Canonical Metrics in Sasakian Geometry

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ABSTRACT

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The aim of this thesis is to study the existence problem for canonical Sasakian metrics, primarily Sasaki-Einstein metrics. We are interested in providing both necessary conditions, as well as sufficient conditions for the existence of such metrics.

We establish several sufficient conditions for the existence of Sasaki-Einstein metrics by studying the Sasaki-Ricci flow. In the process, we extend some fundamental results from the study of the Kähler-Ricci flow to the Sasakian setting. This includes finding Sasakian analogues of Perelman's energy and entropy functionals which are monotonic along the Sasaki-Ricci flow. Using these functionals we extend Perelman's deep estimates for the Kähler-Ricci flow to the Sasaki-Ricci flow. Namely, we prove uniform scalar curvature, diameter and non-collapsing estimates along the Sasaki-Ricci flow. We show that these estimates imply a uniform transverse Sobolev inequality. Furthermore, we introduce the sheaf of transverse foliate vector fields, and show that it has a natural, transverse complex structure. We show that the convergence of the flow is intimately related to the space of global transversely holomorphic sections of this sheaf.

We introduce an algebraic obstruction to the existence of constant scalar curvature Sasakian metrics, extending the notion of K-stability for projective varieties.

Finally, we show that, for regular Sasakian manifolds whose quotients are Kähler-Einstein Fano manifolds, the Sasaki-Ricci flow, or equivalently, the Kähler-Ricci flow, converges exponentially fast to a (transversely) Kähler-Einstein metric.

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To my family

Chapter 1

Motivation

Sasakian geometry has recently garnered a great deal of interest due to its role in the AdS/CFT correspondence [Maldacena, 1998], which predicts that certain superconformal field theories are "dual" to certain string theories on a product of Anti-de Sitter space with a non-compact conical Calabi-Yau manifold. The AdS/CFT correspondence does not, as yet, have a precise mathematical formulation, and so we will refrain from focusing on this aspect of Sasakian geometry. It turns out that conical Calabi-Yau metrics on non-compact manifolds arise exactly as the metric cones over compact Sasaki-Einstein manifolds with positive scalar curvature. As a result, it is important to understand the existence problem for Sasaki-Einstein metrics.

Existence of Sasaki-Einstein metrics, or equivalently, conical Ricci-flat metrics is deep and difficult. In the compact case any Calabi-Yau manifold admits a Ricci-flat metric by Yau's solution of the Calabi conjecture [Yau, 1978]. However, in the non-compact, conical case this is no longer true. For example, if X is a Fano variety, then the cone over X in the total space $K_X\backslash\{0\}$ has a conical Calabi-Yau metric with respect to the natural cone structure if and only if X is Kähler-Einstein with positive scalar curvature.

In turn, existence of Kähler-Einstein metrics with positive scalar curvature has been a subject of intense research over the last 30 years. The fundamental conjecture, made by Yau [Yau, 1993] is that the existence of a Kähler-Einstein metric should be equivalent to some algebro-geometric notion of stability. The precise notion of stability which seems to be appropriate for the Kähler-Einstein problem is the notion of K-stability, formulated by Tian

[Tian, 1997], and later refined by Donaldson [Donaldson, 2002]. The Yau-Tian-Donaldson conjecture states that the existence of a Kähler-Einstein metric on X is equivalent to the K-stability of the polarized variety (X, K_X^{-1}). Recently, there has been a solution to this conjecture put forth by Chen-Donaldson-Sun [Chen *et al.*, 2012a], [Chen *et al.*, 2012b], [Chen *et al.*, 2013], and Tian [Tian, 2013].

On the other hand, there has been comparatively little development of the corresponding problem in the Sasakian setting. As mentioned above, a positive solution to the natural extension of Yau's conjecture, "existence of Sasaki-Einstein metrics is equivalent to stability", implies the solution of the Kähler-Einstein problem, but not conversely. Let us elaborate on this briefly. A conical manifold Y admits a one parameter group of isometries arising from the action of the vector field generating the cone structure. This "Euler vector field" generates a holomorphic vector field on the cone. There are essentially two cases. If the holomorphic vector field integrates to a $\mathbb{C}^*$ action, then the quotient $Y/\mathbb{C}^*$ is a Fano orbifold, and the Sasakian structure is said to be quasi-regular. Otherwise, there is no quotient manifold and the Sasakian structure is said to be irregular.

Initially, Cheeger-Tian [Cheeger and Tian, 1994] conjectured that if Y admits a Ricci flat cone metric, then the Sasakian structure *must* be quasi-regular. This conjecture was shown to be false by Gauntlett-Martelli-Sparks-Waldram [Gauntlett *et al.*, 2004], who constructed infinitely many explicit Sasaki-Einstein metrics with irregular Sasakian structures on $S^2 \times S^3$. In particular, this implies that the methods of projective geometry are not sufficient to address the problem of existence of Sasaki-Einstein metrics in the irregular case.

The aim of this thesis is to extend some of the theory available for Kähler-Einstein metrics to the Sasakian setting. The primary point of view we will take is the heat flow perspective. Namely, we will investigate necessary and sufficient conditions under which the Sasaki-Ricci flow, introduced by Smoczyk-Wang-Zhang [Smoczyk *et al.*, 2010] converges. If the Sasaki-Ricci flow converges, the limit is transversely Kähler-Einstein, and hence Sasaki-Einstein after possibly performing a $\mathcal{D}$-homothetic deformation. In any event, the opposite implication is far from obvious. Namely, if a Sasaki-Einstein metric exists, then is it true that the Sasaki-Ricci flow converges? The tools that we will develop in this thesis allow us to answer this question in some cases, as we will describe below.

In the Kähler setting, it is an unpublished theorem of Perelman that if a Kähler-Einstein metric exists, then the Kähler-Ricci flow converges exponentially fast to a (possibly different) Kähler-Einstein metric. We will give a proof of this theorem in Chapter 5. This material is joint work with Gábor Székelyhidi, and appeared in [Collins and Székelyhidi, 2012b]. In fact, the results here are apply in a more general setting; we introduce a *twisted* version of the Kähler-Ricci flow, and show this flow converges whenever a twisted Kähler-Einstein metric exists; the Kähler-Ricci flow corresponds to taking the trivial twist. Precisely,

Theorem 1 ([Collins and Székelyhidi, 2012b]). *Suppose the Fano manifold (M, J) admits a Kähler-Einstein metric. Then for any $\omega_0 \in c_1(M)$, the Kähler-Ricci flow with initial metric ω_0 converges exponentially fast to a Kähler-Einstein metric.*

The proof of this theorem makes essential use of many results in the Kähler-Ricci flow. The most important ingredients are Perelman's deep and fundamental estimates for the Kähler-Ricci flow [Sesum and Tian, 2008], the smoothing estimates of Székelyhidi-Tosatti [Székelyhidi and Tosatti, 2011], some fundamental results of Phong-Sturm [Phong and Sturm, 2006] and Phong-Song-Sturm-Weinkove [Phong *et al.*, 2009] as well as a nice observation of Tian-Zhu [Tian and Zhu, 2011]. From Perelman's work, the main ingredients we need are the uniform bound for the scalar curvature, and the uniform non-collapsing estimate, together with the monotonicity of the entropy functional.

In order to prove Theorem 1, we must first reprove Perelman's fundamental results on the Kähler-Ricci flow in the twisted setting, and with *effective* control of the constants appearing in the estimates. This requires some new arguments, and we feel that these results are of sufficient independent interest to warrant pointing them out at this early stage of the thesis. Precisely, we prove

Theorem 2 ([Collins and Székelyhidi, 2012b]). *Suppose that $g(t)$ evolves along the twisted Kähler-Ricci flow with $g(0) = g_0$, and let $u(t)$ be the Ricci potential of $g(t)$. Then there exists a constant C depending continuously on the C^3 norm of g_0 (and a uniform lower bound on g_0), such that*

$$|u| + |\nabla u|_{g(t)} + |\Delta_{g(t)} u| \leqslant C.$$

Perhaps most importantly, we are able to deduce these estimates *without* first proving a uniform diameter bound along the Kähler-Ricci flow. This improvement may have applications in the future.

Of course, Theorem 1 implies that the Sasaki-Ricci flow converges on Sasaki-Einstein manifolds which are circle bundles in K_X for X a smooth Fano manifold equipped with a Kähler-Einstein metric of positive scalar curvature. This corresponds to an extremely restrictive assumption on the Reeb vector field; namely, that the Reeb field is *regular*. We would like to relax this assumption, but before this is possible, we need to extend the major results in the Kähler-Ricci flow to the Sasakian setting. This is the task we take on in Chapter 3.

We begin Chapter 3 by defining natural generalizations of Perelman's entropy and energy functionals along the Sasaki-Ricci flow. We show that the Sasaki-Ricci flow has the structure of a gradient flow after composing with some time dependent diffeomorphisms. In particular, we exhibit two functionals which are monotonic along the Sasaki-Ricci flow.

Making use of the transverse entropy functional, we are able to extend Perelman's fundamental estimates to the Sasakian setting. A key step in this extension is to introduce a transverse distance function which is compatible with the foliated structure on the Sasakian manifold induced from the Reeb vector field. This distance function can be quite pathological, especially in the case when the Reeb vector field is irregular. Nevertheless, we are able to find an asymptotic expansion for the volume of the "balls" of this distance function by showing that, for sufficiently small radii the transverse balls are the same as geodesic tubes around imbedded tori, which are precisely the closure of the orbits of the Reeb flow. We prove

Theorem 3 ([Collins, 2013]). *Let $g(t)$ be a solution of the Sasaki-Ricci flow on a Sasaki manifold $(S, g_0, \xi, \eta, \Phi)$ of real dimension $2n+1$ with $c^1_B(S) > 0$. Let $u \in C^\infty_B(S)$ be the transverse Ricci potential. Then there exists a uniform constant C, depending only on the initial metric $g(0) = g_0$ so that*

$$|R^T(g(t))| + |u|_{C^1(g(t))} + \mathrm{diam}(S, g(t)) < C. \tag{1.0.1}$$

We use the Perelman type estimates to prove a uniform Sobolev inequality along the

Sasaki-Ricci flow, extending an estimate of Zhang [Zhang, 2007]. In order to prove this estimate we introduce L^p and Sobolev spaces which are compatible with the Reeb field. An essential difficulty is to show that the ring of smooth functions which are constant along the orbits of the Reeb vector field is dense in the Sobolev spaces. Thia difficulty arises from the fact that mollification immediately moves one out of the space of functions compatible with the Reeb field. Instead, we make use of the foliated structure, and use a heat flow argument to regularize functions. With these developments in hand, we explain how the monotonicity of the entropy functional implies a uniform log Sobolev inequality, which in turn implies a heat kernel estimate, and finally, applying the L^p and Sobolev theory, a bound for the Sobolev constant.

At this point, we are in a position to address the convergence of the flow. In Section 3.3.2 we introduce a sheaf $\mathcal{E}$ called the sheaf transverse foliate vector fields, which has a natural transverse complex structure. We show that the space of global, transversely holomorphic sections of this sheaf is intimately related to the convergence of the Sasaki-Ricci flow. This kind of result was first discovered by Phong-Sturm [Phong and Sturm, 2006] and elaborated upon by Phong-Song-Sturm-Weinkove [Phong *et al.*, 2009] and Zhang [Zhang, 2010]. We adapt these ideas to the Sasakian setting.

Heuristically, the idea is the following. Let us assume that the Riemannian curvature is uniformly bounded in C^∞ along the Sasaki-Ricci flow. An adaptation of Hamilton's compactness theorem implies that, along a subsequence the Sasaki-Ricci flow converges, modulo diffeomorphisms, to a limiting Sasakian metric. If we assume in addition that the transverse Mabuchi energy is bounded below, then the transverse Perelman-type functional μ^T converges along the flow to its maximum value, which is explicit, and topological. Moreover, any metric which achieves this maximum value is necessarily Sasaki-Einstein. Since the μ^T functional is diffeomorphism invariant, it follows that the Sasaki-Ricci flow converges, modulo diffeomorphisms, to a Sasaki-Einstein metric. The problem that needs to be addressed is whether the limiting complex structure is isomorphic to the initial complex structure. That is, we need to rule out the possibility of "jumping complex structures". As long as we permit convergence modulo diffeomorphisms, it is difficult to prove the the limit complex structure is isomorphic. Hence, the main difficulty is to improve the convergence to smooth convergence with respect to a fixed background metric. The key observation of [Phong and Sturm, 2006] is that, if the moduli space of transverse complex structures satisfies a Hausdorff condition, then the boundedness of the Mabuchi functional in fact implies the exponential decay to a constant of the transverse scalar curvature. Exponential decay, together with the Perelman estimates, easily implies a smooth convergence of the metrics by making use of Yau's estimates for the Monge-Ampère equation [Yau, 1978].

In fact, the argument we give is quite different from the heuristic given above. The reason being that we want to avoid assuming any uniform bounds for the Riemann tensor along the flow, which is an overly restrictive assumption. Instead, our arguments follow the ideas of [Phong *et al.*, 2009] and [Zhang, 2010]. Nevertheless, the motivation is the same; one needs to rule out the jumping of complex structures in the limit of the Sasaki-Ricci flow. Instead of imposing the Hausdorff condition on the moduli space of complex structures, we instead impose an assumption on the lowest positive eigenvalue of the $\overline{\partial}$ laplacian on global sections of the sheaf $\mathcal{E}$; namely, we assume that this eigenvalue is bounded uniformly away from zero along the flow. This condition reduces to the Hausdorff condition alluded to above whenever the Riemann curvature tensor is bounded.

Finally, using all of the techniques developed to this point, we prove that if a Sasaki-Einstein metric exists, and the automorphism group of the transverse complex structure is discrete, then the Sasaki-Ricci flow converges exponentially. This work appeared in [Collins and Jacob, 2014], and is joint with Adam Jacob.

In Chapter 4 we present some joint work with Gábor Szkékelyhidi, which appeared in [Collins and Székelyhidi, 2012a]. The objective of this section is to introduce a notion of stability which obstructs the existence of Sasakian metrics with constant scalar curvature. In order to do this, we need to introduce an algebraic structure on a Sasakian manifold. The key observation, which we present in Section 2.2, is that the geometry of Sasakian manifolds should really be regarded as the geometry of affine varieties with isolated singularities at the origin, whose coordinate rings come equipped with a multigrading of positive type. We feel that this observation is important, and should be useful in future developments in the field.

Once we have established the existences of an algebraic structure on a Sasakian manifold, we introduce a notion of K-stability, which extends the notion of K-stability for projective varieties introduced in [Tian, 1997], [Donaldson, 2002], and the extension to projective orbifolds introduced in [Ross and Thomas, 2011]. The central objects in the definition of K-stability are test configurations, which are flat, $\mathbb{C}^*$ equivariant degenerations of the affine variety. We associate to such a test configuration a number, called the Donaldson-Futaki (DF) invariant. We show that the DF invariant can be computed using the multigraded Hilbert series of the affine variety. Moreover, we show that the DF invariant defines a smooth function on the cone of Reeb vector fields. Finally, we can deduce a lower bound for the Calabi functional of a Sasakian manifold, viewed as the link of an affine singularity, by using a simple approximation argument combined with the results of [Donaldson, 2005] [Ross and Thomas, 2011]. As a consequence, we prove

Theorem 4 ([Collins and Székelyhidi, 2012a]). *Let (S, g) be a Sasakian manifold with Reeb vector field ξ. If g has constant scalar curvature, then the cone $(C(S), \xi)$ is K-semistable.*

It turns out that computing the Donaldson-Futaki invariant of a test configuration can be rather easily done in explicit examples, using the framework discussed above. We introduce two simple families of test configurations and show that the resulting obstructions recover the volume minimization results of [Martelli *et al.*, 2008] and the Lichnerowicz obstruction of [Gauntlett *et al.*, 2007]. It is interesting to note that both the volume minimization and Lichnerowicz obstructions have interpretations in terms of restrictions on the AdS/CFT dual conformal field theory. Finally, we compute explicitly the Reeb vector field of the Sasaki-Einstein metric on the cone over the second del Pezzo surface (dP_2). The existence of such a metric is rather surprising, and points to the difference between the Sasakian setting and the Kähler setting. In fact, it is well known dP_2 *does not* admit a Kähler-Einstein metric. Nevertheless, it was shown by Futaki-Ono-Wang [Futaki *et al.*, 2009] that all toric Sasakian manifolds admit Sasaki-Einstein metrics; in particular, the classical Futaki invariant introduced in [Futaki, 1983] does not obstruct in the Sasakian setting. This was observed by Martelli-Sparks-Yau [Martelli *et al.*, 2008]. The methods employed here are rather algorithmic, and could easily be carried out for any toric variety.

Chapter 2

Introduction

2.1 Basic Notions of Sasakian geometry

The remainder of this chapter is spent introducing the basic notions of Sasakian geometry. Most of this material can be found in the references [Boyer and Galicki, 2008], [Sparks, 2011]. The material in Section 2.2 is new. This appeared in joint work with G. Székelyhidi [Collins and Székelyhidi, 2012a] and is crucial for the developments in Chapter 4. Let us begin at the beginning; Sasakian geometry lies at the intersection of many fields of geometry. For example, Sasakian manifolds are foliated manifolds with a contact structure, which are, in a certain sense, both a generalization and a specialization of Kähler manifolds. There are many definitions of a Sasakian manifolds available in the literature. We shall use the following

Definition 2.1.1. *A Sasakian manifold is a Riemannian manifold* (S^{2n+1}, g) *together with a complex structure* J *on the metric cone*

$$(C(S), \bar{g}) = (S \times \mathbb{R}_{>0}, dr^2 + r^2 g),$$

making $(C(S), \bar{g}, J)$ *into a non-compact Kähler manifold. We embed* $S \hookrightarrow C(S)$ *as the level set* $\{r = 1\}$.

We note that, in general, there may be many complex structures making $(C(S), \bar{g})$ into a Kähler manifold, and so our definition fixes one. We refer the reader to to the book of Boyer-Galicki [Boyer and Galicki, 2008] as well as the survey article [Sparks, 2011] for several alternative, but of course equivalent, definitions. These two references also contain proofs and discussion of the basic material which is to follow.

A Sasakian manifold inherits a great deal of structure from the Kähler cone. For example, (S, g) inherits a contact structure, and Reeb vector field in the following way. First, we define

$$\eta := \sqrt{-1}(\overline{\partial} - \partial)\log r \in TC(S)^*$$

as well as the Reeb vector field, which is the $\bar{g}$ dual of η, given in local coordinates by

$$\xi^j := r^2\bar{g}^{jk}\eta_k.$$

The Reeb vector field is extremely important in what is to follow. We note that ξ has a definition which is *independent* of the metrc $\bar{g}$. Alternatively, we can consider the trivial line bundle $S\times\mathbb{R} \to S$. We identify $C(S) \subset S\times\mathbb{R}$, and furthermore the function $r : C(S) \to \mathbb{R}_{>0}$ identifies $S \subset C(S)$ as the set $\{r = 1\}$. We denote by $r\partial_r$ the Euler vector field of the trivial bundle, which we can restrict to $C(S)$.

Definition 2.1.2 (The Reeb Vector Field)**.** *The Reeb vector field of* $(C(S), \bar{g}, J)$ *is* $\xi = J(r\partial_r)$.

It is straight forward to check that ξ lies parallel the level sets $\{r = c\}$, and hence can be restricted to S to define a vector field which we will also denote ξ. Moreover, ξ is a Killing vector field

$$\mathcal{L}_\xi g = 0.$$

It follows immediately from the definitions that

$$\eta(\xi) = 1, \qquad \iota_\xi d\eta = 0$$

where ι denotes the interior derivative. From the above description of ξ as the metric dual of η we obtain the formula

$$\eta(X) = \frac{1}{r^2}\bar{g}(\xi, X)$$

for any $X \in TC(S)$, from which one can show

$$\omega = \frac{1}{2}d(\eta r^2) = \frac{1}{2}i\partial\overline{\partial}r^2$$

where ω is the Kähler form associated to $\bar{g}$ on $C(S)$. Since $\bar{g}$ is Kähler, it follows that $\eta \wedge (d\eta)^n$ defines a volume form on S, and hence S is a contact manifold, and ξ is the unique Reeb vector field for the contact structure.

Let us restrict our attention to S. The vector field ξ is Killing for the metric g, and has $g(\xi, \xi) = 1$; in patrticular, ξ is non-vanishing. Moreover, any Killing vector field with unit length foliates S by geodesics. The properties of this foliation will be very important throughout this thesis. Let $L_\xi \subset TS$ denote the line subbundle generated by the non-vanishing vector field ξ, and let $D = \ker \eta$ be the orthogonal complement with respect to g. We have the exact sequence

$$0 \hookrightarrow L_\xi \hookrightarrow TS \to Q \to 0 \tag{2.1.1}$$

and the (metric dependent) isomorphism $\sigma : Q \to D$. It is a crucial observation that Q is essentially the tangent bundle of a projective Kähler manifold. Before discussing this in general, let us give a concrete example.

Example 2.1.3. *Let (S^3, g) denote the round 3-sphere, embedded isometrically in $\mathbb{C}^2 \cong \mathbb{R}^4$ in the usual way. We equip $\mathbb{C}^2$ with the standard complex structure, making (S^3, g) into a Sasakian manifold. Let (u, v) be coordinates on $\mathbb{C}^2$, so that $r : \mathbb{C}^2 \backslash \{(0,0)\} \to \mathbb{R}_{>0}$ is given by $r(u, v) = \sqrt{|u|^2 + |v|^2}$. The Reeb vector field generates the Hopf fibration*

$$S^1 \hookrightarrow S^3 \to S^2 \cong \mathbb{P}^1.$$

The tangent space of $\mathbb{P}^1$ can be identified non-canonically over S^3 as the orthogonal complement of the Reeb field with respect to the metric g, or canonically as the quotient $Q = TS/L_\xi$.

In general, let us define $\Phi \in \mathrm{End}(TS)$ by

$$\Phi(X) = \begin{cases} JX & \text{for } X \in D \\ 0 & \text{for } X \in L_\xi, \end{cases}$$

which clearly satisfies $\Phi^2 = -\mathbb{I} + \eta \otimes \xi$. We have the formula

$$g(X, Y) = \frac{1}{2} d\eta(X, \Phi(Y)) + \eta(X)\eta(Y). \tag{2.1.2}$$

Definition 2.1.4. *(Sasakian Structure) A Sasakian structure is the tuple (S, g, ξ, η, Φ).*

We note that, from the data of the Sasakian structure, one can uniquely reconstruct the cone $(C(S), \bar{g}, J)$ [Boyer and Galicki, 2008], [Sparks, 2011].

2.1.1 Transverse Kähler Structures and the Reeb Foliation

The Reeb foliation inherits a transverse holomorphic structure and transverse Kähler metric in the following (explicit) way. The leaf space of the foliation induced by the Reeb field can clearly be identified with the leaf space of the *holomorphic* vector field $\xi - iJ(\xi)$ on the cone $C(S)$. Here, J denotes the integrable complex structure on the cone. By using holomorphic, foliated coordinates on $C(S)$, we may introduce a foliation chart $\{U_\alpha\}$ on S, where each U_α is of the form $U_\alpha = I \times V_\alpha$ with $I \subset \mathbb{R}$ and open interval, and $V_\alpha \subset \mathbb{C}^n$. We can find coordinates $(x, z_1, \ldots, z_n)$ on U_α, where $\xi = \partial_x$, and $z_1, \ldots, z_n$ are complex coordinates of V_α. The fact that the cone is complex implies that the transition functions between the V_α are holomorphic. More precisely, if $(y, w_1, \ldots, w_n)$ are similarly defined coordinates on U_β with $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\frac{\partial z_i}{\partial \overline{w}_j} = 0 \ , \quad \frac{\partial z_i}{\partial y} = 0.$$

Recall that the subbundle D is equipped with the almost complex structure $J|_D$, so that on $D \otimes \mathbb{C}$ we may define the $\pm i$ eigenspaces of $J|_D$ as the $(1,0)$ and $(0,1)$ vectors respectively. Then, in the above foliation chart, $(D \otimes \mathbb{C})^{(1,0)}$ is spanned by $\partial_{z_i} - \eta(\partial_{z_i})\xi$. Since ξ is a killing vector field it follows that $g|_D$ gives a well-defined Hermitian metric g^T_α on the patch V_α. Moreover, (2.1.2) implies that

$$d\eta(\partial_{z_i} - \eta(\partial_{z_i})\xi, \partial_{\overline{z}_j} - \eta(\partial_{\overline{z}_j})\xi) = d\eta(\partial_{z_i}, \partial_{\overline{z}_j}).$$

Thus, the fundamental 2-form ω^T_α for the Hermitian metric g^T_α in the patch V_α is obtained by restricting $\frac{1}{2}d\eta$ to a fibre $\{x = \textit{constant}\}$. It follows that ω^T_α is closed, and the transverse metric g^T_α is Kähler. Note that in the chart U_α we may write

$$\eta = dx + i\sum_{i=1}^{n} \partial_{z_i} K_\alpha dz_i - i\sum_{i=1}^{n} \partial_{\overline{z}_i} K_\alpha d\overline{z}_i$$

where K_α is a function of U_α with $\partial_x K_\alpha = 0$. We may identify the collection of transverse metrics $\{g^T_\alpha\}$ with the global tensor field on S given by

$$g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi(Y))$$

and the transverse Kähler form ω^T is similarly defined globally as $\frac{1}{2}d\eta$.

Sasaki manifolds fall in to three categories based on the orbits of the Reeb field. If the orbits of the Reeb field are all closed, then the ξ generates a locally free, isometric $U(1)$ action on (S,g). If the $U(1)$ action is free, then (S,g) is said to be *regular* and the quotient manifold $S/U(1)$ is Kähler. If the action is not free, then (S,g) is said to be *quasi-regular*, and the quotient manifold is a Kähler orbifold. If the orbits of ξ are not closed, then the Sasakian manifold (S,g) is said to be irregular. Carrière [Carrière, 1984] has shown that the leaf closures are diffeomorphic to tori, and that the Reeb flow is conjugate via this diffeomorphism to a linear flow on the torus. The local flow of the Reeb field defines a commutative subgroup of the isometry group of (S,g) whose closure is a torus $\mathfrak{T}$. The dimension of $\mathfrak{T}$ is called the rank of (S,g), denoted $rk(S,g)$, and is an invariant of the Pfaffian structure (S,ξ). If (S,g) has dimension 2n+1, then one can show (see [Boyer and Galicki, 2001]) that $1 \leqslant rk(S,g) \leqslant n+1$. In particular, for any $p \in S$, $\overline{orb_\xi p}$ is an imbedded torus of dimension less or equal $n+1$. Here $orb_\xi p$ denotes the orbit of p under the action generated by ξ. In particular, it follows that any irregular Reeb vector field can be smoothly approximated by quasi-regular Reeb fields. It is not clear, however, that any irregular Sasakian structure can be approximated smoothly by quasi-regular Sasakian structures; this is nevertheless true by [Boyer and Galicki, 2008, Theorem 7.1.10], and we shall outline a proof in the next section.

2.1.2 Transverse Riemannian Geometry and Basic Cohomology

We return now to the subject of transverse Riemannian geometry. Suppose that (S,g,ξ,η,Φ) is a Sasakian manifold. Then the quotient bundle Q defined by (3.0.1) is endowed with a transverse metric g^T. Moreover, there is a unique, torsion-free connection on Q which is compatible with the metric g^T. This connection is defined by

$$\nabla^T_X V = \begin{cases} (\nabla_X \sigma(V))^p, & \text{if } X \text{ is a section of} D \\ [\xi, \sigma(V)]^p, & \text{if } X = \xi \end{cases}$$

where ∇ is the Levi-Civita connection on (S,g), σ is the splitting map induced by g, and p is the projection from TS to the quotient Q. This connection is called the *transverse Levi-Civita connection* as it satisfies

$$\nabla^T_X Y - \nabla^T_Y X - [X,Y]^p = 0,$$

$$Xg^T(V,W) = g^T(\nabla^T_X V, W) + g^T(V, \nabla^T_X W).$$

In this way it is easy to see that the transverse Levi-Civita connection is the pullback of the Levi-Civita connection on the local Riemannian quotient. We can then define the transverse curvature operator by

$$Rm^T(X,Y) = \nabla^T_X \nabla^T_Y - \nabla^T_Y \nabla^T_X - \nabla^T_{[X,Y]},$$

and one can similarly define the transverse Ricci curvature, and the transverse scalar curvature. The geometry of the manifold S is largely controlled by its transverse geometry. For local sections X, Y, Z, W of D, the curvature of (S, g) is related to the curvature of (Q, g^T) by

$$\begin{aligned} Rm(X,Y,Z,W) =& Rm^T(X,Y,Z,W) + g(\Phi(X), Z)g(\Phi(Y), W) \\ &- g(\Phi(X), W)g(\Phi(Y), Z) + 2g(\Phi(X), Y)g(\nabla_\xi X, W). \end{aligned} \tag{2.1.3}$$

The geometry orthogonal to the distribution D is uniform in the sense that for any vector fields $X, Y \in TS$ there holds

$$Rm(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.1.4}$$

$$Rm(X,\xi)Y = \eta(Y)X - g(X,Y)\xi. \tag{2.1.5}$$

We refer the reader to [Boyer and Galicki, 2008], [Futaki *et al.*, 2009] for more on these standard formulae.

There is also a notion of transverse cohomology, which we introduce now.

Definition 2.1.5. *A p-form α on (S,ξ) is called basic if $\iota_\xi \alpha = 0$, and $\mathcal{L}_\xi \alpha = 0$. The sheaf of basic functions will be denoted by $C^\infty_B(S)$.*

On (S,ξ) we let Λ^p_B be the sheaf of basic p-forms, and $\Omega^p_B = \Gamma(S, \Lambda^p_B)$ the global sections. It is clear that the de Rham differential d preserves basic forms, and hence restricts to a well defined operator $d_B : \Lambda^p_B \to \Lambda^{p+1}_B$. We thus get a complex

$$0 \to C^\infty_B(S) \to \Omega^1_B \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Omega^{2n}_B \xrightarrow{d_B} 0$$

whose cohomology groups, denoted by $H^p_B(S)$, are the basic de Rham cohomology groups. Moreover, we can define the basic Laplacian on forms by

$$-\Delta_B = d_B d^\dagger_B + d^\dagger_B d_B.$$

In general, the basic Laplacian *does not* agree with the restriction of the de Rham Laplacian to basic forms [Kamber and Tondeur, 1987]. However, the basic Laplacian on functions *does* agree with the restriction of the metric Laplacian to basic functions, and one can check that

$$\Delta_B = \Delta|_{C^\infty_B(S)}$$

The transverse complex structure Φ allows us to decompose

$$\Lambda^r_B \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_B.$$

We can then decompose $d_B = \partial_B + \overline{\partial}_B$, where

$$\partial_B : \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B, \text{ and } \overline{\partial}_B : \Lambda^{p,q}_B \to \Lambda^{p,q+1}_B.$$

Consider the form $\rho^T = Ric^T(\Phi\cdot,\cdot)$, which is called the *transverse Ricci form.* In analogy with the Kähler case, one can check that

$$\rho^T = -\sqrt{-1}\partial_B\overline{\partial}_B \log\det(g^T)$$

and hence ρ^T defines a basic cohomology class, $\frac{1}{2\pi}[\rho^T]_B$, which is called the *basic first Chern class*, and is independent of the transverse metric.

Definition 2.1.6. *A transverse metric g^T is called a* transverse Einstein metric *if it satisfies*

$$Ric^T = cg^T$$

for some constant c. The transverse metric g^T is said to have transverse constant scalar curvature *if*

$$\mathrm{Tr}_{g^T} Ric^T = c$$

Note that it, in order for a transverse Einstein metric to exists, it is necessary that the basic first Chern class be signed.

2.2 Sasakian Geometry, Kähler Geometry and Affine Varieties

Let us give an alternative formulation of Sasakian geometry in terms of the geometry of affine varieties, and discuss the connection between canonical Sasakian metrics and canonical

Kähler metrics. The perspective presented here turns out to be quite useful, and provides many concrete examples of Sasakian manifolds. Throughout this section we assume the reader is familiar with the basics of orbifolds. To begin, let us state the following theorem.

Theorem 5 ([Boyer and Galicki, 2000] Theorem 2.4)**.** *Let (S, g) be a compact regular or quasi-regular Sasakian manifold of dimension 2n+1. Then the space of leaves of the Reeb foliation Z is a compact, complex Kähler manifold or orbifold, respectively, with a Kähler metric h and a Kähler form ω which defines an integral class $[\omega] \in H^2_{orb}(Z, \mathbb{Z})$ in such a way that $\pi : (S, g) \to (Z, h)$ is a Riemannian submersion.*

In particular, $[\omega] \in H^2_{orb}(Z, \mathbb{Z})$ is a positive class. Let L denote the corresponding positive line bundle on Z. Since S is the total space of the $U(1)$ principal bundle induced by L^{-1}, and S is smooth, it follows from Lemma 4.2.8 of [Boyer and Galicki, 2008] that the local uniformizing groups of the orbifold Z inject into $U(1)$, and hence act faithfully on the fibers of the positive line bundle L. In particular, Z carries an orbiample line bundle in the sense of [Ross and Thomas, 2011, Definition 2.7]. As a result, by [Ross and Thomas, 2011, Proposition 2.11], there is an embedding of Z into a weighted projective space which preserves the orbifold structure. The results of Rukimbira [Rukimbira, 1995a] imply that any irregular Reeb vector field can be approximated by quasi-regular Reeb fields. In particular, every Sasakian manifold admits at least one quasi-regular Reeb vector field. Combining this with Theorem 5, we see that for any Sasakian manifold S, the cone over S is an affine variety with an isolated singularity at 0.

We can now give an algebro-geometric formulation of the notion of a Reeb vector field for a general affine scheme passing through $0 \in \mathbb{C}^N$.

Suppose that $Y \subset \mathbb{C}^N$ is an affine scheme, with a torus $T \subset Aut(Y)$. Let us write $\mathfrak{t} := \mathrm{Lie}(T_{\mathbb{R}})$ for the Lie algebra of the maximal compact sub-torus. Let $\mathcal{H}$ denote the global sections of the structure sheaf of Y, and write

$$\mathcal{H} = \oplus_{\alpha \in \mathfrak{t}^*} \mathcal{H}_\alpha$$

for the weight decomposition under the action of T.

Definition 2.2.1. *A vector $\xi \in \mathfrak{t}$ is a* Reeb vector field *if for each non-empty weight space $\mathcal{H}_\alpha$, with $\alpha \neq 0$, we have $\alpha(\xi) > 0$, i.e. ξ acts with positive weights on the non-constant* *functions on Y. We will often identify the vector ξ with the vector field it induces on Y. We define the* Reeb cone *to be*

$$\mathcal{C}_R := \{\xi \in \mathfrak{t} \big| \xi \text{ is a Reeb field } \} \subset \mathfrak{t}.$$

Since $\mathcal{H}$ is finitely generated, $\mathcal{C}_R$ is a rational, convex, polyhedral cone, and for any $\xi \in \mathcal{C}_R$ there is an $\varepsilon > 0$ such that $\alpha(\xi) \geqslant \varepsilon|\alpha|$ for all non-empty weight spaces. We say that ξ is rational if there exists $\lambda \in \mathbb{R}_{>0}$ such that $\alpha(\lambda\xi) \in \mathbb{N}$ for every non-empty weight space. Otherwise, we say that ξ is irrational.

Note that any homogeneous variety admits a Reeb field generated by the usual $\mathbb{C}^*$ action on $\mathbb{C}$. In analogy with this case, we shall call an affine scheme Y with a holomorphic torus action admitting a Reeb vector field a *polarized affine scheme*. An affine scheme Y may admit more than one Reeb field; choosing a Reeb vector field ξ is analogous to fixing a polarization for a projective scheme. For the most part, we shall consider only polarized affine *varieties*. The next lemma shows that Reeb vector fields are always induced from Lie algebra actions on the ambient space, possibly after increasing the codimension of the embedding.

Lemma 2.2.2. *Let $Y \subset \mathbb{C}^N$ be an affine scheme, and let T be a torus acting holomorphically on Y. Then there exists an embedding $Y \hookrightarrow \mathbb{C}^{N'}$ and a torus $T' \subset GL(N', \mathbb{C})$ such that the multiplicative action of T' on $\mathbb{C}^{N'}$ induces the action of T on Y.*

Proof. Let Y be cut out by the ideal $I \subset \mathbb{C}[x_1, \ldots, x_N]$, so that $Y = \operatorname{Spec} \mathcal{H}$ for $\mathcal{H} = \mathbb{C}[x_1, \ldots, x_N]/I$. The torus T induces a decomposition

$$\mathcal{H} = \oplus_{\alpha \in \mathfrak{t}^*} \mathcal{H}_\alpha,$$

and the images of $x_1, \ldots, x_n$ generate $\mathcal{H}$. In particular, there exists a finite set of homogenous generators $u_1, \ldots u_{N'} \in \mathcal{H}$, with weights $\alpha_1, \ldots, \alpha_{N'}$. Consider the map

$$\begin{array}{ccc} \mathbb{C}[x_1, \ldots, x_{N'}] & \longrightarrow & \mathcal{H} \\ x_i & \longmapsto & u_i \end{array}$$

Define an action of T on $\mathbb{C}[x_1, \ldots, x_{N'}]$, where T acts on x_i with weight α_i. We get an exact sequence

$$0 \longrightarrow I' \longrightarrow \mathbb{C}[x_1, \ldots, x_{N'}] \longrightarrow \mathcal{H} \longrightarrow 0$$

which is equivariant with respect to the torus action. We obtain

$$\text{Spec } \mathcal{H} \cong \text{Spec } \frac{\mathbb{C}[x_1, \ldots, x_{N'}]}{I'} \hookrightarrow \text{Spec } \mathbb{C}[x_1, \ldots, x_{N'}],$$

and hence an embedding $Y \hookrightarrow \mathbb{C}^{N'}$. The action of T on Y is induced by the linear, diagonal action of T on $\mathbb{C}^{N'}$ as desired. □

Because of this lemma, we are essentially dealing with affine schemes defined by ideals $I \subset \mathbb{C}[x_1, \ldots, x_N]$ for some N, which are homogeneous for the action of a torus $T \subset GL(N, \mathbb{C})$. We can even assume that the torus action is diagonal. A choice of an integral vector $\xi \in \mathfrak{t}$ then induces a grading on $\mathbb{C}[x_1, \ldots, x_N]$, which has positive weights when ξ is a Reeb vector.

We will now relate our algebraic Reeb cone to the one defined differential geometrically in [Martelli *et al.*, 2008] (see also [He and Sun, 2012], and the Sasaki Cone in [Boyer *et al.*, 2008]). Suppose that $Y \subset \mathbb{C}^N$ is an affine variety, smooth away from the origin, and Y is defined by an ideal $I \subset \mathbb{C}[x_1, \ldots, x_N]$, homogeneous for the diagonal action of a torus T. We will also assume that Y is not contained in a linear subspace.

Definition 2.2.3. *A Kähler metric Ω on Y is* compatible *with a Reeb vector field $\xi \in \mathfrak{t}$ if there exists a ξ-invariant function $r : Y \to \mathbb{R}_{>0}$ such that $\Omega = \frac{1}{2} i\partial\overline{\partial} r^2$ and $\xi = J(r\frac{\partial}{\partial r})$, where J denotes the complex structure of Y.*

Fixing a Reeb field, and a compatible metric is analogous to fixing an ample line bundle L, and choosing a metric in $c_1(L)$. To see this, let Y be a polarized affine variety with $\dim_{\mathbb{C}} Y = n+1$, and let ξ be a rational Reeb vector field. Let $\xi_{\mathbb{C}}$ be the complexification of ξ and consider the holomorphic action induced by $\xi_{\mathbb{C}} \in \mathfrak{t}_{\mathbb{C}}$. Then $Y \backslash \{0\}$ is a principal $\mathbb{C}^*$ orbibundle over the orbifold $X = Y/\mathbb{C}^*$ corresponding to an ample orbi-line bundle $L \to X$. In particular, $Y \backslash \{0\}$ is the complement of the zero section in the total space of the orbi-line bundle L^{-1}. By the Kodaira-Bailey embedding theorem [Baily, 1957], the ampleness of L is equivalent to the existence of a Hermitian metric h on L^{-1} such that $\omega = i\partial\overline{\partial} \log h$ is a metric on X. We define a function $r : Y \to \mathbb{R}_{>0}$ by $(z, \sigma) \to |\sigma|_{h(z)}$, for σ in the fibre of L^{-1} over $z \in X$. We get a metric on Y by setting

$$\Omega = i\partial\overline{\partial} r^2. \tag{2.2.6}$$

In particular, when ξ is rational, (Y, ξ) always admits a compatible Kähler metric.

Given a rational Reeb vector ξ_0, and compatible metric Ω_0 on Y, the contact 1-form η_0 is defined to be dual to ξ_0. The Reeb cone is defined in [He and Sun, 2012] to be

$$\mathcal{C}'_R = \{\xi \in \mathfrak{t} \,|\, \eta_0(\xi) > 0 \ \text{ on } Y \setminus \{0\}\}. \tag{2.2.7}$$

Proposition 2.2.4. *The cone $\mathcal{C}'_R$ in* (2.2.7) *above coincides with the Reeb cone $\mathcal{C}_R$ that we defined in Definition 2.2.1.*

Proof. We need to relate the condition that $\eta_0(\xi) > 0$ with the weights of the circle action generated by ξ on the ring of functions. As shown in [Martelli *et al.*, 2008], $H = \frac{1}{2}r^2\eta_0(\xi)$ is a Hamiltonian for the vector field ξ with respect to Ω_0. It follows that

$$J\xi = -\nabla H,$$

and moreover $H \to 0$ as we approach the cone point 0.

Suppose first that H is strictly positive, so ξ cannot vanish anywhere. It follows that if we write $\varphi_t : Y \to Y$ for the negative gradient flow of H, then

$$\lim_{t\to\infty} \varphi_t(p) = 0,$$

for any $p \in Y$. Suppose that f is a non-constant regular function on Y (for instance a coordinate function on the ambient $\mathbb{C}^N$), on which ξ acts with weight λ, and p is a point such that $f(p) \neq 0$. Then $J\xi(f) = -\lambda f$, so

$$\frac{d}{dt} f(\varphi_t(p)) = -\lambda f(\varphi_t(p)).$$

Since $f(0) = 0$, we must have $\lambda > 0$. So if $\xi \in \mathcal{C}'_R$, then $\xi \in \mathcal{C}_R$.

Conversely suppose that H is negative somewhere. Since H is homogeneous under $r\frac{\partial}{\partial r}$, we can then find points arbitrarily close to 0, where H is negative. For a suitable point p, the *positive* gradient flow φ_t of H will satisfy $\varphi_t(p) \to 0$ as $t \to \infty$. Then the same argument as above shows that if f is a non-constant homogeneous function for ξ which does not vanish at p, then the weight of ξ on f must be negative. □

Corollary 2.2.5. *If ξ is an irrational Reeb vector field on Y and Ω is a compatible Kähler metric with potential $\frac{1}{2}r_0^2$, then there exists a sequence $\xi_k \in \mathfrak{t}$ of rational Reeb vector fields* *and compatible metrics Ω_k with potentials $\frac{1}{2}r_k^2$ on Y, such that $\xi_k \to \xi$ in $\mathfrak{t}$, the Ω_k converge to Ω smoothly on compact subsets of Y, and $\{r_k = 1\} = \{r_0 = 1\}$.*

Proof. The argument is similar to that given in [He and Sun, 2012, Lemma 2.5]. We include the details for the readers' convenience. Let ξ be an irrational Reeb vector field, and denote by r_0 the potential for the Kähler form Ω on Y. Since $\mathcal{C}_R$ is a rational convex polyhedral cone, we can approximate ξ with a sequence of rational elements $\xi_k \in \mathcal{C}_R$. For ξ_k we define a map $r_k : Y \to S \times \mathbb{R}_{>0}$ as follows; by Proposition 2.2.4, the assumption that $\eta(\xi_k) > 0$ implies that the holomorphic vector field $-J\xi_k - \sqrt{-1}\xi_k$ acts with positive weights on Y. Let $\varphi_k(t, y)$ denote the image of y under the diffeomorphism of Y induced by the vector field $-J\xi_k$. Then for each $y \in Y$ there is a unique time $T_{y,k}$ such that $\varphi(T_{y,k}, y) \in \{r_0 = 1\}$. Define a smooth function by $r_k = e^{-T_{y,k}}$. Clearly $\{r_k = 1\} = \{r_0 = 1\}$. Then r_k defines a diffeomorphism $\Psi_k : Y \to S \times \mathbb{R}_{>0}$ by

$$\Psi_k(y) = (\varphi(T_{y,k}, y), e^{-T_{y,k}}).$$

It follows immediately that the functions r_k generate the Euler vector field of the trivial line bundle $S \times \mathbb{R}$. In particular, on $S \times \mathbb{R}_{>0}$ we have

$$r_k \partial_{r_k} = -J\xi_k.$$

We define

$$\Omega_k = i\partial\overline{\partial}\left(\frac{r_k^2}{2}\right).$$

Arguing as in [He and Sun, 2012, Lemma 2.5], one easily shows that Ω_k define Kähler cone metrics on Y. Finally, it is clear that if ξ_k converge to ξ smoothly on compact sets, then r_k converge smoothly to r_0 on compact sets, and hence Ω_k converges smoothly to Ω. □

2.2.1 Canonical Sasakian and Kähler Metrics

As we discussed above, any polarized Kähler manifold (X, L) gives rise to a Sasakian manifold by taking the circle bundle in $L^{-1} \to X$ defined by a metric h on L with positive curvature. The Reeb field is then nothing more than the $U(1)$ action on the circle fibers. Note that we are free to vary the metric $h \mapsto h_\varphi := e^{-\varphi}h$ for φ a smooth function, provided that the curvature of h_φ is still positive. Moreover, by the $\partial\overline{\partial}$-lemma, any metric in $c_1(L)$ is obtained in precisely this way. These metrics all give rise to deformations the Sasaki metric, while fixing the remaining components of the Sasakian structure. Such deformations are called "transverse Kähler deformations". The following proposition is well-known; see, for example, [Sparks, 2011].

Proposition 2.2.6. *Fix a Sasakian manifold (S, g, ξ, η, Φ). Then any other Sasakian structure on S with the same Reeb field, the same holomorphic structure on the cone $C(S)$ and the same transversely holomorphic structure on the Reeb foliation is related to the original structure via the deformed contact form $\eta' = \eta + d^c_B\varphi$ where φ is a smooth basic function, which is sufficiently small.*

We remark that the converse is also true, by a result of El-Kacimi Aloui [El Kacimi-Alaoui, 1990]. We are primarily interested in the following problem;

Question 2.2.7. Give a Sasakian manifold (S, g, ξ, η, Φ), when does there exist a metric g' so that $(S, g', \xi, \eta', \Phi)$ is Sasakian, and g' is Einstein? That is, when does (S, ξ, Φ) admit a Sasaki-Einstein metric?

The following simple proposition illustrates the importance and difficulty of this question.

