# The arithmetic and geometry of genus four curves

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# Abstract

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We construct a point in the Jacobian of a non-hyperelliptic genus four curve which is defined over a quadratic extension of the base field. We attempt to answer two questions:

1. Is this point torsion?

2. If not, does it generate the Mordell–Weil group of the Jacobian?

We show that this point generates the Mordell–Weil group of the Jacobian of the universal genus four curve. We construct some families of genus four curves over the function field of $\mathbb{P}^1$ over a finite field and prove that half of the Jacobians in this family are generated by this point via the other half are not. We then turn to the case where the base field is a number field or a function field. We compute the Neron–Tate height of this point in terms of the self-intersection of the relative dualizing sheaf of (the stable model of) the curve and some local invariants depending on the completion of the curve at the places where this curve has bad or smooth hyperelliptic reduction. In the case where the reduction satisfies some certain conditions, we compute these local invariants explicitly.

# Contents

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# Chapter 1

# Introduction

## 1.1 Introduction

Let $X$ be a variety defined over a field. Constructing homologically trivial cycles and determining whether they are torsion in the corresponding Chow group is an important problem in algebraic and arithmetic geometry. An explicit construction usually gives deep information on the arithmetic and geometry of $X$. One of the most well-known examples is when $X$ is an elliptic curve over a number field, one can construct the Heegner point on $X$. In this case, whether the point is non-torsion is related to the first derivative of the Hasse–Weil $L$-function of the elliptic curve by the Gross–Zagier formula. This construction, together with other powerful techniques, provides the best results so far towards the famous BSD conjecture.

Another example is the Gross–Schoen cycle on the triple product of a pointed curve and studied in detail in [GS1995,Zha2010]. In the paper of Zhang, the height of the Gross–Schoen cycle is expressed in terms of the self-intersection of the dualizing sheaf of the curve [Zha2010, Theorem 1.3.1]. Various deep arithmetic information is extracted from such a formula. For instance, it is proved that the Northcott property holds for this cycle [Zha2010, Theorem 1.3.5]. This formula is also applied to the effective Bogolomov conjecture, the Beilinson–Bloch conjecture and the non-triviality of the tautological cycles in the Jacobians. The height formula of the Gross–Schoen cycle provides important progress towards these long standing conjectures. For example, by using the Fourier transform on the Chow group, it is proved that if $X$ is a curve, the Ceresa cycle $X - [-1]^*X$ in the Jacobian of $X$ has the Northcott property [Zha2010, Theorem 1.5.5]. Previously, the Ceresa cycle is only known to be non-torsion in the corresponding Chow groups for generic curves.

It is Zhang's remarkable observation that one can construct a degree zero divisor on a non-hyperelliptic genus four curve. More precisely, let $K$ be a field, $X$ a non-hyperelliptic curve genus four over $K$. Then $X$ is a complete intersection of $Q$ and $S$ on $\mathbb{P}^3$, where $Q$ is a (unique) irreducible quadric surface and $S$ is an irreducible cubic surface [Har1977, Chapter IV]. Assume that $Q$ is smooth. For any geometric point $p$ of $X$, there are two lines $l$ and $l'$ of in $Q$ passing through $p$, which intersect $X$ at two degree three divisors $D, D'$. Then $\Xi := D' - D$ is a degree zero divisor on $X$ which gives rise to a point in $\mathrm{Jac}(X)(\overline{K})$. It is clear that this point, i.e. the linear equivalence class of this divisor, up to sign, doesn't depend on the choice of $p$. This divisor class is defined over $K'$, a quadratic extension of $K$ obtained by adding square root of the discriminant of $Q$. Indeed, the base change $Q_{K'}$ of $Q$ to $K'$ is isomorphic to $\mathbb{P} \times \mathbb{P}'$ where $\mathbb{P}$ and $\mathbb{P}'$ are genus zero curve over $K'$. Then the image of the point $\omega_X \otimes \omega_{\mathbb{P}'}|_X \in \mathrm{Jac}(X)(K')$ in $\mathrm{Jac}(X)(\overline{K})$ is $\Xi$. When $Q$ is not smooth, we have the same discussion as above, except that $D$ and $D'$ coincide, hence the point $\Xi \in \mathrm{Jac}(X)(\overline{K})$ is zero. There are two basic questions:

1. Is this point torsion in the Jacobian?
2. When it is non-torsion, does it generate $\mathrm{Jac}(X)(K')$, at least up to torsion?

The aim of this paper is to study these two questions when $K$ is a global field, or when $K$ is the function field of the moduli space of genus four curve and $X$ is the universal genus four curve.

## 1.2 Statement of the main results

We now describe our main theorems. Let $k$ be a field with $\mathrm{char}\, k \neq 2$. Let $\mathcal{M}_4$ be the moduli space of curves of genus four over $k$ and $K$ its function field. Let $\mathcal{M}_{4,1} \to \mathcal{M}_4$ be the universal genus four curve over $\mathcal{M}_4$. Let $X \to \mathrm{Spec}\, K$ be the generic fiber of $\mathcal{M}_{4,1} \to \mathcal{M}_4$. Let $K'$ be the quadratic field extension of $K$ as in the previous section, over which the point $\Xi$ is defined.

**Theorem 1.2.1** (Theorem 2.1.5)**.** *The Mordell–Weil group* $\mathrm{Jac}(X)(K')$ *is of rank one. Moreover* $\mathrm{Jac}(X)(K') \otimes \mathbb{Q}$ *is generated by* $\Xi$.

We now let $k = \mathbb{F}_{p^n}$ be the finite field with $p^n$ elements and $p \neq 2$. The general genus four curve is a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$. The space of all $(3,3)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$ is a projective space $P$ of dimension 15. We consider the line bundle $\mathcal{L} = \mathcal{O}(d,3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the complete linear system $|\mathcal{L}|$ parameterizing hyperplane sections of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then each hyperplane section can be viewed as fibration in genus four curves over $\mathbb{P}^1$ via the first projection. We prove in Lemma 2.2.1 that there is an open subset $V$ of $|\mathcal{L}|$ such that the fibration over $\mathbb{P}^1$ is generically smooth, non-isotrivial, all fibers are irreducible and the total space is smooth. Suppose $s \in V(k)$, we denote by $J_s$ the Jacobian of the generic fiber. We prove in Lemma 2.2.3 that $\Xi$ is always non-torsion in $J_s(K)$ where $K$ it the function field of $\mathbb{P}^1$. Let

$$V(k, \Xi) = \{s \in V(k) \mid \Xi \text{ generates } J_s(K) \}.$$

Then we show that

**Theorem 1.2.2** (Corollary 2.4.5)**.** *Assume that $d \geq 2$. Then*

$$\lim_{\#k \to \infty} \frac{\#V(k, \Xi)}{V(k)} \geq \frac{1}{2}.$$

*If the Tate conjecture holds, then the inequality is an equality.*

We note that similar theorems for the plane curves of a fixed degree have been previously obtained by de Jong and Katz [dJK2000].

Let $k$ be a number field. Then we have a similar construction of the scheme $V$ over $k$ as above. Fix any height function $h$ on $V(\bar{k})$. Let $H$ be any real number. We let

$$V(H) = \{s \in V(\bar{k}) \mid h(s) \leq H\},$$
$$V(H, \Xi) = \{s \in V(H) \mid \Xi \text{ generates } J_s(K)\}.$$

**Theorem 1.2.3** (Theorem 2.4.8)**.** *We have*

$$\lim_{H \to \infty} \frac{\#V(H, \Xi)}{\#V(H)} = 1.$$

Theorem 1.2.2 and 1.2.3 show that the equidistribution behavior over a number field and a finite field is completely different.

From now on, let $K$ be a function field of a smooth projective curve $B$ or a number field in which case we let $B = \operatorname{Spec} \mathfrak{o}_K$ where $\mathfrak{o}_K$ is the ring of integers of $K$. A natural measure of the non-triviality of a point in $\operatorname{Jac}(X)(K)$ is the Neron–Tate height. We are going to give a brief review of the Neron–Tate height in Section 3.1. The last theorem of this paper is an expression of the Neron–Tate height of $\Xi$ in terms of the self-intersection of the relative dualizing sheaf of $X$ and local contributions when the reduction of $X$ satisfies certain conditions.

Let $\mathcal{X}$ be the stable model of $X$ over $B$. If $K$ is a function field, let $\omega$ be the dualizing sheaf of

$\mathcal{X} \to B$. If $K$ is a number field, let $\omega$ be the Arakelov dualizing sheaf, c.f. Section 3.1. Let $v$ be a place of $B$ and $R$ the completion of $B$ at the place $v$. Let $Y$ be the special fiber of $\mathcal{X}$ at $v$. We say that the reduction of $X$ at $v$ is simple if $Y$ is of one of the following types.

1. $Y$ is smooth. We denote by $h(X_v)$ the maximal integer $n$, such that $\mathcal{X}_n := \mathcal{X} \times_R R/\varpi^n$ is still hyperelliptic. In this case, $Y$ is said to be of type SMOOTH. A curve $Y \to R/\varpi^n$ is called hyperelliptic if there is an involution $\sigma : Y \to Y$ over $R/\varpi^n$, such that $Y/\langle\sigma\rangle$ is a $\mathbb{P}^1$-bundle over $R/\varpi^n$.

2. $Y$ is an irreducible nodal curve with a single node $p$ and its normalization is not hyperelliptic. In this case, $Y$ is said to be of type IRRED. We assume that the local equation of $\mathcal{X}$ at the node is $xy - \varpi^{\delta_0}$.

3. $Y$ has two components $C$ and $E$ meeting at a single node $p$, where $C$ is a non-hyperelliptic genus three curve and $E$ is an elliptic curve. In this case, $Y$ is said to be of type ELL. We assume that the local equation of $\mathcal{X}$ at the node $p$ is $xy - \varpi^{\delta_1}$.

4. $Y$ has two components $C_1$ and $C_2$ meeting at a single node $p$. Here both $C_1$ and $C_2$ are of genus two and $p$ is not a Weierstrauss point on either component. In this case, $Y$ is said to be of type TWO. We assume that the local equation of $\mathcal{X}$ at the node $p$ is $xy - \varpi^{\delta_2}$.

**Remark 1.2.4.** The condition that the reduction being simple is an open condition. More precisely, there is an open subscheme $U$ of $\overline{\mathcal{M}}_4$ whose completement is contained in the boundary of $\overline{\mathcal{M}}_4$ and of at least codimension two, such that $X$ has smooth non-hyperelliptic or simple reduction if and only if $Y$ corresponds to a point in $U$.

The theorem is

**Theorem 1.2.5** (Proposition 3.2.1, 3.4.1, 3.4.2)**.** *We have*

$$\operatorname{height} \Xi = 5\omega^2 - \sum_v \varphi_v, \tag{1.1}$$

*where the sum runs over all the places of $K$ and $\varphi_v$ is some constant depending only on $X_v$, the completion of $X$ at the place $v$. If the reduction of $X$ at $v$ is smooth and non-hyperelliptic, then*

$\varphi_v = 0$. *Moreover, if $v$ is non-archimedean and the reduction of $X$ at $v$ is simple, then*

$$\varphi_v = \begin{cases} 36h(X_v), & \textit{Type SMOOTH;} \\ 2\delta_0, & \textit{Type IRRED;} \\ 19\delta_1, & \textit{Type ELL;} \\ 27\delta_2, & \textit{Type TWO.} \end{cases}$$

## 1.3 Consequences of Theorem 1.2.5

As a corollary to Theorem 1.2.5 we get a lower bound of $\omega^2$ for fibred surfaces in genus four.

**Corollary 1.3.1.** *Suppose $B$ is a smooth projective curve and $\mathcal{X} \to B$ is a semistable curve of genus four. Assume that $\mathcal{X}$ is regular, the geometric generic fiber of $\mathcal{X} \to B$ is smooth non-hyperelliptic, the $\mathcal{X} \to B$ is not-isotrivial and all the reductions of $X$ is simple. Then*

$$5\omega^2_{\mathcal{X}/B} \geq 2\delta_0 + 19\delta_1 + 27\delta_2 + 36h.$$

This inequality is sharper than that of Moriwaki [Mor1998] under the stronger assumption that the generic fiber has simple reductions.

Now suppose that $B$ is a smooth projective curve over a finite field and $K$ is its function field. Let $X$ be a non-hyperelliptic curve over $K$ which can be extended to a non-isotrivial smooth family of non-hyperelliptic curves $\mathcal{X} \to B$. Then all the local invariants $\varphi(X_v)$ vanishes, and we obtain a simple identity

$$h(\Xi) = 5\omega^2_{\mathcal{X}/B}.$$

Note that in this case the self-intersection of $\omega_{\mathcal{X}/B}$ is strictly positive [Szp1981, Théorème 2], which means $\Xi$ is not torsion. This shows that

$$\operatorname{rank} \operatorname{Jac}(X)(K') \geq 1.$$

One also looks at the Hasse–Weil $L$-function $L(\operatorname{Jac}(X_{K'}), s)$ of the Jacobian of the curve $X_{K'}$. Since $K$ is a function field, one has

$$\operatorname{ord}_{s=1} L(\operatorname{Jac}(X_{K'}), s) \geq \operatorname{rank} \operatorname{Jac}(X)(K') \geq 1.$$

However, as $X$ has good reduction at all the places of $K$, the sign of the functional equation of $L(\mathrm{Jac}(X_{K'}), s)$ is $+1$. This shows

**Corollary 1.3.2.** *Let $K$ be the function field of a smooth projective curve $B$ over a finite field, and let $X$ be a non-hyperelliptic curve over $K$ which can be extended to a non-isotrivial smooth family of non-hyperelliptic curves over $B$. Then*

$$\mathrm{ord}_{s=1} L(\mathrm{Jac}(X_{K'}), s) \geq 2.$$

In view of Birch–Swinnerton-Dyer conjecture, we make the following conjecture.

**Conjecture 1.3.3.** *Let $K$ be as above and $B$ be as above. Then*

$$\mathrm{rank}\,\mathrm{Jac}(X)(K') \geq 2.$$

It seems difficult to find this "extra" point in $\mathrm{Jac}(X)(K')$.

The non-triviality of $\Xi$ in this case also gives rise to the so-called Northcott property.

**Corollary 1.3.4.** *Let $K$ be a number field Let $H$ and $D$ be two positive real numbers. Suppose $X \to T$ is a non-isotrivial family of smooth non-hyperellitpic genus four curves with $T$ being a proper curve over $K$. Then the set*

$$\{t \in T(\overline{K}) \mid \deg t \leq D,\ \text{height}\,(\Xi_t) \leq H\}$$

*is finite. Here $\deg t$ stands for the degree of the field of the definition of $t$ over $K$ and $\Xi_t$ is the divisor class $\Xi$ on the fiber $X_t$.*

*Proof.* This follows from the fact that $\Xi$ is not torsion on the generic fiber of $X \to T$ and Silverman's specialization theorem [Sil1983, Theorem C]. □

**Remark 1.3.5.** One should note that if $T$ is a proper subvariety of $\mathcal{M}_4$ which does not meet the hyperelliptic locus, then $\dim T \leq 1$.

## 1.4 Organization of this thesis

This thesis has six chapters and is divided into three parts.

The first part is chapter 2 which is devoted to the proof of Theorem 1.2.1 and 1.2.2. The view point we take in this chapter is to put $\Xi$ in families and study how $\Xi$ varies in families. After reviewing some general fact about the moduli space of genus four curves, we prove Theorem 1.2.1 by studying the monodromy of families of $(3,3)$-curves in $\mathbb{P}^1 \times \mathbb{P}^1$. This proof is similar to the proof of Noether's theorem [Del1973, Théorème 1.3]. Then we study the equidistribution of the rank of the Neron–Severi group of the hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ following the technique of de Jong and Katz [dJK2000]. We prove Theorem 1.2.2 at the end of this chapter.

The second part consists of Chapter 3, 4 and 5 and is devoted to the proof of Theorem 1.2.5. We start Chapter 3 by reviewing the notion of heights and recall the Hodge index theorem, which enables us to compute the height of $\Xi$ via intersection theory. Then in Section 3.2 we decompose the height of $\Xi$ into the form (1.1). To compute the constant $\varphi_v$, we describe explicitly the integral model of $X$ in detail when $X$ has simple reductions. This is the content of Section 3.3. Finally in Section 3.4, we reduce the computation of the height of $\Xi$ to two technique propositions, Proposition 3.4.1 and 3.4.2.

The proof of Proposition 3.4.1 and 3.4.2 is the most technical part of this thesis. In Chapter 4, we prove Proposition 3.4.1 via a dedicate computation on surfaces. In Chapter 5, we prove Proposition 3.4.2. We study in Section 5.1 a double cover of the moduli space $\mathcal{M}_4$ whose sheaf of relative differentials is closely related to $\Xi$. The final results amount to give an embedded resolution of the Petri locus in $\mathcal{M}_4$. See Section 2.1 for the definition of the Petri locus. We then prove Proposition 3.4.2 by "pulling back" our results from the moduli space.

The last chapter forms the third part of this thesis. This chapter is more speculative. It summarizes the questions left open in this thesis and raises some conjectures.

## 1.5 Notation and conventions

Through out this thesis, we use the following notation and terminologies.

- A curve is a one-dimensional reduced scheme which is proper over a field.
- Let $X$ be a scheme. Let $\mathrm{Pic}(X)$ be the Picard group of $X$. If $D$ is a Cartier divisor on $X$, then we denote its class in $\mathrm{Pic}(X)$ by $[D]$.
- Let $D$ be a Cartier divisor, then $|D|$ is the complete linear system associated to $D$. Similar notation for any line bundle $L$. We use $\mathfrak{g}^r_d$ to denote a linear system of degree $d$ and dimension $r$.
- For any scheme $S$ and any morphism $S$-scheme $X$ and $T$, we use $X_T$ to denote the base change $X_T = X \times_S T$.

- For any morphism $f : X \to Y$ and any line bundle $L$ on $Y$, we shall denote $f^*L$ by $L|_X$.
- Let $f : X \to S$ be a projective morphism. Let $\mathcal{F}$ be a coherent sheaf on $X$. Let $\det \mathrm{R}f_*\mathcal{F}$ be the line bundle on $S$ defined as in [KM1976]. If $\mathrm{R}^if_*\mathcal{F}$ is locally free for any $i$, then

$$\det \mathrm{R}f_*\mathcal{F} = \bigotimes_i (\det \mathrm{R}^if_*\mathcal{F})^{(-1)^i}.$$

  If $S = \operatorname{Spec} A$ is affine, this is also denoted by $\det \mathrm{H}^*(X, \mathcal{F})$, which is a projective $A$-module.
- For any flat morphism $f : X \to S$ of relative dimension one between normal integral schemes $X$ and $S$, we denote the Deligne pairing [Del1987, Zha1996] by

$$\langle -, - \rangle : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathrm{Pic}(S).$$

  In this case, the Deligne's pairing can be defined as follows. Let $L_1, L_2 \in \mathrm{Pic}(X)$, then

$$\langle L_1, L_2 \rangle := \det \mathrm{R}f_*(L_1 \otimes L_2) \otimes \det \mathrm{R}f_*(L_1^{-1}) \otimes \det \mathrm{R}f_*(L_2^{-1}) \otimes \det \mathrm{R}f_*\mathcal{O}_X.$$

# Chapter 2

# The generic case

## 2.1 Moduli space of genus four curves

Let $\mathcal{M}_4$ be the moduli space of genus four curves. Let $\overline{\mathcal{M}}_4$ be the moduli space of Deligne–Mumford semistable curves [DM1969]. This is a proper smooth stack over $\operatorname{Spec}\mathbb{Z}$. The complement of $\mathcal{M}_4$ in $\overline{\mathcal{M}}_4$ consists of three irreducible components:

$$\overline{\mathcal{M}}_4 \backslash \mathcal{M}_4 = \Delta_0 \cup \Delta_1 \cup \Delta_2.$$

The generic point of $\Delta_0$ corresponds to a curve with a single node whose normalization is connected. The general point of $\Delta_i$ ($i = 1, 2$) corresponds to a curve that has a single node whose normalization consists of two smooth curves with genus $i$ and $4 - i$ respectively.

Let $\pi : \mathcal{M}_{4,1} \to \mathcal{M}_4$ be the universal genus four curve. Then we define the Hodge bundle

$$\lambda = \det \pi_* \omega,$$

where $\omega = \omega_{\mathcal{M}_{4,1}/\mathcal{M}_4}$ stands for the relative dualizing sheaf. It is known that $\operatorname{Pic}(\overline{\mathcal{M}}_4)$ is generated by $\lambda$ and $\Delta_i$ ($i = 0, 1, 2$). It is also known that the natural restriction map

$$\operatorname{Pic}(\mathcal{M}_4) \to \operatorname{Pic}(\mathcal{M}_{4,\mathbb{C}})$$

has finite kernels and cokernels. Therefore it is an isomorphism after tensoring $\mathbb{Q}$.

**Lemma 2.1.1.** *We have*

$$\langle \omega, \omega \rangle \simeq \lambda^{\otimes 12}(-\Delta_0 - \Delta_1 - \Delta_2)$$

*on* $\overline{\mathcal{M}}_4$.

This isomorphism will be referred to as the Mumford isomorphism. See [Mum1977] for a proof.

We recall the following lemma.

