Pricing, Trading and Clearing of Defaultable Claims Subject to Counterparty Risk

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ABSTRACT

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The recent financial crisis and subsequent regulatory changes on over-the-counter (OTC) markets have given rise to the new valuation and trading frameworks for defaultable claims to investors and dealer banks. More OTC market participants have adopted the new market conventions that incorporate counterparty risk into the valuation of OTC derivatives. In addition, the use of collateral has become common for most bilateral trades to reduce counterparty default risk. On the other hand, to increase transparency and market stability, the U.S and European regulators have required mandatory clearing of defaultable derivatives through central counterparties. This dissertation tackles these changes and analyze their impacts on the pricing, trading and clearing of defaultable claims.

In the first part of the thesis, we study a valuation framework for financial contracts subject to reference and counterparty default risks with collateralization requirement. We propose a fixed point approach to analyze the mark-to-market contract value with counterparty risk provision, and show that it is a unique bounded and continuous fixed point via contraction mapping. This leads us to develop an accurate iterative numerical scheme for valuation. Specifically, we solve a sequence of linear inhomogeneous partial differential equations, whose solutions converge to the fixed point price function. We
apply our methodology to compute the bid and ask prices for both defaultable equity and fixed-income derivatives, and illustrate the non-trivial effects of counterparty risk, collateralization ratio and liquidation convention on the bid-ask prices.

In the second part, we study the problem of pricing and trading of defaultable claims among investors with heterogeneous risk preferences and market views. Based on the utility-indifference pricing methodology, we construct the bid-ask spreads for risk-averse buyers and sellers, and show that the spreads widen as risk aversion or trading volume increases. Moreover, we analyze the buyer’s optimal static trading position under various market settings, including (i) when the market pricing rule is linear, and (ii) when the counterparty – single or multiple sellers – may have different nonlinear pricing rules generated by risk aversion and belief heterogeneity. For defaultable bonds and credit default swaps, we provide explicit formulas for the optimal trading positions, and examine the combined effect of heterogeneous risk aversions and beliefs. In particular, we find that belief heterogeneity, rather than the difference in risk aversion, is crucial to trigger a trade.

Finally, we study the impact of central clearing on the credit default swap (CDS) market. Central clearing of CDS through a central counterparty (CCP) has been proposed as a tool for mitigating systemic risk and counterpart risk in the CDS market. The design of CCPs involves the implementation of margin requirements and a default fund, for which various designs have been proposed. We propose a mathematical model to quantify the impact of the design of the CCP on the incentive for clearing and analyze the market equilibrium. We determine the minimum number of clearing participants required so that they have an incentive to clear part of their exposures. Furthermore, we analyze the equilibrium CDS positions and their dependence on the initial margin, risk aversion, and counterparty risk in the inter-dealer market. Our numerical
results show that minimizing the initial margin maximizes the total clearing positions as well as the CCP’s revenue.
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Chapter 1

Introduction

The outstanding notional amount for derivatives traded in the global over-the-counter (OTC) markets has increased drastically from US$142 trillion in December 2002 to US$693 trillion as of December 2012\footnote{According to the Bank for International Settlements (BIS) statistical release available at http://www.bis.org/publ/otc_hy1311.pdf}. A great variety of contracts are traded in these markets, including interest rate swaps, equity-linked contracts, credit derivatives, and foreign exchange derivatives. Associated with each OTC-traded contract, there is a bilateral agreement between traders on the price, quantity, and contractual features. As such, the defaults of counterparties can significantly affect the future cash flow of the contract.

During the 2008 financial crisis, a number of large financial institutions, including Lehman Brothers, Bear Stearns and American International Group (AIG), defaulted on their OTC contracts, such as credit default swaps, causing great losses to their counterparties. In view of these events, new regulations and market conventions have been introduced to incorporate counterparty risk into the valuations of OTC derivatives.

OTC traders commonly update the prices of every traded contract on a daily basis. To this end, they typically develop pricing models which capture
the changing market conditions and various risk factors affecting the future cash flow of the contract. The daily estimated price of a contract is called the mark-to-market (MtM) value. For any contract, the MtM value depends not only on the stochastic dynamics of underlying assets, but also on the counterparties’ changing creditworthiness. According to the Bank for International Settlements (BIS), two-thirds of counterparty risk losses during the crisis were from counterparty risk adjustments in MtM valuations whereas the rest were due to actual defaults. As a result, recent regulatory changes, such as Basel III, incorporate the counterparty risk adjustments in the calculation of capital requirements.

Among all OTC-traded derivatives, credit default swaps have played an important role in the 2008 financial crisis. One notable market phenomenon is the significant widening of their bid-ask spreads during market turbulence. Moreover, higher bid-ask spreads are typically coupled with lower trading volumes. From an OTC trader’s perspective, the optimal trading position would depend on the trader’s risk attitude and market outlook. This has motivated the recent development of pricing models that incorporate potentially different risk preferences and market views into bilateral trading of credit default swaps and other defaultable claims.

On top of pricing issues, more institutional market participants, such as investment banks, have recently started to trade credit default swaps through a central counterparty. A central counterparty stands between any two market participants, acting as the seller to the original buyer and as the buyer to the original seller. As a result, the original contract between the two participants into two contracts with the central counterparty. This mechanism is called central clearing. An important function of the central counterparty is to fulfill the entire financial obligations of any defaulting clearing members, and thus,

\[\text{\footnotesize \textsuperscript{2}}\text{See BIS press release at http://www.bis.org/press/p110601.pdf}\]
reduce the impact of counterparty risk in OTC trading. In order to become a clearing member, a market participant needs to deposit collateral and pay clearing fees. These two features can affect market participants’ incentive to use the clearing service. This gives rise to the problem of understanding the impact of central counterparty policy on the clearing volume of credit default swaps.

Motivated by these observations, we study the issues of pricing, trading and clearing of OTC-traded derivatives in this thesis. In the subsequent sections, we provide a brief summary of our methodologies and related studies for each of these three problems.

### 1.1 Pricing of Defaultable Claims with Counterparty Risk and Collateralization

As a measure to reduce counterparty risk, the use of collateral has also increased dramatically in the OTC markets over the years. According to the survey conducted by the International Swaps and Derivatives Association (ISDA) in 2013[^1], the percentage of all trades subject to collateral agreements in the OTC market increases from 30% in 2003 to 73.7% in 2013. OTC market participants continue to adapt collateralization and counterparty risk adjustments in their valuation methodologies for various contracts. In Chapter 2, we discuss a valuation framework for financial contracts subject to counterparty risk and collateralization.

When an OTC market participant trades a financial claim with a counterparty, the participant is exposed not only to the price change and default risk of the underlying asset but also to the default risk of the counterparty. To

[^1]: Survey available at http://www2.isda.org/functional-areas/research/surveys/margin-surveys/
reflect the counterparty default risk in MtM valuations, three adjustments are calculated in addition to the counterparty-risk free value of the claim. While credit valuation adjustment (CVA) accounts for the possibility of the counterparty’s default, debt valuation adjustment (DVA) is calculated to adjust for the participant’s own default risk. In addition, collateral interest payments and the cost of borrowing generate funding valuation adjustment (FVA). The bilateral credit value adjustment (BCVA) incorporates all three components.

We consider two current market conventions for price computation. The main difference in the two conventions rises in the assumption of the liquidation value – either counterparty risk-free value or MtM value with counterparty risk provision – upon default. Brigo et al. [2012] and Brigo and Morini [2011] show that the values under the two conventions have significant differences and large impacts on net debtors and creditors.

With counterparty risk provision, the MtM contract value is defined implicitly in terms of a risk-neutral expectation. This gives rise to major challenges in analyzing and computing the MtM value. We propose a novel fixed point approach to analyze this problem. A key feature of our methodology is to show that the MtM value is the unique fixed point of a contraction mapping. We analytically construct a sequence of price functions, which are the classical solutions of a sequence of inhomogeneous linear partial differential equations (PDEs), that converges to the fixed point. This approach also directly suggests an iterative numerical scheme to compute the MtM values of a variety of financial claims under different market conventions.

In related studies, Fujii and Takahashi [2013] incorporate BCVA and under/over collateralization, and calculate the MtM value by simulation. Henry-Labordère [2012] approximates the MtM value by numerically solving a related nonlinear PDE through simulation of a marked branching diffusion, and provides conditions to avoid a “blow-up” of the simulated solution. Burgard and
Kjaer [2011] also consider a similar nonlinear PDE and they compute the BCVA for defaultable bonds. In contrast, our fixed point methodology works directly with the price definition in terms of a recursive expectation, rather than heuristically stating and solving a nonlinear PDE. Our contraction mapping result allows us to solve a series of linear PDE problems with bounded classical solutions, and obtain a unique bounded continuous MtM value as a result.

Our model also provides insight on the bid-ask prices of various financial contracts. The CVA or BCVA is asymmetric for the buyer and the seller. As such, the incorporation of adjustment to unilateral or bilateral counterparty risk leads to a non-zero bid-ask spread. In other words, counterparty risk reveals itself as a market friction, resulting in a transaction cost for OTC trades. In addition, we examine the impact of various parameters such as default rate, recovery rate, collateralization ratio and effective collateral interest rate. We find that a higher counterparty default rate and funding cost reduce the MtM value, whereas the market participant’s own default rate and collateralization ratio have positive price effects. For claims with a positive payoff, such as calls and puts, we establish a number of price dominance relationships. In particular, when collateral rates are low, the bid-ask prices are dominated by the counterparty risk-free value. Moreover, the bid-ask prices decrease when we use the MtM value rather than counterparty risk-free value for the liquidation value upon default.

1.2 Trading of Defaultable Claims with Risk Aversion and Belief Heterogeneity

For defaultable claims, such as corporate bonds and credit default swaps, the risks associated with defaults may not be perfectly hedged. In order to value
a defaultable claim, the buyer and seller must quantify the unhedgeable risk based on their partial hedging strategies, subjective market views, and risk preferences. In particular, risk preferences and market views are absent in the classical no-arbitrage pricing framework. In Chapter 3 we propose a utility-indifference approach to study the buyer’s and seller’s pricing rules as well as their optimal trading strategies.

The utility-indifference pricing approach has been applied to credit derivatives valuations in Bielecki and Jeanblanc [2006]; Jaimungal and Sigloch [2012]; Leung et al. [2008]; Shouda [2006]; Sircar and Zariphopoulou [2010], among others. Nevertheless, most existing indifference pricing models commonly focus on the perspective of a single derivative buyer or seller, and do not address the natural question of how multiple risk-averse market participants trade among each other.

Working with exponential utility, we obtain explicit formulas for non-linear bid-ask prices for both defaultable bonds and credit default swaps. Moreover, we prove that the buyers’ indifference prices are strictly concave in trading volume, whereas the sellers’ indifference prices are strictly convex in trading volume. Also, we obtain asymptotic results of the buyer’s and seller’s average bid-ask prices in terms of risk aversion and trading volume. As either risk-aversion or trading volume goes to zero, the average bid-ask prices converge to a single value called zero risk-aversion price. This zero risk-aversion price plays a critical role to initiate a trade. On the other hand, as either risk-aversion or trading volume goes to infinity, the average bid-ask prices converge to the no-arbitrage lower bound and upper bound respectively.

We analyze the optimal static trading strategy to maximize the investor’s benefit defined by the spread between his/her indifference price and the offered prices under two market scenarios, namely, (i) when the offered prices are linear in quantity, and (ii) when the offered prices are the indifference prices set by
the sellers with their heterogeneous risk aversions and beliefs. By using the concavity/convexity property of the indifference prices, we derive formulas for the optimal trading positions of defaultable claims and examine the impact of default risk, risk aversion, and other parameters.

Moreover, our results also help explain the various effects of risk aversion in the trading of defaultable claims. For instance, we show precisely how bid-ask spread widens as risk aversion or trading volume increases. This is consistent with empirical studies that calibrate the risk aversion from market option prices (see, for example, Jackwerth [2000]). Intuitively, a less risk-averse seller tends to offer more competitive prices and gains a larger share of the investor’s total trading volume. Nevertheless, accumulating a large position also in effect makes the seller become more risk-averse and price less aggressively. Hence, the buyer’s optimal position and trading price depend directly on the risk aversion of all sellers.

Furthermore, in establishing the trade/no-trade condition, a special role is played by the zero risk-aversion indifference price limits. The investor’s zero risk-aversion price is directly linked to the investor’s belief on the market conditions and default risk. In the optimal trading problem, a buyer will purchase from a seller if and only if the buyer’s zero risk-aversion price (with respect to the buyer’s belief) exceeds the seller’s. In other words, heterogeneity in beliefs, rather than in risk aversion, is crucial to initiate a trade. If a trade occurs, then the optimal trading position of the investor is determined from maximizing the spread between the buyer’s and seller’s indifference prices.

In a recent related work, German [2011] considers a utility maximization approach to study the trading of a large trader. Garleanu et al. [2009] apply a utility maximization approach to study the demand-pressure effect on option prices. Under exponential utility, he provides a recursive unique pricing rule for an illiquid asset. In our model, we also work with exponential utility,
along with a parametric credit risk model rather than a general semimartingale
market. As such, our model yields analytic formulas for the indifference prices
and optimal trading strategies, and allows for different beliefs among market
participants.

There is also a wealth of literature that investigates the influence of het-
erogeneous risk preferences and beliefs on equilibrium asset prices. Among
others, Scheinkman and Xiong [2003] study the equilibrium prices and the for-
mation of speculative bubbles under a parameterized model of heterogeneous
beliefs. Cvitanić et al. [2012] derive an equilibrium model when investors
have heterogeneous beliefs, risk aversions and time preference rates. Gomes
and Michaelides [2008] and Chabakauri [2010] study the market equilibrium
with heterogeneous agents under various frictions such as uninsurable income
shocks, borrowing or risk constraints. Glosten and Milgrom [1985] show that
the presence of traders with superior information can lead to a positive bid-ask
spread.

There is relatively less research on equilibrium trading volume for options
and other financial derivatives. Benninga and Mayshar [2000] demonstrate
that implied volatility smiles can be generated at equilibrium when options are
traded among heterogeneous agents. Carr and Madan [2001] analyze the opti-
mal position in a stock and the corresponding European options, and find that
heterogeneity in preferences or beliefs induces investors to take a long/short
among traders and examines the effect of disagreement on equilibrium option
trading volume. Xiong and Yan [2010] present an equilibrium model of bond
markets in which two groups of agents hold heterogeneous expectations about
future economic conditions. In our model, we obtain the equilibrium trad-
ing volume and bid-ask spreads for both defaultable bonds and credit default
swaps in a market with multiple traders with heterogeneous risk aversions and
Figure 1.1: The U.S. CDS market breakdowns (in US$ billion)

1.3 Impact of central counterparty design on the credit default swap market

In the current financial market, institutional investors (clients) buy or sell credit default swaps (CDS) from dealers representing major investment banks. In order to hedge their client-dealer positions, CDS dealers establish opposite positions with other dealers in the so-called *inter-dealer market*. Figure 1.1 shows the breakdowns of the U.S. CDS market in terms of outstanding notional amounts.

As is well known, CDS have been repeatedly blamed for causing and exacerabating the credit crisis. The complexity and limited transparency of CDS market have made it difficult especially for CDS dealers to accurately estimate
The U.S. CDS market consists of two types of contracts: inter-dealer contracts and client-dealer contracts. The circle represents the inter-dealer market. After 2009, all inter-dealer market contracts are cleared through a CCP.

The U.S. Treasury Department released a comprehensive financial regulatory reform proposal that would mandate the clearing of inter-dealer CDS contracts through a regulated and qualified central counterparty (CCP).\footnote{“A new foundation: Rebuilding financial supervision and regulation.” Financial Regulatory Reform, U.S. Department of the Treasury, 2009.} We refer the readers to \textcolor{blue}{Stephen et al. 2009}, \textcolor{blue}{Cont 2010} and \textcolor{blue}{Duffie et al. 2010} for more details on the recent regulatory changes in the CDS market. After these changes, all inter-dealer CDS contracts have been cleared in CCPs. Figure 1.2 illustrates the mechanism of central clearing through a CCP.

The main function of a CCP is to assume all the losses whenever clearing members fail to meet their contractual obligations. After every default event,
the defaulted members’ entire clearing positions are auctioned to the remaining members, and the total of the winning bids is the cost of unwinding to the CCP. Since this unwinding process may take five days or more, the CCP is exposed to price fluctuations of the unwinding positions during this period. If the CCP is unable to fulfill this obligation, then many clearing dealers are subject to losses. Therefore, it is crucial to maintain sufficient amount of financial resources for the CCP.

The CCP collects its capital in different ways, such as variation margins, initial margins, and guaranty fund contributions, from its clearing members. The size of each market participant’s initial margin and guaranty fund contribution is reassessed by the CCP on a regular basis according to market conditions and members’ outstanding positions. The CCP also adopts a waterfall structure, which determines the order of absorbing losses in response to defaults of clearing members. Let us explain this mechanism of each capital layer as follows.

- **Variation margin**, also known as *maintenance margin*, is exchanged between the CCP and every clearing member on a daily basis. The variation margin payment is exactly the daily change in the MtM value of the clearing member’s position. As such, it absorbs the short-term losses and first losses when a clearing member defaults.

- **Initial margin**, also known as the *risk margin*, is provided by both clearing members when a trade is cleared with the CCP. This cash amount will be deposited in the CCP until either the contract expires at maturity, or is unwound before maturity. Its main purpose is to absorb the cost of unwinding a defaulting clearing member’s positions by the CCP.

- **Guaranty fund**, also known as *default fund*, is a pool of capital contributed by all the CCP’s clearing members. This absorbs the losses
**Figure 1.3: The waterfall capital structure of a CCP**

The gray area represents the net liability of the defaulting member at its default time. The initial net liability is covered by the variation margin and part of the initial margin of the defaulting member. The shaded area represents the decrease in the value of the defaulting member’s positions during the unwinding process. It is covered by the initial margin and the guaranty fund contribution of the defaulting member, and also part of the guaranty fund contribution of non-defaulting members.

in excess of the defaulting members’ variation and initial margins. The defaulted members’ portion of the guaranty fund is always used to cover the excess losses, followed by the rest of the guaranty fund. This risk sharing feature is the main source of counterparty risk among clearing members in the central clearing process.

- **CCP’s capital** is the capital of last resort to absorb the losses due to clearing members’ defaults.

To summarize, Figure 1.3 illustrates how a CCP orderly allocates its financial resources to absorb losses.

In Chapter 4 we propose a mathematical model to study the equilibrium
demand for clearing CDS among dealers. Our model incorporates various costs and counterparty risks faced by the clearing members (dealers), and also quantify the hedging benefits of clearing through a CCP. Given its initial client-dealer position, each CDS dealer’s problem is to choose the optimal positions with other dealers in order to maximize the expected returns from their respective CDS portfolios, subject to variance risk constraints. The market equilibrium is found from the market clearing condition whereby the sum of all dealers’ optimal (long/short) positions equals to zero. We prove that there exists a unique market equilibrium described by the number of clearing members and their CDS positions. We obtain closed formulas that allow us to study the sensitivities of the equilibrium with respect to model parameters. In particular, we determine the minimum number of CDS dealers in the market to guarantee that the overall demand for clearing is strictly positive. We also find that the CCP can increase the total clearing positions and its profit by reducing its initial margin level.

There is a growing literature on analyzing the roles of CCPs in the OTC markets. Cont [2010] and Stephen et al. [2009] argue qualitatively that introduction of well-designed CCPs can not only increase market transparency, but also help improve the management of counterparty risk and systemic risk. In a general equilibrium setting, Acharya and Bisin [2010] compare OTC and centralized markets. They show that OTC markets yield a counterparty risk externality that leads to ex-ante productive inefficiency. However, this externality is absent in a centralized market that provides transparency of trading positions. Duffie and Zhu [2011] show that adding a CCP to an existing central clearing system can lead to an increase in average exposure to counterparty default based on the assumption that each clearing member’s exposure is independent and normally distributed. They propose that a single CCP should clear credit derivatives and interest-rate derivatives altogether. On the
other hand, Cont and Kokholm [2013] extend their model by incorporating a more sophisticated joint distribution for the exposures of clearing members. In contrast to Duffie and Zhu [2011], they find that clearing of interest rate and credit derivatives separately by two different CCPs can reduce overall exposures. Compared to these models, our framework accounts for the CCP’s capital structure, including initial margins, variation margins, and guaranty fund contributions from its clearing members. We also provide an analysis on the design of a CCP and its impact on the market equilibrium.

Haene and Sturm [2009] conclude that establishing guaranty fund is always optimal for dealers assuming only one representative clearing member. However, they do not explain the impact of the allocation change on dealer’s inter-dealer market CDS demand. Fontaine et al. [2011] derive the optimal inter-dealer market CDS demand of dealers and corresponding equilibrium price based on a circular structure of clients’ demand given as the dealers’ endowments. Nevertheless, among a number of limitations, their model does not explain or include any counterparty risks that arise from CDS trading.
Chapter 2

Pricing of Defaultable Claims with Counterparty Risk and Collateralization

In this chapter, we study a valuation framework for defaultable claims subject to counterparty default risk under collateralization. By using a fixed point approach and contraction mapping, we show that the MtM contract value with counterparty risk provision is a unique bounded and continuous fixed point. To numerically obtain the fixed point value, we devise an accurate iterative algorithm which solves a sequence of linear inhomogeneous PDEs, whose solutions converge to the fixed point. For applications, we numerically compute the bid and ask prices for defaultable claims with counterparty risk provision and analyze the impact of parameters such as counterparty risk, collateralization ratio and effective collateral rates on the bid-ask prices.

In Section 2.1 we formulate the MtM valuation of a generic financial claim with default risk and counterparty default risks under collateralization. In Section 2.2 we provide a fixed point theorem and a recursive algorithm for valuation. In Section 2.3 we compute the MtM values of various default-
able equity claims and derive their bid-ask prices. In Section 2.4, we apply our model to price a number of defaultable fixed-income claims. Section 2.5 concludes this chapter, and Appendix A contains a number of longer proofs.

### 2.1 Model Formulation

In the background, we fix a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\), where \(\mathbb{Q}\) is the risk-neutral pricing measure. In our model, there are three defaultable parties: a reference entity, a market participant, and a counterparty dealer. We denote them respectively as parties 0, 1, and 2. The default time \(\tau_i\) of party \(i \in \{0, 1, 2\}\) is modeled by the first jump time of an exogenous doubly stochastic Poisson process. Precisely, we define

\[
\tau_i = \inf \left\{ t \geq 0 : \int_0^t \lambda^{(i)}_u \, du > E_i \right\},
\]

(2.1)

where \(\{E_i\}_{i=0,1,2}\) are unit exponential random variables that are independent of the intensity processes \((\lambda^{(i)}_t)_{t \geq 0}, i \in \{0, 1, 2\}\). Throughout, each intensity process is assumed to be of Markovian form \(\lambda^{(i)}_t \equiv \lambda^{(i)}(t, S_t, X_t)\) for some bounded positive function \(\lambda^{(i)}(t, s, x)\), and is driven by the pre-default stock price \(S\) and the stochastic factor \(X\) satisfying the SDEs

\[
dS_t = \left( r(t, X_t) + \lambda^{(0)}(t, S_t, X_t) \right) S_t \, dt + \sigma(t, S_t) \, S_t \, dW_t, 
\]

(2.2)

\[
dX_t = b(t, X_t) \, dt + \eta(t, X_t) \, d\tilde{W}_t.
\]

(2.3)

Here, \((W_t)_{t \geq 0}\) and \((\tilde{W}_t)_{t \geq 0}\) are standard Brownian motions under \(\mathbb{Q}\) with an instantaneous correlation parameter \(\rho \in (-1, 1)\). The risk-free interest rate is denoted by \(r_t \equiv r(t, X_t)\) for some bounded positive function. At the default time \(\tau_0\), the stock price will jump to value zero and remain worthless afterwards. This “jump-to-default model” for \(S\) is a variation of those by Merton [1976], Carr and Linetsky [2006], and Mendoza-Arriaga and Linetsky [2011].
2.1.1 Mark-to-Market Value with Counterparty Risk Provision

A defaultable claim is described by the triplet \((g, h, l)\), where \(g(S_T, X_T)\) is the payoff at maturity \(T\), \((h(S_t, X_t))_{0 \leq t \leq T}\) is the dividend process, and \(l(\tau_0, X_{\tau_0})\) is the payoff at the default time \(\tau_0\) of the reference entity. We assume continuous collateralization which is a reasonable proxy for the current market where daily or intraday margin calls are common [see Fujii and Takahashi, 2013]. For party \(i \in \{1, 2\}\), we denote by \(\delta_i\) the collateral coverage ratio of the claim’s MtM value. We use the range \(0 \leq \delta_i \leq 120\%\) since dealers usually require over-collateralization up to 120\% for credit or equity linked notes [see Ramaswamy, 2011, Table 1].

