Proposition 5.1.1 holds only in dimension less than or equal to four.

Theorem 5.1.4. *Let $A(t)$ satisfy the Yang-Mills flow equation on $M \times [0,T)$. For $R < R_0$ and $N \geq 2$, we have the local bound*

$$||F(T)||^2_{L^2(B_{R/N})} \leq ||F(0)||^2_{L^2(B_R)} + \int_0^T \frac{||F^+(t)||_{L^\infty(B_R)}}{\sqrt{\log(N)}} \left(C + ||F^-(t)||^2_{L^2(B_R)}\right) dt \qquad (5.1.2)$$

on concentric geodesic balls in M. Therefore if $||F^+||_{L^\infty(M)} \in L^1([0,T))$, or in particular if F^+ does not concentrate in L^2, then the flow extends smoothly past time T.

Proof. Recall the evolution of the curvature tensor

$$\frac{\partial}{\partial t} F_A = -DD^* F.$$

Multiplying by $\varphi^2 F$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}||\varphi F||^2 + ||\varphi D^* F||^2 = 2(\varphi D\varphi \cdot F, D^* F)$$

where as before we abbreviate $||\cdot|| = ||\cdot||_{L^2}$. On the right-hand side we switch $D^*F = 2D^*F^+$ (2.2.3), and integrate by parts again to obtain

$$\frac{1}{2}\frac{d}{dt}||\varphi F||^2 + ||\varphi D^* F||^2 = 4\int_M \left\langle \left(\nabla_i\varphi\nabla^k\varphi + \varphi\nabla_i\nabla^k\varphi\right) F_{kj} + \varphi\nabla^k\varphi\nabla_i F_{kj}\,, (F^+)^{ij}\right\rangle dV$$

In the inner product with the self-dual 2-form F^+, we may replace the term $\varphi\nabla^k\varphi\nabla_i F_{kj}$ via the identity

$$\begin{aligned}
\left(\nabla^k\varphi\left(\nabla_i F_{kj} - \nabla_j F_{ki}\right)\right)^+ &= \left(\nabla^k\varphi\left((-\nabla_j F_{ik} - \nabla_k F_{ji}) - \nabla_j F_{ki}\right)\right)^+ \\
&= \left(\nabla^k\varphi\nabla_k F_{ij}\right)^+ \\
&= \nabla^k\varphi\nabla_k F^+_{ij}.
\end{aligned}$$

We then write

$$\left\langle \nabla_k F^+_{ij}, (F^+)^{ij}\right\rangle = \frac{1}{2}\nabla_k|F^+|^2$$

and integrate by parts once more, to obtain

$$\frac{1}{2}\frac{d}{dt}||\varphi F||^2 + ||\varphi D^* F||^2 = 4\int_M (\nabla_i\varphi\nabla_k\varphi + \varphi\nabla_i\nabla_k\varphi)\left(\left\langle {F^k}_j, (F^+)^{ij}\right\rangle - g^{ik}\frac{|F^+|^2}{4}\right)\, dV$$

$$= 4\int_M (\nabla_i\varphi\nabla_k\varphi + \varphi\nabla_i\nabla_k\varphi)\left\langle {\left(F^-\right)^k}_j, (F^+)^{ij}\right\rangle dV$$

where the identity follows from a calculation similar to (2.2.6). Removing an L^∞ norm and applying Young's inequality yields

$$\frac{d}{dt}||\varphi F||^2 \leq 8\,||F^+||_{L^\infty(B_r)}\left(\epsilon^{-1}||F^-||^2_{L^2} + \epsilon\left(||\nabla\varphi||^4_{L^4} + ||\varphi\nabla^2\varphi||^2_{L^2}\right)\right).$$

Choose $\epsilon = 8\sqrt{\log(N)}$ and $\varphi = \beta_{N,r}$ from Lemma 5.1.2, to obtain the desired estimate.

By Theorem 2.3.2 (the work of Struwe [32]), to prove the second claim it suffices to show that the full curvature does not concentrate in L^2 at time T. Note that $||F^-(t)||^2$ is decreasing. Therefore if the curvature on B_r is initially less than $\delta/2$, then for N sufficiently large, the estimate implies that the full curvature on $B_{r/N}$ remains less than δ until time T.

Moreover by Proposition 5.1.1, non-concentration of F^+ implies a uniform L^∞ bound, and hence the required $L^1(L^\infty)$ bound at finite time. □

Remark 5.1.5. The proof of Theorem 5.1.4 implies a more refined result, to be used in future work. Let

$$S_{ij} = \langle {F_i}^k, F_{jk}\rangle - g_{ij}\frac{|F|^2}{4}$$

be the stress-energy tensor for Yang-Mills, and

$$N_p(x) = X^i X^j S_{ij}$$

where X^i is the radial vector-field at the point $p \in M$.