Proposition 2.2.8 ([Sparks, 2011], Proposition 1.9)**.** *Let (S, g) be a Sasakian manifold of dimension $2n+1$. Then the following are equivalent;*

1. *(S, g) is Sasaki-Einstein.*
2. *The metric g satisfies $Ric(g) = 2ng$.*
3. *The Kähler cone $(C(S), \bar{g})$ is Ricci-flat. That is, $Ric(\bar{g}) = 0$.*
4. *The transverse Kähler structure to the Reeb foliation is Kähler-Einstein with $Ric^T = 2(n+1)g^T$.*

In particular, we see that the transverse Kähler structure is necessarily Fano, Kähler-Einstein. In particular, if the Reeb field is regular, then (S, g) is Sasaki-Einstein if and only if the underlying Kähler manifold is Fano, Kähler-Einstein. It is well known that Kähler-Einstein metrics on Fano manifolds do not exist in general, and there are many algebraic obstructions. This implies that in general, conical Calabi-Yau metrics are not guaranteed to exist, in contrast with the case of smooth, compact Calabi-Yau manifolds, where the existence of Ricci flat metrics is guaranteed by Yau's solution of the Calabi conjecture [Yau, 1978].

As in the case of Kähler geometry, there are several necessary topological conditions for the existence of Sasaki-Einstein metrics. We list these below.

Proposition 2.2.9 ([Sparks, 2011], Proposition 1.10). *The following necessary conditions for a Sasakian manifold S to admit a transverse Kähler deformation to a Sasaki-Einstein metric are equivalent.*

1. $c_1^B = a[d\eta] \in H_B^{1,1}(S, \xi, \Phi)$
2. $c_1^B > 0$ *and* $c_1(D) = 0 \in H^2(S, \mathbb{R})$
3. *For some* $\ell > 0$ *the* ℓ*-th power of the canonical lines bundle* $K^{\ell}_{C(S)}$ *admits a nowhere vanishing holomorphic section* Ω *with* $\mathcal{L}_\xi = i(n+1)\Omega$*.*

Upon first glance, the conditions mentioned above are weaker than the topological conditions imposed by Proposition 2.2.8 are stronger than those imposed by Proposition 2.2.9. For example, taking cohomology class of condition (4) in Proposition 2.2.8 implies that it is necessary that $a = \frac{n+1}{2\pi}$ in condition (1) above. However, by utilizing the $\mathcal{D}$-homothetic deformations introduced by Tanno [Tanno, 1968], these conditions are easily seen to be equivalent, up to rescaling the Reeb vector field by a constant. Let us describe the deformations now. For $a > 0$, the rescaling

$$g' = ag + (a^2 - a)\eta \otimes \eta, \;\; \eta' = a\eta, \;\; ,\xi' = a^{-1}\xi, \;\; \Phi' = \Phi$$

gives a Sasaki structure $(S, g', \xi', \eta', \Phi')$ with the same holomorphic structure on the cone, but with radial variable $r' = r^a$. One can check that if g^T is a transverse Einstein metric, then a $\mathcal{D}$-homothetic deformation gives a Sasaki-Einstein metric. Throughout this work, we shall assume that $c_1^B(S) > 0$, and $c_1(D) = 0$. By making a $\mathcal{D}$-homothetic deformation we may always assume that $[\frac{1}{2}d\eta]_B = 2\pi c_1^B(S)$. The goal then is to produce a Sasakian metric which is transversely Kähler-Einstein, which can then be $\mathcal{D}$-homothetically transformed to a Sasaki-Einstein metric.

Chapter 3

The Sasaki-Ricci Flow

The primary point of view taken in this thesis is that of geometric heat equations. Since we are interested in finding Sasaki-Einstein metrics, it is natural to introduce a Sasakian analogue of the Kähler-Ricci flow. This was done by Smoczyk-Wang-Zhang [Smoczyk *et al.*, 2010]. In what follows we will consider a slightly more general situation, namely that of Riemannian foliations. Let us briefly recall the definitions we will need, all of which are easily seen to hold in the Sasakian case, and indeed have already been stated in Chapter 2.

Let S be a compact manifold of dimension $2n+1$, and ξ a nowhere vanishing vector field on S. The integral curves of ξ generate a foliation of S by one dimensional submanifolds. Assume now that (S, ξ, g) is a foliated Riemannian manifold, and denote by L_ξ the subbundle of TS generated by ξ.

Definition 3.0.10. *A function $f \in C^\infty(S)$ is said to be* basic *if $\mathcal{L}_\xi f = 0$. The set of smooth basic functions will be denoted by $C^\infty_B(S)$.*

Definition 3.0.11. *A vector field X on S is said to be* foliate *with respect to the foliation induced by ξ if $[X, \xi] \in L_\xi$.*

Definition 3.0.12. *A Riemannian metric g is said to be* bundle-like *with respect to the foliation induced by ξ if for any open set U and foliate vector fields X, Y on U perpendicular to L_ξ, the function $g(X, Y)$ is basic.*

The foliation induced by ξ on S is said to be a Riemannian foliation if S admits a bundle-like metric g. Throughout this thesis we will remain firmly in the category of Riemannian foliations. Riemannian foliations admit global transverse Riemannian structures which reflect the geometry of the local Riemannian quotients. That is, given a foliated Riemannian manifold (S, ξ, g) we obtain an exact sequence

$$0 \to L_\xi \to TS \xrightarrow{p} Q \to 0 \tag{3.0.1}$$

where $Q = TS/L_\xi$. The metric g defines an orthogonal splitting of this sequence $Q \xrightarrow{\sigma} TS$, so that we may write

$$TS = L_\xi^\perp \oplus L_\xi.$$

By identifying $L_\xi^\perp \cong Q$, we get a metric g^T on Q by restricting g to $L_\xi^\perp$. In general, however, this metric will *not* yield a metric on the local Riemannan quotient. In fact, if U is a small open set, and $\pi : U \to \bar{U}$ is the projection to the local Riemannian quotient, then $g^T = \pi^*\tilde{g}$ for some metric $\tilde{g}$ on $\bar{U}$ if and only if g is bundle-like. In the case that (S, ξ, g) is a Riemannian foliation, we see the above procedure produces a global metric g^T on Q which is given locally by the pull-back of a metric from the local Riemannian quotient. If (S, ξ, g) is a Riemannian foliation such that $g(\xi, \xi) = 1$, and the integral curves of ξ are geodesics, then the foliation is said to be a *Riemannian flow*. A key feature of manifolds foliated by a Riemannian flow is that for any point $p \in S$ we can construct local coordinates in a neighbourhood of p which are simultaneously foliated, and Riemann normal coordinates. That is, we can find Riemann normal coordinates $\{x, y_1, \ldots, y_{2n}\}$ on a neighbourhood U of p, such that $\frac{\partial}{\partial x} = \xi$ on U.

We now define the Sasaki-Ricci flow, and the transverse Ricci flow. Given a Riemannian foliation (S, ξ, g) we define the transverse Ricci flow by

$$\frac{\partial g^T}{\partial t} = -Ric^T. \tag{3.0.2}$$

The short time existence for the flow (3.0.2) was established in [Lovrić *et al.*, 2000]. On a Sasakian manifold, we can exploit the transverse Kähler structure to introduce another flow for the transverse metric as follows; fix an initial Sasaki metric $g_o = \frac{1}{2}d\eta_o$ such that $\frac{1}{2}[d\eta_o]_B = 2\pi c_B^1 = [Ric^T]_B$. Using the transverse $\partial\bar{\partial}$-lemma of [El Kacimi-Alaoui, 1990], there is a basic function $F : S \to \mathbb{R}$ such that

$$Ric^T = \frac{1}{2}d\eta_o + d_B d_B^c F.$$

We then define the Sasaki-Ricci flow by

$$\frac{\partial g^T}{\partial t} = -Ric^T_{g(t)} + g^T(t), \tag{3.0.3}$$

which can be expressed locally as a parabolic Monge-Ampère equation on transverse Kähler potentials φ via

$$\frac{\partial \varphi}{\partial t} = \log\det(g^T_{\bar{k}l} + \partial_l \partial_{\bar{k}} \varphi) - \log\det(g^T_{\bar{k}l}) + \varphi - F. \tag{3.0.4}$$

It was proved in [Smoczyk *et al.*, 2010] that this flow is well-posed. Moreover, they showed that this flow preserves (S, ξ, Φ), the solution φ exists for all time, remains basic, and converges exponentially if $c_1^B < 0$ or $c_1^B = 0$. Observe that one can pass from a solution to (3.0.2) and a solution to the Sasaki-Ricci flow (3.0.3) via the usual method of dilating the metric and scaling time. In particular, the results of [Smoczyk *et al.*, 2010] imply that if the initial metric g_o is Sasakian, then the solution to (3.0.2) exists on $[0, \frac{1}{2})$ and remains Sasakian. It will be important for us that these two flows are interchangeable.

We make a brief comment on volume forms. For a contact manifold (S, η), the top form $\eta \wedge (d\eta)^n$ defines a volume form on S. If, on the other hand, (S, ξ, g) is a foliated Riemannian manifold with the foliation induced by ξ, bundle-like metric g and transverse metric g^T, then we can use the standard Riemannian volume form on S. Now, if (S, g) is a Sasaki manifold, then S is both a contact manifold, and a foliated Riemannian manifold, with metric $g = \eta \otimes \eta + \frac{1}{2} d\eta(\cdot, \Phi \cdot)$, and induced transverse metric $g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi Y)$; it is an easy exercise to check that the volume form $\eta \wedge (d\eta)^n$ agrees with the Riemannian volume form. Suppose now that the transverse metric g^T, is evolving by the Sasaki-Ricci flow. Then the evolved contact structure is given, up to a dimensional constant, by

$$\eta_t = \eta_o + d^c_B \varphi(t)$$

for a basic function $\varphi(t)$. The evolved volume form is given by

$$\eta_t \wedge (d\eta_t)^n = (\eta_o + d^c_B \varphi(t)) \wedge (d\eta_t)^n = \eta_o \wedge (d\eta_t)^n + d^c_B \varphi(t) \wedge (d\eta_t)^n$$

By observing that $(d\eta_t)^n \wedge d^c_B \varphi(t)$ is a basic form of degree $2n+1$, we see immediately that

$$\eta_t \wedge (d\eta_t)^n = \eta_o \wedge (d\eta_t)^n = \det(g^T_{\bar{k}l} + \partial_l \partial_{\bar{k}} \varphi).$$

See, for example, Lemma 5.2 of [Smoczyk *et al.*, 2010]. In particular, the canonical volume forms arising from the contact, and foliated Riemannian structures on a Sasaki manifold agree with the local transverse Kähler volume form. The following lemma is an easy consequence of the evolution equation for the Sasaki-Ricci flow.

Lemma 3.0.13. *The volume of S is preserved by the Sasaki-Ricci flow.*

We now make a brief digression on coordinates.

Definition 3.0.14 (Preferred Local Coordinates)**.** *Let (S, g) be a Sasaki manifold, $p \in S$ a point. One can choose local coordinates $(x, z^1, \dots, z^n)$ on a small neighbourhood $p \in U$ such that*

- $\xi = \frac{\partial}{\partial x}$
- $\eta = dx + \sqrt{-1}\sum_{j=1}^n h_j dz^j - \sqrt{-1}\sum_{j=1}^n h_{\bar{j}} d\bar{z}^j$
- $\Phi = \sqrt{-1}\left\{\sum_{j=1}^n \left(\frac{\partial}{\partial z^j} - \sqrt{-1}h_j\frac{\partial}{\partial x}\right)\otimes dz^j - \sum_{j=1}^n \left(\frac{\partial}{\partial \bar{z}^j} + \sqrt{-1}h_{\bar{j}}\frac{\partial}{\partial x}\right)\otimes d\bar{z}^j\right\}$
- $g = \eta\otimes\eta + 2\sum_{j,l=1}^n h_{j\bar{l}} dz^j d\bar{z}^l$

where $h : U \to \mathbb{R}$ is a local, basic function (ie. $\frac{\partial}{\partial x}h = 0$), and we have used $h_j = \frac{\partial}{\partial z^j}h$, and $h_{j\bar{l}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^l}h$.

The existence of preferred local coordinates on a Sasakian manifold is automatic; see, for example, [Godliński *et al.*, 2000]. We can additionally assume that in these coordinates $h_j(p)$=0. An important observation is that the transverse Christoffel symbols with mixed barred and unbarred indices are identically zero. Moreover, the pure barred and unbarred Christoffel symbols are given by the familiar formula for Kähler geometry

$$\Gamma^k_{ij} = (g^T)^{k\bar{p}}\partial_i g^T_{\bar{p}j}.$$

From the local formulae, the preferred local coordinates show immediately that $\frac{\partial}{\partial x}g_{ij} = \frac{\partial}{\partial x}g^T_{ij} = 0$. Moreover, if φ evolves by the transverse parabolic Monge-Ampère equation (3.0.4), then local coordinate expressions for the evolved Sasaki metric $g(t)$ in the preferred local coordinates are obtained by replacing h with $h + \frac{1}{2}\varphi$. We refer the reader to [Smoczyk *et al.*, 2010] for more useful formulae which hold in these coordinates.

3.1 Gradient flow properties of the Sasaki-Ricci flow

We now introduce two new functionals along the Sasaki-Ricci flow, which extend Perelman's well known $\mathcal{F}$ and $\mathcal{W}$ functionals to the Sasakian setting. First, we define the transverse energy functional $\mathcal{F}^T : \mathfrak{Met}^T(S) \times C^\infty_B(S) \to \mathbb{R}$, given by

$$\mathcal{F}^T(g, f) = \int_S (\mathrm{R}^T + |\nabla f|^2) e^{-f} d\mu.$$

We refer the reader to the following sections for the relevant definitions. We then prove:

Theorem 6. *Let (S, g_0) be a Sasaki manifold, foliated by its Reeb field ξ. Suppose that $g(t)$ is a solution of the transverse Ricci flow on $[0, T]$ with $g(0) = g_0$ and let $f_T \in C^\infty_B(S)$. Then there is a solution $f(t)$ to the transverse backward heat equation*

$$\frac{\partial f}{\partial t} = -\Delta_B f + |\nabla f|^2 - R^T,$$

on $[0, T]$ with $f(T) = f_T$, and $f(t)$ basic for each $t \in [0, T]$. Moreover, $\mathcal{F}^T(g(t), f(t))$ is monotonically increasing.

A key feature of this functional $\mathcal{F}^T$ is that we succeed in relating its gradient flow to the transverse Ricci flow via time dependent diffeomorphisms which preserve the foliation. In particular, these diffeomorphisms fix the Reeb field and the integral curves of the Reeb field are geodesics with respect to the pulled-back metric.

Next, we define the transverse entropy functional
$\mathcal{W}^T : \mathfrak{Met}^T(S) \times C^\infty_B(S) \times \mathbb{R}_{>0} \to \mathbb{R}$; it is given by

$$\mathcal{W}^T(g, f, \tau) = (4\pi\tau)^{-n} \int_S \left(\tau(\mathrm{R}^T + |\nabla f|^2) + (f - 2n)\right) e^{-f} d\mu.$$

We then prove:

Theorem 7. *Let (S, g_0) be a Sasaki manifold, foliated by its Reeb field ξ. Suppose that $g(t)$ a solution of the transverse Ricci flow on $[0, T]$ with $g(0) = g_0$. Let $\tau(t)$ be a positive function on $[0, T]$ with $\frac{d}{dt}\tau = -1$. Let $f_T \in C^\infty_B(S)$. Then there is a solution $f(t)$ to the transverse backward heat equation*

$$\frac{\partial f}{\partial t} = -\Delta_B f + |\nabla f|^2 - R^T + \frac{n}{\tau}$$

on $[0, T]$ with $f(T) = f_T$, and $f(t)$ basic for each $t \in [0, T]$. Moreover, $\mathcal{W}^T(g(t), f(t), \tau(t))$ is monotonically increasing.

3.1.1 The transverse energy functional

The aim of this section is to prove Theorem 6. In the course of the proof we will see that the transverse Ricci flow is related to the gradient flow of the $\mathcal{F}^T$ functional via diffeomorphisms. This is in exact analogy with the Ricci flow. However, if (S, g) is Sasakian, the diffeomorphisms which relate the two flows need not preserve the transverse holomorphic structure of the manifold. For this reason it is convenient to consider a larger class of manifolds, namely manifolds foliated by geodesics. Given a Riemannian foliation (S, ξ), the class of metrics for which the foliation is generated by a Riemannian flow is described in the following definition.

Definition 3.1.1. *Let (S, ξ) be a Riemannian foliation. We define the space $\mathfrak{Met}^T(S)$ to be*

$$\mathfrak{Met}^T(S, \xi) = \left\{ g \in C^\infty(S, S^2T^*M) | g > 0, g(\xi, \xi) = 1, \mathcal{L}_\xi g = 0 \right\}.$$

Remark 3.1.2. Note that if $g \in \mathfrak{Met}^T(S, \xi)$, then g is bundle-like. Moreover, it is a fact that if $g \in \mathfrak{Met}^T(S)$, then the integral curves of ξ are geodesics. In particular, for a general one-dimensional foliation the set $\mathfrak{Met}^T(S, \xi)$ may be empty. However, in all applications we consider $\mathfrak{Met}^T(S, \xi)$ will not be empty.

In order to lighten notation, when the vector field ξ generating the foliation is understood, we will denote $\mathfrak{Met}^T(S, \xi)$ by $\mathfrak{Met}^T(S)$.

By the above remarks, we see that every metric $g \in \mathfrak{Met}^T(S)$ induces a transverse metric g^T. In fact, the opposite also holds. Given a transverse metric g^T on the quotient bundle Q, let $g \in \mathfrak{Met}^T(S)$ and define a metric $\tilde{g} \in \mathfrak{Met}^T(S)$ as follows; the metric g defines an orthogonal decomposition $TS = L_\xi \oplus L_\xi^\perp$, hence any vector X may be decomposed as $X = X^\perp + X^\xi$ where $X^\perp \in L_\xi^\perp$, and $X^\xi \in L_\xi$. We then define the metric $\tilde{g}$ by

$$\tilde{g}(X, Y) = g(X^\xi, Y^\xi) + g^T(X^\perp, Y^\perp).$$

It is an easy exercise to check that $\tilde{g} \in \mathfrak{Met}^T(S)$, and that $\tilde{g}$ induces the transverse metric g^T.

Definition 3.1.3. *We define the transverse energy functional* $\mathcal{F}^T : \mathfrak{Met}^T(S) \times C_B^\infty(S) \to \mathbb{R}$ *by*

$$\mathcal{F}^T(g, f) = \int_S (R^T + |\nabla f|^2) e^{-f} d\mu.$$

where $d\mu$ *is the volume form associated to the metric* g.

Remark 3.1.4. We remark that while we have defined the functional $\mathcal{F}^T$ in terms of the full metric $g \in \mathfrak{Met}^T(S)$, it clearly only depends on the induced transverse metric.

Our first task is to compute the variation of $\mathcal{F}^T$. In order to do this, we need to define the space of variations which preserve $\mathfrak{Met}^T(S)$. We observe that if $v \in C^\infty(S, S^2T^*S)$ has $v(\xi, \xi) = 0$, and $\mathcal{L}_\xi v = 0$, then for t sufficiently small, $g + tv \in \mathfrak{Met}^T(S)$. This motivates the following definition:

Definition 3.1.5. *The (formal) tangent space of* $\mathfrak{Met}^T(S)$, *is the set*

$$T\mathfrak{Met}^T(S) = \left\{ v \in C^\infty(S, S^2T^*S) : v(\xi, \xi) = 0, \text{ and } \mathcal{L}_\xi v = 0 \right\}.$$

We can now compute the variation of the transverse energy functional.

Proposition 3.1.6. *If* $g \in \mathfrak{Met}^T$, *and* $v \in T\mathfrak{Met}^T(S)$, *and* h *is a basic function then*

$$\delta \mathcal{F}^T(v, h) = \int_S e^{-f} \left(-v_{ij} \left(Ric^T_{ij} + \nabla_i \nabla_j f \right) + (v - h)(2\Delta_B f - |\nabla f|^2 + R^T) e^{-f} \right) d\mu.$$

Proof. Let $g \in \mathfrak{Met}^T(S)$, and fix $p \in S$. By the remark following Definition 3.1.1, we can find normal coordinates $\{x, y_1, \ldots, y_{2n}\}$ in a neighbourhood U of p for the metric g such that $\frac{\partial}{\partial x} = \xi$, and $\{\frac{\partial}{\partial y_1}|_p, \ldots, \frac{\partial}{\partial y_{2n}}|_p\}$ is an orthonormal basis for $D(p)$. In these coordinates, g takes the form

$$\begin{pmatrix} 1 & g_{01} & g_{02} & \cdots & g_{02n} \\ g_{10} & \widetilde{g_{11}} + g_{01}^2 & \widetilde{g_{12}} + g_{01}g_{02} & \cdots & \widetilde{g_{12n}} + g_{01}g_{02n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{2n0} & \widetilde{g_{2n1}} + g_{02n}g_{01} & \widetilde{g_{2n2}} + g_{02n}g_{02} & \cdots & \widetilde{g_{2n2n}} + g_{02n}^2 \end{pmatrix}$$

where $\widetilde{g_{ij}}$ is the (i,j)-th component of g^T, and we have used the index 0 to denote the dx

components. The variation v is of the form

$$\begin{pmatrix} 0 & v_{01} & \dots & v_{02n} \\ v_{10} & v_{11} & \dots & v_{12n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{2n0} & v_{2n1} & \dots & v_{2n2n} \end{pmatrix}$$

where $\frac{\partial v_{ij}}{\partial x} = 0$. In particular, observe that $\delta g^T_{ij}(p) = v_{ij}(p)$. For $1 \leqslant i, j, k \leqslant 2n$. The connection coefficients for the transverse Levi-Civita connection are

$$\widetilde{\Gamma}^k_{ij} = \frac{1}{2}(g^T)^{kl}\left(\partial_i g^T_{jl} + \partial_j g^T_{il} - \partial_l g^T_{ij}\right) \text{ for } 1 \leqslant i, j, k \leqslant 2n$$
$$\widetilde{\Gamma}^k_{0j} = 0 \text{ for } 1 \leqslant j, k \leqslant 2n.$$

Thus, at p, we have

$$\begin{aligned} \delta R^T &= \sum_{i,j=1}^{2n} -v_{ij}\left(\partial_k \widetilde{\Gamma}^k_{ij} - \partial_j \widetilde{\Gamma}^k_{ik}\right) + \partial_k \partial_i v_{ik} - \partial_k \partial_k v_{ii} \\ &= \sum_{i,j=1}^{2n} -v_{ij} Ric^T_{ij} + \nabla_k \nabla_i v_{ik} - \Delta tr_g v_{i,j} \end{aligned}$$

We now compute the variation of the volume form, to get

$$\delta d\mu = g^{ij} v_{ij} d\mu = tr_g v_{ij} d\mu.$$

Setting $v = tr_g v_{ij} = tr_{g^T} v_{ij}$, it follows easily that

$$\delta \mathcal{F}^T_{(v,h)} = \int_S e^{-f}\left(-v_{ij}\left(Ric^T_{ij} + \nabla_i \nabla_j f\right) + (v-h)(2\Delta_B f - |\nabla f|^2 + R^T)\right) d\mu.$$

□

Note that if g^T and f evolve by

$$\frac{\partial g^T}{\partial t} = -(Ric^T + D^2 f) \tag{3.1.5}$$

$$\frac{\partial f}{\partial t} = -\Delta_B f - R^T \tag{3.1.6}$$

and, $(g(t), f(t)) \in \mathfrak{Met}^T(S) \times C^\infty_B(S)$, then $\mathcal{F}^T((g(t), f(t))$ is increasing, and

$$\frac{d}{dt}\mathcal{F}^T(g(t), f(t)) = \int_S |Ric^T + D^2 f|^2 e^{-f} d\mu.$$

As in the Ricci flow, the gradient flow equations (3.1.5), and (3.1.6), are related to the transverse Ricci flow via time-dependent diffeomorphisms.

Proposition 3.1.7. *Given a solution* $(g(t), f(t)) \in \mathfrak{Met}^T(S) \times C_B^\infty$ *to*

$$\frac{\partial g^T}{\partial t} = -Ric^T,$$

$$\frac{\partial f}{\partial t} = -\Delta_B f + |\nabla f|^2 - R^T \tag{3.1.7}$$

with $(g(t), f(t))$ *defined on* $[0, T]$, $f(t)$ *basic,* $g(0)$ *Sasakian, define a 1-parameter family of diffeomorphisms* $\rho(t) : S \to S$ *by*

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2}\nabla_{g(t)} f(t), \quad \rho(0) = id_S \tag{3.1.8}$$

which is a system of ODE admitting a solution on $[0, T]$*. Then the pulled back metrics* $\bar{g}(t) = \rho(t)^* g(t)$ *are bundle-like, and hence induce transverse metrics* $\bar{g}^T(t)$*. The transverse metrics* $\bar{g}^T(t)$ *and the pulled back dilaton* $\bar{f}(t) = \rho(t)^* f(t)$ *satisfy equations (3.1.5), and (3.1.6) respectively. Moreover,* $\bar{f}$ *is basic, and* $\bar{g}(t) \in \mathfrak{Met}^T(S)$*.*

Remark 3.1.8. The assumption that $g(t) \in \mathfrak{Met}^T(S)$ is not required. It follows from the work in [Smoczyk *et al.*, 2010], that if $g(0)$ is Sasakian, and $g(t)$ solves (3.0.2), then $g(t)$ is Sasakian, and hence $g(t) \in \mathfrak{Met}^T(S)$. That equation (3.1.8) admits a solution of $[0, T]$ is standard; see, for example, Lemma 3.15 in [Chow and Knopf, 2004].

The proof of Proposition 3.1.7 will follow essentially from the following lemma.

Lemma 3.1.9. *Under the assumptions of Proposition 3.1.7, for each* $t \in [0, T]$*, we have*

(i) $\rho(t)_* \xi = \xi$.

(ii) $\rho(t)^* g$ is bundle-like for the foliation induced by the vector field ξ. Moreover, $\rho(t)^*(g^T) = (\rho(t)^* g)^T$.

Proof. For the rest of the proof, fix a point $p \in S$ and choose preferred local coordinates $\{x, z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n\}$ in an open neighbourhood $U \ni p$, for the metric $g(0)$.

We begin by proving *(i)* for small time; fix a time t and asume that t is chosen sufficiently small so that $\rho(s)(x, 0, \ldots, 0)$ remains in the coordinate patch for every $x \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, and $0 \leqslant s \leqslant 2t$. Observe that $f(t)$ and $g(t)$ are independent of x, and so the time dependent vector field $-\frac{1}{2}\nabla_{g(t)} f(t)$ is independent of x. The curves $\rho(s)(x, 0, \ldots, 0)$ have the same tangent vectors for every $x \in [-\varepsilon, \varepsilon]$ and $s \in [0, t]$. Hence $\rho(t)(x, 0, \ldots, 0) = (x, 0, \ldots, 0) + \rho(t)(0, 0, \ldots, 0)$. It is clear then that $\rho(t)_*\xi = \xi$, proving the result for small t. The result for general t is obtained by taking a sequence of preferred local coordinates in open sets U_i which cover the curve $\rho(s)$ where $s \in [0, t]$, and applying the above argument repeatedly.

If h is a smooth, basic, local function in a neighbourhood of $\rho(t)(p)$, then $\rho(t)^*h$ is a smooth, basic function in a neighbourhood of p, since $\xi\rho^*h = \rho_*\xi h = \xi h = 0$. Using the local formula for the evolved contact form, and the above preferred coordinates, this yields

$$\rho(t)^*\eta(t) = dx + \sqrt{-1}\sum_{j=1}^{n} f_j dz^j - \sqrt{-1}\sum_{j=1}^{n} f_{\bar{j}} d\bar{z}^j,$$

where $f_j, f_{\bar{j}}$ are smooth, basic, local functions. A similar argument provides a local formula for $\rho(t)^*g(t)$. They key point is that the component functions of $\rho(t)^*\eta(t)$, and $\rho(t)^*g(t)$ are basic in the preferred coordinates. From now on, we suppress the argument t. We have the following formula for the metric

$$\rho^*g = \rho^*(\eta \otimes \eta + d\eta(\cdot, \Phi\cdot)) = \rho^*\eta \otimes \rho^*\eta + d\eta(\rho_*\cdot, \Phi\rho_*\cdot). \tag{3.1.9}$$

In particular, $\rho^*g(\xi, \cdot) = \rho^*\eta$. We can now prove *(ii)*. Suppose that X is a foliate vector field orthogonal ξ, then

$$X = A(x, z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)\frac{\partial}{\partial x} + Z(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n),$$

where

$$Z = \sum_{i=1}^{n} B_i(z_1, ..., z_n, \bar{z}_1, ..., \bar{z}_n)\frac{\partial}{\partial z_i} + \sum_{i=1}^{n} C_i(z_1, ..., z_n, \bar{z}_1, ..., \bar{z}_n)\frac{\partial}{\partial \bar{z}_i}.$$

Since X is orthogonal ξ, and $\rho_*\xi = \xi$ we have $0 = \rho^*g(X, \xi) = \rho^*\eta(X) = A + \rho^*\eta(Z)$, and so

$$A(x, z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n) = -\rho^*\eta(Z). \tag{3.1.10}$$

By the above arguments, the right hand side of (3.1.10) is a basic function, and thus A is independent of x. It follows that $\rho^*g(X, Y)$ is a basic function for any vector fields X, Y verifying the assumptions of Definition 3.0.12. The last statement is an easy application of the definition of g^T, and property *(i)*. □

Note that in equation (3.1.9), it is precisely the fact that $\Phi\rho_* \neq \rho_*\Phi$ which prevents the pulled back metric ρ^*g from being Sasakian, and necessitates our consideration of the large class of manifolds foliated by Riemannian flows. We can now prove Proposition 3.1.7.

Proof of Proposition 3.1.7. That $\bar{g}$ is bundle-like follows from Lemma 3.1.9. That $\bar{f}$ is basic is clear. That equations (3.1.5) and (3.1.6) are satisfied is a standard computation. That $\bar{g} \in \mathfrak{Met}^T(S)$ follows from Lemma 3.1.9, since $\mathcal{L}_\xi \rho^* g = \mathcal{L}_\xi g$ if $\rho_*\xi = \xi$. □

In order to prove Theorem 6, we are thus reduced to showing that we can solve the coupled equations (3.0.2) and (3.1.7).

Proposition 3.1.10. *If $g^T(t)$ is a transverse Ricci flow on $[0,T]$ with $g(0)$ Sasakian, and f_T is a basic function, then $\exists f(t)$ for $t \in [0,T]$ solving the transverse backward heat equation (3.1.7) with $f(t)$ basic for each time t, and $f(T) = f_T$.*

Proof. Let $\{U_\alpha\}$ be a cover of S by preferred local coordinate charts, constructed with respect to the metric $g(0)$, and let $\{V_\alpha\}$ be also as before. Set $u(s) = e^{-f(s)}$, where $s = T - t$. Then the equation (3.1.7) becomes

$$\frac{\partial u}{\partial s} = \Delta_B u - R^T u, \quad u(0) = \hat{u}.$$

Fix an open set U_β. Note that by the existence and uniqueness results of [Smoczyk *et al.*, 2010], $(g^T)^{-1}$ and R^T are independent of x. By assumption, the initial condition $\hat{u}$ is independent of x also. Thus, if we find a solution $\tilde{u}$ to

$$\frac{\partial \tilde{u}}{\partial s} = \Delta \tilde{u} - R^T \tilde{u}, \quad \tilde{u}(0) = \hat{u}, \tag{3.1.11}$$

on $V_\beta \subset \mathbb{C}^n$ and then set $u(x, z_1, \ldots z_n) = \tilde{u}(z_1, \ldots, z_n)$, then this will solve the problem on U_β. Now, equation (3.1.11) is parabolic as long as g^T remains positive definite, and hence we can solve (3.1.11) on $[0,T]$. We have thus generated a family of local basic solutions $\{U_\alpha, u_\alpha\}$. It remains to show that these solutions glue to a global solution. This is a consequence of the following lemma.

Lemma 3.1.11. *Let $w(t)$ be a basic function solving (3.1.11) on $U_\alpha \cap U_\beta$ with $w(0) = 0$. Then $w(t) = 0$ $\forall t \in [0,T]$.*

Proof. Since R^T is a smooth function on $S \times [0,T]$, we may choose $\lambda > 0$ so that $R^T + \lambda > 0$ on $S \times [0,T]$. Set $v = e^{-\lambda s} w$, and observe that v satisfies

$$\frac{\partial v}{\partial s} = \Delta v - (R^T + \lambda) v$$

where we are considering the equation on $V \subset \mathbb{C}^n$. Applying the maximum principle to v and $-v$, and using that v is basic proves the lemma. □

Now, the lemma shows that the solutions glue to a global solution, and it is clear, by construction, that the solution is basic. □

We now investigate the dependence of the functional $\mathcal{F}^T$ under diffeomorphisms of the type constructed in Proposition 3.1.7, whose existence we have just demonstrated in Proposition 3.1.10. We begin by abstracting the properties of these diffeomorphisms.

Definition 3.1.12. *Let $\mathfrak{Diff}$ be the group of diffeomorphisms of S. We define the class of transverse diffeomorphisms $\mathfrak{Diff}^T$*

$$\mathfrak{Diff}^T = \left\{ \rho \in \mathfrak{Diff} : \rho_* \xi = \xi \text{ and } \rho^*(g^T) = (\rho^* g)^T \right\}$$

First note that $\mathfrak{Diff}^T$ is a subgroup of $\mathfrak{Diff}$ and that $\mathfrak{Diff}^T$ preserves the class $\mathfrak{Met}^T$. We now show that the group $\mathfrak{Diff}^T$ preserves the transverse scalar curvature.

Lemma 3.1.13. *Let $g \in \mathfrak{Met}^T(S)$. If $\rho \in \mathfrak{Diff}^T$ then $\rho^* R^T(g) = R^T(\rho^* g)$*

Proof. Fix a point $s \in S$. Let $\{E_i\}$ be a local orthonormal frame for $L_\xi^\perp$. By [Boyer and Galicki, 2008], we have the following formula for any metric $g \in \mathfrak{Met}^T(S)$,

$$R^T = R + 2 \sum_i g^T(\nabla_{E_i} \xi, \nabla_{E_i}).$$

Where R is the scalar curvature of the metric g. Observe that as $\mathfrak{Diff}^T$ preserves $\mathfrak{Met}^T(S)$, we have

$$\rho^* R^T(g) - R^T(\rho^* g) = 2 \sum_i \rho^*(g^T(\nabla_{E_i} \xi, \nabla_{E_i} \xi)) - (\rho^* g)^T(\tilde{\nabla}_{\rho_*^{-1} E_i} \xi, \tilde{\nabla}_{\rho_*^{-1} E_i} \xi). \quad (3.1.12)$$

where we have used $\tilde{\nabla}$ to denote the Levi-Civita connection of $\rho^* g$, and that $\{\rho_*^{-1} E_i\}$ is an orthonormal frame with respect to $\rho^* g$. It is important to note that the first term on the right hand side of (3.1.12) is the pullback of a *function.* In particular, we have

$$\rho^*(g^T(\nabla_{E_i}\xi, \nabla_{E_i}\xi)) = \rho^* g^T(\rho_*^{-1}\nabla_{E_i}\xi, \rho_*^{-1}\nabla_{E_i}\xi).$$

It is elementary to check that

$$\tilde{\nabla}_X Y = \rho_*^{-1}\left(\nabla_{\rho_* X}\rho_* Y\right).$$

Thus, since $\rho \in \mathfrak{Diff}^T$ we see that the right hand side of (3.1.12) is zero. □

Proposition 3.1.14. *The functional* $\mathcal{F}^T$ *is invariant under the action of* $\mathfrak{Diff}^T$.

Proof. By Lemma 3.1.13, it suffices to show that if $f \in C_B^\infty(S)$ then so is $\rho^* f$. This last fact is obvious as $\rho_*\xi = \xi$. Thus we have

$$|\nabla f|_{g^T} = |\nabla f|_g = |\nabla \rho^* f|_{\rho^* g} = |\nabla \rho^* f|_{(\rho^* g)^T} = |\nabla \rho^* f|_{\rho^*(g^T)}$$

□

We can now prove Theorem 6.

Proof of Theorem 6. By Proposition 3.1.10 we can solve equation (3.1.7). By Proposition 3.1.7 the pulled back pair $(\bar{g}(t), \bar{f}(t))$ satisfy the gradient flow equations (3.1.5) and (3.1.6), and hence $\mathcal{F}^T(\bar{g}(t), \bar{f}(t))$ is increasing. By Lemma 3.1.9, the time dependent diffeomorphisms $\rho(t) \in \mathfrak{Diff}^T$ and so by Proposition 3.1.14, $\mathcal{F}^T(g, f) = \mathcal{F}^T(\bar{g}, \bar{f})$. The theorem is proved. □

3.1.2 The transverse entropy functional

The aim of this section is to prove Theorem 7. As with the Ricci flow, the proof of the theorem is essentially the same as the proof of Theorem 6, and hence we will omit the details. We refer the reader to [Chow *et al.*, 2007], [Perelman, 2002] for the details in the Ricci flow case.

Definition 3.1.15. *Let (S, ξ) be a foliated Riemannian manifold. Define the transverse entropy functional* $\mathcal{W}^T : \mathfrak{Met}^T(S) \times C_B^\infty(S) \times \mathbb{R}_{>0} \to \mathbb{R}$ *by*

$$\mathcal{W}^T(g, f, \tau) = (4\pi\tau)^{-n} \int_S \left(\tau(R^T + |\nabla f|^2) + (f - 2n)\right) e^{-f} d\mu$$

where $d\mu$ is the volume form associated to the metric g.

Remark 3.1.16. Before proceeding, we make a few observations about the scale invariance of $\mathcal{W}^T$. By the remarks following Definition 3.1.3, we see that $\mathcal{W}^T$ depends only on the induced transverse metric. Moreover, it is easy to see that if $g, \tilde{g} \in \mathfrak{Met}^T(S)$ induce the transverse metrics g^T, cg^T respectively, then $\mathcal{W}^T(g, f, \tau) = \mathcal{W}^T(\tilde{g}, f, c\tau)$. By abuse of notation, we will sometimes consider the functional $\mathcal{W}^T(g^T, f, \tau)$, by which we mean $\mathcal{W}^T(g, f, \tau)$ for any metric $g \in \mathfrak{Met}^T(S)$ which induces g^T. Such a metric g exists by the discussion following Definition 3.1.1.

One can compute the variational equations for $\mathcal{W}^T$ easily by using the computation for the $\mathcal{F}^T$ functional and following the same steps as in [Chow *et al.*, 2007]. In particular, one sees that if $(g(t), f(t), \tau(t))$ evolve by

$$\frac{\partial g^T}{\partial t} = -(Ric^T + D^2 f) \tag{3.1.13}$$

$$\frac{\partial f}{\partial t} = -\Delta_B f - R^T + \frac{n}{\tau} \tag{3.1.14}$$

$$\frac{d\tau}{dt} = -1 \tag{3.1.15}$$

then $\mathcal{W}^T(g(t), f(t), \tau(t))$ is increasing, and $\int_S (4\pi\tau)^{-n} e^{-f} d\mu = 1$. By an elementary modification of the proof of Proposition 3.1.7 we have

Proposition 3.1.17. *Suppose* $(g(t), f(t), \tau(t)) \in \mathfrak{Met}^T(S) \times C_B^\infty \times \mathbb{R}_{>0}$ *solves*

$$\frac{\partial g^T}{\partial t} = -Ric^T$$

$$\frac{\partial f}{\partial t} = -\Delta_B f + |\nabla f|^2 - R^T + \frac{n}{\tau} \tag{3.1.16}$$

$$\frac{d\tau}{dt} = -1$$

with $(g(t), f(t))$ defined on $[0,T]$, $f(t)$ basic, $g(0)$ Sasakian, and $\tau(t) > 0$. Define a 1-parameter family of diffeomorphisms $\rho(t) : S \to S$ by

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2}\nabla_{g(t)} f(t), \quad \rho(0) = id_S$$

which is a system of ODE admitting a solution on $[0,T]$. Then the pulled back metrics $\bar{g}(t) = \rho(t)^* g(t)$ are bundle-like, and hence induce transverse metrics $\bar{g}^T(t)$. The transverse metrics $\bar{g}^T(t)$ and the pulled back dilaton $\bar{f}(t) = \rho(t)^* f(t)$ satisfy equations (3.1.13), and (3.1.14) respectively. Moreover, $\bar{f}$ is basic, and $\bar{g}(t) \in \mathfrak{Met}^T(S)$.

Easy modifications of the proofs of Propositions 3.1.7 and 3.1.14 yield

Proposition 3.1.18. *If $g^T(t)$ is a transverse Ricci flow on $[0,T]$ with $g(0)$ Sasakian, $\tau(t)$ a positive solution of (3.1.15), and f_T is a basic function, then $\exists f(t)$ for $t \in [0,T]$ solving the transverse backward heat equation (3.1.16) with f(t) basic for each time t, and $f(T) = f_T$.*

Proposition 3.1.19. *The functional $\mathcal{W}^T$ is invariant under the action of $\mathfrak{Diff}^T$.*

We can now prove Theorem 7 by mimicking the proof of Theorem 6.

Proof of Theorem 7. By Proposition 3.1.18 we can solve equation (3.1.16). By Proposition 3.1.17 the pulled back triple $(\bar{g}(t), \bar{f}(t), \tau(t))$ satisfy the gradient flow equations (3.1.13), (3.1.14) and (3.1.15) and hence $\mathcal{W}^T(\bar{g}(t), \bar{f}(t), \tau(t))$ is increasing. By Lemma 3.1.9, the time dependent diffeomorphisms $\rho(t) \in \mathfrak{Diff}^T$ and so by Proposition 3.1.19, $\mathcal{W}^T(g, f, \tau) = \mathcal{W}^T(\bar{g}, \bar{f}, \tau)$. The theorem is proved. □

We now define the transverse analogue of Perelman's μ-functional.

Definition 3.1.20. *Set*

$$\chi = \left\{ (g, f, \tau) \in \mathfrak{Met}^T(S) \times C^\infty_B(S) \times \mathbb{R}_{>0} : \int_S (4\pi\tau)^{-n} e^{-f} d\mu = 1 \right\}.$$

Then define the functional $\mu^T : \mathfrak{Met}^T(S) \times \mathbb{R}_{>0}$ by

$$\mu^T(g, \tau) = \inf \left\{ \mathcal{W}^T(g, f, \tau) : (g, f, \tau) \in \chi \right\}.$$

Remark 3.1.21. It is easy to check from the definition that μ^T is invariant under the action of $\mathfrak{Diff}^T$. Moreover, from the definition, μ^T shares the same scale invariance as $\mathcal{W}^T$; we refer the reader to the remark following Definition 3.1.15.

In analogy with the Ricci flow, the μ^T-functional is always finite.

Proposition 3.1.22. *For any pair* $(g, \tau) \in \mathfrak{Met}^T(S) \times \mathbb{R}_{>0}$,

$$\mu^T(g, \tau) > -\infty.$$

Proof. The proof is identical to the proof for the general Ricci flow. We refer the reader to [Chow *et al.*, 2007] [Rothaus, 1981], and Section 3.1.3. □

Proposition 3.1.23. *Let* (S, ξ) *be a foliated Riemannian manifold,* $g(t)$ *a solution of the transverse Ricci flow on* $[0, T]$ *with* $g(0)$ *Sasakian. Suppose that* $\tau(t)$ *is a solution of (3.1.15). Then* $\mu^T(g(t), \tau(t))$ *is increasing.*

Proof. Let $t_0 \in [0, T]$, and $f(t_0) \in C_B^\infty(S)$ such that $(g(t_0), f(t_0), \tau(t_0)) \in \chi$. Let $f(t)$ be a solution of (3.1.16). Then, by the monotonicity of $\mathcal{W}^T$ we have

$$\mu^T(g(0), \tau(0)) \leqslant \mathcal{W}^T\left(g(0), f(0), \tau(0)\right) \leqslant \mathcal{W}^T\left(g(t_0), f(t_0), \tau(t_0)\right).$$

Taking the infimum of the last quantity yields the result. □

Throughout this section, and the last, we assumed that the evolving metric $g(t)$ was a solution of the transverse Ricci flow. In the remaining sections, we will be working with the normalized version of the transverse Ricci flow, which we have been referring to as the Sasaki-Ricci flow. Observe that if $g^T(s)$ is a solution of the Sasaki-Ricci flow, then $\tilde{g}^T(t) = (1-t)g^T(-\log(1-t))$ is a solution of the transverse Ricci flow. Thus, if we set $\tau(t) = 1 - t$ and let $f(t)$ evolve according to (3.1.16) then the transverse scale invariance of $\mathcal{W}^T$ implies that

$$\mathcal{W}^T((1-t)g^T(-\log(1-t)), f(t), 1-t) = \mathcal{W}^T(g^T(-\log(1-t)), f(t), 1)$$

is increasing. In particular, we have that $\mu^T(g^T(s), 1)$ is increasing along the Sasaki-Ricci flow. An important property of the μ^T functional is:

Lemma 3.1.24. *Let* (S, g) *be a Sasakian manifold, and let* $\tau > 0$. *There exists* $f_\tau \in C_B^\infty(S)$ *so that* $\mathcal{W}^T(g, f_\tau, \tau) = \mu^T(g, \tau)$.

The proof is given in the next section.

Remark 3.1.25. A consequence of the proof of Lemma 3.1.24 is that

$$\mu^T(g,\tau) = \inf\left\{\mathcal{W}^T(g,f,\tau) : f \in W^{1,2}_B(S), \int_S (4\pi\tau)^{-n} e^{-f} d\mu = 1\right\}.$$

We refer the reader to Section 3.1.3 for the definition of $W^{1,2}_B(S)$.

3.1.3 Rudiments of L^P and Sobolev theory on Sasakian manifolds

We pause briefly to fill in some necessary technical tools which we need to finish the proof of Lemma 3.1.24 in the previous section. These tools will also be necessary for some of the arguments to follow.

We begin somewhat anachronistically, by introducing the basic Sobolev spaces. We will provide two alternative definitions of these spaces, and prove they are equivalent. As mentioned in the introduction, one difficulty is the absence of partitions of unity on Sasaki manifolds, and the lack of a suitable mollifier. It is easy to see, for example, that basic mollifiers cannot exist when the Sasaki structure is irregular. Instead, we shall use the symmetry of the manifold under the action induced by the Reeb field to deduce particularly useful properties of the heat kernel. We remark that the Sobolev theory is somewhat easier to develop than the L^p theory, as the presence of at least one weak derivative provides a particularly simple characterization of the "weakly" basic functions.