We first consider pricing of a defaultable claim without bilateral counterparty risk. We call this value counterparty-risk free (CRF) value. Precisely, the ex-dividend pre-default CRF value of the defaultable claim with \((g, h, l)\) is given by

\[
\Pi(t, s, x) := \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T (r_u + \lambda_u^{(0)}) \, du} g(S_T, X_T) \right. \\
\left. + \int_t^T e^{-\int_u^T (r_v + \lambda_v^{(0)}) \, dv} \left( h(S_u, X_u) + \lambda_u^{(0)} l(u, X_u) \right) \, du \right].
\tag{2.4}
\]

The shorthand notation \(\mathbb{E}_{t, s, x}[\cdot] := \mathbb{E}[\cdot | S_t = s, X_t = x]\) denotes the conditional expectation under \(\mathbb{Q}\) given \(S_t = s, X_t = x\).

Incorporating counterparty risk, we let \(\tau = \min\{\tau_0, \tau_1, \tau_2\}\), which is the first default time among the three parties with the intensity function

\[
\lambda(t, s, x) = \sum_{k=0}^{2} \lambda^{(k)}(t, s, x).
\]

The corresponding three default events \(\{\tau = \tau_0\}, \{\tau = \tau_1\}\) and \(\{\tau = \tau_2\}\) are mutually exclusive. When the reference entity defaults ahead of parties 1 and 2, i.e. \(\tau = \tau_0\), the contract is terminated and party 1 receives \(l(\tau_0, X_{\tau_0})\) from
party 2 at time $\tau_0$. When either the market participant or the counterparty defaults first, i.e. $\tau < \tau_0$, the amount that the remaining party gets depends on unwinding mechanism at the default time. We adopt the market convention where the MtM value with counterparty risk provision, denoted by $P$, is used to compute the value upon the participant’s defaults [see Fujii and Takahashi, 2013; Henry-Labordère, 2012].

Throughout, we use the notations $x^+ = x1_{\{x \geq 0\}}$ and $x^- = -x1_{\{x < 0\}}$. Suppose that party 2 defaults first, i.e. $\tau = \tau_2$. If the MtM value at default is positive ($P_{\tau_2} \geq 0$), then party 1 incurs a loss only if the contract is under-collateralized by party 2 ($\delta_2 < 1$) since the amount $\delta_2 P_{\tau_2}^+$ is secured as a collateral. As a result, with the loss rate $L_2$ (i.e. 1 - recovery rate) for party 2, the total loss of party 1 at $\tau_2$ is $L_2 (1 - \delta_2)^+ P_{\tau_2}^+$. On the other hand, suppose that the MtM value is negative ($P_{\tau_2} < 0$). Party 1 has a loss only if party 1 puts collateral more than the MtM value $P_{\tau_2}$, i.e. the contract is over-collateralized ($\delta_1 \geq 1$). In this case, party 1’s total loss is the product of the party 2’s loss rate and the exposure, i.e. $L_2 (\delta_1 - 1)^+ P_{\tau_2}^-$. Therefore, the remaining value of the party 1’s position at the default time $\tau_2$ is

$$P_{\tau_2} - L_2 (1 - \delta_2)^+ P_{\tau_2}^+ - L_2 (\delta_1 - 1)^+ P_{\tau_2}^-.$$

(2.5)

Next, we consider the case when party 1 defaults first, i.e. $\tau = \tau_1$. We denote by $L_1$ the loss rate of party 1. If the MtM value of party 1’s position at the default is negative ($P_{\tau_1} < 0$) and the contract is under-collateralized ($\delta_1 < 1$), party 2’s loss is $L_1 (1 - \delta_1)^+ P_{\tau_1}^-$. Similarly, when the MtM value is positive ($P_{\tau_1} \geq 0$) and the contract is over-collateralized ($\delta_1 \geq 1$), party 2 incurs a loss of the amount $L_1 (\delta_2 - 1)^+ P_{\tau_1}^+$. Because of the bilateral nature of the contract, party 2’s loss is party 1’s gain. Therefore, at the default time $\tau_1$, the value of party 1’s position is

$$P_{\tau_1} + L_1 (1 - \delta_1)^+ P_{\tau_1}^- + L_1 (\delta_2 - 1)^+ P_{\tau_1}^+.$$

(2.6)
Moreover, the market participant is exposed to funding cost associated with collateralization over the period since the collateral rate and funding rate do not coincide with the risk-free rate. When the liquidation value of the contract $P_t$ is positive to party 1 at time $t$, party 2 posts collateral $\delta_2 P_t^+$ to party 1. To keep the collateral, party 1 continuously pays collateral interest at rate $c_2$ to party 2 until any default time or expiry. On the other hand, when $P_t$ is negative to party 1, party 1 borrows $\delta_1 P_t^-$ to post collateral to party 2. As a result, party 1 receives interest payments at rate $c_1$ proportional to collateral amount. We call $c_i$ the effective collateral rate of party $i$ ($i = 1, 2$), which is the nominal collateral rate minus the funding cost rate of party $i$. The rates $c_1$ and $c_2$ can be both negative in practice if the funding costs are high. Therefore, party 1 has the following cash flow generated by the collateral and effective collateral rates:

$$1_{\{t<\tau\}} \left( c_1 \delta_1 P_t^- - c_2 \delta_2 P_t^+ \right), \quad 0 \leq t \leq T.$$  \hspace{1cm} (2.7)

The aforementioned cash flow analysis implies that the pre-default MtM value with counterparty risk (CR) provision is given by

$$P(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T (r_v + \lambda_v) \, dv} g(S_T, X_T) ight]$$

$$+ \int_t^T e^{-\int_t^u (r_v + \lambda_v) \, dv} \left( h(S_u, X_u) + \lambda_u^{(0)} l(t, X_u) \right) \, du$$

$$+ \int_t^T \lambda_u^{(2)} e^{-\int_t^u (r_v + \lambda_v) \, dv} \left( (1 - L_2 (1 - \delta_2)^+) P_u^+ - (1 + L_2 (\delta_1 - 1)^+) P_u^- \right) \, du$$

$$+ \int_t^T \lambda_u^{(1)} e^{-\int_t^u (r_v + \lambda_v) \, dv} \left( (1 + L_1 (\delta_2 - 1)^+) P_u^+ - (1 - L_1 (1 - \delta_1)^+) P_u^- \right) \, du$$

$$+ \int_t^T e^{-\int_t^u (r_v + \lambda_v) \, dv} \left( c_1 \delta_1 P_u^- - c_2 \delta_2 P_u^+ \right) \, du \right].$$ \hspace{1cm} (2.9)

The first and second line account for the terminal cash flow, the dividend, and the payoff at the reference asset’s default ($\tau = \tau_0$). The third fourth line are the cash flows at party 2’s default ($\tau = \tau_2$) in (2.5) and party 1’s
default \((\tau = \tau_1)\) in \((2.6)\), respectively. The last line results from the collateral and effective collateral rates in \((2.7)\). To simplify, we introduce the following notations

\[
\tilde{r}(t, s, x) = r(t, x) + \lambda(t, s, x),
\]

\[
\alpha(t, s, x) = L_2 \lambda^{(2)}(t, s, x) (1 - \delta_2)^+ - L_1 \lambda^{(1)}(t, s, x) (\delta_2 - 1)^+ + c_2 \delta_2,
\]

\[
\beta(t, s, x) = L_1 \lambda^{(1)}(t, s, x) (1 - \delta_1)^+ - L_2 \lambda^{(2)}(t, s, x) (\delta_1 - 1)^+ + c_1 \delta_1,
\]

\[
f(t, s, x, y) = h(s, x) + \lambda^{(0)}(t, s, x) l(t, x) + (\lambda^{(1)} + \lambda^{(2)} - \beta)(t, s, x)y \\
+ (\beta - \alpha)(t, s, x) y^+.
\]

This allows to express \((2.9)\) in the equivalent but simplified form:

\[
P(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T \tilde{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v \, dv} f(u, S_u, X_u, P_u) \, du \right],
\]

\[
(2.14)
\]

where \(\tilde{r}_t \equiv \tilde{r}(t, S_t, X_t)\) as defined in \((2.10)\).

**Remark 2.1.** As an alternative of MtM value with CR provision, the liquidation value at the time of default can be evaluated as the CRF value of the claim. In other words, at the default time \(\tau < \tau_0\), the liquidation value is evaluated as \(\Pi_\tau\) rather than \(P_\tau\). Replacing \(P_u\) in \((2.14)\) with \(\Pi_u\) for \(t \leq u \leq T\) gives the MtM value without CR provision (see Henry-Labordère [2012]):

\[
\hat{P}(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T \tilde{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v \, dv} f(u, S_u, X_u, \Pi_u) \, du \right].
\]

\[
(2.15)
\]

To conclude this section, we summarize the symbols and their financial meanings in Table 2.1 which we will use frequently throughout this paper.
CHAPTER 2. PRICING OF DEFAULTABLE CLAIMS WITH COUNTERPARTY RISK AND COLLATERALIZATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Symbol</th>
<th>Definition for party $i \in {1, 2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>MtM value with CR provision</td>
<td>$R_i$</td>
<td>Recovery rate</td>
</tr>
<tr>
<td>$\hat{P}$</td>
<td>MtM value without CR provision</td>
<td>$c_i$</td>
<td>Effective collateral rate</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>CRF value</td>
<td>$\delta_i$</td>
<td>Collateralization ratio</td>
</tr>
<tr>
<td>$\tau_0$</td>
<td>Default time of reference asset</td>
<td>$\tau_i$</td>
<td>Default time</td>
</tr>
<tr>
<td>$\lambda^{(0)}$</td>
<td>Default intensity of reference asset</td>
<td>$\lambda^{(i)}$</td>
<td>Default intensity</td>
</tr>
</tbody>
</table>

Table 2.1: Summary of notations

2.1.2 Bid-Ask Prices

In OTC trading, market participants, such as dealers, may take a long position as a buyer or a short position as a seller. Without counterparty risk, the buyer’s CRF bid price $\Pi^b(t, s, x)$ for a claim with payoff $(g, h, l)$ is given by (2.4). The MtM value of the seller’s position satisfies (2.4) by replacing $(g, h, l)$ with $(-g, -h, -l)$, the negative of which gives the seller’s CRF ask price $\Pi^s(t, s, x)$. In fact, the bid-ask prices are identical, i.e. $\Pi^b(t, s, x) = \Pi^s(t, s, x)$.

Similarly for the case with counterparty risk provision, the buyer’s bid price is $P^b(t, s, x) = P(t, s, x)$ as in (2.14). The seller’s ask price is given by

$$P^s(t, s, x) = \mathbb{E}_{t,s,x} \left[ e^{-\int_t^t \hat{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \hat{r}_v \, dv} \tilde{f}(u, S_u, X_u, P^s_v) \, du \right],$$

(2.16)

where

$$\tilde{f}(t, s, x, y) = h(s, x) + \lambda^{(0)}(t, s, x)l(t, x) + (\lambda^{(1)} + \lambda^{(2)} - \beta)(t, s, x) y$$

$$- (\beta - \alpha)(t, s, x) y^-.$$  

(2.17)

Since $\tilde{f}(t, s, x, y)$ is different from $f(t, s, x, y)$ in (2.13), the symmetry observed in the CRF prices generally no longer holds in the presence of bilateral counterparty risk. Most importantly, such an asymmetry generates bid-ask
spreads for defaultable claims. For any contract with counterparty risk provision, the participant can quote two prices: \( P^b(t, s, x) \) as a buyer or \( P^s(t, s, x) \) as a seller. In addition, since the payoff components \((g, h, l)\) can be negative, the bid and/or ask prices also can be negative (see Figure 2.4).

The bilateral credit valuation adjustment (BCVA) is defined as a deviation of the MtM value from the CRF value, namely, \( \Pi - P^b \) for a long position and \( P^s - \Pi \) for a short position. The bid-ask spread accounting for the BCVA with CR provision is defined as \( S(t, s, x) = P^s(t, s, x) - P^b(t, s, x) \).

The two factors \( \alpha \) and \( \beta \) in (2.11) and (2.12) that appear in \( f \) and \( \tilde{f} \) summarize the effects of counterparty risk and collateralization on the bid-ask prices. Specifically, \( \alpha \) explains the effect of positive counterparty exposure of the MtM value \( P^+_u \) while \( \beta \) explains the effect of negative exposure \( P^-_u \). When the two parameters have the same value (\( \alpha = \beta \)), the two functions \( f \) and \( \tilde{f} \) in (2.13) and (2.17) are identical. Therefore, the bid-ask prices \( P^b \) and \( P^s \) are equal. Such a price symmetry also arises in a number of other scenarios: (i) when both parties have perfect collateralization ratio (\( \delta_1 = \delta_2 = 1 \)) and the same effective collateral rate (\( c_1 = c_2 \)); (ii) when both parties have zero collateralization ratio (\( \delta_1 = \delta_2 = 0 \)) with the same effective default rate (\( L_1 \lambda^{(1)} = L_2 \lambda^{(2)} \)), and (iii) when both parties have the same effective collateral rate (\( c_1 = c_2 \)) with the same effective default rate and collateralization ratio (\( L_1 \lambda^{(1)} = L_2 \lambda^{(2)} \), \( \delta_1 = \delta_2 \)).

**Remark 2.2.** When the counterparty risk-free value \( \Pi \) is used to estimate the liquidation value upon default, the seller’s bid price is given by

\[
\tilde{P}^s(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T \tilde{r}_u \, du} \, g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v \, dv} \tilde{f}(u, S_u, X_u, \Pi_u) \, du \right],
\]

(2.18)

where \( \tilde{f} \) is defined in (2.17). In contrast to (2.14), the price function on the LHS does not appear on the RHS.
2.2 Fixed Point Method

The defining equation (2.14) has a recursive form whereby the price function \( P \) appears on both sides. Denote the spacial domain by \( \mathcal{D} := \mathbb{R}^+ \times \mathbb{R} \). For any function \( w \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \), we define the operator \( \mathcal{M} \) by

\[
\mathcal{M}w(t, s, x) = \mathbb{E}_{t,s,x} \left[ e^{-\int_t^T \hat{r}_u\,du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \hat{r}_v\,dv} f(u, S_u, X_u, w(u, S_u, X_u)) \,du \right].
\]

Then, we recognize from (2.14) that the MtM value with counterparty risk provision satisfies \( P = \mathcal{M}P \). This motivates us to show that the operator \( \mathcal{M} \) has a unique fixed point, and therefore, guarantees the existence and uniqueness of the MtM value \( P \).

We discuss our fixed point approach by first showing that the operator \( \mathcal{M} \) defined in (2.19) preserves boundedness and continuity. To this end, we outline a number of conditions according to Heath and Schweizer [2000].

(C1) We define

\[
\Gamma(t, s, x) = \begin{bmatrix} r(t, x) + \lambda^{(0)}(t, s, x) \\ b(t, x) \end{bmatrix},
\]

\[
\Sigma(t, s, x) = \begin{bmatrix} \sigma(t, s) s \\ \rho \eta(t, x) \sqrt{1 - \rho^2 \eta(t, x)} \end{bmatrix}.
\]

The coefficients \( \Gamma \) and \( \Sigma \) are locally Lipschitz-continuous in \( s \) and \( x \), uniformly in \( t \). That is, for each compact subset \( F \) of \( \mathcal{D} \), there is a constant \( K_F < \infty \) such that for \( \psi \in \{\Gamma, \Sigma\}, \)

\[
|\psi(t, s_1, x_1) - \psi(t, s_2, x_2)| \leq K_F|||s_1, x_1) - (s_2, x_2)||
\]

\( \forall t \in [0, T], (s_1, x_1), (s_2, x_2) \in F, \) where \( || \cdot || \) is the Euclidean norm in \( \mathbb{R}^2 \).
(C2) For all \((t, s, x) \in [0, T) \times \mathcal{D}\), the solution \((S, X)\) neither explodes nor leaves \(\mathcal{D}\) before \(T\), i.e.

\[
Q\left(\sup_{t \leq u \leq T} \|(S_u, X_u)\| < \infty\right) = 1 \quad \text{and} \quad Q\left((S_u, X_u) \in \mathcal{D}, \forall u \in [t, T]\right) = 1.
\]

(C3) The functions \(h\) and \(g\) are bounded and continuous, and \(r, l\) and \(\lambda^{(i)}\), \(i \in \{0, 1, 2\}\), are positive, continuous and bounded.
Lemma 2.3. Given any function \( w \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \), it follows that \( v := \mathcal{M}w \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \).

Proof. The boundedness of \( v \) follows directly from that of \( w, h, g, r, \) and \( \lambda(i) \) (see condition \((C3))

To prove the continuity of \( v \), we first observe that

\[
(t, s, x) \mapsto e^{-\int_t^T \tilde{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v \, dv} f(u, S_u, X_u, w(u, S_u, X_u)) \, du
\]

is continuous \( \mathbb{Q} \)-a.s. Indeed, the continuity of \((S, X)\) implies that the mapping \((t, s, x) \mapsto g(S_T, X_T)\) is continuous \( \mathbb{Q} \)-a.s. Also, \((t, s, x, u) \mapsto \tilde{r}(u, S_u, X_u)\) and

\[
(t, s, x, u) \mapsto f(u, S_u, X_u, w(u, S_u, X_u))
\]

are uniformly continuous and bounded \( \mathbb{Q} \)-a.s. on compact subsets of \([0, T] \times \mathcal{D} \times [t, T]\). Hence, the mapping in (2.20) is continuous \( \mathbb{Q} \)-a.s. Taking expectation on the RHS of (2.20) and applying Dominated Convergence Theorem to exchange expectation and continuity limits, we conclude. \( \Box \)

2.2.1 Contraction Mapping

Next, we show that the mapping \( \mathcal{M} \) is a contraction. By the boundedness of \( \alpha(t, s, x), \beta(t, s, x) \) and \( \lambda(i)(t, s, x) \) for \( i \in \{0, 1, 2\} \), we can define a finite positive constant by

\[
L = \sup_{(t,s,x) \in [0,T] \times \mathcal{D}} \{|\lambda^{(1)}(t,s,x) + \lambda^{(2)}(t,s,x) - \beta(t,s,x)| + |\beta(t,s,x) - \alpha(t,s,x)|\}.
\]

Proposition 2.4. The mapping \( \mathcal{M} \) defined in (2.19) is a contraction on the space \( C_b([0,T] \times \mathcal{D}, \mathbb{R}) \) with respect to the norm

\[
\|w\|_\gamma := \sup_{(t,s,x) \in [0,T] \times \mathcal{D}} e^{-\gamma(T-t)} |w(t,s,x)|,
\]

for \( L < \gamma < \infty \). In particular, \( \mathcal{M} \) has a unique fixed point \( w^* \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \).
Proof. From (2.13), we observe that \(|f(t, s, x, y_1) - f(t, s, x, y_2)| \leq L|y_1 - y_2|\), for \((t, s, x) \in [0, T] \times D\). This implies \(f\) is Lipschitz-continuous in \(y\), uniformly over \((t, s, x)\). By Lemma 2.3, the operator \(M\) maps \(C_b([0, T] \times D, \mathbb{R})\) into itself.

For \((t, s, x) \in [0, T] \times D\), \(w_1, w_2 \in C_b([0, T] \times D, \mathbb{R})\), and \(\gamma > 0\), we have

\[
e^{-\gamma(T-t)}|\langle M w_1 \rangle(t, s, x) - \langle M w_2 \rangle(t, s, x)|
= e^{-\gamma(T-t)}\|e^{-\int_t^T \tilde r_v du} (f(u, S_u, X_u, w_1(u, S_u, X_u)) - f(u, S_u, X_u, w_2(u, S_u, X_u))) du\|
\leq e^{-\gamma(T-t)}\|e^{-\int_t^T \tilde r_v du} (f(u, S_u, X_u, w_1(u, S_u, X_u)) - f(u, S_u, X_u, w_2(u, S_u, X_u))) du\|
\leq e^{-\gamma(T-t)}\|e^{-\int_t^T \tilde r_v du} (f(u, S_u, X_u, w_1(u, S_u, X_u)) - f(u, S_u, X_u, w_2(u, S_u, X_u))) du\|
\leq L \gamma \|w_1 - w_2\|.$

We have used the facts that \(\tilde r_v \geq 0\) and \(f\) is Lipschitz in \(y\) in inequalities \((i)\) and \((ii)\) respectively, while \((iii)\) is implied by the norm in (2.21). As a result, for any \(\gamma > L \geq 0\), \(M\) is a contraction.

The norm \(\|\cdot\|_\gamma\) is equivalent to the supremum norm \(\|\cdot\|_\infty\) on the space \(C_b([0, T] \times D, \mathbb{R})\). A similar norm is used in Becherer and Schweizer [2005] and Leung and Sircar [2009] in their studies of reaction diffusion PDEs arising from indifference pricing.

Using the fact that \(M\) is a contraction proved in Proposition 2.4, there exists a sequence of functions \((P^{(n)})_{n \geq 0}\) that satisfy \(P^{(n+1)} = MP^{(n)}, \forall n \geq 0\), and the sequence converges to the fixed point \(P\). The convergence does not rely on the choice of the initial function. Indeed, one can simply pick any bounded continuous function as a starting point, e.g. \(P^{(0)} = 0 \ \forall (t, s, x)\), and iterate to have a sequence \((P^{(n)})_{n \geq 0}\) that resides in \(C_b([0, T] \times D, \mathbb{R})\).
Furthermore, we can show that for each \( n \geq 1 \), \( P^{(n)} \equiv P^{(n)}(t,s,x) \) is a classical solution of the following inhomogeneous PDE problem:

\[
\frac{\partial P^{(n)}}{\partial t} + L P^{(n)} - \tilde{r}(t,s,x) P^{(n)} + f(t,s,x,P^{(n-1)}) = 0,
\]

\[
P^{(n)}(T,s,x) = g(s,x), \tag{2.22}
\]

where the operator \( L \) is defined by

\[
L := \frac{1}{2} \sigma(t,s)^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2} \eta(t,x)^2 \frac{\partial^2}{\partial x^2} + \rho \eta(t,x) \sigma(t,s) s \frac{\partial}{\partial s} \frac{\partial}{\partial x} + \tilde{r}(t,s,x) s \frac{\partial}{\partial s} + b(t,x) \frac{\partial}{\partial x}. \tag{2.23}
\]

In order to prove the result, we need the following additional conditions, adapted in our notation from \((A3') - (A3d')\) of [Heath and Schweizer, 2000].

(C4) There exists a sequence \((D_n)_{n \in \mathbb{N}}\) of bounded domains with closure \(\bar{D}_n \subset D\) such that \(\bigcup_{n=1}^{\infty} D_n = D\) and each \(D_n\) has a \(C^2\)-boundary.

As in [Heath and Schweizer, 2000], one can take \(D_n = [\frac{1}{n}, n] \times [-n,n] \subset \mathbb{R}_+ \times \mathbb{R}\). For each \(n\), we require that

(C5) \( b(t,x), a(t,s,x) := \Sigma(t,s,x) \Sigma^t(t,s,x) \), and \( \tilde{r}(t,s,x) \) be uniformly Lipschitz-continuous on \([0,T] \times \bar{D}_n\), where \(\Sigma^t\) denotes the transpose matrix of \(\Sigma\),

(C6) \( a(t,s,x) \) be uniformly elliptic on \(\mathbb{R}^2\) for \((t,s,x) \in [0,T] \times D_n\), i.e. there is \(\delta_n > 0\) such that \( y^t a(t,s,x) y \geq \delta_n \|y\|^2 \) for all \(y \in \mathbb{R}^2\),

(C7) \( f(t,s,x,y) \) be uniformly Hölder-continuous on \([0,T] \times \bar{D}_n \times \mathbb{R}\).