**Lemma 2.1.2.** *Let $X$ be a smooth non-hyperelliptic genus four curve over field $k$. Let $X \to \mathbb{P}^3$ be the canonical embedding. Then $X$ lies on a unique irreducible quadric surface in $\mathbb{P}^3$ and is a complete intersection of the quadric surface and an irreducible cubic surface.*

*Proof.* Consider the exact sequence

$$0 \to \mathcal{I}_X \otimes \mathcal{O}(2) \to \mathcal{O}(2) \to \mathcal{O}(2)|_X \to 0,$$

where $\mathcal{I}_X$ is the defining ideal of $X$ in $\mathbb{P}^3$. Then taking long exact sequence, we get

$$0 \to \mathrm{H}^0(\mathbb{P}^3, \mathcal{I}_X \otimes \mathcal{O}(2)) \to \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(2)) \to \mathrm{H}^0(X, \mathcal{O}(2)|_X).$$

Note that $\dim \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(2)) = 10$ and $\dim \mathrm{H}^0(X, \mathcal{O}(2)|_X) = \dim \mathrm{H}^0(X, \omega_X^2) = 9$. Therefore there is at least one nonzero section of $\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}_X \otimes \mathcal{O}(2))$. This section defines a quadric surface $Q$ in $\mathbb{P}^3$ that $X$ lies on. Moreover, this surface is integral since a non-hyperelliptic genus four curve can never be a plane curve. The curve $X$ cannot be contained in two distinct irreducible quadric surfaces $Q$ and $Q'$ since the intersection of $Q$ and $Q'$ is a curve of degree four while $X$ is of degree six. This moreover shows that the map $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(2)) \to \mathrm{H}^0(X, \mathcal{O}(2)|_X)$ is surjective.

We consider similarly the exact sequence

$$0 \to \mathcal{I}_X \otimes \mathcal{O}(3) \to \mathcal{O}(3) \to \mathcal{O}(3)|_X \to 0,$$

and conclude that $\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}_X \otimes \mathcal{O}(3))$ is at least five dimensional. The cubic forms consisting of the quadratic form above times a linear form, forms a subspace of dimension four. So $X$ is contained in an irreducible cubic surface $F$. Then $X$ is contained in the intersection of $Q \cap F$. They are both of degree six in $\mathbb{P}^3$. We then conclude that $X$ is this complete intersection. □

Let $\mathcal{H}_4$ be the hyperelliptic locus in $\mathcal{M}_4$ and $\mathcal{M}_4^\circ$ the open substack of $\mathcal{M}_4$ parameterizing non-hyperelliptic curves. Let $Z$ be the Petri locus in $\mathcal{M}_4^\circ$ and $\overline{Z}$ the closure of it in $\overline{\mathcal{M}}_4$. By definition, the divisor $Z$ is the locus in $\mathcal{M}_4^\circ$ consists of non-hyperelliptic curves whose canonical embedding lies on a singular quadric surface in $\mathbb{P}^3$. Note that the closure of $Z$ in $\mathcal{M}_4$ contains the hyperelliptic locus. We shall refer to the closure of $Z$ in $\mathcal{M}_4$ as the Petri locus on $\mathcal{M}_4$ and also denote it by $Z$. There should not be any confusion since the moduli space in question is always clear from the context.

We now give another description of the locus $Z$. It follows from the proof of the above lemma that there is a surjective morphism

$$\mathrm{Sym}^2 \pi_*\omega \to \pi_*\omega^{\otimes 2}$$

on $\mathcal{M}_4^\circ$. Let $\mathcal{L}$ be the kernel of this morphism. Then there is a discriminant morphism

$$\mathrm{disc} : \mathcal{L}^{\otimes 4} \to \lambda^{\otimes 2},$$

which is given as follows. Locally if $fQ$ is a section of $\mathrm{Sym}^2 \pi_*\omega$ where $f$ is a function and

$$Q = \sum_{i,j=1}^{4} q_{ij} \alpha_i \otimes \alpha_j,$$

where $\{\alpha_i : \ i = 1, 2, 3, 4\}$ is a basis of $\pi_*\omega$ and $(q_{ij})$ is a symmetric matrix, then

$$\mathrm{disc}(fQ) = f \det q_{ij} (\wedge_i \alpha_i)^{\otimes 2}.$$

The Petri locus is then the zero locus of the discriminant morphism.

**Lemma 2.1.3.** *We have* $\mathcal{O}(Z) \simeq \lambda^{\otimes 34}$ *on* $\mathcal{M}_4$.

*Proof.* From the above description of $Z$, one sees that

$$\mathcal{O}(Z) \simeq \lambda^2 \otimes (\det \mathrm{Sym}^2 \pi_*\omega)^{-4} \otimes (\det \pi_*\omega^{\otimes 2})^4.$$

There is a canonical isomorphism

$$\det \mathrm{Sym}^2 \pi_*\omega \simeq \lambda^5.$$

It follows from Mumford's isomorphism that

$$\det \pi_*\omega^{\otimes 2} \simeq \lambda^{13}.$$

The lemma then follows. □

The divisor class of $\overline{Z}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_4)$ is obtained by Eisenbud–Harris [EH1987].

**Proposition 2.1.4.** *In* $\mathrm{Pic}(\overline{\mathcal{M}}_4) \otimes \mathbb{Q}$*, one has*

$$[\overline{Z}] = 34\lambda - 4[\Delta_0] - 14[\Delta_1] - 18[\Delta_2].$$

In the following, a stable curve that corresponds to a point in $\overline{Z}$ will be called Petri special, otherwise it will be called Petri general.

From now on, we work over a field $k$ with $\mathrm{char}\, k \neq 2$.

Let $\mathcal{J} \to \mathcal{M}_4$ be the universal Jacobian over $\mathcal{M}_4$. Every point of $\mathcal{M}_4^\circ$ corresponds to a curve whose canonical embedding lines on a smooth quadric surface. It then follows that this curve is of type $(3,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. A $(3,3)$ curve is given by a homogeneous equation

$$\sum_{i,j=0}^{3} a_{ij} x_0^i x_1^{3-i} y_0^j y_1^{3-j} = 0.$$

The space of $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathbb{P}[a_{ij} : i,j : 0, \cdots 3]$ which is isomorphic to $\mathbb{P}^{15}$. The group $\Gamma = \mathrm{PGL}_2 \times \mathrm{PGL}_2 \rtimes \{\pm 1\}$ acts on $\mathbb{P}[a_{ij}]$ where $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts by changing variables and $\{\pm 1\}$ acts by $a_{ij} \mapsto a_{ji}$. There is an open subscheme $P_0$ of $\mathbb{P}[a_{ij}]$ which parameterizes smooth $(3,3)$ curves. Then $\mathcal{M}_4^\circ$ is the quotient of $P_0$ by $\Gamma$ (as a stack).

Let $\mathfrak{X} \to \mathbb{P}[a_{ij}]$ the universal $(3,3)$ curve. There is a embedding $j : \mathfrak{X} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}[a_{ij}]$ over $\mathbb{P}[a_{ij}]$. Let $\mathfrak{X}_0 \to P_0$ be the universal smooth $(3,3)$ curve and $\mathfrak{J} \to P_0$ its Jacobian. Let $L$ be the function field of $\mathbb{P}[a_{ij}]$. Then the line bundle $\Xi_P = j^*\mathcal{O}(1,-1)$ on $\mathfrak{X}$ defines an $L$-point of $\mathfrak{J}$.

**Theorem 2.1.5.** *The Mordell–Weil group* $\mathfrak{J}(L)$ *is of rank one and* $\mathfrak{J}(L) \otimes \mathbb{Q} \simeq \mathbb{Q}.\Xi_P$*.*

**Corollary 2.1.6.** *Let* $K$ *be the function field of* $\mathcal{M}_4$ *and* $\mathcal{J}$ *the generic fiber of the universal Jacobian. Then* $\mathcal{J}(K') \otimes \mathbb{Q} \simeq \mathbb{Q} \simeq \mathbb{Q}.\Xi$*. Here* $K'$ *and* $\Xi$ *are as in the introduction.*

*Proof.* We observe that $L$ is a (transcendental) extension of $K'$. In fact, the field $K'$ is the function field of $[\mathbb{P}[a_{ij}]/\mathrm{PGL}_2 \times \mathrm{PGL}_2]$. Note also that $\mathcal{J} \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \simeq \mathfrak{J}$. Thus there is an injective map $\mathcal{J}(K') \to \mathfrak{J}(L)$. The image of the point $\Xi$ is $\Xi_P \in \mathfrak{J}(L)$ (up to a sign). The corollary then follows from the theorem. $\square$

## 2.2 Curves on quadric surface

In order to prove Theorem 2.1.5, we study the one dimensional families of $(3,3)$-curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}(d,3,3)$ be a line bundle on $Y$. Assume that $d \geq 2$. This is a very ample line bundle on $Y$, which induces an embedding $Y \to \mathbb{P}^N$. The linear system $|\mathcal{L}|$ parameterizes all hyperplane sections of $Y$. Let $U \subset |\mathcal{L}|$ be the open subset parameterizing smooth hyperplane sections of $Y$. Let $\mathbb{X} \to U$ be the universal hyperplane section. Then by construction, there is a tautological morphism $\mathbb{X} \to Y \times U$ over $U$.

Each hyperplane section $X$ of $Y$ is viewed as a fibration over $\mathbb{P}^1$ via the first projection.

**Lemma 2.2.1.** *There is an open subset $V \subset U$, such that any $X \in V$, as a fibration over $B$, satisfies:*

1. *It is not isotrivial.*

2. *All the fibers are irreducible.*

*Proof.* First of all, there is an open subset $U_0$ of $U$, such that all the fibers of $X \in U_0$ are curves, i.e. the morphism $\pi : X \to \mathbb{P}^1$ is flat. There is an open subset $U_1$ of $U_0$, such that all the fibers of $X$ are irreducible. Let $P(m,n)$ be the space of curves of type $(m,n)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then those reducible $(3,3)$ curves form a proper algebraic subset $R$ of $P(3,3)$, namely the union of $P(a,b) \times P(3-a,3-b)$, where $0 \leq a \leq 3$ and $0 \leq b \leq 3$ with $a$ and $b$ not identically 0 or 3.

Any point in $U_0$ corresponds to a morphism from $\mathbb{P}^1$ to $P(3,3)$, thus we get a tautological morphism $U_0 \times \mathbb{P}^1 \to P(3,3)$. This morphism is dominant. Let $Z$ be the inverse image of $R$. This is a proper algebraic subset of $U_0 \times B$ such that $(X,x) \in Z$ if and only if the fiber of $\pi$ over $x \in \mathbb{P}^1$ is reducible. Let $W$ be the image of $Z$ in $U_0$. Since $\mathbb{P}^1$ is proper, $W$ is proper closed in $U_0$. Thus $U_1 = U_0 \backslash W$ is the subset of $U_0$ such that all the fibers of $X \to \mathbb{P}^1$ are irreducible.

Next we show that there's an open subset $U_2$ of $U_0$, such that $X \in U_2$ if and only it is not isotrivial. The fibration $X \to \mathbb{P}^1$ is isotrivial if and only if the image of the corresponding morphism $B \to P(3,3)$ is zero dimensional. This forms a proper algebraic subset of $|\mathcal{L}|$.

Now $V = U_1 \cap U_2$ is the desired subset in the lemma. $\square$

The hyperplane sections $X$ of $Y$ that satisfies the two conditions in the lemma are call good hyperplane sections.

Let $X$ be a hyperplane section of $Y$ which corresponds to a point in $V$. Let $L$ be the function field of $\mathbb{P}^1$, $s = \operatorname{Spec} L$ the generic point of $\mathbb{P}^1$ and $X_s$ the generic fiber. Let $J$ be the Jacobian of $X_s$ and $\rho$ be the rank of the Mordell–Weil group $J(L)$. This is finite since the fibration $X \to \mathbb{P}^1$ is not isotrivial.

The following lemma is well-known, see [Tat1995].

**Lemma 2.2.2.** *For surface $X \in V$, we have*

$$\dim_{\mathbb{Q}} \mathrm{NS}(X) \otimes \mathbb{Q} = 2 + \rho. \tag{2.1}$$

The divisor class $\Xi$ we have introduced in the introduction is $\Xi = \mathcal{O}(0, 1, -1)|_X$.

**Lemma 2.2.3.** *If $X \in V$, then $\Xi$ is not torsion in $J(L)$.*

*Proof.* The self-intersection on $-6d$. The lemma follows from the Hodge index theorem for surfaces and that the fibers of $X \to \mathbb{P}^1$ are all irreducible. □

Let $\eta$ be the generic point of $U$ and $\overline{\eta}$ be the geometric generic point. Let $X_{\overline{\eta}}$ be the geometric generic fiber of $\mathbb{X} \to U$. The tautological morphism $\mathbb{X} \to Y \times U$ over $U$ induces the morphism on the generic fiber $X_{\overline{\eta}} \to Y_{\overline{\eta}}$.

**Proposition 2.2.4.** *The canonical restriction map*

$$\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to \mathrm{NS}(X_{\overline{\eta}}) \otimes \mathbb{Q}$$

*is an isomorphism.*

Proposition 2.2.4 will be proved in the next section.

*Proof of Theorem 2.1.5 assuming Proposition 2.2.4.* It follows from Proposition 2.2.4 that $\mathrm{NS}(X_{\overline{\eta}})$ is of rank three. Let $\mathrm{pr}_1 : X_{\overline{\eta}} \to \mathbb{P}^1_{\overline{\eta}}$ be the first projection and let $J_{\overline{\eta}}$ be the Jacobian of the generic fiber. Let $L'$ be the function field of $\mathbb{P}^1_{\overline{\eta}}$. Then it follows from Lemma 2.2.2 and 2.2.3 that the rank of $J(L')$ is one and is generated by $\Xi$ after tensoring with $\mathbb{Q}$.

By construction, there is a canonical morphism

$$V \times \mathbb{P}^1 \to \mathbb{P}[a_{ij}].$$

This morphism is clearly dominant. Therefore $L'$ is an extension of $L$. It follows that $\mathfrak{J}(L)$ is a subset of $J_{\overline{\eta}}(L')$. It is clear that the point $\Xi_{\overline{\eta}}$ is the image of $\Xi_P$ under the inclusion $\mathfrak{J}(L) \to J_{\overline{\eta}}(L')$. Theorem 2.1.5 is thus proved. □

## 2.3 Monodromy

We keep the notation from the previous section.

We consider the local system $\mathrm{R}^2\phi_*\mathbb{Q}_l(1)$ on $U$ where $\phi : \mathbb{X} \to U$ is the structure morphism. The tautological embedding $\mathbb{X} \to Y \times U$ induces a map

$$\mathrm{H}^2(Y_{\overline{\eta}}, \mathbb{Q}_l(1)) \to \mathrm{H}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1)),$$

which is $\pi_1(U, \overline{\eta})$-equivariant. In fact, $\mathrm{H}^2(Y_{\overline{\eta}}, \mathbb{Q}_l(1))$ is the fixed part of $\mathrm{H}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))$ under the action of $\pi_1(U, \overline{\eta})$. The orthogonal complement of $\mathrm{H}^2(Y_{\overline{\eta}}, \mathbb{Q}_l(1))$ is denoted by $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))$. It is the subspace of $\mathrm{H}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))$ generated by the vanishing cycles.

There is a non-degenerate symmetric pairing $\langle -, - \rangle$ on $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))$. The image of $\pi_1(U, \overline{\eta})$ in $\mathrm{GL}(\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1)))$ actually lies in $\mathrm{O}(\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1)))$. Moreover, this representation is absolutely irreducible. This is classical if $\mathrm{char}\, k = 0$ (c.f. [Voi2007, Corollary 3.28]) and proved by Deligne [Del1974, Corollary 5.5] when $\mathrm{char}\, k > 0$.

We denote by $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}}$ the subspace of $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}}$ generated by algebraic cohomology class. Here "algebraic" means the cohomology classes of algebraic cycles on $X_{\overline{\eta}}$. This subspace is again preserved by the action of $\pi_1(U, \overline{\eta})$. Therefore it is either the whole space or it is empty.

**Lemma 2.3.1.** *If* $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1)) = \mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}}$*, then the image of* $\pi_1(U, \overline{\eta})$ *is finite.*

*Proof.* Recall that $L'$ is the function field of $U$. The canonical map

$$\mathrm{Gal}(\overline{L'}/L') \to \pi_1(U, \overline{\eta})$$

is surjective [Gro1971a, Exposé IX, Corollaire 5.6]. Every algebraic cycle is defined on $X_{\overline{\eta}}$ is defined over some finite extension of $L'$. Therefore it is stabilized by an open subgroup of $\mathrm{Gal}(\overline{L'}/L')$ of finite index. Moreover the space $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))$ is finite dimensional. We then conclude that the action of $\mathrm{Gal}(\overline{L'}/L')$ factors through an open subgroup of finite index. The image is thus finite. □

**Lemma 2.3.2.** *If* $\mathrm{char}\, k = 0$*, then* $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}} = 0$ *and the image of* $\pi_1(U, \overline{\eta})$ *is not finite.*

*Proof.* We may assume that $k = \mathbb{C}$ by the Lefchetz principle. Suppose $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}} \neq 0$, then by the previous lemma, the image of $\pi_1(U, \overline{\eta})$ is finite. We fix an embedding $\iota : \mathbb{Q}_l \to \mathbb{C}$ and denote by $\mathrm{Ev}(X_{\overline{\eta}})$ the subspace of $\mathrm{H}^2(X_{\overline{\eta}}, \mathbb{C})$ generated by $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))$ via this embedding. Then by [Del1973, Proposition 3.4],

$$\mathrm{Ev}^2(X_{\overline{\eta}}) \subset \mathrm{H}^{1,1}(X_{\overline{\eta}}).$$

So to get the contradiction, we only have to check that

$$\mathrm{H}^{2,0}(X_{\overline{\eta}}) \cap \mathrm{Ev}^2(X_{\overline{\eta}}) \neq 0.$$

To simplify notation, we write $X$ (resp. $Y$) instead of $X_{\overline{\eta}}$ (resp. $Y_{\overline{\eta}}$). Denote by $j : X \to Y$ the embedding. Denote by $\mathcal{N}_{X/Y}$ the normal bundle of $X$ in $Y$. Then $\mathcal{N}_{X/Y} \simeq j^*\mathcal{L}$. By adjunction formula, $\omega_X \simeq j^*\mathcal{O}(d-2,1,1)$. Then

$$\mathrm{H}^{2,0} = \mathrm{H}^0(X, \omega_X) \simeq \mathrm{H}^0(Y, \mathcal{O}(d-2,1,1) \otimes j_*\mathcal{O}_X).$$

By definition we have

$$0 \to \mathcal{O}_Y(-X) \to \mathcal{O}_Y \to j_*\mathcal{O}_X \to 0,$$

where $\mathcal{O}_Y(-X) = \mathcal{O}(-d,-3,-3)$. Therefore

$$0 \to \mathcal{O}(-2,-2,-2) \to \mathcal{O}(d-2,1,1) \to \mathcal{O}(d-2,1,1) \otimes j_*\mathcal{O}_{\mathcal{X}} \to 0.$$

Taking long exact sequence gives

$$0 \to \mathrm{H}^0(Y, \mathcal{O}(d-2,1,1)) \to \mathrm{H}^0(Y, \mathcal{O}(d-2,1,1) \otimes j_*\mathcal{O}_{\mathcal{X}}).$$

By Künneth formula we get

$$\dim \mathrm{H}^0(Y, \mathcal{O}(d-2,1,1)) = 4 \dim \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(d-2)) \geq 4.$$

Here we used the assumption that $d \geq 2$. This yields $\dim \mathrm{H}^{2,0}(\mathcal{X}) \geq 4$.

Since $\dim \mathrm{H}^2(Y, \mathbb{C}) = 3$ by Künneth formula, we know that $\mathrm{Ev}^2(X)$ is of codimension 3 in $\mathrm{H}^2(X, \mathbb{C})$. Therefore $\mathrm{H}^{2,0}(X) \cap \mathrm{Ev}^2(X) \neq 0$. This proves the lemma. □

**Lemma 2.3.3.** *If* $\mathrm{char}\, k \neq 2$*, then* $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}} = 0$.

*Proof.* We deduce the lemma from its characteristic zero counterpart. This argument is adapted from [Del1973, § 3.6]. Suppose $\mathrm{Ev}^2(X_{\overline{\eta}}, \mathbb{Q}_l(1))^{\mathrm{alg}} \neq 0$. Let $R$ be a discrete valuation ring of unequal characteristic with residue field $k$ and fraction field $K$. The scheme $Y$ and the line bundle $\mathcal{L}$ are both defined over $R$. We consider the universal smooth hyperplane section $\mathbb{X}_R \to U_R$ of $Y$ over $R$. We denote the universal hyperplane section over $K$ (resp. $k$) by $\mathbb{X}_K \to U_K$ (resp. $\mathbb{X}_k \to U_k$), and the generic fiber by $X_K$ (resp. $X_k$). The generic point of $U_K$ (resp. $U_k$) is denoted by $\eta_K$ (resp. $\eta_k$).

The representation

$$\pi_1(U_k, \overline{\eta_k}) \to \mathrm{O}(\mathrm{Ev}^2(X_k, \mathbb{Q}_l(1)))$$

defines a local system $\underline{\mathrm{Ev}_k}$ on $U_k$. The closure of the image of this monodromy action do not change is we restrict $\underline{\mathrm{Ev}_k}$ to a general Lefchetz pencil $\ell_k$. Here "general" means $\ell_k$ lies in an open subset of the Grassmanian of the lines in $|\mathcal{L}_k|$. The same is true if we replace $k$ by $\overline{K}$ in all the statements above.