Definition 3.1.26. *Define the space $W^{r,2}_B(S)$ to be the closure of $C^\infty_B(S)$ in the norm induced by $W^{r,2}(S)$.*

As promised, we now provide an alternative definition of the basic Sobolev spaces;

Lemma 3.1.27. *Let $H := \left\{f \in W^{r,2}(S) : \int_S f \cdot (\xi\varphi) d\mu = 0 \ \text{ for all } \varphi \in C^\infty(S)\right\}$. Then $H = W^{r,2}_B$.*

Proof. Given $f \in H$, let $f(x,t)$ be the solution to the heat equation on S, with $f(x,0) = f$. Then we can write

$$f(x,t) = \int_S P(x,y,t) f(y) d\mu(y).$$

By the definition of H, and the fact that $P(x,y,t)$ is a smooth function for $t > 0$ with $P(x,y,t) = P(y,x,t)$, we see immediately that $f(x,t)$ is basic for $t > 0$. Moreover, $f(x,t)$

converges to f in $W^{r,2}$ as $t \to 0^+$. Thus $H \subset W^{r,2}_B$. The containment $C^\infty_B \subset H$ is clear. We need only show that H is closed. Suppose $\{f_j\}$ is a Cauchy sequence in H. Then there exists $f \in W^{r,2}$, and a subsequence (not relabeled), such that f_j converges to f in $W^{r,2}$ and $f_j \to f$ pointwise a.e. Fix $\varphi \in C^\infty$, and let $\psi = \xi\varphi$. Then $f_j\psi \to f\psi$ a.e., $f_j\psi \in L^1$, and $|f_j\psi| \leqslant |f_j|^2 + |\psi|^2$. Thus, the dominated convergence theorem yields $0 = \lim_j \int_S f_j(\xi\varphi) = \int_S f(\xi\varphi)$. Thus, $f \in H$, and hence H is closed. This proves the lemma. □

We now proceed to the basic L^p spaces. In a similar fashion as above, we define:

Definition 3.1.28. *Define the space L^p_B to be the closure of C^∞_B in the norm induced by L^p.*

We now give an alternative description of the spaces L^p_B, which is decidedly less attractive than the characterization of the basic Sobolev spaces due to the absence of a weak derivative.

Definition 3.1.29. *We say that a measurable function f is basic if there exists a measurable function g such that for any point x, $g(x) = g(y)$ for every $y \in orb_\xi x$, and $f = g$ a.e. We say that a Borel set $U \subset S$ is a basic set if the indicator function for U, χ_U, is a basic function.*

Remark 3.1.30. We note that in the above description it is clear that a basic measurable function f is almost everywhere equal to a measurable function g with the property that g has an everywhere defined derivative in the ξ direction, and $\xi g \equiv 0$. This is, of course, not an accident.

For the most part, we will be interested in functions obtained by composing with the transverse distance function, whose definition we give now

Definition 3.1.31. *For $x, y \in S$ we define the transverse distance function $d^T : S \times S \to \mathbb{R}$ by*

$$d^T(x,y) = \inf_{\{p \in orb_\xi x, q \in orb_\xi y\}} dist(p,q) \tag{3.1.17}$$

Lemma 3.1.32. *Let $H = \{f \in L^p : f \text{ is basic }\}$. Then $H = L^p_B$.*

Proof. We begin by showing that $L_B^P \subset H$. We clearly have $C_B^\infty \subset H$, and so it suffices to show that H is a closed subspace of L^p. Let $\{f_j\}$ be a Cauchy sequence in H, and without loss of generality, assume that for every j and every $x \in S$, $f_j(x) = f_j(y)$ is verified for every $y \in orb_\xi x$. Let f be the limit point of the f_j's in L^p. By passing to a subsequence we may assume that $f_j \to f$ a.e. Let $x \in S$ be a point where $\lim_j f_j(x) = f(x)$. Since the sequence $f_j(x)$ converges, the sequence $f_j(y)$ converges for any $y \in orb_\xi x$, and we may take $f(y)$ to be given by this limit. Thus, for almost every $x \in S$, we have that $f(x) = f(y)$ for every $y \in orb_\xi x$. By redefining f on a set of measure zero, we see that $f \in H$. To show that $H \subset L_B^p$, it suffices to show that $\chi_E \in L_B^p$ for every basic Borel set E. In fact, we may assume that E is closed, and that if $x \in E$, then $\overline{orb_\xi x} \subset E$, as these modifications are measure zero. For each $0 < \varepsilon < 1$, define the function f_ε by

$$f_\varepsilon(x) = \begin{cases} \varepsilon^{-1} d^T(x,E), & \text{if } d^T(x,E) \leqslant \varepsilon \\ 1, & \text{if } d^T(x,E) > \varepsilon. \end{cases}$$

Note that $f_\varepsilon = 0$ if and only is $x \in E$, since E is closed and basic. Moreover, we clearly have f_ε converging to $1 - \chi_E$ in L^p as $\varepsilon \to 0$. It is easy to check that $|f_\varepsilon(x) - f_\varepsilon(y)| \leqslant \varepsilon^{-1} d(x,y)$. It follows by Rademacher's theorem that f_ε is a.e. differentiable, and $|\nabla f_\varepsilon| \leqslant \varepsilon^{-1}$. It is clear that $\xi f_\varepsilon = 0$ a.e. Thus, $f_\varepsilon \in W_B^{1,2}(S)$, and so there exists a sequence of functions $g_{\varepsilon,j} \in C_B^\infty$ with $g_{\varepsilon,j} \to f_\varepsilon$ in L^2 as $j \to \infty$. The $g_{\varepsilon,j}$ are obtained via heat equation regularization, and so by the maximum principle $|g_{\varepsilon,j}| \leqslant 1$. By passing to a subsequence, we may assume that $g_{\varepsilon,j}$ converges to f_ε pointwise almost everywhere and by the dominated convergence theorem, $g_{\varepsilon,j} \to f_\varepsilon$ in L^p. By passing to a diagonal subsequence, we then obtain the existence of a sequence of smooth, basic functions converging to χ_E in L^p. Since the indicator functions for basic sets are clearly dense in H, we are done. □

We now note a few corollaries of the above proof, which show that L_B^p has many properties in common with the usual L^p space. Denote by Σ_B the set of basic, simple functions.

Corollary 3.1.33. *If $f \in L_B^p$ and $U \subset \mathbb{R}$ is any open set, then $f^{-1}(U)$ is a basic subset of S.*

Corollary 3.1.34. *The set of basic simple functions, denoted Σ_B, is dense in L_B^p for every $1 \leqslant p < \infty$.*

Corollary 3.1.35. *Let p,q be conjugate exponents. Suppose that* $g \in L^q_B(S)$*. Then*

$$\|g\|_q = \sup\left\{\left|\int_S fgd\mu\right| : f \in \Sigma_B \text{ and } \|f\|_p = 1\right\}.$$

The above lemmas allow us to obtain basic versions of the Riesz-Thorin, and the Marcinkiewicz interpolation theorems. The proofs of these theorems are elementary generalizations of the standard proofs. The proofs can be found in any standard book on measure theory, for example [Folland, 1999]. In particular, we have:

Theorem 8 (Riesz-Thorin Interpolation Theorem)**.** *Let* (S, g) *be a Sasaki manifold with measure induced by the standard volume form. Suppose that* $p_0, p_1, q_0, q_1 \in [1, \infty]$*. For* $0 < t < 1$*, define* p_t *and* q_t *by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}\ ,\quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T *is a linear map from* $L^{p_0}_B + L^{p_1}_B$ *into* $L^{q_0}_B + L^{q_1}_B$ *such that* $\|Tf\|_{q_0} \leqslant M_0\|f\|_{p_0}$ *for* $f \in L^{p_0}_B$ *and* $\|Tf\|q_1 \leqslant M_1\|f\|_{p_1}$ *for* $f \in L^{p_1}_B$*, then* $\|Tf\|_{q_t} \leqslant M_0^{1-t}M_1^t\|f\|_{p_t}$*, for each* $0 < t < 1$*.*

Theorem 9 (Marcinkiewicz Interpolation Theorem)**.** *Let* (S, g) *be a Sasaki manifold, with measure induced by the standard volume form. Suppose that* $p_0, p_1, q_0, q_1 \in [1, \infty]$*, such that* $p_0 \leqslant q_0$*,* $p_1 \leqslant q_1$*, and* $q_0 \neq q_1$*.For* $0 < t < 1$*, define* p *and* q *by*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}\ ,\quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T *is a sublinear map from* $L^{p_0}_B$ *to the space of basic measurable functions on S, which is both weak type* (p_0, q_0) *and weak type* (p_1, q_1)*, then* T *is strong type* (p, q)*. That is, if* $[Tf]_{q_j} \leqslant C_j\|f\|_{p_j}$ *for* $j = 0, 1$*, then* $\|Tf\|_q \leqslant B_p\|f\|_p$ *where* B_p *depends only on* p_j, q_j, C_j, p*. Moreover, for* $j = 0, 1$*,* $B_p|p - p_j|$ *(respectively* B_p*) remains bounded as* $p \to p_j$ *if* $p_j < \infty$ *(respectively* $p_j = \infty$*).*

Let us put this material to work. The following estimate plays an important role in the study of estimates along the Sasaki-Ricci flow.

Lemma 3.1.36. *Let* u *satisfy the equation* $g^T_{\bar{k}j} - Ric^T_{\bar{k}j} = \partial_j\partial_{\bar{k}}u$*. Then the following inequality*

$$\frac{1}{Vol(S)}\int_S f^2e^{-u}d\mu \leqslant \frac{1}{Vol(S)}\int_S |\nabla f|^2e^{-u}d\mu + \left(\frac{1}{Vol(S)}\int_S fe^{-u}d\mu\right)^2 \tag{3.1.18}$$

holds for all $f \in C^\infty_B(S)$*.*

The proof is elementary, and well known in the Kähler case. We include a detailed proof, as it highlights some of the technical difficulties of working transversely. We must study the operator

$$Lf = -(g^T)^{j\bar{k}}\nabla_j\nabla_{\bar{k}}f + (g^T)^{j\bar{k}}\nabla_{\bar{k}}f\nabla_j u$$

on $C^\infty_B(S,\mathbb{C})$, the space of smooth, basic, *complex* valued functions, which is *not* elliptic. However, we will still be able to analyze this operator, by observing that L *is* the restriction of an elliptic operator on $C^\infty(S,\mathbb{C})$. In order to prove the lemma, we need to show that L has a complete family of eigenvalues, its kernel is precisely the constants and the lowest strictly positive eigenvalue of L is no smaller than 1. To do this, we employ the basic Sobolev spaces introduced above.

From now on, we suppress the symbol $\mathbb{C}$, and let it be understood that we are considering complex valued functions. In order to show that L has a complete spectrum, it suffices to prove the existence of a Green's function for L. The existence of the Green's function follows from a standard argument, once we have Rellich's theorem, and an elliptic a priori estimate. Rellich's theorem was proved for general foliations in [Kamber and Tondeur, 1987]. We have

Lemma 3.1.37 ([Kamber and Tondeur, 1987] Proposition 4.5)**.** *$\forall r \geqslant 0$ and $t > 0$ the inclusion $H^{r+t}_B(S) \hookrightarrow H^r_B(S)$ is compact.*

We can obtain the required elliptic a priori estimate, and regularity theorems by observing that on basic functions, $-L$ agrees with the elliptic operator

$$\tilde{L} = g^{\alpha\beta}\nabla_\alpha\nabla_\beta - g^{\alpha\beta}\nabla_\alpha u\nabla_\beta.$$

By standard elliptic theory, the operator $\tilde{L}$ has the elliptic a priori estimate. In particular, there is a constant C_r so that, for any $\varphi \in H^{r+1}_B(S) \subset H^{r+1}(S)$ we have

$$\|\varphi\|_{r+2} \leqslant C_r\left(\|L\varphi\|_r + \|\varphi\|_{r+1}\right).$$

We obtain,

Lemma 3.1.38. *Let $\varphi \in H^{r+1}_B$ has $L\varphi \in H^r_B$. Then $\varphi \in H^{r+2}_B$, and*

$$\|\varphi\|_{r+2,B} \leqslant C_r\left(\|L\varphi\|_{r,B} + \|\varphi\|_{r+1,B}\right).$$

In fact, this holds more generally. It is well known (see for example [Kamber and Tondeur, 1987]) that the restriction of the Sobolev norm on $H^r(\Omega)$ to the subspace $H^r(\Omega_B)$ of basic forms induces a norm equivalent to the basic Sobolev norm. In particular, Lemma 3.1.38 holds for basic forms.

Observe now that in the definition of the basic Sobolev spaces we can replace the measure $d\mu$ with $e^{-u}d\mu$ to obtain a *weighted* basic Sobolev space, which we denote by $\widehat{H^r_B}(S)$. It is clear that the weighted basic Sobolev norm is equivalent to the unweighted norm, and hence Lemmas 3.1.37 and 3.1.38 hold for the weighted norm. A corollary of Lemmas 3.1.37 and 3.1.38 is

Corollary 3.1.39. *$KerL \subset H^2_B(S)$ is finite dimensional, and consists of smooth, basic functions.*

Note moreover, that the operator L is self-adjoint with respect to the inner product in $\widehat{H^0_B}(S)$. In particular, if f is basic, L satisfies

$$\int_S |\nabla f|^2 e^{-u} d\mu = \langle Lf, f\rangle.$$

We see immediately that if $f \in KerL \subset C^\infty_B(S)$, then f is constant, and that all the eigenvalues of L are real. Now, $L : \widehat{H^2_B} \to \widehat{L^2_B}$, is self-adjoint and so we can apply the usual argument to obtain the existence of the Green's function $G : \widehat{L^2_B} \to \widehat{L^2_B}$ which is compact and self-adjoint, hence has a spectral decomposition, and yields the spectrum of L. It now suffices to prove that the smallest positive eigenvalue of L is no smaller than 1. This is a straightforward computation. Assume that

$$\int_S f e^{-u} d\mu = 0 \text{ that is} f \perp KerL \text{ in } \widehat{L^2_B}.$$

We use a Bochner type argument. Differentiating the equation yields

$$\begin{aligned}\nabla^T_{\bar{l}}(Lf) = &-(g^T)^{j\bar{k}}\nabla^T_j\nabla^T_{\bar{l}}\nabla^T_{\bar{k}}f + Ric^T_{\bar{l}s}(g^T)^{s\bar{k}}\nabla^T_{\bar{k}}f + (g^T)^{j\bar{k}}\nabla^T_{\bar{l}}\nabla^T_j u\nabla^T_{\bar{k}}f \\ &+ (g^T)^{j\bar{k}}\nabla^T_j u\nabla^T_{\bar{l}}\nabla^T_{\bar{k}}f.\end{aligned}$$

We now multiply by $(g^T)^{m\bar{l}}\nabla^T_m f$ and integrate with respect to the weighted measure $e^{-u}d\mu$. The first term is

$$-\int_S (g^T)^{j\bar{k}}(g^T)^{m\bar{l}}\nabla^T_j\nabla^T_{\bar{l}}\nabla^T_{\bar{k}}f\nabla^T_m f e^{-u}d\mu.$$

Since S is Sasakian, we observe that $\nabla^T_j \nabla^T_{\bar{l}} \nabla^T_{\bar{k}} f = \partial_j \nabla^T_{\bar{l}} \nabla^T_{\bar{k}} f$ as the connection coefficients with mixed barred and unbarred indices are zero. Let $V^{\bar{l}} = (g^T)^{m\bar{l}} \nabla^T_m f$. Integration by parts yields

$$\begin{aligned}\int_S \nabla^T_{\bar{l}} \nabla^T_{\bar{k}} f \big[&- (g^T)^{j\bar{p}} \partial_j g^T_{\bar{p}q} (g^T)^{q\bar{k}} V^{\bar{l}} \\ &+ (g^T)^{j\bar{k}} \partial_j V^{\bar{l}} - (g^T)^{j\bar{k}} V^{\bar{l}} \partial_j u \\ &+ (g^T)^{j\bar{k}} V^{\bar{l}} g^{\alpha\beta} \partial_j g_{\beta\alpha} \big] e^{-u} d\mu.\end{aligned}$$

By computing in preferred local coordinates, we see that $g^{\alpha\beta}\partial_j g_{\beta\alpha} = (g^T)^{q\bar{p}} \partial_j g^T_{\bar{p}q}$. Now, the transverse Kähler condition implies that the first and last terms cancel. Using that $\nabla^T_{\bar{l}} \nabla^T_j u = \partial_{\bar{l}} \partial_j u = g^T_{\bar{l}j} - Ric^T_{\bar{l}j}$ we obtain

$$\begin{aligned}\int_S \nabla_{\bar{l}}(Lf)(g^T)^{m\bar{l}} \nabla_m f e^{-u} d\mu = \int_S \big[&(g^T)^{j\bar{k}} (g^T)^{m\bar{l}} \nabla^T_{\bar{l}} \nabla^T_{\bar{k}} f \nabla^T_j \nabla^T_m f \\ &+ (g^T)^{m\bar{l}} \nabla^T_{\bar{l}} f \nabla^T_m f \big] e^{-u} d\mu.\end{aligned} \tag{3.1.19}$$

Since the first term in (3.1.19) is positive, we see immediately that if f is an eigenfunction of L with eigenvalue $\lambda > 0$, then $\lambda \geqslant 1$. We have succeeded in proving Lemma 3.1.36.

We now turn our attention to the proof of Lemma 3.1.24. The method of proof is essentially that of Rothaus [Rothaus, 1981]. We refer the reader to [Eminenti *et al.*, 2008] for a detailed argument in the Kähler case. We first need the logarithmic Sobolev inequality.

Proposition 3.1.40 (logarithmic Sobolev inequality)**.** *Let (M^n, g) be a closed, Riemannian manifold. For any $a > 0$ there exists a constant $C(a, g)$ such that if $\varphi > 0$ satisfies $\int_M \varphi^2 = 1$, then*

$$\int_M \varphi^2 \log \varphi d\mu \leqslant a \int_M |\nabla \varphi|^2 d\mu + C(a, g).$$

Finally, we give a proof of Lemma 3.1.24, using the estimates described above.

Proof of Lemma 3.1.24. Note that it suffices to show that we can find a minimizer for $\mathcal{W}^T(g, f, 1)$. Setting $w = (4\pi)^{n/2} e^{-f/2}$, we seek to minimize

$$\int_S 4|\nabla w|^2 + \left(R^T - 2\log w - n\log(4\pi) - 2n\right) w^2 d\mu.$$

The logarithmic Sobolev inequality implies any minimizing sequence is uniformly bounded in $W^{1,2}_B(S)$. The same arguments as in [Rothaus, 1981], [Eminenti *et al.*, 2008] yield the existence of a non-negative minimizer $w_1 \in W^{1,2}_B(S)$. The minimizer must be a weak solution of the Euler-Lagrange equation

$$-4\Delta w_1 + R^T w_1 - 2w_1 \log w_1 - (n\log(4\pi) + 2n)w_1 = \mu^T(g,1)w_1.$$

By elliptic regularity we see that $w_1 \in C^\infty$, and an application of the comparison principle shows that $w_1 > 0$. Now, it is clear the w_1 is basic, as $w_1 \in C^\infty(S) \cap W_B^{1,2}(S)$. □

3.2 Estimates along the Sasaki-Ricci flow

In this section we aim to prove several estimates along the Sasaki-Ricci flow. First, we will prove the following theorem, which is a generalization of a result of Perelman. Later in this thesis we will give an effective proof of this theorem in the Kähler case

Theorem 10 ([Collins, 2013]). *Let $g(t)$ be a solution of the Sasaki-Ricci flow on a Sasaki manifold $(S, g_0, \xi, \eta, \Phi)$ of real dimension $2n+1$ with $c_B^1(S) > 0$. Let $u \in C_B^\infty(S)$ be the transverse Ricci potential. Then there exists a uniform constant C, depending only on the initial metric $g(0) = g_0$ so that*

$$|R^T(g(t))| + |u|_{C^1(g(t))} + \mathrm{diam}(S, g(t)) < C. \tag{3.2.20}$$

Later in this thesis we will give an effective proof of this theorem in the Kähler case. It would be very interesting to have an effective version of the above estimate when one also allows the Reeb vector field to vary. As an application of Theorem 10, we will prove a uniform Sobolev inequality along the Sasaki-Ricci flow, generalizing a result of Zhang [Zhang, 2007].

Theorem 11 ([Collins, 2012]). *Let (S, g_0) be a Sasaki manifold of dimension $m = 2n+1$, with basic first Chern class $c_B^1 > 0$. Let $g(t)$ be a solution of the normalized Sasaki-Ricci flow on $[0,\infty)$ with $g(0) = g_0$, and let $d\mu_t$ be the volume form with respect to $g(t)$. Then, for every $v \in W_B^{1,2}(S)$, we have*

$$\left(\int_S v^{2m/(m-2)} d\mu_t\right)^{(m-2)/m} \leqslant C \int_S \left(|\nabla v|^2 + v^2\right) d\mu_t$$

where C depends only on g_0 and m.

Recall the definition of the transverse distance function, which we introduced in Defintition 3.1.31.

Definition 3.2.1. *For $x, y \in S$ we define the transverse distance function $d^T : S \times S \to \mathbb{R}$ by*

$$d^T(x, y) = \inf_{\{p \in orb_\xi x, q \in orb_\xi y\}} dist(p, q) \tag{3.2.21}$$

An important property of the transverse distance function is the following elementary lemma.

Lemma 3.2.2. *Fix $z \in S$, and define a function $h : S \to \mathbb{R}$ by $h(y) = d^T(z, y)$. Then h is basic, and Lipschitz, with $|h(x) - h(y)| \leqslant dist(x, y)$. In particular, h is a.e. differentiable, with $|\nabla h| \leqslant 1$.*

Note that in general, we can not expect any regularity better than Lipschitz. In fact, if the Reeb field is irregular, then the transverse distance function fails to be a metric on the leaf space, and in general, can fail to be differentiable. To see the kind of pathology that can occur in this case, consider a solid torus with a central S^1, foliated by tori, which are the orbits of irrational curves. We regard the solid torus as a subset of $\mathbb{R}^3$ with the induced metric. In this case, it is clear that the distance function fails to be C^1, and that every point on the same torus is distance 0 from every other point, including points *not* on the same orbit. In the regular and quasi-regular cases, the transverse distance function defines a metric on the leaf space. The analogue of a closed geodesic ball is;

Definition 3.2.3. *The closed, transverse tube of radius r about $x \in S$ is the set*

$$T(x, r) = \{y \in S | d^T(x, y) \leqslant r\}.$$

Again, it is instructive to keep in mind the pathology that can occur when the Reeb field has non-closed orbits. If p is a point in S with non-closed orbit, then its closure defines an imbedded torus of dimension $q \leqslant n + 1$. The transverse tube $T(p, r)$, is then the union of transverse tubes about *every* point in the torus. We are led to consider the geometry of such sets. To make this plain, let (M, g) be a Riemannian manifold and P be a topologically imbedded submanifold, possibly with boundary.

Definition 3.2.4. *The closed geodesic tube around P of radius r is the set*

$T_{geo}(P,r) = \{m \in S |\ \exists$ *a geodesic* γ *of length* $L(\gamma) \leqslant r$ *from* m *meeting* P *orthogonally* $\}$.

Observe that when $r \leqslant inj(S)$, then we can write

$$T_{geo}(P,r) = \bigcup_{p \in P} \left\{ exp_p(v) | v \in T_p P^{\perp} \text{ and } \|v\| \leqslant r \right\}$$

It is a perhaps surprising fact that, for sufficiently small radius, transverse tubes and geodesic tubes are equivalent.

Proposition 3.2.5. *Let $p \in S$, and $P = \overline{orb_\xi p}$ be the torus defined by p. If $r \leqslant inj(S)$, then*

$$T(p,r) = T_{geo}(P,r)$$

Proof. First we prove the easy inclusion. Assume that $x \in T_{geo}(P,r)$. Then there is a $y \in P$, and $v \in T_y P^{\perp}$ with $\|v\| \leqslant r$ such that $x = exp_y(v)$. Fix $\varepsilon > 0$, and choose $y_\varepsilon \in orb_\xi p$ such that $d(y_\varepsilon, y) < \varepsilon$. Then,

$$d^T(p,x) = d^T(y_\varepsilon, x) \leqslant d(y_\varepsilon, x) \leqslant r + \varepsilon.$$

Since this holds for every positive ε, we get $x \in T(p,r)$.

To prove the reverse inclusion assume that $x \in T(p,r)$. By definition of d^T, we can find a $y \in P$ and $z \in \overline{orb_\xi x}$ such that $d^T(z,y) \leqslant r$ and $d^T(x,y) = d(z,y)$. Since $r \leqslant inj(S)$, there is a unique geodesic γ joining y to z which has $L(\gamma) = d(z,y)$. By an exercise in differential geometry $\gamma'(0) \in T_y P^{\perp}$. Thus, $z \in T_{geo}(P,r)$. If $orb_\xi x$ is closed, then we are done, so assume otherwise.

We claim that if $z \in \overline{orb_\xi x}$, then $x \in \overline{orb_\xi z}$. In particular, $orb_\xi z$ is not closed. This follows immediately from the fact that the Reeb field generates a Riemannian flow. First, it is clear that if $z \in \overline{orb_\xi x}$, then $orb_\xi z \subset \overline{orb_\xi x}$, as the Reeb field is Killing. Now, given a sequence $\{x_n\} \subset orb_\xi x$ converging to z, we generate a sequence $\{z_n\} \subset orb_\xi z$ converging to x by composing with the diffeomorphism φ_n generated by ξ so that $\varphi_n(x_n) = x$. We then define $z_n = \varphi_n(z)$. That z_n converges to x follows immediately from the fact that φ_n is an isometry.

Since $z \in T_{geo}(P,r)$, it is clear that $orb_\xi z \subset T_{geo}(P,r)$, as ξ is Killing, and thus $\overline{orb_\xi z} \subset T_{geo}(P,r)$, as $T_{geo}(P,r)$ is closed. By the claim, we see that $x \in T_{geo}(P,r)$. □

It is to our advantage that a great deal of work has been done to understand the relationship between the geometry of a tubular set around a submanifold P, and the ambient manifold M. In particular, we have the following asymptotic expansion for the volume of a geodesic tube:

Theorem 12 ([Gray, 2004], Theorem 9.23)**.** *Let M be a manifold of dimension n, and $P \subset M$ a submanifold of dimension q. Let $\mathbb{V}_P^M(r)$ be the volume of a geodesic tube about P. Then the following asymptotic expansion for the volume holds;*

$$\mathbb{V}_P^M(r) = \frac{(\pi r^2)^{\frac{1}{2}(n-q)}}{(\frac{1}{2}(n-q)!)} Vol(P) \left(1 + O(r^2)\right). \tag{3.2.22}$$

The proof of Theorem 10 has essentially two parts. In the first part, we employ maximum principle techniques to show that the conclusion of the theorem follows from a uniform transverse diameter bound along the Sasaki-Ricci flow. In the second part of the proof we use the functionals μ^T, and $\mathcal{W}^T$ to prove a non-collapsing theorem, which is of independent interest. We then employ the non-collapsing theorem to obtain the required diameter bound. The arguments in this section and the next are adapted from Perelman's arguments, which can be found in [Perelman, 2002], [Sesum and Tian, 2008].

Let $\varphi(t)$ be a solution to the Sasaki-Ricci flow (3.0.3) with initial condition $\varphi(0) = 0$. Using the transverse $\partial\bar{\partial}$-lemma of [El Kacimi-Alaoui, 1990], there is a basic function u such that

$$\partial_l \partial_{\bar{k}} \dot{\varphi} = \frac{\partial g^T_{\bar{k}l}}{\partial t} = g^T_{\bar{k}l} - R^T_{\bar{k}l} = \partial_l \partial_{\bar{k}} u$$

We may assume that $\dot{\varphi} = u$ and that u is normalized by the condition

$$\int_S e^{-u(t)} d\mu = (4\pi)^n. \tag{3.2.23}$$

We compute that

$$\partial_l \partial_{\bar{k}} (\frac{\partial u}{\partial t}) = \partial_l \partial_{\bar{k}} u + \partial_l \partial_{\bar{k}} \Delta_B u.$$

Thus, we may take u to evolve by $\dot{u} = \Delta_B u + u - a$, where

$$a(t) = \frac{1}{Vol(S)} \int_S u e^{-u} d\mu \tag{3.2.24}$$

Lemma 3.2.6. *The quantity a is monotone under the Sasaki-Ricci flow. In particular, there is a uniform constant C_1 depending only on $g(0)$ such that $a \geqslant C_1$*

Proof. Using the evolution equation for u, we compute

$$\dot{a} = \frac{1}{Vol(S)} \int_S \left(\Delta_B u + u - a - u^2 - u\Delta_B u + ua + u\Delta_B u\right) e^{-u} d\mu.$$

Using now the definition of a and the normalization (3.2.23), we get upon integration by parts

$$\dot{a} = a^2 + \frac{1}{Vol(S)} \int_S |\nabla u|^2 e^{-u} d\mu - \frac{1}{Vol(S)} \int_S u^2 e^{-u} d\mu.$$

The result follows from the Poincaré type inequality of Lemma 3.1.36. □

The quantity a is trivially bounded above, as xe^{-x} is bounded above on $\mathbb{R}$.

The following lemma shows that it is enough to obtain a bound from above for the scalar curvature.

Lemma 3.2.7. *The transverse scalar curvature R^T is uniformly bounded from below along the Sasaki-Ricci flow.*

Proof. We compute the evolution equation for the transverse scalar curvature

$$\dot{R^T} = -R^T + |Ric^T|^2 + \Delta_B R^T.$$

As R^T is basic, an application of the minimum principle yields the result. □

We now use the uniform bound for a to obtain a uniform lower bound for u, which reduces the problem to obtaining uniform upper bounds for the transverse scalar curvature, and u.

Lemma 3.2.8. *The function $u(t)$ is uniformly bounded below.*

Proof. The proof is identical to the proof in [Sesum and Tian, 2008], and we may use that $\Delta_B = \Delta$ on basic functions. □

Proposition 3.2.9. *There is a uniform constant C such that $|\nabla u|^2 + |R^T| \leqslant C(u + C)$.*

Proof. The computations and arguments in the proof of this proposition in [Sesum and Tian, 2008] are completely local. Thus they carry over verbatim to the Sasaki setting. □

Lemma 3.2.10. *Let $x \in S$ be such that $u(x,t) = min_{y \in S} u(y,t)$. There is a uniform constant C such that*

$$u(x,t) \leqslant C d^T(x,y)^2 + C$$

$$R^T(x,t) \leqslant C d^T(x,y)^2 + C$$

$$|\nabla u| \leqslant C d^T(x,y)^2 + C$$

Proof. By Lemma 3.2.8 we can assume $u \geqslant \delta > 0$. From Proposition 3.2.9, we have that $\sqrt{u}$ is uniformly Lipschitz bounded and basic. Thus,

$$|\sqrt{u(y,t)} - \sqrt{u(z,t)}| \leqslant \frac{|\nabla u|(p,t)}{2\sqrt{u}} d_t^T(y,z) \leqslant C d_t^T(y,z)$$

Thus,

$$u(y,t) \leqslant C_1 d_t^T(x,y)^2 + C_1 u(x,t)^2.$$

Now, $u(x,t) \leqslant K$ for some K independent of t, for if not, then

$$(4\pi)^n = \int_S e^{-u} dV_t \leqslant e^{-u(x,t)} Vol(S) \to 0.$$

In particular, $u(y,t) \leqslant C d_t^T(y,x)^2 + C'$ for C, C' independent of t. The conclusion of the lemma follows from this and Proposition 3.2.9. □

3.2.1 Non-collapsing and uniform diameter bounds along the Sasaki-Ricci flow

In this section we use the transverse entropy functional to prove a non-collapsing theorem for the Sasaki-Ricci flow, which we then use to obtain uniform diameter bounds along the Sasaki-Ricci flow. The proof of the non-collapsing theorem is essentially that of Perelman [Perelman, 2002]; see also [Chow *et al.*, 2007], [Kleiner and Lott, 2008], [Sesum and Tian, 2008].

Proposition 3.2.11. *Let $g^T(t)$ be a solution of the Sasaki-Ricci flow. There exists a positive constant C, depending only on $g(0)$, such that for every $p \in S$, $Vol(T_{g(t)}(x,1)) \geqslant C$, for time t where the metric $g(t)$ satisfies $|R^T| \leqslant 1$ on $T_{g(t)}(x,1)$.*

Note that if the condition $|R^T| \leqslant 1$ holds at a point $x \in S$, then it holds everywhere on $orb_\xi x$ as the transverse scalar curvature is constant along ξ. Thus, the conditions in Proposition 3.2.11 are local. That is, after fixing preferred local coordinates, it suffices to check the condition on a single fibre $\{x = const\}$. The proof follows from the following useful proposition.

Proposition 3.2.12. *Let $g^T(t)$ be a solution of the unnormalized Sasaki-Ricci flow $(d/dt)g(t) = -Ric^T(g(t))$. There is a constant $\kappa = \kappa(g(0)) > 0$ so that if, $|R^T(g(t))| \leqslant 1/r^2$ in a tube $T_{g(t)}(p,r)$ around a point p with $\overline{orb_\xi p}$ a torus of dimension q then $Vol_{g_{(t)}}(T_{g(t)}(p,r)) > \kappa r^{2n}$.*

The proof requires the following technical lemma, which relies crucially on the properties of the transverse distance function.

Lemma 3.2.13. *Fix $p \in S$ and $t \in [0,T)$, and suppose $\exists r > 0$ such that $|R^T| = |R^T_{g(t)}| \leqslant \frac{C}{r^2}$ on $T(p,r)$. Then there exists $r' \in (0,r]$ such that*

(i) $|R^T| = |R^T_{g(t)}| \leqslant \frac{C}{r'^2}$ *on* $T(p,r')$

(ii) $(r')^{-2n}Vol(T(p,r')) \leqslant 3^{2n} r^{-2n} Vol(T(p,r))$

(iii) $Vol_{g(t)}(T_{g(t)}(p,r')) - Vol_{g(t)}(T_{g(t)}(p,r'/2)) \leqslant C(n,q)$.

Proof. By Proposition 3.2.5 and the volume expansion in Theorem 12, we know that

$$\lim_{k\to\infty} \frac{Vol_{g(t)}(T_{g(t)}(p,r/2^k))}{Vol_{g_{(t)}}(T_{g(t)}(p,r/2^{k+1}))} = 2^{2n+1-q}$$

Hence, there is a $k < \infty$ such that

- $$\frac{Vol_{g(t)}(T_{g(t)}(p,r/2^k))}{Vol_{g(t)}(T_{g(t)}(p,r/2^{k+1}))} \leqslant 3^{2n}$$

- if $l < k$, then
$$\frac{Vol_{g(t)}(T_{g(t)}(p,r/2^l))}{Vol_{g(t)}(T_{g(t)}(p,r/2^{l+1}))} > 3^{2n} \tag{3.2.25}$$

By iterating the inequality in (3.2.25) we obtain the result with $r' = r/2^{k+1}$. □

Proof of Proposition 3.2.12. We argue by contradiction. Suppose there exists sequence of points $p_k \in S$, times $t_k \to T$, and radii r_k so that $\dim \overline{orb_\xi p_k} = q$, and

$$|R^T_k| = |R^T_{g(t_k)}| \leqslant \frac{C}{r_k^2}, \text{ but } r_k^{-2n} Vol_{g_{(t_k)}}(T_{g(s_k)}(p_k, r_k)) = r_k^{-2n} Vol(T_k) \to 0 \tag{3.2.26}$$

By Lemma 3.2.13, we may assume that $\{r_k\}$ is chosen so that property (*iii*) holds. Let $\psi \in C^\infty(\mathbb{R})$ be such that ψ is 1 on $[0, 1/2]$, decreasing on $[1/2, 1]$ and 0 on $[1, \infty)$. Define functions $\{u_k\}$ by

$$u_k = e^{C_k} \psi \left(r_k^{-1} d^T(x, p_k) \right) \tag{3.2.27}$$

where C_k is chosen so that

$$(4\pi)^n = r_k^{-2n} \int_{T_k} u_k^2 d\mu \leqslant e^{2C_k} r_k^{-2n} Vol(T_k)$$

By assumption $r_k^{-2n} Vol(T_k) \to 0$, and so we necessarily have $C_k \to \infty$. It is easy to see that the function u_k defined in (3.2.27) is basic, and $u_k \in W^{1,2}_B$. We now apply the same argument as Perelman, using instead the transverse entropy functional, see [Kleiner and Lott, 2008],[Perelman, 2002], [Sesum and Tian, 2008] or Chapter 4 below. Proposition 3.2.12 is proved. □

Proposition 3.2.11 follows easily by rescaling. It is standard to check that if $g^T(t)$ is a solution of the Sasaki-Ricci flow, then $\tilde{g}^T(s) = (1-s) g^T(t(s))$ is a solution of the unnormalized Sasaki-Ricci flow for $t(s) = -\ln(1-s)$. It is easy to see that the curvature assumption in Proposition 3.2.11 implies the curvature assumption in Proposition 3.2.12 at radius $\sqrt{1-s}$; it follows that

$$\kappa(1-s)^n \leqslant Vol_{\tilde{g}(s)}(T_{\tilde{g}(s)}(p, \sqrt{1-s})) = (1-s)^n Vol_{g(t(s))}(T_{g(t(s))}(p, 1)) \tag{3.2.28}$$

which shows that Proposition 3.2.11 follows from Proposition 3.2.12.

Remark 3.2.14. One might hope that in Proposition 3.2.12 the exponent $2n$ could be replaced with the exponent $2n + 1 - q$ which is optimal in light of Theorem 12. However, the scaling argument above, along with the fact that the normalized Sasaki-Ricci flow preserves the volume of S, shows that this is not possible unless $q = 1$

We now employ these non-collapsing results to obtain uniform transverse diameter bounds along the Sasaki-Ricci flow, which in light of Lemma 3.2.10 will prove Theorem 10.

Proposition 3.2.15. *There is a uniform constant C such that $diam^T(S, g^T(t)) \leqslant C$.*

The proof is essentially identical to Perelman's, using the same adaptations as before. We argue by contradiction. Assume that the diameters are unbounded in time. Denote by $d_t^T(z) = d_t^T(x, z)$ where $u(x,t) = \min_{y \in S} u(y,t)$. Proposition 3.2.15 is then a consequence of the following two lemmas.

Lemma 3.2.16. *For every $\varepsilon > 0$, we can find $T(k_1, k_2)$, $k_1 < k_2$, such that if $diam^T(S, g(t))$ is sufficiently large, then*

(i) $Vol(T(k_1, k_2)) < \varepsilon$

(ii) $Vol(T(k_1, k_2)) \leqslant 2^{10n} Vol(T(k_1 + 2, T_2 - 2))$

The proof of the the lemma is identical to the proof given in [Sesum and Tian, 2008], and thus we omit it. We note, that the non-collapsing theorem is crucial to the proof.

Lemma 3.2.17. *Let k_1, k_2 be as in Lemma 3.2.16. Then there exists r_1, r_2 and a constant C independent of time such that $2^{k_1} \leqslant r_1 \leqslant 2^{k_1+1}$, $2^{k_2-1} \leqslant r_2 \leqslant 2^{k_2}$, and*

$$\int_{T(r_1, r_2)} R^T \leqslant C Vol(T(k_1, k_2)) = CV$$

Proof. By the co-area formula we have

$$Vol(T(r)) = \int_{\{0 \leqslant d^T \leqslant r\}} 1 d\mu \geqslant \int_{\{0 \leqslant d^T \leqslant r\}} |\nabla d^T| d\mu = \int_0^r \mathcal{H}^{2n}\{d^T(x) = t\} dt.$$

In analogy with the Euclidean case, denote $S(t) = \mathcal{H}^{2n}\{d^T(x) = t\}$. First, note that there exists $r_1 \in [2^{k_1}, 2^{k_1+1}]$ such that

$$S(r_1) \leqslant 2\frac{V}{2^{k_1}}.$$

Indeed, if this was not the case, then

$$Vol(T(k_1, k_1 + 1) \geqslant \int_{2^{k_1}}^{2^{k_1+1}} S(t) dt > 2V = 2Vol(T(k_1, k_2))$$

which is not possible as $k_2 \gg k_1$. In a similar fashion we obtain the existence of $r_2 \in [2^{k_2-1}, 2^{k_2}]$ such that

$$S(r_2) \leqslant 2\frac{V}{2^{k_2}}.$$

Now we have

$$\begin{aligned}\int_{T(r_1,r_2)} R^T &= \int_{T(r_1,r_2)} (R^T - n) + nVol(T(r_1,r_2)) \\ &= -\int_{T(r_1,r_2)} \Delta u + nVol(T(r_1,r_2)) \\ &\leqslant \int_{\{d^T=r_1\}} |\nabla u| + \int_{\{d^T=r_2\}} |\nabla u| + nV \\ &\leqslant C'(2^{k_1+1}+1)2\frac{V}{2^{k_1}} + C'(2^{k_2}+1)2\frac{V}{2^{k_2}} + nV \leqslant CV\end{aligned}$$

where we have used the estimates in Lemma 3.2.10 in the last line. Clearly C is independent of t.

□

Proof of Proposition 3.2.15. Suppose that the diameter is not uniformly bounded. By a result of Rukimbira [Rukimbira, 1995b], [Rukimbira, 1999] the Reeb vector field ξ has at least $n+1$ closed orbits. Moreover, for every $t \geqslant 0$, we have $g(t)(\xi,\xi) = 1$. In particular, if $p \in S$ has $orb_\xi p$ closed, then the length of the curve defined by $orb_\xi p$ is independent of t. In particular, if the diameter is not bounded, there must exist points $p_t, q_t \in S$ with $d^T_{g(t)}(p_t,q_t) \to \infty$. Let ε_i be a sequence of positive real numbers with $\varepsilon_i \to 0$. By Lemmas 3.2.16 and 3.2.17, we can find sequences k_1^i, and k_2^i, such that

$$Vol_{t_i}(T_{t_i}(k_1^i,k_2^i)) < \varepsilon_i$$

$$Vol_{t_i}(T_{t_i}(k_1^i,k_2^i)) \leqslant 2^{10n} Vol_{t_i}(T_{t_i}(k_1^i+2,k_2^i-2))$$

For each i, find r_1^i and r_2^i as in Lemma 3.2.17. Let φ_i be a sequence of cutoff functions such that $\varphi(z) = 1$ on $[2^{k_1^i+2}, 2^{k_2^i-2}]$, and $\varphi = 0$ on $(-\infty, r_1^i] \cup [r_2^i, \infty)$. Let $u_i = e^{C_i}\varphi_i(d^T_{t_i}(x,p_i))$ such that $\int_S u_i^2 = (4\pi)^n$. We have

$$(4\pi)^n = e^{2C_i}\int_S \varphi_i^2 \leqslant e^{2C_i}\varepsilon_i.$$

Since $\varepsilon_i \to 0$, we have $C_i \to \infty$. We plug the functions u_i into the $\mathcal{W}^T$ functional, and follow the proof in [Sesum and Tian, 2008]. □

Combining Lemma 3.2.10, and Proposition 3.2.15, we have proved Theorem 10. Moreover, combining the uniform bound for the transverse scalar curvature with the non-collapsing estimate 3.2.11, we obtain

Proposition 3.2.18. *Let $g^T(t)$ be a solution of the normalized Sasaki-Ricci flow with $\kappa g^T(0) \in c_B^1(S) > 0$, and let $\rho > 0$ be fixed. Then there exists a constant $c > 0$, depending only on $g(0)$ and ρ, such that for every $p \in S$, $t \geqslant 0$, and r with $0 < r \leqslant \rho$,*

$$\int_{\{y:d^T(p,y)<r\}} d\mu > cr^{2n}.$$

Let us turn our attention to the proof of Theorem 11. We begin with a brief study of the basic heat kernel.

3.2.2 The basic heat kernel

We now examine the heat kernel on L^2 associated to any elliptic operator $L = \Delta - \Psi$ for Ψ a positive, smooth, basic potential function. We will show that the restriction of the heat kernel to L_B^2 is defined by a smooth, basic function for $t > 0$. Our construction depends strongly on the Lemma 3.1.37 and Lemma 3.1.38, which we apply to the *positive* operator $-L$. As we noted before, these results yield the existence of a complete, orthonormal set $\{\varphi_i, \lambda_i\}$ of eigenfunctions and eigenvalues for $-L$, spanning L_B^2, such that each φ_i is smooth and basic. We can then define the basic heat kernel of the operator $L - \partial_t$ to be

$$P^B(x, y, t) = \sum_i e^{-\lambda_i t} \varphi_i(x) \varphi_i(y). \tag{3.2.29}$$

By the usual elliptic theory, the set $\{\varphi_i, \lambda_i\}$ can be completed to a complete set of eigenfunctions and eigenvalues for the space L^2. We may choose eigenfunctions $\{\psi_i, \sigma_i\}$ so that $\{\varphi_i, \lambda_i\} \cup \{\psi_i, \sigma_i\}$ span L^2, and $\psi_j \perp \varphi_i$ for each pair i, j with respect to the L^2 inner-product. The heat kernel for the parabolic operator $L - \partial_t$ is then given by

$$P(x, y, t) = P^B(x, y, t) + P^{\perp}(x, y, t) = \sum_i e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) + \sum_i e^{-\sigma_i t} \psi_i(x) \psi_i(y). \tag{3.2.30}$$

We remark that the spectral representation in (3.2.30) is generally formal. However, in the present case, the defining sum converges uniformly to a smooth function on $S \times S \times (0, \infty]$. This follows from the following lemmas, whose proofs we only sketch.

Lemma 3.2.19. *Suppose $L\varphi = -\lambda\varphi$, and $\|\varphi\|_{L^2} = 1$. Then $\|\varphi\|_\infty < C(|\lambda| + 1)^m$ for constants C and m depending only on the dimension of S and the Sobolev imbedding theorem. Moreover, $\lambda \geqslant 0$, with equality if and only if $\varphi \equiv 0$.*

Proof. Apply the elliptic $W^{p,2}$ estimate inductively, until p is sufficiently large to apply the Sobolev imbedding theorem. The second statement follows from the positivity of Ψ, and integration by parts. □

Lemma 3.2.20. *The defining sum (3.2.30) for the heat kernel associated to the operator L converges uniformly to a smooth function for each time $t > 0$. In particular, the basic heat kernel (3.2.29) is a smooth, basic function.*

Proof. Fix a time $T > 0$. We apply the bound in Lemma 3.2.19, and Hörmander's estimate [Hörmander, 1968] for the spectral counting function to show that the spectral representations for P, P^B and $P^\perp$ converge uniformly for every $t > T$ to a continuous function. We now employ Bernstein's principle to extend the C^0 bound, $\|\varphi\|_\infty < C(|\lambda| + 1)^m$ in Lemma 3.2.19, to a C^k bound for $\|\varphi\|_{C^k} \leqslant Ch(|\lambda|)$ for a constant C independent of λ, and a polynomial h of order m_k independent of λ. Again, Hörmander's estimate for the spectral counting function yields convergence. □

From the defining formula (3.2.30), we see that, for any $f \in L^2_B$ we have

$$\int_S P^\perp(x, y, t) f(y) d\mu(y) = 0.$$

That is, the restriction of P to L^2_B is P^B. One obtains still more information about the integral kernel P^B by applying the maximum principle to the equation $L - \partial_t = 0$.