The conditions (C1) – (C7) are quite general, and they allow for various models, including the Heston, CEV, and thus, geometric Brownian motion models for equity, and the Ornstein-Uhlenbeck and Cox-Ingersoll-Ross models for the stochastic factor \(X\) [Heath and Schweizer, 2000 Sect. 2]. The triplet \((g,h,l)\), default intensities \(\lambda^{(i)}\) and interest rate \(r\) can be easily chosen to satisfy the boundedness and continuity conditions in (C3), as we will do in our examples in Sections 2.3 and 2.4.
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Theorem 2.5. Under conditions $(C_1) - (C_7)$, there exists a sequence of bounded classical solutions $(P^{(n)}) \subset C^{1,2}_b([0,T] \times \mathcal{D}, \mathbb{R})$ of the PDE problem (2.22) that converges to the fixed point $P \in C_b([0,T] \times \mathcal{D}, \mathbb{R})$ of the operator $\mathcal{M}$.

We provide the proof in Appendix A.1. The insight of Proposition 2.4 and Theorem 2.5 is that we can construct and solve a series of inhomogeneous but linear PDEs whose classical solutions converge to a unique fixed point price function $P$ as in (2.14). Recent studies by Burgard and Kjaer [2011] and Henry-Labordère [2012] evaluate the MtM value $P(t,s,x)$ by working with the associated nonlinear PDE of the form:

$$\frac{\partial P}{\partial t} + \mathcal{L}P - \tilde{r}(t,s,x)P + f(t,s,x,P) = 0,$$  \hspace{1cm} (2.24)

for $(t,s,x) \in [0,T] \times \mathcal{D}$, with terminal condition $P(T,s,x) = g(s,x)$, for $(s,x) \in \mathcal{D}$. The nonlinearity of (2.24) poses major challenges on analyzing and numerically solving for $P$. Henry-Labordère [2012] provides a method to approximate the solution that involves replacing the nonlinear term $f$ with a polynomial and simulating a marked branching diffusion. This method, however, does not guarantee that the solution from simulation will resemble the solution of the nonlinear PDE, and does not ensure any regularity, such as continuity or boundedness of, either solution. Henry-Labordère [2012] provides conditions on the chosen polynomial to avoid a “blow-up” of the simulation algorithm. In contrast, our fixed point methodology circumvents this issue by establishing that the pricing definition in (2.14) is a contraction mapping, as opposed to working with the nonlinear PDE. As a result, we solve a series of linear PDE problems with bounded classical solutions. In the limit, a unique bounded continuous MtM value $P$ is obtained.
2.2.2 Numerical Implementation

Our contraction mapping methodology lends itself to a recursive numerical algorithm. As mentioned in the previous section, we iteratively solve a sequence of linear inhomogeneous PDEs (2.22). At each iteration, the error is measured in terms of the maximum difference between two consecutive solutions \( P(n) \) and \( P(n-1) \) over the entire domain \([0, T] \times \mathcal{D}\). We continue the iteration procedure until the error is less than the pre-defined tolerance level \( \bar{\epsilon} \).

For implementation, we use the standard Crank-Nicolson finite difference method (FDM) to obtain the values (see, among others, [Wilmott et al. 1995] and [Strikwerda 2007]). We restrict the domain \([0, T] \times \mathcal{D}\) to a finite domain \( \bar{\mathcal{D}} = \{(t, s, x) : 0 \leq t \leq T, X \leq x \leq \bar{X}, 0 \leq s \leq \bar{S}\} \). The parameters \( \bar{S}, X \) and \( \bar{X} \) are sufficiently large enough to preserve the accuracy of the numerical solutions. We discretize the function \( P^{(n)}(t, s, x) \) as \( P^{(n)}(t_i, s_j, x_k) \) where \( i \in \{0, ..., N\} \), \( j \in \{0, ..., M\} \) and \( k \in \{0, ..., L\} \) with \( \Delta t = T/N, \Delta s = \bar{S}/M, \Delta x = (\bar{X} - X)/L \) and \( t_i = i \Delta t, s_j = j \Delta s, x_k = k \Delta x \). Our numerical procedure is summarized in Algorithm 1.

**Algorithm 1** Fixed Point Algorithm for Evaluating the MtM Value \( P \)

1. set \( n = 1, \ P^+ = P^{(0)} \)
2. solve for \( P^{(1)} \) from PDE (2.22)
3. set \( \epsilon = \| P^{(1)} - P^{(0)} \|_{\infty} \)
4. while \( \epsilon > \bar{\epsilon} \) do
   1. set \( n = n + 1, \ P^+ = P^{(n-1)} \)
   2. solve for \( P^{(n)} \) from PDE (2.22)
   3. set \( \epsilon = \| P^{(n)} - P^{(n-1)} \|_{\infty} \)
5. end while
6. return \( P^{(n)} \)

For the CRF value, we solve the linear PDE associated with \( \Pi \equiv \Pi(t, s, x) \)
2.3 Defaultable Equity Derivatives with Counterparty Risk

We now apply our valuation methodology to value a number of defaultable equity claims. Specifically, we will derive and compare the MtM values with and without counterparty risk provisions as well as the CRF value. Moreover, we will analyze and illustrate the bid-ask prices.

As a special case of (2.2), we model the pre-default stock price process by

\[ dS_t = \left( r + \lambda(0) \right) S_t dt + \sigma S_t dW_t , \]  

(2.27)

where we assume constant interest rate \( r \) and default rates \( \lambda(i), i \in \{0,1,2\} \).

In addition, we let \( \lambda = \sum_{k=0}^{2} \lambda^{(k)} \), and set

\[ \alpha = L_2 \lambda^{(2)}(1 - \delta_2)^+ - L_1 \lambda^{(1)}(\delta_2 - 1)^+ + c_2 \delta_2 , \]  

(2.28a)

\[ \beta = L_1 \lambda^{(1)}(1 - \delta_1)^+ - L_2 \lambda^{(2)}(\delta_1 - 1)^+ + c_1 \delta_1 . \]  

(2.28b)
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2.3.1 Call Spreads

Let us consider a generic call spread with the terminal payoff:

\[ g(S_T) = \begin{cases} 
  m_2 & \text{if } S_T > K + \epsilon_2, \\
  \frac{(m_1+m_2)}{\epsilon_1+\epsilon_2} (S_T - K) & \text{if } K - \epsilon_1 \leq S_T \leq K + \epsilon_2, \\
  -m_1 & \text{if } S_T < K - \epsilon_1,
\end{cases} \]  

(2.30)

with \( m_1, m_2, \epsilon_1, \epsilon_2 > 0 \), where \( m_1/\epsilon_1 = m_2/\epsilon_2 =: M \). The payoff resembles that of a long position of \( M \) call options with strike \( K - \epsilon_1 \), a short position of \( M \) call options with strike \( K + \epsilon_2 \) and short \( m_1 \) notional of zero coupon bond with the same maturity. Similar positions can be achieved when two OTC traders buy and sell call options with different strikes, plus/minus some cash. As \( \epsilon_1 \) and \( \epsilon_2 \) in (2.30) go to zero, the payoff converges to that of a digital call position covered in Henry-Labordère [2012].

With the terminal payoff \( g \) in (2.30), dividend \( h = 0 \), and value at reference default \( l(\tau_0) = -m_1 e^{-r(T-\tau_0)} \), the CRF value of the spread contract admits the formula

\[ \Pi(t,s) = M \left( C_{BS}(t,s;T,K - \epsilon_1,r + \lambda(0),\sigma) - C_{BS}(t,s;T,K + \epsilon_2,r + \lambda(0),\sigma) \right) - e^{-r(T-t)} m_1, \]

where \( C_{BS}(t,s;T,K,r,\sigma) \) is the Black-Scholes call option price at time \( t \) with spot price \( s \), maturity \( T \), strike price \( K \), risk-free rate \( r \) and volatility \( \sigma \). From (2.14), the MtM value with counterparty risk provision is given by

\[ P(t,s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} f(u,S_u,P_u) \, du \right], \]  

(2.31)

where \( f(t,s,y) := \lambda(0) l(t) + (\lambda^{(1)} + \lambda^{(2)} - \beta) y + (\beta - \alpha) y^+ \). The MtM value without counterparty risk provision \( \hat{P}(t,s) \) is similarly obtained replacing \( P_u \) in (2.31) with \( \Pi_u \).
The model for $S$ in (2.27), the triple $(g,h,l)$, and other (constant) coefficients satisfy the conditions (C1)-(C7) with domain $\mathcal{D} = \mathbb{R}_+$ (see also [Heath and Schweizer 2000, Sect.2]). We numerically compute the MtM value $P(t,s)$ by Algorithm 1 from Section 2.2.2. For the iterative PDE (2.22), we adopt the coefficients in this section and the terminal payoff $g(S_T)$ given in (2.30). In Table 2.2 we show the convergence of the MtM values with provision for three different contracts where $\epsilon_1 = \epsilon_2 = \{2, 1, 0.01\}$. The first column of each contract shows the value of the MtM value of the contract at each step $0 \leq n \leq 5$. The second column of each contract shows the supremum norm $\epsilon = \|P^{(n)} - P^{(n-1)}\|_\infty$ for each step $0 \leq n \leq 5$. The algorithm stops at $n = 5$ for all three contracts.

<table>
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Table 2.2: Convergence of the MtM values of a call spread

Convergence of the MtM values with provision $P(0,s)$ of call spread contract at spot price $s = 10$ (at-the-money) and $m_1 = m_2 = 1$. Parameters: $K = 10$, $T = 2$, $t = 0$, $r = 2\%$, $\sigma = 25\%$, $\lambda^{(0)} = 3\%$, $\lambda^{(1)} = 5\%$, $\lambda^{(2)} = 15\%$, $R_1 = 40\%$, $R_2 = 40\%$, $\delta_1 = \delta_2 = 0$, $\bar{\epsilon} = 10^{-5}$, $\bar{S} = 40$, $\Delta S = 0.01$, $\Delta t = 1/1000$.

Let us visualize the convergence of the MtM value with CR provision $P^{(n)}(0,s)$ in Figure 2.1 (left). Using the tolerance level $\bar{\epsilon} = 10^{-5}$ for the maximum difference over each iteration, the algorithm stops after 4 iterations. As we can see, the price functions $P^{(3)}(0,s)$ and $P^{(4)}(0,s)$ over $0 \leq s \leq \bar{S} = 40$
are not visibly distinguishable.

Figure 2.1: The MtM values of a call spread
(Left) Convergence of the MtM values with provision of a call spread \( P(0,s) \) for \( s \in [10,18] \). (Right) Comparison of the three MtM values of a call spread \( \{\Pi(0,s), \hat{P}(0,s), P(0,s)\} \) over the spot price. Parameters are given in Table 2.2.

In Figure 2.1 (right), we plot three different values \( \Pi(0,s), \hat{P}(0,s) \) and \( P(0,s) \) for \( 0 \leq s \leq \bar{S} \). As we can see, the ordering of these three values can change completely depending on the spot price. For example, for large spot prices, we observe that the CRF value dominates the other two MtM values, but it is lowest when the spot price is small. Furthermore, the value without provision dominates the value with provision for high spot prices, and the opposite holds true for low spot prices.

Next, we look at the sensitivity of the MtM values with respect to the counterparty’s or own default risk, collateralization ratio and effective collateral rate. In Figure 2.2, the MtM values are decreasing in the counterparty default rate (left) and increasing in the participant’s own default rate (right), as is intuitive. Note that the MtM value with provision moves more rapidly with respect to the counterparty default rate, but the MtM value without provision is more sensitive in the participant’s own default rate.

In Figure 2.3 (left), an increase in \( \delta_2 \) reduces counterparty-risk exposure,
and therefore, increases the MtM values with and without counterparty risk provision. The rate of increase in contract value slows down when the collateralization ratio exceed 1. In the over-collateralized range $[1, 1.2]$, party 1 is no longer exposed to the counterparty’s default risk. The increase in the contract value (from party 1’s perspective) results from the possibility of collecting the excess collateral upon party 1’s own default.

In practice, if the participant’s funding cost rate is high, the effective collateral rate can be negative (see Burgard and Kjaer [2011]). This implies a net interest payment by the participant for the long position due to collateralization. As the effective collateral rate becomes more negative, the contract values with and without provision decrease as we observe on the right panel of Figure 2.3.

Figure 2.2: The MtM values of a call spread in terms of $\lambda^{(2)}$ and $\lambda^{(1)}$. (Left) The MtM values with and without provision are decreasing in the counterparty default rate $\lambda^{(2)}$ with $\lambda^{(1)} = 15\%$. (Right) The MtM values are increasing in $\lambda^{(1)}$ with $\lambda^{(2)} = 15\%$. The CRF value stays constant as $\lambda^{(2)}$ or $\lambda^{(1)}$ varies. Parameters: $\epsilon_1 = \epsilon_2 = 0.01, m_1 = m_2 = 1, s = 15, K = 10, T = 2, t = 0, r = 2\%, \sigma = 25\%, \lambda^{(0)} = 3\%, R_1 = R_2 = 40\%, \delta_1 = \delta_2 = 0\%, c_1 = c_2 = 1\%, \bar{\epsilon} = 10^{-5}, \Delta S = 0.05, \Delta t = 1/250$.

We illustrate the bid-ask prices $P^b$ and $P^s$ of a call spread in Figure 2.4. On
Figure 2.3: The MtM values of a call spread in terms of $\delta^{(2)}$ and $c_1$

(Left) The MtM values of a call spread with and without provision are decreasing in counterparty’s collateralization ratio $\delta_2$. (Right) The MtM values of a call spread are increasing in the participant’s effective collateral rate $c_1 \in [-5\%,0\%]$. Parameters: $\epsilon_1 = \epsilon_2 = 0.01$, $m_1 = m_2 = 1$, $s = 15$, $K = 10$, $T = 2$, $t = 0$, $r = 2\%$, $\sigma = 25\%$, $\lambda^{(0)} = 3\%$, $\lambda^{(1)} = 5\%$, $\lambda^{(2)} = 15\%$, $R_1 = R_2 = 40\%$, $\delta_1 \in \{0\% \text{ (left)}, 100\% \text{ (right)}\}$, $\delta_2 = 100\%$, $c_1 = c_2 = 1\%$, $\bar{\epsilon} = 10^{-5}$, $\Delta S = 0.05$, $\Delta t = 1/250$.

the left panel where the participant is assumed to be default-free, we observe the dominance of the three prices: $P_s \geq \Pi \geq P_b$. However, in the bilateral counterparty-risk case, the ordering of prices is different in in-the-money (ITM) and out-of-money (OTM) ranges. We see that $\Pi \geq P_s \geq P_b$ in the ITM range, but $P_s \geq P_b \geq \Pi$ in the OTM range.

2.3.2 Equity Forwards

Equity forward contracts are commonly traded in the OTC market. With stock $S$ as the underlying asset, we consider a forward with maturity $T$. The initial forward price $F_0$ is set so that the contract has zero value at inception.

When the underlying stock defaults, the stock price goes to zero, and the buyer has to pay the discounted value $e^{-r(T-\tau_0)}F_0$ at the default time. The contract cash flow is described by the triplet $(g,h,l) = (S_T - F_0, 0, -e^{-r(T-\tau_0)}F_0)$. As
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Call spread bid and ask prices with counterparty risk provision under (left) unilateral counterparty risk \( \lambda^{(1)} = 0\% \), \( \lambda^{(2)} = 15\% \), and (right) bilateral counterparty risk \( \lambda^{(1)} = 5\% \), \( \lambda^{(2)} = 15\% \). Other parameters are the same as in Figure 2.2.

At time \( t \), when the stock price is \( s \), the CRF value of a long forward is given by

\[
\Pi(t, s) = (s - e^{-r(T-t)} F_0).
\]

In order to compute the MtM value of a long forward contract \( \hat{H}^b \) without
CHAPTER 2. PRICING OF DEFAULTABLE CLAIMS WITH COUNTERPARTY RISK AND COLLATERALIZATION

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<th>( \tilde{S} = 40 )</th>
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Table 2.3: Convergence of the MtM values of a forward contract when spot price \( S_1 = 20 \) at \( t = 1 \), with maximum stock price \( \tilde{S} \in \{30 \text{ (left column)}, 40 \text{ (right column)} \} \). Other common parameters: \( F_0 = 10 \), \( T = 3 \), \( r = 2\% \), \( \sigma = 25\% \), \( \lambda(0) = 3\% \), \( \lambda(1) = 5\% \), \( \lambda(2) = 15\% \), \( R_1 = 40\% \), \( R_2 = 40\% \), \( \delta_1 = \delta_2 = 0 \), \( \bar{\epsilon} = 10^{-5} \), \( \Delta S = 0.05 \), \( \Delta t = 1/500 \).

To simplify the above equation, we notice that

\[ E_{t,s} \left[ e^{-r(T-u)} \left( S_u - e^{-r(T-u)} F_0 \right) \right] = C_{BS} \left( t, s; T, e^{-r(T-u)} F_0, r + \lambda(0), \sigma \right). \]

We apply similar arguments to the seller’s MtM value, and summarize as follows.
Proposition 2.6. The bid-ask prices without counterparty risk provision, \( \hat{P}^b(t, s) \) and \( \hat{P}^s(t, s) \), of a stock forward without counterparty risk provision are given by

\[
\hat{P}^b(t, s) = \Pi(t, s) + \int_t^T e^{-\lambda(u-t)} \left( (\beta - \alpha) C_{BS}(u, s; T, e^{-r(T-u)}F_0, r + \lambda(0), \sigma) \right. \\
\left. - \beta(s - e^{-r(T-t)}F_0) \right) du,
\]

\[
\hat{P}^s(t, s) = \Pi(t, s) + \int_t^T e^{-\lambda(u-t)} \left( (\alpha - \beta) C_{BS}(u, s; T, e^{-r(T-u)}F_0, r + \lambda(0), \sigma) \right. \\
\left. - \alpha(s - e^{-r(T-t)}F_0) \right) du,
\]

with \( \Pi(t, s) \) satisfying (2.32).

The fair forward price makes the MtM value of the contract equal to zero at the start of the contract. The CRF value in (2.32) implies that \( F_0 = e^{rT}S_0 \). However, the fair forward price in presence of bilateral counterparty risk is found implicitly. Precisely, the fair forward price \( F^*_0 \) of a long (resp. short) position makes the MtM value with counterparty risk provision satisfies \( P^b(0, S_0; F^b_0) = 0 \) (resp. \( P^s(0, S_0; F^s_0) = 0 \)).

In Figure 2.5, we plot the bid-ask prices \( P^b \) and \( P^s \) of the forward contract with counterparty risk provision together with the CRF value. On the left panel, all three values increase as the underlying stock price increases. Similar to the call spread case, the price ordering changes from \( \Pi \geq P^s \geq P^b \) in the ITM range to \( P^s \geq P^b \geq \Pi \) in the OTM range. On the right panel, both MtM values decrease significantly as the counterparty default rate increases. However, the MtM with provision moves more rapidly, similar to Figure 2.2.

Remark 2.7 (Total Return Swap). Total return swaps (TRS) are also traded over the counter as an alternative to equity stock forwards. The swap buyer will receive the increase in equity value at swap expiration date \( T, S_T - S_0 \). On the other hand, the buyer continuously pays a premium rate \( p \geq r \) until
Figure 2.5: The MtM value of a stock forward in terms of spot price and $\lambda^{(2)}$ (Left) The value of a stock forward at time 0 is increasing in the spot price. The fair forward price is the spot price at which the contract value is zero. (Right) The MtM forward contract values of a stock forward are decreasing in the counterparty’s default rate $\lambda^{(2)}$, with $F_0 = 10$, $t = 1$ and $T = 3$. Parameters: $r = 2\%$, $\sigma = 25\%$, $\lambda^{(0)} = 3\%$, $\lambda^{(1)} = 5\%$, $R_1 = 40\%$, $\lambda^{(2)} = 15\%$, $R_2 = 40\%$, $\delta_1 = \delta_2 = 0$, $\bar{\epsilon} = 10^{-5}$, $\Delta S = 0.05$, $\Delta t = 1/500$.

the expiration date and also pays the decrease in the equity value at the expiration date to the swap seller. The TRS is represented as the triplet $(S_T - S_0, -p, -S_0 e^{-r(T-t_0)})$.

2.3.3 Claims with Positive Payoffs

Some derivatives have positive payoffs, i.e. the triplet $g, h, l \geq 0$, including calls, puts and digital options. For both conventions, the nonlinear property of the price function disappears since we can substitute the nonlinear functions $P^+(t, s)$ and $\Pi^+(t, s)$ in (2.24) and (2.26) by the linear functions $P(t, s)$ and $\Pi(t, s)$. From this observation, we derive the formulae for the bid-ask prices of derivatives with positive payoffs.

**Proposition 2.8.** For any claim with $g, h, l \geq 0$, the bid-ask prices of with
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40

counterparty risk provision satisfy

\[ P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda^0)(T-t)} g(S_T) + \int_t^T e^{-(r+\alpha+\lambda^0)(u-t)} (h(S_u) + \lambda^0 I(u, S_u)) \, du \right], \]

(2.33)

\[ P^s(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\beta+\lambda^0)(T-t)} g(S_T) + \int_t^T e^{-(r+\beta+\lambda^0)(u-t)} (h(S_u) + \lambda^0 I(u, S_u)) \, du \right]. \]

(2.34)

Without counterparty risk provision (see Remark 2.1), the bid-ask prices satisfy

\[ \hat{P}^b(t, s) = \Pi(t, s) - \mathbb{E}_{t,s} \left[ \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right], \]

(2.35)

\[ \hat{P}^s(t, s) = \Pi(t, s) - \mathbb{E}_{t,s} \left[ \int_t^T \beta e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right]. \]

(2.36)

We apply these results to call options on a defaultable stock.

Example 2.9. The bid price of a European call option with provision is given by

\[ P^b(t, s) = e^{-\alpha(T-t)} C^{BS}(t, s; T, K, r + \lambda^0, \sigma). \]

(2.37)

Moreover, the bid price of a European call option without provision is given by

\[ \hat{P}^b(t, s) = \left( 1 - \frac{\alpha}{\lambda^{(1)}} \left( 1 - e^{-\lambda^{(1)}(T-t)} \right) \right) C^{BS}(t, s; T, K, r + \lambda^0, \sigma). \]

(2.38)

In both (2.37) and (2.38), the bid prices have two components: the CRF value of the call option, i.e. Black-Scholes price, and a multiplier that depends on the parameter \( \alpha \) (see (2.28)). First, suppose that \( \alpha = 0 \). This happens, for example, when the contract is perfectly collateralized and effective collateral rate is zero, i.e. \( \delta_2 = c_2 = 0 \). In this case, both bid prices are identical to the CRF value \( \Pi(t, s) = C^{BS}(t, s; T, K, r + \lambda^0, \sigma) \). The parameter \( \alpha \) is positive when the contract is under-collateralized by party 2, i.e. \( \delta_2 < 1 \). In
this case, the call option buyer is exposed to counterparty default risk, and consequently, the bid prices are less than the CRF value $\Pi(t, s)$. On the other hand, $\alpha$ becomes negative when the contract is over-collateralized by party 2 and the collateral rate is negligible, i.e. $\delta_2 > 1$ and $c_2 = 0$. The call buyer receives additional financial benefit from excess collateral since the buyer only returns a fraction $1 - L_1$ of the over-collateralized amount at the buyer’s own default. As a result, the bid prices are greater than the CRF value.

**Figure 2.6**: The bid prices of a call option

The bid prices of a call option with and without counterparty risk provision are increasing in stock price $s \in [5, 15]$ (left) and in maturity $T \in [0, 5]$ (right). Parameters: $\lambda^{(0)} = 5\%$, $\lambda^{(2)} = 10\%$, $S_0 = 10$, $T = 1$, $t = 0$, $r = 2\%$, $\sigma = 25\%$, $R_1 = R_2 = 40\%$, $K = S_0 = 10$ and $\delta_1 = \delta_2 = 0$.

Figure 2.6 illustrates the two bid prices (with and without provision) for various maturities and spot prices. We observe the dominance of the price without provision over the price with provision. Moreover, the difference of two prices increase as maturity or spot price increases. This wider level of difference is attributed to the fact that the difference of counterparty risk exposure increases as the maturity increases.