We consider a general Lefchetz pencil $\ell$ over $R$. The exceptional locus $T \subset \ell$ is etale over $R$. Note that we have used the fact that $\operatorname{char} k \neq 2$ here. The monodromy group of $\ell$ over the generic point of $\operatorname{Spec} R$ and the closed point of $\operatorname{Spec} R$ are the same. We conclude then that image of

$$\pi_1(U_{\overline{K}}, \overline{\eta_k}) \to \mathrm{O}(\mathrm{Ev}^2(X_{\overline{K}}, \mathbb{Q}_l(1)))$$

is finite. This contradicts Lemma 2.3.2. □

*Proof of Proposition 2.2.4.* To simplify notation, we write $X$ (resp. $Y$) instead of $X_{\overline{\eta}}$ (resp. $Y_{\overline{\eta}}$). By the weak Lefchetz theorem, the map

$$\mathrm{NS}(Y) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$$

is injective. The surjectivity of this map follows from the orthogonal decomposition

$$\mathrm{H}^2(X, \mathbb{Q}_l(1)) = \mathrm{H}^2(Y, \mathbb{Q}_l(1)) \oplus \mathrm{Ev}^2(X, \mathbb{Q}_l(1))$$

and the fact that there is no algebraic cohomology class in $\mathrm{Ev}^2(X, \mathbb{Q}_l(1))$. □

## 2.4 Equidistribution

We work over the a field $k$ with $\operatorname{char} k \neq 2$ in this section.

Recall that $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}(d, 3, 3)$. Let $X$ be a hyperplane section of $Y$ in $|\mathcal{L}|$.

**Lemma 2.4.1.** *For any $d \geq 1$, we have*

$$\dim \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_l(1)) = 34d - 14, \quad \dim \mathrm{Ev}^2(X_{\overline{k}}, \mathbb{Q}_l(1)) = 34d - 17.$$

*Proof.* Note that $\dim \mathrm{H}^0(X_{\overline{k}}, \mathbb{Q}_l(1)) = \dim \mathrm{H}^4(X_{\overline{k}}, \mathbb{Q}_l(1)) = 1$, $\dim \mathrm{H}^1(X_{\overline{k}}, \mathbb{Q}_l(1)) = \mathrm{H}^3(X_{\overline{k}}, \mathbb{Q}_l(1)) =$

0 by the weak Lefchetz theorem. Therefore

$$\chi(X_{\overline{k}}, \mathbb{Q}_l) = 2 + \dim \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_l(1)).$$

Let $d = 1$. Then we may view $X_{\overline{k}}$ has the total space of the Lefchetz pencil of hyperplane sections of $\mathbb{P}^1 \times \mathbb{P}^1$ embeded in $\mathbb{P}^{15}$ via the very ample line bundle $\mathcal{O}(3,3)$. Therefore $X$ is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 18 points. We conclude then that

$$\chi(X_{\overline{k}}, \mathbb{Q}_l) = \chi(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}_l) + 18 = 22.$$

Therefore $\dim \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_l(1)) = 20$.

We can also compute $\chi(X_{\overline{k}}, \mathbb{Q}_l)$ via the Lefchetz fibration $X_{\overline{k}} \to \mathbb{P}^1$. Let $S$ be the number of singular fibers. Then

$$\chi(X_{\overline{k}}, \mathbb{Q}_l) = (2 - S)\chi(\text{smooth fiber}) + S \cdot \chi(\text{singular fiber}).$$

Since the singular fiber is smooth except for an ordinary double point, the Euler characteristic is one plus the Euler characteristic of the smooth fiber. Then we have

$$\chi(X_{\overline{k}}, \mathbb{Q}_l) = 2 \cdot \chi(\text{smooth fiber}) + S = S - 12.$$

Therefore $S = 34$.

Now let $d \geq 2$. We may compute $\chi(X_{\overline{k}}, \mathbb{Q}_l)$ via the pullback of a Lefchetz pencil via a degree $d$ finite flat morphism $\mathbb{P}^1 \to \mathbb{P}^1$ which is etale over all the exceptional locus on the Lefchetz pencil. This fibration then has $34d$ singular fibers. Each singular fiber is irreducible and has a unique ordinary double point. Then

$$\begin{aligned}\chi(X_{\overline{k}}, \mathbb{Q}_l) &= (2 - 34d)\chi(\text{smooth fiber}) + 34d \cdot \chi(\text{singular fiber}) \\ &= 34d - 12.\end{aligned}$$

Therefore

$$\dim \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_l(1)) = 34d - 14.$$

□

From now on, we take $k = \mathbb{F}_{p^n}$ to be a finite field with $p^n$ elements. Assume that $d \geq 2$ as before. Let $\phi : \mathbb{X} \to V$ be the universal good hyperplane section of $Y$, c.f. 2.2.1. We consider the local system $E$ on $V$ which is the orthogonal complement of $\mathrm{R}^2 \operatorname{pr}_* \mathbb{Q}_l(1)$ in $\mathrm{R}^2\phi_*\mathbb{Q}_l(1)$, where $\mathrm{pr} : Y \times V \to V$ is the second projection. The cup product pairing on $\mathrm{R}^2\phi_*\mathbb{Q}_l(1)$ restricted to $E$ is non-degenerate. Let $s \in V(k)$ be any point and $\overline{s} = \operatorname{Spec} \overline{k} \to V$ be a geometric point over $s$. Then the image of the monodromy action

$$\pi_1(V, \overline{s}) \to \mathrm{GL}(E_{\overline{s}})$$

actually lies in $\mathrm{O}(E_{\overline{s}})$.

**Theorem 2.4.2** ([Del1980, Théorème 4.4.1, dJK2000, Theorem 7.5])**.** *The closure of the image of* $\pi_1(V_{\overline{k}}, \overline{s})$ *is* $\mathrm{O}(E_{\overline{s}})$.

Let $\mathrm{Frob}_s$ be the image of the Frobenius conjugacy class in $\mathrm{O}(E_{\overline{s}})$. By [Del1974], the reverse characteristic polynomial $\det(1 - t\,\mathrm{Frob}_s \mid E_{\overline{s}})$ has coefficient in $\mathbb{Q}$ and is independent of $l$. If we fix an embedding $\mathbb{Q}_l \to \mathbb{C}$, the roots of this polynomial all lie on the unit circle. Therefore, there is a unique conjugacy class $\theta(s)$ in $\mathrm{O}(34d-17)$ with the same reverse characteristic polynomial. Here $\mathrm{O}(34d-17)$ is the compact real orthogonal group of rank $34d - 17$.

**Lemma 2.4.3.** *Let* $V(k,+)$ *(resp.* $V(k,-)$*) be the subset of* $V(k)$ *such that* $\det(-\theta(s)) = 1$ *(resp.* $\det(-\theta(s)) = -1$*). Then*

$$\lim_{\#k\to\infty} \frac{\#V(k,+)}{V(k)} = \lim_{\#k\to\infty} \frac{\#V(k,-)}{V(k)} = \frac{1}{2}.$$

*Proof.* The character

$$\pi_1(V, \overline{s}) \to \mathrm{O}(34d-17) \xrightarrow{\det} \{\pm 1\}$$

is not trivial on $\pi_1(V_{\overline{k}}, \overline{s})$. In fact, the image of $\pi_1(V_{\overline{k}}, \overline{s})$ in $\mathrm{O}(34d-17)$ is Zariski dense. Now the lemma follows from Chebotarev density theorem. □

We now apply Deligne's equidistribution theorem [Del1980, Théorème 3.5.3], see also [dJK2000, § 6.5–6.9].

**Proposition 2.4.4.** *Let*

$$V(k,\theta=1)=\left\{ s\in V(k)\ \middle|\ \begin{array}{c}\theta(s)\ \textit{has a eigenvalue } 1 \textit{ and no other}\\ \textit{eigenvalues which are roots of unity}\end{array}\right\},$$

$$V(k,\theta=-1)=\left\{ s\in V(k)\ \middle|\ \begin{array}{c}\theta(s)\ \textit{has a eigenvalue } -1 \textit{ and no other}\\ \textit{eigenvalues which are roots of unity}\end{array}\right\}.$$

*Then*

$$\lim_{\#k\to\infty}\frac{\#V(k,\theta=1)}{V(k)}=\lim_{\#k\to\infty}\frac{\#V(k,\theta=-1)}{V(k)}=\frac{1}{2}.$$

*Proof.* The proof is the same as that of [dJK2000, Theorem 6.11]. Let $f_1$ be the characteristic function of $\mathrm{SO}(34d-17)$. For any root of unity $\zeta$, let $Z(\zeta)$ be the subset of $\mathrm{O}(34d-17)$ consisting of matrix $A$ such that $\zeta$ is a root of $\det(1-tA)/(1-t\det A)$. Let $Z$ be the union of all $Z(\zeta)$ where $\zeta$ runs over all roots of unity whose minimal polynomial has degree less or equal to $34d-17$. Then $Z$ is a Zariski closed subset of $\mathrm{O}(34d-17)$. Let $f_2$ be the characteristic function of $\mathrm{O}(34d-17)\backslash Z$.

Let $\mathrm{d}A$ be the Haar measure on $\mathrm{O}(34d-17)$ with total mass 1. The equidistribution theorem of Deligne gives

$$\lim_{\#k\to\infty}\frac{1}{V(k)}\sum_{s\in V(k)}f(\theta(s))=\int_{\mathrm{O}(34d-17)}f(A)\mathrm{d}A=\frac{1}{2}.$$

Since $34d-17$ is odd, any matrix $A\in\mathrm{O}(34d-17)$ has an eigenvalue 1. It then follows that

$$\lim_{\#k\to\infty}\frac{\#V(k,\theta=1)}{V(k)}=\frac{1}{2}.$$

The other limit formula follows similarly. ☐

Combining this Proposition with the Tate conjecture for $X_s$, we have the following observation. Recall that the Tate conjecture claims that there is a bijection

$$\mathrm{NS}(X_s)\otimes\mathbb{Q}_l\to\mathrm{H}^2(X_{\overline{s}},\mathbb{Q}_l(1))^{\mathrm{Frob}=1},$$

where $\mathrm{H}^2(X_{\overline{s}},\mathbb{Q}_l(1))^{\mathrm{Frob}=1}$ is the eigenspace of Frobenius Frob with eigenvalue 1.

We view $X_{\overline{s}}$ as a fibration over $\mathbb{P}^1_{\overline{k}}$ and let $J_{\overline{s}}$ be the Jacobian of the generic fiber. Let $K$ be the function field of $\mathbb{P}^1_{\overline{k}}$. Let

$$V(k,\Xi)=\{s\in V(k)\mid \Xi_s \text{ generates } J_{\overline{s}}(K)\}.$$

**Corollary 2.4.5.** *We have*

$$\lim_{\#k\to\infty} \frac{\#V(k,\Xi)}{\#V(k)} \geq \frac{1}{2}.$$

*If the Tate conjecture holds for $X_s$, then the above inequality is an equality.*

*Proof.* This follows from the fact that $\Xi$ is not trivial on $X_{\overline{s}}$, Lemma 2.2.2 and Lemma 2.2.3. □

Having looked at the equidistribution behavior over a finite field, we now look at the equidistribution behavior over a number field. So from now on we take $k$ to be number field.

Let $\eta$ be the generic point of $V$ and $\overline{\eta}$ be any geometric point over $\eta$. Recall that we have shown in Proposition 2.2.4 that

$$\operatorname{rank} \mathrm{NS}\, X_{\overline{\eta}} = 3.$$

We now make use of the specialization theorem of the Neron–Severi group. Recall that by [Gro1971b, Exposé X, App 7, MP2012, Proposition 3.6], for any geometric point $\overline{s} \in V(\overline{k})$, one has

$$\operatorname{rank} \mathrm{NS}\, X_{\overline{s}} \geq \operatorname{rank} \mathrm{NS}\, X_{\overline{\eta}} = 3.$$

**Theorem 2.4.6** ([And1996, MP2012])**.** *Let*

$$V_{\text{jump}}(k) = \{s \in V(k) \mid \operatorname{rank} \mathrm{NS}\, X_{\overline{s}} > 3\}.$$

*Then $V_{\text{jump}}(k)$ is a thin set* [Ser1997, Chapter 9].

Recall that a subset $\Omega$ of a projective space $\mathbb{P}^N(k)$ is called thin if there is a generically finite morphism $f : A \to \mathbb{P}^N$ which admits no section and $\Omega \subset f(A(k))$. The property of the thin set that we are going to use is the following. This is a combination of the results of [Coh1981] and [Sch1964].

**Lemma 2.4.7.** *Let $\Omega \subset \mathbb{P}^N(k)$ be a thin set. Let $h$ be a height function on $\mathbb{P}^N(\overline{k})$. Then*

$$\lim_{H\to\infty} \frac{\#\{s \in \Omega \mid h(s) \leq H\}}{\#\{s \in \mathbb{P}^N(k) \mid h(s) \leq H\}} = 0.$$

We now fix any height function $h$ on $V(\overline{k})$. Let $H$ be any real number. Let

$$V(H) = \{s \in V(\overline{k}) \mid h(s) \leq H\},$$

$$V(H,\Xi) = \{s \in V(H) \mid \Xi \text{ generates } J_s(K)\}.$$

**Theorem 2.4.8.** *We have*

$$\lim_{H\to\infty} \frac{\#V(H,\Xi)}{\#V(H)} = 1.$$

*Proof.* This is a combination of Lemma 2.2.2, Theorem 2.4.6 and Lemma 2.4.7. □

# Chapter 3

# Towards a height formula

## 3.1 Review of heights

Let $K$ be the function field of a smooth projective curve $B$ over some base field $k$ or a number field in which case we let $B = \operatorname{Spec} \mathfrak{o}_K$. By a metrized line bundle on $B$ we mean a projective module $M$ over $\mathfrak{o}_K$ of rank one, together with a metric $\|\cdot\|_v$ on $M \otimes \overline{K_v}$ for each archimedean place $v$. We define the degree of $\overline{M}$ by

$$\deg \overline{M} = \log \#M/l\mathfrak{o}_K - \sum_{v|\infty} \epsilon_v \log\|l\|_v,$$

where $l \in M$ is any element and $\epsilon_v = 1$ if $v$ is real and 2 if $v$ is complex. The following lemma is clear from this definition.

**Lemma 3.1.1.** *Let $L$ and $M$ be two metrized line bundles on $B$. Suppose there is an open subset $B^\circ \subset B$ and an isomorphism $f : L \simeq M$ over $B_0$. If $v$ is archimedean, then let $\varphi_v = \log(\|s\|_{L_v}/\|f(s)\|_{M_v})$. If $v$ is non-archimedean, let $R_v$ be the local ring at $v$. We view $f$ as a function on $B$ with zeros and poles outside $B_0$. Let $\varphi_v = \operatorname{ord}_v(f)$ where $\operatorname{ord}_v$ is the valuation at the place $v$. Then*

$$\deg L = \deg M + \sum_v \varphi_v \log \mathrm{N}v.$$

*If $K$ is a number field, then $\mathrm{N}_v$ is the number of element in the residue field of $v$ if $v$ is non-archimedean and $\log \mathrm{N}_v = 1$ if $v$ is real and $\log \mathrm{N}_v = 2$ if $v$ is complex. If $K$ is the function field, then $\log \mathrm{N}_v = 1$.*

Let $f : \mathcal{X} \to B$ be a generically smooth projective flat morphism of relative dimension one. Let $L$ be a line bundle on $\mathcal{X}$. If $K$ is a number field, then we endow $L$ with a metric at each archimedean place. A line bundle $L$ together with the metric at each archimedean place is referred to as a metrized line bundle. Then there is a canonical way to endow $\det \mathrm{R}f_*L$ with a metric so that it is a metrized line bundle on $B$ [Del1987, Fal1984]. Let $L$ and $M$ be two metrized line bundles. We define a metric on the Deligne pairing $\langle L, M\rangle$ so that

$$\langle L, M\rangle = \det \mathrm{R}f_*(L \otimes M) \otimes (\det \mathrm{R}f_*L)^{-1} \otimes (\det \mathrm{R}f_*M)^{-1} \otimes \det \mathrm{R}f_*\mathcal{O}$$

is an isometry. Then the arithmetic intersection number $L.M$ is defined to be $\deg\langle L, M\rangle$.

We now review the notion of heights and the Hodge index theorem. Let $X$ be a smooth projective curve over $K$. Let $x \in \mathrm{Jac}(X)(\overline{K})$. Then we define the Neron–Tate height of $x$ as in [Ser1997]. Then $x$ is torsion if and only if the height of $x$ is zero. The Hodge index theorem provides us with a way to compute the height of $x$ in terms of (Arakelov) algebraic geometry.

Suppose the field of definition of $x$ is some finite extension $K'$ of $K$. Suppose $B$ is a smooth projective curve with function field $K'$ or $B = \operatorname{Spec} \mathfrak{o}_{K'}$ if $K'$ is a number field. By replacing $K'$ by a finite extension, we may assume that $X$ has semistable reduction at all the places of $K'$. Let $f : \mathcal{X} \to B$ be a regular minimal semistable model of $X$. Let $\overline{x}$ be a line bundle on $\mathcal{X}$ (possibly with $\mathbb{Q}$ coefficients, i.e. an element in $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Q}$) whose restriction to $X$ is $x$ and that the restriction to the normalization of any vertical divisor on $\mathcal{X}$ is of divisor is of degree zero. Here a divisor $V \subset \mathcal{X}$ is vertical if $V$ does not dominate $B$ under the morphism $f$.

If $K'$ is a number field and $v$ an archimedean place of $K'$. Then the completion $X_v = X \times_{K'} \overline{K'_v}$ is a Riemann surface and $x_v$ is a line bundle on $X_v$ of degree zero. It then follows from the work of Arakelov [Ara1974] that there is a metric on $x_v$, unique up to a constant, whose curvature is zero. We denote by $\widehat{x}$ the line bundle $\overline{x}$, together with such a metric at each archimedean place. If $K'$ is a function field, then we let $\widehat{x} = \overline{x}$.

The Hodge index theorem [Fal1984, Hri1985], in this context, asserts that

$$\operatorname{height} x = -\frac{2}{(L : K)}\widehat{x}.\widehat{x},$$

where $\widehat{x}.\widehat{x}$ is the arithmetic intersection number on the arithmetic surface $\mathcal{X}$ if $L$ is a number field, or the usual intersection number on the fibered surface $\mathcal{X}$ [Har1977].

We now recall the notation the Arakelov dualizing sheaf [Ara1974, Fal1984]. Let $X$ be a Riemann surface of genus $g$. Let $\alpha_1, \cdots, \alpha_g$ be an orthogonal basis of $\mathrm{H}^0(X, \omega_X)$. Let

$$\mathrm{d}\mu = \frac{i}{2g}\sum_{i=1}^{g} \alpha_i \wedge \overline{\alpha_i}$$

be a measure on $X$ with total mass one. Let $L$ be line bundle on $X$. A metric on $L$ is called admissible if its curvature equals $\deg L \cdot \mathrm{d}\mu$. Let $p \in X$. We endow the line bundle $\mathcal{O}(p)$ with the metric $\|\cdot\|$ such that $\log\|1_p\|(q) = g(p,q)$ where $g(p,q)$ is the Arakelov Green's function. By definition, it satisfies the condition that $\partial_q \overline{\partial}_q g(p,\cdot) = \mu - \delta_p$ and $\int_X g(p,q)\mathrm{d}\mu_q = 0$. We put a metric on any line bundle of the form $\mathcal{O}(D)$ where $D$ is a divisor on $X$ so that for any $p,q \in X$, the isomorphism $\mathcal{O}(p+q) \simeq \mathcal{O}(p)\otimes\mathcal{O}(q)$ is an isomorphism. This metric is admissible. We then put a metric on $\omega_X$ such that the canonical residue map

$$\omega_X \simeq \mathcal{O}(-p)|_p$$

is an isometry. This is an admissible metric.

Now let $X$ be a curve defined over the number field $K$ and $\mathcal{X} \to B$ be the minimal regular semistable model. Let $\omega_{\mathcal{X}/B}$ be the relative dualizing sheaf. For each archimedean place, we endow $\omega_{X_v} = \omega_{\mathcal{X}/B}|_{X_v}$ with the metric described as above. This is the Arakelov dualizing sheaf. We sometimes call $\omega_{\mathcal{X}/B}$ the (Arakelov) dualizing sheaf of $X$. We denote it by $\omega_X$ and its self-intersection by $\omega_X^2$.

## 3.2 Decomposition of the height

Let $k$ be a field with $\operatorname{char} k \neq 2$. Let $K$ be a function field of a smooth projective curve $B$ over $k$ or a number field in which case we put $B = \operatorname{Spec} \mathfrak{o}_K$ where $\mathfrak{o}_K$ is the ring of integers of $K$. Let $X$ be a smooth Petri general genus four curve over $K$. We assume that $X$ has semistable reduction over $K$.

Replacing $K$ be a quadratic extension if necessary, we may assume that $X$ is a complete intersection of a quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and an irreducible cubic surface $C$ in $\mathbb{P}^3$. Let $j : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the embedding. Then by definition, the point $\Xi = j^*\mathcal{O}(1,-1) \in \operatorname{Jac} X(K)$.

The goal of this section is to prove the following proposition.

**Proposition 3.2.1.** *There is a $\varphi_v$ for each place $v$, which depends only on the completion of $X$ at the place $v$, such that*

$$h(\Xi) = 5\omega^2 - \sum \varphi_v,$$

*where the sum is over all places of $B$ and $\omega$ is the dualizing sheaf of $X$. Moreover, if $X$ has good non-hyperelliptic reduction at $v$ then $\varphi_v = 0$.*

We first prove a technical lemma which implies Proposition 3.2.1.