Lemma 3.2.21. *If φ solves $L\varphi - \partial_t \varphi = 0$, with $\varphi(x,t) = \varphi(x,0)$ smooth, then $\max_x \varphi^+(x,t)$, and $\max_x \varphi^-(x,t)$ are strictly decreasing. If $\varphi(0) > 0$, then $\varphi(x,t) > 0$ for all t.*

By the density of C^∞_B in the basic Sobolev and Lebesgue spaces, we see that the integral kernel $P^B(x, y, t)$ is a positivity-preserving contraction. Now, since the function $f \equiv 1$ is trivially basic, we have $1 = \int_S P(x, y, 0) d\mu(y) = \int_S P^B(x, y, 0) d\mu(y)$. Thus, for any $t > 0$,

$$\frac{\partial}{\partial t} \int_S P(x, y, t) d\mu(y) = \frac{\partial}{\partial t} \int_S P^B(x, y, t) d\mu(y) = - \int_S P^B(x, y, t) \Psi(y) d\mu(y) < 0$$

Where the last inequality uses that $\Psi > 0$, and P^B is positivity preserving. We thus see $\int_S P^B(x, y, t) d\mu(y) \leqslant 1$.

3.2.3 Proof of the uniform Sobolev inequality

Throughout this section we will assume, without loss of generality, that $\kappa = 1$, so that the transverse Ricci flow exists on $[0,1)$. We begin by proving a restricted log Sobolev inequality, which will allow us to obtain a uniform upper bound for the basic heat kernel of a certain natural heat operator.

Proposition 3.2.22. *Let $g(t)$ be a solution of the Sasaki-Ricci flow defined on $[0,\infty)$, with $g(0)$ Sasaki. For any $0 < \varepsilon \leqslant 2$, $t \in [0,\infty)$ and any $v \in W^{1,2}_B$ with $\|v\|_{L^2(g(t))} = 1$, we have*

$$\int_S v^2 \log v^2 d\mu_t \leqslant \varepsilon^2 \int_S \left(|\nabla v|^2 + \frac{1}{4} R^T v^2 \right) d\mu_t - (2n+1)\log\varepsilon + C_1 + \max_{x\in S} R^T_-(x,0)$$

for a constant C_1 depending only on n and $g(0)$.

The proof of this theorem relies on the monotonicity of μ^T along the transverse Ricci flow on a Sasaki manifold, and on previously known log Sobolev inequalities. Before beginning the proof of Proposition 3.2.22 we state the precise version of the log Sobolev inequality we will need, which is well known; see, for instance, [Ye, 2007] Theorem 3.3 for a proof.

Proposition 3.2.23. *Let (M^n, g) be a Riemannian manifold, and $\lambda \in (0,2]$. For any $v \in W^{1,2}(M)$ with $\|v\|_2 = 1$, there exists a positive constant C_0 depending only on g such that*

$$\int_S v^2 \log v^2 d\mu \leqslant \lambda^2 \int_S |\nabla v|^2 d\mu - n\log\lambda + C_0.$$

Proof of Proposition 3.2.22. Fix $\varepsilon \in (0,2]$. Recall that if $g^T(s)$ is a solution of the Sasaki-Ricci flow on $[0,\infty)$, then $\tilde{g}^T(t) = (1-t)g^T(-\log(1-t))$ is a solution of the transverse Ricci flow on $[0,1)$. Fix a time $s \in [0,\infty)$, and define t by $s = -\log(1-t)$. By Proposition 3.1.23, we have (upon suppressing the superscript T)

$$\mu(g(s),\varepsilon^2) = \mu(g(-\log(1-t)),\varepsilon^2) = \mu(\tilde{g}(t),(1-t)\varepsilon^2) \geqslant \mu(g(0),(1-t)\varepsilon^2 + t). \quad (3.2.31)$$

Let $\sigma(t) = t + (1-t)\varepsilon^2$, and observe that $\sigma(t) \in [\varepsilon^2, 1]$; combining this with equation (3.2.31), and Proposition 3.2.23 we obtain, for every $v \in W^{1,2}_B$ with $\|v\|_{L^2(g(s))} = 1$ we have

$$\begin{aligned}
&\int_S \varepsilon^2 \left(R^T_{g(s)} v^2 + 4|\nabla v|^2_{g(s)} \right) - v^2 \log v^2 d\mu_s - n \log(\varepsilon^2) \\
&\quad \geqslant \int_S \sigma(t) \left(R^T_{g(0)} v_0^2 + 4|\nabla v_0|_{g(0)} \right) - v_0^2 \log v_0^2 d\mu_0 - n \log(\sigma(t)) \\
&\quad \geqslant -\sup_{x \in S} R^T_-(x,0) + \frac{1}{2} \log(\sigma(t)) - C_1.
\end{aligned}$$

where $R^T = R^T_+ - R^T_-$, and we have used $\lambda = \sqrt{\sigma(t)}$ in our application of Proposition 3.2.23. Note that C_1 depends only on $g(0)$ and n. Rearranging this equation, and using that $\sigma(t) \in [\varepsilon^2, 1]$ yields,

$$\int_S v^2 \log v^2 d\mu_t \leqslant \varepsilon^2 \int_S \left(4|\nabla v|^2 + R^T v^2\right) d\mu_t - (2n+1)\log\varepsilon + C_1 + \max_{x\in S} R^T_-(x,0).$$

Redefining ε, we are done. □

Fix a time t_0 along the Sasaki-Ricci flow. We use the Proposition 3.2.22 to obtain a uniform upper bound for the operator

$$-\frac{\partial}{\partial t} + \Delta_{g(t_0)} - \frac{1}{4} R^T(x,t_0) - \Theta - 1, \tag{3.2.32}$$

where $\Theta = \sup_{x\in S} R^T_-(x,0)$. It is standard to compute that $\sup R^T_-(x,t)$ is increasing under the flow, and hence $\frac{1}{4}R^T(x,t) + \Theta > 0$ for all $t > 0$. Let P_Θ be the heat kernel of the operator (3.2.32). By the preceding discussion we may decompose this operator as $P_\Theta = P^B_\Theta + P^\perp_\Theta$. Let $u(x,t_0)$ be a positive, basic solution of the equation

$$\Delta_{g(t_0)} u(x,t_0) - (\frac{1}{4} R^T(x,t_0) + \Theta + 1) u(x,t_0) = 0.$$

Note that any solution to this equation with smooth, positive, basic initial data will be positive and basic for all time by Lemma 3.2.21. For simplicity, we now suppress the argument t_0, with the understanding that it is fixed, and all computations will be done with respect to the *fixed* metric $g(t_0)$; we will set $\Psi(x) = \frac{1}{4}R^T(x,t_0) + \Theta + 1$. We now follow the computation in [Zhang, 2007], which carries over verbatim for the basic function $u(x,t)$. For completeness, and as our operator is a slight modification of the operator in [Zhang, 2007] we include a few steps in the computation. Choose $T \in (0,1]$ and set $p(t) = T/(T-t)$, so that $p(0) = 1$, $p(T) = \infty$. For convenience we define the function

$v(x,t) = u^{p/2}/\|u^{p/2}\|_{L^2}$, which has the property that $v \in W^{1,2}_B(dx)$ and $\|v\|_{L^2(dx)} = 1$. One then computes

$$\begin{aligned} &p^2(t)\partial_t \log \|u\|_{p(t)} \\ &= p'(t)\int_S v^2 \log v^2 dx - (4p-1)\int_S |\nabla v|^2 dx - p^2 \int_S \Psi(x) v^2 dx \\ &= p'\left\{\int_S v^2 \log v^2 - \frac{4(p-1)}{p'}\left(\int_S |\nabla v|^2 + \Psi(x)v^2 dx\right)\right\} + \left((4p-1) - p^2\right)\int_S \Psi v^2 dx. \end{aligned} \tag{3.2.33}$$

It is easy to check that $4(p-1)/p' \leqslant T \leqslant 1$, and $-T \leqslant \left(4(p-1) - p^2\right)/p' \leqslant 0$. Since $\Psi > 0$, the final terms in (3.2.33) is negative. We apply Proposition 3.2.22 with $\varepsilon = 4(p-1)/p'$ to obtain

$$\partial_t \log \|u\|_{p(t)} \leqslant \frac{1}{T}\left(-\frac{2n+1}{2}\log\left(\frac{4t(T-t)}{T}\right) + C_1\right)$$

Integrating in t from 0 to T and rearranging yields that for every smooth function $u \in C^\infty_B$, and every $0 < T \leqslant 1$ we have

$$|u(x,T)| = \left|\int_S P_\Theta(x,y,T)u(y)dy\right| = \left|\int_S P^B_\Theta(x,y,T)u(y)dy\right| \leqslant T^{-(2n+1)/2}\Lambda\|u\|_{L^1(dx)}.$$

Since $P^B_\Theta(x,y,t)$ is basic, it follows that for every $T \in (0,1]$ we have the bound

$$P^B_\Theta(x,y,T) \leqslant CT^{-(2n+1)/2}. \tag{3.2.34}$$

where C is a constant that depends only on $g(0)$ and n.

We now show that the short time upper bound (3.2.34) extends to a bound for every positive time t. Fix $y \in S$ and consider the function $f(x,t) = P^B_\Theta(x,y,1+t)$, which is a smooth, basic function. Equation (3.2.34) shows the $f(x,0) \leqslant C$, and we have

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t) - \Psi(x) f(x,t),$$

where $\Psi \geqslant 1$. We then compute $\partial_t \left(e^t f(x,t)\right)$ and apply the maximum principle to see that $f(x,t) \leqslant e^{-t}C$, whence

$$P^B_\Theta(x,y,t) \leqslant CT^{-(2n+1)/2}, \quad \text{for every } t > 0. \tag{3.2.35}$$

By the discussion following Lemma 3.2.21, P_Θ is a contraction on L^1, and so P^B_Θ is a contraction on L^1_B. Hölder's inequality, combined with the long time upper bound (3.2.35) yields that, for any $f \in L^2_B$, we have

$$\left| \int_S P^B_\Theta(x, y, t) f(y) d\mu(y) \right| \leqslant \Lambda^{1/2} t^{-(2n+1)/4} \|f\|_2. \tag{3.2.36}$$

In the Riemannian case, passing from a bound of the type (3.2.36) to the Sobolev inequality is by now a standard argument; we refer the reader to [Davies, 1989], [Ye, 2007]. In our case, the argument is similar, with the added complication that the bound (3.2.36) only holds on L^2_B, which is a closed subspace of L^2. However, since the operator defined by the integral kernel preserves the basic spaces, we can restrict our attention to the *basic* Lebesgue and Sobolev spaces. The ingredients in the proof which don't immediately carry over are none other than the Riesz-Thorin interpolation theorem and the Marcinkiewicz interpolation theorem, which we established in section 3.1.3. With these considerations, we do not feel it is irresponsible to omit the details. We thus obtain the existence of constants $A, B > 0$ depending only on n and g_0 such that, for each time t in the Sasaki-Ricci flow, we have

$$\left(\int_S v^{2m/(m-2)} d\mu_t \right)^{(m-2)/m} \leqslant A \int_S \left(|\nabla v|^2 + \frac{1}{4} R^T v^2 \right) d\mu_t + B \int_S v^2 d\mu_t.$$

By applying Theorem 10 to bound R^T above uniformly, we establish Theorem 11.

3.3 Sufficient conditions for the convergence of the Sasaki-Ricci flow

We now aim to use the estimates developed above to prove several theorems about the convergence of the Sasaki-Ricci flow. In general, the existence of a Sasaki-Einstein metric is obstructed, and so the convergence of the flow is not guaranteed. It is expected that the existence of a Sasaki-Einstein metric should be equivalent to the algebraic notion of K-Stability, which we will introduce in Chapter 3. In the Kähler setting, there is a large body of work relating various notions of algebraic stability to convergence of the Kähler-Ricci flow; see for example [Phong and Sturm, 2009], [Székelyhidi, 2010], [Tosatti, 2010] and the references therein. It is desirable to determine whether analogues of these results hold in the Sasakian case. A particular form of stability which arises in the Kähler setting concerns the degeneration of eigenvalues of various Laplacians along the flow. Phong and

Sturm [Phong and Sturm, 2006], and Phong, Song, Sturm and Weinkove [Phong *et al.*, 2009] proved convergence of the Kähler-Ricci flow assuming a bound below for the Mabuchi functional and stability conditions for the lowest positive eigenvalue of the $\overline{\partial}$ Laplacian on $T^{1,0}$ vector fields (cf. condition(B) in [Phong and Sturm, 2006], and condition (S) in [Phong *et al.*, 2009]). In [Zhang, 2010], Zhang proved convergence of the flow under a non-degeneracy condition for the 'second' eigenvalue of a modified Laplacian on smooth functions.

In the Sasakian case, it is natural to ask whether forms of stability analogous to those studied in [Phong and Sturm, 2006], [Phong *et al.*, 2009] [Zhang, 2010] are available, and whether they imply the convergence of the Sasaki-Ricci flow to a transverse Kähler-Einstein metric when the Futaki invariant vanishes or the Mabuchi functional is bounded below. We aim to address these questions presently. We would like to point out a few simple observations which hint at the difficulties ahead. Assume for simplicity that the Sasaki structure is regular, so that the Sasakian manifold S is diffeomorphic to a $U(1)$ principal bundle over the projective Kähler manifold $S/U(1)$. In this case, the stability conditions we seek are necessarily the pull back of the Kähler stability conditions under the quotient map $\pi : S \to S/U(1)$. The first observation is that pulling back sections of the tangent bundle by π does not yield a module over the ring of smooth functions. Thus, if we seek to generalize condition (B) of [Phong and Sturm, 2006] or condition (S) of [Phong *et al.*, 2009] we must work in the realm of locally free sheaves of modules over the ring of *basic* functions. More generally, when the Sasaki structure is irregular, so that the leaf space of the Reeb foliation does not have the structure of a Kähler orbifold, how do we identify the space of "holomorphic vector fields"? A general approach to this problem is to extend Kähler notions of stability to Kähler orbifolds and then formulate some related notion of stability which behaves well under approximation by quasi-regular Sasaki structures. Presently, we shall take a point of view which avoids these approximation techniques. In [Boyer *et al.*, 2008], [Futaki *et al.*, 2009], [Nitta and Sekiya, 2012], the Lie algebra of holomorphic Hamiltonian vector fields was identified as a central object of study in the existence of Sasaki-Einstein and extremal Sasakian metrics. Is it possible to view these vector fields as the kernel of a $\overline{\partial}$ operator on the global sections of some sheaf $\mathcal{E}$? More importantly, if such an $\mathcal{E}$ exists, can we relate the convergence of the Sasaki-Ricci flow to the eigenvalues of the $\overline{\partial}$ Laplacian on the global sections of $\mathcal{E}$? In this paper we answer these questions in the affirmative. We identify a sheaf $\mathcal{E}$, called the sheaf of transverse foliate vector fields, which has a well defined transverse $\overline{\partial}$- type operator, with the property that the global holomorphic sections of $\mathcal{E}$ correspond precisely to the Hamiltonian holomorphic vector fields. We consider the following notions of stability;

(M) The Mabuchi energy is bounded below.

(F) The Futaki invariant vanishes.

(C) Let (ξ, η, Φ) be a Sasaki structure on S. Then the C^∞ closure of the orbit of the triple (ξ, η, Φ) under the diffeomorphism group of S does not contain any Sasaki structure $(\xi_\infty, \eta_\infty, \Phi_\infty)$ with the property that the dimension of the space of global holomorphic sections of the sheaf of transverse foliate vector fields with respect to $(\xi_\infty, \eta_\infty, \Phi_\infty)$ has dimension strictly higher than the dimension of the space of global holomorphic sections of the sheaf of transverse foliate vector fields with respect to (ξ, η, Φ).

Condition (C) generalizes condition (B) of [Phong and Sturm, 2006]. In light of Proposition 3.3.15 below, condition (F) is at least *a priori* weaker than condition (M). We refer the reader to Section 3.3.2 for details on the sheaf $\mathcal{E}$, and its transverse holomorphic structure. Our first theorem extends Theorem 1 in [Phong and Sturm, 2006].

Theorem 13 ([Collins, 2011]). *Let $(S, g_0, \xi, \eta_0, \Phi)$ be a compact Sasakian manifold with $c_1^B(S) > 0$. Assume that $g(t)$ is a solution of the Sasaki-Ricci flow with $g(0) = g_0$, and $d\eta_0 \in 4\pi c_1^B(S)$. Assume that the transverse Riemann curvature is bounded along the flow.*

(i) If condition (M) holds, then we have for any $s \geqslant 0$

$$\lim_{t\to\infty} \|Ric^T(t) - g^T(t)\|_{(s)} = 0$$

where $\|\cdot\|_{(s)}$ denotes the Sobolev norm of order s with respect to the metric $g(t)$.

(ii) If both conditions (M) and (C) hold, then the Sasaki-Ricci flow converges exponentially fast in C^∞ to a transversely Kähler-Einstein metric.

We remove the condition on the boundedness of Rm^T by introducing the stability condition (T), which generalizes condition (S) of [Phong *et al.*, 2009]. The following theorem extends the results of [Phong *et al.*, 2009], [Zhang, 2010].

Theorem 14 ([Collins, 2011])**.** *Let $(S, g_0, \xi, \eta_0, \Phi)$ be a compact Sasakian manifold with $c_1^B(S) > 0$. Assume that $g(t)$ is a solution of the Sasaki-Ricci flow with $g(0) = g_0$, and $d\eta_0 \in 4\pi c_1^B(S)$. Let λ_t be the lowest strictly positive eigenvalue of the Laplacian $\Box_{\mathcal{E}} := -(g^T)^{j\bar{k}}\nabla_j^T\nabla_{\bar{k}}^T$ acting on smooth global sections of $\mathcal{E}^{1,0}$.*

(i) If condition (F) and condition

(T) $\inf_{t\in[0,\infty)} \lambda_t > 0$

hold, then the metrics $g(t)$ converge exponentially fast in C^∞ to a transversely Kähler-Einstein metric.

(ii) Conversely, if the metrics $g(t)$ converge in C^∞ to a transversely Kähler-Einstein metric, then conditions (F) and (T) hold.

(iii) In particular, if the metrics $g(t)$ converge in C^∞ to a transversely Kähler-Einstein metric, then they converge exponentially fast in C^∞ to this metric.

In order to reduce the convergence of the flow to the problem of estimating the transverse Kähler potentials, it is necessary to fix a normalization, which is determined by a choice of the initial value for the Sasaki-Ricci flow. Let us discuss this choice now.

We now discuss the choice for the initial value of the transverse Kähler potential. Suppose that a given flow φ satisfies $\varphi(0) = c_0$, then one can easily check that $\tilde{\varphi} := \varphi + (\tilde{c}_0 - c_0)e^t$ satisfies the same flow with initial condition $\tilde{c_0}$. This underlines the importance of choosing the initial value properly; any two solutions with different initial value differ by terms diverging exponentially in time. We introduce the quantity

$$c_0 = \int_0^\infty e^{-t}\|\nabla\dot{\varphi}\|_{L^2}^2 dt + \frac{1}{Vol(S)}\int_S u(0)d\mu_0. \tag{3.3.37}$$

One easily checks that this does not depend on $\varphi(0)$. The first indication that this is the correct choice for $\varphi(0)$ is that the bound in Theorem 10 implies

$$\sup_{t\geqslant 0}\|\dot{\varphi}\|_{C^0} \leqslant C. \tag{3.3.38}$$

To see this, observe that $\partial_B\overline{\partial}_B(u-\dot{\varphi})=0$, and so by the uniform bound for u it suffices to bound, $\alpha(t) := Vol(S)^{-1}\int_S \dot{\varphi} d\mu_t$. This is easily done by computing the evolution of α; see the computations in [Phong *et al.*, 2007]. The upshot of this is contained in the following;

Proposition 3.3.1. *Let $(S, g_0, \xi, \eta_0, \Phi)$ be a compact Sasakian manifold with $[\frac{1}{2}d\eta_0]_B = 2\pi c_1^B(S)$. Consider the Sasaki-Ricci flow defined by (3.0.4), with $\varphi(0) = c_0$, as defined in equation (3.3.37). Then we have the a priori estimates*

$$\sup_{t\geqslant 0}\|\varphi\|_{C^0} \leqslant A_0 < \infty \iff \sup_{t\geqslant 0}\|\varphi\|_{C^k} \leqslant A_k < \infty \quad \forall k \in \mathbb{N}.$$

The estimates in Proposition 3.3.1 are the transverse parabolic version of Yau's famous estimates [Yau, 1978], or equivalently the parabolic version of El Kacimi-Alaoui's generalization of Yau's estimates [El Kacimi-Alaoui, 1990]. For the Kähler-Ricci flow, these estimates are well known, and can be found, for example, in [Cao, 1985], [Phong *et al.*, 2007]. For the Sasaki-Ricci flow, the estimates can be found in [Smoczyk *et al.*, 2010]. In light of the uniform bound for $\dot{\varphi}$, it is straight forward to check that Proposition 3.3.1 holds.

3.3.1 Bernstein-Bando-Hamilton-Shi estimates for the transverse curvature along the Sasaki-Ricci flow

In this section we prove the following theorem, which is the transverse extension of well-known estimates of Hamilton [Hamilton, 1995b] and Shi [Shi, 1989] for the Ricci flow.

Theorem 15. *Let (S, g_0) be a compact Sasakian manifold of dimension $2n+1$, and suppose that $g(t)$ is a solution of the normalized Sasaki-Ricci flow, with $g(0) = g_0$. Then, for each $\alpha > 0$, and every $m \in \mathbb{N}$, there exists a constant C_m depending only on m, n and $\max\{\alpha, 1\}$ such that if K satisfies*

$$|Rm^T(x,t)|_{g^T(x,t)} \leqslant K \textit{ for every } x \in S, \textit{ and } t \in [0, \frac{\alpha}{K}]$$

then, the bound

$$\max\left\{|\nabla^m Rm(x,t)|_{g(t)},\ |\nabla^m Rm^T(x,t)|_{g^T(x,t)}\right\} \leqslant \frac{C_m \max\{K^{1/2}, K\}}{t^{m/2}}$$

holds for every $x \in S$ and $t \in (0, \frac{\alpha}{K}]$.

The proof of this theorem follows essentially from the arguments in the Riemannian case, and the curvature identities (2.1.3), (2.1.4), and (2.1.5). As the techniques used to prove this theorem in the Riemannian case are purely local, one hopes that this result will carry over to the Sasakian case. However, we must proceed with some care. For example, the standard commutation formulae for the covariant derivative and the Laplacian do not apply to 'tensors' on the bundle Q. In fact, the commutation relation which does hold involves not only the transverse Riemann tensor, but also the full Riemann curvature; here the curvature identities (2.1.3), (2.1.4), and (2.1.5) are crucial. In order to avoid being swamped by indices we introduce the following notation; if A and B are two sections of $TS^{*\otimes p} \otimes Q^{*\otimes q}$, we denote by $A * B$ any quantity obtained from $A \otimes B$ by summation over paired indices, contraction with g, g^{-1}, g^T, or $(g^T)^{-1}$, and multiplication by constants depending only on n, p and q.

Lemma 3.3.2. *Suppose $A \in TS^{*\otimes p} \otimes Q^{*\otimes q}$. Then there is a constant C, depending only on p, q, and n, such that the following commutation relation holds;*

$$[\nabla, \Delta]A = Rm^T * \nabla A + A * \nabla Rm^T + C\nabla A.$$

Proof. The argument is elementary, and so we only provide a sketch. The key point is that commuting covariant derivatives yields an expression involving both the full Riemann tensor, and the transverse Riemann tensor. In particular,

$$[\nabla, \Delta]A = Rm^T * \nabla A + Rm * \nabla A + A * \nabla Rm + A * \nabla Rm^T.$$

We now use the curvature relations (2.1.3), (2.1.4), and (2.1.5) to replace all the terms involving the full curvature tensor with terms involving only the metric and Rm^T. Collecting terms, and observing that any constants which appear depend only on n, p and q, the lemma is proved. □

Proof of Theorem 15. We begin by computing

$$\partial t Rm^T - \Delta Rm^T = Rm^T * Rm^T. \tag{3.3.39}$$

We can now compute the evolution equation for $|\nabla Rm^T|^2$. Before proceeding, we point out that the quantity $|\nabla Rm^T|^2$ involves both g^T and g, as the covariant derivative ∇

takes arguments in TS. However, by looking in preferred coordinates we clearly have $\nabla_{\partial x} Rm^T = 0$, and so we see that the norm $|\nabla Rm^T|^2$ agrees with the norm when we replace g by g^T, regarded as a bilinear form on TS. With this in mind, we obtain

$$\partial t|\nabla Rm^T|^2 = \Delta|\nabla Rm^T|^2 - 2|\nabla^2 Rm^T|^2 + Rm^T * (\nabla Rm^T)^{*2} + (\nabla Rm^T)^{*2}. \qquad (3.3.40)$$

The last line follows by Lemma 3.3.2. We now consider the quantity $F = t|\nabla Rm^T|^2 + \beta|Rm^T|^2$. Using equations (3.3.39) and (3.3.40), we compute that

$$\begin{aligned} \partial t F &\leqslant |\nabla Rm^T|^2 + t\partial t|\nabla Rm^T|^2 + \beta(Rm^T)^{*3} + \beta\Delta|Rm^T| - 2\beta|\nabla Rm^T|^2 \\ &\leqslant \Delta F + \big(1 + c_1 t|Rm^T| + c_2 t - 2\beta\big)\,|\nabla Rm^T|^2 + c_3|Rm^T|^3. \end{aligned}$$

By assumption, $|Rm^T| \leqslant K$ if $t \in [0, \alpha/K]$. Set $2\beta = 1 + c_1\alpha + c_2\frac{\alpha}{K}$, then

$$\partial t F - \Delta F \leqslant c_3 K^3.$$

Applying the maximum principle yields $F(x,t) \leqslant \beta K^2 + c_3\beta K^3 t$. Using the definition of β we obtain that there is a constant C_4 depending only on n, and $\max\{\alpha, 1\}$, such that for every $t \in (0, \alpha/K]$ we have

$$|\nabla Rm^T| \leqslant \sqrt{\frac{F}{t}} \leqslant C_4 t^{-1/2} \max\{K^{1/2}, K\}.$$

This proves the theorem in the case $m = 1$. The general case of $m > 1$ follows by making similar adaptations to the Kähler, or Riemannian case. See [Chow *et al.*, 2007] for details in these cases. The curvature equations (2.1.3), (2.1.4), and (2.1.5) show that our bounds for Rm^T extend to bounds for the full Riemann tensor. □

Corollary 3.3.3. *If Rm^T is uniformly bounded in C^0 along the normalized Sasaki-Ricci flow, then for any time $A > 0$, there is a constant $C_{A,k}$ depending only on $|Rm^T|_{C^0}$, k and A, so that $|Rm^T|_{C^k} \leqslant C_{A,k}$ for all $t \geqslant A$.*

3.3.2 The $\overline{\partial}$-operator on foliate vector fields and a notion of stability on Sasakian manifolds

For the remainder of the paper we will be concerned with employing various types of stability to prove the convergence of the Sasaki-Ricci flow. We begin by presenting two notions of stability on Sasakian manifolds which generalize notions on Kähler manifolds. Later, we shall introduce the sheaf of transverse foliate vector fields, which seems central to the problem of stability on Sasakian manifolds. We first define the Futaki invariant of a Sasakian manifold. In the Sasakian case, the Lie algebra on which the Futaki invariant acts is the space of Hamiltonian, holomorphic vector fields.

Definition 3.3.4 ([Futaki *et al.*, 2009] Definition 4.5)**.** *Let $U_\alpha = I \times V_\alpha$ be a foliated coordinate patch, with $I \subset \mathbb{R}$ an open interval, and $V_\alpha \subset \mathbb{C}^n$. Let $\pi_\alpha : U_\alpha \to V_\alpha$ be the projection. A complex vector field X on a Sasakian manifold is called a Hamiltonian holomorphic vector field if*

(i) $d\pi_\alpha X$ is a holomorphic vector field on V_α

(ii) the complex valued function $u_X := \sqrt{-1}\eta(X)$ satisfies

$$\partial_{\bar{B}} u_X = -\frac{\sqrt{-1}}{2}\iota_X d\eta.$$

Such a function u_X is called a Hamiltonian function.

Remark 3.3.5. If X is a Hamiltonian holomorphic vector field, then in preferred local coordinates

$$X = \eta(X)\partial_x + \sum_{i=1}^{n} X^i \partial z^i - \eta\left(\sum_{i=1}^{n} X^i \partial z^i\right)\partial_x, \tag{3.3.41}$$

where the X^i are local, holomorphic basic functions.

The Futaki invariant was originally defined for Sasakian manifolds by Boyer, Galicki and Simanca in [Boyer *et al.*, 2008], where it was considered to be a character on a quotient of the Lie algebra of "transversally holomorphic" vector fields (see [Boyer *et al.*, 2008] Definition 4.5). In [Futaki *et al.*, 2009] Futaki, Ono and Wang recast the Futaki invariant as a character of the Lie algebra on Hamiltonian holomorphic vector fields as defined above. For our purposes, we are only concerned with the case when the distribution D has $c_1(D) = 0$.

Theorem 16 ([Boyer *et al.*, 2008] Proposition 5.1, [Futaki *et al.*, 2009] Theorem 4.9)**.** *Let (S, g) be a Sasakian manifold with $c_1^B(S) > 0$, and $c_1(D) = 0$. Assume $g^T \in 2\pi c_1^B(S)$*

and let u be the transverse Ricci potential for the metric g, and let X be a holomorphic, Hamiltonian vector field. We define the Futaki invariant $fut_S(X)$ by the equation

$$fut_S(X) := \int_S X u d\mu.$$

Then $fut_S(X)$ is independent of the choice of Sasakian metric in $2\pi c_1^B(S)$.

Remark 3.3.6. It is clear that the vanishing of the Futaki invariant is necessary for the existence of a Sasaki-Einstein metric.

Later we will provide an alternative characterization of the Futaki invariant as a character on a certain subspace of the global sections of the soon-to-be-defined sheaf $\mathcal{E}$, and show that this characterization is equivalent to the above. We now introduce the Mabuchi energy on a Sasakian manifold in the special case that $c_1(D) = 0$. First, we define the space of transverse Kähler potentials.

Definition 3.3.7. *We define the space of transverse Kähler potentials to be*

$$\mathcal{H}_\eta(S) = \{\varphi \in C_B^\infty(S) \,|\, \eta_\varphi := \eta + d_B^c \varphi \text{ is a contact 1 form}\}.$$

Theorem 17 ([Futaki *et al.*, 2009] Theorem 4.12)**.** *Let (S, g) be a Sasakian manifold, with $2\pi c_1^B(S) = [\frac{1}{2} d\eta_0]_B$. Let η' be in the Kähler class of η_0. Let φ_t, $t \in [a, b]$ be a path in $\mathcal{H}_{\eta_0}$ connecting η, η'. Then the Mabuchi K-energy,*

$$\nu_{\eta_0}(\eta') = -\int_a^b \int_S \dot{\varphi}_t (R^T - \bar{R}^T) d\mu_t dt,$$

is independent of the path φ_t.

In the Kähler theory, the boundedness below of the Mabuchi K-energy is crucial to the existence of canonical metrics. It is known that a bound below for the K-energy is not sufficient to guarantee the existence of a Kähler-Einstein metric; a counterexample is given by Tian's unstable deformation of the Mukai-Umemura threefold [Donaldson, 2008],[Tian, 1997]. However, Bando and Mabuchi proved that any manifold that admits a Kähler-Einstein metric ω necessarily has the K-energy bounded below on the Kähler class of ω [Bando, 1987], [Bando and Mabuchi,]. Their result extends to Sasakian manifolds.

Proposition 3.3.8. *Suppose (S, ξ, η, Φ, g) is a Sasakian manifold with $c_1(D) = 0$, and $2\pi c_1^B(S) = [\frac{1}{2}d\eta]_B$. Assume that S admits a Sasakian metric which is transversely Kähler-Einstein in the Kähler class of g. Then the Mabuchi K-energy is bounded below on $[\frac{1}{2}d\eta]_B$.*

Proof. The proof is a consequence of the work of Nitta and Sekiya [Nitta and Sekiya, 2012]. The estimates in [Nitta and Sekiya, 2012] imply that the result of Bando and Mabuchi in [Bando and Mabuchi,] holds on Sasakian manifolds. That is, the Mabuchi K-energy is bounded below on the set of Sasakian metrics with positive transverse Ricci curvature. It is straight forward to check that the extension of the results of [Bando and Mabuchi,], due to Bando in [Bando, 1987] carries over verbatim to the Sasakian setting. □

If $g^T(t)$ is evolving by the Sasaki-Ricci flow (3.0.3), then

$$\nu_{\eta_0}(\eta(a)) = -\int_0^a \int_S u(-\Box_B u) d\mu_t dt = -\int_0^a \int_S (g^T)^{j\bar{k}} \partial_j u \overline{\partial_k u} d\mu_t dt. \tag{3.3.42}$$

The last term in the above expression is precisely the integral of the negative of the L^2 norm of $\partial_B u$ regarded as a section of $\Lambda^{1,0}Q^*$.

A significant difficulty in extending the results of [Phong and Sturm, 2006] is identifying the appropriate operator to study. As noted in the introduction, we are guided primarily by the model case of a regular Sasakian manifold.

Definition 3.3.9. *On an open subset $U \subset S$, let $\Xi(U)$ be the Lie algebra of smooth vector fields on U and let $\mathcal{N}_\xi(U)$ be the normalizer of the Reeb field in $\Xi(U)$,*

$$\mathcal{N}_\xi(U) = \{X \in \Xi(U) : [X, \xi] \in L_\xi\}.$$

We define a sheaf $\mathcal{E}$ on S by

$$\mathcal{E}(U) := N_\xi(U)/L_\xi.$$

The sheaf $\mathcal{E}$ will be referred to as the sheaf of transverse foliate vector fields.

When there is some chance of confusion, for $V \in TS$ we denote by $[V]$ the equivalence class of V in Q. Recalling the exact sequence (3.0.1), the inclusion $\mathcal{N}_\xi \subset TS$ induces an inclusion of sheaves $\mathcal{E} \subset Q$. The sheaf $\mathcal{E}$ is easily seen to be a locally free sheaf of C_B^∞-modules. When the Reeb field is regular or quasi-regular, the sheaf $\mathcal{E}$ descends through the quotient to the sheaf of smooth sections of the tangent bundle of the Kähler manifold or orbifold. $\mathcal{E}$ inherits a great deal of structure from the vector bundle Q; the metric g^T restricts to a metric on $\mathcal{E}$, and it is easy to check that the transverse complex structure Φ on Q restricts to an endomorphism of $\mathcal{E}$, and hence splits $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$. The key observation is that $\mathcal{E}$ has a well defined $\bar{\partial}$ operator. Define a map $\mathfrak{e} : Q \to Q$ by $\mathfrak{e}(V) := \nabla^T_{\xi} V$. By the definition of ∇^T, it is clear that $\mathcal{E}$ is precisely the kernel of the map $\mathfrak{e}$. In particular, on $\mathcal{E}$ the covariant derivative ∇^T descends to a well defined map $d_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes Q^*$. This map is defined for $[V] \in \mathcal{E}$ and $[X] \in Q$ by

$$d_{\mathcal{E}}[V]([X]) = \nabla^T_X [V],$$

and this is clearly independent of the representative of the equivalence class $[X]$. Moreover, the transverse complex structure Φ yields a splitting $\mathcal{E} \otimes Q^* = \mathcal{E} \otimes (Q^*)^{1,0} \oplus \mathcal{E} \otimes (Q^*)^{0,1}$. The map $d_{\mathcal{E}}$ then splits as $\partial_{\mathcal{E}} + \bar{\partial}_{\mathcal{E}}$. Hence, we have a well defined operator $\bar{\partial}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes (Q^*)^{(0,1)}$. We can extend the operator $\overline{\partial}_{\mathcal{E}}$ to a differential operator. Dualizing the exact sequence (3.0.1) we have an operator, also denoted by $\overline{\partial}_{\mathcal{E}}$ satisfying

$$\overline{\partial}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes p^{\dagger}(Q^*) \hookrightarrow \mathcal{E} \otimes TS^*.$$

The metric g^T induces L^2 inner products on $\Gamma(X, \mathcal{E}^{1,0})$ and $\Gamma\left(X, \mathcal{E}^{1,0} \otimes p^{\dagger}(Q^*)\right)$. We can then define the formal adjoint of the $\overline{\partial}_{\mathcal{E}}$-operator, denoted $\overline{\partial}^{\dagger}_{\mathcal{E}}$, on smooth sections by the usual formula. One easily computes using integration by parts that $\bar{\partial}^{\dagger}_{\mathcal{E}}[W] = -(g^T)^{j\bar{k}} \nabla^T_j W^p_{\bar{k}}$, and hence we can define the Laplacian of $\mathcal{E}$ by

$$\Box_{\mathcal{E}}[V] = -(g^T)^{j\bar{k}} \nabla^T_j \nabla^T_{\bar{k}} V^p.$$

We comment that at first glance this operator does not appear to be elliptic. However, by recalling the definition of the bundle $\mathcal{E}$, we see that

$$\Box_{\mathcal{E}}[V] = -\nabla^T_{\xi} \nabla^T_{\xi} V^p - (g^T)^{j\bar{k}} \nabla^T_j \nabla^T_{\bar{k}} V^p,$$

which is clearly elliptic. Thus we can apply the usual elliptic theory. In a similar fashion we may also define the $\partial_{\mathcal{E}}$ Laplacian $\overline{\Box}_E$. A standard integration by parts computation proves;

Proposition 3.3.10. *For every $[V] \in \Gamma(S, \mathcal{E}^{1,0})$, we have the Bochner-Kodaira formula for the Laplacians $\Box_E$ and $\overline{\Box}_E$*

$$\|\partial_{\mathcal{E}}[V]\|^2_{L^2} = \|\overline{\partial}_{\mathcal{E}}[V]\|^2_{L^2} + \int_S R^T_{\bar{k}j} V^j \overline{V^k} d\mu,$$

Proposition 3.3.11. *We define the space $H^0(\mathcal{E}^{1,0})$, which we refer to as the space global holomorphic sections of the sheaf of transverse foliate vector fields, by*

$$H^0(\mathcal{E}^{1,0}) := Ker \bar{\partial}_{\mathcal{E}}\big|_{\Gamma(S, \mathcal{E}^{(1,0)})}.$$

- *$H^0(\mathcal{E}^{1,0})$ has the structure of a finite dimensional Lie algebra over $\mathbb{C}$.*
- *If $c_1^B(S) > 0$, $H^0(\mathcal{E}^{1,0})$ is isomorphic as a Lie algebra to the Lie algebra of holomorphic, Hamiltonian vector fields on S.*
- *The space $H^0(\mathcal{E}^{1,0})$ depends only on the complex structure J on the cone, the Reeb field ξ, and the transverse holomorphic structure. In particular,* $\dim H^0(\mathcal{E}^{1,0})$ *is invariant along the Sasaki Ricci flow.*

Remark 3.3.12. The Lie algebra $H^0(\mathcal{E}^{1,0})$ has appeared in the literature before, under a number of different guises. In [Nishikawa and Tondeur, 1988] it was proved that if the transverse scalar curvature of the Sasakian metric g is constant, then $H^0(\mathcal{E}^{1,0})$ is reductive. $H^0(\mathcal{E}^{1,0})$ also played an important role in the work of Boyer, Galicki and Simanca on extremal Sasakian metrics [Boyer *et al.*, 2008]. In [Nitta and Sekiya, 2012] it was shown that if S admits a Sasaki-Einstein metric g_{SE}, $\mathcal{G}$ is the identity component of the automorphism group of the transverse holomorphic structure, and $\mathcal{O}$ is the orbit of g_{SE} under the action of $\mathcal{G}$, then the tangent space to $\mathcal{O}$ at the point g_{SE} is isomorphic to $H^0(\mathcal{E}^{1,0})$.

Finally, let $\mathrm{Aut}(S)^0$ be the connected component of the identity in the Lie group of biholomorphic automorphisms of the Kähler cone $(C(S), J)$ which commute with the holomorphic flow generated by $\xi - i\, J(\xi)$. The reader can easily check that $H^0(\mathcal{E}^{1,0})$ is isomorphic to $\mathrm{Aut}(S)^0$.

We delay the proof of Proposition 3.3.11 for a moment in order that we may discuss the local structure of $\mathcal{E}$. We feel this local picture is essential to understanding the structure we are describing abstractly and so we shall be very explicit in this part of the development. Let

$U \subset S$ be an open subset of S on which we have a preferred coordinate system, and suppose that $[V] \in \Gamma(U, \mathcal{E})$ is a section of $\mathcal{E}$ over U. Over U we can write $V = V^i[\partial_{z^i}] + V^{\bar{i}}[\partial_{\bar{z}^i}]$. Let V be any lift of $[V]$ to $\mathcal{N}_\xi(U)$. In preferred local coordinates

$$V = f\partial_x + \sum_{i=1}^{n} V^i \frac{\partial}{\partial z^i} + \sum_{i=1}^{n} V^{\bar{i}} \frac{\partial}{\partial \bar{z}^i}.$$

As $[V] \in \mathcal{E}$, we necessarily have $V^i, V^{\bar{i}}$ basic functions. If we restrict to $[V] \in \mathcal{E}^{1,0}$, then $V^{\bar{i}} = 0$. A basis of $Q^{0,1}$ over U is given by $\{[\partial_{\bar{z}^j}]\}$ where $1 \leqslant j \leqslant n$. By definition, we compute

$$\overline{\partial}_{\mathcal{E}}[V] = \sum_{i=1}^{n} \partial_{\bar{j}} V^i \left[\partial z^i\right] \otimes d\bar{z}^j. \tag{3.3.43}$$

Proof of Proposition 3.3.11. Since S is compact, the kernel of the self-adjoint elliptic operator $\Box_{\mathcal{E}}$ is finite dimensional, and hence $H^0(\mathcal{E}^{1,0})$ is a finite dimensional vector space over $\mathbb{C}$. The Jacobi identity shows that the Lie bracket on $\Xi(S)$ descends to a Lie bracket on $\Gamma(U, \mathcal{E})$. Moreover, the local formulae show that the bracket preserves the decomposition $\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$. Computing over an open set shows that the Lie bracket preserves the space $H^0(\mathcal{E}^{1,0})$, and establishes the first statement. We turn our attention to the second and third statements. Suppose $[V] \in H^0(\mathcal{E}^{1,0})$, then equation (3.3.43) shows that condition *(i)* of Definition 3.3.4 is satisfied for any lift V of $[V]$ to $N_\xi(S)$. It remains to show condition *(ii)* holds. Let us define $\alpha = \frac{1}{2} d\eta(V, \cdot)$. Since $V \in N_\xi(S)$, it is clear that $\alpha \in \Lambda^{0,1}_B$. Moreover, since $[V] \in H^0(\mathcal{E}^{1,0})$, we have $\overline{\partial}_B \alpha = 0$, and so α defines an element of the basic Dolbeault cohomology group $H^{0,1}_B(X, \mathbb{C})$. Since $c_1^B(S) > 0$, a standard Bochner formula argument, together with the foliated Hodge decomposition of Kamber-Tondeur [Kamber and Tondeur, 1987] implies that $\alpha = \overline{\partial}_B \varphi$ for some $\varphi \in C^\infty_B(S, \mathbb{C})$. Define a lift of $[V]$ to $N_\xi(S)$ by setting $\tilde{V} = (g^T)^{i\bar{j}} \partial_{\bar{j}} \varphi$. Using the formula for η in preferred local coordinates yields $u_{\tilde{V}} = -\sum_{i=1}^n h_i V^i$. As $\overline{\partial}_{\mathcal{E}}[V] = 0$, equation (3.3.43) shows that $\tilde{V}$ satisfies property *(ii)* of Definition 3.3.4. Conversely, suppose V is a Hamiltonian, holomorphic vector field. The expression (3.3.41) for V, along with the remarks following Definition 3.3.4 show that V is in the normalizer of ξ and that $[V] \in H^0(\mathcal{E}^{1,0})$. Computing locally we see that the Lie bracket commutes with these identifications. Finally, by Proposition 2.2.6 any Sasaki structure on S with the same complex structure J on the cone, Reeb field ξ and transverse holomorphic structure is related to the original Sasaki structure by transverse Kähler deformations. One can then easily check via the above computations that $H^0(\mathcal{E}^{1,0})$ is unchanged under these deformations. □

Corollary 3.3.13. *The Futaki invariant defines a character on the Lie algebra* $H^0(E^{1,0})$.

Remark 3.3.14. Corollary 3.3.13 is essentially a restatement of the original definition in [Boyer *et al.*, 2008]. However, the identification of the Lie algebra which appears in [Boyer *et al.*, 2008] as the global holomorphic sections of the sheaf $\mathcal{E}$ provides what we feel to be a particularly attractive definition of the Futaki invariant, which is seen to generalize the Kähler setting.

Proposition 3.3.15. *Let* (S, g_0) *be a Sasakian manifold with* $c_1^B(S) > 0$*, and* $c_1(D) = 0$*. Suppose that* $[\frac{1}{2}d\eta_0]_B = 2\pi c_1^B(S)$*. If* $\nu_{\eta_0}(\eta) > -C$ *for every* η *in the Kähler class of* η_0*, then* $fut_S(X) \equiv 0$*.*

Proof. Suppose $\nu_{\eta_0}(\eta) > -C > -\infty$, but $fut([X]) \neq 0$ for some $[X] \in H^0(\mathcal{E}^{1,0})$. We can assume that $Re(fut([X])) < 0$, by replacing $[X]$ with $[-X]$, or $[iX]$. Let $X = \sigma_0([X])$. The vector field X is foliate, orthogonal to ξ, and holomorphic. In preferred local coordinates we have $X = \sum_{i=1}^n X^i(\partial z^j - \sqrt{-1}h_j \partial x)$ for X^i basic, holomorphic functions, and h a local, real valued function. Define the vector field $\tilde{X}$ on the cone $C(S)$, by $\tilde{X}(z,r) = Re(X)(z)$, where $z \in S$, and r is the radial variable on the cone. The local formula shows $\tilde{X}$ is real holomorphic, and so $\mathcal{L}_{\tilde{X}}J = 0$, where J is the complex structure on the cone. Let ρ_t be the local flow of $Re(X)$ on S, and $\tilde{\rho}_t$ be the local flow of $\tilde{X}$. Then $\tilde{\rho}_t$ is a biholomorphism, and it is clear that $\tilde{\rho}_t(z,r) = (\Phi_t(z), r)$. In particular, $(S, \rho_t^* g)$ is a Sasakian manifold with the same Reeb field, the same complex structure on the cone and the same transversely holomorphic structure on the Reeb foliation. By Proposition 2.2.6 we have that $\rho_t^* d\eta = d\eta + \partial_B \partial_{\bar{B}} \psi_t$. We can now follow the argument in [Tian, 2000] to obtain the proposition. □

Our primary concern will be sections of $\mathcal{E}$ induced from basic functions. In order to have our theory sufficiently well adapted for our future applications we discuss this now. Given a basic function h, $\overline{\partial}_B h$ is a section of $\Lambda_B^{0,1}$. We then define $V^j = (g^T)^{j\bar{k}} \partial_{\bar{k}} h$. V defines a section of the quotient bundle Q, and the splitting map σ satisfies $\sigma([V]) = V$. Moreover,

V lies in the normalizer of ξ in TS. Thus, $[V]$ defines a global section of $\mathcal{E}^{1,0}$ over S. We now compute that

$$\partial_{\mathcal{E}}[V] = \sum_{l=1}^{n} \nabla_l^T \left((g^T)^{j\bar{k}} \partial_{\bar{k}} h \right) \left[\partial z^j\right] \otimes dz^l = \sum_{l=1}^{n} (g^T)^{j\bar{k}} \partial_l \partial_{\bar{k}} h \left[\partial z^j\right] \otimes dz^l,$$

and so

$$\|\partial_{\mathcal{E}}[V]\|_{L^2(\mathcal{E}^{1,0}\otimes TS^*)} = \int_S (g^T)^{l\bar{p}} (g^T)^{j\bar{k}} \left(\nabla_l^T \nabla_{\bar{k}}^T h\right) \left(\overline{\nabla_p^T \nabla_{\bar{j}}^T h}\right) d\mu. \tag{3.3.44}$$

Expressions such as these shall appear repeatedly in what is to follow. In order to simplify our notation, we will use ∇ and $\overline{\nabla}$ to denote covariant derivative in the unbarred and barred directions. For example, equation (3.3.44) can then be written as $\|\partial_{\mathcal{E}}[V]\|_{L^2} = \int_S |\nabla\overline{\nabla} h|^2 d\mu$.