Theorem 5.1.6. *If, for some* $r_0 > 0$, *there holds*

$$\sup_{0<r<r_0} \left| \fint_{S^3_r} N_p(x) dS \right| < C$$

then no singularity occurs at p within finite time.

We now return to the consequences of Theorem 5.1.4.

Corollary 5.1.7. *If the maximal existence time is finite, then both* F^+ *and* F^- *must concentrate.*[1]

Remark 5.1.8. In view of the Corollary, one can modify the standard rescaling argument [27] at a finite-time singularity to obtain a weak limit which has either nonzero F^+ or nonzero F^-. Proposition 5.1.1 implies that this energy cannot be lost in the limit. Thus one cannot have a finite-time singularity for which every weak blowup limit is strictly self-dual, or anti-self-dual. Since any stable Yang-Mills connection on an $SU(2)$ or $SU(3)$-bundle over S^4 is either self-dual, anti-self-dual, or reducible, Theorem 5.1.4 in this case should imply that finite-time singularities are unstable.

Corollary 5.1.9. *For* δ *as in Proposition 5.1.1, if an initial* H^1 *connection has self-dual curvature* $||F^+_{A(0)}||_{L^2(M)} < \delta$ *then the Yang-Mills flow exists for all time and blows up at most exponentially, with asymptotic rate bounded uniformly for* M. *On any* $SU(2)$*-bundle, there exists a nonempty* H^1*-open set of initial connections for which the Yang-Mills flow exists for all time, and converges exponentially if* $H^{2+}(M) = 0$.

Proof. The connection is smooth after a short time, modulo gauge (see §2.3). Proposition 5.1.1 then implies a uniform bound on F^+ for all future time, and long-time existence follows from Theorem 5.1.4.

Following Freed and Uhlenbeck [13], for any δ_1 one can construct smooth pointlike $SU(2)$-connections with $||F^+||_{L^2} < \delta_1$ and $||F^+||_{L^\infty} < C$ (p. 124). Provided $H^{2+}(M) = 0$, Theorem

[1] Since the singularities are isolated, $F^\pm$ clearly must concentrate at the same point. This is easily shown by adding a boundary term to Proposion 5.1.1.

5.3.1 (below) yields convergence at infinite time, which holds in an H^1-open neighborhood of the resulting instantons (Theorem 5.3.4). □

Remark 5.1.10. Finite-time blowup of equivariant harmonic map flow $S^2 \to S^2$, as constructed by [5], occurs even with low holomorphic energy (see [36] for definitions).

5.2 Convergence at infinite time

In this section we assume that all connections have globally small self-dual energy

$$||F_A^+||_{L^2(M)} < \delta.$$

By (2.2.4), this condition is preserved by the flow, which exists for all time by Corollary 5.1.9. It is also attained for a nonempty set of connections on bundles with $c_2(E) \geq 0$ and structure group $SU(2)$, and in this case should represent the generic end-behavior of the flow.

We first recall and adapt several standard pieces of Yang-Mills theory. For an open set $\Omega \subset M$, we will write

$$\Omega_r = \{x \in \Omega \mid d(x, \Omega^c) > r\} \subset\subset \Omega.$$

Lemma 5.2.1. *There exists $\epsilon_0 > 0$ as follows. For $R < R_0$, if the energy*

$$||F(t)||^2_{L^2(B_R)} < \epsilon_0 \qquad \big(-R^2 \leq t < 0\big)$$

then there holds

$$||\nabla_A^k F(t)||_{L^\infty\left(B_{R_k}\right)} < \frac{C_k}{R^{2+k}} \qquad \big(-R_k^2 \leq t < 0\big)$$

for all $k \geq 0$, where $R_k = R/2^{k+1}$.

Proof. See [6], [18] for standard proofs of the $k = 0$ estimate via monotonicity formulae. For $k \geq 1$, this is the result of the Bernstein-Hamilton-type derivative estimates of [39].[2] □

[2] The $k = 0$ bound also follows simply from the derivative estimates.

Lemma 5.2.2. *Assume* $||F^+(t)||_{L^\infty(\Omega)} < K^+$ *for* $0 \le t \le \tau$. *Let* ϵ_0 *be as above, and assume that for some* $r_0 < R_0$ *there holds*

$$||F(\tau)||^2_{L^2(B_{r_0}(x))} < \epsilon_0/3 \tag{5.2.1}$$

for all $x \in \Omega_{r_0}$, *with* $0 < r_0^2 < \tau$. *If*

$$||F(0)||^2_{L^2(M)} - ||F(\tau)||^2_{L^2(M)} \le \epsilon_0/3 \tag{5.2.2}$$

then we have

$$||\nabla_A^k F(\tau)||_{L^\infty(\Omega_{r_0})} < \frac{C_{5.2.2}}{r_0^{2+k}}$$

for $k \ge 0$. *The constant depends on* $K^+, ||F(0)||$, *and* k.