Let us make some remarks on the degenerate locus of a morphism. Let $S$ be a regular scheme and

$M$ and $N$ are two vector bundles on $S$ of the same rank. Let $f : M \to N$ be an injective morphism. Let $I$ be the sheaf of ideals on $S$ generated by the determinant of $f$. Then the degenerate locus $Z$ of $f$ is the subscheme of $S$ defined by $I$. The degenerate locus $Z$ is a Cartier divisor and there is an isomorphism $\det N \simeq \det M \otimes \mathcal{O}(Z)$ on $S$.

We shall work with in the following situation.

Let $S$ be a connected regular scheme and $f : X \to S$ a projective flat morphism with the total space $X$ being normal. Assume that

1. All the fibers of $f$ are geometrically reduced and the generic fiber of $f$ is a smooth genus four curve.

2. The morphism $f$ is local complete intersection. This implies that the dualizing sheaf $\omega_{X/S}$ is a line bundle.

3. The relative dualizing sheaf $\omega_{X/S}$ is very ample.

Under these conditions, there is a canonical embedding $X \to \mathbb{P}(f_*\omega_{X/S})$. This embedding factors through a quadratic space bundle $Q$ over $S$. We assume

4. The generic fiber of $Q \to S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The fibers of $Q \to S$ are geometrically integral and has at most one ordinary double point.

This quadratic surface is defined by the kernel of the morphism $\mathrm{Sym}^2 \pi_*\omega_{X/S} \to \pi_*\omega_{X/S}^{\otimes 2}$, say $\mathcal{L}$. Just as in section 2.1, we can form the discriminant morphism disc : $\mathcal{L}^4 \to \lambda^2$. Let $Z$ be the zero locus of the discriminant morphism.

Let $S^\circ = S\backslash Z$ be the open subscheme of $S$. By assumption, the quadric surface $Q_{S^\circ} \simeq \mathbb{P} \times \mathbb{P}'$ over $S^\circ$ where $\mathbb{P}$ and $\mathbb{P}'$ are $\mathbb{P}^1$-bundles over $S^\circ$. By the theory of the compactifiaction of the Picard functor using torsion free sheaves [AK1980], we see there is a torsion free sheaf $F$ on $Q$, which is isomorphic to $\mathcal{O}(1,0)$ when restricted to $Q_{S^\circ}$. More concretely, there is a $\mathbb{P}^1$-bundle $P$ over $S$ which is a closed subscheme of $Q$. For any geometric point $s$ of $S$, the fiber $P_s$ of $P$ is a reduced line on the quadric surface $Q_s$. let $I$ be the defining ideal of $Q$. Then $F$ is isomorphic to $\underline{\mathrm{Hom}}(I, \mathcal{O}_Q)$ or $\mathcal{O}(1) \otimes I$ where $\mathcal{O}(1)$ is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^3$ to $Q$. We denote the pullback of $F$ to $X$ by $L$. Then $L_s$ is a torsion free sheaf of degree three on $X_s$. If $X_s$ does not pass through the singular point on $Q_s$, then $L_s$ is locally free.

**Lemma 3.2.2.** *For any geometric point $s \in S$, one has* $\dim \mathrm{H}^0(Q_s, F_s) = 2$.

*Proof.* The lemma is clear if $Q_s$ is smooth. Assume that $Q_s$ is singular. Let $\nu : \widetilde{Q_s} \to Q_s$ be the desingularization of $Q_s$. Then $Q_s$ is isomorphic to a ruled surface $\pi : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2)) \to \mathbb{P}^1$. Let $l$ be any fiber of $Q_s$ over $\mathbb{P}^1$, then $\nu_* \mathcal{O}(-l) \simeq I_s$. Let $e$ be the section of $\pi$ such that $e^2 = -2$ on $\widetilde{Q_s}$. Then $\nu^* \mathcal{O}(1) \simeq \mathcal{O}(2l + e)$. Here $\mathcal{O}(1)$ stands for the pullback of $\mathcal{O}(1)$ of $\mathbb{P}^3$ to $Q_s$. One has $F_s = \nu_* \mathcal{O}(l + e)$ and

$$\mathrm{H}^0(Q_s, F_s) = \mathrm{H}^0(\widetilde{Q_s}, \mathcal{O}(l + e)) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1) \otimes \pi_* \mathcal{O}(e)).$$

The result then follows from the fact that $\pi_* \mathcal{O}(e) = \mathcal{O} \oplus \mathcal{O}(-2)$. □

**Lemma 3.2.3.** *For any geometric point $s \in S$, we have $\mathrm{H}^0(X_s, L_s) \simeq \mathrm{H}^0(Q_s, F_s)$. So in particular, the sheaf $f_* L$ on $S$ is a vector bundle of rank two and is compatible with any base change.*

*Proof.* The lemma is clear if $Q_s$ is smooth. Assume that $Q_s$ is singular. Let $j : X_s \to Q_s$ be the embedding. Then

$$\mathrm{H}^0(X_s, L_s) = \mathrm{H}^0(Q_s, j_* \mathcal{O}_{X_s} \otimes F).$$

Consider the exact sequence

$$0 \to \mathcal{O}(-X_s) \otimes F \to F \to j_* \mathcal{O}_{X_s} \otimes F \to 0.$$

Since $\mathcal{O}(-X_s) \simeq \mathcal{O}(-3)$, where $\mathcal{O}(-3)$ is the pullback of $\mathcal{O}(-3)$ on $\mathbb{P}^3$ to $Q_s$, taking long exact sequence of the short exact sequence above shows in order to prove the lemma, we need to prove $\mathrm{H}^1(Q_s, F_s \otimes \mathcal{O}(-3)) = 0$.

We use the notation from (the proof) of the previous lemma. Note that $\mathrm{R}^1 \nu_* \mathcal{O}_{\widetilde{Q_s}} = 0$. Then

$$\mathrm{H}^1(Q_s, F_s \otimes \mathcal{O}(-3)) = \mathrm{H}^1(\widetilde{Q_s}, \mathcal{O}(-5l - 2e)).$$

The cohomology group $\mathrm{H}^1(Q_s, F_s \otimes \mathcal{O}(-3))$ sits in the exact sequence

$$0 \to \mathrm{H}^1(\mathbb{P}^1, \pi_* \mathcal{O}(-5l - 2e)) \to \mathrm{H}^1(Q_s, F_s \otimes \mathcal{O}(-3)) \to \mathrm{H}^0(\mathbb{P}^1, \mathrm{R}^1 \pi_* \mathcal{O}(-5l - 2e)).$$

We have $\pi_* \mathcal{O}(-5l - 2e) = \mathcal{O}(-5) \otimes \pi_* \mathcal{O}(-2) = 0$ since $\pi_* \mathcal{O}(-2e) = 0$. We also have $\mathrm{R}^1 \pi_* \mathcal{O}(-5l - 2e) = \mathcal{O}(-5) \otimes \mathrm{R}^1 \mathcal{O}(-2e)$. Consider the short exact sequence

$$0 \to \mathcal{O}(-2e) \to \mathcal{O}(-e) \to \mathcal{O}(-e)|_e \to 0.$$

We have $\pi_*\mathcal{O}(-e) = 0$ and $\mathrm{R}^1\pi_*\mathcal{O}(-e) = 0$. Therefore $\mathrm{R}^1\pi_*\mathcal{O}(-2e) = \pi_*\mathcal{O}(-e)|_e$. The self-intersection of $e$ is $-2$ and $\pi$ is an isomorphism when restricted to $e$. Therefore $\mathrm{R}^1\pi_*\mathcal{O}(-2e) = \mathcal{O}(2)$. Thus $\mathrm{H}^0(\mathbb{P}^1, \mathrm{R}^1\pi_*\mathcal{O}(-5l-2e)) = 0$. We conclude that $\mathrm{H}^1(\widetilde{Q_s}, \mathcal{O}(-5l-2e)) = 0$. The lemma is thus proved. □

**Lemma 3.2.4.** *The sheaf $f_*\underline{\mathrm{Hom}}(L, \omega_{X/S})$ is a vector bundle of rank two and is compatible with arbitrary base change.*

*Proof.* This follows from duality and the lemma above. □

Consider the cup product morphism

$$f_*L \otimes f_*\underline{\mathrm{Hom}}(L, \omega_{X/S}) \to f_*\omega_{X/S}. \tag{3.1}$$

Both the source and the target are vector bundles of rank four. Moreover, it is an isomorphism at the generic point of $S$. Therefore it is injective.

**Lemma 3.2.5.** *Let $Z_i : i = 1, \cdots, r$ be the irreducible components of $Z$ and $Z = \sum_{i=1}^n a_i Z_i$ as the divisor on $S$. Then $a_i$'s are even.*

*Proof.* Let $\eta$ be the generic point of $Z_i$, $R$ the local ring of $S$ at $\eta$ and $t$ a local uniformizer. Then by assumption, we can choose coordinates $x, y, w, z$ of $\mathbb{P}^3$ so that the equation of the $Q$ over $R$ is

$$xy = z(z + t^r w).$$

Then the discriminant is $t^{2n}$. This shows $a_i$ is even. □

We set $Z' = \frac{1}{2}Z$ as a divisor on $S$.

**Lemma 3.2.6.** *The degenerate locus of the cup product (3.1) is $Z'$.*

*Proof.* We note that

$$f_*\underline{\mathrm{Hom}}(L, \omega_{X/S}) \simeq g_*\underline{\mathrm{Hom}}(F, \mathcal{O}(1)), \quad f_*\omega_{X/S} \simeq g_*\mathcal{O}(1),$$

where $g : Q \to S$ is the structure morphism. The line bundle $\mathcal{O}(1)$ on $Q$ is the pullback of $\mathcal{O}(1)$ from $\mathbb{P}(f_*\omega_{X/S})$. So we are reduced to show that the degeneration locus of

$$g_*F \otimes g_*\underline{\mathrm{Hom}}(F, \mathcal{O}(1)) \to g_*\mathcal{O}(1)$$

is $Z'$. Let $t$ be a uniformizer at the generic point of any irreducible component of $Z$. Then we can choose the isomorphisms

$$g_*F \simeq \mathcal{O}_S u_1 \oplus \mathcal{O}_S u_2, \quad g_*\underline{\mathrm{Hom}}(F, \mathcal{O}(1)) \simeq \mathcal{O}_S v_1 \oplus \mathcal{O}_S v_2$$

so that the cup product is given by

$$u_1v_1 \mapsto w_1,\ u_1v_2 \mapsto w_2,\ u_2v_1 \mapsto w_3,\ u_2v_2 \mapsto w_3 + t^n w_4$$

The leading term of the discriminant of $Q$ at $\eta$ is thus $t^{2n}$. The Petri locus is thus defined by $t^{2n}$ and $Z'$ is defined by $t^n$. □

*Proof of Proposition 3.2.1.* Let $f : \mathcal{X} \to B$ be a minimal regular semistable model of $X \to K$. Let $B^\circ \subset B$ be the open subset of $B$ such that for any point $b \in B$, the fiber $\mathcal{X}_b$ is a smooth non-hyperelliptic curve. Since $X$ is Petri general, we see that $B^\circ$ is not empty. We denote by $Z$ the locus on $B^\circ$ where $Q$ is singular and $Z' = \frac{1}{2}Z$ as in Lemma 3.2.6.

Let $\mathcal{X}^\circ = \mathcal{X} \times_B B^\circ$. According to Lemma 3.1.1, we only have to give an isomorphism

$$\langle \widehat{\Xi}, \widehat{\Xi} \rangle \simeq \langle \omega, \omega \rangle^{-5}$$

over $B^\circ$, where $\widehat{\Xi}$ is some extension of $\Xi$ to $\mathcal{X}^\circ$ and $\omega$ is the dualizing sheaf of $\mathcal{X}^\circ \to B^\circ$.

By assumption, the space $\mathcal{X}^\circ$ embeds in $Q_{B^\circ}$ where the generic fiber of $Q_{B^\circ} \to B^\circ$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We are therefore in the situation at the beginning of this section. Let $s$ be any geometric point of $B^\circ$ such that $Q_s$ is singular, then $\mathcal{X}_s$ does not pass through the singular point on $Q_s$. Therefore $L$ is a line bundle on $\mathcal{X}^\circ$ and $\omega \otimes L^{-2}$ is an extension of $\Xi$ on $\mathcal{X}^\circ$. Thus

$$\langle \omega \otimes L^{-2}, \omega \otimes L^{-2} \rangle \simeq \langle \omega, \omega \rangle \otimes \langle L, \omega \otimes L^{-1} \rangle^{-4}.$$

By the Riemann–Roch formula, we have

$$\langle L, \omega \otimes L^{-1} \rangle \simeq (\det \mathrm{R} f_* \mathcal{O})^2 \otimes (\det \mathrm{R} f_* L)^{-2}.$$

Note that $(\mathrm{R}^1 f_* L)^\vee \simeq f_* \mathrm{Hom}(L, \omega_{X/S})$ by duality. By Lemma 3.2.6,

$$\det(f_* L \otimes (\mathrm{R}^1 f_* L)^\vee) \otimes \mathcal{O}(Z') \simeq \det f_* \omega_{X/S}.$$

Note that

$$\det(f_*L\otimes(\mathrm{R}^1f_*L)^\vee)\simeq(\det\mathrm{R}f_*L)^2,\quad \det f_*\omega_{X/S}\simeq\det\mathrm{R}f_*\mathcal{O}_{\mathcal{X}},\quad \mathcal{O}(Z')\simeq(\det\mathrm{R}f_*\mathcal{O}_{\mathcal{X}})^{17}$$

where the last isomorphism follows from Lemma 2.1.3. The proposition then follows from the Mumford isomorphism $(\det\mathrm{R}f_*\mathcal{O}_{\mathcal{X}})^{12}\simeq\langle\omega,\omega\rangle$. □

## 3.3 The integral model

In order to compute the height of $\Xi$, or equivalently the constant $\varphi_v$, we need to describe explicitly the integral model of $X$ and the extension $\Xi$ to the integral model. This is the goal of this section.

Let $\mathcal{Y}$ be the stable model of $X$ over $B$ and $\mathcal{X}$ be the minimal regular semistable model of $X$. There is a morphism $\mathcal{X}\to\mathcal{Y}$ over $B$ which contracts all rational curves on $\mathcal{X}$ whose self-intersection is $(-2)$. We choose a divisor $D$ on $X$ so that $L=\mathcal{O}(D)$ on $X$ and let $\overline{D}$ the closure of $D$ in $\mathcal{X}$. Let $V$ be a rational linear combination of vertical divisors on $\mathcal{X}$ so that $\widehat{\Xi}:=\omega_{\mathcal{X}/B}(-2\overline{D}+V)$ has the property that it is of degree zero on any irreducible vertical component on $\mathcal{X}$. If $K$ is a number field, we endow $\widehat{\Xi}$ on $X(\overline{K_v})$ with a metric for each place $v$ at infinity of $K$. The metric we choose is the Arakelov metric for the dualizing sheaf and the admissible metric given by the Arakelov Green's function for $\mathcal{O}(D)$, c.f. Section 3.1. The curvature of this metric is zero.

We shall compute $\varphi_v$ when $v$ is non-archimedean and the reduction of $X$ at the place $v$ satisfies certain restrictions. We contend ourselves with the following remark concerning the archimedean places.

**Remark 3.3.1.** If $K$ is a number field and $v$ is a place at infinity, then we can write $\varphi_v$ as a sum of some other (less complicated) constants. It follows from the computation in section 2.1 that there is an isomorphism $\mathcal{O}(Z)\simeq\lambda^{34}$ over $\mathcal{M}_{4,\mathbb{C}}$. The line bundle $\mathcal{O}(Z)$ has a canonical section which we denote by $1_Z$. The Hodge bundle $\lambda$ on $\mathcal{M}_4$ has a canonical metric which gives a metric on $\mathcal{O}(Z)$ via the isomorphism $\mathcal{O}(Z)\simeq\lambda^{34}$. We denote the norm of the section by $\mathrm{disc}_v$. Its evaluation at $X_v$ is $\mathrm{disc}_v(X_v)$. This is a continuous function on $\mathcal{M}_4^\circ(\mathbb{C})$.

The cup product morphism 3.1 takes the form

$$\mathrm{H}^0(X_v,L_v)\otimes\mathrm{H}^0(X_v,\omega_{X_v}\otimes L_v^{-1})\to\mathrm{H}^0(X_v,\omega_{X_v}).$$

This is an isomorphism by the assumption that $X$ is Petri general. Then the cup product induces an isomorphism $(\det \mathrm{H}^*(X_v, L_v))^2 \simeq \det \mathrm{H}^*(X_v, \mathcal{O}_{X_v})$. Both sides are endowed with the canonical metric by Faltings [Fal1984, Theorem 1]. The norm of this isomorphism is denoted by $c_v(X_v)$. The the proof of Proposition 3.2.1, combined with the Noether's formula for arithmetic surface [Fal1984, MB1989] yields

$$\varphi_v = -4 \log c_v(X_v) + 2 \log \mathrm{disc}_v(X_v) - 96 \log 2\pi + 6\delta_v(X_v),$$

where $\delta_v(X_v)$ is the Faltings' delta constant [Fal1984]. Since $\varphi_v$, as a function on $\mathcal{M}_{4,\mathbb{C}}$, is bounded near the non-hyperelliptic, but Petri special curves. This in particular implies that $c_v(X_v)^4/\mathrm{disc}_v(X_v)^2$ is bounded. This is not obvious from the definition as both of them acquire singularities on the Petri locus. The nature of $c_v$ and $\mathrm{disc}_v$ deserves further study. However, we will not address this issue in this paper.

From now on, we focus on the case where $v$ is non-archimedean.

Let $v$ be a finite place of $K$, and $R_v$ be the strict henselization of $B$ at $v$. Let $k = \overline{k}$ be the residue field of $R_v$ and $Y_v = \mathcal{Y} \times_B k$ be the fiber of $\mathcal{Y}$ at $v$.

**Definition 3.3.2.** We say that the reduction of $X$ at $v$ is simple if $Y_v$ is of one of the following cases.

1. (SMOOTH) $Y_v$ is smooth and hyperelliptic.
2. (IRRED) $Y_v$ is an irreducible nodal curve with a single node $p$. The normalization of $Y_v$ is non-hyperelliptic.
3. (ELL) $Y_v$ has two components $C$ and $E$ meeting at a single node $p$. Here $C$ is a non-hyperelliptic genus three curve and $E$ is an elliptic curve.
4. (TWO) $Y_v$ has two components $C_1$ and $C_2$ meeting at a single node $p$. Here both $C_1$ and $C_2$ are of genus two and $p$ is not a Weierstrauss point on either component.

We refer to these four cases SMOOTH, IRRED, ELL, TWO respectively. They stand for "smooth", "irreducible", "elliptic tail" and "genus two" respectively.

We shall describe the integral model at the place $v$ and the divisor $\overline{D}$ more explicitly. We deal with one $v$ at a time. So we supress the subscript $v$ to simplify notation. We shall denote by $R$ the strict henselization of $B$ at the place $v$. Let $\varpi$ be a uniformizer of $R$.

**<u>The case IRRED</u>**: In this case, $Y$ is an irreducible curve with a single node $p$. Let $\nu : Y' \to Y$ be the normalization. Then $Y'$ is not hyperelliptic. Let $p_1, p_2 \in Y'$ be the inverse image of $p \in Y$.

**Lemma 3.3.3.** *There are at most two line bundles $M$ on $Y$ with $\dim \mathrm{H}^0(Y, M) \geq 2$ and $\deg \nu^* M = 3$. They can be describe as follows. Let $j : Y' \to \mathbb{P}^2$ be the canonical embedding. Let $l$ be line that passes* *through $p_1$ and $p_2$ which intersect $Y'$ at other two point $p_3, p_4$ (They might coincide and might coincide with $p_1$ or $p_2$). Then $\nu^*M \simeq \omega_{Y'}(-p_3)$ or $\nu^*M \simeq \omega_{Y'}(-p_4)$.*

*Proof.* We note that there is an injective map

$$\mathrm{H}^0(Y, M) \to \mathrm{H}^0(Y', \nu^*M).$$

Therefore $\dim \mathrm{H}^0(Y', \nu^*M) \geq 2$. Since $\deg \nu^*M = 3$, we must have $\dim \mathrm{H}^0(Y', \nu^*M) = 2$. Thus $\nu^*M$ is of the form $\omega_{Y'}(-p_3)$ for some $p_3 \in Y'$.

The space $\mathrm{H}^0(Y, M)$ consists of sections of $\nu^*M$ on $Y'$ whose value at $p_1$ and $p_2$ coincide. This means when $Y'$ is canonically embedded in $\mathbb{P}^2$, the points $p_3, p_1$ and $p_2$ lie on the same line $l$. This line intersects $Y'$ another point $p_4$. If $p_3 \neq p_4$, there are two line bundles $M$ and $M'$ with the desired property and $\nu^*M = \omega_{Y'}(-p_3)$, $\nu^*M' = \omega_{Y'}(-p_4)$. If $p_3 = p_4$, the line bundle $M$ with $\nu^*M = \omega_{Y'}(-p_3)$ is the only choice. □

By the lemma above, there is a choice of the divisor $D$ on $X$, such that the closure $D$ on $\mathcal{Y}$ intersects $Y$ at three points $q_1 + q_2 + q_3$. Their inverse image in $Y'$ is linearly equivalent to $p_1 + p_2 + p_3$ or $p_1 + p_2 + p_4$.