3.3.3 Proof of Theorem 13 part *(i)*

In this section we use the bound below for the Mabuchi functional to show that the L^2 norm of $\partial_B u$ goes to zero as $t \to \infty$, where u is the transverse Ricci potential, $R^T_{\bar{k}j} - g^T_{\bar{k}j} = \partial_j \partial_{\bar{k}} u$. The uniform bounds for the transverse Riemann tensor allow us to employ an inductive argument to obtain the decay to zero of all Sobolev norms of $\partial_B u$. The Mabuchi K-energy along the normalized Sasaki-Ricci flow is given by (3.3.42). Thus, if the Mabuchi energy is bounded below on $\mathcal{H}_{\eta_0}$ then there exists times $t_k \to \infty$ such that $\|\nabla u\|_{L^2}(t_k) \to 0$. We can obtain convergence for the full sequence by computing the evolution equation for the quantity $Y(t) = \|\nabla u\|^2_{L^2}$. Following the computations in [Phong and Sturm, 2006] we obtain

$$\dot{Y}(t) = (n+1)Y(t) - \int_S |\partial_B u|^2 R^T d\mu - \int_S |\overline{\nabla}\nabla u|^2 d\mu - \int_S |\nabla\nabla u|^2 d\mu. \tag{3.3.45}$$

Applying the uniform bound for R^T in Theorem 10, the argument from [Phong and Sturm, 2006] carries over verbatim to yield;

Lemma 3.3.16. *Assume the Mabuchi K-energy is bounded from below on the Kähler class of η_0. Then $Y(t) \to 0$ along the Kähler-Ricci flow as $t \to \infty$.*

Proof of Theorem 13 part (i). In light of Lemma 3.3.16, rearranging (3.3.45) and integrating with respect to t gives,

$$\int_0^\infty dt \int_S |\overline{\nabla}\nabla u|^2 d\mu_t + \int_0^\infty dt \int_S |\nabla\nabla u|^2 d\mu_t < \infty.$$

We are in a position to apply our previous argument inductively. Define

$$Y_{r,s}(t) = \int_S |\nabla^s \overline{\nabla}^r u|^2 d\mu_t. \tag{3.3.46}$$

Following the computations in [Phong and Sturm, 2006], and making use of the uniform C^∞ bounds on Rm^T and Rm guaranteed by Theorem 15, we compute that

$$\begin{aligned} \dot{Y}_{r,s}(t) \leqslant & C_1 Y_{r,s}(t) + C_2 \left(\int_S |D^{r+s-p} u|^2 d\mu_t \right)^{1/2} Y_{r,s}^{1/2}(t) \\ & - \int_S |\nabla^{s+1} \overline{\nabla}^r u|^2 d\mu_t - \int_S |\overline{\nabla} \nabla^s \overline{\nabla}^r u|^2 d\mu_t, \end{aligned} \tag{3.3.47}$$

where summation over $1 \leqslant p \leqslant r+s-1$ is understood. We now employ the argument in [Phong and Sturm, 2006], which carries over verbatim. □

3.3.4 Convergence in presence of stability

We begin this section by manipulating the equation (3.3.45) into a more suggestive form.

$$\dot{Y}(t) = -\int_S |\partial_B u|^2 (R^T - n) d\mu_t - \int_S \nabla^j u \nabla^{\bar{k}} u (R^T_{\bar{k}j} - g^T_{\bar{k}j}) d\mu_t - 2 \int_S |\bar{\nabla} \bar{\nabla} u|^2 d\mu_t. \tag{3.3.48}$$

This follows by applying the Bochner-Kodaira formula obtained in Proposition 3.3.10 to the section of $\mathcal{E}$ defined by $V^j = (g^T)^{j\bar{k}} \partial_{\bar{k}} u$. For large time t, the first two terms on the right hand side can easily be bounded by εY, by Theorem 13 part *(i)*. In order to obtain the exponential decay of the quantity Y, we must bound the last term in equation (3.3.48). Let λ_t be the smallest, strictly positive eigenvalue of the Laplacian $\square_{\mathcal{E},t}$. We include the subscript t to enforce that the metric $g(t)$ is evolving. By the elliptic theory we have

$$\lambda_t \|V - \pi_t V\|_{L^2(\mathcal{E}^{1,0})} \leqslant \int_S |\overline{\partial}_{\mathcal{E}} V|^2 d\mu_t,$$

where π_t is the L^2 projection onto $H^0(\mathcal{E}^{1,0})$ with respect to the metric $g^T(t)$. As in the Kähler case, we observe that for the section $V \in \mathcal{E}$ in question, we have by Corollary 3.3.13

$$\|\pi_t V\|^2 = fut_S(\pi_t V).$$

Thus, equation (3.3.48) yields the inequality

$$\begin{aligned} \dot{Y} \leqslant & -2\lambda_t Y + 2\lambda_t fut_S \left(\pi_t (g^T)^{j\bar{k}} \partial_{\bar{k}} u \right) \\ & - \int_S |\partial_B u|^2 (R^T - n) d\mu - \int_S \nabla^j u \nabla^{\bar{k}} u \left(R^T_{\bar{k}j} - g^T_{\bar{k}j} \right) d\mu \end{aligned} \tag{3.3.49}$$

We remark that equation (3.3.49) is completely general.

Proof of Theorem 13 part (ii). Since the Mabuchi functional is bounded below, Proposition 3.3.15 implies that the Futaki invariant is zero. The uniform transverse curvature bound, and the conclusion of Theorem 13 part *(i)* imply that for any $\varepsilon > 0$, there is a T_ε such that, for every $t \in [T_\varepsilon, \infty)$ we have $\dot{Y}(t) \leqslant (-\lambda_t + \varepsilon)Y(t)$. It suffices to find a positive lower bound for λ_t. Condition (C) is tailor made for the task.

Theorem 18. *Let (S, ξ, η, Φ, g) be a compact Sasakian manifold of dimension $2n + 1$. Assume that the Sasaki structure (ξ, η, Φ, g) satisfies stability condition (C). Fix $V, D, \delta > 0$, and constants C_k. Then there exists an integer N and a constant $C(V, D, \delta, C_k, n, N) > 0$ such that*

$$C\|V\|^2 \leqslant \|\overline{\partial}_{\mathcal{E}} V\|^2, \quad \forall \quad V \perp H^0(\mathcal{E}^{1,0})$$

for all Sasaki structures (S, ξ, η, Φ, g) with $Vol_g(S) < V$, and $diam_g(S) < D$, and whose injectivity radius is bounded below by δ, and the k-th derivative of whose curvature tensors are uniformly bounded by C_k for all $k \leqslant N$.

The proof of Theorem 18 is taken up in the next section. It follows that for t sufficiently large, we have $Y(t) \leqslant Ce^{-ct}$. With the exponential decay of the L^2 norm of $|\nabla u|$ established, a straight forward adaptation of the arguments in [Phong and Sturm, 2006] yield the exponential decay of the L^2 norms of $\bar{\nabla}^r \nabla^s u$ where all norms are computed with respect to the evolving metric $g^T(t)$. The Sobolev imbedding theorem with uniform constants then gives the exponential decay of the C^k norm of u for any k. Since $\dot{g}^T_{\bar{k}j} = \partial_j \partial_{\bar{k}} u$, we have $\int_0^\infty \sup_S |\dot{g}^T_{\bar{k}j}|_{g^T(t)} dt < \infty$. A lemma of Hamilton [Hamilton, 1982, Lemma 14.2] allows us to conclude that the metrics $g^T_{\bar{k}j}$ on Q are uniformly equivalent. While Hamilton's proof is for metric tensors on the tangent bundle, one can easily check that the argument holds for vector bundles. The uniform equivalence of the metrics g^T imply that for any section $W \in Q$ we have

$$|g^T_{\bar{k}j}(\mathcal{T}) W^j \overline{W^k} - g^T_{\bar{k}j}(\mathcal{S}) W^j \overline{W^k}| \leqslant C|W|^2_{t=0} \left(e^{-c\mathcal{T}} - e^{-c\mathcal{S}}\right)$$

As the last term goes to zero exponentially as $\mathcal{S}, \mathcal{T} \to \infty$, we obtain the exponential convergence of the transverse metrics to some metric $g^T_{\bar{k}j}$ which is equivalent to all the metrics $g^T_{\bar{k}j}(t)$. Iteration yields exponential convergence in C^∞. Since $\partial_j \partial_{\bar{k}} u$ tends to zero, the limiting metric $g^T_{\bar{k}j}(\infty)$ is Sasaki-Einstein. □

3.3.5 Compactness theorems and the proof of Theorem 18

Our main objective in this section is to prove Theorem 18, which will finish the proof of Theorem 13. We begin by stating and proving a Sasakian version of Gromov compactness. This theorem is well known, and follows easily from Hamilton's compactness theorem [Hamilton, 1995a] but we include the short proof for completeness.

Theorem 19. *Let (S, g) be a compact Sasakian manifold. Let $g(t)$ be a sequence of Sasakian metrics on S, and $J(t)$ a sequence of complex structures on the cone $C(S)$ such that $(C(S), \bar{g}(t) := dr^2 + r^2 g(t), J(t))$ is Kähler. Assume that the $g(t)$'s have bounded geometry in the sense that their volumes, diameters, curvatures and covariant derivatives of their curvature tensor are all bounded from above, and their injectivity radii are bounded below. Then there exists a subsequence $\{t_j\}$, and a sequence of diffeomorphisms $F_{t_j} : S \to S$ such that the pulled back metrics $F_{t_j}^* g(t_j)$ converge in C^∞ to a smooth metric $\tilde{g}(\infty)$. Moreover, on the cone, the lifted diffeomorphisms defined by $\tilde{F}_{t_j}(r, z) := (r, F_{t_j}(z))$ have that the sequence $\tilde{F}_{t_j}^* J(t_j))$ converges in C^∞ to an integrable complex structure $\tilde{J}(\infty)$ on $C(S)$. Furthermore, the metric $\tilde{g}(\infty)$ is Sasakian with respect to the complex structure $\tilde{J}(\infty)$. In particular, the Sasaki structures $(\xi(t_j), \eta(t_j), \Phi(t_j), g(t_j))$ converge in C^∞ to a Sasaki structure $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$.*

Proof. The C^∞ convergence part of this theorem is just Hamilton's compactness theorem [Hamilton, 1995a]. Thus, we are reduced to showing that the complex structures converge. This follows essentially from the proof of [Phong and Sturm, 2006] Theorem 4, with the wrinkle that the cone $C(S)$ is not compact. Consider instead the truncated cone $\tilde{C}(S) = ((\frac{1}{2}, 1) \times S)$. The argument in [Phong and Sturm, 2006] shows that the complex structures converge to an integrable complex structure $\tilde{J}(\infty)$ on compact sets making $(\tilde{C}(S), \tilde{J}(\infty), dr^2 + r^2\tilde{g}(\infty))$ into a Kähler manifold. We extend the complex structure to the whole cone by using the fact that $\mathcal{L}_{r\partial_r} J(t_j) = 0$, and that $\tilde{F}_{t_j *} r\partial_r = r\partial_r$. □

Proof of Theorem 18. Let λ_t be the smallest positive eigenvalue of $\Box_{\mathcal{E}}$, defined with respect to the Sasaki structure $\mathfrak{s}(t) := (\xi(t), \eta(t), \Phi(t), g(t))$. Suppose $\mathfrak{s}(t)$ converges in C^∞ to a Sasaki structure $\mathfrak{s}_\infty := (\xi_\infty, \eta_\infty, \Phi_\infty, g_\infty)$ and the dimension of the space of global holomorphic sections of the sheaf of transverse foliate vector fields is the same for every $N \leqslant t \leqslant \infty$. The perturbation theory for the Laplacian used in the proof of Theorem 3 of [Phong and

Sturm, 2006] extends to the global sections of the sheaf $\mathcal{E}$, and we obtain

$$\lim_{t\to\infty} \lambda_t = \lambda_\infty. \tag{3.3.50}$$

We can now prove by contradiction: assume there exists a sequence of metrics $g(t)$ with $\lambda_t \to 0$. By passing to a subsequence (not relabeled), we can apply Theorem 19 to obtain the existence of diffeomorphisms F_t so that the pulled back Sasaki structure

$$\tilde{\mathfrak{s}}(t) := (F_t^* g(t), F_t^*\xi(t), F_t^*\eta(t), F_t^*\Phi(t))$$

converges in C^∞ to a Sasaki structure $\tilde{\mathfrak{s}}_\infty = (\tilde{\xi}_\infty, \tilde{\eta}_\infty, \tilde{\Phi}_\infty, \tilde{g}_\infty)$. By equation (3.3.50), the lowest positive eigenvalue of $\tilde{\mathfrak{s}}(t)$ converges to a strictly positive limit. Let $\mathcal{E}(t), \tilde{\mathcal{E}}(t)$ be the sheaves defined by the Sasaki structures $\mathfrak{s}(t), \tilde{\mathfrak{s}}(t)$ respectively. Observe that the diffeomorphism F_t induces an isomorphism of sheaves $\mathcal{E}(t) \cong \tilde{\mathcal{E}}(t)$. Moreover, F_t is an isometry which preserves the transverse holomorphic structure, and hence descends to an isometry of the quotient bundles $Q(t), \tilde{Q}(t)$. Using the computations in Section 3.3.2, it is then clear that the sheaves $\mathcal{E}(t), \tilde{\mathcal{E}}(t)$ are isospectral, providing a contradiction.

□

3.3.6 The proof of Theorem 14

Note that in the proof of Theorem 13 we only needed a bound on the smallest positive eigenvalue of $\Box_{\mathcal{E}}$ restricted to sections of $\mathcal{E}^{1,0}$ induced by basic functions. Rather than study the $\overline{\partial}_{\mathcal{E}}$ Laplacian on global sections of $\mathcal{E}^{1,0}$ we are thus motivated to study the following operator on C^∞_B;

$$L := -(g^T)^{j\bar{k}}\nabla_j\nabla_{\bar{k}} + (g^T)^{j\bar{k}}\nabla_j u \nabla_{\bar{k}}.$$

Recall that the operator L appeared in section 3.1.3, where it was shown to be elliptic and self-adjoint with respect to the L^2 inner-product on H^2_B induced by the probability measure $d\rho := Vol(S)^{-1}e^{-u}d\mu$. Moreover, it was shown that L has a complete spectrum of smooth, basic eigenfunctions $\{\psi_j\}_{j\in\mathbb{N}}$ spanning L^2_B, with eigenvalues $\lambda_j \geqslant 1$. This yields the Poincaré inequality in Lemma 3.1.36

Observe that if $\lambda_t \geqslant 1+\delta > 1$, then the computation for the operator L carried out in section 3.1.3 suggests that the final term in (3.3.48) can be controlled by $-\delta Y(t)$. We refer the reader to equation (3.1.19). In the present setting, as we are not assuming a lower bound for the Mabuchi energy, it is no longer natural to work with $Y(t)$. Instead, we define

$$W(t) := \frac{1}{Vol(S)} \int_S (u-a)^2 e^{-u} d\mu,$$
$$Z(t) := \partial_t a = \frac{1}{Vol(S)} \int_S \left(|\nabla u|^2 - (u-a)^2\right) e^{-u} d\mu,$$

where $a(t)$ is defined by (3.2.24). The main result in this section is the following proposition.

Proposition 3.3.17. *Assume that condition (F) holds on $2\pi c_1^B(S)$, and that condition (T) holds along the Sasaki-Ricci flow with initial value $g_0 \in 2\pi c_1^B(S)$. Then there are constants $b, C > 0$ independent of t so that $W(t) \leqslant Ce^{-bt}$, for every $t \in [0, \infty)$. Moreover,*

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R^T - n\|_{C^0} \leqslant Ce^{-\frac{1}{2(n+1)}bt}, \quad t \in [0, \infty).$$

The proof of Proposition 3.3.17 proceeds in several steps. First, we describe a smoothing lemma which will reduce the proof of the proposition to proving the exponential decay of W. The idea of a smoothing lemma was first introduced by Bando [Bando, 1987], and appeared in [Phong *et al.*, 2009]; it was subsequently improved in [McFeron, 2014].

Lemma 3.3.18. *There exist positive constants δ, K depending only on n with the following property; for any $\varepsilon \in (0, \delta]$, and any $t_0 > 0$, if $\|u(t_0)\|_{C^0} \leqslant \varepsilon$, then*

$$\|\nabla u(t_0 + 2)\|_{C^0} + \|R(t_0 + 2) - n\|_{C^0} \leqslant K\varepsilon.$$

The proof of Lemma 3.3.18 is identical to the Kähler case, and can be found in [Phong *et al.*, 2009]. In order to prove Proposition 3.3.17, we must first establish the exponential decay of W. We now describe a condition under which such decay holds.

Proposition 3.3.19. *Suppose there is a uniform constant $\delta > 0$ independent of time so that*

$$\frac{(1+\delta)}{V} \int_S (u(t) - a(t))^2 e^{-u(t)} d\mu_t \leqslant \frac{1}{V} \int_S |\nabla u(t)|^2 e^{-u(t)} d\mu_t. \tag{3.3.51}$$

Then there are constants $b, C > 0$ independent of t so that $W(t) \leqslant Ce^{-bt}$.

The proof of this proposition requires the following lemma, which is a consequence of the developments in Section 3.2.

Lemma 3.3.20. *The transverse Ricci potential $u(t)$, and its average $a(t)$ satisfy the following inequalities, where the constants C_1 and C_2 depend only on g_0.*

(i) $0 \leqslant -a \leqslant \|u-a\|_{C^0}$

(ii) $\|u-a\|_{C^0}^{n+1} \leqslant C_1\|\nabla u\|_{C^0}^2\|u-a\|_{L^2} \leqslant C_2\|\nabla u\|_{L^2}\|\nabla u\|_{C^0}^n$

Proof. We combine Lemma 3.1.36, Proposition 3.2.18 and follow the proof in [Phong *et al.*, 2009]. □

Proof of Proposition 3.3.19. The proof follows essentially from the arguments in [Zhang, 2010], and so we provide only a sketch. We first claim that $W(t) \to 0$ as $t \to \infty$. To see this, observe that $Z(t) \geqslant \delta W(t)$ by assumption. From Theorem 10 we have

$$\int_0^\infty Z(t)dt = \lim_{t\to\infty} a(t) - a(0) < \infty.$$

Thus, $Z(t) \to 0$ along a subsequence $t_k \to \infty$. The convergence of the full sequence is obtained as in Section 3.3.3, by computing the evolution equation for W. Lemma 3.3.20, combined with the uniform bounds in Theorem 10 imply that $\|u-a\|_{C^0} \leqslant AW(t)^{1/(2n+2)}$, and so $u \to a$ in C^0 as t goes to infinity. The result follows from elementary modifications to the proof of Lemma 2.4 in [Zhang, 2010]. □

Lemma 3.3.20 and the uniform bounds for u in Theorem 10 imply that if $W(t)$ decays exponentially, then $\|u\|_{C^0}$ decays exponentially. By Lemma 3.3.18, we see that the second statement in Proposition 3.3.17 follows from the exponential decay of $W(t)$. Our task is now reduced to showing that when conditions (T) and (F) hold, the assumptions of Proposition 3.3.19 are satisfied. We begin by showing that if the Futaki invariant vanishes and we have a non-degeneracy condition on the 'second' eigenvalue of L, then (3.3.51) holds.

Proposition 3.3.21. *Let $\nu(t)$ be the smallest eigenvalue of L larger than one. Assume that $\nu(t) \geqslant 1+\delta$ for some $\delta > 0$ uniformly along the flow. If the Futaki invariant vanishes, then (3.3.51) holds.*

Proof. Fix a time t, and from now on suppress the t variable. Recall that in S5 it was pointed out the $(g^T)^{i\bar{j}}\partial_{\bar{j}}u$ defines a section of $\mathcal{E}^{1,0}$. For simplicity we denote this section by $\nabla u \in \mathcal{E}^{1,0}$. Since $fut_S \equiv 0$, we necessarily have $\nabla u \perp H^0(\mathcal{E}^{1,0})$ in $L^2(\mathcal{E}^{1,0}, d\mu)$. Let π denote the orthogonal projection to $H^0(\mathcal{E}^{1,0})$ in the space in $L^2(\mathcal{E}^{1,0}, d\rho)$, where we recall that $d\rho := Vol(S)^{-1}e^{-u}d\mu$. We decompose $\nabla u = \pi(\nabla u) + V$, then we have

$$\langle \pi(\nabla u), \pi(\nabla u)\rangle_{L^2(\mathcal{E}^{1,0},d\mu)} \leqslant \langle V, V\rangle_{L^2(\mathcal{E}^{1,0},d\mu)}. \tag{3.3.52}$$

Let $\nu_0 = 1 < \nu_1 < \nu_2 < \ldots$ be the distinct eigenvalues of L acting on the function space $H^2_B(d\rho)$, and let E_k be the eigenspace of ν_k. E_0 and may be empty, and corresponds to those basic functions which induce sections of $H^0(\mathcal{E}^{1,0})$. Let $u - a = u_0 + u_1 + \ldots$ be the unique decomposition of $u - a$ into eigenfunctions $u_k \in E_k$ so that u_i and u_j are orthogonal in $L^2(d\rho)$ for $i \neq j$. For $k \geqslant 1$ we have $\nu_k \geqslant \nu_1 > 1 + \delta$ uniformly along the flow, and so

$$\begin{aligned}\int_S (u-a)^2 d\rho &= \sum_k \int_S |u_k|^2 d\rho = \sum_k \lambda_k^{-1} \int_S |\nabla u_k|^2 d\rho \\ &\leqslant \int_S |\nabla u_0|^2 d\rho + \sum_{k=1}^{\infty} (1+\delta)^{-1} \int_S |\nabla u_k|^2 \\ &= \int_S |\nabla u_0|^2 + (1+\delta)^{-1} \int_S |V|^2 d\rho\end{aligned}$$

We claim that $\pi(\nabla u) = \nabla u_0$ as sections of $\mathcal{E}^{1,0}$. Assuming this for the moment, we obtain from (3.3.52)

$$\int_S |\nabla u_0|^2 d\rho \leqslant \frac{e^{-\inf u}}{Vol(S)} \langle V, V\rangle_{L^2(\mathcal{E}^{1,0},d\mu)} \leqslant e^{osc(u)} \int_S |V|^2 d\rho.$$

It follows that

$$\int_S (u-a)^2 d\rho \leqslant \int_S |\nabla u|^2 d\rho - \frac{\delta}{1+\delta} \int_S |V|^2 d\rho \leqslant \left(1 - \frac{\delta}{(1+\delta)(1+e^{osc(u)})}\right) \int_S |\nabla u|^2 d\rho.$$

From this, one easily shows that equation (3.3.51) holds. It suffices to establish the claim. In fact, we shall prove something more general. Suppose that ψ is an eigenfunction of L with eigenvalue $\nu > 1$. By elliptic regularity, $\psi \in C^\infty_B$. Denote by $\nabla\psi$ the global section of $\mathcal{E}^{1,0}$ induced by $\overline{\partial}_{\mathcal{E}}\psi$; we claim that $\nabla\psi \perp H^0(\mathcal{E}^{1,0})$ in $L^2_B(d\rho)$. To see this, let $V \in H^0(\mathcal{E}^{1,0})$, and compute

$$\begin{aligned}\langle V, \nabla\psi\rangle_{L^2(d\rho)} &= \int_S V^i \partial_i \bar{\psi} e^{-u} d\mu \\ &= \nu^{-1} \int_S V^i (g^T)^{l\bar{k}} \left(-\nabla_i^T \partial_l \partial_{\bar{k}} \bar{\psi} + \partial_i \partial_{\bar{k}} u \partial_l \bar{\psi} + \partial_{\bar{k}} u \nabla_i^T \partial_l \bar{\psi}\right) e^{-u} d\mu \\ &= \nu^{-1} \int_S V^i (g^T)^{l\bar{k}} \left(-\nabla_l^T \nabla_{\bar{k}}^T \partial_i \bar{\psi} + (-Ric^T_{\bar{k}i} + g^T_{\bar{k}i}) \partial_l \bar{\psi} + \partial_{\bar{k}} u \nabla_i^T \partial_l \bar{\psi}\right) e^{-u} d\mu \\ &= \nu^{-1} \int_S V^i \partial_i \bar{\psi} d\rho,\end{aligned}$$

where the last line follows by commuting covariant derivatives and integrating by parts. Since $\nu > 1$, we obtain the claim. □

Proof of Proposition 3.3.17. Let ν denote the first eigenvalue of L strictly larger than 1. In light of Proposition 3.3.21, it suffices to show that when condition (T) holds there is a $\delta > 0$ such that $\nu > 1 + \delta$ along the flow. Let $\lambda, \tilde{\lambda}$ be the smallest positive eigenvalues of the $\overline{\partial}_{\mathcal{E}}$ operator acting on $L^2(\mathcal{E}^{1,0}, d\mu)$ and $L^2(\mathcal{E}^{1,0}, d\rho)$ respectively. That is, we have

$$\int_S |\overline{\nabla} V|^2 d\mu \geqslant \lambda \int_S |V|^2 d\mu, \quad \text{for all } V \perp H^0(\mathcal{E}^{1,0}) \text{ in } L^2(d\mu)$$

$$\int_S |\overline{\nabla} V|^2 d\rho \geqslant \tilde{\lambda} \int_S |V|^2 d\rho, \quad \text{for all } V \perp H^0(\mathcal{E}^{1,0}) \text{ in } L^2(d\rho)$$

One can easily check that $e^{-osc(u)}\lambda \leqslant \tilde{\lambda} \leqslant e^{osc(u)}\lambda$, see for example [Phong *et al.*, 2008a], [Zhang, 2010]. We now show that $\nu > 1 + e^{-osc(u)}\lambda$, which suffices to establish Proposition 3.3.17. Let $\psi \in L^2_B$ be an eigenfunction of L with eigenvalue λ, and let $\nabla\psi$ denote the section of $\mathcal{E}^{1,0}$ induced by $\overline{\partial}_{\mathcal{E}}\psi$. From the proof of Proposition 3.3.21, we know that $\nabla\psi \perp H^0(\mathcal{E}^{1,0})$ in $L^2_B(d\rho)$. By equation (3.1.19), we have

$$(\nu - 1) \int_S |\nabla\psi|^2 d\rho = \int_S |\overline{\partial}_{\mathcal{E}} \nabla\psi|^2 d\rho \geqslant \tilde{\lambda} \int |\nabla\psi|^2 d\rho.$$

It follows that $\nu > 1 + e^{-osc(u)}\lambda$. By the uniform C^0 bound for u in Theorem 10 we see that if condition (T) holds, then there is $\delta > 0$ such that $\nu > 1 + \delta$ uniformly along the flow. □

The following general lemma gives a condition under which the Sasaki-Ricci flow converges; in light of Proposition 3.3.17, it finishes the proof of Theorem 14.

Lemma 3.3.22. *Assume that the transverse scalar curvature $R^T(t)$ along the Sasaki-Ricci flow satisfies*

$$\int_0^\infty \|R^T(t) - n\|_{C^0} dt < \infty.$$

Then the metrics $g(t)$ converge exponentially fast to a Sasaki-Einstein metric.

Proof. The Sasaki potential $\varphi(t)$, satisfies equation (3.0.4), with $F = u(0)$, and $\varphi(0) = c_0$, where c_0 is defined by (3.3.37). For this particular choice of initial condition, Theorem 10 implies that $\|\dot{\varphi}\|_{C^0} \leqslant C$ uniformly along the flow. Now,

$$\partial_t \log \frac{(\frac{1}{2}d\eta)^n \wedge \eta_0}{(\frac{1}{2}d\eta_0)^n \wedge \eta_0} = -(R^T - n),$$

and so our assumption implies

$$\left|\log \frac{(\frac{1}{2}d\eta)^n \wedge \eta_0}{(\frac{1}{2}d\eta_0)^n \wedge \eta_0}\right| = \left|\int_0^t (R^T - n)dt\right| \leqslant \int_0^\infty \|R^T - n\|_{C^0} dt < \infty.$$

Rearranging equation (3.0.4) as an equation for φ, and using the uniform bound for $\dot{\varphi}$ we obtain that φ is uniformly bounded in C^0. By Proposition 3.3.1, $\|\varphi\|_{C^k}$ is uniformly bounded for each $k \in \mathbb{N}$, where the C^k norm is with respect to the initial metric $g^T(0)$. The uniform bounds on φ imply that the metrics $g^T(t)$ are uniformly equivalent and uniformly bounded in C^∞; in particular, Rm^T is uniformly bounded. It follows that there exists a subsequence of times $t_m \to \infty$ with $\varphi(t_m)$ converging in C^∞ to smooth basic function $\varphi(\infty)$. By uniform equivalence we have

$$\left|\Box_{B,g(0)} u(t_m)\right| \leqslant C \left|\Box_{B,g(t)} u(t_m)\right| \leqslant C|R^T(t_m) - n|_{C^0} \longrightarrow 0.$$

Thus, $\varphi(\infty)$ is a potential for a transversely Kähler-Einstein metric. Let λ_t be the smallest positive eigenvalue of $\Box_{\mathcal{E}}$ acting on smooth global sections of $\mathcal{E}^{1,0}$. We claim that $\lambda_t \geqslant \lambda > 0$. If this were not the case, then there is a further subsequence (not relabeled) such that $g(t_m)$ converges in C^∞ to a Sasakian metric $\tilde{g}$, and $\lambda_{t_m} \to 0$. We can now apply the arguments in the proof of Theorem 18 in the special case that the Reeb field ξ and the transverse complex structure Φ are fixed, and $\eta(t) = \eta_0 + 2d^c_B\varphi(t)$. In particular, by Proposition 3.3.11 the dimension of the space of global holomorphic sections of $\mathcal{E}^{1,0}$ is constant. We then obtain $0 = \lim_{m\to\infty} \lambda_{t_m} = \lambda(\tilde{g}) > 0$, which is a contradiction. Proposition 3.3.8 implies that the

Mabuchi K-energy is bounded below and so by Lemma 3.3.17 we obtain the exponential decay to zero of $Y(t) = \|\nabla u\|^2_{L^2}$. We claim that this implies the exponential decay to zero of $\|\nabla u\|_{(s)}$ for any Sobolev norm $\|\cdot\|_{(s)}$. This follows essentially from our previous computations. For example, rearranging equation (3.3.45), we get

$$(n+1)Y(t) - \int_S |\partial_B u|^2 R^T d\mu - \dot{Y(t)} = \int_S |\overline{\nabla}\nabla u|^2 d\mu + \int_S |\nabla\nabla u|^2 d\mu. \tag{3.3.53}$$

Thus, the uniform bound for R^T and the exponential decay of Y yields the exponential decay of the right hand side of equation (3.3.53). We then proceed inductively, using the functions $Y_{r,s}(t)$ as defined in (3.3.46) and employing the uniform curvature bounds (3.3.47). Since the transverse Ricci potential u is basic, and the metrics $g^T(t)$ are uniformly equivalent, the Sobolev imbedding theorem yields the exponential decay to zero of $\|u\|_{C^k}$ for any k, and hence $\|\dot{g}^T_{\bar{k}j}\|_{C^k} = \|R^T_{\bar{k}j} - g^T_{\bar{k}j}\|_{C^k}$ decays exponentially to zero for any k. □

Proof of Theorem 14. Part *(i)* follows from Proposition 3.3.17, and Lemma 3.3.22. Part *(ii)* follows from the argument in the proof of Lemma 3.3.22. Part *(iii)* follows from part *(ii)* and part *(i)*. □

3.4 Convergence of the Sasaki-Ricci flow on Sasaki-Einstein manifolds with finite automorphism group

3.4.1 Important Functionals

Here we introduce some functionals which will be important in our development, all of which are defined in [Nitta and Sekiya, 2012]. For notational simplicity throughout the paper we denote the volume form on S defined by η as $d\mu = (d\eta)^n \wedge \eta$. Given a potential φ, the volume form with respect to $\eta_\varphi := \eta + d^c_B\varphi$ is given by $d\mu_\varphi = (d\eta_\varphi)^n \wedge \eta$. Now, consider the following functionals on $\mathcal{H}_\eta(S)$

$$\begin{aligned} I_\eta(\varphi) &:= \frac{1}{V}\int_S \varphi\,(d\mu - d\mu_\varphi) \\ J_\eta(\varphi) &:= \frac{1}{V}\int_0^1\int_S \dot{\varphi}_t\,(d\mu - d\mu_{\varphi_t})\,dt, \end{aligned}$$

where φ_t is any path with $\varphi_0 = c$ and $\varphi_1 = \varphi$. Various forms of these functionals exist (for details see [Nitta and Sekiya, 2012]), and we can use these formulations to prove:

$$\frac{1}{n} J_\eta \leqslant \frac{1}{n+1} I_\eta \leqslant J_\eta. \tag{3.4.54}$$

The time derivatives of these functionals along any path φ_t can now be computed easily:

$$\begin{aligned}
\partial_t I_\eta(\varphi_t) &:= \frac{1}{V}\int_S \dot{\varphi}_t\,(d\mu - d\mu_{\varphi_t}) - \frac{1}{2V}\int_S \varphi_t\,\partial_t\,d\mu_{\varphi_t},\\
\partial_t J_\eta(\varphi_t) &:= \frac{1}{V}\int_S \dot{\varphi}_t\,(d\mu - d\mu_{\varphi_t}).
\end{aligned}$$

Thus, the time derivative of the difference is given by:

$$\partial_t(I_\eta - J_\eta)(\varphi_t) = -\frac{1}{V}\int_S \varphi_t\,\partial_t\,d\mu_{\varphi_t}.$$

Next, we consider the following two functionals, which differ only by the last term:

$$\begin{aligned}
F^0_\eta(\varphi) &:= J_\eta(\varphi) - \frac{1}{V}\int_S \varphi\,d\mu. \qquad (3.4.55)\\
F_\eta(\varphi) &:= J_\eta(\varphi) - \frac{1}{V}\int_S \varphi\,d\mu - \log\left(\frac{1}{V}\int_X e^{-u-\varphi} d\mu\right).
\end{aligned}$$

As before, u is the transverse Ricci potential of η. Finally we define the transverse K-energy, which once again is defined along any path φ_t with $\varphi_0 = c$ and $\varphi_1 = \varphi$:

$$K_\eta(\varphi) = -\frac{1}{V}\int_0^1 \int_S \dot{\varphi}_t(R^T_{\varphi_t} - n)\,d\mu_{\varphi_t}.$$

3.4.2 Convergence of the Sasaki-Ricci flow

In this section we apply all the results we have obtained so far to to extend a theorem of Perelman to the Sasaki-Ricci flow. Let $Aut^0(S) = H^0(\mathcal{E}^{1,0})$ denote the identity component of the automorphism group of the transverse holomorphic structure of (S, ξ, η, g_0). We then prove the following theorem:

Theorem 20 ([Collins and Jacob, 2014]). *If $(S, g_0, \xi, \eta_0, \Phi)$ admits a Sasakian metric in $\frac{1}{2}[d\eta_0]_B = 2\pi c_1^B(S)$ which is transversely Kähler-Einstein and $Aut^0(S) = \{e\}$, then the Sasaki-Ricci flow converges exponentially fast to a transversely Kähler-Einstein metric.*

The proof follows the ideas of [Phong and Sturm, 2006] closely. We utilize an inequality of Moser-Trudinger type, proved in the Sasaki case by Zhang [Zhang, 2011] and in the Kähler-Einstein case by Tian [Tian, 1997], and subsequently improved by Tian-Zhu [Tian and Zhu, 2000], and Phong-Song-Sturm-Weinkove [Phong *et al.*, 2008b]. For fixed contact form η on S, consider the following functionals defined on all basic potentials φ:

$$\begin{aligned} J_\eta(\varphi) &:= \frac{1}{V}\int_0^1\int_S \dot\varphi_t\,(d\mu - d\mu_\varphi)\,dt, \\ F_\eta(\varphi) &:= J_\eta(\varphi) - \frac{1}{V}\int_S \varphi\,d\mu - \log\left(\frac{1}{V}\int_X e^{-u-\varphi}d\mu\right), \end{aligned}$$

where φ_t is any path with $\varphi_0 = c$ and $\varphi_1 = \varphi$ and h is the transverse Ricci-potential associated to η. We need the following inequality:

Theorem 21 ([Zhang, 2011], Theorem 6.1)**.** *If (S, ξ, η, g_0) admits a Sasaki-Einstein metric g_{SE} and $Aut^0(S) = 0$, then there exists positive constants A,B such that following inequality holds for all potentials:*

$$F_{\eta_0}(\varphi) \geqslant A\, J_{\eta_0}(\varphi) - B.$$

With this inequality in hand, and the uniform Sobolev inequality in Theorem 11, an application of the results in section 3.3 allow us to obtain exponential convergence of the Sasaki-Ricci flow to a Sasaki-Einstein metric.

We want to exploit the Moser-Trudinger inequality to deduce a uniform C^0 bound for the potential φ along the Sasaki-Ricci flow. The first step is to establish some relations between the functionals defined in the previous subsection, and in particular we make explicit use the Sasaki-Ricci flow.

Lemma 3.4.1. *There exists constants C_1, C_2, depending only on g_0, so that if $\varphi = \varphi(t)$ is evolving along the Sasaki-Ricci flow, we have*

$$\begin{aligned} &i) \qquad K_{\eta_0}(\varphi) - F^0_{\eta_0}(\varphi) - \frac{1}{V}\int_S \dot\varphi\, d\mu_\varphi = C_1, \\ &ii) \qquad |F_{\eta_0}(\varphi) - K_{\eta_0}(\varphi)| + |F^0_{\eta_0}(\varphi) - K_{\eta_0}(\varphi)| \leqslant C_2 \end{aligned}$$

Proof. We begin with i). First we compute the time derivative of $F^0_{\eta_0}$ along the flow. Using the formula for the variation of J_{η_0} from Section 3.4.1, we have:

$$\partial_t F^0_{\eta_0}(\varphi) = -\frac{1}{V}\int_S \dot\varphi(t)\, d\mu_\varphi.$$

On the other hand, since $\ddot{\varphi} = \Delta_B \dot{\varphi} + \dot{\varphi}$, we obtain

$$\frac{1}{V}\int_S \dot{\varphi}\, d\mu_\varphi = \frac{1}{V}\int_S \ddot{\varphi}\, d\mu_\varphi = \partial_t \left(\frac{1}{V}\int_S \dot{\varphi}\; d\mu_\varphi \right) - \frac{1}{V}\int_S \dot{\varphi}\Delta_B \dot{\varphi}\, d\mu_\varphi.$$

Computing the evolution equation for K_{η_0} we have:

$$\partial_t K_{\eta_0}(\varphi) = -\frac{1}{V}\int_S \dot{\varphi}(R^T - n)\, d\mu_\varphi = \frac{1}{V}\int_S \dot{\varphi}\Delta_B \dot{\varphi}\, d\mu_\varphi.$$

Combining the above equations, we obtain

$$\partial_t F^0_{\eta_0}(\varphi) = -\partial_t \left(\frac{1}{V}\int_S \dot{\varphi}\, d\mu_\varphi \right) + \partial_t K_{\eta_0}(\varphi),$$

from which *i*) follows. To prove *ii*), observe that, by Theorem 10, $\dot{\varphi}$ is uniformly bounded by a constant depending only on g_0, and so

$$|F^0_{\eta_0}(\varphi) - K_{\eta_0}(\varphi)| < C(g_0).$$

To establish the second inequality, we use the definition of F_η and employ the uniform bound for $\dot{\varphi}$ again to obtain

$$\begin{aligned} |F_{\eta_0}(\varphi) - K_{\eta_0}(\varphi)| &\leqslant \left|F^0_{\eta_0}(\varphi) - K_{\eta_0}(\varphi)\right| + \left| \log \left(\frac{1}{V}\int_S e^{-u(0)-\varphi}\, d\mu_0 \right) \right| \\ &\leqslant C(g_0) + \left| \log \left(\frac{1}{V}\int_S e^{-\dot{\varphi}}\, d\mu_\varphi \right) \right| \\ &\leqslant C(g_0) + C'(g_0), \end{aligned}$$

where in the second line we used the Sasaki-Ricci flow equation for potentials (3.0.4). This establishes *ii*). $\square$

Lemma 3.4.2. *There exists a constant C so that the following estimates hold uniformly along the Sasaki-Ricci flow:*

$$\begin{aligned} &iii) \qquad \frac{1}{nV}\int_S (-\varphi)\, d\mu_\varphi - C \leqslant J_{\eta_0}(\varphi) \leqslant \frac{1}{V}\int_S \varphi\, d\mu_0 + C \\ &iv) \qquad \frac{1}{V}\int_S \varphi\, d\mu_0 \leqslant \frac{n}{V}\int_S (-\varphi)\, d\mu_\varphi - (n+1)K_{\eta_0}(\varphi) + C. \end{aligned}$$

Proof. Since the Mabuchi K-energy decreases monotonically along the Sasaki-Ricci flow, the second inequality in Lemma 3.4.1 implies that $F_{\eta_0}(\varphi) \leqslant C$ uniformly along the flow.

Rearranging equation (3.4.55), combined with the upper bound for $F^0_{\eta_0}(\varphi)$ yields the right hand inequality in *iii*). Now, by definition of the $F^0_{\eta_0}$ we have the following equation:

$$F^0_{\eta_0}(\varphi) = -\left[(I_{\eta_0} - J_{\eta_0})(\varphi) + \frac{1}{V}\int_S \varphi\, d\mu_\varphi\right]. \tag{3.4.56}$$

Rearranging this equation, and applying the upper bound for $F^0_{\eta_0}(\varphi)$ yields:

$$\frac{1}{V}\int_S (-\varphi)\, d\mu_\varphi - C \leqslant (I_{\eta_0} - J_{\eta_0})(\varphi).$$

Applying estimate (3.4.54) establishes the left hand inequality in *iii*). To establish *iv*), we apply Lemma 3.4.1 inequality *ii*) and equation (3.4.55) to obtain:

$$\frac{1}{V}\int_S \varphi\ d\mu_0 \leqslant J_{\eta_0}(\varphi) - K_{\eta_0}(\varphi) + C \leqslant \frac{n}{n+1} I_{\eta_0}(\varphi) - K_{\eta_0}(\varphi) + C,$$

where the second inequality follows from (3.4.54). Applying the definition of the functional I_{η_0} and rearranging terms yields the result. □

The following proposition is a corollary of the uniform Sobolev inequality.

Proposition 3.4.3. *Along the Sasaki-Ricci flow we have the following inequality*

$$osc(\varphi) \leqslant \frac{A}{V}\int_S \varphi\, d\mu_0 + B,$$

where A and B are constants depending only on g_0.

Proof. Define the function $f = \max_S \varphi - \varphi + 1 \geqslant 1$. We now apply the standard Moser iteration technique. Let $\alpha > 0$ and write

$$\begin{aligned}\int_S f^{\alpha+1}(d\eta_\varphi)^n \wedge \eta_0 &\geqslant \int_S f^{\alpha+1}(d\eta_\varphi - d\eta_0) \wedge d\eta_\varphi^{n-1} \wedge \eta_0 \\ &= -\frac{i}{2}\int_S f^{\alpha+1}\partial\overline{\partial} f \wedge (d\eta_\varphi)^{n-1} \wedge \eta_0.\end{aligned}$$

We now integrate by parts

$$\begin{aligned}-\frac{i}{2}\int_S f^{\alpha+1}\partial\overline{\partial} f \wedge (d\eta_\varphi)^{n-1} \wedge \eta_0 &= -\frac{i}{2}\int_S f^{\alpha+1} d_B\overline{\partial} f \wedge (d\eta_\varphi)^{n-1} \wedge \eta_0 \\ &= \frac{i(\alpha+1)}{2}\int_S f^\alpha \partial f \wedge \overline{\partial} f \wedge (d\eta_\varphi)^{n-1} \wedge \eta_0 \\ &= \frac{i(\alpha+1)}{2(\frac{\alpha}{2}+1)^2}\int_S \partial(f^{\frac{\alpha}{2}+1}) \wedge \overline{\partial}(f^{\frac{\alpha}{2}+1}) \wedge (d\eta_\varphi)^{n-1} \wedge \eta_0.\end{aligned}$$

Thus, we obtain

$$\|\nabla(f^{\frac{\alpha}{2}+1})\|_{L^2(S,\eta_\varphi)} \leqslant \frac{n(\frac{\alpha}{2}+1)^2}{\alpha+1}\int_S f^{\alpha+1}\,d\mu_\varphi. \tag{3.4.57}$$

Set $\beta = \frac{2n+1}{2n-1}$ and $p = \alpha + 2 \geqslant 2$. Since Sobolev constant is uniformly bounded along the flow by Theorem 11, we obtain

$$\left[\int_S f^{p\beta}\,d\mu_\varphi)\right]^{\frac{1}{\beta}} \leqslant Cp\int_S f^p\,d\mu_\varphi$$

for a constant C depending only on g_0. Taking $p = 2$ and iterating in the usual fashion we obtain

$$\log\left(\|f\|_{L^\infty(S)}\right) \leqslant \sum_{k=1}^{\infty}\frac{\log(2C\beta^k)}{2\beta^k} + \log\left(\|f\|_{L^2(S,\eta_\varphi)}\right) = C_1 + \log\left(\|f\|_{L^2(S,\eta_\varphi)}\right).$$

It remains only to bound the L^2 norm of f. By the Poincaré inequality Lemma 3.1.36 we have that

$$\frac{1}{V}\int_S f^2 e^{-u}\,d\mu_\varphi \leqslant \frac{1}{V}\int_S |\nabla f|^2 e^{-u}\,d\mu_\varphi + \left(\frac{1}{V}\int_S fe^{-u}d\mu_\varphi\right)^2,$$

where $u = u(t)$ is the transverse Ricci potential. Moreover, by the uniform bounds for u along the Sasaki-Ricci flow from Theorem 10, the measures $e^{-u}\,d\mu_\varphi$ and $d\mu_\varphi$ are equivalent. We obtain

$$\begin{aligned}\|f\|^2_{L^2(S,\eta_\varphi)} &\leqslant C\frac{1}{V}\int_S |\nabla f|^2\,d\mu_\varphi + C\left(\frac{1}{V}\int_S f\,d\mu_\varphi\right)^2\\ &\leqslant C'\left[1 + \frac{1}{V}\int_S f\,d\mu_\varphi\right]^2\end{aligned}$$

where the final inequality follows by applying equation (3.4.57) with $\alpha = 0$. Finally, since $\Delta_{g_0}\varphi > -n$, a standard argument with the Green's function of g_0 yields

$$\sup_S \varphi \leqslant \frac{1}{V}\int_S \varphi\,d\mu_0 + C''.$$

Moreover, by Lemma 3.4.2 part *iii*) we have

$$\frac{1}{V}\int_S(-\varphi)\,d\mu_\varphi \leqslant \frac{n}{V}\int_S \varphi\,d\mu_0 + C'''.$$

Applying the definition of f, the proposition follows. □

Combining Proposition 3.4.3 with the argument in Lemma 3.3.22, we can prove the following corollary:

Corollary 3.4.4. *Let (S, η_0, ξ, g_0) be a compact Sasaki manifold, with $d\eta_0 \in c_1^B(S)$, and consider the Sasaki-Ricci flow with initial value given by* (3.3.37)*. If there exists a constant C with*

$$\sup_{t\in[0,\infty)} \frac{1}{V}\int_S \varphi\, d\mu_0 \leqslant C < \infty, \tag{3.4.58}$$

then the Sasaki-Ricci flow converges exponentially fast in C^∞ to a Sasaki-Einstein metric.