Next, we derive a number of price dominance relationships for the bid-ask prices and the CRF value of claims with positive payoffs.
Proposition 2.10. Consider any claim with the triplet $g, h, l \geq 0$. If $\alpha, \beta \geq 0$, then the following price dominance relationships hold:

\[ \hat{P}^b(t, s), \hat{P}^s(t, s) \leq \Pi(t, s) \quad \text{and} \quad P^b(t, s), P^s(t, s) \leq \Pi(t, s). \] (2.39)

Furthermore, if $\lambda^{(1)} + \lambda^{(2)} \geq \alpha, \beta \geq 0$, then we have

\[ P^b(t, s) \leq \hat{P}^b(t, s) \leq \Pi(t, s) \quad \text{and} \quad P^s(t, s) \leq \hat{P}^s(t, s) \leq \Pi(t, s). \] (2.40)

We provide a proof in Appendix A.3. In market conditions where the effective collateral rates $c_1, c_2$ are much lower than the counterparty/own default rates $\lambda^{(1)}, \lambda^{(2)}$, the conditions for (2.39) and (2.40) are satisfied. In turn, since the buyer’s (resp. seller’s) MtM value with provision assumes $P^b$ (resp. $P^s$) as the liquidation value, which is lower than the liquidation value $\Pi$ in the case without provision according to (2.39), this implies the price dominance relationships in (2.40), as is presented in Figure 2.6.

2.4 Defaultable Fixed-Income Derivatives with Counterparty Risk

We now consider defaultable fixed-income claims under bilateral counterparty risk setting. As a special case, we assume that the risk free rate is a time-deterministic function $r(t)$, and the default intensities $\lambda^{(i)}, i \in \{0, 1, 2\}$, are of the form $\lambda^{(i)}(t, X_t) = \psi_i(t) + w_i X_t$. We model the stochastic factor $X$ by two affine diffusions: Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR) processes. Under the OU model, the factor process $X$ follows

\[ dX_t = \kappa(\theta - X_t) dt + \sigma d\tilde{W}_t, \]

with positive constant parameters $\kappa, \theta, \sigma > 0$ that represent the speed of mean reversion, long-term mean, and volatility, respectively. In the CIR model, the
factor process $X$ follows
\[dX_t = \kappa(\theta - X_t) \, dt + \sigma \sqrt{X_t} \, d\tilde{W}_t,\]
where the positive parameters $\{\kappa, \theta, \sigma\}$ satisfy $\kappa \theta > \sigma^2$ so that $X$ stays positive at all times a.s. Both models for $X$ satisfy the corresponding conditions among (C1)-(C7) [Heath and Schweizer, 2000, Sections 2.1-2.2], and we will consider here swaps whose payoff are bounded continuous.

A generic fixed-income contract is described by the triplet $(g(x), h(x), l(t, x))$, with default times $\{\tau_i\}_{i=0,1,2}$ as in (2.1). From (2.14), the MtM value with counterparty risk provision is given by
\[P(t, x) = \mathbb{E}_{t,x} \left[ e^{-\int_t^T (r(u) + \lambda(u, X_u)) \, du} g(X_T) + \int_t^T e^{-\int_u^T (r(v) + \lambda(v, X_v)) \, dv} f(u, X_u, P_u) \, du \right], \tag{2.41}\]
where $f(t, x, y) = h(x) + \lambda^{(0)}(t, x) l(t, x) + (\lambda^{(1)}(t, x) + \lambda^{(2)}(t, x) - \beta(t, x)) y + (\beta(t, x) - \alpha(t, x)) y^+$. The parameters $\alpha(t, x)$ and $\beta(t, x)$ are defined similarly to the definitions (2.11) and (2.12). The MtM value without counterparty risk provision $\tilde{P}(t, x)$ is similarly obtained by replacing $P(u, X_u)$ with $\Pi(u, X_u)$ in the right hand side of (2.41). Noticing that $P(t, x)$ in (2.41) is a special case of $P(t, s, x)$ in (2.14), we use Algorithm 1 to find the MtM value by solving iteratively the following PDE:
\[\frac{\partial P^{(n)}}{\partial t} + \mathcal{L}_X P^{(n)} - (r(t) + \lambda(t, x)) P^{(n)} + f(t, x, P^{(n-1)}) = 0, \tag{2.42}\]
for $(t, x) \in [0, T) \times \mathbb{R}$, with $P^{(n)}(T, x) = g(x)$ for $x \in \mathbb{R}$ ($x \in \mathbb{R}_+$ for the CIR case).

The computation of the MtM values involves zero-coupon bond prices. In the OU model, the pre-default zero-coupon bond price with maturity $T$ and
zero recovery meets the formula [Schönbucher, 2003, Chap. 7.1.1]

\[ C_1(t, x; T) = e^{-\int_t^T r(u) \, du} e^{-\int_t^T \psi_0(u) \, du} E_t [ e^{-\int_t^T w_0 X_s \, ds} ] = e^{-\int_t^T \left( r(u) + \psi_0(u) \right) du} A_1(T-t) e^{A_1(T-t)} - B_1(T-t) w_0 x, \]  

(2.43)

where

\[ A_1(u) = \int_0^u \left( \frac{1}{2} \sigma^2 B_1(v)^2 - \kappa \theta B_1(v) \right) dv, \quad B_1(u) = \frac{1 - e^{\kappa u}}{\kappa}, \quad 0 \leq u \leq T. \]

In the CIR model, the bond price is given by [Schönbucher, 2003, Chap. 7.2]

\[ C_2(t, x; T) = e^{-\int_t^T r(u) \, du} e^{-\int_t^T \psi_0(u) \, du} E_t [ e^{-\int_t^T w_0 X_u \, du} ] = e^{-\int_t^T \left( r(u) + \psi_0(u) \right) du} A_2(T-t) e^{-B_2(T-t)} x, \]  

(2.44)

where

\[ A_2(u) = \left[ \frac{2w_0 \left( \Xi + \kappa \right)}{\left( \Xi + \kappa \right) \left( e^{\Xi u} - 1 \right) + 2\Xi} \right]^{2\theta/\sigma^2}, \quad B_2(u) = \left[ \frac{2 \left( e^{\Xi u} - 1 \right) w_0}{\left( \Xi + \kappa \right) \left( e^{\Xi u} - 1 \right) + 2\Xi} \right], \]

(2.45)

for 0 \leq u \leq T with constant \( \Xi = \sqrt{\kappa^2 + 2 \sigma^2 w_0} \).

2.4.1 Credit Default Swaps (CDS)

After the 2008 financial crisis, CDS contracts are traded in a new standardized way whereby the protection buyer pays at a fixed premium rate, along with a non-zero (positive/negative) upfront payment at the start of the contract. A long position of a CDS contract written on the default event \{\tau_0 \leq T\} is summarized by the triplet notation (0, −p, 1). Under the OU model, the buyer pays the pre-default upfront price for the CDS:

\[
\Pi(t, x) = \int_t^T C_1(t, x; u) \left( w_0 x e^{-\kappa(u-t)} + (\kappa \theta - \frac{\sigma^2}{\kappa})(u-t) + \frac{\sigma^2}{\kappa} (1 - e^{-\kappa(u-t)}) - p \right) du.
\]  

(2.46)
CHAPTER 2. PRICING OF DEFAULTABLE CLAIMS WITH COUNTERPARTY RISK AND COLLATERALIZATION

Under the CIR model, the pre-default upfront price of a CDS with maturity $T$ and premium rate $p$ is given by

$$\Pi(t, x) = \int_t^T C_2(t, x; u) \left[ w_0 \kappa \theta B_2(u - t) + B'_2(u - t)x - p \right] du,$$  \hspace{1cm} (2.47)

with $C_2(t, x; u)$ in (2.44) and $B_2(\cdot)$ in (2.45). See Chap. 7 of Schönbucher [2003].

For numerical examples, we assume the constant interest rate $r$ and constant default rates $\lambda^{(1)}$ and $\lambda^{(2)}$. The reference default intensity is modeled by $\lambda^{(0)}_t = X_t$, i.e. $\psi_0 = 0$ and $w_0 = 1$. For implementation, we apply the FDM and Algorithm 1 in Section 2.2.2 to (2.42) to obtain the MtM value with counterparty risk provision. In Table 2.4, we show the convergence of the CDS MtM values. The first and second columns for each model show the values of the CDS at $x = 2\%$ and $x = 8\%$, respectively. In addition, the third column shows the error in terms of the supremum norm $\epsilon = \|P^{(n)} - P^{(n-1)}\|_{\infty}$ over the entire grid for each step $n = 1, 2, \ldots, 7$. The algorithm stops at $n = 5$ under the OU model, and at $n = 7$ under the CIR model with the common tolerance level $\bar{\epsilon} = 10^{-5}$.

In Figure 2.7, we compare the three MtM values in terms of the counterparty recovery rate $R_2 = 1 - L_2$ and the counterparty default rate $\lambda^{(2)}$. The CRF value is given by (2.47) and is independent of $R_2$ and $\lambda^{(2)}$. An increase in counterparty recovery rate or a decrease in counterparty default rate improves both MtM values, whereas the CRF value does not change. In both cases, the MtM value without provision dominates the MtM value with provision.

2.4.2 Total Return Swaps (TRS)

Total return swaps (TRS) on defaultable bonds are OTC traded, and their MtM values are subject to counterparty risk. A TRS is also referred to as a bond forward [see Schönbucher, 2003, Chap. 2.5]. Fix a maturity of $T$ years,
### Table 2.4: Convergence of the MtM values of a CDS

Convergence of MtM values of a CDS in the OU/CIR model with reference default rate \( x = 2\% \) and \( x = 10\% \). For the OU model: \( \theta = 3\% \), \( \sigma = 3.5\% \), \( \kappa = 5\% \). For the CIR model: \( \theta = 3\% \), \( \sigma = 5\% \), \( \kappa = 5\% \). Other common Parameters: \( T = 5 \), \( t = 0 \), \( p = 100\ bps \), \( r = 2\% \), \( \psi_0 = 0 \), \( \psi_1 = 5\% \), \( \psi_2 = 25\% \), \( w_0 = 1 \), \( w_1 = w_2 = 0 \), \( \lambda^{(1)} = 5\% \), \( \lambda^{(2)} = 25\% \), \( R_1 = R_2 = 40\% \), \( \delta_1 = \delta_2 = 0 \), \( \bar{\epsilon} = 10^{-5} \), \( \Delta X = 0.001 \), \( \Delta t = 1/500 \).

we consider a TRS on a zero-recovery defaultable bond with maturity \( T' \geq T \). The value of the defaultable bond, denoted by \( C \), is given in (2.43) for the OU model and in (2.44) for the CIR model. Given no default up to the swap maturity \( T \), the swap buyer receives the difference between the bond price \( C(T, X_T; T') \) and the pre-specified strike \( K = C(0, X_0; T') \) from the swap seller. If the bond defaults before time \( T \), the swap buyer pays the strike \( K \) to the seller. Until the first default time \( \tau \) or expiration date \( T \), the buyer continues to pay at the risk-free rate plus a spread \( r(t) + p \). The CRF upfront

<table>
<thead>
<tr>
<th>( n = 0 )</th>
<th>( P^{(n)}(0, 2%) )</th>
<th>( P^{(n)}(0, 10%) )</th>
<th>( \epsilon )</th>
<th>( n = 1 )</th>
<th>( P^{(n)}(0, 2%) )</th>
<th>( P^{(n)}(0, 10%) )</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>-0.1326</td>
<td>0.05218</td>
<td>0.9048</td>
<td>-0.1130</td>
<td>0.1569</td>
<td>0.3950</td>
<td></td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>-0.1526</td>
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<td>0.0992</td>
<td>-0.1131</td>
<td>0.1176</td>
<td>0.0911</td>
<td></td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>-0.1515</td>
<td>0.04562</td>
<td>0.0060</td>
<td>-0.1130</td>
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<td>0.0159</td>
<td></td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>-0.1516</td>
<td>0.04560</td>
<td>&lt; 10^{-5}</td>
<td>-0.1130</td>
<td>0.1238</td>
<td>0.0022</td>
<td></td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.1130</td>
<td>0.1239</td>
<td>&lt; 10^{-4}</td>
<td></td>
</tr>
<tr>
<td>( n = 7 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.1130</td>
<td>0.1239</td>
<td>&lt; 10^{-5}</td>
<td></td>
</tr>
</tbody>
</table>
Figure 2.7: The CDS upfront prices under the CIR model

The CDS upfront prices under the CIR Model are increasing in the counterparty recovery rate $R_2$ (left) and decreasing in the counterparty default rate $\lambda^{(2)}$ (right). Other parameters are same as in Table 2.4 along with $x = 8\%$.

For the MtM upfront value $P(t, x)$ with counterparty risk provision, we follow (2.41) with the triplet

$$g(x) = C(T, x ; T') - K, \quad h(t, x) = -K(r(t) + p), \quad l(t, x) = -K.$$
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Figure 2.8: The CDS and TRS bid-ask upfront prices under the CIR model

(Left) CDS bid-ask upfront prices under the CIR model. (Right) TRS bid-ask upfront prices under the CIR model. Other common parameters: \( x = 8\% \), \( T = 3 \), \( T' = 10 \), \( t = 0 \), \( p = 100\text{bps} \), \( r = 2\% \), \( \theta = 3\% \), \( \kappa = 5\% \), \( \psi_0 = 0 \), \( \psi_1 = 5\% \), \( \psi_2 = 25\% \), \( w_0 = 1 \), \( w_1 = w_2 = 0 \), \( \lambda^{(1)} = 5\% \), \( R_1 = 40\% \), \( \lambda^{(2)} = 25\% \), \( R_2 = 40\% \), \( \delta_1 = \delta_2 = 0 \), \( \bar{\epsilon} = 10^{-8} \), \( \Delta X = 0.001 \), \( \Delta t = 1/500 \).

the reference default intensity (left). On the other hand, since the TRS buyer is shorting default risk, the TRS upfront value decreases as the reference asset’s default intensity \( \lambda^{(0)} \) increases (right).

2.5 Conclusion

In summary, we have discussed a valuation framework to analytically study and numerically compute financial contracts subject to reference and counterparty default risks with bilateral collateralization. In addition to the underlying price dynamics used in this chapter, our fixed point approach and the corresponding iterative numerical algorithm can potentially be adapted to price derivatives with counterparty risk in other models. The challenge lies in efficiently and accurately solving a sequence of inhomogeneous PDE problems. Our model also sheds light on the role played by counterparty risk and collateralization.
in the formation of bid-ask spreads.

Several interesting and practically important problems remain for future investigation. For example, what are the price and risk impacts of utilizing \textit{multiple} counterparties for OTC trading? A recent study by [Bo and Capponi 2013] derives the BCVA of a CDS portfolio with a large number of entities. Apart from pricing, counterparty risk should also be incorporated into static or dynamic trading strategies for credit derivatives (see e.g. [Jiao and Pham 2011; Leung and Liu 2012; Leung 2012]).

Recently, a new contractual feature, termed the early termination clause, has been introduced to a number of credit derivatives to mitigate the impact of counterparty default risk. [Giada and Nordio 2012] study the valuation of an early termination option for swaps in presence of bilateral counterparty risk. Also, [Leung and Yamazaki 2013] and [Egami et al. 2013] incorporate the timing option to increase or reduce the position in the valuation of CDS. We can also apply our counterparty risk pricing framework and methodologies to these studies and analyze the potential impacts of bilateral counterparty risk on the optimal timing. In view of the financial crisis, counterparty risk has also become a key component in the design of clearing houses. Answers to these questions will be useful not only for individual or institutional investors, but also for regulators.
Chapter 3

Trading of Defaultable Claims
under Risk Aversion and Belief Heterogeneity

In this chapter, we study the problem of pricing and trading of defaultable claims among investors with heterogeneous risk preferences and market views. First, based on the utility-indifference pricing methodology, we construct the bid-ask spreads for risk-averse buyers and sellers as a function of trading volume. In addition, we show that the spreads are nonlinear functions with respect to trading volume and widen as risk aversion or trading volume increases. Moreover, we analyze the optimal trading of defaultable claims for risk-averse buyers and sellers under various market settings, including (i) when the market pricing rule is linear, and (ii) when the counterparty – single or multiple sellers – may have different nonlinear bid-ask prices generated by risk aversion and belief heterogeneity. For applications, we provide explicit formulas for the optimal trading positions of defaultable bonds and CDS, respectively, and examine the properties and sensitivities of the optimal positions with respect to risk aversions and beliefs. In particular, we find that belief heterogeneity,
rather than the difference in risk aversion, is crucial to trigger a trade.

In Section 3.1, we formulate the optimal static trading problem for defaultable claims based on linear and nonlinear pricing rules. In Section 3.2, we establish the investors’ utility-indifference pricing rules for defaultable bonds, and derive the optimal positions under different market settings. In Section 3.3, we analyze the pricing and trading of CDS. Section 3.4 concludes this chapter.

### 3.1 Problem Overview

When an investor considers buying defaultable claims from a single or multiple sellers, he/she has to determine how many units to trade from each seller, given their offered prices. Intuitively, the investor should take a long (resp. short) position from a seller if the offered price is lower (resp. higher) than the investor’s reservation price. If both the investor and sellers adopt the classical risk-neutral pricing rules, which are linear in quantity, then the resulting trading position will be either zero or infinity. This motivates us to study the trading problem when the investor’s reservation price is nonlinear in quantity while the seller(s) may have a linear or nonlinear pricing rules. Our objective is to (i) investigate the critical factors that trigger a trade between the investor and the sellers, (ii) determine the optimal trading positions, and (iii) examine the impact of risk aversion and other market parameters on trade sizes.

Given any contingent claim, the buyer’s pricing rule is defined by the mapping $p : \mathbb{R}_+ \mapsto \mathbb{R}$ where $p(c)$ represents the maximum amount of cash the buyer is willing to pay for $c \geq 0$ units of the claim. Similarly, the seller’s pricing rule is defined by the mapping $\tilde{p} : \mathbb{R}_+ \mapsto \mathbb{R}$ where $\tilde{p}(c)$ is the minimum amount of cash the seller requires for selling $c \geq 0$ units of the claim. In other words, a buyer takes a long position in the claim with a pricing rule
$p(c)$ and a seller takes a short position with a pricing rule $\tilde{p}(c)$. In the case of zero-coupon defaultable bonds, to be discussed in Section 3.2, the buyer’s and seller’s prices should both be positive since the bonds have non-negative terminal payoffs. Nevertheless, the trading of CDS involves more complex pricing rules characterized by both upfront fee and premium rate. We will discuss the CDS problem in Section 3.3.

3.1.1 Optimal Trading with Linear Prices

Let us begin with the optimal static trading problem when the bid-ask prices of a defaultable claim are linear with respect to quantity. The investor contemplates a long or short position in the claim at per-unit bid-ask prices $(\pi, \bar{\pi})$ offered by the sellers. At the moment, it is not necessary to specify a model that generates the market bid-ask prices $(\pi, \bar{\pi})$ as long as they do not lead to arbitrage opportunities. For instance, we can simply assume that $\pi \leq \bar{\pi}$ and that they fall within the no-arbitrage price interval. The investor has no influence over the market bid-ask prices, so $(\pi, \bar{\pi})$ are independent of the investor’s risk aversion and market belief. This setting is most relevant to markets with high liquidity where bid-ask prices are quoted per-unit regardless of transaction quantity.

Given the ask price $\bar{\pi}$, the investor pays $c \bar{\pi}$ to the sellers for a long position of $c$ units of the claim. Since the investor is willing to pay at most $p(c)$ for the position, the benefit from the trade is $B_b := p(c) - c \bar{\pi}$, which is the price difference from a buyer’s perspective. The investor selects the optimal long position $c^*(\bar{\pi})$ that maximizes the benefit, namely,

$$c^*(\bar{\pi}) = \arg \max_{c \geq 0} p(c) - c \bar{\pi}. \quad (3.1)$$

From (3.1), we observe that the optimal position is found from the Fenchel-Legendre transform of $p(c)$ as a function of $c$ evaluated at $\bar{\pi}$. By similar
arguments, if the investor decides to sell \( c \) units of the claim, the benefit from the trade is \( B_s := c\bar{p} - \tilde{p}(c) \) given the bid price \( \bar{p} \). The optimal short position \( \tilde{c}^*(\bar{p}) \) is determined by

\[
\tilde{c}^*(\bar{p}) = \arg \max_{\tilde{c} \geq 0} c\bar{p} - \tilde{p}(\tilde{c}) .
\]

We note that the buyer’s and seller’s optimal positions directly depend on the market bid-ask prices \((\bar{p}, \tilde{p})\).

### 3.1.2 Optimal Trading with Heterogenous Sellers

In a different setting, we consider the trading of a defaultable claim between a risk-averse investor and a single or multiple risk-averse sellers. The investor and the sellers may have different indifference pricing rules, reflecting their differences in risk preferences and market beliefs.

First, let us discuss the case with a single seller. If the investor is considering to buy a defaultable claim, then the seller’s ask prices represent the cost of acquiring the claim. The resulting trading benefit to the investor (buyer) is the difference \( B_b := p(c) - \tilde{p}(c) \), and the optimal long position is given by

\[
c_b^* = \arg \max_{c \geq 0} p(c) - \tilde{p}(c) .
\]

On the other hand, if the investor decides to sell certain units of the claim and receive the seller’s bid prices, then the trading benefit is \( B_s := p(c) - \tilde{p}(c) \) and the optimal short position is

\[
c_s^* = \arg \max_{c \geq 0} p(c) - \tilde{p}(c) .
\]

As a result, the investor’s optimal trading position maximizes the spread between his/her pricing rule and the pricing rule offered by the seller.

Next, we proceed to a market with \( N \) heterogeneous sellers. Suppose the investor is a potential buyer of the claim, and each seller \( i \) has a pricing rule
$p_i$, for $i \in \{1, \ldots, N\} =: \mathcal{N}$. The buyer’s static position is described by the vector $\vec{c} = (c_1, \ldots, c_N)$, where each $c_i$ represents the number of units of the claim purchased from seller $i$, and the total holding is $\bar{c} = \sum_{i=1}^{N} c_i$. The buyer determines the optimal position $c^*$ by maximizing the trading benefit:

$$c^* = \arg \max_{c_i \geq 0, i=1,\ldots,N} p(\bar{c}) - \sum_{i=1}^{N} \tilde{p}_i(c_i).$$

In particular, if the trading benefit $B_b(\bar{c}) = p(\bar{c}) - \sum_{i=1}^{N} \tilde{p}_i(c_i)$ is strictly concave in $\vec{c} \in \mathbb{R}_+^N$, then it has a unique maximizer. Note that it is possible to have $c_i^* = 0$ for one or more $i \in \mathcal{N}$, which means the buyer does not purchase from these sellers.

If the buyer seeks to buy a pre-specified $\alpha > 0$ units of the claim from $N$ sellers, then an additional constraint, $\sum_{i=1}^{N} c_i = \alpha$, is imposed on the buyer’s optimization problem:

$$c^{\alpha*} = \arg \max_{\substack{c_i \geq 0, i=1,\ldots,N; \\ \sum_{i=1}^{N} c_i = \alpha}} p(\alpha) - \sum_{i=1}^{N} \tilde{p}_i(c_i)$$

$$= \arg \min_{\substack{c_i \geq 0, i=1,\ldots,N; \\ \sum_{i=1}^{N} c_i = \alpha}} \sum_{i=1}^{N} \tilde{p}_i(c_i). \quad (3.2)$$

Therefore, this amounts to minimizing the total cost of the claim from all sellers. In particular, if all sellers’ prices are linear, then the investor will always purchase all $\alpha$ units from the seller with the lowest per-unit price. Henceforth, we will focus on sellers with nonlinear pricing rules.

### 3.2 Utility-Indifference Pricing and Optimal Trading of Defaultable Bonds

We now discuss the trading problems described in Section 3.1 for defaultable bonds. To this end, we apply the utility-indifference pricing methodology
to establish the investor’s and sellers’ pricing rules. Our goal is to provide explicit formulas and sensitivity analysis for the optimal trading positions under various settings.