Suppose the local equation at the node $p$ is given by $xy - \varpi^e$. Consider the desingularization $\mathcal{X} \to \mathcal{Y}$. The component in the special fiber of $\mathcal{X}$ that dominates the special fiber of $\mathcal{Y}$ is also denoted by $Y$. Then the exceptional divisor is a chain of $(e-1)$ $\mathbb{P}^1$'s. We denote them consequtively by $E_i$ $(i = 1, \cdots, e-1)$ with $E_i.E_{i+1} = 1$ and $E_1.Y = E_{e-1}.Y = 1$.

Note that $\omega_{\mathcal{X}/R}(-2\overline{D})$ is of degree zero on each vertical component of the special fiber of $\mathcal{X}$.

**<u>The case ELL</u>**: In this case, the special fiber $Y$ has two components $C$ and $E$ where $C$ is a non-hyperelliptic genus three curve and $E$ is an elliptic curve. Suppose the local equation of $\mathcal{Y}$ at $p$ is $xy - \varpi^e$. Consider the desingularization $\mathcal{X} \to \mathcal{Y}$. The inverse image of $p$ in $\mathcal{X}$ consists of $(e-1)$ $\mathbb{P}^1$'s, denoted by $E_1, \cdots, E_{e-1}$, such that $E_i.E_{i+1} = 1$ and $C.E_1 = E.E_{e-1} = 1$.

Let $\mathcal{L}$ be an extension of $L$ on $\mathcal{X}$.

**Lemma 3.3.4.** *Let $L_s$ be the restriction of $\mathcal{L}$ on $Y$ with the property that $\deg L|_C = 3$, $\deg L_s|_E = 0$ and $\deg L_s|_{E_i} = 0$ $(i = 1, \cdots, e-1)$. Then $L_s|_E \simeq \mathcal{O}_E$ and $L_s|_C$ be can described as follows. Let $j : C \to \mathbb{P}^2$ be the canonical embedding. Let $l$ be the tangent line of $C$ at the point $p$, which intersect $C$ at two other points $q_1, q_2$ (they might coincide and they might coincide with $p$). Then $L_s|_C \simeq \omega_C(-q_1)$ or $L_s|_C \simeq \omega_C(-q_2)$.*

*Proof.* let $p_1$ be the intersection point of $C$ and $E_1$ and $p_2$ be the intersection point of $E$ and $E_{e-1}$. By semi-continuity, for any divisor $V$ supported on $Y$, we must have

$$\dim \mathrm{H}^0(Y, L_s(V)|_Y) \geq 2.$$

We first claim that $L_s|_E$ must be effective, hence $L_s|_E = \mathcal{O}_E$. If not, then

$$\mathrm{H}^0(Y, L_s|_Y) \subset \mathrm{H}^0(C, L_s|_C(-p_1)).$$

As $C$ is not hyperelliptic, we have $\dim \mathrm{H}^0(C, L_s|_C(-p_1)) \leq 1$. This is a contradiction. This in addition shows that $\dim \mathrm{H}^0(C, L_s|_C) \geq 2$.

Now there is a divisor $V$ supported in $Y$, such that $L_s(V)|_C = 1$, $L_s(V)|_E = 2$ and $L_s(V)|_{E_i} = 0$. Then $L_s(V)|_C = L_s|_C(-2p_1)$ and $L_s(V)|_E = L_s|_E(2p_2)$. Since $\dim \mathrm{H}^0(E, L_s(V)_E) = 2$, we conclude that $L_s(V)|_C = L_s|_C(-2p_1)$ must be effective on $C$. Therefore $L_s|_C = \mathcal{O}(2p_1 + q)$ for some $q$ on $C$ and $\dim \mathrm{H}^0(C, \mathcal{O}(2p_1 + q)) \geq 2$. The rest of the argument is similar to the proof of Lemma 3.3.3. □

By this lemma, there is a choice of $D$ on $\mathcal{X}$, so that the closure $D$ on $\mathcal{X}$ only intersects $C$ at three points. The location of these three points are described as in the lemma.

Let $V$ be a divisor supported on the special fiber of $\mathcal{X}$ defined by

$$V = E_1 + 2E_2 + \cdots + (e-1)E_{e-1} + eE.$$

Then it is easy to check that $\omega_{\mathcal{X}/R}(-2\overline{D} + V)$ has degree zero on each irreducible component of the special fiber of $\mathcal{X}$. The self-intersection of $V$ is

$$V^2 = -e.$$

**The case TWO**: In this case, the special fiber $Y$ has two components $C_1$ and $C_2$. They are both of genus two and the node is not a Weierstrauss point on either component.

Suppose the equation of $\mathcal{Y}$ at the node $p$ is $xy - \varpi^e$. Then the inverse image $p$ in the desingularization $\mathcal{X} \to \mathcal{Y}$ consists of $(e-1)$ $\mathbb{P}^1$'s. We denote them by $E_1, \cdots, E_{e-1}$ so that $E_i.E_{i+1} = 1$, $C_1.E_1 = 1$ and $C_2.E_{e-1} = 1$.

Let $\mathcal{L}$ be an extension of $L$ on $\mathcal{X}$ with $\deg \mathcal{L}|_{C_1} = 2$, $\deg \mathcal{L}|_{C_2} = 1$ and $\deg \mathcal{L}|_{E_i} = 0$. Let the intersection of $C_1$ and $E_1$ by $p_1$ and the intersection of $C_2$ and $E_{e-1}$ by $p_2$.

**Lemma 3.3.5.** *One of the following two cases happens.*

1. *One has* $\dim \mathrm{H}^0(C_1, \mathcal{L}|_{C_1}) = 2$ *and* $L|_{C_2} \simeq \mathcal{O}(q)$ *where* $q \in C_2$ *is the hyperelliptic involution of* $p_2$ *on* $C_2$*.*

2. *One has* $L|_{C_1} = \mathcal{O}(2p_1)$ *and* $L|_{C_2} \simeq \mathcal{O}(p_2)$*.*

*If the second case happens, the line bundle* $\omega_{\mathcal{X}/R} \otimes \mathcal{L}^{-2}(-E_1 - \cdots - E_{e-1})$ *is an extension of* $\omega_X(-D)$ *and satisfies the first condition.*

*Proof.* The proof is similar to that of Lemma 3.3.4. □

By this lemma, there are two possibilities.

1. There is a choice of $D$ on $X$ so that its closure on $\mathcal{X}$ intersects with $C_1$ at two points $q_1 + q_2$ such that $\dim \mathrm{H}^0(C_1, \mathcal{O}(q_1 + q_2)) = 2$ and intersect $C_2$ at the hyperelliptic involution of $p$. Let $V$ be the divisor supported on the special fiber of $\mathcal{X}$ defined by

$$V = E_1 + 2E_2 + \cdots + (e-1)E_{e-1} + eC_2.$$

   Then $\omega_{\mathcal{X}/R}(-2\overline{D} + V)$ has degree zero on each irreducible component of the special fiber of $\mathcal{X}$. The self-intersection of $V$ is

$$V^2 = -e.$$

2. If such a $D$ does not exist. Then we can find an effective divisor $D'$ such that $\omega_X \simeq \mathcal{O}(D + D')$ and similar properties for $D'$ hold.

From now on, we fix the choice of $D$ as described above in each case. In case TWO, we assume that the first of the two possibilities happens.

**Lemma 3.3.6.** 1. *The closure of* $D$ *in* $\mathcal{X}$ *does not meet the rational component in the special fiber whose self-intersection is* $-2$*. Equivalently, this mean the closure of* $D$ *in* $\mathcal{Y}$ *does not pass through the singular point on the special fiber of* $\mathcal{Y}$*.*

2. *Both* $\mathrm{H}^0(\mathcal{X}, \mathcal{O}(\overline{D}))$ *and* $\mathrm{H}^0(\mathcal{X}, \omega_{\mathcal{X}/R}(-\overline{D}))$ *are free* $R$*-modules of rank two and are compatible with arbitrary base change.*

*Proof.* This follows from the explicit description of the integral model in this section. □

## 3.4 The cokernel of the cup product

We keep the notation from the previous section. We always work at a fixed place $v$, so the subscript $v$ is always suppressed. We always assume that $X$ has simple reduction at the place $v$.

We study the cup product

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}(\overline{D})) \otimes \mathrm{H}^0(\mathcal{X}, \omega_{\mathcal{X}/R}(-\overline{D})) \to \mathrm{H}^0(\mathcal{X}, \omega_{\mathcal{X}/R}). \tag{3.2}$$

By the assumption that $X$ is Petri general, this cup product is injective. Let $W$ be the degenerate locus. Then $W$ is supported on the closed point of $\operatorname{Spec} R$. We would like to study the length of $W$.

Note that the morphism $\mathcal{X} \to \operatorname{Spec} R$ induces a canonical morphism $\operatorname{Spec} R \to \overline{\mathcal{M}}_4$. The pullback of $\mathcal{O}(\overline{Z})$ to $\operatorname{Spec} R$ is a free $R$-module generated by $\varpi^m$. We define $p(X) = m$.

**Proposition 3.4.1.** *Suppose we are in case IRRED, ELL or TWO. Suppose the local equation of $\mathcal{Y}$ at the singular point on the special fiber is $xy - \varpi^e$. Then*

$$\operatorname{length} W = \begin{cases} \frac{1}{2}p(X) & \textit{Case IRRED} \\ \frac{1}{2}p(X) + e & \textit{Case ELL} \\ e & \textit{Case TWO.} \end{cases}$$

Now consider the case SMOOTH.

First let us make some remarks on families of hyperelliptic curves. Let $S$ be a scheme and $C \to S$ a smooth projective morphism of relative dimension one. We say that $C$ is a hyperelliptic curve over $S$ if one of the following equivalent conditions holds.

1. There is an involution $\sigma : C \to C$ over $S$, such that the quotient $C/\langle\sigma\rangle \simeq P$ where $P \to S$ is smooth projective and all the fibers are genus zero curves.

2. There is a finite flat degree two morphism $C \to P$ over $S$ where $P \to S$ is smooth projective and all the fibers are genus zero curves.

We call the fix point of $\sigma$ the Weierstrauss subscheme of $C$.

We define the hyperelliptic multiplicity $h(X)$ of $X$ as

$$h(X) = \max\{n \mid \mathcal{X} \times_R R/\varpi^n \text{ is a hyperelliptic curve over } R/\varpi^n\}.$$

**Proposition 3.4.2.** *Suppose we are in the case SMOOTH. Then*

$$\operatorname{length} W = \frac{1}{2}p(X) - 9h(X).$$

The proof of these two propositions will be the theme of the next two chapters.

# Chapter 4

# Proof of Proposition 3.4.1

## 4.1 Some preliminaries

The goal of this chapter is to prove Proposition 3.4.1. We keep the notation from Chapter 3 Section 3.3. We recall that $\mathcal{X} \to \operatorname{Spec} R$ is a minimal regular semistable model of $X$ and $\mathcal{Y} \to \operatorname{Spec} R$ the stable model of $X$. The special fiber of $\mathcal{Y}$ is denoted by $Y$. There is a unique node on $Y$ which we denote by $p$. The local equation of $\mathcal{Y}$ at $p$ is $xy - \varpi^e$ for some integer $e$.

We denote by $\mu : \mathcal{X} \to \mathcal{Y}$ be the canonical morphism. The divisor $D$ on $X$ is chosen as was described in Section 3.3. We shall denote by $\overline{D}$ the closure of $D$ on any model of $X$ over $\operatorname{Spec} R$. This is a slight abuse of notation, but we are going to specify the model whenever there is some possible confusion.

**Lemma 4.1.1.** *We have*

$$\mu_* \mathcal{O}_{\mathcal{X}}(\overline{D}) \simeq \mathcal{O}_{\mathcal{Y}}(\overline{D}), \quad \mu_* \omega_{\mathcal{X}} \simeq \omega_{\mathcal{Y}}, \quad \mu_* \omega_{\mathcal{X}}(-\overline{D}) \simeq \omega_{\mathcal{Y}}(-\overline{D}).$$

*Proof.* This is clear since $\overline{D}$ does not meet the rational components with self-intersection $-2$ in the special fiber of $\mathcal{X}$. □

For later use, we recall the Grothendieck duality theorem and some of its consequences here following Artin [Art1986]. We temporarily change the notation. Let $R$ be a discrite valuation ring and $X \to \operatorname{Spec} R$ a flat projective morphism of relative dimension one. Assume that $X$ is integral and normal. This implies that the generic fiber of $X$ is smooth and the special fiber is reduced. Let $E$ be a connected reduced subscheme of $X$ supported on the special fiber of $X$. A contraction of $E$

is a morphism $\pi : X \to \overline{X}$ over $\operatorname{Spec} R$, such that $\overline{X}$ is normal, the morphism $\overline{X} \to \operatorname{Spec} R$ is flat projective, the image of $E$ is a point $p \in \overline{X}$ and the restriction

$$\pi|_{X\backslash E} : X\backslash E \to \overline{X}\backslash\{p\}$$

is an isomorphism.

**Lemma 4.1.2** ([Art1986, 1.5]). *1. The sheaf $\mathrm{R}^q\pi_*F = 0$ if $q > 1$ and $\mathrm{R}^1\pi_*F$ is a finite length $\mathcal{O}_{\overline{X}}$ module supported at $p$.*

*2. If $q \neq 0$, then $\mathrm{R}^q\pi_*\omega_X = 0$.*

*3. If $F$ is an $\mathcal{O}_{\overline{X}}$ module of finite length, then $\underline{\mathrm{Ext}}^q(F, \omega_{\overline{X}}) = 0$ if $q \neq 2$. Moreover, $\underline{\mathrm{Ext}}^2(F, \omega_{\overline{X}})$ is dual to $F$ as a finite length $\mathcal{O}_{\overline{X}}$-module. Therefore they are of the same length.*

*4. If $F$ locally free, then $\underline{\mathrm{Ext}}^q(F, \omega_X) = 0$ for any $q \neq 0$.*

The Grothendieck duality theorem [Har1966], applied to the morphism $\pi$, yields that for any coherent sheaf $F$ on $X$, there is a canonical isomorphism

$$\mathrm{R}\pi_*\mathrm{R}\underline{\mathrm{Hom}}(F, \omega_X) \simeq \mathrm{R}\underline{\mathrm{Hom}}(\mathrm{R}\pi_*F, \omega_{\overline{X}}).$$

in the derived category of the coherent sheaves on $\overline{X}$. It then follows from Lemma 4.1.2 that

**Lemma 4.1.3** ([Art1986, 1.6]). *Suppose $F$ is locally free. Then there is an exact sequence*

$$0 \to \pi_*\underline{\mathrm{Hom}}(F, \omega_X) \to \underline{\mathrm{Hom}}(\pi_*F, \omega_{\overline{X}}) \to \underline{\mathrm{Ext}}^2(R^1\pi_*F, \omega_{\overline{X}}) \to R^1\pi_*\underline{\mathrm{Hom}}(F, \omega_X) \to 0. \tag{4.1}$$

## 4.2 The case of IRRED

We are going to prove Proposition 3.4.1 in the case IRRED in this section. Recall that $Y$ is irreducible and has single node $p$. The normalization $Y'$ of $Y$ is not hyperelliptic.

We first study the relative dualizing sheaf $\omega_{\mathcal{Y}/R}$.

**Lemma 4.2.1.** *The dualizing sheaf $\omega_{\mathcal{Y}/R}$ of $\mathcal{Y}$ is very ample. The canonical embedding of $Y$ is a complete intersection of a quadric surface $Q$ and a cubic surface $C$ in $\mathbb{P}^3$. Moreover, if $Q$ is singular, then $Y$ does not pass through the singular point of $Q$.*

*Proof.* What we need to prove is that for any points $x, y \in Y$, the canonical morphisms

$$\mathrm{H}^0(Y, \omega_Y) \to \mathcal{O}_{Y,x}/\mathfrak{m}_x \oplus \mathcal{O}_{Y,y}/\mathfrak{m}_y, \quad \mathrm{H}^0(Y, \omega_Y) \to \mathcal{O}_{Y,x}/\mathfrak{m}_x^2$$

are surjective. Since $Y'$ is not hyperelliptic, the lemma is clear if neither $x$ nor $y$ is the node.

Suppose $x = p$ and $y \neq p$. There is a holomorphism differential form on $Y'$ that is not zero $y$. This differential form descends to a section of $\omega_Y$ on $Y$ that vanishes on $p$ and is not zero at $y$. We can also find a form $\alpha$ on $Y'$ that is has first order pole at $q_1$ and $q_2$ with $\mathrm{Res}_{q_1} \alpha + \mathrm{Res}_{q_2} \alpha = 0$. If this form is not zero at $y$, then let $\beta$ be a holomorphic form on $Y'$ that is not zero at $y$ and some suitable combination of $\alpha$ and $\beta$ gives a form on $Y'$ that has first order pole at $q_1$ and $q_2$ and vanishes at $y$. This gives a sections of $\omega_Y$ on $Y$ that vanishes on $y$ but not at $p$.

To show that $\mathrm{H}^0(Y, \omega_Y) \to \mathcal{O}_{Y,p}/\mathfrak{m}_p^2$ is surjective, it is enough to show that $\mathrm{H}^0(Y, \omega_Y) \to \mathfrak{m}_p/\mathfrak{m}_p^2$ is surjective. we note that there is an isomorphism $\mathfrak{m}_p/\mathfrak{m}_p^2 \simeq \mathfrak{m}_{q_1}/\mathfrak{m}_{q_1}^2 \oplus \mathfrak{m}_{q_2}/\mathfrak{m}_{q_2}^2$. A holomorphic form $\alpha$ on $X'$ that vanishes at $q_1$ but not at $q_2$ gives a surjection to $\mathfrak{m}_{q_1}/\mathfrak{m}_{q_1}^2$. Similar for $\mathfrak{m}_{q_2}/\mathfrak{m}_{q_2}^2$.

The description of the canonical embedding of $Y$ is obtained similarly as Lemma 2.1.2. □

*Proof of Proposition 3.4.1.* Proposition 3.4.1 follows from Lemma 3.2.6. □

## 4.3 The case of ELL

In this section, we prove Proposition 3.4.1 in the case ELL. The special fiber $Y$ consist of two components $C$ and $E$ of genus three and one respectively, and they meet at a node $p$. The component $C$ is not hyperelliptic.

By our description of the integral model, the divisor $\overline{D}$ intersects with $C$ at three point $p_1 + p_2 + p_3$ on $C$ and does not meet $E$. It does not pass through the node $p$. Let $C \to \mathbb{P}^2$ be the canonical embedding of $C$ and $l$ the tangent line of $C$ at $p$. Then $l$ intersects $C$ at two other points $q_1$ and $q_2$ (which might coincide). Then the divisor $p_1 + p_2 + p_3$ on $C$ is linearly equivalent to $2p + q_1$ or $2p + q_2$.

We define

$$\overline{\mathcal{X}} = \mathrm{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{Y}, \mathcal{O}(n\overline{D}))$$

and let $f : \mathcal{Y} \to \mathrm{Spec}\, R$ and $g : \overline{\mathcal{X}} \to \mathrm{Spec}\, R$ the structure morphism. There is a canonical morphism $\pi : \mathcal{Y} \to \overline{\mathcal{X}}$, which contracts $E$. The special fiber of $\overline{X}$ is now an irreducible reduced curve $\overline{C}$ acquiring a cusp at $p$. Since divisor $\overline{D}$ does not meet $E$, its image in $\overline{\mathcal{X}}$, which we still denote by $\overline{D}$ is still an effective Cartier divisor. It is also clear that $\pi_* \mathcal{O}_{\mathcal{Y}}(\overline{D}) = \mathcal{O}_{\overline{\mathcal{X}}}(\overline{D})$ and $\pi^* \mathcal{O}_{\mathcal{Y}}(\overline{D}) = \mathcal{O}_{\overline{\mathcal{X}}}(\overline{D})$.

**Lemma 4.3.1.** *The relative dualizing sheaf $\omega_{\overline{\mathcal{X}}/R}$ is relatively very ample.*

*Proof.* The special fiber $\overline{C}$ is of arithmetic genus four and normalization $C$ is of genus three. This means that the delta invariant of the cusp is 1 [Har1977, Chapter 4]. Note that this is a unibranch singularity therefore locally it must be of the form $y^2 = x^3$. A local generator of $\omega_{\overline{C}}$ is $\mathrm{d}x/y$. A similar argument as the proof of Lemma 4.2.1 proves the lemma. □

**Lemma 4.3.2.** *The $R$-module $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}$ is torsion of length $e$.*

*Proof.* We use the formal function theorem to compute the length of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}$. Let $\mathcal{I}$ be the defining ideal of $E$ on $\mathcal{Y}$, and $E_n$ be the closed subscheme of $\mathcal{Y}$ defined by $\mathcal{I}^n$. We have

$$\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}} \simeq \widehat{\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}} \simeq \varprojlim_n \mathrm{H}^1(E_n, \mathcal{I}^n).$$

Here $\widehat{\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}}$ is the completion of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}$ as an $R$-module. It is isomorphic to $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}$ because it is a torsion $R$-module. The second isomorphism follows from the formal function theorem.