Proof. Proposition 3.4.3 implies that $\mathrm{osc}(\varphi)$ is uniformly bounded along the Sasaki-Ricci flow. Moreover, we have:

$$1 = \frac{1}{V}\int_S d\mu_\varphi = \frac{1}{V}\int_S e^{-\varphi+\dot{\varphi}-u(0)}\, d\mu_0.$$

Now, since $\|\dot{\varphi}\|_{C^0}$ is uniformly controlled along the Sasaki-Ricci flow by Theorem 10, we have

$$0 < C_1 \leqslant \frac{1}{V}\int_S e^{-\varphi}\, d\mu_0 \leqslant C_2,$$

which easily implies a lower bound for $\sup_S \varphi$. Combined with the uniform bound for $\mathrm{osc}(\varphi)$ we obtain we obtain a uniform bound for $\|\varphi\|_{C^0}$. By Proposition 3.3.1 we obtain uniform bounds for φ in $C^k(S, g_0)$. We can now apply the argument in the proof of Lemma 3.3.22 to obtain the exponential convergence of φ to a Sasaki-Einstein potential. □

We are now ready to prove our main result:

Proof of Theorem 20*.* By assumption, the Moser-Trudinger inequality holds along the Sasaki-Ricci flow. Since $K_{\eta_0}(\varphi)$ is decreasing along the flow, by Lemma 3.4.1 we know F_{η_0} is bounded from above. It follows from Theorem 21, applied with the reference metric equal to η_0, that J_{η_0} is uniformly bounded from above. Thus by Lemma 3.4.2 inequality *iii*) we have:

$$\int_S (-\varphi)\, d\mu_\varphi \leqslant C.$$

Since $J_{\eta_0} \geqslant 0$, applying the Moser-Trudinger inequality we know that $F_{\eta_0}(\varphi)$ is uniformly bounded below. Then again applying Lemma 3.4.1 we see the Mabuchi K-energy $K_{\eta_0}(\varphi)$ is uniformly bounded from below. By Lemma 3.4.2, part *ii*) we obtain:

$$\frac{1}{V}\int_S \varphi\, d\mu_0 < C.$$

The desired result follows from Corollary 3.4.4. □

Chapter 4

K-stability for Sasakian manifolds

In this chapter we introduce an algebro-geometric obstruction to the existence of Sasaki-Einstein metrics, called K-stability. The notion of K-stability was introduced by Tian [Tian, 1997], and refined by Donaldson [Donaldson, 2002]. It was extended by Ross-Thomas to the setting of projective orbifolds polarized by orbi-ample line bundles [Ross and Thomas, 2011].

Our primary interest is to develop a notion of K-stability for a Sasakian manifold S. When the Sasakian manifold is quasi-regular, then this is equivalent to the work of Ross-Thomas [Ross and Thomas, 2011] on K-stability for orbifolds. The question of whether there is a suitable extension of this to the irregular case has been posed in several places in the literature [Sparks, 2011, Problem 7.1], [Martelli *et al.*, 2008], [Futaki and Ono, 2011]. In this section we provide such an extension, and prove that the existence of a constant scalar curvature Sasakian metric implies the K-semistability of S. Our main result is thus the following corollary of Theorem 25.

Corollary 4.0.5. *Let (S, g) be a Sasakian manifold with Reeb vector field ξ. If g has constant scalar curvature, then the cone $(C(S), \xi)$ is K-semistable.*

As already suggested by Sparks [Sparks, 2011], the obvious approach is to approximate a given irregular Sasakian manifold with a sequence of regular ones, and attempt to take a limit of the obstructions provided by the results of Ross-Thomas in the orbifold case. In particular one can always approximate an irregular Reeb field with a sequence of quasi-regular ones. It is difficult to deal with the varying orbifold quotients as the Reeb field changes, so instead we work on the cone over the Sasakian manifold, which remains unchanged. The central ingredient of K-stability is the Futaki invariant, and to define this we use the Hilbert series as opposed to the usual Riemann-Roch expansions. This point of view was already used in [Martelli *et al.*, 2008] in the form of the index character, to compute the volume of a Sasakian manifold. The main advantage is that for orbifolds the Riemann-Roch expansions contain periodic terms, which become unmanageable as we approach an irrational Reeb field, whereas in the index character these periodic terms are not visible. This allows us to show the continuity of the Futaki invariant with respect to the Reeb field, and thus carry out the approximation argument.

As an application, we recover the results of [Martelli *et al.*, 2008] algebraically by showing that in the situation they considered, volume minimization is equivalent to K-semistability for product test configurations. As a second application, we show that the Lichnerowicz obstruction to existence of Sasaki-Einstein metrics studied in [Gauntlett *et al.*, 2007] can be interpreted in terms of K-semistability for deformations arising from the Rees algebra of a principal ideal of the coordinate ring. Indeed, for rational Reeb fields, the Lichnerowicz obstruction was interpreted in terms of slope stability for the quotient orbifold in [Ross and Thomas, 2011]. Our computations recover this result, and extend it to irrational Reeb fields by establishing an explicit formula for the Donaldson-Futaki invariant of the Rees deformation.

4.1 Orbifold K-stability

For a review of the basic properties of orbifolds with constant scalar curvature metrics in mind, see Ross-Thomas [Ross and Thomas, 2007]. Similarly to them, we will only be interested in polarized orbifolds, and as explained in [Ross and Thomas, 2007] Remark 2.16., these can be viewed as global $\mathbb{C}^*$-quotients of affine schemes. More precisely, given a finitely generated graded ring

$$R = \bigoplus_{k \geqslant 0} R_k$$

over $\mathbb{C}$, the grading induces a $\mathbb{C}^*$-action on $\mathrm{Spec}(R)$. When $\mathrm{Spec}(R)$ is smooth, the corresponding orbifold is the quotient of $\mathrm{Spec}(R) \setminus \{0\}$ by this $\mathbb{C}^*$-action. More generally the quotient is a Deligne-Mumford stack. In our terminology below, the grading corresponds to a choice of rational Reeb field on the affine scheme $\mathrm{Spec}(R)$.

Differential geometrically the affine scheme $Y = \mathrm{Spec}(R)$, if smooth away from the origin, arises as the blowdown of the zero section in the total space of L^{-1} for an orbifold X with orbiample line bundle L. In Section 4.2 below we will express the Calabi functional on the orbifold X in terms of a cone metric on Y. In the rest of this section we will review the work of [Ross and Thomas, 2007] which gives a lower bound for the Calabi functional on the orbifold X in terms of the Futaki invariants of test-configurations.

Roughly speaking a test-configuration for a polarized orbifold (X, L) is a polarized, flat, $\mathbb{C}^*$-equivariant family over $\mathbb{C}$, whose generic fiber is (X, L^r) for some $r > 0$. In greatest generality the family should be allowed to be a Deligne-Mumford stack. For computations it is useful to reformulate this more algebraically. Let

$$R = \bigoplus_{k \geqslant 0} H^0(X, L^k)$$

be the homogeneous coordinate ring of (X, L). Any set of homogeneous generators $f_1, \ldots, f_k$ of R give rise to an embedding of $X \hookrightarrow \mathbb{P}$ into a weighted projective space. Assigning weights to the $f_1, \ldots, f_k$ induces a $\mathbb{C}^*$-action on the weighted projective space $\mathbb{P}$. Acting on $X \hookrightarrow \mathbb{P}$ we obtain a family $X_t \subset \mathbb{P}$ for $t \neq 0$. Taking the flat completion of this family across $t = 0$ is a test-configuration χ. The central fiber of this test-configuration is a polarized Deligne-Mumford stack (X_0, L_0), with a $\mathbb{C}^*$-action. It is convenient to allow L_0 to be a $\mathbb{Q}$-line bundle, so that on the generic fiber we recover L instead of a power of L. Let us write $d_k = \dim H^0(L_0^k)$ and let w_k be the total weight of the $\mathbb{C}^*$-action on $H^0(L_0^k)$. As explained in [Ross and Thomas, 2007], the Riemann-Roch theorem from Toën [Toen, 1999] implies that for large k we have expansions

$$\begin{aligned} d_k &= a_0 k^n + (a_1 + \rho_1(k)) k^{n-1} + \ldots, \\ w_k &= b_0 k^{n+1} + (b_1 + \rho_2(k)) k^n + \ldots, \end{aligned} \tag{4.1.1}$$

where ρ_1, ρ_2 are periodic functions with average zero. The Futaki invariant of the test-configuration is then defined to be

$$Fut(\chi) = \frac{a_1}{a_0} b_0 - b_1.$$

Writing A_k for the infinitesimal generator of the $\mathbb{C}^*$-action on $H^0(L_0^k)$, there is also an expansion

$$\mathrm{Tr}(A_k^2) = c_0 k^{n+2} + O(k^{n+1}),$$

and the norm of the test-configuration is defined by

$$\|\chi\|^2 = c_0 - \frac{b_0^2}{a_0}.$$

The main result that we need is the extension by Ross-Thomas [Ross and Thomas, 2011] of Donaldson's lower bound for the Calabi functional [Donaldson, 2005], to orbifolds.

Theorem 22 (Donaldson [Donaldson, 2005], Ross-Thomas [Ross and Thomas, 2011])**.** *Suppose that (X, L) is a polarized orbifold of dimension n, and let $\omega \in c_1(L)$ be an orbifold metric. In addition suppose that χ is a test-configuration for (X, L). Then*

$$\|\chi\| \cdot \|R_\omega - \hat{R}\|_{L^2(\omega)} \geqslant -c(n)Fut(\chi),$$

where R_ω is the scalar curvature of ω, and $\hat{R}$ is its average.

Although this result is not stated explicitly in [Ross and Thomas, 2011], it follows easily from their proofs. In particular using the notation in [Ross and Thomas, 2011], in their Theorem 6.6 the constant C can be taken to be $\frac{1}{2}\left(\mathrm{vol}\sum_i c_i\right)^{1/2}$, while in the proof of Theorem 6.8 the constant c equals $a_0 \sum_i c_i k^{n+1}$ to highest order. Combining these, the last inequality in the proof of Theorem 6.8 gives the result we need.

4.2 The Calabi Functional on a Polarized Affine Variety

Let us suppose as in Section 2.2 that $Y \setminus \{0\}$ is the complement of the zero section in the total space of an orbi-line bundle L^{-1} over X, and h is a Hermitian metric on L^{-1} such that $\omega = i\partial\overline{\partial}\log h$ is positive on X. Letting r be the fiberwise norm, define the metric

$$\Omega = i\partial\overline{\partial} r^2$$

on $Y \setminus \{0\}$. We will compute the Calabi functional of ω in terms of the metric Ω on Y. We assume, for convenience, that L is primitive. That is, there is no line bundle L' such that $L'^{\otimes k} = L$ for $k \in \mathbb{N}$. This assumption can easily be removed by determining the precise scaling of the Calabi functional, as we point out at the end of this section.

Fix local coordinates (z, w) where $z \in X$ and w is a local holomorphic section of L^{-1} in a neighborhood of $p = (z_0, w_0)$, and assume that $dh = 0$ at p. At p we compute

$$\Omega = i\partial\overline{\partial}h(z)|w|^2 = r^2 i\partial\overline{\partial}\log h + h(z)i\partial\overline{\partial}|w|^2 = r^2\left(\pi^*\omega + \frac{idw \wedge d\overline{w}}{|w|^2}\right).$$

Here $\pi : Y \to X$ is the natural projection map. It follows that the Ricci form and scalar curvature of Ω are given by

$$Ric(\Omega) = \pi^* Ric(\omega) - (n+1)\pi^*\omega, \qquad R_\Omega = r^{-2}(\pi^* R_\omega - (n+1)n).$$

On a fixed fibre, the cylinder metric $|w|^{-2}(dw \wedge d\overline{w})$ can also be written as $\frac{1}{r}dr \wedge d\theta$, where $d\theta$ is given by the $U(1)$ action on the fibres of L^{-1}. Hence, the volume form of Ω is

$$\Omega^{n+1} = r^{2n+1}(\pi^*\omega)^n \wedge dr \wedge d\theta.$$

Let $\{U_i, \Gamma_i\}, i = 1, \ldots, n$ be a family of open sets $U_i \subset \mathbb{C}^n$ together with local uniformizing groups Γ_i, so that $U_i/\Gamma_i \cong V_i \subset X$ gives an open cover of X, and so that L^{-1} is trivial on each V_i. Let φ_i be a partition of unity subordinate to the cover V_i. Note that the set $S := \{r = 1\} \subset Y$ is a smooth submanifold of Y, which is the total space of a principal $U(1)$ orbibundle over X. Thus, by Lemma 4.2.8 of [Boyer and Galicki, 2008], we have that the local uniformizing groups inject into $U(1)$. In particular, the maps

$$U(1) \times U_i \xrightarrow{\psi_i} V_i$$

are exactly $|\Gamma_i|$-to-one on the complement of the orbifold locus. Let $\hat{R}_\omega$ denote that average scalar curvature of X. We compute

$$\begin{aligned} Cal_X(\omega)^2 := 2\pi \int_X (R_\omega - \hat{R}_\omega)^2\omega^n &= \sum_i \frac{2\pi}{|\Gamma_i|}\int_{U_i} \varphi_i (R_\omega - \hat{R}_\omega)^2\omega^n \\ &= \sum_i \frac{1}{|\Gamma_i|}\int_{U(1)\times U_i} \pi^*\varphi_i(\pi^*R_\omega - \hat{R}_\omega)^2\pi^*\omega^n \wedge d\theta \\ &= \sum_{i=1}^N \int_{V_i} \pi^*\varphi(\pi^*R_\omega - \hat{R}_\omega)^2\pi^*\omega^n \wedge d\theta \\ &= \int_S (\pi^*R_\omega - \hat{R}_\omega)^2 \iota_{\frac{\partial}{\partial r}}(\Omega^{n+1}). \end{aligned}$$

Let us write $\hat{R}_\Omega$ for the average of R_Ω when restricted to S. Then we have the relation

$$\hat{R}_\Omega = \hat{R}_\omega - (n+1)n.$$

Finally, we can compute

$$\begin{aligned}\int_{\{r\leqslant 1\}\subset Y}(r^2R_\Omega - \hat{R}_\Omega)^2\Omega^{n+1} &= \int_0^1\int_S(\pi^*R_\omega - \hat{R}_\omega)^2\iota_{\frac{\partial}{\partial r}}(\Omega^{n+1})r^{2n+1}dr \\ &= \frac{1}{2n+2}Cal_X(\omega)^2.\end{aligned}$$

Definition 4.2.1. *Let Y be an affine variety with isolated singular point at 0, and Reeb field ξ and let Ω be a Kähler metric on Y compatible with ξ, with scalar curvature R_Ω. Define*

$$Cal_Y(\Omega) := \left(\int_{\{r\leqslant 1\}}(r^2R_\Omega - \hat{R}_\Omega)^2\Omega^{n+1}\right)^{1/2}$$

where,

$$\hat{R}_\Omega := \frac{\int_S R_\Omega\iota_{\frac{\partial}{\partial r}}(\Omega^{n+1})}{\int_S \iota_{\frac{\partial}{\partial r}}(\Omega^{n+1})}.$$

In order to relate this to the Sasakian setting, let (S, g) be a Sasakian manifold. Observe that when ξ is rational,

$$\pi^*R_\omega = \frac{1}{4}R^T = \frac{1}{4}(R+2n), \tag{4.2.2}$$

where R^T is the transverse scalar curvature of the Reeb foliation and R is the scalar curvature of the Sasakian metric g. In this case, we have

$$\begin{aligned}\int_S(\pi^*R_\omega - \hat{R}_\omega)^2\iota_{\frac{\partial}{\partial r}}(\Omega^{n+1}) &= \frac{1}{16}\int_S(R^T - \hat{R}^T)^2\iota_{\frac{\partial}{\partial r}}(\Omega^{n+1}) \\ &= \frac{1}{16}\int_S(R-\hat{R})^2d\mu.\end{aligned}$$

Here $\hat{R}$ is the average scalar curvature of (S, g). We note that this agrees, up to a constant, with the functional studied in [Boyer *et al.*, 2008]. We have shown that for rational Reeb fields we have the equality

$$Cal_Y(\Omega) = \frac{1}{4(2n+2)^{1/2}}Cal_S(g).$$

Since both sides of this equality depend continuously on the Reeb vector field, this also holds for irrational Reeb fields by approximation. We record this in the following proposition.

Proposition 4.2.2. *Let Y be an affine variety polarized by a rational Reeb field ξ. Then,*

$$Cal_Y(\Omega) = \frac{1}{(2n+2)^{1/2}} Cal_X(\omega).$$

Moreover, when Y is the cone over a Sasakian manifold (S, g), then

$$Cal_Y(\Omega) = \frac{1}{4(2n+2)^{1/2}} Cal_S(g),$$

and this holds for any Reeb vector field.

Before proceeding, we make a few brief remarks about the scaling of the Calabi functional as a function of the Reeb field. More precisely, suppose that Y is an affine cone with Reeb vector field ξ, and a compatible Kähler metric $\Omega = i\partial\overline{\partial}r^2$. Scaling the Reeb vector field by a factor $\lambda > 0$, corresponds to changing r by $r \mapsto r^\lambda$. This scaling yields a new metric $\Omega_\lambda = i\partial\overline{\partial}r^{2\lambda}$. It is straight forward to check that under a deformation of this type we have

$$Cal_Y(\Omega_\lambda) = \lambda^{\frac{n-1}{2}} Cal_Y(\Omega). \tag{4.2.3}$$

4.3 The Index Character and the Donaldson-Futaki Invariant

The main difficulty in extending the definition of K-stability to irregular Sasakian manifolds is the absence of a suitable Riemann-Roch formula when the Reeb field is irrational. When the Reeb field is rational, Ross-Thomas showed in [Ross and Thomas, 2011] that the relevant coefficients are the non-periodic terms of the orbifold Riemann-Roch expansion (see Section 4.1), so we would like to define the relevant coefficients by approximating an irrational Reeb vector field ξ by a sequence of rational ones ξ_k. Unfortunately the periodic terms in the expansions (4.1.1) corresponding to the ξ_k become unmanageable as $k \to \infty$. The key observation is that the Riemann-Roch coefficients are determined by the Hilbert series, or equivalently the index character introduced by Martelli-Sparks-Yau [Martelli *et al.*, 2008]. For the leading term (the volume), this was also used by [Martelli *et al.*, 2008].

In this section and the next we take $Y \subset \mathbb{C}^N$ to be an affine scheme of dimension $n+1$, defined by the ideal $I = (f_1, \ldots, f_r) \subset R = \mathbb{C}[x_1, \ldots, x_N]$. Let $T \subset GL(N, \mathbb{C})$ be a torus of dimension s acting diagonally, holomorphically and effectively on Y. We make this assumption without loss of generality by Lemma 2.2.2. Denote by $\mathfrak{t}$ the Lie algebra of T and let $\mathcal{H} = R/I$ be the ring of regular functions on Y. Since T fixes Y, the ideal I is homogeneous for the torus action. By Corollary 2.2.5 we may always assume that T contains at least one rational Reeb vector field. Let

$$\mathcal{H} = \oplus_{\alpha \in \mathfrak{t}^*} \mathcal{H}_\alpha$$

be the weight space decomposition of $\mathcal{H}$.

Definition 4.3.1. *In the above situation, we define the T-equivariant index character $F(\xi, t)$ for $\xi \in \mathcal{C}_R$ and $t \in \mathbb{C}$ with* $\mathrm{Re}(t) > 0$*, by*

$$F(\xi, t) := \sum_{\alpha \in \mathfrak{t}^*} e^{-t\alpha(\xi)} \dim \mathcal{H}_\alpha. \tag{4.3.4}$$

Lemma 4.3.2. *The defining sum for $F(\xi, t)$ converges if ξ is a Reeb vector field and* $\mathrm{Re}(t) > 0$.

Proof. The dimensions $\dim \mathcal{H}_\alpha$ are bounded by the corresponding dimensions for $\mathbb{C}^N$. As ξ acts by positive weights, $\dim \mathcal{H}_\alpha < C|\alpha|^N$. Moreover, since ξ is a Reeb vector field, there is a $c > 0$ such that $\alpha(\xi) > c|\alpha|$ for all α with non-zero $\mathcal{H}_\alpha$. We obtain

$$\sum_{\alpha \in \mathfrak{t}^*} \left| e^{-t\alpha(\xi)} \right| \dim \mathcal{H}_\alpha \leqslant C \sum_{\alpha \in \mathfrak{t}^*} e^{-c|\alpha| \mathrm{Re}(t)} |\alpha|^N,$$

which converges if $\mathrm{Re}(t) > 0$. □

Suppose that ξ is rational, and it is minimal satisfying the condition that $\alpha(\xi)$ is integral for each α with non-zero weight space. Then as before we can think of Y as the total space of a line bundle L over the orbifold $X = Y/\mathbb{C}^*$, and

$$H^0(X, L^k) = \bigoplus_{\alpha;\, \alpha(\xi) = k} \mathcal{H}_\alpha.$$

By the orbifold Riemann-Roch theorem [Kawasaki, 1979], [Toen, 1999], we have

$$\dim H^0(X, L^k) = a_0 k^n + (a_1 + \rho) k^{n-1} + \cdots$$

for some periodic function ρ with average zero. In this case we have the following.

Proposition 4.3.3. *The T-equivariant index character $F(\xi, t)$ as a function of t has a meromorphic extension to a neighborhood of the origin, and it has Laurent expansion*

$$F(\xi, t) = \frac{a_0 n!}{t^{n+1}} + \frac{a_1(n-1)!}{t^n} + O(t^{1-n}),$$

near $t = 0$.

Proof. By definition we have

$$\begin{aligned} F(\xi, t) &= \sum_{k=0}^{\infty} e^{-kt} \dim H^0(L^k) \\ &= \sum_{k=0}^{\infty} e^{-kt} \left(a_0 k^n + (a_1 + \rho)k^{n-1} + O(k^{n-2})\right). \end{aligned}$$

Note that

$$\sum_k e^{-tk} = \frac{1}{1 - e^{-t}} = \frac{1}{t} + f(t),$$

where f is analytic, so differentiating n times with respect to t, we get

$$\sum_k e^{-tk} k^n = \frac{n!}{t^{n+1}} + (-1)^n f^{(n)}(t).$$

Moreover, $G(t) = \sum_k \rho(k) e^{-tk}$ is analytic near $t = 0$ since ρ has average zero. Indeed if d is the period of ρ then we have

$$\sum_k (\rho(k) + \rho(k+1) + \ldots + \rho(k+d-1))e^{-tk} = 0,$$

and so

$$G(t) + e^t(G(t) - \rho(0)) + \ldots + e^{(d-1)t}\left(G(t) - \sum_{k=0}^{d-2} \rho(k) e^{-kt}\right) = 0,$$

and therefore

$$G(t) = \frac{H(t)}{1 + e^t + e^{2t} + \ldots + e^{(d-1)t}}$$

where $H(t)$ is analytic since it is a finite sum. It follows that $G(t)$ is also analytic near 0, with poles at $t = \frac{2\pi i k}{d}$ for non-zero integers $k \neq 0$. Finally it follows that $F(\xi, t)$ is meromorphic near $t = 0$ with a pole at the origin, and we have

$$F(\xi, t) = \frac{a_0 n!}{t^{n+1}} + \frac{a_1(n-1)!}{t^n} + O(t^{1-n}).$$

□

In particular, we can read off the Riemann-Roch coefficients from the index character, and the periodic terms do not appear. Our goal is to study how these coefficients change as we vary ξ. We will now assume that the embedding $Y \subset \mathbb{C}^N$ is obtained through an application of Lemma 2.2.2. The corresponding ideal $I \subset R$ is then homogeneous with respect to a multigrading on R. More precisely, let $E := \{e_1^*, \ldots, e_s^*\}$ be an integral basis of $\mathbb{R}^s \cong \mathfrak{t}^*$ and let α_i be the weight of the representation on the generator x_i of R. Expressing the α_i in the basis E yields an $s \times N$ matrix

$$W = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{s,1} & \alpha_{s,2} & \ldots & \alpha_{s,N} \end{pmatrix} \tag{4.3.5}$$

with integer entries. Since, R is graded by W, and I is homogenous, it follows that R/I is a W-graded R module, generated in degree zero.

Definition 4.3.4. *Let $s \geqslant 1$, let R be graded by a matrix W of rank s in $Mat_{s,N}(\mathbb{Z})$, and let $\alpha_1, \ldots, \alpha_s$ be the rows of W. The grading on R given by W is of* positive type *if there exists $a_1, \ldots, a_s \in \mathbb{Z}$ such that all the entries of $a_1\alpha_1 + \cdots + a_s\alpha_s$ are positive.*

Lemma 4.3.5. *If there exists a Reeb vector field in $\mathfrak{t}$, then the grading induced by W is of positive type.*

Proof. We first need to show that W has rank s. Observe that if $v^T \cdot W = 0$, then the action induced by v is trivial. In particular, the action of T is not effective. Secondly, by Corollary 2.2.5 we can assume that there is an integral Reeb field $\xi \in \mathfrak{t}$, given in terms of the dual basis $\{e_1, \ldots, e_s\}$ by a vector $(a_1, \ldots, a_s)$ with $a_i \in \mathbb{Z}$. The entries of $a_1\alpha_1 + \ldots a_s\alpha_s$ are the weights of the action induced by ξ on the generators $x_1, \ldots, x_N$. By definition of a Reeb field these are all positive. □

We will now recall some results about multigradings and the multigraded Hilbert function.

Lemma 4.3.6 ([Kreuzer and Robbiano, 2005], Proposition 4.1.19)**.** *Let $R = \mathbb{C}[x_1, \ldots, x_N]$ be graded by a matrix $W \in Mat_{m,N}(\mathbb{Z})$ of positive type, and let M be a finitely generated graded R-module. Then,*

1. $R_0 = \mathbb{C}$. *That is, the degree zero elements in R are precisely the constants.*

2. *For every $d \in \mathbb{Z}^m$, we have* $\dim_{\mathbb{C}}(M_d) < \infty$

The previous lemma indicates that the following definition makes sense;

Definition 4.3.7 ([Kreuzer and Robbiano, 2005], Definition 5.8.8, 5.8.11)**.** *Let R be graded by a matrix $W \in Mat_{m,N}(\mathbb{Z})$, and let M be a finitely generated, graded R module. Then the map*

$$HF_{M,W} : \mathbb{Z}^m \to \mathbb{Z}$$

$$(i_1, \ldots, i_m) \mapsto \dim_{\mathbb{C}}(M_{i_1,\ldots,i_m})$$

for all $(i_1, \ldots, i_m) \in \mathbb{Z}^m$ is called the multigraded Hilbert function *of M with respect to the grading W. We may define the multivariate power Hilbert series of M with respect to the grading W by*

$$HS_{M,W}(z_1, \ldots z_m) = \sum_{(i_1,\ldots,i_m)\in\mathbb{Z}^m} HF_{M,W}(i_1, \ldots, i_m) z_1^{i_1} \cdots z_m^{i_m} \in \mathbb{Z}[[\mathbf{z}, \mathbf{z}^{-1}]] \qquad (4.3.6)$$

The following lemma provides a convenient characterization of multivariate Hilbert series under changes in the grading.

Lemma 4.3.8 ([Kreuzer and Robbiano, 2005], Proposition 5.8.24)**.** *Let $W \in Mat_{m,N}(\mathbb{Z})$, and $A = (a_{ij}) \in Mat_{l,m}(\mathbb{Z})$ be two matrices such that the gradings on $R = \mathbb{C}[z_0, \ldots, z_N]$ given by W and $A \cdot W$ are both of positive type. Let M be a finitely generated R-module which is graded with respect to the grading given by W. Then the Hilbert series of M with respect to the grading given by $A \cdot W$ is given by*

$$HS_{M,A\cdot W}(z_1, \ldots, z_l) = HS_{M,W}(z_1^{a_{11}} \cdots z_l^{a_{l1}}, \ldots, z_1^{a_{1m}} \cdots z_l^{a_{lm}})$$

If R is graded by $W = (w_{ij}) \in Mat_{m,n}(\mathbb{Z})$ of positive type, and ξ is a Reeb field, then the grading induced by $\xi^T \cdot W$ is clearly of positive type, and so the above lemma describes the relation between the multigraded Hilbert series and the index character. The next proposition describes the general shape of multivariable Hilbert series.

Proposition 4.3.9 ([Kreuzer and Robbiano, 2005], Corollary 5.8.19)**.** *Let R be graded by $W \in Mat_{m,N}(\mathbb{Z})$, a matrix of positive type. Let M be a finitely generated, graded R-module,* *and $(m_1, \ldots, m_r)$ be a tuple of non-zero homogeneous elements of M which form a minimal system of generators. For $i = 1, \ldots, r$, let $d_i = deg_W(m_i)$. Then the multivariate Hilbert series of M has the following form;*

$$HS_{M,W}(z_1, \ldots z_m) = \frac{z_1^{\alpha_1} \cdots z_m^{\alpha_m} \cdot HN(z_1, \ldots, z_m)}{\prod_{j=1}^{N}(1 - z_1^{w_{1j}} \cdots z_m^{w_{mj}})}$$

where $(\alpha_1, \ldots, \alpha_m)$ is the component wise minimum of $\{d_i\}$, and $HN_M(z_1, \ldots, z_m)$ is a polynomial in $\mathbb{Z}[z_1, \ldots, z_m]$.

We can now translate this result to the language of index characters.

Theorem 23. *Let $Y \subset \mathbb{C}^N$ be an affine scheme of dimension $n+1$, and suppose that $T \subset GL(N, \mathbb{C})$ is a torus acting effectively, diagonally and holomorphically on Y. Let $\mathfrak{t}$ be the Lie algebra of T, and $\mathcal{C}_R \subset \mathfrak{t}$ be the Reeb cone. For fixed $\xi \in \mathcal{C}_R$ the index character $F(\xi, t)$ has a meromorphic extension to $\mathbb{C}$ with poles along the imaginary axis. Near $t = 0$ it has a Laurent series*

$$F(\xi, t) = \frac{a_0(\xi) n!}{t^{n+1}} + \frac{a_1(\xi)(n-1)!}{t^n} + \ldots, \tag{4.3.7}$$

where $a_i(\xi)$ depend smoothly on $\xi \in \mathcal{C}_R$, and $a_0(\xi) > 0$.

Proof. As above, with a basis of $\mathfrak{t} \cong \mathbb{R}^s$ fixed, write $\xi = (\xi_1, \ldots, \xi_s)$ for an element of $\mathfrak{t}$. By Proposition 4.3.9, the Hilbert series of the grading induced by W is given by

$$HS_W(e^{-t_1}, \ldots, e^{-t_s}) = \frac{e^{-t_1\alpha_1} \cdots e^{-t_s\alpha_s} \cdot HN(e^{-t_1}, \ldots, e^{-t_s})}{\prod_{j=1}^{N}(1 - e^{-t_1 w_{1j}} \cdots e^{-t_s w_{sj}})}$$

where $\alpha_i \geqslant 0$ for every i. By Lemma 4.3.8, we obtain

$$F(\xi, t) = HS_{\xi^T.W}(e^{-t}) = \frac{e^{-t(\xi_1\alpha_1 + \cdots + \xi_s\alpha_s)} \cdot HN(e^{-t\xi_1}, \ldots, e^{-t\xi_s})}{\prod_{j=1}^{N}(1 - e^{-t(\xi_1 w_{1j} + \cdots + \xi_s w_{sj})})}$$

From this formula it follows that $F(\xi, t)$ is a meromorphic function with coefficients depending smoothly on the Reeb field. More precisely for fixed $\xi \in \mathcal{C}_R$ there are no poles other than the origin in the ball where

$$|t| < \frac{2\pi}{\max_j\{\xi_1 w_{1j} + \ldots + \xi_s w_{sj}\}},$$

so we can compute the coefficients of the Laurent series using the Cauchy integral formula on a small circle around the origin. As long as ξ varies in a bounded subset of $\mathcal{C}_R$, we can use the same circle around the origin, and the coefficients will vary smoothly with ξ. It follows also that the order of the pole at $t = 0$ is determined by the order of the pole for rational ξ, which is $n + 1$ by Proposition 4.3.3. Note that the coefficients blow up at the boundary of the Reeb cone, since as ξ approaches the boundary, there will be a j such that $\xi_1 w_{1j} + \ldots + \xi_s w_{sj} \to 0$. □

In some special cases, we can recover this result by computing the index character explicitly. For example, we have

Proposition 4.3.10. *Let Y be a complete intersection, determined by the regular sequence $f_1 = \cdots = f_k = 0$. Let α_i be the weight of the generators x_i, and let β_j be the weight of f_j. Then we have*

$$F(\xi, t) = \frac{\prod_{j=1}^{k}(1 - e^{-t\beta_j(\xi)})}{\prod_{i=0}^{N}(1 - e^{-t\alpha_i(\xi)})}$$

Proof. Use the degree shifted Koszul complex resolution of R/I, and compute the Hilbert series. □

In order to define the Futaki invariant, we need equivariant versions of the index character, taking into account an extra $\mathbb{C}^*$-action.

Definition 4.3.11. *In the situation of Theorem 23 with $\xi \in \mathcal{C}_R$, suppose $\eta \in \mathfrak{t}$, and define the weight characters*

$$C_\eta(\xi, t) = \sum_{\alpha \in \mathfrak{t}^*} e^{-t\alpha(\xi)}\alpha(\eta),$$

$$C_{\eta^2}(\xi, t) = \sum_{\alpha \in \mathfrak{t}^*} e^{-t\alpha(\xi)}(\alpha(\eta))^2.$$

The convergence of these weight character follows from the arguments in Lemma 4.3.2. As before, when ξ is rational, we obtain a line bundle L over the orbifold $X = Y/\mathbb{C}^*$ with a $\mathbb{C}^*$-action on L generated by η. Adapting the computations preceding Proposition 4.3.3 proves the following.

Proposition 4.3.12. *In the situation of Theorem 23, with ξ rational, write A_k for the infinitesimal action of η on $H^0(X, L^k)$, and define b_0, b_1, c_0 by the expansions*

$$\mathrm{Tr}(A_k) = b_0 k^{n+1} + (b_1 + \rho)k^n + O(k^{n-1}),$$
$$\mathrm{Tr}(A_k^2) = c_0 k^{n+2} + O(k^{n+1}),$$

where ρ is a periodic function with average zero, and $c_0 \geqslant 0$. Then the weight characters have the asymptotic expansions

$$C_\eta(\xi, t) = \frac{b_0(n+1)!}{t^{n+2}} + \frac{b_1 n!}{t^{n+1}} + O(t^{-n}),$$
$$C_{\eta^2}(\xi, t) = \frac{c_0(n+2)!}{t^{n+3}} + O(t^{-n-2}).$$

We remark that the inequality $c_0 \geqslant 0$ follows from equation (2.20) in [Ross and Thomas, 2011]. The results of Theorem 23 can also be extended quite easily.

Theorem 24. *In the situation of Theorem 23, with $\eta \in \mathfrak{t}$, the weight characters admit meromorphic expansions to a small neighborhood of $0 \in \mathbb{C}$ of the form*

$$C_\eta(\xi, t) = \frac{b_0(\xi)(n+1)!}{t^{n+2}} + \frac{b_1(\xi)n!}{t^{n+1}} + O(t^{-n})$$
$$C_{\eta^2}(\xi, t) = \frac{c_0(\xi)(n+2)!}{t^{n+3}} + O(t^{-n-2}),$$

where b_0, b_1, c_0 depend smoothly on $\xi \in \mathcal{C}_R$. Moreover, we have

$$b_i(\xi) = \frac{-1}{(n+1-i)} D_\eta a_i(\xi) \quad \text{for } i = 0, 1$$
$$c_0(\xi) = \frac{1}{(n+2)(n+1)} D_\eta^2 a_0(\xi),$$

where D_η denotes the directional derivative along η in $\mathbb{R}^s \cong \mathfrak{t}$.

Proof. We define

$$G(\xi, s, t) = \sum_{\alpha \in \mathfrak{t}^*} e^{-t\alpha(\xi - s\eta)} \dim H_\alpha.$$

For s sufficiently small, $\xi - s\eta$ is a Reeb vector field and so the defining sum for $G(\xi, s, t)$ converges uniformly for $t > 0$, and we have $G(\xi, s, t) = F(\xi - s\eta, t)$. It is clear that

$$tC_\eta(\xi, t) = \frac{\partial}{\partial s} G(\xi, s, t)\bigg|_{s=0} = \frac{\partial}{\partial s}\left(\frac{a_0(\xi - s\eta)n!}{t^{n+1}} + \frac{a_1(\xi - s\eta)(n-1)!}{t^n} + \cdots\right)\bigg|_{s=0}.$$

By Theorem 23 the coefficients $a_0, a_1, \dots$ depend smoothly on the Reeb field and so we can differentiate term by term to obtain

$$C_\eta(\xi, t) = \frac{b_0(\xi)(n+1)!}{t^{n+2}} + \frac{b_1(\xi)n!}{t^{n+1}} + \cdots ,$$

where, for example, $b_0(\xi) = \frac{-1}{n+1} D_\eta a_0(\xi)$ and D_η denotes the directional derivative along η. The argument for C_{η^2} is identical. □

4.4 Test Configurations for Polarized Affine Varieties

Our first task is to define a test configuration for an affine variety Y polarized by a Reeb field ξ. Recall that we can assume that $Y \subset \mathbb{C}^N$ is invariant under the linear action of a torus T and the Reeb field ξ is in the Lie algebra $\mathfrak{t}$ of the maximal compact subtorus. Let $\mathcal{H}$ be the coordinate ring of Y.

Definition 4.4.1. *A T-equivariant test-configuration for Y consists of the following data.*

1. *A set of T-homogeneous elements $\{f_1, \dots, f_k\} \in \mathcal{H}$, which generate $\mathcal{H}$ in sufficiently high degrees.*
2. *Integers w_i for $i = 1, \dots, k$.*

This corresponds to the usual, more geometric definition of test-configurations. Namely we can embed Y into $\mathbb{C}^k$ using the functions $\{f_1, \dots, f_k\}$, and then act on $\mathbb{C}^k$ by the $\mathbb{C}^*$-action with weights w_i. Taking the flat limit across 0 of the $\mathbb{C}^*$-orbit of Y we obtain a flat family of affine schemes over $\mathbb{C}$. It is in this form that we will construct our test configurations in Section 4.5. The central fiber Y_0 still has an action of T as well as a new $\mathbb{C}^*$-action commuting with T (if we have a product configuration, then this new $\mathbb{C}^*$ is actually a subgroup of T). Note that when ξ is rational, then we can take T to be the 1-dimensional torus generated by ξ, and a test-configuration for Y is the same as a test-configuration for the quotient orbifold as we defined it in Section 4.1.

It is important to note that as a T-representation, the ring of functions on the central fiber Y_0 is isomorphic to $\mathcal{H}$, it is only the multiplicative structure that changes. In particular if $\xi \in \mathfrak{t}$ is a Reeb field on Y, then it is also a Reeb field on Y_0. We can therefore apply our results on the index character to Y_0. By Theorem 23, the index character expands asymptotically as

$$F(\xi, t) = \frac{a_0(\xi)n!}{t^{n+1}} + \frac{a_1(\xi)(n-1)!}{t^n} + O(t^{1-n})$$

where $a_0, a_1 : \mathcal{C}_R \to \mathbb{R}$ are smooth functions. Moreover, Y inherits an extra $\mathbb{C}^*$ action generated by $\eta \in \mathfrak{t}' = \mathrm{Lie}(T'_{\mathbb{R}})$ for some torus $T' \subset GL(N, \mathbb{C})$ with $T \subset T'$. By Theorem 24, the weight characters expand as

$$C_\eta(\xi, t) = \frac{b_0(\xi)(n+1)!}{t^{n+2}} + \frac{b_1(\xi)n!}{t^{n+1}} + O(t^{-n})$$
$$C_{\eta^2}(\xi, t) = \frac{c_0(\xi)(n+2)!}{t^{n+3}} + O(t^{2-n})$$

where $b_0, b_1, c_0 : \mathcal{C}_R \to \mathbb{R}$ are smooth functions, and $c_0 \geqslant 0$.

Definition 4.4.2. *In the above situation, we define the Donaldson-Futaki invariant of the test configuration, with respect to the Reeb field ξ, by*

$$Fut(Y_0, \xi, \eta) := \frac{a_1(\xi)}{a_0(\xi)}b_0(\xi) - b_1(\xi) = \frac{a_0(\xi)}{n}D_\eta(\frac{a_1}{a_0})(\xi) + \frac{a_1(\xi)D_\eta a_0(\xi)}{n(n+1)a_0(\xi)}. \tag{4.4.8}$$

Here, as in Theorem 24, $D_\eta a_i, i = 0, 1$ denotes the directional derivative of a_i along η, and the second equality follows from Theorem 24. We also define the norm of η, with respect to the Reeb field ξ by

$$\|\eta\|_\xi^2 = c_0(\xi) - \frac{b_0(\xi)^2}{a_0(\xi)}.$$

Propositions 4.3.3 and 4.3.12 show that the above definition of the Donaldson-Futaki invariant extends Ross-Thomas's orbifold Donaldson-Futaki invariant to irrational Reeb vector fields.

Definition 4.4.3. *We say that (Y, ξ) is K-semistable if, for every torus $T \ni \xi$, and every T-equivariant test configuration with central fibre Y_0, we have*

$$Fut(Y_0, \xi, \eta) \geqslant 0$$

where $\eta \in T'$ is the induced $\mathbb{C}^*$ action on the central fibre.

K-stability could also be defined along similar lines. Since there is usually a positive dimensional torus of automorphisms, one natural way would be to use the notion of relative stability following [Székelyhidi, 2007]. This would also allow us to consider the analogs of extremal metrics (called canonical Sasakian metrics in [Boyer *et al.*, 2008]). Since we do not use these notions, we will not define them. We are now in a position to prove our main theorem;

Theorem 25 ([Collins and Székelyhidi, 2012a]). *Let (Y, ξ) be a polarized affine variety of dimension $n+1$ with a torus of automorphisms T, containing the Reeb field. Suppose that we have a T-equivariant test-configuration for Y and let Y_0 be the central fibre with induced $\mathbb{C}^*$-action η. For any Kähler metric Ω on Y compatible with ξ,*

$$\|\eta\|_\xi \cdot Cal_Y(\Omega) \geqslant -c(n)Fut(Y_0, \xi, \eta), \tag{4.4.9}$$

where $c(n)$ is a strictly positive constant depending only on n.

Proof. When ξ is rational and minimal satisfying the condition that $\alpha(\xi)$ is integral for each $\alpha \in \mathfrak{t}^*$ with non-empty weight space, this theorem is just a restatement of the results of Donaldson [Donaldson, 2005] and Ross-Thomas [Ross and Thomas, 2011]. However, from the definitions of a_i, b_i, c_0 for $i = 0, 1$, and the scaling of the Calabi functional in equation (4.2.3), the inequality is invariant under scaling the Reeb field. In particular, it holds for all rational Reeb vector fields.

Assume that ξ is irrational. According to Corollary 2.2.5 we can approximate ξ with a sequence of rational Reeb fields $\xi_k \in \mathfrak{t}$, and find corresponding compatible Kähler metrics Ω_k, which converge to Ω smoothly on compact sets. For the rational ξ_k we already know that

$$\|\eta\|_{\xi_k} \cdot Cal_Y(\Omega_k) \geqslant -c(n)Fut(Y_0, \xi_k, \eta).$$

All the terms in this inequality depend smoothly on the Reeb vector field by Theorems 23 and 24. Moreover, $\Omega_k \to \Omega$ smoothly on compact sets, and hence $Cal_Y(\Omega_k) \to Cal_Y(\Omega)$. For this last statement one can either observe that the integrand in the definition of $Cal_Y(\Omega_k)$ is uniformly bounded as $r_k \to 0$, or by applying the second formula of Proposition 4.2.2 to the link $S = \{r_k = 1\}$, which is independent of k by Corollary 2.2.5. We can therefore take a limit as $k \to \infty$ to obtain the inequality for the irrational Reeb field ξ. □

Corollary 4.0.5, stated in the introduction, follows immediately from Theorem 25, since $c(n) > 0$.

4.5 Applications and Examples

As an application of our techniques, we will show that the volume minimization results of [Martelli *et al.*, 2008] and the Lichnerowicz obstruction of [Gauntlett *et al.*, 2007] can be obtained directly from K-stability considerations as obstructions to existence of Sasaki-Einstein metrics. More precisely, we will show that for Calabi-Yau cones with isolated Gorenstein singularities, and a torus action inducing a Reeb vector field, K-stability for product test configurations implies the volume minimization results of [Martelli *et al.*, 2008]. Martelli, Sparks and Yau noticed that when the Reeb field minimizing the volume functional was rational, the Futaki invariant on the quotient orbifold vanished. Secondly, we will apply the Rees deformation to interpret the Lichnerowicz obstruction of [Gauntlett *et al.*, 2007] in terms of K-stability. For rational Reeb vector fields, the Lichnerowicz obstruction was shown to imply the slope instability, and hence K-instability, of the quotient orbifold in [Ross and Thomas, 2011]. Our results recover this theorem, and extend it to the setting of irrational Reeb fields.