Proof. Let φ be the cutoff of Lemma 5.1.2 for $B_{r_0/N}(x) \subset B_{r_0}(x)$, and apply the proof of Theorem 5.1.4 using $\overline{\varphi} = 1 - \varphi$. This gives

$$||F(\tau)||^2_{L^2(M \setminus B_{r_0})} - ||F(t)||^2_{L^2(M \setminus B_{r_0/N})} < \epsilon_0/3 \tag{5.2.3}$$

for N large enough based on $||F||^2$ and K^+ (but independent of x, r_0). Adding (5.2.2), with t in place of zero, and (5.2.3), we obtain

$$||F(t)||_{L^2(B_{r_0/N})} - ||F(\tau)||_{L^2(B_{r_0})} < 2\epsilon_0/3.$$

By (5.2.1), we have

$$||F(t)||_{L^2(B_{r_0/N})} < \epsilon_0$$

for $0 \le t \le \tau$, and the desired L^∞ bounds from Lemma 5.2.1. □

Definition 5.2.3. For a sequence $t_j \to \infty$, we say that (A_∞, E_∞) is an *Uhlenbeck limit* for the flow if the following holds. There exists a subsequence of times t_{j_k} and smooth bundle isometries $u_k : E \to E_\infty$ defined on an exhaustion of open sets

$$U_1 \subset \cdots \subset U_k \subset \cdots \subset M_0 = M \setminus \{x_1, \ldots x_n\}$$

such that on any open set $\Omega \subset\subset M_0$, we have $u_k^*(A_{t_k}) \to A_\infty$ smoothly.

Theorem 5.2.4. *Assuming $||F^+|| < \delta$, any sequence $t_j \to \infty$ necessarily contains an Uhlenbeck limit which is a Yang-Mills connection on E_∞.*

Proof. This is a standard improvement of the detailed arguments found in [27], by analogy with the Kahler case (see [12], Ch. 6).

The existence of weak H^1 limits on a countable family of balls in M_0 is the result of compactness theory for connections with bounded L^2 curvature ([30], [37]) in Coulomb gauge. By Lemma 5.2.2, we in fact have L^∞ bounds on the curvature of $A(t_{j_k})$ and all its derivatives on each ball, for k large enough. By [12], Lemma 2.3.11, upon taking further subsequences, the weak limit can be taken to be a smooth limit over each ball, and by [12], Corollary 4.4.8, the gauge transformations can be patched together over the open sets U_i.[3]

The fact that the limiting connection is Yang-Mills away from the bubbling points, and therefore extends to a smooth Yang-Mills connection on E_∞, follows from the energy inequality, [38], and the next estimate. □

Lemma 5.2.5. *Assume $||F(t)||_{L^\infty(B_R(x_0))} < K$ for $0 \le t < T$. Then for $\tau > 0, R < R_0$, we have*

$$||\nabla^k D^* F(t)||^2_{L^\infty(B_{R_k})} \le C_{5.2.5} ||D^* F||^2_{L^2(B_R \times [t-\tau, t])}$$

$$||\nabla^k F(t)||^2_{L^\infty(B_{R_k})} \le C_{5.2.5} \left(||D^* F||^2_{L^2(B_R \times [t-\tau, t])} + ||F(t)||^2_{L^2(B_R)} \right)$$

for $k \ge 0$ and $k\tau \le t < T$. The constants depend on K, R, τ, and k.

[3] Note that Theorem 1.3(ii) of Schlatter [27] does not include any patching, because this may not be possible with H^2 gauge transformations.

Proof. One computes the evolution

$$\begin{aligned}\frac{\partial}{\partial t}\left(D^{*}F_{i}\right) &= -\frac{\partial}{\partial t}\nabla^{k}F_{ki} \\ &= -\left[(-D^{*}F)^{k}, F_{ki}\right] + D^{*}(-DD^{*}F)_{i} \\ &= \left[F_{i}{}^{k}, D^{*}F_{k}\right] - \Delta_{A}\left(D^{*}F\right)_{i} \\ &= \nabla^{k}\nabla_{k}D^{*}F_{i} + 2\left[F_{i}{}^{k}, D^{*}F_{k}\right] + Rm\# D^{*}F_{i}.\end{aligned} \tag{5.2.4}$$

In the third line we used the Bianchi identity $(D^*)^2F = 0$ to obtain the Hodge Laplacian Δ_A. Multiplying (5.2.4) by D^*F gives

$$\left(\partial_t + \Delta\right)|D^*F|^2 \leq C\left(1+K\right)|D^*F|^2.$$

The first estimate, with $k = 0$, then follows again from standard Moser iteration (Li [20], 19.1) applied to (5.2.4).