The defaultable bond pays $1 on the expiration date $T$ if there is no default. Throughout, we shall work with discounted cash flows. If the underlying asset defaults at time $\tau$ before $T$, then the bond holder receives the discounted recovery value $R e^{-rT}$ with constant recovery rate $R \in [0,1)$. The investor can dynamically trade the defaultable underlying, and a correlated non-defaultable asset, along with the money market account with a constant interest rate $r \geq 0$.

The discounted values of the non-defaultable asset price $S^{(1)}$ and defaultable underlying $S^{(2)}$ evolve according to

$$dS^{(i)}_t = \mu_i S^{(i)}_t dt + \sigma_i S^{(i)}_t dW^{(i)}_t, \quad i = 1, 2,$$

where the expected excess return and the volatility vectors are defined by $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2_+$. The Brownian motions $(W^{(1)}, W^{(2)})$ are defined on the probability space $(\Omega, \mathcal{F}, Q)$, and are correlated with correlation parameter $\rho \in (-1, 1)$. In addition, the default time of stock $S^{(2)}$ is modeled by $\tau \sim \exp(\lambda)$, an exponential random variable with rate parameter $\lambda > 0$. We assume that $\tau$ is independent of $(W^{(1)}, W^{(2)})$.

With initial capital $X_0$ at time 0, the discounted trading wealth at $t \in [0, \tau \wedge T]$ satisfies

$$dX_t = \sum_{i=1}^{2} \theta^{(i)}_t (\mu_i dt + \sigma_i dW^{(i)}_t), \quad (3.3)$$

where $\theta_t = (\theta^{(1)}_t, \theta^{(2)}_t)$ represents the discounted cash amounts invested in $S^{(1)}$ and $S^{(2)}$ respectively. A trading strategy is deemed admissible if it is self-financing, $\mathcal{F}_t$-adapted and satisfies the integrability condition $\mathbb{E}\{\int_0^T ||\theta||^2 dt\} < \infty$. We denote by $\Theta_{s,t}$ the set of admissible strategies $\theta$ over the horizon $[s,t]$.

We model the investor’s risk preferences by the exponential utility function $U(x) = -e^{-\gamma x}$, with constant risk aversion coefficient $\gamma > 0$. If the
investor trades only the non-defaultable asset \( S^{(1)} \) and the money market account throughout the horizon \([t, T]\), then the discounted wealth follows
\[
d\hat{X}_t = \theta_t^{(1)} \left( \mu_1 dt + \sigma_1 dW_t^{(1)} \right).
\]
Here, \( \theta_t^{(1)} \) is the discounted cash amount invested in \( S^{(1)} \), and we denote by \( \hat{\Theta}_{s,t} \) the set of admissible strategies in \( S^{(1)} \) over the horizon \([s, t]\). In order to maximize the expected utility from dynamically trading, the investor solves the well-known Merton [see Merton, 1969] portfolio optimization problem:
\[
M(t, x; \gamma) := \sup_{\hat{\Theta}_{t,T}} \mathbb{E}\left\{ U(\hat{X}_T) \mid \hat{X}_t = x \right\}
\]
\[
= -e^{-\gamma x} e^{-\frac{1}{2} \eta_1^2 (T-t)}, \quad 0 \leq t \leq T, \tag{3.5}
\]
with
\[
\eta_1 = \frac{\mu_1}{\sigma_1}, \tag{3.6}
\]
and the optimal trading strategy is given by \( \hat{\theta}_s^{*} = \frac{\mu_1}{\gamma} \) for \( 0 \leq t \leq s \leq T \).

### 3.2.1 Bid-Ask Prices

We use utility indifference pricing approach to derive the pricing rules of the risk-averse investor. We consider the utility maximization problem of the investor holding the defaultable bond in addition to a dynamic portfolio. As soon as \( S^{(2)} \) defaults, the investor can only trade \( S^{(1)} \) thereafter. After default, the investor faces the Merton’s portfolio optimization problem solved in (3.5). With \( c \in \mathbb{R} \) units of the defaultable bond, the investor’s discounted trading wealth \( (X_t)_{0 \leq t \leq \tau \wedge T} \) follows (3.3) prior to default, and he/she solves the utility maximization problem:
\[
V(t, x; \gamma, c) = \sup_{\hat{\Theta}_{t,T}} \mathbb{E}\left\{ U(X_T + ce^{-rT}) \mathbf{1}_{\{\tau > T\}} + M(\tau, X_\tau + c Re^{-r\tau}) \mathbf{1}_{\{\tau \leq T\}} \mid X_t = x \right\}.
\]
\[
(3.7)
\]
In particular, if the investor holds no claim \((c = 0)\), then the value function \(V(t, x; \gamma, 0)\) is the maximal expected utility from dynamically investing in the stocks without the defaultable bond.

Before stating the result, we define
\[
A(t) = e^{-\alpha(T-t)}, \quad B(t) = \frac{\lambda}{\alpha - \frac{1}{2} \eta_1^2} \left( e^{-\frac{1}{2} \eta_1^2(T-t)} - e^{-\alpha(T-t)} \right),
\]
where \(\eta_1\) is defined in (3.6). We observe that \(\eta_2 \geq \eta_1^2\) for \(\rho \in (-1, 1)\) since
\[
\eta_2 - \eta_1^2 = \frac{(\rho \mu_1 \sigma_2 - \mu_2 \sigma_1)^2}{(1 - \rho^2) \sigma_1^2 \sigma_2^2} \geq 0.
\]
This implies that \(\alpha > \frac{\eta_1^2}{2}\) for \(\lambda > 0\), and \(B(t)\) is well-defined.

**Proposition 3.1.** The value function \(V(t, x; \gamma, c)\) is given by
\[
V(t, x; \gamma, c) = -e^{-\gamma(x+ce^{-rT})} w(t; \gamma, c),
\]
where
\[
w(t; \gamma, c) = A(t) + e^{\gamma c(1-r)} e^{-rT} B(t), \quad 0 \leq t \leq T.
\]
The corresponding optimal trading strategy \(\theta^*_u\) prior to default is
\[
\theta^*_u = \frac{\Sigma^{-1} \mu}{\gamma}, \quad t \leq u < \tau \wedge T.
\]

**Proof.** We first write down the Hamilton-Jacobi-Bellman (HJB) PDE associated with \(V \equiv V(t, x; \gamma, c)\) in (3.7):
\[
\frac{\partial V}{\partial t} - \lambda \left( e^{-\gamma(x+ce^{-rT})} - \frac{1}{2} \eta_1^2(T-t) + V \right) + \sup_{\theta} \left\{ \theta' \mu \frac{\partial V}{\partial x} + \frac{\theta' \Sigma \theta}{2} \frac{\partial^2 V}{\partial x^2} \right\} = 0,
\]
where \(\eta_1\) is defined in (3.6). We observe that \(\eta_2 \geq \eta_1^2\) for \(\rho \in (-1, 1)\) since
\[
\eta_2 - \eta_1^2 = \frac{(\rho \mu_1 \sigma_2 - \mu_2 \sigma_1)^2}{(1 - \rho^2) \sigma_1^2 \sigma_2^2} \geq 0.
\]
This implies that \(\alpha > \frac{\eta_1^2}{2}\) for \(\lambda > 0\), and \(B(t)\) is well-defined.
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for \((t, x) \in [0, T) \times \mathbb{R}\), with \(V(T, x; \gamma, c) = -e^{-\gamma(x+ce^{-rT})}\). Maximizing over \(\theta\) in (3.13) yields

\[
\frac{\partial V}{\partial t} - \lambda \left( e^{-\gamma(x+ce^{-rT})} - \frac{1}{2} \eta^2 (T-t) + V \right) - \frac{\mu^T \Sigma^{-1} \mu (\partial V/\partial x)^2}{2} = 0. \tag{3.14}
\]

Substituting (3.10) into (3.13) gives an ODE for \(w(t) \equiv w(t; \gamma, c)\):

\[
w'(t) - \left( \frac{\eta^2}{2} + \lambda \right) w(t) + \lambda e^{\gamma c(1-R)e^{-rT}-\frac{1}{2} \eta^2 (T-t)} = 0, \tag{3.15}
\]

with \(w(T) = 1\). The explicit solution is given by (3.11). Finally, applying (3.10) to the optimal \(\theta^*\) in (3.13), we get \(\theta^*(t, x) = (\Sigma^{-1} \mu)/\gamma\), for \((t, x) \in [0, T] \times \mathbb{R}\).

Definition 3.2. At time \(t\), the indifference bid (resp. ask) price for buying (resp. selling) \(c \geq 0\) units of the defaultable claim is defined by the discounted cash amount \(p(t; \gamma, c)\) (resp. \(\bar{p}(t; \gamma, c)\)) that satisfies the equations

\[
V(t, x; \gamma, 0) = V(t, x - p(t; \gamma, c); \gamma, c), \tag{3.16}
\]

\[
V(t, x; \gamma, 0) = V(t, x + \bar{p}(t; \gamma, c); \gamma, -c). \tag{3.17}
\]

The buyer’s indifference bid price \(p\) can also be interpreted as the highest price that the buyer is willing to pay for a long position in \(c\) units of the defaultable bond. Similarly, the seller’s indifference ask price \(\bar{p}\) is the lowest price that the seller is willing to receive for a short position in \(c\) units of the defaultable bond. Applying (3.10) to (3.16) and (3.17) leads to the formulas:

\[
p(t; \gamma, c) = ce^{-rT} - \frac{1}{\gamma} \log \frac{w(t; \gamma, c)}{w(t; 0, 0)} \quad \text{and} \quad \bar{p}(t; \gamma, c) = ce^{-rT} - \frac{1}{\gamma} \log \frac{w(t; 0, 0)}{w(t; \gamma, -c)}. \tag{3.18}
\]

For notational convenience, we write the indifference prices and other quantities at time 0 as

\[
p(\gamma, c) \equiv p(0; \gamma, c), \quad \bar{p}(\gamma, c) \equiv \bar{p}(0; \gamma, c), \quad w(\gamma, c) \equiv w(0; \gamma, c), \quad A = A(0) \quad \text{and} \quad B = B(0). \tag{3.19}
\]
Next, we study several properties of the indifference prices, especially their dependence on risk aversion \( \gamma \) and volume \( c \).

**Proposition 3.3.** The buyer’s and seller’s indifference prices have the following properties:

(i) Monotonicity and concavity/convexity in trading volume \( c \): for \( c \geq 0 \), 
\( p(\gamma, c) \) is increasing strictly concave in \( c \) and \( \tilde{p}(\gamma, c) \) is increasing strictly convex in \( c \).

(ii) Monotonicity in risk aversion \( \gamma \): \( p(\gamma, c) \) is decreasing in \( \gamma \) and \( \tilde{p}(\gamma, c) \) is increasing in \( \gamma \).

(iii) Risk aversion asymptotics:
\[
\lim_{\gamma \to \infty} \frac{p(\gamma, c)}{c} = 0, \quad \lim_{\gamma \to \infty} \frac{\tilde{p}(\gamma, c)}{c} = e^{-rT}, \quad \lim_{\gamma \to 0} \frac{p(\gamma, c)}{c} = p^*, \quad \lim_{\gamma \to 0} \frac{\tilde{p}(\gamma, c)}{c} = \tilde{p}^*,
\]

where the zero risk-aversion (unit) prices \( p^* \) and \( \tilde{p}^* \) are given by
\[
p^* = \tilde{p}^* = e^{-rT} \left( 1 - \frac{(1 - R)B}{A + B} \right). \tag{3.21}
\]

Therefore, for any \( c, \gamma \geq 0 \), the indifference prices satisfy
\[0 \leq p(\gamma, c) \leq c p^* \leq \tilde{p}(\gamma, c) \leq c e^{-rT}.
\]

(iv) Volume-scaling property: \( p(\gamma, c) = c p(c \gamma, 1) \) and \( \tilde{p}(\gamma, c) = c \tilde{p}(c \gamma, 1) \). This leads to the following volume asymptotics:
\[
\lim_{c \to \infty} \frac{p(\gamma, c)}{c} = 0, \quad \lim_{c \to \infty} \frac{\tilde{p}(\gamma, c)}{c} = e^{-rT}, \quad \lim_{c \to 0} \frac{p(\gamma, c)}{c} = p^*, \quad \lim_{c \to 0} \frac{\tilde{p}(\gamma, c)}{c} = \tilde{p}^*.
\]

**Proof.** (i) The first and second derivatives of \( p(\gamma, c) \) w.r.t. \( c \) are
\[
\frac{\partial p(\gamma, c)}{\partial c} = e^{-rT} \left( 1 - \frac{(1 - R)e^{\gamma c}B}{A + e^{\gamma c}B} \right) > 0, \quad \text{and}
\frac{\partial^2 p(\gamma, c)}{\partial c^2} = -e^{-rT} \left( \frac{\gamma(1 - R)Ae^{\gamma c}B}{(A + e^{\gamma c}B)^2} \right) < 0.
\]
Therefore, \( p(\gamma, c) \) is an increasing and strictly concave function of \( c \). Similar straightforward calculations give the conclusion for the seller’s price.

(ii) Differentiating \( p(\gamma, c) \) in (3.18) w.r.t. \( \gamma \), we get

\[
\frac{\partial p(\gamma, c)}{\partial \gamma} = \frac{1}{\gamma^2} \left[ -\frac{\bar{\gamma} c e^{\bar{\gamma} c} B}{A + e^{\bar{\gamma} c} B} + \log \left( \frac{A + e^{\bar{\gamma} c} B}{A + B} \right) \right],
\]

with \( \bar{\gamma} := \gamma(1 - R)e^{-rT} \).

The function \( h(\gamma) \) is negative since \( h(0) = 0 \) and \( h'(\gamma) = -e^{2\bar{\gamma}}e^{\bar{\gamma} c}AB(A + e^{\bar{\gamma} c}B)^2 \leq 0 \). Hence, \( p(\gamma, c) \) is decreasing in \( \gamma \). The proof for \( \tilde{p}(\gamma, c) \) is similar and thus omitted.

(iii) All the limits can be obtained by direct computation with l’Hopital’s rule.

(iv) The volume-scaling property follows directly from the indifference price formulas (3.18) via the expression for \( w(\gamma, c) \) in (3.11). This in turn leads to the volume limits using (3.20).

Proposition 3.3 is best illustrated in Figure 3.1. On the left panel, we present the buyer’s and seller’s indifference prices in terms of volume \( c \) for various risk aversions. The seller’s price curve dominates the buyer’s curve for all \( c \). However, as the risk aversion reduces to zero, the buyer’s and seller’s price curves bend towards to the straight line with slope \( p^* \), the zero risk-aversion limit price in (3.21). At the same time, the price difference shrinks to zero. Moreover, at zero volume \( (c = 0) \), the buyer’s and seller’s price curves both have slope \( p^* \). On the right panel, we present the buyer’s and seller’s indifference average prices in terms of \( c \). As volume goes to infinity, the average price \( \frac{p}{c} \) converges to the lower bound 0 and \( \tilde{p}/c \) converges to the upper bound \( e^{-rT} \). On the other hand, as volume goes to zero, both average prices converge to the zero risk-aversion price \( p^* \).

For any fixed number \( c \), the investor’s bid-ask spread is defined by

\[
S(\gamma, c) := \tilde{p}(\gamma, c) - p(\gamma, c).
\]
Figure 3.1: The buyer’s and seller’s indifference prices in terms of trading volume

(Left) As $\gamma$ increases, with $(0.2, 0.35, 0.5)$ along the arrows, $\tilde{p}(\gamma, c)$ increases while $p(\gamma, c)$ decreases, leading to a wider bid-ask spread. The dotted line is the zero risk-aversion price line. (Right) As trading volume $c$ increases, the average indifference prices, $p(\gamma, c)/c$ and $\tilde{p}(\gamma, c)/c$, go to the limits 0 and $e^{-rT}$ respectively ($\gamma = 0.3$). For both cases, the other parameters are $T = 10$, $r = 2\%$, $\lambda = 5\%$, $\eta = 0.5774$ ($\mu_2 = 10\%$, $\sigma_2 = 20\%$), $\eta_1 = 0.5$ ($\mu_1 = 8\%$, $\sigma_1 = 16\%$), $\rho = 0.5$ and $R = 40\%$.

In Figure 3.1 the bid-ask spread is the vertical distance between the buyer’s and seller’s indifference prices. We can infer the bid-ask spread behavior from Proposition 3.3, namely,

**Corollary 3.4.** The bid-ask spread $S(\gamma, c)$ is increasing in $\gamma$ and $c$.

The insight of this corollary is that the bid-ask spread widens as the investor becomes more risk averse or takes on a larger position. As a result, our model allows us to quantify the contribution of risk aversion and trading volume to the formation of bid-ask spreads for defaultable bonds. Lastly, we emphasize that the indifference price properties are based on the investor’s belief ($Q$). When dealing with multiple investors, their beliefs, described by the parameters ($\mu_1, \sigma_1, \mu_2, \sigma_2, \rho, \lambda$), can differ. Then, their corresponding volume or risk aversion limits are computed with their individual set of parameters.
3.2.2 Optimal Trading with Linear Prices

We now investigate the buyer’s and seller’s optimal positions in the defaultable bond when the market bid-ask prices are linear in quantity as described in Section 3.1.1. To distinguish the buyer and the seller, we use the subscript notation: $b$ for the buyer and $s$ for the seller. For example, we denote the buyer’s risk aversion, default rate, market belief and value function by $\gamma_b$, $\lambda_b$, $Q_b$ and $V_b$, respectively. The notation also applies to the other parameters such as $\eta$ and $\eta_1$ which are subject to the buyer’s market belief $Q_b$. Given the ask price $\bar{\pi}$, the buyer selects the optimal long position $c^*$ that maximizes the value function, namely,

$$
c^*(\gamma_b, \bar{\pi}) = \arg \max_{c \geq 0} V_b(0, x - c\bar{\pi}; \gamma_b, c)
\quad = \arg \max_{c \geq 0} V_b(0, x + p(\gamma_b, c) - c\bar{\pi}; \gamma_b, 0)
\quad = \arg \max_{c \geq 0} p(\gamma_b, c) - c\bar{\pi}.
$$

(3.22)

The second equality follows from (3.16) and the last one from the increasing monotonicity of the value function with respect to the wealth argument. Similarly, given the bid price $\pi$, the optimal short position for the seller with risk aversion $\gamma_s$ is given by

$$
\bar{c}^*(\gamma_s, \pi) = \arg \max_{c \geq 0} c\pi - \bar{p}(\gamma_s, c).
$$

(3.23)

The buyer’s and seller’s indifference prices $p$ and $\bar{p}$ in (3.22) and (3.23) are computed from (3.19) based on their own market beliefs $Q_b$ and $Q_s$, respectively. Also, the buyer’s and seller’s zero risk-aversion prices $p^*$ (w.r.t. $Q_b$) and $\bar{p}^*$ (w.r.t. $Q_s$) are computed from (3.21).

Proposition 3.5. Given the market bid-ask prices $(\pi, \bar{\pi})$, with $0 \leq \pi \leq \bar{\pi} \leq e^{-rT}$, the buyer’s optimal position $c^*(\gamma_b, \pi)$ and the seller’s optimal position $\bar{c}^*(\gamma_s, \pi)$ at time 0 are given as follows:
(i) If $\bar{\pi} < p^*$, then
\[ c^*(\gamma_b, \bar{\pi}) = \frac{1}{\gamma_b(1 - R)e^{-rT}} \log \left( \frac{(e^{-rT} - \bar{\pi})A_b}{(-Re^{-rT} + \bar{\pi})B_b} \right) > 0. \]
If $\bar{\pi} \geq p^*$, then $c^*(\gamma_b, \bar{\pi}) = 0$.

(ii) If $\bar{\pi} < \pi^*$, then
\[ \tilde{c}^*(\gamma_s, \pi) = \frac{1}{\gamma_s(1 - R)e^{-rT}} \log \left( \frac{(-Re^{-rT} + \pi)B_s}{(e^{-rT} - \pi)A_s} \right) > 0. \]
If $\bar{\pi} \geq \pi^*$, then $\tilde{c}^*(\gamma_s, \pi) = 0$.

The constants $(A_b, B_b)$ and $(A_s, B_s)$ are defined, respectively, corresponding to buyer and seller according to (3.8) at time zero.

**Proof.** By Proposition 3.3, $p(\gamma_b, c)$ is increasing, strictly concave, with derivative at $c = 0$: $\frac{\partial p}{\partial c}(\gamma_b, 0) = p^*$. In case (i), if $p^* \leq \bar{\pi}$, then the function $p(\gamma_b, c) - c\bar{\pi}$ in (3.22) is always negative, so the buyer’s optimal position is zero. If $\bar{\pi} < p^*$, then the optimal quantity $c^*(\gamma, \bar{\pi})$ is strictly positive and can be found from the first-order condition. By similar arguments, since the seller’s price $\tilde{p}(\gamma_s, c)$ is increasing strictly convex with $\frac{\partial \tilde{p}}{\partial c}(\gamma_s, 0) = \tilde{p}^*$, the seller will sell if and only if $\tilde{p}^* < \pi$, which corresponds to case (ii).

By differentiating the optimal positions $c^*$ and $\tilde{c}^*$, we obtain the following properties.

**Corollary 3.6.** The buyer’s optimal position $c^*$ is decreasing in $\gamma_b$, $\lambda_b$ and $\bar{\pi}$. The seller’s optimal position $\tilde{c}^*$ is decreasing in $\gamma_s$, but increasing in $\lambda_s$ and $\pi$.

We observe that the risk-averse investor will never simultaneously take a non-zero long and a non-zero short position in the defaultable bond even if the market bid-ask spread is zero, i.e. $\pi = \bar{\pi}$. This is due to the mutual exclusiveness of the trade/no-trade conditions for the two opposite positions.
Figure 3.2: The buyer’s and seller’s option positions
(Left) The buyer’s optimal position when the $\bar{\pi} = 0.2$. (Right) The seller’s optimal position when $\bar{\pi} = 0.5$. For both cases, $\gamma_b = \gamma_s = 0.2$ and the other parameters are the same as in Figure 3.1.

Since $p^* = \tilde{p}^*$ for a particular investor herself, the two conditions $\bar{\pi} < p^*$ and $\tilde{p}^* < \bar{\pi}$ are mutually exclusive.

Alternatively, we can visualize the procedure of determining the optimal position in Figure 3.2. As implied by (3.22), the buyer should search along the indifference price curve for a point where the slope is $\bar{\pi}$, and the corresponding volume on the x-axis is the optimal position. Suppose the market price increases (holding other model parameters constant), then the buyer will have to find a point on the price curve with a steeper slope. Due to the concavity in $c$ of the buyer’s indifference price, the optimal long position will move to the left (i.e. reduce) as market price increases. Similarly, due to the convexity in $c$ of the seller’s indifference price, the optimal short position will move to the right (i.e. increase) as market price increases. In summary, the buyer will buy less and the seller will sell more defaultable bonds when the trading price increases, and vice versa.

As the buyer optimally purchases $c^*(\gamma_b, \bar{\pi})$ defaultable bonds at the total price of $\bar{\pi}c^*(\gamma_b, \bar{\pi})$, the benefit from taking this position is $B_b = p(\gamma_b, c^*) -$
$c^*\bar{\pi} \geq 0$. In Figure 3.2 (left), with $\gamma_b = 0.2$, $c^* = 5.0261$ and $\bar{\pi} = 0.2$, the benefit is $B_b = 0.3526$ which also coincides the intercept of the zero risk-aversion price line. Similarly, as the investor optimally sells $c^*(\gamma_s, \bar{\pi})$ units of the bond at the total price of $\$c^*(\gamma_s, \bar{\pi})\bar{\pi}$, the benefit is $B_s = c^*\bar{\pi} - \tilde{p}(\gamma_s, c^*) \geq 0$.

In Figure 3.2 (right), with $\gamma_s = 0.2$, $c^* = 4.6206$ and $\bar{\pi} = 0.5$, the benefit is $B_s = 0.3516$ which coincides the absolute value of the intercept of the zero risk-aversion price line.

Figure 3.3: The buyer’s and seller’s optimal positions with the linear bid-ask prices
(Left) The buyer’s optimal position as a function of $\bar{\pi} \in [0, e^{-rT}]$, (Right) The seller’s optimal position as a function of $\bar{\pi} \in [0, e^{-rT}]$. For both cases, the other parameters are the same as in Figure 3.1.