Let Y be an affine, Calabi-Yau variety with an isolated singularity at 0, and a torus T, acting holomorphically, and effectively on Y, admitting a Reeb vector field $\xi \in \mathfrak{t}$. We suppose that $0 \in Y$ is a Gorenstein singularity, by which we mean that the canonical bundle is trivial on $X := Y - \{0\}$. According to section 2.7 of [Martelli *et al.*, 2008], we fix a non-vanishing section $\Theta \in H^0(X, K_X)$ which is homogeneous of degree $n+1$ for the action of the Reeb field. More precisely, we fix a cross-section $\Sigma \subset \mathcal{C}_R$ so that for each $\xi \in \Sigma$, we have $\mathcal{L}_\xi \Theta = i(n+1)\Theta$. According to [Martelli *et al.*, 2008], Σ is a compact, convex polytope. By the computations in Section 3.1 of [Martelli *et al.*, 2008]

$$\int_S R(g_S) d\mu = 2n(2n+1)Vol(S).$$

Assuming for the moment that $\xi \in \Sigma$ is rational, the orbifold Riemann-Roch theorem implies that

$$a_1(\xi) = \frac{1}{8\pi^n} \int_X R_\omega \frac{\omega^n}{n!} = \frac{n!}{16\pi^{n+1}} \left(\int_S R(g_S) d\mu + 2nVol(S) \right).$$

This follows from a computation similar to the computation in section 4 for the Calabi functional, and the relation between the *complex* transverse scalar curvature of the Reeb foliation and the *real* scalar curvature of the Sasakian metric given by equation (4.2.2). For a similar computation, see [Martelli *et al.*, 2008]. We also have

$$a_0(\xi) = \frac{n!}{2\pi^{n+1}} Vol(S),$$

which follows easily by a similar argument. Since both of these identities are continuous in the Reeb field, they extend from the rational Reeb fields to all of Σ. As a result, we have

$$a_1(\xi) = \frac{n(n+1)}{2} a_0(\xi). \tag{4.5.10}$$

Consider now a product test configuration $Y \times \mathbb{C}$, with a $\mathbb{C}^*$ action generated by $\eta \in \mathfrak{t}$. We assume additionally that η is tangent to Σ. Applying equation (4.4.8), the Donaldson-Futaki invariant is given by

$$Fut(Y, \xi, \eta) = \frac{1}{2} D_\eta a_0(\xi).$$

Since we could replace η with $-\eta$, it follows from Theorem 25 that If ξ is the Reeb vector field of a Sasaki-Einstein metric, then we must have

$$D_\eta a_0(\xi) = 0$$

for every rational η, and hence ξ must be a critical point of the volume functional. Moreover, it was shown in [Martelli *et al.*, 2008] that the volume functional of a Sasakian manifold is strictly convex when restricted to Σ, so a critical point is necessarily a minimum. In particular, we have

Theorem 26 ([Collins and Székelyhidi, 2012a])**.** *Let (Y, Θ) be an isolated Gorenstein singularity with link L, and Reeb vector field ξ satisfying $\mathcal{L}_\xi \Theta = i(n+1)\Theta$. If ξ does not minimize the volume functional of the link L, then (Y, ξ) is K-unstable.*

From the argument it is clear that more generally in any family of Reeb fields ξ for which the ratio a_1/a_0 is constant, a K-semistable Reeb field must be a critical point of the volume a_0. In the case of Gorenstein singularities we obtain the following corollary, which was first pointed out in [Martelli *et al.*, 2008].

Corollary 4.5.1. *Let (Y, Θ) be an isolated Gorenstein singularity with link L, and Reeb vector field ξ satisfying $\mathcal{L}_\xi \Theta = i(n+1)\Theta$. If ξ does not minimize the volume functional* *of the link L, then (Y,ξ) does not admit a compatible Kähler metric with constant scalar curvature. In particular, the link L with Reeb field ξ does not admit a Sasaki-Einstein metric.*

Next, we aim to show how the Lichnerowicz obstruction of Gauntlett, Martelli, Sparks and Yau [Gauntlett *et al.*, 2007] can be interpreted in terms of K-stability by computing explicitly the Donaldson-Futaki invariant of a test configuration arising from the Rees algebra for a principal ideal. These test configurations, which we call the Rees deformation, are a simplified version of the deformation to the normal cone test configurations studied by Ross-Thomas [Ross and Thomas, 2011],[Ross and Thomas, 2007]. Let $R = \mathbb{C}[x_1,\ldots,x_N]/(f_1,\ldots,f_d)$, and $Y = \text{Spec } R$ be an affine variety with an effective, holomorphic action of a torus T, and let $V \subset Y$ be an invariant subscheme, corresponding to a homogenous ideal $I \subset R$. Suppose that $\xi \in \mathfrak{t}$ is a Reeb vector field. We consider the Rees algebra of R with respect to I, given by

$$\mathcal{R} = \mathcal{R}(R,I) := \bigoplus_{n\in\mathbb{Z}} t^{-n}I^n = R[t,t^{-1}I] \subset R[t,t^{-1}] \tag{4.5.11}$$

where $I^n := R$ for $n \leqslant 0$. For ease of notation we set $\mathcal{Y} = \text{Spec } \mathcal{R}$. Note that $\mathcal{Y}$ admits a $\mathbb{C}^*$ action induced by $\lambda\cdot t = \lambda^{-1}t$ for $\lambda \in \mathbb{C}^*$. The canonical inclusion $\mathbb{C}[t] \hookrightarrow \mathcal{R}$ gives a map $\pi : \mathcal{Y} \to \mathbb{C}$, and this map is clearly $\mathbb{C}^*$ equivariant with respect to the above action. The scheme $\mathcal{Y}$ carries a natural action of T by acting on the t-graded components, and hence commuting with the $\mathbb{C}^*$ action. For $\alpha \in \mathbb{C}-\{0\}$, the fibre $\pi^{-1}(\alpha) \cong Y$, as $\mathcal{R}/(T-\alpha)\mathcal{R} \cong R$, and so the generic fibre is isomorphic to Y. The T action on $\mathcal{Y}$ clearly preserves the fibres, and restricts to the action of T on Y away from the central fibre. Moreover, we have

$$\mathcal{Y}_0 := \pi^{-1}(0) = \text{Spec } \bigoplus_{n\geqslant 0} I^n/I^{n+1},$$

and so the central fibre is precisely the normal cone of V in Y. The $\mathbb{C}^*$ action on the central fibre is determined by the grading giving I^n/I^{n+1} degree n. Moreover, if $\xi \in \mathfrak{t}$ is the Reeb field, then ξ induces a Reeb field on $\mathcal{Y}_0$. To see this, observe that if ξ induces a positive grading on R, and $I \subset R$ is homogeneous, then ξ also induces a positive grading on

$$R/I \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots$$

Finally, it is well known that $\mathcal{R}(R,I)$ is flat over $\mathbb{C}[t]$; see for instance [Eisenbud, 1995].

In order to obtain the Lichnerowicz obstruction, we consider the simplest family of Rees deformations; namely, those obtained from principal ideals. Fix a holomorphic function $f : Y \to \mathbb{C}$, which is homogeneous for the torus action. We denote by $\alpha \in \mathfrak{t}^*$ the weight of f under T. Consider the ideal $I = (f) \subset R$, and the test configuration given by the Rees algebra $\mathcal{R}(R,I)$. The central fibre, which we denote by Y_0, of this test configuration is determined by the ring

$$\bigoplus_{n \geqslant 0} I^n/I^{n+1} \cong R/I \otimes_{\mathbb{C}} \mathbb{C}[w].$$

The grading on the latter ring is induced by the torus T on the first factor. The torus action on the second factor is by weight α on w. Finally, the induced $\mathbb{C}^*$ action, denoted η, on the central fibre is trivial on R/I, and acts with weight 1 on w. We can compute the Donaldson-Futaki invariant of this test configuration entirely in terms of the weight of the torus action on f, and the Hilbert series of R. First, we observe that if $H_R(z_0,\ldots,z_s)$ is the Hilbert series of R with multigrading induced by T, and $\alpha_0,\ldots,\alpha_s$ denote the weight of f under multigrading, then

$$H_{R/I}(z_0,\ldots,z_s) = (1 - z_0^{\alpha_0}\cdots z_s^{\alpha_s})H_R(z_0,\ldots,z_s)$$

is the Hilbert series of R/I. This follows immediately from the degree shifted exact sequence

$$0 \longrightarrow R^{[\alpha_0,\ldots,\alpha_s]} \xrightarrow{f} R \longrightarrow R/I \longrightarrow 0.$$

Since the Hilbert series is multiplicative on tensor products, we have

$$H_{R/I\otimes_R\mathbb{C}[w]}(z_0,\ldots,z_s,\tilde{z}) = \frac{(1 - z_0^{\alpha_0}\cdots z_s^{\alpha_s})}{(1 - z_0^{\alpha_0}\cdots z_s^{\alpha_s}\tilde{z})}H_R(z_0,\ldots,z_s).$$

Suppose that the index character of Y with Reeb field $\xi \in \mathfrak{t}$ expands as

$$F(\xi,t) = \frac{a_0(\xi)n!}{t^{n+1}} + \frac{a_1(\xi)(n-1)!}{t^n} + O(t^{1-n}).$$

then one easily obtains that the index character of the central fibre is given by

$$\begin{aligned} F(\xi - s\eta, t) &= \frac{1 - e^{-t\alpha(\xi)}}{(1 - e^{-t(\alpha(\xi)-s)})}\left(\frac{a_0(\xi)n!}{t^{n+1}} + \frac{a_1(\xi)(n-1)!}{t^n} + O(t^{1-n})\right) \\ &= \frac{a_0(\xi)\alpha(\xi)n!}{(\alpha(\xi)-s)t^{n+1}} + \frac{\alpha(\xi)(n-1)!}{(\alpha(\xi)-s)t^n}\left[a_1(\xi) - \frac{s}{2}a_0(\xi)n\right] + \ldots. \end{aligned}$$

Applying Theorem 24, the Donaldson-Futaki invariant is given by

$$Fut(Y_0, \xi, \eta) = \frac{-1}{n(n+1)} \left[\frac{a_1(\xi)}{\alpha(\xi)} - \frac{n(n+1)}{2} a_0(\xi) \right]. \tag{4.5.12}$$

Until now, our developments have been completely general, and equation (4.5.12) is the formula for the Donaldson-Futaki invariant of the Rees algebra for a homogeneous principal ideal. We now employ the assumption that the Y is Gorenstein and Calabi-Yau, and $\Theta \in H^0(X, K_X)$ is a non-vanishing section satisfying $\mathcal{L}_\xi \Theta = i(n+1)\Theta$; equation (4.5.10) applies, and so

$$Fut(Y_0, \xi, \eta) = -\frac{1}{2} \left[\frac{1}{\alpha(\xi)} - 1 \right].$$

In particular, we have the following theorem, which was proved for rational Reeb vector fields in [Ross and Thomas, 2011].

Theorem 27 ([Collins and Székelyhidi, 2012a])**.** *Let (Y, Θ) be an isolated Gorenstein singularity with link L, and Reeb vector field ξ satisfying $\mathcal{L}_\xi \Theta = i(n+1)\Theta$. If Y admits a holomorphic function f with $\mathcal{L}_\xi f = i\lambda f$, and $\lambda < 1$, then (Y, ξ) is K-unstable.*

This gives the following corollary, which was first observed in [Gauntlett *et al.*, 2007].

Corollary 4.5.2. *Let (Y, Θ) be an isolated Gorenstein singularity with link L, and Reeb vector field ξ satisfying $\mathcal{L}_\xi \Theta = i(n+1)\Theta$. If Y admits a holomorphic function f with $\mathcal{L}_\xi f = i\lambda f$, and $\lambda < 1$, then (Y, ξ) does not admit a compatible constant scalar curvature Kähler metric. In particular, L does not admit a Sasaki-Einstein metric with Reeb field ξ.*

Note that even if Y is not a Gorenstein singularity, from (4.5.12) we obtain a lower bound on $\alpha(\xi)$ in terms of the ratio a_1/a_0 whenever ξ is a K-semistable Reeb field on Y.

Finally, as an example, we indicate how our methods can be used to compute stable Reeb fields. The example we are interested in is the canonical cone over dP_2, the second del Pezzo surface, although these techniques apply in any situation where the equations defining the affine variety are known. It is well known that the automorphism group of dP_2 is not reductive, and hence does not admit a Kähler-Einstein metric by a result of Matsushima [Matsushima, 1957]. However, by a recent result of Futaki-Ono-Wang [Futaki *et al.*, 2009], it is known that there exists an irregular Sasaki-Einstein metric on the circle bundle of a power of the anti-canonical bundle. We illustrate how our techniques can be used to determine this stable Reeb vector field explicitly. We point out that this Reeb vector field was also computed in [Martelli *et al.*, 2008], using different methods. First, the affine scheme corresponding to the complement of the zero section in the canonical bundle embeds into $\mathbb{C}^8$, equipped with the variables x_i for $1 \leqslant i \leqslant 8$. Explicitly, it is given by $Y = \text{Spec } \mathbb{C}[x_1, \ldots, x_8]/I$, where $I =$

$$\begin{aligned}\big(x_4^2 - x_3x_5, \quad x_4^2 - x_1x_7, \quad x_4^2 - x_6x_2, \quad x_6^2 - x_3x_8, \quad x_7^2 - x_5x_8,\\ x_3x_2 - x_4x_1, \quad x_6x_4 - x_7x_3, \quad x_4x_2 - x_5x_1, \quad x_6x_5 - x_7x_4,\\ x_3x_4 - x_6x_1, \quad x_4x_5 - x_7x_2, \quad x_6x_7 - x_4x_8, \quad x_6x_4 - x_1x_8, \quad x_7x_4 - x_2x_8\big).\end{aligned}$$

These equations can be determined using standard theory of toric varieties. Note that this ideal is preserved by three linear, diagonal $\mathbb{C}^*$ actions, which we take to be

$$\begin{aligned} e_1 &= (1,1,1,1,1,1,1,1)\\ e_2 &= (0,1,-1,0,1,-1,0,-1)\\ e_3 &= (1,0,1,0,-1,0,-1,-1). \end{aligned}$$

Here, the j-th entry defining e_i denotes the weight of the action of e_i on the variable x_j. As a result, we find that the Reeb cone is given by

$$\mathcal{C}_R = \left\{ a_1e_1 + a_2e_2 + a_3e_3 \middle| a_1 > \max\{|a_2|, |a_3|, |a_2 - a_3|\}, \quad a_1 > a_2 + a_3 \right\}$$

Now, $\mathcal{H} = \mathbb{C}[x_1, \ldots, x_8]/I$ is multigraded by the e_i's. Using Macaulay2 [Grayson and Stillman,] we compute that the multivariate Hilbert series is given by

$$\begin{aligned} H(T_0,T_1,T_2) =& \frac{1 + T_0 + T_0T_1^{-1} + T_0T_2^{-1} - T_0^2T_1 - T_0^2T_1T_2^{-1}}{P(T)}\\ &+ \frac{-T_0^2T_2 - 2T_0^2 - T_0^2T_2^{-1} - T_0^2T_1^{-1}T_2}{P(T)}\\ &+ \frac{-T_0^2T_1^{-1} + T_0^3 + T_0^3T_1 + T_0^3T_2 + T_0^4}{P(T)}, \end{aligned}$$

where the function $P(T) := P(T_0, T_1, T_2)$ is given by

$$P(T) = (1 - T_0T_1)(1 - T_0T_1T_2^{-1})(1 - T_0T_2)(1 - T_0T_1^{-1}T_2)(1 - T_0T_1^{-1}T_2^{-1}).$$

For a Reeb vector field $\xi = b_1e_1 + b_2e_2 + b_3e_3$, the index character is given by $F(\xi, t) = H(e^{-b_1t}, e^{-b_2t}, e^{-b_3t})$. Expanding up to order t^{-2} we obtain

$$F(\xi,t) = \frac{(7b_1^2 + 2b_1b_2 - b_2^2 + 2b_1b_3 + 2b_2b_3 - b_3^2)t^{-3}}{(b_1+b_2)(b_1-b_2-b_3)(b_1+b_2-b_3)(b_1+b_3)(b_1-b_2+b_3)}$$

$$+ \frac{(7b_1^3 + 2b_1^2b_2 - b_1b_2^2 + 2b_1^2b_3 + 2b_1b_2b_3 - b_1b_3^2)t^{-2}}{2(b_1+b_2)(b_1-b_2-b_3)(b_1+b_2-b_3)(b_1+b_3)(b_1-b_2+b_3)}.$$

We can read off the gauge fixing condition from this expression and equation (4.5.10) as $b_1 = 3$.

If the link of (Y, ξ) admits a Sasaki-Einstein metric, then necessarily it is K-semistable. In particular, by Corollary 4.5.1, ξ must be a minimum for a_0. In order to determine ξ we must minimize the function

$$a_0(\xi) = \frac{(7b_1^2 + 2b_1b_2 - b_2^2 + 2b_1b_3 + 2b_2b_3 - b_3^2)}{(b_1+b_2)(b_1-b_2-b_3)(b_1+b_2-b_3)(b_1+b_3)(b_1-b_2+b_3)},$$

subject to the constraints $b_1 = 3$, and $(b_1, b_2, b_3) \in \mathcal{C}_R$. Computing, we find

$$b_1 = 3, \qquad b_2 = b_3 = \frac{-57 + 9\sqrt{33}}{16}$$

which agrees exactly with the result found in [Martelli *et al.*, 2008].

Chapter 5

The twisted Kähler-Ricci flow

The Kähler-Ricci flow, introduced by Hamilton [Hamilton, 1982] has been studied extensively in recent years. In this section we study a generalization of the Kähler-Ricci flow, which we call the *twisted* Kähler-Ricci flow. Fix a compact Kähler manifold (M, J).

Definition 5.0.3. *Let α be a closed, non-negative $(1,1)$-form on (M, J), and suppose that $2\pi c_1(M) - \alpha > 0$ is a Kähler class. Fix $\omega_0 \in 2\pi c_1(M) - \alpha$. The* normalized *twisted Kähler-Ricci flow is the evolution equation*

$$\frac{\partial}{\partial t}\omega = \omega + \alpha - \mathrm{Ric}(\omega), \tag{5.0.1}$$

with $\omega(0) = \omega_0$ at $t = 0$.

The normalized twisted Kähler-Ricci flow preserves the cohomology class of ω, and the short and long-time existence results follow from the standard arguments for the Kähler-Ricci flow on a Fano manifold (see e.g. Cao [Cao, 1985], or the book [Chow *et al.*, 2007]). Below we will also consider an unnormalized version of the flow. When $\alpha = 0$, both of these flows reduce to the usual Kähler-Ricci flow on the underlying Fano manifold (M, J). Our main object of study in this section is the convergence of the flow (5.0.1) when a solution of the equation

$$\mathrm{Ric}(\omega) = \omega + \alpha \tag{5.0.2}$$

exists. Solutions of this equation are called twisted Kähler-Einstein metrics, and they arise in various settings, for instance in Fine [Fine, 2004] and Song-Tian [Song and Tian, 2009]. Of particular interest recently has been the generalization where α is a multiple of the current of integration along a divisor, in relation with Kähler-Einstein metrics which have conical singularities (see for example Donaldson [Donaldson, 2012], Jeffres-Mazzeo-Rubinstein [Jeffres *et al.*, 2011]). However, we will focus on the case when α is a smooth form.

Our main result is the following.

Theorem 28. *Suppose that there is a solution ω of equation* (5.0.2). *Then for any $\omega_0 \in [\omega]$, the flow* (5.0.1) *with initial metric ω_0 converges exponentially fast to a (perhaps different) solution of* (5.0.2).

Similar results can be proved in the case when $c_1(M) - \alpha \leqslant 0$, but this is essentially contained in the work of Cao [Cao, 1985]. In the case when $\alpha = 0$, our main theorem reduces to an unpublished result of Perelman. Namely, we obtain as a corollary;

Corollary 5.0.4. *Suppose the Fano manifold (M, J) admits a Kähler-Einstein metric. Then for any $\omega_0 \in c_1(M)$, the Kähler-Ricci flow with initial metric ω_0 converges exponentially fast to a Kähler-Einstein metric.*

This theorem has been addressed several times in the literature, most notably by Tian-Zhu and collaborators (see [Tian and Zhu, 2011], [Tian and Zhu, 2007], [Tian *et al.*, 2011]). Our approach is based on the ideas in [Tian and Zhu, 2011], in particular we make strong use of the result of Tian-Zhu [Tian and Zhu, 2011] that Perelman's entropy functional increases to a fixed topological constant along the Kähler-Ricci flow.

The first step in the proof of Theorem 28 is an extension of Perelman's estimates [Sesum and Tian, 2008] to the twisted flow. For our later applications, we require uniform control of the constants appearing Perelman's estimates for a family of twisted Kähler-Ricci flows with initial metrics lying in a bounded family in C^3. This requires us to reformulate the arguments in [Sesum and Tian, 2008] in order to obtain effective bounds. The presence of the extra form α causes little difficulty, although at various points it is important that α is closed and non-negative. In addition, we extend to the twisted case the uniform Sobolev inequality along the Kähler-Ricci flow, proved by Zhang [Zhang, 2007]. These developments appear in Sections 5.1 and 5.2. With these results established, we show in Section 5.3

that Perelman's entropy functional increases along the twisted KRF to a fixed topological constant, extending a result of Tian-Zhu [Tian and Zhu, 2011] to the twisted setting.

Finally, in Section 5.4 we prove Theorem 28. The proof is by a method of continuity argument for the initial metric, similar to the method of Tian-Zhu in [Tian and Zhu, 2011]. The main difference between the two approaches is the use of different norms to measure the distance from a metric to a Kähler-Einstein metric. In [Tian and Zhu, 2011], the distance between two metrics is measured by setting

$$||g - g'||_{C^\ell(M)} = \inf_{\Phi} |g - \Phi^*(g')|_{C^\ell(M)},$$

where the norm on the right hand side is computed with respect to a fixed metric, and the infimum runs over all diffeomorphisms of M. Instead, we measure the distance between an evolving metric and a twisted Kähler-Einstein metric, using instead a C^0-norm on Kähler potentials. More precisely, fixing a twisted Kähler-Einstein metric g_{tKE}, we can write $\tau^* g = g_{tKE} + i\partial\overline{\partial}\varphi_\tau$ for any biholomorphism τ of (M, J) which fixes α. Then, we set

$$d(g) = \inf_{\tau} \text{osc}\varphi_\tau.$$

We believe that using this norm makes some of the arguments more transparent. Moreover, working with Kähler potentials allows us to avoid using a result analogous to Chen-Sun's generalized uniqueness theorem [Chen and Sun, 2010] similarly to Tian-Zhang-Zhang-Zhu [Tian *et al.*, 2011]. Instead, we only need the extension of Bando-Mabuchi's result [Bando and Mabuchi,] to the twisted case, which was given by Berndtsson [Berndtsson, 2011].

Before proceeding, we make a note about conventions. In Section 5.1 we work in the Riemannian setting. We take this approach, since our results hold in the case of the *real* Ricci flow, twisted by a $(0, 2)$ tensor satisfying a "contracted Bianchi identity". In particular, all geometric quantities are the *Riemannian* quantities. In all subsequent sections, we work in complex coordinates, with the corresponding complex quantities.

5.1 The twisted $\mathcal{W}$-functional

In this section we introduce the twisted analog of some of Perelman's functionals. These functionals will play a crucial role in our later estimates. For this section only, we will consider the *unnormalized* twisted Kähler-Ricci flow, which is the evolution equation

$$\frac{\partial}{\partial t}\omega = -2(\mathrm{Ric}(\omega) - \alpha). \tag{5.1.3}$$

This will alow us to use calculations in the existing literature more readily. Note that if $\omega(t)$ is a solution of the normalized flow (5.0.1), then $\tilde{\omega}(t) = (1-2t)\omega(-\log(1-2t))$ is a solution of the unnormalized flow with the same initial condition. In particular in our situation the existence time of the flow (5.1.3) is $t \in [0, \frac{1}{2})$.

Definition 5.1.1. *Let (M, g, J) be a compact Kähler manifold of complex dimension n, and let α be a closed, non-negative $(1,1)$-form. Let $\beta := \alpha(\cdot, J\cdot)$ be the induced, symmetric, non-negative $(0,2)$ tensor. Define the twisted entropy functional $\mathcal{W} : \mathfrak{Met} \times C^\infty(\mathbb{R}) \times \mathbb{R}_{>0} \to \mathbb{R}$ by*

$$\mathcal{W}(g, f, \tau) := (4\pi\tau)^{-n} \int_M \big(\tau(R - \mathrm{Tr}_g\beta + |\nabla f|^2) + (f - 2n)\big)\, e^{-f} dm$$

where $dm = \sqrt{\det g}$ denotes the Riemannian volume form of g, and all quantities are the real quantities.

Theorem 29. *Suppose the $(g(t), f(t), \tau(t)) \in \mathfrak{Met} \times C^\infty(\mathbb{R}) \times \mathbb{R}_{>0}$ solves the coupled system of partial differential equations*

$$\frac{\partial}{\partial t}g = -2(Ric(g) - \beta) \tag{5.1.4}$$

$$\frac{\partial}{\partial t}f = -\Delta_g f + |\nabla f|_g^2 - R(g) + \mathrm{Tr}_g\beta + \frac{n}{\tau}, \tag{5.1.5}$$

$$\frac{d}{dt}\tau = -1, \tag{5.1.6}$$

on the interval $[0, T]$. Then, $\mathcal{W}(t) := \mathcal{W}(g(t), f(t), \tau(t))$ satisfies

$$\frac{d}{dt}\mathcal{W}(t) = \\ 2\tau \int_M \left(\left| Ric(g) + \nabla\nabla f - \beta - \frac{g}{2\tau} \right|_g^2 + \beta(\nabla f, \nabla f) \right) (4\pi\tau)^{-n} e^{-f} dm.$$

In particular, $\mathcal{W}(g(t), f(t), \tau(t))$ is monotonically increasing in t.

Proof. For the proof, we work in *real* coordinates. From now on, we shall suppress the dependence on g, with the understanding that all Laplacians, curvatures, traces and inner products are computed with respect to g, unless otherwise noted. Denote by $\mathcal{W}(t) := \mathcal{W}(g(t), f(t), \tau(t))$. Following the computation of the variational formula for Perelman's entropy functional [Cao and Zhu, 2006], [Chow *et al.*, 2007] we find the variational formula for the twisted entropy functional is given by

$$\begin{aligned}(4\pi\tau)^n \frac{d}{dt}\mathcal{W}(t) \\ &= \int_M -\tau \left\langle \frac{\partial}{\partial t} g, Ric - \beta + \nabla\nabla f - \frac{1}{2\tau} g \right\rangle e^{-f} dm \\ &+ \int_M \left(\frac{1}{2}\mathrm{Tr}\frac{\partial}{\partial t} g - \frac{\partial}{\partial t} f + \frac{n}{\tau}\right) \Big[\tau(R - \mathrm{Tr}\beta + 2\Delta f - |\nabla f|^2) \\ &\qquad + f - 2n - 1\Big] e^{-f} dm \\ &- \int_M \left(R - \mathrm{Tr}\beta + |\nabla f|^2 - \frac{n}{\tau}\right) e^{-f} dm \end{aligned} \tag{5.1.7}$$

Plugging in the evolution equations (5.1.4), (5.1.5), (5.1.6), we obtain

$$\begin{aligned}(4\pi\tau)^n \frac{d}{dt}\mathcal{W}(t) \\ &= \int_M -2\tau \left\langle Ric - \beta, Ric - \beta + \nabla\nabla f - \frac{1}{2\tau} g \right\rangle e^{-f} dm \\ &+ \int_M (\Delta f - |\nabla f|^2)(\tau(R - \mathrm{Tr}\beta + 2\Delta f - |\nabla f|^2) + f] e^{-f} dm \\ &- \int_M \left(R - \mathrm{Tr}\beta + |\nabla f|^2 - \frac{n}{\tau}\right) e^{-f} dm \end{aligned}$$

where in the second line we have used that $\int_M (\Delta f - |\nabla f|^2) e^{-f} dm = 0$. Since α is a closed, $(1,1)$-form, the tensor β satisfies the "contracted Bianchi identity"

$$\nabla_i \mathrm{Tr}\beta = 2g^{jp}\nabla_p \beta_{ij}. \tag{5.1.8}$$

Using this identity, the second term can be manipulated as follows;

$$
\begin{aligned}
&\int_M (\Delta f - |\nabla f|^2)(\tau(R - \mathrm{Tr}\beta + 2\Delta f - |\nabla f|^2) + f]e^{-f} dm \\
&= \int_M (\Delta f - |\nabla f|^2)(2\tau\Delta f - \tau|\nabla f|^2)e^{-f} dm \\
&- \int_M |\nabla f|^2 e^{-f} dm - \tau \int_M \langle \nabla f, \nabla(R - \mathrm{Tr}\beta)\rangle e^{-f} dm \\
&= \tau \int_M \langle -\nabla f, \nabla(2\Delta f - |\nabla f|^2)\rangle e^{-f} dm \\
&- \int_M \Delta f e^{-f} dm - 2\tau \int_M g^{jp} g^{ki} (\nabla_p R_{ij} - \nabla_p \beta_{ij}) \nabla_k f e^{-f} dm \\
= &-2\tau \int_M g^{ij} \nabla_i f(\nabla_j \Delta f - \langle \nabla f, \nabla_j \nabla f\rangle) e^{-f} dm \\
+ &2\tau \int_M g^{jp} g^{ki} [(R_{ij} - \beta_{ij}) \nabla_p \nabla_k f - \nabla_p f \nabla_k f (R_{ij} - \beta_{ij})] e^{-f} dm \\
+ &2\tau \int_M \left\langle \frac{g}{2\tau}, \nabla\nabla f \right\rangle e^{-f} dm \\
= &2\tau \int_M \left[\left\langle \nabla\nabla f, Ric - \beta + \nabla\nabla f - \frac{1}{2\tau} g \right\rangle + \beta(\nabla f, \nabla f) \right] e^{-f} dm.
\end{aligned}
$$

Moreover, the third term can be written as

$$
\begin{aligned}
&- \int_M \left(R - \mathrm{Tr}\beta + |\nabla f|^2 - \frac{n}{\tau} \right) e^{-f} dm \\
&= 2\tau \int_M \left\langle \frac{-1}{2\tau} g, Ric - \beta + \nabla\nabla f - \frac{1}{2\tau} g \right\rangle e^{-f} dm.
\end{aligned}
$$

Combining these three expressions we obtain

$$
\frac{d}{dt}\mathcal{W}(t) = 2\tau \int_M \left(\left| Ric - \beta + \nabla\nabla f - \frac{1}{2\tau} g \right|_g^2 + \beta(\nabla f, \nabla f) \right) (4\pi\tau)^{-n} e^{-f} dm,
$$

which proves the theorem.

□

Remark 5.1.2. Note that this computation works for any *real* Ricci flow twisted by any $(0, 2)$-tensor β satisfying the contracted Bianchi identity (5.1.8) with respect to all metrics along the flow. However, monotonicity may fail if β is not semi-positive.

We define the twisted μ functional as follows;

Definition 5.1.3. *The functional* $\mu : \mathfrak{Met} \times \mathbb{R}_{>0} \to \mathbb{R}$ *is defined by*

$$
\mu(g, \tau) := \inf \{\mathcal{W}(g, f, \tau) : f \in C^\infty(M, \mathbb{R}) \text{ satisfies } (g, f, \tau) \in \mathcal{X}\}
$$

where

$$\mathcal{X} := \left\{ (g, f, \tau) : \int_M (4\pi\tau)^{-n} e^{-f} dm = 1 \right\}$$

Many of the properties of the twisted μ functional carry over from the standard Ricci flow. We list them, without proof, in the following proposition. For details, we refer the reader to [Chow *et al.*, 2007].

Proposition 5.1.4. *The twisted μ functional has the following properties;*

(i) *For any $c \in \mathbb{R}_{>0}$ we have $\mu(cg, c\tau) = \mu(g, \tau)$.*

(ii) *For a fixed $\tau \in \mathbb{R}_{>0}$, the function $\mu(g, \tau)$ is continuous in the C^2 topology on $\mathfrak{Met}$.*

(iii) *For any (g, τ) there exists a function $f \in C^\infty(M, \mathbb{R}) \cap \mathcal{X}$ such that $\mathcal{W}(g, f, \tau) = \mu(g, \tau)$. In particular, $\mu(g, \tau) > -\infty$. Moreover, f satisfies the nonlinear elliptic equation*

$$\tau(2\Delta f - |\nabla f|^2 + R - \mathrm{Tr}_g \beta) + f - 2n = \mu(g, \tau) \tag{5.1.9}$$

(iv) *Let $(g, \tau) \in \mathfrak{Met} \times \mathbb{R}_{>0}$ satisfy equations (5.1.4) and (5.1.6) on $[0, T]$. Then functional $\mu(g, \tau)$ is monotonically increasing. More precisely, for any $t_0 \in [0, T]$, if $f(t_0) \in \mathcal{X}$ is the minimizer whose existence is guaranteed by (iii) we have*

$$\frac{d}{dt}\Big|_{t=t_0} \mu(g, \tau) \geqslant 2\tau(t_0) \int_M \left| Ric(t_0) + \nabla\nabla f(t_0) - \beta - \frac{g(t_0)}{2\tau(t_0)} \right|^2 (4\pi\tau(t_0))^{-n} e^{-f(t_0)} dm_{t_0}$$

in the sense of $\liminf$ *backwards difference quotients. In particular, for $r > 0$ we have*

$$\mu(g(t_0), r^2) \geqslant \mu(g(0), r^2 + t_0)$$

which follows by taking $\tau(t) = t_0 + r^2 - t$.

It follows from Proposition 5.1.4 (*i*) and (*iv*), and the relation between the normalized and unnormalized twisted Kähler Ricci flow, that $\mu(g, \frac{1}{2})$ is monotonically increasing along the normalized flow. For convenience, we record this in the following lemma.

Lemma 5.1.5. *The quantity $\mu(g, \frac{1}{2})$ is monotonically increasing along the normalized twisted Kähler-Ricci flow. Moreover, $\mu(g, \frac{1}{2})$ is uniformly bounded above by a topogical constant,*

$$\mu(g, \frac{1}{2}) \leqslant \log\left[(2\pi)^{-n} Vol(M)\right].$$

Proof. This follows immediately by substituting

$$f = -n \log(2\pi) + \log(Vol(M)),$$

into the $\mathcal{W}$-functional, using the relation between the *real* scalar curvature and the complex scalar curvature, and the semi-positivity of β. □

In the remainder of this section we prove a non-collapsing estimate along the twisted Kähler-Ricci flow, extending Perelman's estimates for the Kähler-Ricci flow. In the untwisted case, this result is an improvement (due to Perelman), of Perelman's original non-collapsing result [Kleiner and Lott, 2008], [Sesum and Tian, 2008]. For our later applications, it will be important to prove *effective* estimates, with clear dependence on the geometry of the initial metric g_0. First, we need the following easy lemma.

Lemma 5.1.6. *Fix $x \in M$ and $t \in [0, \frac{1}{2})$, and suppose there is an $r > 0$ such that $|R(g(t)) - Tr_{g(t)}\beta| \leqslant \frac{K}{r^2}$ on $B(x, r)$. Then there exists $r' \in (0, r]$ such that*

(i) $|R - Tr_g \beta| \leqslant \frac{K}{r'^2}$ *on* $B(x, r')$

(ii) $(r')^{-2n} Vol(B(x, r')) \leqslant r^{-2n} Vol(B(x, r))$

(iii) $Vol(B(x, r')) \leqslant 3^{2n} Vol(B(x, r'/2))$.

Proof. The first item holds for any $r' \leqslant r$. By the standard expansion of the volume of a geodesic ball we have

$$\lim_{k \to \infty} \frac{Vol(B(x, r/2^k))}{Vol(B(x, r/2^{k+1}))} = 2^{2n}$$

Hence, there is a $k < \infty$ such that

$$\frac{Vol(B(x, r/2^k))}{Vol(B(x, r/2^{k+1}))} \leqslant 3^{2n}, \tag{5.1.10}$$

and if $l < k$, then

$$\frac{Vol(B(x, r/2^l))}{Vol(B(x, r/2^{l+1}))} > 3^{2n}. \tag{5.1.11}$$

Choosing $r' = r/2^k$, item (iii) follows from (5.1.10), while item (ii) follows by iterating the inequality (5.1.11) □

We first prove the non-collapsing estimate along the *unnormalized* twisted Kähler-Ricci flow. The corresponding estimate along the tKRF is then obtained by rescaling.

Proposition 5.1.7. *Let $\tilde{g}(s)$ be a solution of the unnormalized twisted Kähler-Ricci flow with $\tilde{g}(0) = g_0$. Fix a number $\rho > 0$, and define*

$$A(\rho) := \inf_{\tau \in [0, \frac{1}{2} + \rho^2]} \mu(g(0), \tau) > -\infty.$$

Then, for all $(x, t) \in M \times [0, \frac{1}{2})$ and $0 < r \leqslant \rho$ such that

$$r^2 |R(\tilde{g}(s)) - \mathrm{Tr}_{\tilde{g}(s)} \beta| \leqslant K \text{ on } B_{\tilde{g}(s)}(x, r),$$

there holds

$$Vol_{\tilde{g}(s)}(B(x, r)) \geqslant \kappa(K, \rho) r^{2n},$$

where $\kappa(K, \rho) = \exp\big(A(\rho) + 2n + n\log(4\pi) - 3^{2n+2} - K\big)$.

Proof. For simplicity, we suppress the dependence on $\tilde{g}(s)$. Fix a point $x \in M$, a radius $r \in (0, \rho]$ and a time $t \in [0, 1/2)$. Suppose that

$$Vol(B(x, r)) < \kappa r^{2n}.$$

Let $r' \in (0, r]$ be the number provided by Lemma 5.1.6. Then we have
$r'^2 |R - \mathrm{Tr}\beta| \leqslant K$ on $B(x, r')$, and

$$Vol(B(x, r')) < \kappa r'^{2n}.$$

In order to ease notation, we now set $r = r'$. Let $\varphi : [0, \infty) \to \mathbb{R}$ be the function which is 1 on $[0, 1/2)$, decreases linearly to zero on $(1/2, 1]$, and is identically zero on $[1, \infty)$. Then we set

$$u(y) = e^C \varphi(r^{-1} d(x, y))$$

where C is chosen so that

$$(4\pi)^n = e^{2C} r^{-2n} \int_{B(x,r)} \varphi(r^{-1} d(x, y))^2 dm(y).$$

It follows immediately from the definition of φ that

$$C > \frac{n}{2} \log(4\pi) - \frac{1}{2} \log(\kappa).$$

We now plug the function u into the twisted entropy functional to get

$$\begin{aligned}\mathcal{W}(g(t),u,r^2) &\leqslant 8(4\pi)^{-n}r^{-2n}e^{2C}\left[Vol(B(x,r)) - Vol(B(x,r/2))\right] \\ &\quad + K - 2n - 2C \\ &\leqslant 3^{2n+2}(4\pi)^{-n}r^{-2n}e^{2C}Vol(B(x,r/2)) + K - 2n - 2C \\ &\leqslant 3^{2n+2}(4\pi)^{-n}r^{-2n}e^{2C}\int_M \varphi^2(y)dm(y) + K - 2n - 2C \\ &< 3^{2n+2} + K - 2n - n\log(4\pi) + \log(\kappa).\end{aligned}$$

Since $A(\rho) \leqslant \mu(g(0), t + r^2) \leqslant \mu(g(t), r^2)$ for any $r \in [0, \rho)$, we obtain

$$A(\rho) < 3^{2n+2} + K - 2n - n\log(4\pi) + \log(\kappa),$$

from which it follows that $\kappa > \exp(A(\rho) + 2n + n\log(4\pi) - 3^{2n+2} - K)$. □

Along the unnormalized flow we obtain;

Proposition 5.1.8. *Let $g(t)$ be a solution of the normalized twisted Kähler-Ricci flow. Fix a number $\rho > 0$. Then, for all $(x,t) \in M \times [0,\infty)$ and $0 < r \leqslant e^{t/2}\rho$ such that $r^2|R(g(t)) - \mathrm{Tr}_{g(t)}\beta| \leqslant K$ on $B(x,r)$ there holds*

$$Vol_{g(t)}(B(x,r)) \geqslant \kappa(K,\rho)r^{2n}$$

where $\kappa(K,\rho)$ is defined in Proposition 5.1.7.

Proof. The proof is just an exercise in scaling. We include the details for completeness. Define a solution of the unnormalized tKRF by setting $\tilde{g}(s) = (1-2s)g(t(s))$ where $t(s) = -\ln(1-2s)$. Set $\tilde{r} = e^{-t/2}r$. Then $0 < r \leqslant \rho$, and we have

$$\tilde{r}^2|R(\tilde{g}) - \mathrm{Tr}_{\tilde{g}}\beta| = r^2|R(g) - \mathrm{Tr}_g\beta| \leqslant K$$

on $B_{\tilde{g}}(x,\tilde{r}) = B_g(x,r)$. Thus, we can apply the non-collapsing estimate of Proposition 5.1.7 to $\tilde{g}(s)$ to obtain

$$Vol_{\tilde{g}}(B(x,\tilde{r})) \geqslant \kappa(K,\rho)\tilde{r}^{2n}.$$

Replacing $\tilde{g}$ with g proves the proposition. □

5.2 Perelman type estimates and the Sobolev inequality

In this section we extend Perelman's bounds [Sesum and Tian, 2008] on the scalar curvature and diameter along the Kähler-Ricci flow to the twisted flow. We will need effective estimates in our application, so we are careful to keep track of constants. In addition one difference with the arguments in [Sesum and Tian, 2008] is that we bound u independently of the diameter. This may be of independent interest in situations when the diameter is not bounded.

Since the twisted Kähler-Ricci flow (tKRF) preserves the cohomology class of ω, we can write it in terms of the Kähler potential φ defined by $\omega(t) = \omega_0 + i\partial\overline{\partial}\varphi(t)$. Let $u(t)$ be the twisted Ricci potential defined by

$$i\partial\overline{\partial}u = \omega + \alpha - \mathrm{Ric}(\omega).$$

Then on the level of potentials the twisted Kähler-Ricci flow is given by

$$\dot{\varphi} = \log\left(\frac{(\omega_0 + i\partial\overline{\partial}\varphi)^n}{\omega_0^n}\right) + \varphi + u(0), \tag{5.2.12}$$

up to the addition of a time dependent constant. At each time we can normalize the twisted Ricci potential u, so that

$$\int_M e^{-u}\, dm = \int_M dm = Vol(M).$$

This normalization implies that u must vanish somewhere. We can then normalize the potentials along the flow by setting $\varphi(0) = 0$, and

$$\frac{\partial}{\partial t}\varphi = u,$$

which gives an equation equal to (5.2.12), up to adding a time dependent constant.

Differentiating (5.2.12), the evolution of u is given by

$$\frac{\partial}{\partial t}u = \Delta u + u - c(t), \tag{5.2.13}$$

where $c(t)$ is a time-dependent constant. We can compute c from the normalization condition, since

$$0 = \frac{d}{dt}\int_M e^{-u} dm = \int_M (-\Delta u - u + c + \Delta u)e^{-u}\, dm,$$

so we need

$$c = \int_M ue^{-u}\, dm \leqslant 0.$$

where the inequality follows from Jensen's inequality.

The following is analogous to the weighted Poincaré inequality of Futaki [Futaki, 1983](see also, [Tian and Zhu, 2007]). The proof is identical, using that α is a non-negative form.

Lemma 5.2.1. *For any f on M we have*

$$\frac{1}{V}\int_M f^2 e^{-u}\, dm \leqslant \frac{1}{V}\int_M |\nabla f|^2 e^{-u}\, dm + \left(\frac{1}{V}\int_M f e^{-u}\, dm\right)^2.$$

A consequence is the following monotonicity.

Lemma 5.2.2.

$$\frac{d}{dt}c = \frac{d}{dt}\int_M ue^{-u}\, dm \geqslant 0.$$

Proof. We simply compute

$$\begin{aligned}\frac{d}{dt}\int_M ue^{-u}\, dm &= \int_M (\Delta u + u - c - u\Delta u - u^2 + cu + u\Delta u)e^{-u}\, dm \\ &= \int_M (\Delta u - u^2 + cu)e^{-u}\, dm \\ &= \int_M |\nabla u|^2 e^{-u}\, dm - \int_M u^2 e^{-u}\, dm + \frac{1}{V}\left(\int_M ue^{-u}\, dm\right)^2,\end{aligned}$$

but this last expression is non-negative by the weighted Poincaré inequality applied to $f = u$. □

Next we need to find evolution equations for $|\nabla u|^2$ and Δu. Standard calculations give the following.

Lemma 5.2.3. *We have*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla u|^2 &= \Delta|\nabla u|^2 + |\nabla u|^2 - \alpha^{j\bar{k}}\nabla_j u \nabla_{\bar{k}} u - |\nabla\nabla u|^2 - |\nabla\overline{\nabla} u|^2 \\ \frac{\partial}{\partial t}\Delta u &= \Delta(\Delta u) + \Delta u - |\nabla\overline{\nabla} u|^2.\end{aligned}$$

Before proceeding, we note the following lemma, which follows immediately from the maximum principle applied to the second equation in the preceding lemma .

Lemma 5.2.4. *Along the twisted Kähler-Ricci flow we have*

$$R - \mathrm{Tr}_g\alpha \geqslant -\max(R - \mathrm{Tr}_g\alpha)^-(0).$$

In particular, $\Delta u \leqslant n + \max(R - \mathrm{Tr}_g\alpha)^-(0)$.

Lemma 5.2.5. *Along the twisted Kähler-Ricci flow we have the lower bound*

$$u(t) > -n + c(0) - \max(R - \mathrm{Tr}_g\alpha)^-(0)).$$

Proof. The argument of Perelman, as communicated by Sesum-Tian [Sesum and Tian, 2008] carries over to prove that $u(t) > -B$ for some constant $B > 0$. In order to obtain the effective estimate, we observe that by equation (5.2.13) and the monotonicity of $c(t)$ we have

$$\frac{\partial}{\partial t}u = n - c(t) + \mathrm{Tr}_g\alpha - R + u \leqslant n - c(0) + \max(R - \mathrm{Tr}_g\alpha)^-(0) + u.$$

Set $C = n - c(0) + \max(R - \mathrm{Tr}_g\alpha)^-(0)$, then it follows that, for any t_0

$$u(t) < e^{t-t_0}(u(t_0) + C) - C$$

The lower bound for u implies that, at t_0, we have

$$u(t_0) \geqslant -C = -n + c(0) - \max(R - \mathrm{Tr}_g\alpha)^-(0).$$

Since t_0 was arbitrary, the lemma follows. □

Using the evolution equations in Lemma 5.2.3 the proof of the following lemma is identical to that in [Sesum and Tian, 2008], again using that α is non-negative. The effective bounds can be read off directly from the proof.

Lemma 5.2.6. *Let* $B = n - c(0) + \max_M(R - \mathrm{Tr}_g\alpha)^-(0)$. *Then we have*

$$|\nabla u|^2 < 200B(u + 200B)$$
$$|\Delta u| < 200B(u + 200B).$$

Let us assume now that we have a constant K such that

$$|\Delta u|, |\nabla u|^2 < Ku, \text{ wherever } u > K.$$

We can take $K = 400B$ with the B as in the previous lemma. For any two numbers $a < b$ define the set

$$M(a,b) = \{x \in M \mid a < u(x) < b\}.$$

These sets will be used instead of the geodesic annuli in [Sesum and Tian, 2008].