Applying a cutoff for $B_{3R_1/2} \subset B_{R_0}$ to (5.2.4) also gives

$$\begin{aligned}\int_{t-\tau/2}^{t} ||\nabla D^*F(t')||^2_{L^2\left(B_{3R_1/2}\right)}dt' &\leq C||D^*F||^2_{L^\infty\left(B_{R_0}\times[t-\tau/2,t]\right)} \\ &\leq C||D^*F||^2_{L^2(B_R\times[t-\tau,t])}.\end{aligned} \tag{5.2.5}$$

Note that we have an evolution equation

$$\left(\partial_t + \nabla^*\nabla\right)\left(\nabla D^*F\right) = F\#\nabla D^*F + Rm\#\nabla D^*F + \nabla F\# D^*F + \nabla Rm\# D^*F.$$

From Lemma 5.2.1,[4] all derivatives of F are bounded in terms of K. Again by Moser iteration, we may bound $||\nabla D^*F(t)||_{L^\infty\left(B_{R_1}\right)}$, for $t \geq \tau$, by the LHS of (5.2.5), which concludes the $k = 1$ case. The higher derivative estimates proceed by induction.

[4] It is possible to recover a version of Lemma 5.2.1 independently using Moser iteration (combining Prop. 5.1.1 and this argument), although not the fully local statement.

The argument for the second inequality is identical, beginning with the $k = 1$ case

$$||\nabla F||^2_{L^2(B_{R_0})} \leq C \left(||D^* F||^2_{L^2(B_R)} + ||F||^2_{L^2(B_R)} \right)$$

following from the Weitzenbock formula (2.2.2). □

5.2.1 Sobolev and Poincaré inequality for self-dual forms

As before, we abbreviate $||\cdot|| = ||\cdot||_{L^2(M)}$.

Assuming $||F_A^+|| < \delta$, Hölder's inequality applied to the Weitzenbock formula (2.2.7) implies, for $\omega \in \Omega^{2+}(\mathrm{End}E)$, the Sobolev inequality

$$\begin{aligned} ||\omega||^2_{L^4(M)} + ||\nabla_A \omega||^2_{L^2(M)} &\leq C_M \left(||D_A \omega||^2 + ||D_A^* \omega||^2 + ||\omega||^2 \right) \\ &\leq C_M \left(||D_A^* \omega||^2 + ||\omega||^2 \right). \end{aligned} \tag{5.2.6}$$

In the second line we used the pointwise identity

$$|D_A \omega| = |-*D_A * \omega| = |D_A^* \omega|.$$

Recall the basic instanton complex

$$0 \to \mathfrak{g}_E \xrightarrow{D_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{\pi_+ D_A} \Omega^{2+}(\mathfrak{g}_E) \to 0.$$

Under the assumption $H_A^{2+} = 0$, there are no nonzero L^2 self-dual two-forms with $D_A^* \omega = 0$ in the distributional sense. Therefore, by the standard compactness argument, we have

$$||\omega||^2 \leq C_A ||D_A^* \omega||^2.$$

Hence this term can be dropped from the RHS of (5.2.6), yielding

$$||\omega||^2_{L^4} + ||\omega||^2 + ||\nabla_A \omega||^2 \leq C_A ||D_A^* \omega||^2 \tag{5.2.7}$$

for $\omega \in \Omega^{2+}(\mathfrak{g}_E)$. We will always take $C_A \geq C_M$.

Lemma 5.2.6. *Let A_0 be a connection on a bundle E_0 over M which satisfies the Poincaré inequality*

$$||\omega||^2_{L^4} + ||\omega||^2 \leq C_{A_0}||D^*_A\omega||^2 \tag{5.2.8}$$

for $\omega \in \Omega^{2+}(\mathfrak{g}_E)$. Assume A is a connection on E with $||F^+_A|| < \delta$, for which there exists a smooth bundle isometry $u : E_0 \to E$ defined over $M_r = M \setminus \bar{B}_r(x_1) \cup \cdots \cup \bar{B}_r(x_n)$ with $||u^*(A) - A_0||_{L^4} \leq \epsilon$. Then if r, ϵ are sufficiently small, A satisfies (5.2.8) with constant $8C_{A_0}$.