Figure 3.3 illustrates the dependence of the optimal trading position on the bid or ask price. As the risk aversion increases from 0.2 to 0.5, both $c^*$ and $c^*$ decrease. Moreover, $c^*$ also decreases as the default intensity $\lambda_b$ increases, while $c^*$ increases as the default rate $\lambda_s$ increases. All these patterns are consistent with Corollary 3.6 above.
3.2.3 Optimal Trading with a Single Seller

We proceed to the trading position problem with a single risk-averse seller as in Section 3.1.2. When the investor acts as a potential buyer, given the bid pricing rule $\tilde{p}(\gamma_s, c)$ of the seller, he/she selects the optimal long position $c_b^*$ that maximizes the value function:

$$ c_b^* = \arg \max_{c \geq 0} V_b(0, x - \tilde{p}(\gamma_s, c), \gamma_b, c). $$

(3.24)

Similarly, the seller’s optimal short position $c_s^*$ is given by

$$ c_s^* = \arg \max_{c \geq 0} p(\gamma_b, c) - p_s(\gamma_s, c). $$

(3.25)

The optimal position depends on both buyer’ and seller’s risk aversions $(\gamma_b, \gamma_s)$ as well as their beliefs $(Q_b, Q_s)$. As we show next, if the buyer and the seller have the same belief, then it is optimal for both of them not to trade for any (possibly different) risk aversions. In other words, risk aversion difference alone does not trigger a trade between a buyer and a seller – the buyer and the seller must have different views of the market in order for a trade to occur.

**Proposition 3.7.** Assume that the buyer and the seller have the same market belief, i.e. $Q_b = Q_s$. For any arbitrarily fixed values of $\gamma_b, \gamma_s > 0$, the buyer and seller find it optimal not to trade with each other.

**Proof.** Assume $Q_b = Q_s$. By Proposition 3.3, $\tilde{p}(\gamma_s, c)$ dominates $p(\gamma_b, c)$ for all $c \geq 0$, and the spread $p(\gamma_b, c) - \tilde{p}(\gamma_s, c)$ decreases from zero as $c$ increases from zero. Therefore, it follows from (3.24) and (3.25) that $c_b^* = c_s^* = 0$.

Next, we establish the necessary and sufficient condition for a trade to occur and provide a formula for the optimal trading position. To this end, we denote by $p^*$ and $\tilde{p}^*$ respectively the buyer’s and seller’s zero risk-aversion prices with respect to their own reference measures $Q_b$ and $Q_s$. 
Proposition 3.8. The buyer’s optimal trading position is given as follows:

(i) If \( p^* > \tilde{p}^* \), then

\[
c_b^* = \frac{1}{(1 - R) e^{-rT} (\gamma_s + \gamma_b)} \log \left( \frac{A_bB_s}{A_sB_b} \right) > 0.
\] (3.26)

(ii) If \( p^* \leq \tilde{p}^* \), then \( c_b^* = 0 \).

The constants \((A_b, B_b)\) and \((A_s, B_s)\) are defined according to (3.8) with parameters corresponding to buyer and seller at time zero, respectively.

Proof. Since the function \( p(\gamma_b, c) - p_s(\gamma_s, c) \) in (3.24) is strictly concave in \( c \), the optimal solution \( c_b^* > 0 \) if and only if \( \frac{\partial}{\partial c} [p(\gamma_b, c) - p_s(\gamma_s, c)]_{c=0} > 0 \) or equivalently \( p_b^* > p_s^* \) according to (3.20). Finally, the first-order optimality condition \( \frac{\partial}{\partial c} [p(\gamma, c) - p_s(\gamma, c)] = 0 \) yields the explicit formula for \( c_b^* \) in (3.26).

As we can see, the trade condition \( p^* > \tilde{p}^* \) plays a critical role to trigger a trade between the buyer and the seller.

Corollary 3.9. The optimal trading position \( c_b^* \) is decreasing in \( \gamma_b, \gamma_s, \lambda_b \), but increasing in \( \lambda_s \).

The financial intuition of these results is illustrated in Figures 3.1 and 3.4. First, we recall that, in the case with identical belief, the seller’s price always dominates the buyer’s price, as previously shown in Figure 3.1 and therefore no trade will occur. In Figure 3.4, the trade condition \( p^* > \tilde{p}^* \) implies that the seller’s price curve lies below the buyer’s price curve for a range of \( c \), so the optimal position is positive and is found within this range. As indicated by (3.26), a higher risk aversion of the buyer and/or the seller reduces the optimal trading position. In Figure 3.4, as the buyer’s risk aversion increases from 0.2 to 0.5, the optimal position decreases from 2.9329 to 1.6934.
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0 1 2 3 4 5 6 7 8 9 10
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Trading Volume

Figure 3.4: The buyer’s optimal positions with a single seller
(Left) The buyer’s optimal position $c^* = 2.9329$ when $\gamma_b = \gamma_s = 0.2$. (Right) The
buyer’s optimal position $c^* = 1.6934$ when $\gamma_b = 0.5$ and $\gamma_s = 0.2$. For both cases,
$\lambda_b = 5\%$, $\lambda_s = 10\%$ and the other parameters are the same as in Figure 3.1.

Remark 3.10. The formula (3.26) holds even if $\gamma_b$ or $\gamma_s$ takes value zero. If
$\gamma_s = 0$, then we get

$$c^*_b = \frac{1}{\gamma_b(1 - R)e^{-rT}} \log \left( \frac{(e^{-rT} - \bar{p}^*) A_b}{(-R e^{-rT} + \bar{p}^*) B_b} \right) > 0,$$

and $c^*_s(\gamma, \pi) = 0$. (3.27)

When $\gamma_s = 0$, the seller’s pricing rule is linear. Therefore, we can also apply
Proposition 3.5 to get (3.26) by substituting $\pi = \bar{p}^*$. By the same arguments,
when $\gamma_b = 0$, we obtain

$$c^*_b = 0,$$

and

$$c^*_s = \frac{1}{\gamma_s(1 - R)e^{-rT}} \log \left( \frac{(e^{-rT} - p^*) B_s}{(-R e^{-rT} + p^*) A_s} \right) > 0. (3.28)$$

In the special case of $\gamma_b = \gamma_s = 0$, (3.26) indicates that $c^*_b = +\infty$ if
$p^*_b > p^*_s$. This is consistent with (3.24) because the buyer’s per-unit price
dominates the seller’s, and therefore, it is optimal for the buyer to take an
infinite long position.
3.2.4 Optimal Trading with Multiple Sellers

Turning to the case with multiple sellers, we apply the subscript notation \( b \) and \( s_i, i \in \mathcal{N} \) to distinguish the buyer and the multiple sellers. The buyer’s optimal long position is found from

\[
\begin{align*}
    c^* &= \arg\max_{c_i \geq 0, i = 1, \ldots, N} V_b \left( 0, x - \sum_{i=1}^N \tilde{p}_i(\gamma_{s_i}, c_i); \gamma_b, \bar{c} \right) \\
    &= \arg\max_{c_i \geq 0, i = 1, \ldots, N} p(\gamma_b, \bar{c}) - \sum_{i=1}^N \tilde{p}_i(\gamma_{s_i}, c_i),
\end{align*}
\]

where \( \bar{c} = \sum_{i=1}^N c_i \).

In preparation for the following results, we denote \( p^* \) and \( \tilde{p}_i^* \) as the buyer’s and seller \( i \)'s zero risk-aversion prices based on their respective beliefs. Define \( \mathcal{N}^0 := \{ i \in \mathcal{N} : p^* \leq \tilde{p}_i \} \) which is the set of sellers whose zero risk-aversion prices dominate the buyer’s. Denote the set of remaining sellers by \( \mathcal{N}^1 = \mathcal{N} \setminus \mathcal{N}^0 \). We call the set \( \mathcal{N}^1 \) and \( \mathcal{N}^0 \) the trade and no-trade sets, respectively. As we will show below, the buyer will purchase from seller \( i \) if and only if seller \( i \in \mathcal{N}^1 \).

**Proposition 3.11.** The buyer’s optimal position in each seller \( i \in \mathcal{N} \), denoted by \( c_i^* \), satisfies that

\[
    c_i^* > 0 \quad \text{if} \quad i \in \mathcal{N}^1 \quad \text{or} \quad c_i^* = 0 \quad \text{if} \quad i \in \mathcal{N}^0.
\]

The non-zero positions are found from the first-order conditions:

\[
    \frac{\partial p(\gamma_b, \bar{c})}{\partial c} = \frac{\partial \tilde{p}_i(\gamma_{s_i}, c_i^*)}{\partial c}, \quad \forall i \in \mathcal{N}^1. \quad (3.29)
\]

**Proof.** Recall that \( p(\gamma_b, c) \) is strictly concave in \( c \) and \( \tilde{p}_i(\gamma_{s_i}, c_i) \) is strictly convex in \( c_i \). If \( p^* \leq \tilde{p}_i \), with other components \( c_j, j \neq i \), arbitrarily fixed, then the trading benefit \( B_b(\bar{c}) := p(\gamma_b, \bar{c}) - \sum_{i=1}^N \tilde{p}_i(\gamma_{s_i}, c_i) \) decreases as \( c_i \) increases from 0. Therefore, \( c_i^* = 0 \) for \( i \in \mathcal{N}^0 \). If \( i \in \mathcal{N}^1 \), then \( \frac{\partial B_b(\bar{c})}{\partial c_i}|_{c_i=0} > 0 \), so \( c_i^* > 0 \) and is determined from the first-order condition. \( \square \)
Proposition 3.12. The buyer’s optimal trading position \( c^*_i \) is decreasing in \( \gamma_b \) and \( \gamma_{s_i} \).

Proof. The partial derivative \( \partial_c p(\gamma_b, c) \) is decreasing in \( \gamma_b \) and \( \partial_c \tilde{p}(\gamma_{s_i}, c) \) is increasing in \( \gamma_{s_i} \) since

\[
\frac{\partial^2 p(\gamma_b, c)}{\partial c \partial \gamma} = \frac{\gamma_b}{c} \frac{\partial^2 p(\gamma_b, c)}{\partial c^2} \leq 0, \quad \frac{\partial^2 \tilde{p}(\gamma_{s_i}, c)}{\partial c \partial \gamma} = \frac{\gamma_{s_i}}{c} \frac{\partial^2 \tilde{p}(\gamma_{s_i}, c)}{\partial c^2} \geq 0.
\]

Applying these properties and the concavity/convexity of \( p \) and \( \tilde{p} \) in \( c \) to the first-order condition in (3.29), we conclude that \( c^*_i \) is decreasing in \( \gamma_b \) and \( \gamma_{s_i} \). \( \square \)

Next, we will discuss some interesting market phenomena implied by this result. Denote \( \mathcal{M}(P) \subseteq \mathcal{N} \) to be the set of all sellers who share a certain market view \( P \), so \( Q_{s_i} = P \) for every \( i \in \mathcal{M}(P) \). In particular, \( \mathcal{M}(Q_b) \) is the set of sellers whose beliefs coincide with the buyer’s, but they may differ in risk aversion.

Corollary 3.13. For each seller \( i \) whose belief coincides with the buyer’s i.e. \( i \in \mathcal{M}(Q_b) \), the risk-averse buyer’s optimal position is \( c^*_i = 0 \).

Furthermore, if all the sellers have the same market view as the buyer’s, i.e. \( \mathcal{M}(Q_b) = \mathcal{N} \), then even with heterogeneous risk aversions, it is optimal for all parties not to trade, yielding \( c^* = (0, \ldots, 0) \). In other words, belief heterogeneity is a necessary condition for a trade to occur in the multi-seller market.

Suppose some sellers with a certain market view also have identical risk aversion (both possibly different from the buyer’s). These sellers have their common indifference price function, and therefore, by (3.29) of Proposition 3.11 the buyer will acquire the same quantity from each of the sellers.

Corollary 3.14. Suppose that more than one seller share their beliefs \( (Q_s) \) and risk aversions \( \gamma_s > 0 \), i.e. \( |\mathcal{M}(Q_s)| \geq 2 \) and \( \gamma_{s_i} = \gamma_s \forall i \in \mathcal{M}(Q_s) \). If
the sellers satisfy the trade condition i.e. \( \mathcal{M}(Q_s) \subseteq \mathcal{N}^1 \), then every seller in \( \mathcal{M}(Q_s) \) has equal and positive trading volume with the buyer, i.e. \( c_j^* = c_k^* > 0 \) for \( j, k \in \mathcal{M}(Q_s) \).

In essence, the buyer will equally allocate his trading volume in any group of identical sellers. Nevertheless, the computation of \( c_i^* \) requires the information of all other sellers in the market.

Furthermore, in a market with \( N \) homogeneous sellers (with the same risk aversion and market view), we infer from Corollary 3.14 that the buyer will either not trade with any of these sellers, or purchase the same quantity from each seller. In the latter case, the \( N \) sellers have equal market shares, but the total trading volume depends on the number of sellers in the market.

**Proposition 3.15.** In a market of homogeneous sellers, i.e. \( \gamma_{si} = \gamma_s \forall i \in \mathcal{N} \) and \( |\mathcal{M}(Q_s)| = N \) for some \( \gamma_s \) and \( Q_s \), the buyer’s optimal position in each seller is given as follows:

(i) If \( p^* > \tilde{p}^* \), then

\[
 c_b^*(N) = \frac{1}{(1 - R) e^{-rT} (\gamma_s + N\gamma_b)} \log \left( \frac{A_b B_s}{A_s B_b} \right) > 0.
\]

(ii) If \( p^* \leq \tilde{p}^* \), then \( c_b^*(N) = 0 \).

**Corollary 3.16.** When the number of homogeneous sellers goes to the infinity, the buyer’s total optimal position satisfies that \( \lim_{N \to \infty} Nc_b^*(N) = c_b^* \), where \( c_b^* \) is given in (3.27).
Proof. The volume-scaling property of Proposition 3.3 (iv) implies that
\[
\lim_{N \to \infty} N c^*_b(N) = \lim_{N \to \infty} N \left( \arg \max_{c \geq 0} p(\gamma_b, N c) - \tilde{p}(\frac{\gamma_s}{N}, N c) \right) = \arg \max_{c' \geq 0} \left( p(\gamma_b, c') - \lim_{N \to \infty} \tilde{p}(\frac{\gamma_s}{N}, c') \right) = \arg \max_{c' \geq 0} p(\gamma_b, c') - \tilde{p}^* c' = c^*_b,
\]
where the third equality follows from setting \(c' = Nc\). \(\square\)

Therefore, in the case with infinitely many homogeneous sellers, the risk-averse buyer’s total volume is the same as the scenario when he/she trades with a single seller with zero risk-aversion and the same market belief as the homogeneous sellers. Clearly, the quantity \(c^*_b(N)\) for each homogeneous seller decreases to zero as \(N\) increases to infinity. Nevertheless, the buyer’s total volume \(N c^*_b(N)\) actually increases with \(N\) and admits the limit given by Corollary 3.16.

Finally, we provide further analytic results and numerical examples for the case with two heterogeneous sellers. Recall from (3.21) that \(p^*\) and \(\tilde{p}^*_i\), \(i = 1, 2\), represent the buyer’s and seller \(i\)’s zero risk-aversion prices.

Proposition 3.17. (Two-Seller Case) The buyer’s optimal static position \(c^* = (c^*_1, c^*_2)\) is given as follows:

(i) If \(p^* > \tilde{p}^*_1\), for \(i = 1, 2\), then
\[
c^*_1 = \frac{1}{(1 - R) e^{-rT} (\gamma_b(\gamma_{s1} + \gamma_{s2}) + \gamma_s \gamma_{s2})} \left[ \gamma_{s2} \log \left( \frac{A_b B_{s1}}{A_{s1} B_b} \right) + \gamma_b \log \left( \frac{A_{s2} B_{s1}}{A_{s1} B_{s2}} \right) \right] > 0,
\]
\[
c^*_2 = \frac{1}{(1 - R) e^{-rT} (\gamma_b(\gamma_{s1} + \gamma_{s2}) + \gamma_s \gamma_{s2})} \left[ \gamma_{s1} \log \left( \frac{A_b B_{s2}}{A_{s2} B_b} \right) + \gamma_b \log \left( \frac{A_{s1} B_{s2}}{A_{s2} B_{s1}} \right) \right] > 0.
\]

(ii) If \(p^* > \tilde{p}^*_1\) and \(p^* \leq \tilde{p}^*_2\), then
\[
c^*_1 = \frac{1}{(1 - R) e^{-rT} (\gamma_{s1} + \gamma_b)} \log \left( \frac{A_b B_{s1}}{A_{s1} B_b} \right) > 0, \quad \text{and} \quad c^*_2 = 0.
\]
(iii) If \( p^* \leq \tilde{p}_1^* \) and \( p^* > \tilde{p}_2^* \), then
\[
c_1^* = 0, \quad \text{and} \quad c_2^* = \frac{1}{(1-R) e^{-rT} (\gamma s_2 + \gamma b)} \log \left( \frac{A_b B s_2}{A s_2 B_b} \right) > 0.
\]

(iv) If \( p^* \leq \tilde{p}_i^* \), for \( i = 1, 2 \), then
\[
c_1^* = c_2^* = 0.
\]

Here, the constants \((A_b, B_b)\) and \((A_{s_i}, B_{s_i})\), \( i = 1, 2 \) are defined according to (3.8) with parameters corresponding to buyer and seller \( i, \ i = 1, 2 \), at time zero, respectively.

The proof follows from that of Proposition 3.11 and is thus omitted. Here, each trade condition \( p^* > \tilde{p}_i^* \), for \( i = 1, 2 \), checks whether seller \( i \) is in the trade set \( \mathcal{N}^1 \) and guarantees that the optimal solution is in the interior of the domain \([0, \infty)^2\). In this case, the optimal positions with both sellers are strictly positive. If one of these inequalities is violated, then the problem is reduced to the single-seller case which has been solved in Proposition 3.8.

In case (i), each seller’s trading volume depends on the other competing seller’s market view and risk aversion. In particular, if the competing seller becomes less risk averse, then she tends to get a larger share of the buyer’s position. Nevertheless, in cases (ii) and (iii) where only one seller trades with the buyer, the trading volume is not affected by the other non-trading seller.

Figures 3.5 shows the buyer’s optimal position under two cases (i) and (ii). Interestingly, if the sellers have the same risk aversion and market view, but their common market view does not coincide with the buyer’s, then the two sellers will equally share the sales volume (left panel). In the case that the buyer does not buy from one of the sellers (right panel), we see that the optimal solution lies on the boundary of the \((c_1, c_2)\) domain. In Figure 3.6 (left), we see that a less risk averse seller will gain a larger share of the total volume because she offers more competitive (or lower) prices to the buyer. On
the other hand, when a seller believes a higher probability of default, she will be willing to sell the defaultable bonds at lower prices, and thus capturing a larger market share, as shown in Figure 3.6 (right).

We end this section by studying the case when the buyer pre-specifies $\alpha > 0$ units of the defaultable bond to purchase from the sellers. As discussed in Section 3.1 (see (3.2)), the buyer’s optimal position solves the cost optimization problem:

$$
c_1^{* \alpha} = \arg \min_{0 \leq c_1 \leq \alpha} \tilde{p}_1(\gamma_{s1}, c_1) + \tilde{p}_2(\gamma_{s2}, \alpha - c_1),
$$

$$
c_2^{* \alpha} = \alpha - c_1^{* \alpha}.
$$

Note that the function $g(c_1) := \tilde{p}_1(\gamma_{s1}, c_1) + \tilde{p}_2(\gamma_{s2}, \alpha - c_1)$ is strictly convex in $c_1$. The buyer will purchase all from seller 1 if $g'(\alpha) \leq 0$; purchase all from seller 2 if $g'(0) \geq 0$; or buy positive units of the defaultable bond from both sellers. Re-writing these conditions using the explicit indifference price expressions, we obtain the buyer’s optimal static position

$$
c_1^{* \alpha} = \begin{cases} 
\alpha & \text{if } h(\alpha) \geq \alpha, \\
h(\alpha) & \text{if } 0 < h(\alpha) < \alpha, \\
0 & \text{if } h(\alpha) \leq 0,
\end{cases}
$$

(3.30)

where

$$
h(\alpha) = \frac{\bar{\gamma}_{s2}}{\bar{\gamma}_{s1} + \bar{\gamma}_{s2}} \alpha - \frac{1}{\bar{\gamma}_{s1} + \bar{\gamma}_{s2}} \log \left( \frac{A_{s1} B_{s2}}{A_{s2} B_{s1}} \right),
$$

with $\bar{\gamma}_{si} = \gamma_{si} (1 - R) e^{-rT}$, $i = 1, 2$. In particular, the buyer will purchase all $\alpha$ units of defaultable bonds from seller 1 in the first case, and from seller 2 in the third case in (3.30). As $\alpha$ increases from 0, the location of the optimal position $(c_1^{* \alpha}, c_2^{* \alpha})$ will trace out a straight line from an intercept, as shown by the dark straight line at the bottom of the graphs in Figures 3.5 and 3.6. Each point on the blue line indicates the buyer’s optimal trading positions with the two sellers for a specific $\alpha$ ($0 \leq \alpha \leq 10$).
Figure 3.5: The buyer’s optimal positions with two sellers
(Left) When both sellers are identical, the buyer purchases from both sellers $c_1^* = 0.817$, $c_2^* = 0.817$ with $\lambda_{s1} = \lambda_{s2} = 10\%$, $\lambda_b = 5\%$. (Right) When the buyer purchases from seller 1 only: $c_1^* = 1.23$, $c_2^* = 0$ with $\lambda_b = \lambda_{s2} = 5\%$, $\lambda_{s1} = 10\%$. For both cases, $\gamma_b = \gamma_{s1} = \gamma_{s2} = 0.5$ and the other parameters are the same as in Figure 3.1.
Figure 3.6: The buyer’s optimal positions with two sellers

(Left) The two sellers differ only in risk aversion, with $\gamma_{s1} = 0.2$, $\gamma_{s2} = 1$ and $\lambda_{s1} = \lambda_{s2} = 10\%$. $\gamma_b = 0.5$ and $\lambda_b = 5\%$. The less risk-averse seller captures a larger market share: $c_1^* = 1.532, c_2^* = 0.306$. (Right) The two sellers differ only in their market views, with $\lambda_{s1} = 8\%$, $\lambda_{s2} = 10\%$ while $\lambda_b = 5\%$ and $\gamma_b = \gamma_{s1} = \gamma_{s2} = 0.5$. Seller 2 gains a larger market share: $c_1^* = 0.2479, c_2^* = 1.1016$. For both cases, the other parameters are the same as in Figure 3.1.
3.3 Utility-Indifference Pricing and Optimal Trading of CDS

Credit default swaps (CDS) played a critical role in the 2008 financial crisis, and their market has undergone significant changes since then. Before 2009, market participants could trade CDS without initial upfront fees. Currently, CDS contracts are traded in a new standardized way whereby the protection buyer pays at a fixed premium rate and pay/receive a non-zero upfront payment at the start of the contract. The fixed standardized premium rate is 100 basis points (bps) for investment grade references and 500bps for non-investment grade references in North America\(^1\).

Let us consider a CDS with unit notional, a fixed premium rate \(\kappa\), and expiration date \(T\). If the reference entity defaults at time \(\tau \leq T\), then the buyer stops paying the premium and receives the discounted amount \$(1 - R)e^{-rT}\) from the protection seller. We use the same notations and assumptions for the tradable assets and default risk as in Section 3.2. Prior to default, the investor’s discounted trading wealth satisfies

\[
d\tilde{X}_t = -c \kappa dt + \sum_{i=1}^{2} \theta^{(i)}_t (\mu_i dt + \sigma_i dW^{(i)}_t), \quad t \in [0, \tau \wedge T],
\]

where \(\theta_t = (\theta^{(1)}_t, \theta^{(2)}_t)\) represents the discounted cash amounts invested in \(S^{(1)}\) and \(S^{(2)}\) respectively. After default, the investor only trades the non-defaultable asset \(S^{(1)}\) and solves the Merton problem (3.4). With \(c \in \mathbb{R}\) units of the CDS, the investor solves the utility maximization problem:

\[
\tilde{V}(t, x; \gamma, c, \kappa) = \sup_{\Theta_{t,T}} \mathbb{E} \left\{ U(\tilde{X}_T) \mathbf{1}_{\{\tau > T\}} + M \left( \tau, \tilde{X}_\tau + (1 - R)c e^{-rT} \right) \mathbf{1}_{\{\tau \leq T\}} \mid \tilde{X}_t = x \right\}.
\]

(3.31)

3.3.1 Bid-Ask Upfront Prices

We apply the utility indifference pricing approach to derive the pricing rules for the protection buyer and seller. To facilitate the presentation, we define

\[ F(t; c, \kappa) = e^{(\gamma c \kappa - \alpha)(T-t)} \quad \text{and} \quad G(c, \kappa) = \frac{\lambda e^{-\gamma (1-R)c e^{-rT}}}{\alpha - \gamma c \kappa - \frac{1}{2} \eta^2} . \]

We note that \( G(c, \kappa) \) does not depend on time \( t \).