Lemma 5.2.7. *There is a constant $\kappa_1 > 0$ such that if $a > K$ and $b > a+2$ then*

$$Vol(M(a,b)) > \kappa_1 a^{-n},$$

as long as there is a point x with $u(x) = a+1$.

Proof. On the set $M(a,a+2)$ we have $|\nabla u| < \sqrt{K(a+2)}$, so $M(a,a+2)$ contains the ball of radius

$$\frac{1}{\sqrt{K(a+2)}}$$

around the point x. At the same time, on this ball we have

$$|\Delta u| < K(a+2),$$

so by non-collapsing estimate in Proposition 5.1.8 the volume of this ball is at least $\kappa(n+1,1)[K(a+2)]^{-n}$. This in turn is at least $\kappa_1 a^{-n}$ for some other constant κ_1. □

Lemma 5.2.8. *Let $0 < \varepsilon < 1$ and $k > \max\{\log_2 \kappa_1^{-1/n}, 2\}$. Suppose that*

$$Vol(M(2^k, 2^{10k})) < \varepsilon,$$

and $u(x) > 2^{10k}$ for some x. Then there exist integers $k_1, k_2 \in [k, 10k]$ with $k_2 > k_1 + 4$ such that

$$Vol(M(2^{k_1}, 2^{k_2})) < \varepsilon,$$
$$Vol(M(2^{k_1+2}, 2^{k_2-2})) > 2^{-3n} Vol(M(2^{k_1}, 2^{k_2})).$$

Proof. The first condition is true for any $k_1, k_2 \in [k, 10k]$ by hypothesis. We claim that for some integer $p \in [0, 2k-1]$ we also have

$$Vol(M(2^{k+2p}, 2^{9k+2-2p})) < 2^{3n} Vol(M(2^{k+2p+2}, 2^{9k-2p})),$$

in which case we could choose $k_1 = k + 2p$ and $k_2 = 9k + 2 - 2p$. Otherwise we would have

$$1 > \varepsilon > Vol(M(2^k, 2^{9k+2})) \geqslant 2^{3n} Vol(M(2^{k+2}, 2^{9k})) \geqslant \dots$$
$$\geqslant 2^{6nk} Vol(M(2^{5k}, 2^{5k+2})) \geqslant 2^{6nk} \kappa_1 2^{-5nk} = 2^{nk} \kappa_1.$$

Since by our assumption $2^{nk}\kappa_1 > 1$, we get a contradiction. □

Lemma 5.2.9. *If $k_2 > k_1 + 1$, then we can find $r_1 \in [2^{k_1}, 2^{k_1+1}]$ and $r_2 \in [2^{k_2-1}, 2^{k_2}]$ such that*

$$\int_{M(r_1,r_2)} -\Delta u \, dm < CV,$$

for some fixed constant C, where

$$V = Vol(M(2^{k_1}, 2^{k_2})).$$

Proof. On $M(2^{k_1}, 2^{k_1+1})$ we have $|\nabla u|^2 < 2^{k_1+1}K$, so

$$\int_{M(2^{k_1},2^{k_1+1})} |\nabla u|^2 \, dm < 2^{k_1+1} KV.$$

By the coarea formula we have

$$\int_{M(2^{k_1},2^{k_1+1})} |\nabla u|^2 \, dm = \int_{2^{k_1}}^{2^{k_1+1}} \int_{\{u=t\}} |\nabla u| \, dS \, dt.$$

Note that by Sard's theorem, the level sets $\{u = t\}$ are smooth for almost all values of t, so that the integral on the right hand side makes sense. It follows that there exists $r_1 \in [2^{k_1}, 2^{k_1+1}]$ such that $\{u = r_1\}$ is smooth and

$$\int_{\{u=r_1\}} |\nabla u| \, dS < 2KV.$$

Similarly there exists $r_2 \in [2^{k_2-1}, 2^{k_2}]$ such that $\{u = r_2\}$ is smooth and

$$\int_{\{u=r_2\}} |\nabla u| \, dS < 2KV.$$

It follows that

$$\int_{M(r_1,r_2)} -\Delta u \, dm \leqslant \int_{\{u=r_1\}} |\nabla u| \, dS + \int_{\{u=r_2\}} |\nabla u| \, dS$$
$$\leqslant 4KV.$$

□

Proposition 5.2.10. *There exists a constant $\varepsilon > 0$ such that if for some*

$$k > \max\{\log_2 \kappa_1^{-1/n}, 2\}$$

we have

$$Vol(M(2^k, 2^{10k})) < \varepsilon,$$

then $u < 2^{10k}$ everywhere.

Proof. Suppose by contradiction that u takes on larger values, so we can find numbers k_1, k_2 satisfying the conclusions of Lemma 5.2.8, and then also r_1, r_2 using Lemma 5.2.9. Let us choose a cutoff function φ on $\mathbb{R}$ such that

$$\varphi = \begin{cases} 1 & \text{in} \quad [2^{k_1+2}, 2^{k_2-2}] \\ 0 & \text{outside} \quad [2^{k_1+1}, 2^{k_2-1}]. \end{cases}$$

We can do this in such a way that on the interval $[2^{k_1+1}, 2^{k_1+2}]$ we have

$$|\varphi'(x)| < \frac{2}{2^{k_1+1}} \leqslant \frac{4}{x},$$

and also on the interval $[2^{k_2-2}, 2^{k_2-1}]$ we have

$$|\varphi'(x)| < \frac{2}{2^{k_2-2}} \leqslant \frac{4}{x}.$$

So in sum we have $|\varphi'(x)| < 4/x$ for all x.

Let us now set $f = A\varphi(u)$ for some constant A, normalised so that

$$\int_M f^2 \, dm = (2\pi)^n,$$

and we use this as a test function in the $\mathcal{W}$ functional, with $\tau = \frac{1}{2}$. We have

$$\begin{aligned}(2\pi)^n = \int_M f^2 \, dm &> A^2 Vol(M(2^{k_1+2}, 2^{k_2-2})) \\ &> A^2 2^{-3n} Vol(M(2^{k_1}, 2^{k_2})) =: A^2 2^{-3n} V,\end{aligned}$$

so

$$A^2 V < 2^{3n}(2\pi)^n.$$

The monotonicity of $\mu\left(g, \frac{1}{2}\right)$ implies that

$$\int_M (-\Delta u) f^2 + 4|\nabla f|^2 - f^2 \ln f^2 - 2n f^2 \, dm > -C_1 \tag{5.2.14}$$

for some constant C_1. Letting $\hat{C} = \hat{C}(g_0)$ be the constant in Lemma 5.2.4, so that $-\Delta u + \hat{C} \geqslant 0$. We have

$$\begin{aligned}\int_M (-\Delta u) f^2 \, dm &\leqslant \int_M (-\Delta u + \hat{C}) f^2 \, dm \\ &< A^2 \int_{M(r_1,r_2)} (-\Delta u + \hat{C}) \, dm \\ &< A^2(CV + \hat{C}V) < 2^{3n}(2\pi)^n(C + \hat{C}),\end{aligned}$$

with the constant C from Lemma 5.2.9. Also

$$|\nabla f|^2 = A^2|\varphi'(u)||\nabla u| < A^2 \frac{4}{u}\sqrt{Ku} < 4A^2\sqrt{K},$$

assuming $u > 1$ on $M(r_1, r_2)$, ie. if $r_1 > 1$. So we get

$$\int_M 4|\nabla f|^2 \, dm < 16A^2\sqrt{K}V < 2^{3n+4}(2\pi)^n\sqrt{K}.$$

Also

$$\begin{aligned}\int_M -f^2 \ln f^2 \, dm &= A^2 \int_M -\varphi(u)^2 \ln \varphi(u)^2 \, dm - \ln A^2 \int_M f^2 \, dm \\ &< A^2 V - 2\ln A \\ &< 2^{3n}(2\pi)^n - 2\ln A.\end{aligned}$$

We used that $-1 < x \ln x \leqslant 0$ for $x \in [0, 1]$. In sum from (5.2.14) we get

$$2^{3n}(2\pi)^n(C + \hat{C}) + 2^{3n+4}(2\pi)^n\sqrt{K} + 2^{3n}(2\pi)^n - 2n - 2\ln A > -C_1.$$

This gives an upper bound $A < C_2$. At the same time

$$(2\pi)^n = \int_M f^2 \, dm < A^2 V,$$

so we get $V > (2\pi)^n A^{-2} > C_2^{-2}$. Note that C_2 only depends on the initial metric of the flow via the entropy and the constant $\hat{C}$. We can therefore choose $\varepsilon = C_2^{-2}$. □

Corollary 5.2.11. *Along the flow, u is uniformly bounded.*

Proof. Fix a $k > \max\{\log_2 \kappa_1^{-1/n}, 2\}$. For any integer $N > 1$ we have

$$\sum_{i=1}^{N} Vol(M(2^{10^{i-1}k}, 2^{10^i k})) < V,$$

where V is the volume of M, so if $N > V/\varepsilon$, then there is an $i \leqslant N$ for which

$$Vol(M(2^{10^{i-1}k}, 2^{10^i k})) < \varepsilon.$$

Using the previous Proposition, we must therefore have

$$u < 2^{10^N k}.$$

□

Now, since u is uniformly bounded above, we obtain uniform upper bounds for $|\nabla u|$ and $|\Delta u|$ from Lemma 5.2.6. We summarize our results in the following

Theorem 30. *Suppose that $g(t)$ evolves along the twisted Kähler-Ricci flow with $g(0) = g_0$, then there exists a constant C depending continuously on the C^3 norm of g_0 (and a uniform lower bound on g_0), such that*

$$|u| + |\nabla u|_{g(t)} + |\Delta_{g(t)} u| \leqslant C.$$

Notice that our estimates *did not* require a diameter bound. In this sense our approach differs from that in [Sesum and Tian, 2008]. The diameter bound can now be obtained via a simple covering argument.

Lemma 5.2.12. *Along the twisted Kähler-Ricci flow the diameter of M is uniformly bounded. More precisely,*

$$diam(M, \omega(t)) \leqslant \frac{2^{2n} Vol(M)}{\kappa(K, 1/2)},$$

where K denotes the uniform upper bound of $|R - \mathrm{Tr}_g \alpha|$ along the flow, and $\kappa(K, 1/2)$ is defined in Proposition 5.1.7

Proof. Fix points $p_1, p_2 \in M$ with $d(p_1, p_2) = R$. Let $\gamma : [0, R] \to M$ be a length minimizing geodesic connecting p_1 to p_2. Let $B_0, B_1, \ldots, B_{\lfloor R \rfloor}$ be balls of radius 1/2, centered at the points $\gamma(0), \gamma(1), \ldots, \gamma(\lfloor R \rfloor)$. These balls are disjoint since γ is length minimizing. By the bound on $|R - Tr_g \alpha|$ and Proposition 5.1.7, we have

$$Vol(M) \geqslant \sum_{0}^{\lfloor R \rfloor} Vol(B_i, \omega(t)) \geqslant R \cdot 2^{-2n} \kappa(K, 1/2).$$

It follows that

$$R \leqslant \frac{2^{2n}Vol(M)}{\kappa(K,1/2)}.$$

□

The above estimates allow us to deduce a uniform Sobolev inequality along the tKRF. Since the proof follows closely the arguments of [Hsu, 2007], [Ye, 2007], [Zhang, 2007] we list only the key ingredients. The main observation is that the monotonicity of the μ functional implies the following uniform, restricted log Sobolev inequality.

Proposition 5.2.13. *Let $g(t)$ be a solution of the twisted Kähler-Ricci flow defined on $[0,\infty)$, with $g(0)=g_0$. Define*

$$C_1 = C(n) + 4n\log C_S(M,g(0)) + 4\frac{Vol(M)^{-n}}{C_S(M,g_0)^2} + \max_M (R(0) - Tr_{g_0}\alpha)^-,$$

where $C_S(M,g_0)$ denotes the Sobolev constant of (M,g_0). Then, for all $\varepsilon \in (0,2]$, $t\in[0,\infty)$ and $u \in W^{1,2}(M)$ satisfying $||u||_{L^2(g(t))}=1$, we have

$$\int_M v^2 \log v^2 dm(t) \leqslant \varepsilon^2 \int_M \left(|\nabla v|^2 + \frac{(R(t) - \mathrm{Tr}_{g(t)}\alpha)}{4} u^2 \right) dm(t) - 2n\log\varepsilon + C_1.$$

There is a close relationship between log Sobolev inequalities and estimates for certain heat kernels. Heat kernel estimates are in turn closely related to Sobolev inequalities. In the case of the Ricci flow, the uniform log Sobolev inequality of the above proposition implies a uniform Sobolev inequality along the flow. Of course, a similar estimates hold in the case of the twisted Kähler-Ricci flow. Since the techniques are by now well documented, we omit the details and record only the result. For details in the Kähler case, we refer the reader to [Hsu, 2007], [Ye, 2007], [Zhang, 2007].

Proposition 5.2.14. *Let $g(t)$ be a solution of the twisted Kähler-Ricci flow defined on $[0,\infty)$, with $g(0)=g_0$. Define*

$$C_2 = \sup_M (R(0) - \mathrm{Tr}_{g_0}\alpha)^- + C_1$$

where C_1 is the constant defined in Proposition 5.2.13. Then there are positive constants $C(n), \beta(n)$, depending only on n, such that for all $u \in W^{1,2}(M)$ we have

$$\left(\int_M u^{2n/(n-1)} dm(t)\right)^{(n-1)/n} \leqslant C(n)e^{\beta(n)C_2} \int_M \left(|\nabla u|^2 + \frac{(R - Tr_g\alpha)}{4} u^2 \right) dm$$
$$+ C(n)e^{\beta(n)C_2}\left(1 + \max(R(0) - \mathrm{Tr}_{g_0}\alpha)^-\right) \int_M u^2 dm(t).$$

5.3 The twisted Mabuchi energy

In this section we study the tKRF under the assumption that a twisted Kähler-Einstein metric exists. More generally it is enough to assume that the twisted Mabuchi energy is bounded below on the Kähler class $2\pi c_1(M) - \alpha$. First, we recall the definition of the twisted Mabuchi energy

Definition 5.3.1 ([Stoppa, 2009] Definition 1.8)**.** *Fix $\omega_0 \in 2\pi c_1(M) - \alpha$. For any Kähler form $\omega_\varphi = \omega_0 + i\partial\overline{\partial}\varphi$, we define the twisted Mabuchi energy by its variation at φ*

$$\delta\mathcal{M}_\alpha(\delta\varphi) = -\int_M \delta\varphi \left(R(\omega_\varphi) - \mathrm{Tr}_{g_\varphi}\alpha - n\right)\omega_\varphi^n dt$$

where $\delta\varphi \in C^\infty(M, \mathbb{R})$.

The Mabuchi energy of a metric ω is obtained by fixing a base point ω_0 and integrating the variation along a path of metrics connecting ω_0 to ω. It is a basic fact that the result is independent of the path chosen. The connection with twisted Kähler-Einstein metrics is the following.

Theorem 31. *Suppose that* $\mathrm{Ric}(\omega) = \omega + \alpha$. *Then the twisted Mabuchi energy is bounded below on the Kähler class $2\pi c_1(M) - \alpha$.*

This is a generalization of the result of Bando-Mabuchi [Bando and Mabuchi,]. In the twisted case it follows from the results of Chen-Tian [Chen and Tian, 2008] (see e.g. Stoppa [Stoppa, 2009] or Székelyhidi [Székelyhidi, 2011]). Along the twisted Kähler-Ricci flow the Mabuchi energy takes on a particularly convenient form.

Lemma 5.3.2. *Suppose that $\omega(t)$ evolves along the twisted Kähler-Ricci flow with $\omega(0) = \omega_0$. Then the Mabuchi energy, with base point ω_0 is given by*

$$\mathcal{M}_\alpha(\omega_0, \omega(t)) = -\int_0^t \int_M |\nabla u|^2(s)\omega(s)^n ds.$$

In particular, if the Mabuchi energy is bounded below, then

$$\lim_{t\to\infty} \int_M |\nabla u|^2(t)\omega(t)^n = 0.$$

Proof. The first assertion follows directly from computation, so we omit the details. For the second assertion, observe that since the Mabuchi energy is bounded below, there exists times $t_i \in [i, i+1]$ such that

$$\lim_{i\to\infty} \int_M |\nabla u|^2(t_i)\omega(t_i)^n = 0.$$

This extends to the full sequence by observing that using Theorem 30, we have the differential inequality

$$\frac{\partial}{\partial t} Y(t) \leqslant CY(t)$$

where $Y(t) = \int_M |\nabla u|^2(t)\omega(t)^n$ and C is independent of time. For details in the untwisted case, see Phong-Sturm [Phong and Sturm, 2006]. □

This estimate allows us to improve the estimates in Theorem 30.

Proposition 5.3.3. *Suppose that the twisted Mabuchi energy is bounded below on* $2\pi c_1(M) - \alpha$*. Then*

$$\lim_{t\to\infty} |u(t)| + |\nabla u(t)| + |\Delta u(t)| = 0.$$

Proof. This is identical to the KRF. See, for instance Phong-Song-Sturm-Weinkove [Phong *et al.*, 2009]. □

We can now employ these estimate to study the behaviour of the twisted μ functional along the tKRF.

Proposition 5.3.4. *Let* f_t *be the function achieving* $\mu(g(t), \frac{1}{2})$*, where* $g(t)$ *is a solution of the twisted Kahler-Ricci flow with initial value* $g(0) = g_0$*. Then the following estimates hold along the twisted Kähler-Ricci flow.*

(i) There exists a constant $C_1 = C_1(g_0)$ *such that* $\sup_M |f_t| \leqslant C_1$*.*

(ii) There exists a subsequence of times $t_i \in [i, i+1]$ *such that*

$$\lim_{i\to\infty} \left(\int_M |\nabla f_{t_i}|^2 dm_{t_i} \right)^2 + \lim_{i\to\infty} \int_M |\Delta f_{t_i}|^2 dm_{t_i} = 0.$$

(iii) Along the sequence t_i *we have*

$$\lim_{i\to\infty} \int_M f_{t_i} e^{-f_{t_i}} dm_{t_i} = (2\pi)^n \log\left((2\pi)^{-n} Vol(M)\right)$$

Proof. The proof of (i) follows [Tian and Zhu, 2011] closely, so we will only outline the argument. We begin by proving the first bound. For ease of notation, we suppress the dependence on t. From (5.1.9) together with the bounds on $R - \mathrm{Tr}_g\alpha$ and $\mu(g, 1/2)$, the minimizer f satisfies

$$\Delta f - \frac{1}{2}|\nabla f|^2 < C - f,$$

from which we get

$$\Delta e^{-f/2} = -\frac{1}{2}e^{-f/2}\left(\Delta f - \frac{1}{2}|\nabla f|^2\right) > -\frac{1}{2}(C - f)e^{-f/2}.$$

Letting $h = e^{-f/2}$ we then have a constant C_δ for any $\delta > 0$ such that

$$\Delta h \geqslant -h^{1+\delta} - C_\delta.$$

Moser iteration, together with the normalization $\int h^2\, dm = (2\pi)^n$ implies an upper bound for h, i.e. a lower bound $f \geqslant -C_1$ for f.

We turn our attention now to the upper bound. Define

$$E_A = \{x \in M : f(x) < A\}.$$

Using the bound on u from Theorem 30 and the normalization of f, we have

$$\int_M e^{-f} e^{-u}\, dm > C^{-1},$$

for some C. Using the normalization $\int_M e^{-u}\, dm = V$ together with the lower bound $f \geqslant -C_1$, this implies that there exists a sufficiently large A, and a $\delta > 0$ such that

$$\int_{E_A} e^{-u}\, dm > \delta.$$

Using the bound on u again, we have

$$\begin{aligned}\int_M f e^{-u} dm &= \int_{E_A} f e^{-u}\, dm + \int_{M\setminus E_A} f e^{-u}\, dm \\ &\leqslant AC + (V - \delta)^{\frac{1}{2}}\left(\int_M f^2 e^{-u}\, dm\right)^{\frac{1}{2}}.\end{aligned} \tag{5.3.15}$$

It follows that for some C_2 we have

$$\left(\int_M f e^{-u}\, dm\right)^2 \leqslant C_2 + (V - \delta/2)\int_M f^2 e^{-u}\, dm. \tag{5.3.16}$$

Multiplying equation (5.1.9) by e^{-u}, integrating, and using the uniform bounds on u, $R-\mathrm{Tr}_g\alpha$ and $\mu(g)$, we obtain

$$\begin{aligned}\int_M |\nabla f|^2 e^{-u}\,dm &\leqslant \int_M f e^{-u} dm + C_3\\ &\leqslant \frac{\delta}{4V}\int_M f^2 e^{-u}\,dm + C_4,\end{aligned}$$

for some C_3, C_4. Substituting this into the weighted Poincaré inequality of Lemma 5.2.1, and using (5.3.16) yields

$$\begin{aligned}\int_M f^2 e^{-u} dm &\leqslant \int_M |\nabla f|^2 e^{-u} dm + \frac{1}{V}\left(\int_M f e^{-u} dm\right)^2\\ &\leqslant \frac{\delta}{4V}\int_M f^2 e^{-u} dm + \left(1-\frac{\delta}{2V}\right)\int_M f^2 e^{-u} dm + C_5.\end{aligned}$$

Rearranging this, we get an upper bound

$$\int_M f^2 e^{-u} dm \leqslant C_6$$

where C_6 can be chosen to depend only on $g(0)$. By equation (5.1.9) we have $\Delta f \geqslant -f - C$, so the upper bound for f follows from Moser iteration and the L^2 bound.

We now prove the second and third items. We begin by observing that

$$\mu(g(T)) - \mu(g(0)) \geqslant \int_0^T \left(\int_M |Ric(g) + \nabla\nabla f - \alpha - g|_g^2\,(2\pi)^{-n} e^{-f} dm\right)(s)ds \qquad (5.3.17)$$

To see this, we fix a partition $P_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0, T]$, and write

$$\mu(T) - \mu(0) = \sum_{i=1}^N \frac{\mu(t_i) - \mu(t_{i-1})}{t_i - t_{i-1}}(t_i - t_{i-1}).$$

Let f_i be the smooth function satisfying $\mathcal{W}(g_i, f_i, \frac{1}{2}) = \mu(t_i)$, and let $f_i(t)$ be the solution to the backwards heat equation (5.1.5) on $[t_{i-1}, t_i]$ with $f_i(t_i) = f_i$. Then by the mean value theorem we have

$$\begin{aligned}\frac{\mu(t_i) - \mu(t_{i-1})}{t_i - t_{i-1}} &\geqslant \frac{\mathcal{W}(g_i, f_i, \frac{1}{2}) - \mathcal{W}(g_{i-1}, f_i(t_{i-1}), \frac{1}{2})}{t_i - t_{i-1}}\\ &= (t_i - t_{i-1})\frac{d}{dt}\bigg|_{t=t_i^*} \mathcal{W}\left(g(t), f_i(t), \frac{1}{2}\right),\end{aligned}$$

for some $t_i^* \in (t_{i-1}, t_i)$. Using the result of the computation in the proof of Theorem 29 we have

$$\mu(T) - \mu(0) \geqslant \sum_{i=1}^N (t_i - t_{i-1})\left(\int_M \left|Ric(g) + \nabla\overline{\nabla} f_i - \alpha - g\right|_g^2 e^{-f_i} dm\right)(t_i^*).$$

Taking the $\liminf$ as $N \to \infty$ proves the result. Since μ is increasing and bounded above, it follows immediately that that the bracketed term on the right had side of (5.3.17) goes to zero along a subsequence of times $t_i \in [i, i+1]$. In particular, we have

$$\lim_{i\to\infty} \int_M |\nabla\overline{\nabla} f_i - \nabla\overline{\nabla} u(t_i)|^2 e^{-f_i} dm = 0.$$

Now observe that $|\nabla\overline{\nabla} f_i - \nabla\overline{\nabla} u(t_i)|^2 \geqslant n^{-1}|\Delta(f-u)|^2$. Applying Proposition 5.3.3, we obtain

$$\lim_{i\to\infty} \int_M |\Delta f_i|^2 e^{-f_i} dm = 0.$$

The second item follows, using the upper bound for f, and the observation

$$\int_M |\nabla f|^2 dm = -\int_M f\Delta f dm \leqslant C\left(\int_M |\Delta f|^2 dm\right)^{1/2}$$

where C depends only on the bound for f.

Finally, we prove the third item. Here our argument differs somewhat from [Tian and Zhu, 2011]. First, it follows from Jensen's inequality that

$$\int_M f e^{-f} \leqslant (2\pi)^n \log\left((2\pi)^{-n} V\right).$$

Thus, it suffices to prove a lower bound. Set

$$\tilde{f} := f - V^{-1}\int_M (f-u)e^{-u} dm.$$

By the weighted Poincaré inequality of Lemma 5.2.1, and the choice of normalization we have

$$\begin{aligned}\int_M (\tilde{f}-u)^2 dm &\leqslant C\int_M (\tilde{f}-u)^2 e^{-u} dm \\ &\leqslant C\int_M \left(|\nabla f|^2 + |\nabla u|^2\right) e^{-u} dm\end{aligned}$$

for a constant C depending only on g_0. From this and the upper bound for $|f|$ we obtain

$$\begin{aligned}\int_M |\tilde{f}-u| e^{-f} dm &\leqslant C\int_M |\tilde{f}-u| e^{-u} dm \\ &\leqslant C'\int_M (\tilde{f}-u)^2 e^{-u} dm \\ &\leqslant C''\int_M (|\nabla f|^2 + |\nabla u|^2) e^{-u} dm.\end{aligned}$$

In particular, applying item (*ii*), we have that

$$\lim_{i\to\infty}\int_M |\tilde{f}_i - u|e^{-f_i}dm = 0. \tag{5.3.18}$$

Moreover, by Jensen's inequality we have

$$\begin{aligned}\tilde{f} - f &= V^{-1}\int_M \log(e^{u-f})e^{-u}dm \\ &\leqslant \log\left(V^{-1}\int_M e^{-f}dm\right) \\ &= \log((2\pi)^n V^{-1}).\end{aligned}$$

As a result, we have

$$(2\pi)^{-n}\int_M (u-f)e^{-f}dm \leqslant \log((2\pi)^n V^{-1}) + \int_M (u-\tilde{f})e^{-f}dm$$

Rearranging this equation gives

$$(2\pi)^{-n}\int_M fe^{-f} \geqslant -||u||_{C^0} + \log((2\pi)^{-n}V) - (2\pi)^{-n}\int_M |\tilde{f}-u|e^{-f}dm.$$

In particular, by Propostion 5.3.3 and equation (5.3.18) we obtain that

$$\lim_{i\to\infty}\int_M f_i e^{-f_i} \geqslant (2\pi)^n \log((2\pi)^{-n}V),$$

which finishes the proof. □

As a result we obtain the following important corollary;

Corollary 5.3.5. *Suppose that the twisted Mabuchi energy of (M, J) is bounded below on the Kähler class $2\pi c_1(M) - \alpha$. Let $\omega(t)$ be a solution of the tKRF, with $\omega(0) = \omega_0 \in 2\pi c_1(M) - \alpha$. Then*

$$\lim_{t\to\infty}\mu\left(\omega(t), \frac{1}{2}\right) = \log\left((2\pi)^{-n}Vol(M)\right).$$

Moreover, if $\omega_0 \in 2\pi c_1(M) - \alpha$ satisfies

$$\mu(\omega_0, \frac{1}{2}) = \log\left((2\pi)^{-n}Vol(M)\right)$$

then ω_0 is a twisted Kähler-Einstein metric.

Proof. The first statement follows immediately from Proposition 5.3.4. We prove the second statement. Let $\omega(t)$ be a solution of the tKRF with $\omega(0) = \omega_0$. Then by the monotonicity of μ, and Lemma 5.1.5 we have

$$\log\left((2\pi)^{-n}Vol(M)\right) = \mu(\omega_0, \frac{1}{2}) \leqslant \mu(\omega(t), \frac{1}{2}) \leqslant \log\left((2\pi)^{-n}Vol(M)\right),$$

Thus, $\mu(t) = \log\left((2\pi)^{-n}Vol(M)\right)$ for all t. Let f be any minimizer of $\mathcal{W}(g(0), \cdot, \frac{1}{2})$. Then by Proposition 5.1.4 we have that

$$\left|Ric(t_0) + \nabla\overline{\nabla}f - \alpha - g(0)\right|^2 = 0.$$

By the definition of the Ricci potential, we must have

$$\nabla\overline{\nabla}f = \nabla\overline{\nabla}u,$$

and hence $f = u + c$ for some constant c. However, we clearly have that $\tilde{f} = -n\log(2\pi) + \log(Vol(M))$ is a minimizer of $\mathcal{W}(g(0), \cdot, \frac{1}{2})$, which follows by direct computation. As a result we have that u is a constant. It follows from the normalizations that $u = 0$, and ω_0 is twisted Kähler-Einstein. □

5.4 The convergence of the twisted Kähler-Ricci flow

In this section we prove Theorem 28. The argument builds on Tian-Zhu [Tian and Zhu, 2011], in that the behaviour of Perelman's entropy along the flow is exploited. One important difference is that our distance function $d(g)$ below measures the oscillation of the Kähler potentials rather than a $C^{3,\alpha}$ norm of the metric.

Suppose that g_{tKE} is a twisted Kähler-Einstein metric on M, satisfying the equation

$$\mathrm{Ric}(g_{tKE}) = g_{tKE} + \alpha, \tag{5.4.19}$$

where α is a non-negative, closed, (1,1)-form. Write $G \subset Aut_0(M)$ for the connected component of the group of biholomorphisms preserving α, i.e.

$$G = \{\tau \in Aut_0(M) \,:\, \tau^*(\alpha) = \alpha\}.$$

By Berndtsson's generalization [Berndtsson, 2011] of Bando-Mabuchi's uniqueness result [Bando and Mabuchi,], every solution of (5.4.19) is given by $\tau^* g_{tKE}$ for some $\tau \in G$.

If $g = g_{tKE} + i\partial\overline{\partial}\varphi$, and $\tau \in G$, let us define φ_τ by

$$\tau^* g = g_{tKE} + i\partial\overline{\partial}\varphi_\tau,$$

and let

$$d(g) = \inf\{\operatorname{osc}\varphi_\tau \,:\, \tau \in G\}.$$

Note that $d(g)$ is independent of the normalization of the Kähler potentials.

Let us write u_g for the twisted Ricci potential of g, normalized in any way we like; i.e.

$$i\partial\overline{\partial}u_g = \omega + \alpha - \operatorname{Ric}(g).$$

Note that we can take $u_{\tau^* g} = \tau^* u_g$ for any $\tau \in G$. We will work with $\operatorname{osc} u_g$, which is independent of the normalization, and $\operatorname{osc} u_{\tau^* g} = \operatorname{osc} u_g$.

The normalized twisted Kähler-Ricci flow (5.0.1) is given by

$$\frac{\partial}{\partial t}\varphi(t) = u_{g(t)} + c(t),$$

where $c(t)$ is a time dependent constant depending on our normalizations.

Theorem 30 implies that there is a constant K depending on $g(0)$, such that $\operatorname{osc} u_{g(t)} < K$ for all t. Moreover K can be chosen uniformly as long as $g(0)$ is bounded in C^3 relative to g_{tKE}.

We will need the following smoothing result for the twisted Kähler-Ricci flow.

Theorem 32. *Suppose that $\omega(0) = g_{tKE} + i\partial\overline{\partial}\varphi(0)$ for some fixed background metric g_{tKE}, and $\operatorname{osc}\varphi(0), \operatorname{osc} u(0) < K$ for some K. Then there exist $s, C > 0$ depending on K (and g_{tKE}), such that at time s along the twisted Kähler-Ricci flow starting with $\omega(0)$, we have*

$$C^{-1} g_{tKE} < g(s) < C g_{tKE},$$
$$\|g(s)\|_{C^3} < C,$$

where the C^3 norm is measured using g_{tKE}.

Proof. This follows from Proposition 2.1 in Székelyhidi-Tosatti [Székelyhidi and Tosatti, 2011], and is similar to the result of Song-Tian [Song and Tian, 2009] for the Kähler-Ricci flow. One just has to normalize φ and u_g first in order to bound $\sup|\varphi(0)|$ and $\sup|u(0)|$. □

In addition we will use the following result, which follows from the work of Phong-Song-Sturm-Weinkove [Phong *et al.*, 2009].

Theorem 33. *Suppose that along the twisted Kähler-Ricci flow $g(t)$ we have $d(g(t)) < K$ for a constant independent of time. Then $g(t)$ converges to a twisted Kähler-Einstein metric exponentially fast.*

Proof. By Perelman's estimate we can assume that also $\operatorname{osc} u_{g(t)} < K$ for all t, increasing K if necessary. Fix a time T, and let $\tau \in G$ such that

$$\tau^* g(T) = g_{tKE} + i\partial\overline{\partial}\varphi,$$

with $\operatorname{osc}\varphi < K$. The Ricci potential still satisfies $\operatorname{osc} u_{\tau^* g(T)} < K$, so the smoothing property applied to the twisted KR flow starting with $\tau^* g(T)$ implies that for some s, C (depending on K) we have

$$C^{-1} g_{tKE} < \tau^* g(T+s) < C g_{tKE}$$
$$\|\tau^* g(T+s)\|_{C^3(g_{tKE})} < C.$$

Since T was arbitrary, this shows that up to the action of G, the metrics along the flow are uniformly bounded in C^3, relative to g_{tKE}. It follows that a subsequence converges in $C^{2,\alpha}$, and the limit is necessarily a twisted Kähler-Einstein metric, since u tends to a constant by Proposition 5.3.3. We can also obtain exponential convergence without needing the action of G, by following the argument of Phong-Song-Sturm-Weinkove [Phong *et al.*, 2009]. Indeed the uniform C^3 bound implies that the first eigenvalue of $\overline{\partial}$ on TM (which is invariant under the action of G) is bounded away from zero uniformly. □

We now show that the distance $d(g)$ is continuous with respect to the C^3 metric on g.

Lemma 5.4.1. *If $g_k \to g$ in C^3 (with respect to g_{tKE}), then $d(g_k) \to d(g)$.*

Proof. Recall that for $\tau \in G$ we wrote

$$\tau^* g = g_{tKE} + i\partial\overline{\partial}\varphi_\tau,$$

and

$$d(g) = \inf\{\operatorname{osc}\varphi_\tau \,:\, \tau \in G\}.$$

We first prove that there exists a $\tau \in G$ realizing this infimum. Let $\tau_k \in G$ be a sequence so that $d(g) = \lim \operatorname{osc} \varphi_{\tau_k}$. We can choose a constant K such that

$$\begin{aligned} \operatorname{osc} u_g &< K \\ \operatorname{osc} \varphi_{\tau_k} &< K. \end{aligned} \tag{5.4.20}$$

Using the smoothing property (applied to the twisted KR flow $\tau_k^* g(t)$) we can find s, C such that

$$C^{-1} g_{tKE} < \tau_k^* g(s) < C g_{tKE} \tag{5.4.21}$$

Fix a point $p \in M$. Choosing a subsequence of the τ_k we can assume that $\tau_k(p) \to q$ for some $q \in M$. From (5.4.21) it follows that for large enough k, each τ_k maps an open coordinate neighborhood B_p about p to a coordinate neighborhood about q. The component functions of these τ_k are then given by uniformly bounded holomorphic functions on B_p, so after choosing a further subsequence, we can assume that the τ_k converge when restricted to the half ball $\frac{1}{2}B_p$. The open sets $\frac{1}{2}B_p$ cover M, so we can choose a finite subcover, and a subsequence of the τ_k will then converge over all of M to a holomorphic map $\tau_\infty : M \to M$. Taking the limit in (5.4.21) we see that τ is injective, and it is an open map so it is also surjective. Moreover, τ clearly preserves α, and the connected component of the identity in $Aut(M)$ is closed, so $\tau_\infty \in G$. It follows that $d(g) = \operatorname{osc} \varphi_{\tau_\infty}$.

Suppose now that $g_k \to g$ in C^3, and $\tau \in G$ realizes the infimum $d(g)$. Since $\tau^* g_k \to \tau^* g$, using the same τ to bound each $d(g_k)$ we find that

$$\limsup d(g_k) \leqslant d(g).$$

For the converse inequality suppose that for each k, $\tau_k \in G$ realizes the infimum $d(g_k)$. By the same argument as above (we can choose a uniform K in (5.4.20) for all the g_k), up to choosing a subsequence we can assume that $\tau_k \to \tau_\infty$. Then $\tau_k^* g_k \to \tau_\infty^* g$, and using τ_∞ to bound $d(g)$ we get

$$d(g) \leqslant \liminf d(g_k).$$

This shows that $d(g) = \lim d(g_k)$. □

We will now write $\mu(g) = \mu\left(g, \frac{1}{2}\right)$ for Perelman's entropy. We collect here a few of the previous results about μ, from Lemma 5.1.5 and Corollary 5.3.5:

1. μ is continuous in the C^3-norm, measured relative to g_{tKE}.

2. For any initial metric $g(0)$, $\mu(g(t))$ is monotonically increasing along the twisted Kähler-Ricci flow $g(t)$, and $\lim \mu(g(t)) = \Lambda$, where we fix $\Lambda = \log\left((2\pi)^{-n} Vol(M)\right)$. In particular, Λ is independent of $g(0)$.

3. $\mu(g) = \Lambda$ if and only if g is a twisted KE metric. By Berndtsson's uniqueness theorem this is equivalent to: $\mu(g) = \Lambda$ if and only if $g = \tau^* g_{tKE}$ for a biholomorphism $\tau \in G$ of (M, J) fixing α.

The following is a consequence of the smoothing result.

Lemma 5.4.2. *Fix $K > 0$, and suppose that*

$$1 \leqslant d(g) < K$$
$$\operatorname{osc} u_{g(t)} < K,$$

for all t along the twisted KR flow $g(t)$ with initial metric g. There exist $s, C > 0$ depending on K, and a $\tau \in G$ such that

$$d(g(s)) \geqslant \frac{1}{2},$$
$$C^{-1} g_{tKE} < \tau^* g(s) < C g_{tKE},$$
$$\|\tau^* g(s)\|_{C^3} < C,$$

where the C^3-norm is measured with respect to the fixed metric g_{tKE}.

Proof. For any $\tau \in G$, the flow $\tau^* g(t)$ is a solution of the twisted KR flow, and it follows by our assumption that $\operatorname{osc} u_{\tau^* g(t)} < K$ along the flow with initial metric $\tau^* g$. Write

$$\tau^* g(t) = g_{tKE} + i\partial\overline{\partial}\varphi(t),$$

so that $\operatorname{osc}\varphi(0) \geqslant 1$ by the assumption that $d(g) \geqslant 1$. Along the twisted KR flow,

$$\dot{\varphi}(t) = u_{\tau^* g(t)} + c(t),$$

where $c(t)$ is a time dependent constant. So if $s < (4K)^{-1}$ we have

$$\sup \varphi(s) > \sup \varphi(0) - Ks + A > \sup \varphi(0) - \frac{1}{4} + A$$
$$\inf \varphi(s) < \inf \varphi(0) + Ks + A < \inf \varphi(0) + \frac{1}{4} + A,$$

for some constant A (the integral of $c(t)$), and so

$$\operatorname{osc}\varphi(s) > \operatorname{osc}\varphi(0) - \frac{1}{2} \geqslant \frac{1}{2}.$$

Since this is true for any $\tau \in G$, by taking infimum we have

$$d(g(s)) \geqslant \frac{1}{2}.$$

For the smoothing result, we first choose a $\tau \in G$ such that

$$\tau^* g = g_{tKE} + i\partial\overline{\partial}\varphi,$$

with $\operatorname{osc}\varphi < K$. Then we can use the smoothing theorem, applied to the flow $\tau^* g(t)$ with initial metric $\tau^* g$, and choose s even smaller than we did in the previous step if necessary. □

Proposition 5.4.3. *Fix $K > 0$. There exists a $c > 0$ depending on K, such that if*

$$\mu(g) > \Lambda - c,$$
$$d(g), \operatorname{osc} u_{g(t)} < K,$$

for all time t along the twisted KR flow $g(t)$, then $d(g) < 1$.

Proof. We argue by contradiction. Suppose there is a $K > 0$ for which there is no suitable c. This means that we can choose a sequence g^k for which

$$\mu(g^k) > \Lambda - 1/k, \tag{5.4.22}$$

and $d(g^k), \operatorname{osc} u_{g_k(t)} < K$ for all t, but $d(g^k) \geqslant 1$.

We apply Lemma 5.4.2. We get $s, C > 0$, and a $\tau_k \in G$ such that

$$\begin{gathered} d(g^k(s)) \geqslant \frac{1}{2} \\ C^{-1} g_{tKE} < \tau_k^* g^k(s) < C g_{tKE}, \\ \|\tau_k^* g^k(s)\|_{C^3(g_{tKE})} < C, \\ \mu(\tau_k^* g^k(s)) = \mu(g^k(s)) \geqslant \mu(g^k) > \Lambda - 1/k, \end{gathered} \tag{5.4.23}$$

where we also used the monotonicity of μ along the twisted KR flow. We can choose a subsequence of the $\tau_k^* g^k(s)$ converging to some g_∞ in $C^{2,\alpha}$ and in particular $\mu(g_\infty) = \Lambda$, so

$g_\infty = \tau^* g_{tKE}$ for some $\tau \in G$. The metric g_∞ is uniformly equivalent to g_{tKE} (we can use the constant C given by the bounds (5.4.23)) and so

$$\begin{aligned}\|(\tau^{-1})^*\tau_k^* g^k(s) - g_{tKE}\|_{C^{2,\alpha}(g_{tKE})} &= \|(\tau^{-1})^*\tau_k^* g^k(s) - (\tau^{-1})^* g_\infty\|_{C^{2,\alpha}(g_{tKE})} \\ &= \|\tau_k^* g^k(s) - g_\infty\|_{C^{2,\alpha}(\tau^* g_{tKE})} \to 0.\end{aligned}$$

Writing

$$(\tau^{-1})^*\tau_k^* g^k(s) = g_{tKE} + i\partial\overline{\partial}\varphi_k(s)$$

with $\varphi_k(s)$ normalized to have zero mean, we obtain

$$\varphi_k(s) \to 0 \text{ in } C^{4,\alpha}(g_{tKE}),$$

which contradicts

$$\operatorname{osc}(\varphi_k(s)) \geqslant d(g^k(s)) \geqslant \frac{1}{2}.$$

□

Next we prove a stability result for the twisted Kähler-Ricci flow. In the untwisted case, a similar result was proved by Sun-Wang [Sun and Wang, 2013], using different techniques.

Proposition 5.4.4. *There is a $\delta > 0$ such that if $\|g - g_{tKE}\|_{C^3(g_{tKE})} < \delta$, then the twisted Kähler-Ricci flow starting with g converges to a twisted Kähler-Einstein metric.*

Proof. We can choose a constant $K > 1$ such that if $\|g - g_{tKE}\|_{C^3(g_{tKE})} < \frac{1}{2}$, then $d(g) < K$, and also by Perelman's estimate $\operatorname{osc} u_{g(t)} < K$ for all t. Let $c = c(K)$ be the constant given by Proposition 5.4.3. We can now choose $\delta < 1$ sufficiently small so that

$$\|g - g_{tKE}\|_{C^3(g_{tKE})} < \delta$$

implies that $\mu(g) > \Lambda - c$. By the monotonicity we have

$$\mu(g(t)) > \Lambda - c$$

for all $t > 0$. The previous Proposition implies that $d(g) < 1$ and then also $d(g(t)) < 1$ for all $t > 0$ (since at any time if $d(g(t)) = 1$, then Proposition 5.4.3 says $d(g(t)) < 1$). Theorem 33 implies that $g(t)$ converges to a twisted KE metric. □

Let S be the set of C^5 metrics g in the class $[g_{tKE}]$ such that the twisted KR flow starting with g converges to a twisted KE metric in $C^5(g_{tKE})$. Our goal is to prove that S is both open and closed. We claim that this implies our main theorem. To see this, observe that the set of C^5 metrics in the class $[g_{tKE}]$ is convex, and hence connected. It follows that either S contains all metrics, or S is empty. But $g_{tKE} \in S$, and hence Theorem 28 follows.

Proposition 5.4.5. *S is open in the C^5 topology.*

Proof. Suppose that $g \in S$. Then for sufficiently large T there exists a $\tau \in G$ such that $\|\tau^* g(T) - g_{tKE}\|_{C^3(g_{tKE})} < \delta/2$ with the δ from the previous result. For a finite time $t \in [0, T]$ the solution of the twisted KR flow depends smoothly on the initial data, so, since $\tau \in G$ is fixed, we can choose c small so that if $\|h - g\|_{C^5(g_{tKE})} < c$, then $\|\tau^* h(T) - \tau^* g(T)\|_{C^3(g_{tKE})} < \delta/2$. Then

$$\|\tau^* h(T) - g_{tKE}\|_{C^3(g_{tKE})} < \delta,$$

so the stability result implies that the flow starting with $\tau^* h(T)$, and hence also the flow starting with h, converges to a twisted Kähler-Einstein metric. □

Proposition 5.4.6. *S is closed in C^5.*

Proof. Suppose that $g_k \in S$, and

$$g_k \to g \text{ in } C^5.$$

By Perelman's estimate, we can choose a $K > 2$ such that $\operatorname{osc} u_{g_k(t)} < K$ for all k, t (using that the g_k are in a bounded set of metrics in C^5). Let c be the constant given by Proposition 5.4.3 corresponding to K. By the properties of μ, there exists a T such that

$$\mu(g(T)) > \Lambda - c.$$

Since the tKRF is stable for finite time, and μ is continuous for a C^3 family of metrics, there exists an N such that

$$\mu(g_k(T)) > \Lambda - c$$

for all $k > N$. By monotonicity, it follows that

$$\mu(g_k(t)) > \Lambda - c$$

for all $k > N$ and $t \geqslant T$. Since we know that $g_k(t)$ converges to a twisted KE metric $\tau_k^* g_{tKE}$ for some $\tau_k \in G$, it follows that $(\tau_k^{-1})^* g_k(t)$ converges to g_{tKE}. Now, since $(\tau_k^{-1})^* g_k(t)$ converges to g_{tKE} we can apply Lemma 5.4.1 to obtain that $d((\tau_k^{-1})^* g_k(t)) = d(g_k(t))$ converges to $d(g_{tKE}) = 0$. In particular, there must be a first time $t_k \geqslant T$ for which

$$d(g_k(t_k)) \leqslant K/2.$$

By Proposition 5.4.3, at this time we have $d(g_k(t_k)) < 1$, so since $K > 2$ and $t_k \geqslant T$ was chosen to be minimal, $t_k = T$. But then as before, we have $d(g_k(t)) < 1$ for all $t > T$ and $k > N$. For any fixed t we have $g_k(t) \to g(t)$ in $C^3(g_{tKE})$ as $k \to \infty$, and so apply Lemma 5.4.1 again, we obtain

$$d(g(t)) \leqslant 1$$

for all $t \geqslant T$. Theorem 33 then implies that $g(t)$ converges to a twisted KE metric. □

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