Proof. Assume first that $\mathrm{Supp}(\omega) \subset M_r$. Write $\tilde{A} = u^*(A), \tilde{\omega} = u^*(\omega)$, $a = A_0 - \tilde{A}$. We then have

$$||D^*_A\omega||^2 = ||D^*_{\tilde{A}}\tilde{\omega}||^2 = ||D^*_{A_0}\tilde{\omega} + a\#\tilde{\omega}||^2$$

and

$$||D^*_{A_0}\tilde{\omega}||^2 \leq 2\left(||D^*_A\omega||^2 + ||a||^2_{L^4}||\omega||^2_{L^4}\right).$$

On the other hand, if $\mathrm{Supp}(\omega) \subset B_r(x_1) \cup \cdots \cup B_r(x_n)$, then

$$||\omega||^2 \leq cnr^2||\omega||^2_{L^4}.$$

Choose ϵ, r, N such that

$$4\epsilon^2 + cnr^2 + 2L^2/\log(N) < (8C_{A_0})^{-1}.$$

Let $\varphi = \sum \beta_{N,r}(x - x_i)$ be a sum of the logarithmic cutoffs of Lemma 5.1.2, and $\overline{\varphi} = 1 - \varphi$. Combining the above observations, we have

$$\begin{aligned}
||\omega||^2_{L^4} + ||\omega||^2 &\leq 2\left(||\varphi\omega||^2_{L^4} + ||\varphi\omega||^2 + ||\overline{\varphi}\omega||^2_{L^4} + ||\overline{\varphi}\omega||^2\right)\\
&\leq 2C_M\left(||D^*_A(\varphi\omega)||^2 + ||\varphi\omega||^2\right) + 2C_{A_0}||D^*_{A_0}(\overline{\varphi}\tilde{\omega})||^2\\
&\leq 4C_{A_0}\left(||\varphi D^*_A\omega||^2 + ||\overline{\varphi}D^*_A\omega||^2 + 2||D\varphi\#\omega||^2 + \left(4||a||^2_{L^4} + cnr^2\right)||\omega||^2_{L^4}\right)\\
&\leq 4C_{A_0}\left(||D^*_A\omega||^2 + \left(2||D\varphi||^2_{L^4} + 4\epsilon^2 + cnr^2\right)||\omega||^2_{L^4}\right).
\end{aligned}$$

Upon rearranging, this yields the claim (replacing r/N by r in the statement). □

5.2.2 Convergence

We now proceed to the proof of our main convergence result.

Proposition 5.2.7. *Assume $||F(t)||_{L^\infty(\Omega)} < K$ for $0 \leq t < T$. Then we have the L^∞ bound*

$$||A(T) - A(\tau)||^2_{L^\infty(\Omega_r)} \leq C_{5.2.7} \left(||F(0)||^2 - ||F(T)||^2\right)(T - \tau)$$

as well as the derivative bounds

$$||\nabla^k_{ref} \left(A(T) - A(\tau)\right)||^2_{L^\infty(\Omega_{kr})} \leq C_{5.2.7} \left(||F(0)||^2 - ||F(T)||^2\right)(T - \tau)\left(1 + ||A||^{2k}_{L^\infty_{k-1}(\Omega_{(k-1)r})}\right).$$

for $k > 0$. The constants depend on $K, r, \tau, k,$ and $\Omega \subset M$.

Proof. For the first bound, write

$$\begin{aligned} ||A(T) - A(\tau)||_{L^\infty(\Omega_r)} &\leq \int_\tau^T ||D^*F(t)||_{L^\infty} dt \\ &\leq C_{5.2.5} \int_\tau^T ||D^*F||_{L^2(\Omega \times [t-\tau,t])} dt \\ &\leq C(T-\tau)^{1/2} \left(\int_\tau^T ||D^*F||^2_{L^2(\Omega \times [t-\tau,t]dt)}\right)^{1/2} \\ &\leq C(T-\tau)^{1/2}\tau^{1/2} \left(\int_0^T ||D^*F||^2 dt\right)^{1/2} \\ &\leq C(T-\tau)^{1/2} \left(||F(0)||^2 - ||F(T)||^2\right)^{1/2} \end{aligned}$$

by Lemma 5.2.5, as desired. The first derivative bound follows from

$$\begin{aligned} \partial_t \nabla_{ref} A &= -\nabla_{ref} D^*F \\ &= -\nabla_A D^*F + A \# D^*F \end{aligned} \tag{5.2.9}$$

and the same computation. The higher derivative bounds proceed similarly. □

Theorem 5.2.8. *Assume $||F^+(0)|| < \delta$, and there exists an Uhlenbeck limit A_∞ on (M, E_∞) which is an instanton with $H^{2+}_{A_\infty} = 0$. Then $E = E_\infty$, and the flow converges smoothly to a connection which is gauge-equivalent to A_∞.*

More precisely, if A_∞ is a connection satisfying (5.2.8), then for any $\tau_1 \geq \tau_0 > 0$ there exist δ_1, ϵ_1, and $r_1 > 0$ as follows. If for some $\tau \geq \tau_1$, $||F^+(\tau - \tau_0)|| < \delta_1$ and $A(\tau)$ is within ϵ_1 of A_∞ in $H^1(M_{r_1})$ modulo gauge, then for $t \geq \tau$ the flow converges exponentially (in the sense below). The constants δ_1 and ϵ_1 depend on A_∞, τ_0, and $||F^+(0)||_{L^4}$, but can be taken independent of the latter for τ_1 sufficiently large.