We note that $E$ is not a Cartier divisor on $\mathcal{Y}$, but $eE$ is a Cartier divisor. Let $\mathcal{I}_e$ be the defining ideal of $eE$. This is an ideal locally generated by a nonzero divisor. We observe that the system of ideals $\{\mathcal{I}^n\}$ is cofinal to the system $\{\mathcal{I}_e^n\}$. So to calculate $\varprojlim \mathrm{H}^1(E_n, \mathcal{I}^n)$, we may use the system $\{\mathcal{I}_e^n\}$ instead. Let $E'_n$ be the Cartier divisor defined by the ideal $\mathcal{I}_e^n$. There is an exact sequence

$$0 \to \mathcal{I}_e^n/\mathcal{I}_e^m \to \mathcal{O}_{E'_m} \to \mathcal{O}_{E'_n} \to 0$$

for any $m \geq n$. It follows that

$$\mathrm{H}^1(E'_{m-n}, \mathcal{I}_e^n/\mathcal{I}_e^m) \to \mathrm{H}^1(E'_m, \mathcal{O}_{E'_m}) \to \mathrm{H}^1(E'_n, \mathcal{O}_{E'_n}) \to 0.$$

We claim that for any $n \geq 1$, there is an isomorphism $\mathrm{H}^1(E'_n, \mathcal{I}_e^n) \simeq \mathrm{H}^1(E'_{n+1}, \mathcal{I}_e^{n+1})$. By definition, $\mathcal{I}_e$ is locally generated by a nonzero divisor. Therefore

$$\mathcal{I}_e^n/\mathcal{I}_e^{n+1} \simeq \mathrm{Sym}^n\, \mathcal{I}_e/\mathcal{I}_e^2, \quad n \geq 1.$$

When $n = 1$, the scheme $E'$ is an elliptic curve over $R/\varpi^e$ with a section $s$ and $\mathcal{I}_e/\mathcal{I}_e^2 \simeq \mathcal{O}_{E'}(s)$. Therefore

$$\mathrm{H}^1(E', \mathcal{I}_e^n/\mathcal{I}_e^{n+1}) \simeq \mathrm{H}^1(E', \mathcal{O}_{E'}(ns)) = 0$$

for all $n \geq 1$. This shows our claim. Moreover, when $n = 1$, $\mathrm{H}^1(E', \mathcal{O}_{E'}) \simeq R/\varpi^e$. Then $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{X}} \simeq R/\varpi^e$ by formal function theorem. The lemma is thus proved. $\square$

**Lemma 4.3.3.** *The cokernel of $f_*\omega_{\mathcal{Y}} \to g_*\omega_{\overline{\mathcal{X}}}$ is of length $e$.*

*Proof.* Consider the exact sequence (4.1)

$$0 \to \pi_*\omega_{\mathcal{Y}} \to \omega_{\overline{\mathcal{X}}} \to \underline{\mathrm{Ext}}^2(\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{Y}}) \to 0.$$

It gives rise to the long exact sequence

$$0 \to f_*\omega_{\mathcal{Y}} \to g_*\omega_{\overline{\mathcal{X}}} \to g_*\underline{\mathrm{Ext}}^2(\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}, \omega_{\overline{\mathcal{X}}}) \to \mathrm{R}^1g_*\pi_*\omega_{\mathcal{Y}}.$$

It follows from the spectral sequence $\mathrm{R}^pg_*\mathrm{R}^q\pi_*\omega_{\mathcal{Y}} \Rightarrow \mathrm{R}^{p+q}f_*\omega_{\mathcal{Y}}$ that there is an exact sequence

$$0 \to \mathrm{R}^1g_*\pi_*\omega_{\mathcal{Y}} \to \mathrm{R}^1f_*\omega_{\mathcal{Y}} \to g_*\mathrm{R}^1\pi_*\omega_{\mathcal{Y}} \to 0.$$

Therefore $\mathrm{R}^1g_*\pi_*\omega_{\mathcal{Y}} \simeq \mathrm{R}^1f_*\omega_{\mathcal{Y}} \simeq R$ since $\mathrm{R}^1\pi_*\omega_{\mathcal{Y}} = 0$.

The sheaf $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}$ is supported at $p$ and $\underline{\mathrm{Ext}}^2(\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}, \omega_{\overline{\mathcal{X}}})$ and $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}$ are dual as finite length $\mathcal{O}_{\overline{\mathcal{X}}}$-modules. Thus $\underline{\mathrm{Ext}}^2(\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}}, \omega_{\overline{\mathcal{X}}}) \to \mathrm{R}^1g_*\pi_*\omega_{\mathcal{Y}}$ is the zero map. The lemma then follows. $\square$

**Lemma 4.3.4.** *The $R$-module $\mathrm{R}^1f_*\omega_{\mathcal{Y}}(-\overline{D})$ is a vector bundle of rank two and compatible with any base change. The cokernel of $f_*\omega_{\mathcal{Y}}(-\overline{D}) \to g_*\omega_{\overline{\mathcal{X}}}(-\overline{D})$ is also of length $e$.*

*Proof.* The proof is similar to that of Lemma 4.3.2. We leave the details to the interested reader. $\square$

*Proof of Proposition 3.4.1 in the case ELL.* Consider the cup product

$$\mathrm{H}^0(\overline{\mathcal{X}}, \mathcal{O}(\overline{D})) \otimes \mathrm{H}^0(\overline{\mathcal{X}}, \omega_{\overline{\mathcal{X}}/R}(-\overline{D})) \to \mathrm{H}^0(\overline{\mathcal{X}}, \omega_{\overline{\mathcal{X}}/R}) \tag{4.2}$$

whose cokernel is denoted by $\overline{Q}$. It follows from Lemma 3.2.6 that the length of $c_1(\overline{Q})$ as a zero dimensional $R$ scheme is $\frac{1}{2}p(X)$.

There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{O}_{\mathcal{Y}}(\overline{D}) \otimes f_*\omega_{\mathcal{Y}}(-\overline{D}) & \longrightarrow & f_*\omega_{\mathcal{Y}} & \longrightarrow & Q & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_*\mathcal{O}_{\overline{\mathcal{X}}}(\overline{D}) \otimes g_*\omega_{\overline{\mathcal{X}}}(-\overline{D}) & \longrightarrow & g_*\omega_{\overline{\mathcal{X}}} & \longrightarrow & \overline{Q} & \longrightarrow & 0
\end{array}$$

Note that $f_*\mathcal{O}_{\mathcal{Y}}(\overline{D})$ and $g_*\mathcal{O}_{\overline{\mathcal{X}}}(\overline{D})$ are free of rank two. The Proposition 3.4.1 then follows from the snake lemma and Lemma 4.3.3 and Lemma 4.3.4. □

## 4.4 The case of TWO

In this section, we prove Proposition 3.4.1 in the case TWO. The special fiber $Y$ consists of two genus two components $C_1$ and $C_2$ meeting at a node $p$. By assumption, the node $p$ is not a Weierstrauss point on either component. The local equation of $\mathcal{Y}$ at the node $p$ is $xy - \varpi^e$. The closure $\overline{D}$ of $D$ on $\mathcal{Y}$ intersects $C_1$ at two points $p_1 + p_2$ and $C_2$ at a point $q$. Moreover, $\dim \mathrm{H}^0(C_1, \mathcal{O}(p_1 + p_2)) = 2$ and $q$ is the hyperelliptic involution of $p$ on $C_2$.

**Lemma 4.4.1.** *The base point of $\overline{D}$ is a length $e$ zero dimensional subscheme of $\mathcal{Y}$ supported at $q$. In other words, the cokernel $K$ of*

$$\mathrm{H}^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\overline{D})) \to \mathcal{O}_{\mathcal{Y}}(\overline{D})$$

*is supported at $q$ and is of length $e$ as an $R$-module.*

*Proof.* By our description of the integral model $\mathcal{Y}$ and $\overline{D}$, it is clear that the base point of $\overline{D}$ is supported at $q$. We first note that the length of the base point is at least $e$. This is because $\mathcal{Y} \otimes_R R/\varpi^e$ is reducible and consists of two genus two curves $C_{1,e}$ and $C_{2,e}$ over $R/\varpi^e$. There are sections $s_i : R/\varpi^e \to C_{i,e}$ of $C_{i,e}$, $i = 1, 2$, along which $C_{1,e}$ and $C_{2,e}$ are glued to obtain $\mathcal{Y} \otimes R/\varpi^e$. The intersection of $\overline{D}$ and $\mathcal{Y} \otimes R/\varpi^e$ is a subscheme of the base locus of $\overline{D}$. The base locus of $\overline{D}$ is thus at least of length $e$.

Let $\mathcal{I}$ be the defining ideal of the Cartier divisor $eC_2$ on $\mathcal{Y}$ and $C_{2,n}$ be the closed subscheme of $\mathcal{X}$ defined by $\mathcal{I}^n$. We consider the exact sequence

$$\mathrm{H}^0(C_{2,n}, \mathcal{O}(\overline{D})|_{C_{2,n}}) \to \mathcal{O}(\overline{D})|_{C_{2,n}} \to K_n \to 0.$$

If $m \geq n$, then the canonical morphism $K_m \to K_n$ is surjective. We claim that if for any $n \geq 1$, then the morphism $K_{n+1} \to K_n$ is an isomorphism. Moreover, $K_1$ is of length $e$. In fact, $K_1$ is isomorphic to the structure sheaf of the intersection of $\overline{D}$ and $\mathcal{Y} \otimes R/\varpi^e$. So $K_n$ is of length $e$ for any $n \geq 1$.

To prove that $K_{n+1} \to K_n$ is an isomorphism, we make use of the exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}(\overline{D}) \to \mathcal{O}(\overline{D})|_{C_{2,n+1}} \to \mathcal{O}(\overline{D})|_{C_{2,n}} \to 0$$

Since $\mathcal{I}$ is locally generated by a nonzero divisor, one has $\mathcal{I}^n/\mathcal{I}^{n+1} \simeq \operatorname{Sym}^n \mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}(ns_2)$ on $C_{2,1}$. It follows that the morphism

$$\mathrm{H}^0(C_{2,1}, \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}(\overline{D})) \to \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}(\overline{D})$$

is surjective. Moreover, $\mathrm{H}^1(C_{2,1}, \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}(\overline{D})) = 0$ since $s_2$ is not a Weierstrauss point on $C_{2,1}$. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}(\overline{D}) & \longrightarrow & \mathcal{O}(\overline{D})|_{C_{2,n+1}} & \longrightarrow & \mathcal{O}(\overline{D})|_{C_{2,n}} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{H}^0(C_{2,1}, \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}(\overline{D})) & \longrightarrow & \mathrm{H}^0(C_{2,n+1}, \mathcal{O}(\overline{D}|_{C_{2,n+1}})) & \longrightarrow & \mathrm{H}^0(C_{2,n}, \mathcal{O}(\overline{D})|_{C_{2,n}}) & \longrightarrow & 0
\end{array}.$$

It follows then from the snake lemma that $K_{n+1} \to K_n$ is an isomorphism.

We now relate $K_n$ with $K$. Since $K$ is supported on $q \in C_2$ and the morphism $\mathcal{Y} \to \operatorname{Spec} R$ is smooth at $q$. Since $K$ is of finite length, we have $K \otimes \mathcal{O}/\mathcal{I}^n \simeq K$ for sufficiently large $n$. Moreover the image of $\mathrm{H}^0(\mathcal{X}, \mathcal{O}(D))$ in $\mathcal{O}(\overline{D}) \otimes \mathcal{O}/\mathcal{I}^n$ is a subsheaf of the image of $\mathrm{H}^0(C_{2,n}, \mathcal{O}(\overline{D})|_{C_{2,n}})$ in $\mathcal{O}/\mathcal{I}^n$ since the morphism $\mathrm{H}^0(\mathcal{X}, \mathcal{O}(\overline{D})) \to \mathcal{O}(\overline{D}) \otimes \mathcal{O}/\mathcal{I}^n$ factors through $\mathrm{H}^0(C_{2,n}, \mathcal{O}(\overline{D})|_{C_{2,n}}) \to \mathcal{O}(\overline{D}) \otimes \mathcal{O}/\mathcal{I}^n$. Thus $K \otimes \mathcal{O}/\mathcal{I}^n$ is a subsheaf of $K_n$. It follows that $K \simeq K_1$ which is of length $e$. □

Let $B$ be the base locus of $\overline{D}$ on $\mathcal{Y}$ and $\mathcal{I}_B$ its defining ideal, i.e. the image of $\mathrm{H}^0(\mathcal{Y}, \mathcal{O}(\overline{D})) \otimes \mathcal{O}(-\overline{D}) \to \mathcal{O}_{\mathcal{Y}}$. It follows from (the proof of) Lemma 4.4.1 that $B$ is flat over $R/\varpi^e$. Let $\nu : \mathcal{Y}' \to \mathcal{Y}$ be the blowup of $\mathcal{Y}$ along $\mathcal{I}_B$. Then the closure $\overline{D}$ of $D$ in $\mathcal{Y}'$ is base point free. The exceptional divisor of $\mathcal{Y}' \to \mathcal{Y}$ is isomorphic to $\mathbb{P}^1$ over $B$. We denote the reduced part of the exceptional divisor by $E$. Note that this not a Cartier divisor unless $e = 1$. Let $\mathcal{I}_E$ be its defining ideal and $E_n$ the closed subscheme defined by $\mathcal{I}_E^n$. Then $E_e$ is the exceptional divisor of $\nu$ which is a Cartier divisor. Denote the intersection of $E$ and $C_2$ by $p'$. Then local equation of $\mathcal{Y}'$ at $p'$ is $xy - \varpi^e$. We denote the intersection of $\overline{D}$ and $E$ by $p_3$. We denote the structure morphism $\mathcal{Y}' \to \operatorname{Spec} R$ by $f$ and the structure morphism $\mathcal{Y} \to \operatorname{Spec} R$ by $g$.

**Lemma 4.4.2.** *Let $Q'$ be the cokernel of*

$$\mathrm{H}^0(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}(\overline{D})) \otimes \mathrm{H}^0(\mathcal{Y}', \omega_{\mathcal{Y}'/R}(-\overline{D})) \to \mathrm{H}^0(\mathcal{Y}', \omega_{\mathcal{Y}'/R}).$$

*Then $Q' \simeq Q$.*

*Proof.* It follows from the general theory of blowup that

$$\nu_*\mathcal{O}_{\mathcal{Y}'}(\overline{D}) \simeq \mathcal{I}_B, \quad \nu_*\mathcal{O}_{\mathcal{Y}'}(-\overline{D}) \simeq \mathcal{O}_{\mathcal{Y}}(-\overline{D}), \quad \nu_*\omega_{\mathcal{Y}'/R} \simeq \omega_{\mathcal{Y}/R}.$$

We also note that $\mathrm{H}^0(\mathcal{Y}, \mathcal{O}(\overline{D})) \simeq \mathrm{H}^0(\mathcal{Y}, \mathcal{O}(\overline{D} \otimes \mathcal{I}_B))$ since $B$ is the base locus of $\overline{D}$ on $\mathcal{Y}$. The lemma then follows. $\square$

**Lemma 4.4.3.** *There is an $R$-module isomorphism* $\mathrm{R}^1 f_*\omega_{\mathcal{Y}'}(-\overline{D}) \simeq R^{\oplus 2} \oplus R/\varpi^e$.

*Proof.* It follows from the spectral sequence $\mathrm{R}^p g_*\mathrm{R}^q\nu_*\omega_{\mathcal{Y}'}(-\overline{D}) \Rightarrow \mathrm{R}^{p+q} f_*\omega_{\mathcal{Y}'}(-\overline{D})$ that there is an exact sequence

$$0 \to \mathrm{R}^1 g_*\omega_{\mathcal{Y}}(-\overline{D}) \to \mathrm{R}^1 f_*\omega_{\mathcal{Y}'}(-\overline{D}) \to f_*\mathrm{R}^1\nu_*\omega_{\mathcal{Y}'}(-\overline{D}) \to 0 \tag{4.3}$$

We have $\mathrm{R}^1 g_*\omega_{\mathcal{Y}}(-\overline{D}) \simeq R^{\oplus 2}$. We then need to compute $\mathrm{R}^1\nu_*\omega_{\mathcal{Y}'}(-\overline{D})$. By the projection formula,

$$\mathrm{R}^1\nu_*\omega_{\mathcal{Y}'}(-\overline{D}) \simeq \omega_{\mathcal{Y}}(-\overline{D}) \otimes \mathrm{R}^1\nu_*\mathcal{O}(E_{2e}).$$

We use the formal function theorem to compute $\mathrm{R}^1\nu_*\mathcal{O}(E_{2e})$. There's an exact sequence

$$0 \to \mathcal{I}^{ne}/\mathcal{I}^{(n+1)e} \otimes \mathcal{O}(E_{2e}) \to \mathcal{O}(E_{2e})|_{E_{(n+1)e}} \to \mathcal{O}(E_{2e})|_{E_{ne}} \to 0.$$

Taking long exact sequence for the global section on $\mathcal{Y}'$ gives an isomorphism

$$H^1(E_{(n+1)e}, \mathcal{O}(E_{2e})|_{E_{(n+1)e}}) \simeq H^1(E_{ne}, \mathcal{O}(E_{2e})|_{E_{ne}}).$$

This is because $E_e$ is an local complete intersection in $\mathcal{Y}'$ and is isomorphic to $\mathbb{P}^1$ over $B$. Therefore $\mathcal{I}^{ne}/\mathcal{I}^{(n+1)e} \simeq \mathrm{Sym}^n(\mathcal{I}^e/\mathcal{I}^{2e}) \simeq \mathcal{O}(n)$. It follows from the formal function theorem that

$$\mathrm{R}^1\nu_*\omega_{\mathcal{Y}'}(-\overline{D}) \simeq \mathrm{H}^1(E_e, \mathcal{O}(E_{2e})|_{E_e}) \simeq R/\varpi^e.$$

We are left to show that extension (4.3) splits. We note that $\mathrm{R}^1 f_*\omega_{\mathcal{Y}'}(-\overline{D})$ is compatible with any base change since the relative dimension of $f$ is one. Therefore $\mathrm{R}^1 f_*\omega_{\mathcal{Y}'}(-\overline{D}) \otimes_R R/\varpi^e$ is a free $R/\varpi^e$-module of rank three. The split extension is the only extension of $R/\varpi^e$ by $R^{\oplus 2}$ such that the base change to $R/\varpi^e$ gives a free $R/\varpi^e$ moduli of rank three. $\square$

Let

$$\pi : \mathcal{Y}' \to \overline{\mathcal{X}} := \operatorname{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}(n\overline{D}))$$

This is the morphism that contracts the component $C_2$ on $\mathcal{Y}'$. We denote by $h : \overline{\mathcal{X}} \to \operatorname{Spec} R$ the structure morphism and by $\overline{X}$ the special fiber of $\overline{\mathcal{X}}$. The closure $\overline{D}$ of $D$ on $\overline{\mathcal{X}}$ does not pass through the singular point on the special fiber of $\mathcal{X}$, hence it is an effective Cartier divisor on $\overline{\mathcal{X}}$. We denote the image of the components $C_1$ and $E$ by $\overline{C_1}$ and $\overline{E}$ respetively. The point where they meet is denoted by $\overline{p}$.

**Lemma 4.4.4.** *The components $\overline{C_1}$ and $\overline{E}$ of $\overline{X}$ are both smooth.*

*Proof.* To prove this lemma, we only need to exhibit a function $f$ in a formal neighborhood $U$ of $C_2$ on $\mathcal{X}'$, so that $f$ vanishes at along $C_2$ and vanishes exactly to the order one at the point $C_1 \cap U$ and $E \cap U$.

Let $1_D \in \mathrm{H}^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\overline{D}))$ be the canonical section of $\overline{D}$. Then we claim that $1_D$ vanishes along $C_2$. Indeed, $\mathrm{H}^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\overline{D}))$ is a vector bundle of rank two over $\operatorname{Spec} R$ and is compatible with base change. Since

$$\mathrm{H}^0(\overline{X}, \mathcal{O}(\overline{D})|_{\overline{X}}) \simeq \mathrm{H}^0(C_1, \mathcal{O}(\overline{D})|_{C_1}),$$

the section $1_D$ must be constant along $C_2$. Moreover, $1_D|_{\overline{X}}$ vanishes at $q$. Therefore $1_D$ vanishes along $C_2$. This proves our claim. It also follows from this argument that $1_D|_{C_1}$ vanishes exactly to the order one at $p$, since

$$\mathrm{H}^0(C_1, \mathcal{O}(\overline{D})|_{C_1}(-2p)) = 0.$$

We have $\mathrm{H}^0(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}(\overline{D})) \simeq \mathrm{H}^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\overline{D}))$ since $B$ is the base locus of $\overline{D}$ on $\mathcal{Y}$. Let $1'_D$ be the image of $1_D$ under this isomorphism. Then $1'_D$ vanishes along $C_2$. Moreover, $1'_D|_E$ vanishes exactly to the order one at $p'$. Otherwise $1'_D|_E$ vanishes to the order two at $p'$ and has at most an order one pole at $p'$. This is not possible. Therefore $1'_D|_U$ is the desire function on any formal neighborhood $U$ of $C_2$. The lemma is thus proved. □

**Lemma 4.4.5.** *The cup product $h_*\mathcal{O}_{\overline{\mathcal{X}}}(\overline{D}) \otimes h_*\omega_{\overline{\mathcal{X}}}(-\overline{D}) \to h_*\omega_{\overline{\mathcal{X}}}$ is surjective.*

*Proof.* To prove the surjectivity, we only need to show that the cup product on the special fiber is surjective and $h_*\mathcal{O}_{\overline{\mathcal{X}}}(\overline{D})$ and $h_*\omega_{\overline{\mathcal{X}}}(-\overline{D})$ are both compatible with base change.