**Proposition 3.18.** The value function \( \tilde{V}(t, x; \gamma, c, \kappa) \) is given by

\[ \tilde{V}(t, x; \gamma, c, \kappa) = -e^{-\gamma x} v(t; \gamma, c, \kappa) , \]

where \( v(t) \equiv v(t; \gamma, c, \kappa) \), \( 0 \leq t \leq T \) satisfies

\[
v(t) = \begin{cases} e^{-\frac{1}{2} \eta^2 (T-t)} \left( 1 + e^{-\gamma (1-R)c e^{-rT}} \lambda(T-t) \right) & \text{if } \alpha = \gamma c \kappa + \frac{\eta^2}{2} (3.33) \\ (1 - G(c, \kappa)) F(t; c, \kappa) + G(c, \kappa) e^{-\frac{1}{2} \eta^2 (T-t)} & \text{if } \alpha \neq \gamma c \kappa + \frac{\eta^2}{2} (3.34) \end{cases}
\]

The optimal trading strategy \( \tilde{\theta}^* \) prior to default is

\[ \tilde{\theta}^*_u = \frac{\Sigma^{-1} \mu}{\gamma} , \quad t \leq u < \tau \wedge T . \]

**Proof.** We first write down the HJB PDE associated with \( \tilde{V} \equiv \tilde{V}(t, x; \gamma, c, \kappa) \) in (3.31):

\[
\frac{\partial \tilde{V}}{\partial t} - \lambda \left( e^{-\gamma (x+c(1-R)e^{-rT}) - \frac{1}{2} \eta^2 (T-t) + \tilde{V}} \right) - c \kappa \frac{\partial \tilde{V}}{\partial x} + \sup_\theta \left\{ \theta' \mu \frac{\partial \tilde{V}}{\partial x} + \frac{\theta' \Sigma \theta}{2} \frac{\partial^2 \tilde{V}}{\partial x^2} \right\} = 0 ,
\]

(3.35)

for \((t, x) \in [0, T) \times \mathbb{R} \), with \( \tilde{V}(T, x; \gamma, c, \kappa) = -e^{-\gamma x} \). Maximizing over \( \theta \) in (3.35) yields

\[
\frac{\partial \tilde{V}}{\partial t} - \lambda \left( e^{-\gamma (x+c(1-R)e^{-rT}) - \frac{1}{2} \eta^2 (T-t) + \tilde{V}} \right) - c \kappa \frac{\partial \tilde{V}}{\partial x} - \frac{\mu' \Sigma^{-1} \mu}{2} \frac{(\partial \tilde{V}/\partial x)^2}{\partial^2 \tilde{V}/\partial x^2} = 0 .
\]

(3.36)

Substituting (3.32) into (3.35) gives an ODE for \( v(t) \equiv v(t; \gamma, c, \kappa) \):

\[
v'(t) + (\gamma c \kappa - \alpha)v(t) + \lambda e^{\gamma c(1-R)e^{-rT} - \frac{1}{2} \eta^2 (T-t)} = 0 ,
\]

(3.37)
with \( v(T) = 1 \). The explicit solution is given by (3.33) and (3.34). We can easily check that \( v(t; \gamma, c, \kappa) \) is continuous in \( c \) and well-defined. Finally, applying (3.32) to the optimal \( \tilde{\theta}^* \) in (3.35), we get \( \tilde{\theta}^*(t, x) = (\Sigma^{-1} \mu)/\gamma \), for \((t, x) \in [0, T] \times \mathbb{R} \).

We now define the indifference bid-ask upfront prices of a CDS.

**Definition 3.19.** At time \( t \), the protection buyer’s (resp. seller’s) indifference bid (resp., ask) upfront price for \( c \geq 0 \) units of a CDS with premium rate \( \kappa \) is defined by the cash amount \( p(t; \gamma, c, \kappa) \equiv p(t, x; \gamma, c, \kappa) \) (resp. \( \tilde{p}(t; \gamma, c, \kappa) \)) satisfying

\[
\tilde{V}(t, x, y; \gamma, 0, \kappa) = \tilde{V}(t, x - p(t; \gamma, c, \kappa), y; \gamma, c, \kappa),
\]

(3.38)

\[
\tilde{V}(t, x, y; \gamma, 0, \kappa) = \tilde{V}(t, x + \tilde{p}(t; \gamma, c, \kappa), y; \gamma, -c, \kappa).
\]

(3.39)

Applying (3.32) to (3.38) and (3.39), we express the indifference bid and ask upfront prices for fixed premium \( \kappa \) at time \( t \):

\[
p(t; \gamma, c, \kappa) = \frac{1}{\gamma} \log \frac{v(t; 0)}{v(t; \gamma, c, \kappa)} \quad \text{and} \quad \tilde{p}(t; \gamma, c, \kappa) = \frac{1}{\gamma} \log \frac{v(t; \gamma, -c, \kappa)}{v(t; 0)},
\]

(3.40)

where \( v(t; 0) \equiv v(t; 0, 0, 0) \). As long as \( c = 0 \), the value of \( v(t; \gamma, c, \kappa) \) does not depend on \( \gamma \) or \( \kappa \). By direction substitution of (3.40) into (3.37) yields the ODEs for the indifference prices.

**Proposition 3.20.** The buyer’s and seller’s indifference upfront prices \( p(t) \equiv p(t; \gamma, c, \kappa) \) and \( \tilde{p}(t) \equiv \tilde{p}(t; \gamma, c, \kappa) \), respectively, solve the ODEs:

\[
p'(t) - ck - \frac{\lambda}{\gamma v(t; 0)} e^{-\frac{1}{2} \eta_t^2 (T-t)} (e^{\gamma p(t) - \gamma c (1-R) e^{-rT}} - 1) = 0,
\]

(3.41)

\[
\tilde{p}'(t) - ck + \frac{\lambda}{\gamma v(t; 0)} e^{-\frac{1}{2} \eta_t^2 (T-t)} (e^{-\gamma \tilde{p}(t) + \gamma c (1-R) e^{-rT}} - 1) = 0,
\]

(3.42)

with the terminal conditions \( p(T) = 0 \) and \( \tilde{p}(T) = 0 \).
For notational convenience, we write in this section
\[ p(\gamma, c, \kappa) \equiv p(0; \gamma, c, \kappa), \quad \tilde{p}(\gamma, c, \kappa) \equiv \tilde{p}(0; \gamma, c, \kappa), \]
(3.43)
\[ v(\gamma, c, \kappa) \equiv v(0; \gamma, c, \kappa), \quad v(0) \equiv v(0; 0, 0, 0) \quad \text{and} \quad F(c, \kappa) \equiv F(0; c, \kappa). \]
(3.44)

For our results below, we define the function
\[ v^{(1)}(\gamma, c, \kappa) \equiv \frac{1}{\gamma} \frac{\partial v(\gamma, c, \kappa)}{\partial c} \]
which has the following explicit expression:
\[ v^{(1)} = \begin{cases} 
-(1 - R) \lambda T e^{-(r + \frac{1}{2} \eta \kappa^2)T - \gamma(1 - R)e^{-rT}} & \text{if } \alpha = \gamma c \kappa + \frac{\eta^2}{2}, \\
T \kappa \left(1 - G(c, \kappa)\right) F(c, \kappa) & \text{if } \alpha \neq \gamma c \kappa + \frac{\eta^2}{2}, \\
+G(c, \kappa) \left(e^{-\frac{1}{2} \eta \kappa^2 T - F(c, \kappa)}\right) \left(\frac{\kappa}{\alpha - \gamma c \kappa - \frac{\eta^2}{2}} - (1 - R) e^{-rT}\right) & \end{cases} \]

**Proposition 3.21.** The CDS protection buyer’s and seller’s indifference upfront prices have the following properties:

(i) Concavity/convexity in \( c \): for \( c \geq 0 \), \( p(\gamma, c, \kappa) \) is strictly concave in \( c \), and \( \tilde{p}(\gamma, c, \kappa) \) is strictly convex in \( c \).

(ii) Monotonicity in \( \gamma \): \( p(\gamma, c, \kappa) \) is decreasing and \( \tilde{p}(\gamma, c, \kappa) \) is increasing in \( \gamma \).

(iii) Risk aversion asymptotics:
\[ \lim_{\gamma \to \infty} \frac{p(\gamma, c, \kappa)}{c} = -\kappa T, \quad \lim_{\gamma \to \infty} \frac{\tilde{p}(\gamma, c, \kappa)}{c} = e^{-rT}(1 - R), \]
\[ \lim_{\gamma \to 0} \frac{p(\gamma, c, \kappa)}{c} = p^*, \quad \lim_{\gamma \to 0} \frac{\tilde{p}(\gamma, c, \kappa)}{c} = \tilde{p}^*, \]
(3.45)
(3.46)

where the zero risk-aversion upfront prices \( p^* \) and \( \tilde{p}^* \) are given by
\[ p^* = \tilde{p}^* = \frac{-v^{(1)}(0, 0, \kappa)}{v(0)}, \]
(3.47)

where \( v(0) \) and \( v(0, 0, \kappa) \) are defined in (3.44). Therefore, for any \( \gamma, c \geq 0 \), the following inequalities hold:
\[ -c \kappa T \leq p(\gamma, c, \kappa) \leq c p^* \leq \tilde{p}(\gamma, c, \kappa) \leq c e^{-rT}(1 - R). \]
(iv) Volume-scaling property: \( p(\gamma, c, \kappa) = c p(\gamma, 1, \kappa) \) and \( \bar{p}(\gamma, c, \kappa) = c \bar{p}(\gamma, 1, \kappa) \).

This leads to the following volume asymptotics:

\[
\lim_{c \to \infty} \frac{p(\gamma, c, \kappa)}{c} = -\kappa T, \quad \lim_{c \to \infty} \frac{\bar{p}(\gamma, c, \kappa)}{c} = e^{-rT}(1 - R) ,
\]

\[
\lim_{c \to 0} \frac{p(\gamma, c, \kappa)}{c} = p^*, \quad \lim_{c \to 0} \frac{\bar{p}(\gamma, c, \kappa)}{c} = \bar{p}^*.
\]

**Proof.** (i) For any \( 0 \leq c_1, c_2 < \infty \) and \( \theta \in (0, 1) \), we define \( \bar{p}(t) := \theta p(t; c_1) + (1 - \theta) p(t; c_2) \) and \( \bar{c} := \theta c_1 + (1 - \theta) c_2 \). To show the strict concavity of \( p(t; c) \equiv p(t; \gamma, c, \kappa) \), it is sufficient to show that the convex combination \( \bar{p}(t) \) is a super-solution of the ODE for \( p(t; \bar{c}) \):

\[
p'(t; \bar{c}) - \bar{c} \kappa = \frac{\lambda e^{-\frac{1}{2} \eta^2(T-t)}}{\gamma v(t; 0)} (e^{\gamma(p(t; \bar{c}) - \bar{c} l)} - 1) = 0, \tag{3.48}
\]

with \( l := (1 - R)e^{-rT} \). Substituting \( \bar{p}(t; \theta, c_1, c_2) \) into (3.48), we get

\[
\theta p'(t; c_1) + (1 - \theta)p'(t; c_2) - \bar{c} \kappa = \frac{\lambda e^{-\frac{1}{2} \eta^2(T-t)}}{\gamma v(t; 0)} (e^{\gamma(p(t; c_1) - c_1 l)} + (1 - \theta)e^{\gamma(p(t; c_2) - c_2 l)} - e^{\gamma(p(t; c_1) - c_1 l) + (1 - \theta)(p(t; c_2) - c_2 l)})
\]

\[
> 0.
\]

The equality comes from the fact that both \( p(t; c_1) \) and \( p(t; c_2) \) satisfy the ODE (3.41). The inequality holds since \( v(t; 0) > 0 \) and \( e^{\gamma x} \) is a strictly convex function of \( x \) for \( \gamma > 0 \). Hence, \( \bar{p}(t; \theta, c_1, c_2) \) is a super-solution of (3.48) and thus, \( p(t; c) \) is strictly concave in \( c \) (see [Khalil 2002, Lemma 3.4]). Similar arguments yield the strict convexity of \( \bar{p}(t; \gamma, c, \kappa) \).

(ii) Differentiating with respect to \( \gamma \), we obtain

\[
\frac{\partial p(\gamma, c, \kappa)}{\partial \gamma} = \frac{1}{\gamma^2} \left[ -\frac{\gamma}{v(\gamma, c, \kappa)} \frac{\partial v(\gamma, c, \kappa)}{\partial \gamma} + \log \left( \frac{v(\gamma, c, \kappa)}{v(0)} \right) \right] .
\]

The function \( h(\gamma) \) is negative since \( h(0) = 0 \) and \( h'(\gamma) = c^2 \frac{\partial^2 p(\gamma, c, \kappa)}{\partial c^2} \leq 0 \). Therefore, \( p(\gamma, c, \kappa) \) is decreasing in \( \gamma \). Similar arguments conclude that \( \bar{p}(\gamma, c, \kappa) \) is increasing in \( \gamma \).
(iii) All the limits can be obtained by direct computation using l’Hospital’s rule.

(iv) The volume-scaling property can be easily inferred from the indifference upfront formulas (3.40) via the expression for $v(\gamma, c, \kappa)$ in (3.34).

Figure 3.7: The indifference upfronts in terms of trading volume. As $\gamma \in \{0.2, 0.35, 0.5\}$ increases along the arrows, $p(\gamma, c, \kappa)$ decreases while $\tilde{p}(\gamma, c, \kappa)$ increases, leading to a wider bid-ask spread. The dotted line is the zero risk-aversion upfront line with slope $p^\ast$. (Left) High default rate case with $\lambda = 5\%$. (Right) Low default rate case with $\lambda = 1\%$. For both cases, $\kappa = 0.01$, $T = 5$ and the other parameters are the same as in Figure 3.1.

We can visualize Proposition 3.21 in Figure 3.7. The seller’s upfront price curve dominates the buyer’s curve for all volume $c$. Moreover, at zero volume ($c = 0$), the buyer’s and seller’s upfront curves both have slope $p^\ast$ if they have the same market belief ($\mathbb{Q}$).

Unlike the defaultable bond case, the indifference upfront prices $p$ and $\tilde{p}$ both can be negative depending on the parameters, especially the relative values of the default rate $\lambda$ and premium rate $\kappa$. If the default risk is very high, the seller may require the buyer to pay initial upfront on top of the fixed premium. However, if the default rate is very low, then the buyer may need to receive an upfront fee to compensate the relatively high fixed premium.
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The protection seller may also find it acceptable to pay a positive upfront fee (meaning a negative indifference price) in exchange for the premium payments from the buyer if the likelihood of future liability from default is very small. On the right panel of Figure 3.7, both $p$ and $\tilde{p}$ take negative values for some intervals of $c$.

Moreover, as either $c$ or $\gamma$ goes to $\infty$, the buyer’s indifference upfront price $p$ decreases to $-\infty$, since the average bid upfront $p/c$ converges to the negative lower bound $-\kappa T < 0$. On the other hand, the seller’s indifference upfront price $\tilde{p}$ increases to $\infty$, since the the average ask upfront $\tilde{p}/c$ converges to the positive upper bound $e^{-rT}(1 - R) > 0$.

3.3.2 Optimal Trading with Linear Upfront Prices

Given a linear pricing rule with an ask upfront fee $\bar{\pi}$, the buyer’s optimal long position is given by

$$c^*(\gamma_b, \bar{\pi}, \kappa) = \arg \max_{c \geq 0} p(\gamma_b, c, \kappa) - c\bar{\pi},$$

(3.49)

where $\gamma_b$ denotes the buyer’s risk aversion. Similarly, given the per-unit bid upfront price $\bar{\pi}$, the seller selects the optimal short position via

$$\tilde{c}^*(\gamma_s, \bar{\pi}, \kappa) = \arg \max_{c \geq 0} c\bar{\pi} - \tilde{p}(\gamma_s, c, \kappa).$$

(3.50)

We denote by $p^*$ and $\tilde{p}^*$ the buyer’s and the seller’s zero risk-aversion upfront prices (see (3.47)) based on their own reference measures $Q_b$ and $Q_s$, respectively. We again use the subscripts $b$ and $s$ to distinguish the buyer and seller.

Proposition 3.22. Given the market bid-ask upfronts $(\pi, \bar{\pi})$, with $-\kappa T \leq \pi \leq \bar{\pi} \leq e^{-rT}(1 - R)$, the buyer’s optimal position $c^*(\gamma_b, \bar{\pi}, \kappa)$ and the seller’s optimal position $\tilde{c}^*(\gamma_s, \bar{\pi}, \kappa)$ are given as follows:
(i) If $\bar{\pi} < p^*$, then $c^*(\gamma_b, \bar{\pi}, \kappa) > 0$ is a unique solution of the equation:
\[ v_b^{(1)}(\gamma_b, c^*, \kappa) + \bar{\pi} v_b(\gamma_b, c^*, \kappa) = 0. \] (3.51)
Otherwise, $c^*(\gamma_b, \bar{\pi}, \kappa) = 0$.

(ii) If $\tilde{p}^* < \pi$, then $\tilde{c}^*(\gamma_s, \pi, \kappa) > 0$ is a unique solution of the equation:
\[ v_s^{(1)}(\gamma_s, -c^*, \kappa) + \pi v_s(\gamma_s, -c^*, \kappa) = 0. \] (3.52)
Otherwise, $\tilde{c}^*(\gamma_s, \pi, \kappa) = 0$.

**Proof.** The arguments for the trade/no-trade scenarios follow from the proof of Proposition 3.5. By Proposition 3.21, $p(\gamma_b, c, \kappa)$ is strictly concave with $\frac{\partial p}{\partial c}(\gamma_b, 0, \kappa) = p^*$ and $\tilde{p}(\gamma_s, c, \kappa)$ is strictly convex with $\frac{\partial \tilde{p}}{\partial c}(\gamma_s, 0, \kappa) = \tilde{p}^*$. Differentiating $p(\gamma_b, c, \kappa) - c \bar{\pi}$ in (3.49) and $c \pi - \tilde{p}(\gamma_s, c, \kappa)$ in (3.50) w.r.t. $c$ and applying first-order conditions give (3.51) and (3.52).

From (3.51) and (3.52) above, we can deduce by differentiation the following properties.

**Corollary 3.23.** The buyer’s optimal position $c^*$ is decreasing in $\gamma_b$ and $\bar{\pi}$. The seller’s optimal position $\tilde{c}^*$ is decreasing in $\gamma_s$ but increasing in $\pi$.

Figure 3.8 demonstrates how the buyer’s (resp. seller’s) optimal CDS position is where the indifference upfront curve has a slope of value $\bar{\pi}$ (resp. $\pi$). Also, the buyer will buy less and the seller will sell more CDS when the market upfront increases, and vice versa. When the buyer purchases $c^*(\gamma_b, \bar{\pi}, \kappa)$ units of CDS at the total upfront payment of $c^*(\gamma_b, \bar{\pi}, \kappa) \bar{\pi}$, the benefit is measured by $B_b := p(\gamma_b, c^*, \kappa) - c^* \bar{\pi} \geq 0$. On the left panel of Figure 3.8 with $\gamma_b = 0.2$, $c^* = 5.8173$, $\bar{\pi} = 0.03$ and $\kappa = 0.01$, the benefit is $p(0.2, 0.03, 0.01) - 0.03 \cdot 5.8173 = 0.1549$ which has the same value as the intercept of the market upfront line. Similarly, the seller’s benefit can also be read off from the absolute value of the market upfront line intercept on the right panel of Figure 3.8.
Figure 3.8: The buyer’s and seller’s optimal positions with linear bid-ask up-fronts
(Left) The buyer’s optimal position $c^*$ when $\pi = 0.03$. (Right) The seller’s optimal position $\tilde{c}^*$ when $\pi = 0.15$. For both cases, $\gamma = 0.2$ and the other parameters are the same as in Figure 3.7.

### 3.3.3 Optimal Trading with a Single Seller

When trading with a single seller, the protection buyer pays the upfront price $\tilde{p}(\gamma_s, c, \kappa)$ required by the seller and selects the optimal long position $c^*_b$ that maximizes the value function:

$$c^*_b = \arg \max_{c \geq 0} \tilde{V}_b(0, x - p_s(\gamma_s, c, \kappa); \gamma_b, c, \kappa)$$

$$= \arg \max_{c \geq 0} p(\gamma_b, c, \kappa) - p_s(\gamma_s, c, \kappa).$$

From the protection seller’s perspective, her optimal short position $c^*_s$ also maximizes the price spread, namely,

$$c^*_s = \arg \max_{c \geq 0} p(\gamma_b, c, \kappa) - p_s(\gamma_s, c, \kappa).$$

**Proposition 3.24.** If $p^* > \tilde{p}^*$, then the buyer’s optimal trading position $c^*_b$ is found from

$$v_b(\gamma_b, c, \kappa) v_s^{(1)}(\gamma_s, -c, \kappa) = v_s(\gamma_s, -c, \kappa) v_b^{(1)}(\gamma_b, c, \kappa). \quad (3.53)$$

Otherwise, $c^*_b = c^*_s = 0$. 
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Proof. Since the function \( p(\gamma_b, c, \kappa) - p_s(\gamma_s, c, \kappa) \) is strictly concave in \( c \), the optimal solution \( c_b^* > 0 \) if and only if \( \frac{\partial}{\partial c} [p(\gamma_b, c, \kappa) - p_s(\gamma_s, c, \kappa)] \big|_{c=0} > 0 \) or equivalently \( p^* > \tilde{p}^* \) according to (3.46). Finally, the first-order optimality condition \( \frac{\partial}{\partial c} [p(\gamma_b, c, \kappa) - p_s(\gamma_s, c, \kappa)] = 0 \) yields the equation (3.53). \( \square \)

Corollary 3.25. The optimal trading position \( c_b^* \) is decreasing in \( \gamma_b \) and \( \gamma_s \).

Corollary 3.26. Let \( Q_b = Q_s \). For any arbitrarily fixed values of \( \gamma_b, \gamma_s > 0 \), the buyer and seller find it optimal not to trade with each other.

Regardless of the difference in risk aversion, if the buyer and the seller have the same market belief \( Q_b = Q_s \), then the seller’s upfront price always dominate the buyer’s (see Figure 3.7) and no trade occurs. In Figure 3.9, the condition \( p^* > \tilde{p}^* \) in Proposition 3.24 implies that the seller’s upfront curve lies below the buyer’s upfront curve for a range of \( c \), so the optimal position is positive. A higher risk aversion of the buyer and/or the seller reduces the optimal trading position. In Figure 3.9, as the buyer’s risk aversion increases from 0.2 to 0.5, the optimal position decreases from 2.9329 to 1.6934.