Proof. Let $M_0 = M \setminus \{x_1, \ldots, x_n\}$ be as in Definition 5.2.3. Let $r_1 = r/3$ (where r is as in Proposition 5.2.6), and choose $r_0 < \min(r_1, R_0, \sqrt{\tau_0})$ such that for every $x \in M_{2r_1}$, we have

$$||F_{A_\infty}||^2_{L^2(B_{r_0}(x))} < \epsilon_0/3.$$

Now, let $\tau \geq \tau_1$ be such that

$$||F^+(\tau - \tau_0)||^2 < \delta_1^2$$

and there exists a smooth isometry u such that

$$||u^*(A(\tau)) - A_\infty||_{H^1(M_{r_1})} < \epsilon_1. \tag{5.2.10}$$

By the local Sobolev inequality,[5] we have

$$||u^*(A(\tau)) - A_\infty||_{L^4(M_{2r_1})} \leq C\epsilon_1.$$

Choosing ϵ_1 such that $C\epsilon_1 < \epsilon/2$ (where ϵ is as in Proposition 5.2.6), the Poincaré inequality holds for $A(t)$ with constant $C_\infty = C_{A_\infty}$ on some maximal interval $[\tau, T)$. We will argue that if $\delta_1 > 0$ is small enough, then $T = \infty$ and the flow converges.

[5] applied with respect to a smooth reference connection for E_∞

Applied to the global energy inequality for F^+, the Poincaré inequality

$$||F^+||^2 \leq C_\infty ||D^* F^+||^2$$

yields

$$\partial_t ||F^+||^2 + C_\infty^{-1} ||F^+||^2 \leq \partial_t ||F^+||^2 + ||D^* F^+||^2 = 0.$$

This implies the exponential decay for $t \geq \tau$

$$||D^* F||^2_{L^2(M\times[t,T])} \leq ||F^+(t)||^2 \leq \delta_1^2 e^{-(t-\tau)/C_\infty}. \tag{5.2.11}$$

By Proposition 5.1.1, we have the global L^∞ bound

$$||F^+(t)||^2_{L^\infty(M)} \leq K^+(t)^2 := C_{5.1.1}\delta_1^2 e^{-(t-\tau)/C_\infty} \tag{5.2.12}$$

for $t \geq \tau$. Therefore, if δ_1 is sufficiently small we have

$$\left(C + ||F(t)||^2\right) \int_\tau^T K^+(t) dt < \epsilon_0/3. \tag{5.2.13}$$

By Theorem 5.1.4, the full curvature cannot concentrate on M_{2r_1} before time T, and we have a uniform bound

$$||F(t)||_{L^\infty(M_{2r_1})} < K \tag{5.2.14}$$

for $\tau + r_0^2 < t < T$.

In order to apply Proposition 5.2.7, we need this curvature bound on M_{2r_1} also from time $\tau - r_0^2/2$. Note that

$$\delta_1^2 > ||F^+(\tau - r_0^2)||^2 \geq \frac{1}{2}\left(||F(\tau - r_0^2)||^2 - ||F(T)||^2\right).$$

By Lemma 5.2.2, provided $\delta_1^2 < \epsilon_0/6$, we in fact have a larger uniform bound (5.2.14) on M_{2r_1} for $\tau - r_0^2/2 < t \leq \tau + r_0^2$.

With this curvature bound, we may now apply Proposition 5.2.7 and (5.2.11) at each time $\tau + i$, to conclude

$$||A(\tau + i + 1) - A(\tau + i)||^2_{L^4(M_{3r_1})} \leq C^0_{5.2.7} \left(K^+(\tau + i)\right)^2.$$

By the triangle inequality and geometric series, we have

$$||A(T) - A(\tau)||_{L^4(M_{3r_1})} \leq C \sum_i K^+(\tau + i) \leq CK^+(\tau) = C\delta_1.$$

If δ_1 is small enough that $C\delta_1 < \epsilon/2$, we conclude

$$\begin{aligned} ||u^*(A(T)) - A_\infty||_{L^4(M_{3r_1})} &\leq ||u^*(A(T)) - u^*(A(\tau))||_{L^4(M_{3r_1})} + ||u^*(A(\tau)) - A_\infty||_{L^4(M_{2r_1})} \\ &\leq C\delta_1 + C\epsilon_1 < \epsilon. \end{aligned}$$

Therefore $T = \infty$, and the above estimates continue as $t \to \infty$.