Consider the special fiber $\overline{X}$. The components $\overline{C_1}$ and $\overline{E}$ are both smooth. Therefore $\overline{X}$ has planar singularity at $\overline{p}$. With a suitably choosing the coordinates, the local equation of the singularity is $y(y - x^r)$. We see that the $r = 3$ since the arithmetic genus of $\overline{X}$ is four while the arithmetic genus of $\overline{C_1}$ is two and $\overline{E}$ is rational. Therefore by the description of $\omega_{\overline{X}}$ as the sheaf of Rosenlisch differentials [BHPVdV2004, Chapter II Proposition 6.2], one sees that $\omega_{\overline{X}}|_{\overline{C_1}} = \omega_{\overline{C_1}}(3p)$ and $\omega_{\overline{X}}|_{\overline{E}} = \omega_{\overline{E}}(3p)$. The local generator of $\omega_{\overline{X}}$ at the singularity is $\mathrm{d}x/x^3$.

We have $\mathrm{H}^0(\overline{E}, \omega_{\overline{E}}(3p))$ is one dimensional, as $\overline{E}$ is isomorphic to $\mathbb{P}^1$. We choose a global section $\eta \in \mathrm{H}^0(\overline{X}, \omega_{\overline{\mathcal{X}}}(-\overline{D})|_{\overline{X}})$. Assume that the restriction of $\eta$ to $\overline{E}$ does not vanish. Then by the description of the Rosenlisch differential, the restriction of $\eta$ to $\overline{C_1}$ at the point $p$ must have an order three pole at $p$. On the other hand, from the Riemann–Roch theorem, one sees that

$$\dim \mathrm{H}^0(\overline{C_1}, \omega_{\overline{C_1}}(-\overline{D}|_{\overline{C_1}} + 3p)) - \dim \mathrm{H}^0(\overline{C_1}, \omega_{\overline{C_1}}(-\overline{D}|_{\overline{C_1}} + 2p)) = 1$$

This shows that there is a unique Rosenlisch differential form, up to a constant multiple, that is nonzero when restricted to $\overline{E}$. If the restriction of $\eta$ to $\overline{E}$ is zero, then the restriction of $\eta$ to $\overline{C_1}$ is holomorphic at $p$. One has $\dim \mathrm{H}^0(\overline{C_1}, \omega_{\overline{C_1}}(-\overline{D}|_{\overline{C_1}})) = 1$. This shows $\dim \mathrm{H}^0(\overline{X}, \omega_{\overline{X}}(-\overline{D})) = 2$. This implies by duality and base change theorem that $h_*\mathcal{O}_{\overline{\mathcal{X}}}(\overline{D})$ and $h_*\omega_{\overline{\mathcal{X}}}(-\overline{D})$ are both vector bundles and are compatible with base change.

We now prove that the cup product on the special fiber is an isomorphism. As we have seen above, the space $\mathrm{H}^0(\overline{X}, \omega(-\overline{D})|_{\overline{X}})$ is generated by two Rosenlisch differentials $\eta_1$ and $\eta_2$, where $\eta_1$ is zero when restricted to $\overline{E}$ and $\eta_2$ has an order three pole when restricted to $\overline{E}$. We note also that $\mathrm{H}^0(\overline{X}, \mathcal{O}(\overline{D})|_{\overline{X}})$ is generated by two elements 1 and $s$ where 1 is the constant function 1 when restricted to each component while $s$ is a meromorphism function which has precisely poles at $\overline{D}|_{\overline{X}}$ when restricted to each component. One can choose $s$ so that it has a simple zero at $\overline{p}$ when restricted to both components.

Then by looking at the zero's and poles of the Rosenlisch differentials $\eta_1$, $\eta_2$, $\eta_1 s$ and $\eta_2 s$, we see that they are linearly independent in $\mathrm{H}^0(\overline{X}, \omega_{\overline{X}})$. This show that the cup product is injective, hence an isomorphism. □

**Lemma 4.4.6.** *1. The sheaf* $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Y}'}$ *is supported at* $\overline{p}$ *and is of length* $3e$.

*2. The morphism* $f_*\omega_{\mathcal{X}'} \to h_*\omega_{\overline{\mathcal{X}}}$ *is injective and the cokernel is of length* $3e$.

*Proof.* This is the similar computation as in Lemma 4.3.3 using the formal function theorem and Lemma 4.1.3. We leave the details to interested readers. □

**Lemma 4.4.7.** *The morphism* $f_*\omega_{\mathcal{Y}'}(-\overline{D}) \to h_*\omega_{\overline{\mathcal{X}}}(-\overline{D})$ *is injective and the cokernel is of length* $2e$.

*Proof.* It follows from the projection formula and Lemma 4.4.6 that there is a short exact sequence

$$0 \to \pi_* \omega_{\mathcal{Y}'}(-\overline{D}) \to \omega_{\overline{\mathcal{X}}}(-\overline{D}) \to T \to 0,$$

where $T$ is a torsion sheaf supported at $\overline{p}$ and is of length $3e$. Pushing forward via $h_*$, we get

$$0 \to f_* \omega_{\mathcal{Y}'}(-\overline{D}) \to h_* \omega_{\overline{\mathcal{X}}}(-\overline{D}) \to h_* T \to \mathrm{R}^1 f_* \omega_{\mathcal{Y}'}(-\overline{D}) \to \mathrm{R}^1 h_* \omega_{\overline{\mathcal{X}}}(-\overline{D}) \to 0.$$

Here we have used the fact that $\mathrm{R}^1 \pi_* \omega_{\mathcal{Y}'}(-\overline{D}) = 0$ to conclude $\mathrm{R}^1 h_* \pi_* \omega_{\mathcal{Y}'}(-\overline{D}) \simeq \mathrm{R}^1 f_* \omega_{\mathcal{Y}'}(-\overline{D})$.

It follows from Lemma 4.4.3 that $\mathrm{R}^1 f_* \omega_{\mathcal{Y}'}(-\overline{D})$ is isomorphic to $R^{\oplus 2} \oplus R/\varpi^e$. We have also proved in Lemma 4.4.5 that $h_* \mathcal{O}(\overline{D})$ is a rank two vector bundle over $R$. It follows from duality that $\mathrm{R}^1 h_* \omega_{\overline{\mathcal{X}}}(-\overline{D})$ is a rank two vector bundle over $R$. The lemma then follows. $\square$

*Proof of Proposition 3.4.1 in the case TWO.* Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_* \mathcal{O}_{\mathcal{X}'}(D') \otimes f_* \omega_{\mathcal{X}'}(-D') & \longrightarrow & f_* \omega_{\mathcal{X}'} & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & g_* \mathcal{O}_{\overline{X}}(\overline{D}) \otimes g_* \omega_{\overline{X}}(-\overline{D}) & \longrightarrow & g_* \omega_{\overline{X}} & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{4.4}$$

Proposition 3.4.1 in the case TWO then follows from the snake lemma and Lemma 4.4.6 and Lemma 4.4.7 $\square$

# Chapter 5

# Proof of Proposition 3.4.2

## 5.1 Fitting ideals

For the convenience of the reader, we recall the notion of the Fitting ideal here. Let $X$ be a scheme and $F$ a coherent sheaf on $X$. Let

$$M \xrightarrow{\gamma} N \to F \to 0$$

be a presentation of $F$, with $M$ and $N$ being locally free sheaves on $X$ of rank $m$ and $n$ respectively. Then the $i$-th Fitting ideal $I_i$ of $F$, is by definition the sheaf of ideals on $X$ generated by the $(n-i+1) \times (n-i+1)$ minor of the morphism $\gamma$. More precisely, this is the image of

$$\wedge^{n-i+1} M \otimes \wedge^{n-i+1} N^{\vee} \to \mathcal{O}_X.$$

The Fitting ideal is independent of the choice of the presentation.

## 5.2 Double cover of the local moduli space

We are going to work on the moduli space $\mathcal{M}_4$ over $\mathbb{Z}[\frac{1}{2}]$. We are going to construct a double cover of $\mathcal{M}_4$ around each point. This double cover is closely related to the cup product map (3.1) via deformation theory.

Let $k$ be an algebraically closed field with char $k \neq 2$. Let $X$ be a hyperelliptic curve over $k$ which gives $k$-point $[X]$ of the moduli space $\mathcal{M}_4$. Let $\mathsf{M}$ be a the strict henselization of $\mathcal{M}_4$ at $[X]$. Let $\mathsf{C} \to \mathsf{M}$ be the universal curve and $\mathsf{J} \to \mathsf{M}$ be the universal Jacobian. Note that in this case, $\mathsf{J} \to \mathsf{M}$ is representable since $\mathsf{C} \to \mathsf{M}$ has a section. The universal Jacobian $\mathsf{J}$ is thus a scheme. We denote by $\mathsf{J}^d \to \mathsf{M}$ the universal Jacobian over $\mathsf{M}$ parameterizing degree $d$ line bundles on $\mathsf{C}$. Let $\mathsf{L}$ be the universal line bundle of degree three on $\mathsf{C} \times_{\mathsf{M}} \mathsf{J}^3$. Let $\mathsf{N}$ be a closed subscheme of $\mathsf{J}^3$ defined by the second Fitting ideal of $\mathrm{R}^1 p_* \mathsf{L}$ where $p : \mathsf{C} \times_{\mathsf{M}} \mathsf{J}^3 \to \mathsf{J}^3$ is the projection to the second factor. So set theoretically, the fiber of $\mathsf{N} \to \mathsf{M}$ consists of the degree three line bundles $L$ on curve $C$ with $\dim \mathrm{H}^0(C, L) \geq 2$. We may in fact replace "$\geq$" by "$=$" as $\dim \mathrm{H}^0(C, L) \leq 2$ for any degree three line bundles on a genus four curve.

We call $\mathsf{Z} = \mathsf{M} \times_{\mathcal{M}_4} Z$ the Petri locus on $\mathsf{M}$. We denote the hyperelliptic locus on $\mathsf{M}$ by $\mathsf{H}$. Then $\mathsf{H} \subset \mathsf{Z}$. Let $\mathsf{M}^\circ = \mathsf{M} \backslash \mathsf{H}$ be the open subscheme of $\mathsf{M}$. Then $\mathsf{N}^\circ = \mathsf{N} \times_{\mathsf{M}} \mathsf{M}^\circ \to \mathsf{M}^\circ$ is finite flat of degree two. Since $\mathsf{Z}$ is regular away from the hyperelliptic locus, the scheme $\mathsf{N}^\circ$ is regular. The fiber of $\mathsf{N} \to \mathsf{M}$ over the closed point $[X]$ of $\mathsf{M}$ is isomorphic to $X$, embedded in $\mathrm{Jac}(X)$ via the morphism

$$X \to \mathrm{Jac}(X), \quad p \mapsto L_0 \otimes \mathcal{O}(p),$$

where $L_0$ is the unique degree two line bundle on $X$ with $\dim \mathrm{H}^0(X, L_0) = 2$ and $p$ is some point on $X$.

We first study the tangent space of $\mathsf{N}$ at $(X, L)$. We denote by $\pi : \mathsf{N} \to \mathsf{M}$ and by $\mathrm{d}\pi : T_{\mathsf{N},(X,L)} \to T_{\mathsf{M},X}$ the tangent morphism. The kernel of $\mathrm{d}\pi$ is at least one dimensional. Consider the following cup product map

$$\mu_0 : \mathrm{H}^0(X, L) \otimes \mathrm{H}^0(X, \omega \otimes L^{-1}) \to \mathrm{H}^0(X, \omega),$$

where $\omega$ stands for the dualizing sheaf of $X$. There is a morphism

$$\mu_1 : \mathrm{Ker}\, \mu_0 \to \mathrm{H}^0(X, \omega_X^{\otimes 2})$$

whose cokernel is dual to the image of $\mathrm{d}\pi$, c.f. [ACG2011, Chapter XXI § 6, p. 808]. We briefly recall the construction of $\mu_1$ here. Recall that $T_{\mathsf{M},X} \simeq \mathrm{H}^1(X, T_X)$. A little deformation theory shows that $T_{\mathsf{N},(X,L)}$ can be interpreted in the following way. The line bundle $L$ gives a cohomology class in $\mathrm{H}^1(X, \omega_X) \simeq \mathrm{Ext}^1(T_X, \mathcal{O}_X)$. This is via taking the logarithmic differential of the transition function of $L$. This class defines a rank two vector bundle on $X$ such that

$$0 \to \mathcal{O}_X \to \Sigma_L \to T_X \to 0.$$

There is a cup product homomorphism, c.f. [ACG2011, chapter XXI, § 5]

$$\mathrm{H}^1(X, \Sigma_L) \to \mathrm{Hom}(\mathrm{H}^0(X, L), \mathrm{H}^1(X, L)).$$

The kernel is identified with the tangent space of $\mathcal{N}$ at $(X, L)$. The dual of this homomorphism "lifts" the usual cup product homomorphism

$$\mu_0 : \mathrm{H}^0(X, L) \otimes \mathrm{H}^0(X, \omega_X \otimes L^{-1}) \to \mathrm{H}^0(X, \omega_X).$$

One can define the following homomorphism

$$\mu_1 : \mathrm{Ker}\,\mu_0 \to \mathrm{H}^0(X, \omega_X^{\otimes 2}),$$

whose cokernel is dual to the image of $\mathrm{d}\pi$ at the point $(X, L)$. This homomorphism fits in the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Ker}\,\mu_0 & \xrightarrow{\mu_1} & \mathrm{H}^0(\omega^{\otimes 2}) & \longrightarrow & \mathrm{coker}\,\mu_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{H}^0(L) \otimes \mathrm{H}^0(\omega L^{-1}) & \longrightarrow & \mathrm{H}^0(\omega \Sigma_L^\vee) & \longrightarrow & T^\vee_{\mathsf{N},(X,L)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^0(L) \otimes \mathrm{H}^0(\omega L^{-1})/\mathrm{Ker}\,\mu_0 & \xrightarrow{\mu_0} & \mathrm{H}^0(\omega) & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (5.1)$$

Here all the global sections are over $X$. The homomorphism $\mathrm{H}^0(\omega^{\otimes 2}) \to T^\vee_{\mathsf{N},(X,L)}$ is dual to $\mathrm{d}\pi$. The kernel of it is thus the orthogonal complement of image of $\mathrm{d}\pi$. By snake lemma, the homomorphism $\mathrm{coker}\,\mu_1 \to T^\vee_{\mathsf{N},(X,L)}$ is injective. Therefore the kernel of $\mathrm{H}^0(\omega^{\otimes 2}) \to T^\vee_{\mathsf{N},(X,L)}$ is identified with the image of $\mu_1$.

**Lemma 5.2.1.** *The scheme* $\mathsf{N}$ *is regular.*

*Proof.* We only need to show that $\mathsf{N}$ is regular at the points over the closed point $[X]$ of $\mathsf{M}$. Such a point corresponds to a pair $(X, L)$ where $L$ is a degree three line bundle on $X$ and $\dim \mathrm{H}^0(X, L) = 2$.

We are going to show that the image of $T_{\mathsf{N},(X,L)}$ under $\mathrm{d}\pi$ is eight dimensional. This implies the regularity of $\mathsf{N}$ at $(X, L)$. From the discussion of the homomorphism $\mu_1$ above, we see that it is enough to show that $\operatorname{Ker}\mu_1$ is one-dimensional.

We now choose charts and calculate $\mu_1$ using Cech cocycles. Choose a chart on $X$ as follows. Let $X = U_0 \cup V_0$, where $U_0$ is the subscheme of $\mathbb{A}^2$ given by the equation $y^2 = f(x)$, and $V_0$ is the subscheme of $\mathbb{A}^2$ given by the equation $v^2 = g(u)$ and $x = u^{-1}$, $y = u^{-5}v$. In this chart, one has

1. A basis of $\mathrm{H}^1(X, T_X)$ is
$$\{\theta_i,\ i = 1, \cdots, 7;\ \eta_j,\ j = 1, 2\},$$
where
$$\theta_i = x^{-i}y^2\frac{\partial}{\partial x}, \quad \eta_j = x^{-j}y\frac{\partial}{\partial x}.$$

2. A basis of $\mathrm{H}^0(X, \omega_X^{\otimes 2})$ is
$$\{\rho_i,\ i = 1, \cdots, 7;\ \sigma_j,\ j = 1, 2\},$$
where
$$\rho_i = x^{i-1}\left(\frac{\mathrm{d}x}{y}\right)^2, \quad \sigma_j = x^{j-1}\frac{(\mathrm{d}x)^2}{y}.$$

They are dual basis to each other. It should be noted that $\theta_i,\ i = 1, \cdots, 7$ (resp. $\rho_i$) span the tangent (resp. cotangent) space of $H_4$ at $[X]$.

Let $f : X \to \mathbb{P}^1$ be the canonical morphism, given by $(x, y) \mapsto x$. Let $(x_0, y_0) = p \in X$. Suppose that the line bundle $L$ is isomorphic to $\mathcal{O}(f^{-1}(\infty) + p)$. For simplicity assume that $f$ is not ramified at $p$. The case that $f$ ramifies at $p$ can be treated in a similar way. We choose a refined chart on $X$ as follows. Let $X = U \cup V$. Here $U = U_0 \backslash \{p\}$ and $V = V_0 \backslash \{p'\}$ where $p' = (x_0, -y_0)$. Using this chart, the restriction of $\theta_i$'s, etc. still give a basis of corresponding cohomology.

A basis of $\mathrm{H}^0(X, L)$ is $\{1, x\}$, and a basis of $\mathrm{H}^0(X, \omega \otimes L^{-1})$ is $\{(x - x_0)\frac{\mathrm{d}x}{y},\ x(x - x_0)\frac{\mathrm{d}x}{y}\}$. The kernel of $\mu_0$ is thus one dimensional and is generated by
$$1 \otimes x(x - x_0)\frac{\mathrm{d}x}{y} - x \otimes (x - x_0)\frac{\mathrm{d}x}{y}.$$

By the description of $\mu_1$ given in [ACG2011, p. 809-810], the image of $\mu_1$ is given by
$$(x - x_0)\frac{(\mathrm{d}x)^2}{y} = \sigma_2 - x_0\sigma_1.$$

Its orthogonal complement in $\mathrm{H}^1(X, T_X)$, i.e. the image of $T_{\mathsf{N},(X,L)}$ in $T_{\mathsf{M},X}$ is thus generated by
$$\theta_i,\ i = 1, \cdots, 7; \quad x_0\eta_2 + \eta_1.$$

To summarize, we have shown that the image of $T_{\mathsf{N},(X,L)}$ in $T_{\mathsf{M},X}$ is eight dimensional and it contains $T_{\mathsf{H},X}$. □

**Lemma 5.2.2.** *There is an isomorphism $\mathsf{C} \times_{\mathsf{M}} \mathsf{H} \to \mathsf{N} \times_{\mathsf{M}} \mathsf{H}$, i.e. the fiber of $\mathsf{H}$ of $\mathsf{N}$ is isomorphic to the universal hyperelliptic curve over $\mathsf{H}$.*

*Proof.* To simplify notation, we write $\mathsf{N_H} = \mathsf{N} \times_{\mathsf{M}} \mathsf{H}$ and $\mathsf{C_H} = \mathsf{C} \times_{\mathsf{M}} \mathsf{H}$. There is an hyperelliptic involution $\iota : \mathsf{C_H} \to \mathsf{C_H}$ over $\mathsf{H}$. As $\mathsf{C_H} \to \mathsf{H}$ has a section $s$, the line bundle $\mathsf{F} = \mathcal{O}(s + \iota^* s)$ on $\mathsf{C_H}$ restricts to the unique line bundle of degree two that has two dimensional global sections on each fiber of $\mathsf{C_H} \to \mathsf{H}$.

Consider the morphism $\nu_i : \mathsf{C_H} \times_{\mathsf{H}} \mathsf{C_H} \to \mathsf{C_H}$ $(i = 1, 2)$ where $\nu_i$ is the projection to the $i$-th factor. Consider the line bundle $\mathsf{F}^+ := \nu_1^* \mathsf{F} \otimes \mathcal{O}(\Delta)$ on $\mathsf{C_H} \times \mathsf{C_H}$ where $\Delta$ is the diagonal morphism. Now view $\mathsf{C_H} \times \mathsf{C_H}$ as a family of curves over $\mathsf{C_H}$ via $\nu_2$. The line bundle $\mathsf{F}^+$ is of degree three on each fiber, and has the property that $\operatorname{rank} \mathrm{R}^1 \nu_* \mathsf{F}^+ = 2$. So by the universal property of $\mathsf{N_H}$, there is a morphism $\mathsf{C_H} \to \mathsf{N_H}$ over $\mathsf{H}$. It is not hard to see that this morphism is in fact an isomorphism over each geometric point of $\mathsf{H}$. This shows that it is in fact an isomorphism. □

Let $\widetilde{\mathsf{M}}$ be the blowup of $\mathsf{M}$ with the center $\mathsf{H}$. Since $\mathsf{N}$ is regular, the inverse image $\mathsf{N_H}$ is thus a Cartier divisor on $\mathsf{N}$, hence the morphism $\mathsf{N} \to \mathsf{M}$ factors through $\widetilde{\mathsf{M}} \to \mathsf{M}$. Let $\mathsf{P} \to \mathsf{H}$ be the exceptional divisor of the blowup. It is a $\mathbb{P}^1$-bundle over $\mathsf{H}$. Then the morphism $\mathsf{N} \to \widetilde{\mathsf{M}}$ gives a morphism $\mathsf{N_H} \to \mathsf{P}$. From the proof of lemma 5.2.1, one sees that this morphism is identified with the universal double cover $\mathsf{C_H} \to \mathsf{P}$. The morphism $\mathsf{N} \to \widetilde{\mathsf{M}}$ is thus finite flat of degree two. Let $\mathsf{Z}'$ be the ramification locus of $\mathsf{N} \to \widetilde{\mathsf{M}}$ on $\mathsf{N}$.