Note that Theorem 5.1.4 and (5.2.13) imply that the curvature does not concentrate anywhere on M as $t \to \infty$. Therefore the flow converges globally and strongly in H^1 (and by Proposition 5.2.7 and (5.2.11) applied on M, at least exponentially). This proves the second statement.

In the case that $F^+_{A_\infty} = 0$, by taking r_1 and ϵ_1 smaller in the second statement, we can clearly satisfy the assumption $||F^+(\tau - \tau_0)|| < \delta_1$. Hence the second statement implies the first. □

5.3 Further results

Theorem 5.3.1. (Taubes's grafting theorem, parabolic version.) *Let (E_0, A_0) be a flat bundle on M with $H^{2+}_{A_0} = 0$. For any K^+, and points $x_1, \ldots, x_n \in M$, there exist $\delta_1, \epsilon_1, r_1 > 0$ such that if A is a connection on E with $||F^+_A|| < \delta_1$, $||F^+_A||_{L^\infty(M)} < K^+$, and*

$$||A - A_0||_{H^1(M_{r_1})} < \epsilon_1 \tag{5.3.1}$$

then the flow with initial data $A(0) = A$ converges and remains L^4-close to A_0 modulo gauge on $M_{r_1} = M \setminus \bar{B}_{r_1}(x_1) \cup \cdots \cup \bar{B}_{r_1}(x_n)$.

Proof. By assumption, a Poincaré estimate (5.2.8) holds, and we choose $\epsilon_1 \leq \epsilon/2$, $r_1 = r/2$ according to Lemma 5.2.6.

By (5.3.1), we have $||F(0)||_{L^2} < C\epsilon_1$. Applying the maximum principle to the evolution (5.1.1) of $|F^+|^2$, we have $||F^+(t)||_{L^\infty(M)} < 2K^+$ for $0 \leq t < \tau < 1$. Therefore, taking δ_1 sufficiently small, Proposition 5.1.1 and Theorem 5.1.4 imply

$$||F(t)||_{L^2(M_{2r_1})} < 2C\epsilon_1$$

for $0 \leq t \leq \tau$. Assume first that M is simply-connected, so we may take $A_0 = 0$. Note that from Proposition 5.2.5 and the energy inequality, the curvature at time τ and all its derivatives are bounded by a constant times δ_1. According to [12], Proposition 4.4.10, for δ_1 sufficiently small there exists a gauge transformation u on M_{2r_1} (also simply-connected) with[6]

$$||u^* A(\tau)||_{L^4(M_{2r_1})} < C\epsilon_1.$$

The claim now follows from the precise statement of Theorem 5.2.8.

If M is not simply-connected, we argue as follows. Let $\pi : \tilde{M} \to M$ be the universal cover, and choose a simply-connected domain $\Omega \subset \tilde{M}$ covering M_{2r_1}, which is a finite union of preimages of $B_i \subset M_{r_1}$, with $B_i \cap B_j$ connected.[7] Assume that $\pi^* A_0 = 0$, and let $\tilde{A} = \pi^* A(\tau)$. As before, we may choose a gauge u on Ω such that

$$||u^* \tilde{A}||_{L^4(\Omega)} < C\epsilon_1. \tag{5.3.2}$$

If this is done using Coulomb gauges on the B_i, then $u^{-1}du$ is well-defined on M.

[6] Here Proposition 5.2.5 exactly replaces Theorem 2.3.8 of [12]. In both cases, bounds on all derivatives of the connection in Coulomb gauge are supplied by [12], Lemma 2.3.11, which are used in the gluing argument of Proposition 4.4.10.

[7] This can be done for instance by lifting the geodesic balls B_i to $\tilde{M}$ using a set of based paths which form a spanning tree for their incidence graph.

Note that we also have

$$\begin{aligned} ||A(\tau) - A_0||_{L^2(M_{r_1})} &\leq ||A(\tau) - A(0)||_{L^2(M_{r_1})} + C||A(0) - A_0||_{L^4(M_{r_1})} \\ &\leq \tau^{1/2} \left(\int ||D^* F||^2_{L^2(M_{r_1})} dt \right)^{1/2} + C\epsilon_1 \\ &\leq \delta_1 + C\epsilon_1. \end{aligned} \tag{5.3.3}$$

Over Ω, combining (5.3.2) and (5.3.3) yields

$$||du|| = ||u^{-1}du|| \leq ||u^*\tilde{A}|| + ||u\tilde{A}u^{-1}|| < C\epsilon_1.$$

By the Poincaré inequality, in each ball

$$||u - \bar{u}||_{L^2(B_i)} < C\epsilon_1.$$

We may therefore choose points $p_i \in B_i$ such that $d(p_i, p_j) \geq c > 0$ and

$$|u(\tilde{p}_i) - u(\tilde{p}_j)| < C\epsilon_1$$

for each $\tilde{p}_i, \tilde{p}_j \in \Omega$ such that $\pi(\tilde{p}_i) = p_i$ and $\pi(\tilde{p}_j) = p_j$.

It is clearly possible to construct a frame v over Ω such that $v(\tilde{p}_i) = u(\tilde{p}_i) \,\forall\, i$, $||dv||_{L^\infty} < C\epsilon_1$, (depending on Ω) and $v^{-1}dv$ is well-defined on M_{r_1}. The frame $w = v^{-1}u$ then satisfies $w(\tilde{p}_i) = 1$ for all $\tilde{p}_i$, and descends to a frame on E over M_{r_1}. Note that

$$||w^*\tilde{A}||_{L^4(\Omega)} \leq ||v^{-1}dv + v^{-1}(u^*\tilde{A})v||_{L^4(\Omega)} \leq 2C\epsilon_1$$

and so downstairs

$$||w^*A(\tau) - A_0||_{L^4(M_{r_1})} \leq C\epsilon_1.$$

Convergence follows for ϵ_1 and δ_1 sufficiently small as before. □

Remark 5.3.2. A similar argument can be used to recover the gluing theorem for connected sums with long necks of small volume, i.e. [12], Theorem 7.2.24.

Corollary 5.3.3. *Assume that* $\pi_1(M)$ *has no nontrivial representations in* $SU(2)$, *and* $H^{2+}(M) = 0$. *For any initial connection on the bundle* E *with structure group* $SU(2)$ *and* $c_2(E) = 1$, *assuming* $||F^+(0)|| < \delta_1$, *no bubbling occurs and the flow has a smooth subsequential limit as* $t \to \infty$. *If this limit is an irreducible instanton, then it is unique and the flow converges exponentially.*

Proof. Assume, by way of contradiction, that bubbling occurs as $t \to \infty$. The blowup limits of [27] at a presumed singularity, as well as the Uhlenbeck limit, preserve the structure group. Due to the L^∞ bound on F^+, the blowup limit at a bubble must be anti-self-dual, and therefore contains all but $2\delta_1$ of the energy. If the Uhlenbeck limit A_∞ obtained from Theorem 5.2.4 on the same sequence of times is also anti-self-dual, it must be flat by integrality of κ. By the assumption on $\pi_1(M)$, A_∞ is equal to the product connection on the trivial bundle. But then its cohomology is exactly $H^{2+}(M) = 0$, and by the Theorem the flow converges, which is a contradiction. If the Uhlenbeck limit is not anti-self-dual, it must nonetheless be L^4-close to a flat connection (arguing as in the previous Theorem), which is still a contradiction.

Therefore a smooth Uhlenbeck limit exists. If it is irreducible then $H^{2+} = 0$, and again by Theorem 5.2.8 we have exponential convergence. □

Theorem 5.3.4. *The instantons with* $H^{2+} = 0$ *are asymptotically stable in the* H^1 *topology. In other words, given an* H^1 *neighborhood* U *of* A, *there exists a neighborhood* $U' \subset U$ *of initial connections for which the limit under the flow will again be an instanton with* $H^{2+} = 0$, *lying in* U *modulo smooth gauge transformations.*

Moreover, there exists an H^1*-open neighborhood* N *for which the flow gives a deformation retraction from* $N \cap H^k$, $k >> 1$, *onto the moduli space of instantons with* $H^{2+} = 0$.

Proof. By Struwe's construction [32] (see Chapter 2, §2.3), choosing the instanton A itself as the connection D_1, the gauge-equivalent flow (2.3.3) remains in U for a time τ, long enough for ϵ-regularity to take effect. This gives a uniform bound on the curvature at time τ, including on $||F^+||_{L^\infty}$. Choosing U' small enough, we also obtain $||F^+|| < \delta_1$. We are then in the situation of Theorem 5.2.8, which can be applied with $\{x_i\} = \emptyset$.

The latter refinement follows from standard parabolic theory. For, two connections in N which are initially H^k-close remain so under the gauge-equivalent flow. As this exists on any interval $[t, t+\tau]$, they must remain close, modulo gauge, for a long time; but then both are close to their respective limits under the Yang-Mills flow. □

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