There is an involution

$$\sigma : \mathsf{N} \to \mathsf{N}, \quad (Y \to S, L) \mapsto (Y \to S, \omega_{Y/S} \otimes L^{-1}),$$

on $\mathsf{N}$ over $\mathsf{M}$, where $S$ is an $\mathsf{M}$-scheme. This is precisely the "deck transformation" on the double cover $\mathsf{N}^\circ \to \mathsf{M}^\circ$. Its fixed point in the locus where $\pi$ ramifies, which is nothing but $\mathsf{Z}'$. Therefore the intersection of the closure of $\mathsf{Z}'$ in $\mathsf{N}$ and $\mathsf{N_H}$ is the Weierstrauss subscheme on $\mathsf{N_H}$. It is finite etale over $\mathsf{H}$ of degree ten. This is equivalent to that the intersection of the strict transform of $\mathsf{Z}$ in $\widetilde{\mathsf{M}}$ and $\mathsf{P}$ is the universal branching divisor on $\mathsf{P}$.

We summarize the above discussion in the following lemma.

**Lemma 5.2.3.** *The multiplicity of $\mathsf{Z}$ at the generic point of $\mathsf{H}$ is ten. All components of $\mathsf{Z}$ are regular at the generic point of $\mathsf{H}$.*

## 5.3 Computing the length of the cokernel of the cup product

There is an universal line bundle $\mathsf{L}$ over $\mathsf{C_N} := \mathsf{C} \times_{\mathsf{M}} \mathsf{N}$ such that $\operatorname{rank} \mathrm{R}^1 p_* \mathsf{L} = 2$. Here $p$ is the projection to the second factor. We consider the "universal" cup product

$$p_* \mathsf{L} \otimes p_* \omega \mathsf{L}^{-1} \to p_* \omega.$$

Here $\omega = \omega_{\mathsf{C_N/N}}$ is the dualizing sheaf. Let $\mathsf{Q}$ be its cokernel.

**Proposition 5.3.1.** *We have an isomorphism* $\mathsf{Q} \otimes \det \mathrm{R} p_* \mathsf{L} \simeq \Omega^1_{\mathsf{N/M}}$.

*Proof.* We recall that the defining ideal of $\mathsf{N}$ in $\mathsf{J}^3$ is the second Fitting ideal $I$ of $\mathrm{R}^1 p_* \mathsf{L}$, where $q : \mathsf{C} \times_{\mathsf{M}} \mathsf{J}^3 \to \mathsf{J}^3$ is the projection to the second factor and $\mathsf{L}$ is the universal line bundle. Let $\mathsf{E}$ be a section of $q$ and $n$ a sufficiently large integer. Then there is an exact sequence

$$0 \to K_0 \to K_1 \to \mathrm{R}^1 q_* \mathsf{L} \to 0,$$

where $K_0 \simeq q_* \mathsf{L}(n\mathsf{E})$, $K_1 \simeq q_* \mathsf{L}(n\mathsf{E})/\mathsf{L}$. The morphism $K_0 \to K_1$ is injective since generically a degree three line bundle on $\mathsf{C}$ has no section.

The sheaf $K_0$ and $K_1$ are both locally free, say of rank $r$. The defining ideal $I$ of $\mathsf{N}$ is generated by all the minors of the morphism $K^0 \to K^1$ of size $(r-1) \times (r-1)$. If we restrict this presentation to $\mathsf{N}$, then one has

$$0 \to p_* \mathsf{L} \to K^0|_{\mathsf{N}} \to K^1|_{\mathsf{N}} \to \mathrm{R}^1 p_* \mathsf{L} \to 0.$$

Note that $\mathrm{R}^1 q_* \mathsf{L}$ is compatible with base change and $\mathrm{R}^1 q_* \mathsf{L}|_{\mathsf{N}} \simeq \mathrm{R}^1 p_* \mathsf{L}$. Thus there is a morphism

$$p_* \mathsf{L} \otimes (\mathrm{R}^1 p_* \mathsf{L})^{\vee} \to (K^0 \otimes K^{1,\vee})|_{\mathsf{N}}.$$

Moreover since $K^0$ and $K^1$ are locally free of rank $r$, we have $K^i \simeq \wedge^{r-1} K^{i,\vee} \otimes \wedge^r K^i$, $(i = 0, 1)$. It then follows from the definitions of the Fitting ideal that there is a morphism

$$p_* \mathsf{L} \otimes (\mathrm{R}^1 p_* \mathsf{L})^{\vee} \to I/I^2 \otimes \det \mathrm{R} p_* \mathsf{L}.$$

By duality, this is

$$p_* \mathsf{L} \otimes p_* \omega \mathsf{L}^{-1} \to I/I^2 \otimes \det \mathrm{R} p_* \mathsf{L}.$$

Here $\omega$ stands for the dualizing sheaf of $\mathsf{C_N} \to \mathsf{N}$.

Consider exact sequence

$$0 \to I/I^2 \to \Omega^1_{\mathsf{J}^3/\mathsf{M}}|_{\mathsf{N}} \to \Omega^1_{\mathsf{N}/\mathsf{M}} \to 0.$$

Note that $\Omega^1_{\mathsf{J}^3/\mathsf{M}}|_{\mathsf{N}} \simeq p_*\omega \otimes \det \mathrm{R}p_*\mathsf{L}$. We then get a surjective morphism

$$\mathsf{Q} \otimes \det \mathrm{R}p_*\mathsf{L} \to \Omega^1_{\mathsf{N}/\mathsf{M}}.$$

By the deformation theory [ACGH1985, Chapter IV, Proposition 4.2], this morphism is an isomorphism at each geometric point of $\mathsf{N}$. It is thus an isomorphism of sheaves since $\mathsf{N}$ is reduced. □

We have factorized $\mathsf{N} \to \mathsf{M}$ as $\mathsf{N} \to \widetilde{\mathsf{M}} \to \mathsf{M}$ where $\widetilde{\mathsf{M}} \to \mathsf{M}$ is the blowup of $\mathsf{H}$ and $\mathsf{N} \to \widetilde{\mathsf{M}}$ is a double cover branching along the strict transform of $\mathsf{Z}$. Therefore

$$\Omega^1_{\widetilde{\mathsf{M}}/\mathsf{M}}|_{\mathsf{N}} \to \Omega^1_{\mathsf{N}/\mathsf{M}} \to \Omega^1_{\mathsf{N}/\widetilde{\mathsf{M}}} \to 0.$$

**Lemma 5.3.2.** *The morphism* $\Omega^1_{\widetilde{\mathsf{M}}/\mathsf{M}}|_{\mathsf{N}} \to \Omega^1_{\mathsf{N}/\mathsf{M}}$ *is injective.*

*Proof.* The scheme $\mathsf{N}$ (resp. $\widetilde{\mathsf{M}}$) is a union of a closed subscheme $\mathsf{N_H} = \mathsf{N} \times_{\mathsf{M}} \mathsf{H}$ (resp. $\mathsf{P}$) and its complement $\mathsf{N}^\circ$ (resp. $\mathsf{M}^\circ$). Here we consider $\mathsf{M}^\circ$ as an open subscheme of $\widetilde{\mathsf{M}}$. Let $i : \mathsf{N_H} \to \mathsf{N}$ be the closed immersion and $j : \mathsf{N}^\circ \to \mathsf{N}$ the open immersion. Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!j^*\Omega^1_{\mathsf{N}/\mathsf{M}} & \longrightarrow & \Omega^1_{\mathsf{N}/\mathsf{M}} & \longrightarrow & i_*\Omega^1_{\mathsf{N_H}/\mathsf{H}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!j^*\Omega^1_{\mathsf{N}/\widetilde{\mathsf{M}}} & \longrightarrow & \Omega^1_{\mathsf{N}/\widetilde{\mathsf{M}}} & \longrightarrow & i_*\Omega^1_{\mathsf{N_H}/\mathsf{P}} & \longrightarrow & 0 \end{array}$$

The left vertical arrow is an isomorphism since they are both isomorphic to $j_*\Omega^1_{\mathsf{N}^\circ/\mathsf{M}^\circ}$. The middle arrow is surjective by definition and the right arrows is surjective since $i$ is a closed immersion.

Note that we have $\Omega^1_{\widetilde{\mathsf{M}}/\mathsf{M}}|_{\mathsf{N}} \simeq i_*\left(\Omega^1_{\mathsf{P}/\mathsf{H}}|_{\mathsf{N_H}}\right)$. It then follows from diagram chasing that $\Omega^1_{\widetilde{\mathsf{M}}/\mathsf{M}}|_{\mathsf{N}} \to \Omega^1_{\mathsf{N}/\mathsf{M}}$ is injective if and only if $\Omega^1_{\mathsf{P}/\mathsf{H}}|_{\mathsf{N_H}} \to \Omega^1_{\mathsf{N_H}/\mathsf{H}}$ is injective. The injectivity of $\Omega^1_{\mathsf{P}/\mathsf{H}}|_{\mathsf{N_H}} \to \Omega^1_{\mathsf{N_H}/\mathsf{H}}$ is clear since they are both line bundles and the morphism is nontrivial. □

**Corollary 5.3.3.** *Keep the notation as above. Then*

$$\det \mathsf{Q} \simeq \mathcal{O}_{\mathsf{N}}(\mathsf{Z}' + \mathsf{N_H}).$$

*Proof.* It follows from the above lemma and Proposition 5.3.1 that

$$\det \mathsf{Q} \simeq \det \Omega^1_{\mathsf{N}/\widetilde{\mathsf{M}}} \otimes \det \Omega^1_{\widetilde{\mathsf{M}}/\mathsf{M}}|_{\mathsf{N}}.$$

Here we have used the fact that $\mathsf{Q}$ is zero at the generic point of $\mathsf{N}$. Then $\det \Omega^1_{\mathsf{N}/\widetilde{\mathsf{M}}} \simeq \mathcal{O}(\mathsf{Z}')$ since $\mathsf{Z}'$ is the ramification locus of $\mathsf{N} \to \widetilde{\mathsf{M}}$ and $\det \Omega^1_{\widetilde{\mathsf{M}}/\mathsf{M}}|_{\mathsf{N}} \simeq \mathcal{O}(\mathsf{N}_{\mathsf{H}})$ since $\widetilde{\mathsf{M}}/\mathsf{M}$ is the blowup of $\mathsf{M}$ centered at $\mathsf{H}$. □

*Proof of Proposition 3.4.2.* We let $R$ be a strict henselizian discrete valuation ring. Choose a uniformizer $\varpi$ of $R$. Let $K$ be the function field of $R$ and $k$ the residue field. By assumption, $k$ is not of characteristic two. We assume that the curve $X$ is Petri general and has smooth hyperelliptic reduction over $R$. Let $f : \mathcal{X} \to \operatorname{Spec} R$ be an integral model of $X$ and $Y$ be the special fiber. Then $Y$ is a smooth hyperelliptic curve. The divisor $D$ on $X$ is of degree three and $\dim \mathrm{H}^0(X, \mathcal{O}(D)) = 2$. We denote by $\overline{D}$ its closure on $\mathcal{X}$.

By the moduli interpretation of $\mathsf{N}$, there is a unique morphism $i : \operatorname{Spec} R \to \mathsf{N}$ and $\mathcal{X} \simeq \operatorname{Spec} R \times_{\mathsf{N}} \mathsf{C}_{\mathsf{N}}$. Let $\mathcal{L} = \mathsf{L}|_{\mathcal{X}}$. Then it is a line bundle on $\mathcal{X}$ which extends $\mathcal{O}(D)$ on the generic fiber.

We need to compute the cup product over $\operatorname{Spec} R$, i.e.

$$f_*\mathcal{L} \otimes f_*\omega\mathcal{L}^{-1} \to f_*\omega, \tag{5.2}$$

where $\omega$ is the dualizing sheaf of $\mathcal{X}$ over $R$. Let $W \subset \operatorname{Spec} R$ be the degenerate locus of the cup product. Then $W$ is a finite length scheme supported at the closed point of $\operatorname{Spec} R$. Then

$$\det \mathrm{R} f_*\omega \simeq (\det \mathrm{R} f_*\mathcal{L})^2 \otimes \mathcal{O}(W).$$

This is the pullback of the isomorphism

$$\det \mathrm{R} p_*\omega_{\mathsf{C}_{\mathsf{N}}/\mathsf{N}} \simeq (\det \mathrm{R} p_*\mathsf{L})^2 \otimes \det \mathsf{Q}.$$

The length of $\mathsf{N}_{\mathsf{H}} \times_{\mathsf{N}} \operatorname{Spec} R$ as a finite length $R$ scheme is $h(X)$ by the lemma below. The length of $\mathsf{Z}' \times_{\mathsf{N}} \operatorname{Spec} R$ is $\frac{1}{2}p(X) - 10h(X)$. This follows from Lemma 5.2.3 and the fact that $\widetilde{\mathsf{M}} \to \mathsf{M}$ is the blowup centered at $\mathsf{H}$.

We then conclude from Corollary 5.3.3 that the length of $W$ is the $\frac{1}{2}p(X) - 9h(X)$. The Proposition 3.4.2 is thus proved. □

**Lemma 5.3.4.** *Let $M$ be a scheme and $Z \subset M$ be a closed subscheme. Let $\widetilde{M} \to M$ be the blowup centered at $Z$ and $E \subset \widetilde{M}$ be the exceptional divisor. Let $R$ be a discrete valuation ring with function field $K$, residue field $k$ and a uniformizer $\varpi$. Let $i : \operatorname{Spec} R \to M$ be a morphism such that the image of $\operatorname{Spec} K$ does not lie in $Z$ while the image of $\operatorname{Spec} k$ lies in $Z$. Let $\widetilde{i} : \operatorname{Spec} R \to \widetilde{M}$ be the strict transform. Let $i_n : \operatorname{Spec} R/\varpi^n \to M$ (resp. $\widetilde{i}_n : \operatorname{Spec} R/\varpi^n \to \widetilde{M}$) be the morphism induced by $i$ (resp. $\widetilde{i}_n$). Let*

$$z = \max\{n \mid i_n \textit{ is a closed immersion}\},$$

*and*

$$\widetilde{z} = \max\{n \mid \widetilde{i}_n \textit{ is a closed imersion}\}.$$

*Then $z = \widetilde{z}$.*

*Proof.* Let $I_Z$ (resp. $I_E$) be the defining ideal of $Z$ (resp. $E$). The pullback of $I_Z$ (resp. $I_E$) to $\operatorname{Spec} R$ is an ideal in $R$ generated by $\varpi^z$ (resp. $\varpi^{\widetilde{z}}$). But $I_E$ is the inverse image ideal sheaf of $I_Z$. The lemma then follows. $\square$

# Chapter 6

# Speculations

This chapter is largely speculative. We summarize the problems left open in this thesis and provide some hints to the solutions.

## 6.1 Double cover of $\overline{\mathcal{M}}_4$

As we have seen in Chapter 5, for any smooth genus four curve $X$, there is a double cover of a neighbourhood of $[X]$ in $\mathcal{M}_4$. The construction of this double cover can be summarizes as follows. The singularity of the Petri locus $Z$ in $\mathcal{M}_4$ is hyperelliptic locus $\mathcal{H}_4$. Blowing up $\mathcal{H}_4$ in $\mathcal{M}_4$ gives an embedded resolution of the Petri locus $Z$. Let $\widetilde{Z}$ be the strict transform of $Z$. The double cover precisely ramifies along $\widetilde{Z}$.

Now the question is:

Can we make similarly constructions on $\overline{\mathcal{M}}_4$?

We have also used the double cover of $\mathcal{M}_4$ in a crucial way to compute the cokernel of the cup product (3.1). If the answer to the above question is YES, then the next question is:

What is the relation between the double cover of $\overline{\mathcal{M}}_4$ and cup product?

These questions seem to be inaccessible in its full generality at this moment. We only give some tentative description of the geometry of $\overline{\mathcal{M}}_4$ near the point $[X] \in \overline{Z}$.

**Expectation 6.1.1.** If $X$ is of the type IRRED, ELL Section 3.3, then the closure of the Petri locus is smooth at $[X]$ if $[X]$

**Remark 6.1.2.** If $X$ is of type TWO, then it does not lies in the closure of the Petri locus.

**Expectation 6.1.3.** Suppose $X$ is irreducible and has a single node $p$. Assume that the normalization $X' \to X$ is hyperelliptic. Let $q_1$ and $q_2$ be the inverse image of $p$. If $q_1$ and $q_2$ are not hyperelliptic involution of each other, then $\overline{Z}$ at $[X]$ is smooth.

*Explanation.* If $X$ is irreducible, then $X$ is a complete intersection of a singular quadric and a cubic surface in $\mathbb{P}^3$. That means, the Petri locus near the point $[X]$ is isomorphic to an open part of the projective space, which is smooth.

Suppose $X$ has two components $C$ and $E$ of genus three and one respectively, the component $C$ being non-hyperelliptic. Let $Q$ be a singular quadric in $\mathbb{P}^3$ and we look at the linear system $P$ on $Q$ cut out by degree three hypersurfaces. The locus in $P$ where the divisor acquires at most a cusp is smooth if the singularity of the divisor is not on the singularity of $Q$. The locus $C$ where the divisor acquires a cusp is also smooth. Then locally on the moduli space $\overline{\mathcal{M}}_4$ near the point $[X]$, the Petri locus is isomorphic to the blowup of $P$ along $C$, which is again smooth. $\square$

**Expectation 6.1.4.** Let

$$\Sigma = \left\{ X \in \Delta_1 \;\middle|\; \begin{array}{c} \text{X consists of two components } C \text{ and } E \text{ meeting at one point} \\ \text{where } C \text{ is a hyperelliptic genus three curve and } E \text{ is an elliptic curve} \end{array} \right\}.$$

Let $\eta$ be the generic point of $\Sigma$. Then the Petri locus $\overline{Z}$ at $\eta$ has multiplicity four.

*Explanation.* Suppose the geometric generic point of $\Sigma$ corresponds to a curve $X = C \cup E$. We then take the strict henselization of $\overline{\mathcal{M}}_4$ at the generic point of $\Sigma$. We then get a two dimensional scheme $M$ and $\eta$ is the only closed point. Then $\overline{Z}$ is the divisor on $M$ which has three components at $\eta$ and each component is smooth.

Denote the universal curve over $M$ by $\mathcal{X}$ and the degree three Jacobian by $\mathcal{J}^3$. The Jacobian is the connected component of the relative Picard scheme that is of multi-degree $(2,1)$ on $X$, i.e. of degree two on $C$ and degree one on $E$. Still consider the locus $W \subset \mathcal{J}$ consisting of line bundles whose global section has dimension at least two. A little computations show that the closed fiber of $W$ over $\eta$ is of dimension one and is isomorphic to $E$.

It is expected that $W$ is regular. Moreover, the morphism $W \to M$ admits a factorization $W \to \widetilde{M} \to M$ where $\widetilde{M} \to M$ is the blowup of $M$ at the closed point and $W \to \widetilde{M}$ is finite flat of degree two. Let $P$ be the exceptional divisor. Then the morphism $E \to P$ is the usual double cover defined by the linear system $|2p|$ on $E$. The strict transform of $Z$ intersects $P$ precisely at the branching point of $E \to P$. $\square$

## 6.2 The local invariant and the Northcott property

We have shown in Corollary 1.3.4 that for a complete family of smooth non-hyperelliptic genus four curves, the Northcott property holds. The obvious question is, does this holds over the whole moduli space $\mathcal{M}_4^\circ$ parameterizing non-hyperelliptic genus four curve?

**Conjecture 6.2.1.** *Let $K$ be a number field Let $H$ and $D$ be two positive real numbers. Then the set*

$$\{t \in \mathcal{M}_4^\circ(\overline{K}) \mid \deg t \leq D,\ \text{height}\,(\Xi_t) \leq H\}$$

*is finite.*

This conjecture is closely related to the lower bound of $\omega_X^2$.

**Expectation 6.2.2.** There is some $\epsilon > 0$, such that

$$5\omega_X^2 \geq (1+\epsilon)\sum_v \varphi_v.$$

The inequality we obtained from the non-negativity of the height is inequality above with $\epsilon = 0$. Note that this equality implies Conjecture 6.2.1.

Recall that the effective Bogomolov conjecture predicts that there is an effectively computable constant $c > 0$ such that $\omega_X^2 \geq c$.

**Expectation 6.2.3.** If $v$ is non-archimedean, the constant $\varphi_v$ is effectively computable. If $v$ is archimedean, then $\varphi_v > 0$.

*Explanation.* Suppose $v$ is non-archimedean. We expect that there is an absolute constant $c > 0$, such that

$$\varphi_v \geq c(h(X) + \delta(X)).$$

Here $h(X)$ is the number of nodes on the minimal regular semistable model of $X$ over $B$ and $h(X)$ is the number of hyperelliptic fibers, counting multiplicity. Moreover, we expect that we could take $c = \frac{2}{5}$. This means that when getting into the deeper strata of the the boundary of $\overline{\mathcal{M}_4}$, the constant $\varphi_v$ increases.

□

Our result on the height of $\Xi$ is somewhat a reminiscence of [Zha2010, Theorem 1.3.5]. But we should point out that there is an important difference. We expect that when $v$ is archimedean, the local constant $\varphi_v$ is not an expression involving only the Green's functions. If we consider the Green's function as some "distance" function on the moduli space to the boundary of $\overline{\mathcal{M}}_4$, then the expression of $\varphi_v$ should involve some other function which expresses the "distance" to the hyperelliptic locus. I do not know at this moment what this function should be. This seems to be an interesting phenomena. One perhaps should try to look for similar distance function for curves of other genera.

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