3.3.4 Optimal Trading with Multiple Sellers

For the buyer’s trading problem involving multiple sellers, we use the subscript notation \( b \) and \( s_i, i \in N \) to distinguish the buyer and the multiple sellers. The buyer selects the optimal long position \( c^* \) that maximizes the value function. Precisely,

\[
c^* = \arg \max_{c_i \geq 0, i = 1, \ldots, N} \tilde{V}_b \left( 0, x - \sum_{i=1}^{N} \tilde{p}_i(\gamma_{s_i}, c_i, \kappa) ; \gamma_b, \bar{c}, \kappa \right)
\]

\[
= \arg \max_{c_i \geq 0, i = 1, \ldots, N} p(\gamma_b, \bar{c}, \kappa) - \sum_{i=1}^{N} \tilde{p}_i(\gamma_{s_i}, c_i, \kappa),
\]
Figure 3.9: The CDS buyer’s and seller’s optimal positions with a single dealer. The optimal position $c^*$ locates the maximum difference between the buyer’s upfront (solid) and the seller’s upfront (dashed) curves. As the buyer’s risk aversion $\gamma_b$ increases from 0.2 to 0.5 (left panel to right panel), the buyer’s upfront curve moves downward, and $c^*$ shifts to the left from 2.9329 to 1.6934. For both cases, $\gamma_s = 0.2$ and the other parameters are the same as in Figure 3.7.
where \( \bar{c} = \sum_{i=1}^{N} c_i \). Similarly, the seller’s optimal short position \( \tilde{c}^* \) is given by

\[
\tilde{c}^* = \arg \max_{c_i \geq 0, i = 1, \ldots, N} \sum_{i=1}^{N} p_i(\gamma_{b_i}, c_i, \kappa) - \tilde{p}(\gamma_{s_i}, \bar{c}, \kappa).
\]

Since the buyer’s (resp. seller’s) indifference upfront \( p(\gamma_{b_i}, c, \kappa) \) (resp. \( \tilde{p}(\gamma_{s_i}, c, \kappa) \)) is strictly concave (resp. convex) in \( c \), Proposition 3.11, and Corollaries 3.13 and 3.14 hold as well if we replace the buyer’s indifference bond price with the buyer’s indifference upfront price, and the seller \( i \)’s indifference price with the seller \( i \)’s indifference upfront. The buyer’s trade condition for CDS seller \( i \in \mathcal{N} \) is \( p^* > \tilde{p}^*_i \), where \( p^* \) and \( \tilde{p}^*_i \) are the buyer’s and seller \( i \)’s zero risk-aversion upfront prices based on their respective beliefs.

As a special case, in a market with \( N \) homogeneous sellers (with identical risk aversion and market view), Corollary 3.14 suggests that the buyer will either not trade with any of these sellers, or purchase the same quantity of the CDS from each seller.

**Proposition 3.27.** In a market of homogeneous sellers, i.e. \( \gamma_{s_i} = \gamma_s, \forall i \in \mathcal{N} \) and \( |\mathcal{M}(Q_s)| = N \) for some \( \gamma_s \) and \( Q_s \), the buyer’s optimal position in every seller \( c_b^*(N) \) is a unique solution of the equation

\[
v_b(\gamma_{b}, c, \kappa) v_s^{(1)}(N \gamma_{s}, -c, \kappa) = v_s(N \gamma_{s}, -c, \kappa) v_b^{(1)}(\gamma_{b}, c, \kappa).
\]

Recall from (3.47) that \( p^* \) and \( \tilde{p}^*_i \), \( i = 1, 2 \), represent the buyer’s and seller \( i \)’s zero risk-aversion upfront prices. The proof is omitted since it follows from that of Proposition 3.11 by replacing \( p(\gamma_{b_i}, c) \) and \( \tilde{p}_i(\gamma_{s_i}, c) \) with \( p(\gamma_{b_i}, c, \kappa) \) and \( \tilde{p}_i(\gamma_{s_i}, c, \kappa) \).

**Proposition 3.28.** (Two-Seller Case) The buyer’s optimal static position \( c^* = (c_1^*, c_2^*) \) in two sellers is given as follows:

(i) If \( p^* > \tilde{p}^*_i \), for \( i = 1, 2 \), then \( c^* = (c_1^*, c_2^*) \) is the unique solution of the
following system

\[
v_b(\gamma_b, c_1 + c_2, \kappa) v_s^{(1)}(\gamma_{s_1}, -c_1, \kappa) = v_s(\gamma_{s_1}, -c_1, \kappa) v_b^{(1)}(\gamma_b, c_1 + c_2, \kappa),
\]

\[
v_b(\gamma_b, c_1 + c_2, \kappa) v_s^{(1)}(\gamma_{s_2}, -c_2, \kappa) = v_s(\gamma_{s_2}, -c_2, \kappa) v_b^{(1)}(\gamma_b, c_1 + c_2, \kappa).
\]

(ii) If \( p^* > \tilde{p}_1^* \) and \( p^* \leq \tilde{p}_2^* \), then \( c_1^* \) is the unique solution of the equation

\[
v_b(\gamma_b, c_1, \kappa) v_s^{(1)}(\gamma_{s_1}, -c_1, \kappa) = v_s(\gamma_{s_1}, -c_1, \kappa) v_b^{(1)}(\gamma_b, c_1, \kappa) \quad \text{and} \quad c_2^* = 0.
\]

(iii) If \( p^* \leq \tilde{p}_1^* \) and \( p^* > \tilde{p}_2^* \), then \( c_2^* \) is the unique solution of the equation

\[
v_b(\gamma_b, c_2, \kappa) v_s^{(1)}(\gamma_{s_2}, -c_2, \kappa) = v_s(\gamma_{s_2}, -c_2, \kappa) v_b^{(1)}(\gamma_b, c_2, \kappa) \quad \text{and} \quad c_1^* = 0.
\]

(iv) If \( p^* \leq \tilde{p}_i^* \), for \( i = 1, 2 \), then \( c_1^* = c_2^* = 0 \).

For each seller \( i \in \{1, 2\} \), the trade condition \( p^* > \tilde{p}_i^* \) guarantees a non-zero trade with the buyer. If one of these inequalities is violated, then the problem is reduced to the single-seller case, and the CDS buyer only trades with the seller who satisfies the trade condition. In cases (ii) and (iii), only one seller trades with the buyer, and the trading volume is not affected by the other non-trading seller. Otherwise, in case (i), each seller’s trading volume depends on the other seller’s market view and risk aversion.

In Figure 3.10 both sellers satisfy the trade condition. A less risk averse seller will gain a larger share of the total volume due to a more competitive (or lower) upfront price (left panel). On the other hand, if one seller believes in a higher default rate than the other, a higher upfront fee will be charged to the buyer, leading the buyer to purchase less from this expensive seller (right panel).

As discussed in Section 3.1 (see (3.2)), we can also determine the buyer’s optimal position when the position is pre-specified at \( \alpha > 0 \) units of CDS. The buyer minimizes the total trading cost \( \hat{g}(c_1) := \tilde{p}_1(\gamma_{s_1}, c_1, \kappa) + \tilde{p}_2(\gamma_{s_2}, \alpha - c_1, \kappa) \)
Figure 3.10: The CDS buyer’s optimal positions with two sellers
(Left) The two sellers differ only in risk aversion and have same views on the other
parameters, with $\gamma_{s1} = 0.2$, $\gamma_{s2} = \gamma_{b} = 0.5$ and $\lambda_{s1} = \lambda_{s2} = 3\%$, $\lambda_{b} = 5\%$. The less
risk-averse seller captures a larger market share: $c_{1}^{*} = 1.1011, c_{2}^{*} = 0.4304$. (Right) The
two sellers differ only in their market views with $\lambda_{s1} = 3\%$, $\lambda_{s2} = 3.5\%$ and
have the same risk aversion with $\gamma_{b} = \gamma_{s1} = \gamma_{s2} = 0.5$. Seller 1 gains a larger market
share: $c_{1}^{*} = 0.7207, c_{2}^{*} = 0.5205$. For both cases, the other parameters are the same
as in Figure 3.7.
which is strictly convex in $c_1 \in [0, \alpha]$. Therefore, the buyer will purchase all from seller 1 if $\tilde{g}'(\alpha) \leq 0$; purchase all from seller 2 if $\tilde{g}'(0) \geq 0$; or buy positive units of the CDS from both sellers. Expressing these inequalities in terms of $v$ and $v^{(1)}$ yields the buyer’s optimal trade allocation

$$c_\alpha^* = \begin{cases} 
\alpha & \text{if } v_{s_2}(\gamma_{s_2}, 0, \kappa) v_{s_1}^{(1)}(\gamma_{s_1}, -\alpha, \kappa) \geq v_{s_1}(\gamma_{s_1}, -\alpha, \kappa) v_{s_2}^{(1)}(\gamma_{s_2}, 0, \kappa), \\
0 & \text{if } v_{s_1}(\gamma_{s_1}, 0, \kappa) v_{s_2}^{(1)}(\gamma_{s_2}, -\alpha, \kappa) \geq v_{s_2}(\gamma_{s_2}, -\alpha, \kappa) v_{s_1}^{(1)}(\gamma_{s_1}, 0, \kappa), \\
\hat{h}(\alpha) \in (0, \alpha) & \text{otherwise},
\end{cases}$$

and $c_2^* = \alpha - c_1^*$. In the third case, for any fixed positive $\alpha$, $\hat{h}(\alpha)$ is uniquely determined from the equation

$$v_{s_1}(\gamma_{s_1}, -\hat{h}, \kappa) v_{s_2}^{(1)}(\gamma_{s_2}, \hat{h} - \alpha, \kappa) = v_{s_2}(\gamma_{s_2}, -\hat{h} - \alpha, \kappa) v_{s_1}^{(1)}(\gamma_{s_1}, -\hat{h}, \kappa).$$

In particular, the buyer will purchase all $\alpha$ units of CDS from seller 1 in the first case, and all from seller 2 in the second case in (3.3.4). As $\alpha$ increases from 0, the location of the optimal position $(c_1^*, c_2^*)$ will trace out a straight line from an intercept, as shown by the dark straight line at the bottom of the graphs in Figure 3.10.

### 3.3.5 Bid-Ask Premia

We close this section by briefly discussing the prior industry standard of setting a fixed upfront fee (e.g. zero for unfunded CDS) followed by determining the premium. In this case, we define the bid-ask premia via utility-indifference.

**Definition 3.29.** At time 0, the protection buyer’s (resp. seller’s) indifference premium for $c \geq 0$ units of a CDS with a fixed unit upfront fee $f_0$ is defined by the total spread amount $\kappa^* \equiv \kappa^*(\gamma, c, f_0)$ (resp. $\tilde{\kappa}^* \equiv \tilde{\kappa}^*(\gamma, c, f_0)$) satisfying

$$\tilde{V}(0, x ; \gamma, 0, 0) = \tilde{V} \left(0, x - c f_0; \gamma, c, \frac{\kappa^*}{c}\right),$$

(3.54)

$$\tilde{V}(0, x ; \gamma, 0, 0) = \tilde{V} \left(0, x + c f_0; \gamma, -c, \frac{\tilde{\kappa}^*}{c}\right),$$

(3.55)
where $\bar{V}$ is defined in (3.31).

Applying (3.32) to (3.54) and (3.55), the buyer’s and seller’s indifference premia can be found from

$$v(\gamma, c, \frac{\kappa^*(\gamma, c, f_0)}{c}) = v(0) e^{-\gamma c f_0} \quad \text{and} \quad v(\gamma, -c, \frac{\bar{\kappa}^*(\gamma, c, f_0)}{c}) = v(0) e^{\gamma c f_0},$$

where $v$ is defined in (3.44). Among various properties of the utility-indifference premium, we highlight the risk aversion asymptotics:

$$\lim_{\gamma \to \infty} \frac{\kappa^*(\gamma, c, f_0)}{c} = 0, \quad \lim_{\gamma \to \infty} \frac{\bar{\kappa}^*(\gamma, c, f_0)}{c} = \infty, \quad \lim_{\gamma \to 0} \frac{\kappa^*(\gamma, c, f_0)}{c} = \lim_{\gamma \to 0} \frac{\bar{\kappa}^*(\gamma, c, f_0)}{c} = \bar{\kappa},$$

where the zero risk-aversion premium is

$$\bar{\kappa} = \frac{(\alpha - \frac{1}{2} \eta_1^2) \left( (1 - R) e^{-r T} - f_0 (\alpha - \frac{1}{2} \eta_1^2 + \lambda (e^{-\frac{1}{2} \eta_1^2 T} - e^{-\alpha T})) \right)}{\left( (\alpha - \frac{1}{2} \eta_1^2 - \lambda) T e^{-\alpha T} + \lambda (e^{-\frac{1}{2} \eta_1^2 T} - e^{-\alpha T}) \right)}.$$

In the special case when the upfront fee is zero ($f_0 = 0$) and the buyer and the seller do not incorporate investment in stocks ($\eta = \eta_1 = 0$), the zero risk-aversion premium reduces to

$$\bar{\kappa} = \lambda (1 - R) e^{-r T},$$

which is the well known credit triangle formula for risk-neutral CDS premium (see, for example, O’Kane [2011] Chap. 3.10). Finally, we also have the volume-scaling property: $\kappa^*(\gamma, c, f_0) = c \kappa^*(c \gamma, 1, f_0)$ and $\bar{\kappa}^*(\gamma, c, f_0) = c \bar{\kappa}^*(c \gamma, 1, f_0)$, which then yields the same large and small volume asymptotics as in (3.56).

### 3.4 Conclusion

In summary, we have discussed a utility-based trading mechanism for defaultable claims in markets with different pricing rules and agents with heterogeneous risk aversions and market views. Working with exponential utility, we
obtain formulas for the buyer’s and seller’s pricing rules and optimal trading positions for both defaultable bonds and CDS. These results allow us to better understand the interaction between market prices and risk aversion, market view, as well as trading volume. Most interestingly, the belief heterogeneity and the zero-risk aversion prices are important inputs for determining whether the buyer will trade with any seller and vice versa.

Our framework can be applied to study the trading of other financial derivatives, such as equity options, volatility derivatives, and insurance products. One extension of the static trading problem is to consider the optimal timing to buy or sell assets or derivatives (see Leung and Ludkovski [2012]). For these applications, explicit formulas for the buyer’s and seller’s indifference prices, if available, can greatly facilitate the analysis. To design alternative pricing rules, one can also apply results from risk measures, prospect theory, and other utility maximization approaches to the investor’s portfolio optimization problem, as long as the model remains amenable for analysis or numerically tractable.
Chapter 4

Impact of Central Counterparty Design on the Credit Default Swap Market

In this chapter, we consider a model for an inter-dealer market of credit default swaps (CDS) with a central counterparty (CCP). We analyze the equilibrium demands for clearing CDS among all dealers, especially its dependence on the design of the CCP. This involves solving the dealers’ portfolio optimization problems under the market clearing condition. We determine the minimum number of clearing participants required to create sufficient incentive for the dealers to use the clearing service. We also find that the CCP can increase the total clearing positions and its profit by reducing its initial margin level.

In Section 4.1 we formulate the equilibrium model and provide solution to the dealers’ mean-variance optimization problems. In Section 4.2 we characterize the equilibrium in closed-form formulas in the case with two heterogeneous groups. In Section 4.3 we implement our model using empirical data. In Section 4.4 we propose an optimal design of a CCP’s capital structure to maximize its profit.
4.1 Equilibrium of the CDS Inter-Dealer Market

We consider a CDS market with \( n_0 \) dealers. For each dealer \( i \), let \( \psi^{(i)} \) be the net client-dealer position of dealer \( i \). A positive (resp. negative) value of \( \psi^{(i)} \) represents net long (resp. short) positions. We denote the market-wide client-dealer positions by the vector

\[
\psi = (\psi^{(1)}, ..., \psi^{(n_0)}) \in \mathbb{R}^{n_0}.
\] (4.1)

We assume that the client-dealer positions are static, given as initial inputs.

In order to partially hedge their CDS positions, dealers establish CDS positions with peer dealers in the CDS inter-dealer market. We let \( \phi^{(i)} \) be the inter-dealer position of dealer \( i \) within the inter-dealer market. Among all \( n_0 \) dealers, the inter-dealer market positions are represented by the vector

\[
\phi = (\phi^{(1)}, ..., \phi^{(n_0)}) \in \mathbb{R}^{n_0}.
\] (4.2)

All inter-dealer positions have to satisfy the market clearing condition, defined by

\[
\sum_{i=1}^{n_0} \phi^{(i)} = 0.
\] (4.3)

To summarize, there are client-dealer positions \( \psi \) and inter-dealer market positions \( \phi \), but the inter-dealer market positions have to meet the market clearing condition \[4.3\].

4.1.1 Design of the CCP’s Capital Structure

Since inter-dealer contracts have to be cleared with the CCP, dealers, as the CCP’s clearing members, must post initial margins and guaranty funds. We let \( IM^{(i)} \) and \( GF^{(i)} \) be dealer \( i \)’s initial margin and guaranty fund contribution,
respectively. In practice, the initial margin is commonly set as the value-at-risk (VaR) of the CDS portfolio. The CCP approximates the 5-day return of dealer $i$’s position by the 5-day CDS spread return multiplied by the per-contract notional value $N$ and the absolute position $|\phi^{(i)}|$. In turn, the dealer $i$’s initial margin imposed by the CCP is specified as

$$IM^{(i)}(\phi^{(i)}) = c N |\phi^{(i)}|,$$  \hspace{1cm} (4.4)

where $c$ is the initial margin level\footnote{More details on the market conventions can be found in the document “CDS Clearing at ICE: A Unique Methodology” available at https://www.theice.com/publicdocs/ice_trust/FIA_magazine_CDS_risk_management_article.pdf}. The CCP has the freedom to choose the initial margin level based on empirical data. For instance, the CCP may use a parametric model that assumes a Gaussian distribution for the 5-day CDS return, and set the initial margin level $c$ as the 5-day volatility multiplied by 1.96 (99% quantile). Alternatively, the CCP may pick $c$ proportional to the 99% quantile of the empirical 5-day CDS returns.

When dealer $i$ defaults, the CCP has to find other participating dealers to assume the positions of the defaulting dealer $i$ by holding an auction. This auction will determine the liquidation value of dealer $i$’s positions, and the CCP will have to pay the auction winner(s) the cash amount

$$LC^{(i)}(\phi^{(i)}) = b N |\phi^{(i)}|,$$  \hspace{1cm} (4.5)

where $b$ is called the liquidation cost level. The constant $b$ is usually determined from historical CDS returns.

The liquidation cost will be partially covered by the defaulting dealer’s initial margin. The remaining cost will be paid for by the guaranty fund in the CCP. We name this amount the excess loss over margin associated with dealer $i$. This excess loss over margin is very important in terms of systemic risk, since it can indirectly affect other members’ wealth. Indeed, the CCP may
utilize others’ guaranty fund contributions to cover excess losses. Therefore, regulators now require that the initial margin level $c$ to be set between a lower bound $c_{min} > 0$ and the upper bound $b$. As a result, the excess loss over margin of dealer $i$ is given by

$$EL^{(i)}(\phi^{(i)}) = (b - c) N |\phi^{(i)}|, \quad \text{for } c \in [c_{min}, b]. \quad (4.6)$$

The Bank for International Settlements suggests that the total guaranty fund, $GF$, be the maximum over all existing bilateral counterparties the sum of two members’ excess losses conditioned on the two members’ simultaneous default. As such, the total guaranty fund is defined by

$$GF(\phi) = \max_{(i,j), i \neq j} \left\{ EL^{(i)}(\phi^{(i)}) + EL^{(j)}(\phi^{(j)}) \right\}. \quad (4.7)$$

This fund is obtained by clearing members’ contributions. A rule that is currently adopted by many CCPs specifies the guaranty fund contribution of dealer $i$ by

$$GF^{(i)}(\phi) = GF(\phi) \frac{EL^{(i)}(\phi^{(i)})}{\sum_{j=1}^{\infty} EL^{(j)}(\phi^{(j)})}. \quad (4.8)$$

Under this rule, the guaranty fund contribution is calculated based on individual and aggregate excess losses. One can interpret this rule as capturing each clearing member’s contribution to systemic risk that triggers other members’ losses in extreme market situations. Also, note that dealer $i$’s guaranty fund contribution is a function of the positions of all dealers $\phi$ in the inter-dealer market.

In addition, the CCP charges a fee rate $f$ for its clearing service. Dealer $i$’s total amount of clearing fee is given by $f N |\phi^{(i)}|$. In summary, the design of the CCP’s clearing mechanism can be described by the control variables: the initial margin level $c$, the liquidation cost $b$, the clearing fee $f$. 
4.1.2 Mean-Variance Optimization of Dealers

We now turn to the dealers’ mean-variance optimization problems and determine their optimal clearing positions. For each dealer $i$, the 1-period return of each CDS contract is given by the increment in the CDS spread. We model the future CDS spread by a Gaussian random variable $\mathcal{N}(\mu^{(i)}, \sigma^2)$ with mean $\mu^{(i)}$ and variance $\sigma^2$, and the current spread is denoted by $\xi$. Accounting for dealer $i$’s total CDS positions $(\psi^{(i)} + \phi^{(i)})$ and the notional value $N$, the total return to dealer $i$ is given by

$$
(\psi^{(i)} + \phi^{(i)}) N \mathcal{N}(\mu^{(i)} - \xi, \sigma^2).
$$

(4.9)

Since every dealer has to post an initial margin and guaranty fund to the CCP by cash or cash equivalent assets, a funding cost arises from depositing the initial margin and guaranty fund, in addition to the clearing fee. In total, we have

$$
r(IM^{(i)}(\phi^{(i)}) + GF^{(i)}(\phi)) + f |\phi^{(i)}| N,
$$

(4.10)

where $r$ is funding cost rate over a single period.

Moreover, in the case of one or more participant dealers’ defaults, the non-defaulting dealers may lose their guaranty fund contributions due to the losses caused by the defaults while their initial margins are secure. As a consequence, the presence of a CCP also introduces counterparty risk faced by the member dealers. For each dealer $i$, we let random variable $L^{(i)}$ be the loss of its guaranty fund contribution. We assume it is uniformly distributed between 0 and $GF^{(i)}(\phi)$ when such a loss occurs with probability $p$, namely,

$$
L^{(i)} = \begin{cases} 
Unif(0, GF^{(i)}(\phi)) & \text{with probability } p, \\
0 & \text{with probability } 1 - p.
\end{cases}
$$

(4.11)
Combining (4.10), (4.11) and (4.12), the increment of dealer $i$’s wealth over one period is given by

$$\Delta W^{(i)} = (\psi^{(i)} + \phi^{(i)})N(\mu^{(i)} - \xi, \sigma^2) N - f|\phi^{(i)}| N - r(IM^{(i)} + GF^{(i)}(\phi)) - L^{(i)}.$$  

(4.12)

Each dealer selects the optimal inter-dealer position $\phi^{(i)}$ to maximize the objective function

$$E[\Delta W^{(i)}] - \frac{\alpha^{(i)}}{2} Var[\Delta W^{(i)}],$$

where $\alpha^{(i)}$ is the positive risk aversion parameter for dealer $i$. We denote the dealers’ risk aversions by $\alpha = (\alpha^{(1)}, ..., \alpha^{(n_0)})$. From (4.10), (4.11) and (4.12), we get

$$E[\Delta W^{(i)}] = (\psi^{(i)} + \phi^{(i)}) \mu^{(i)} - \xi N - f|\phi^{(i)}| N - r(IM^{(i)} + GF^{(i)}(\phi)) - p\frac{GF^{(i)}(\phi)}{2},$$

$$Var[\Delta W^{(i)}] = (\psi^{(i)} + \phi^{(i)})^2 \sigma^2 N^2 + \left(\frac{p}{3} - \frac{p^2}{4}\right) (GF^{(i)}(\phi))^2.$$

**Problem 4.1. (Mean-Variance Optimization)** The optimal inter-dealer position for dealer $i$, $i = 1, \ldots, n_0$, is given by

$$\phi^{*(i)} = \arg \max_{\phi^{(i)}} (\psi^{(i)} + \phi^{(i)}) \mu^{(i)} - \xi N - f|\phi^{(i)}| N - r(IM^{(i)} + GF^{(i)}(\phi))$$

$$- p\frac{GF^{(i)}(\phi)}{2} - \frac{\alpha^{(i)}}{2} \left( (\psi^{(i)} + \phi^{(i)})^2 \sigma^2 N^2 + \left(\frac{p}{3} - \frac{p^2}{4}\right) (GF^{(i)}(\phi))^2\right).$$  

(4.13)

We remark that the dealers solve their mean-variance optimization problems based on the inputs $(b, c, f)$ that are specified by the CCP. As defined in Section 4.1.1 dealer $i$’s guaranty fund contribution $GF^{(i)}(\phi)$ depends not only dealer $i$’s positions $\phi^{(i)}$ but also others dealers’ positions $\phi^{\setminus(i)} = (\phi^{(1)}, ..., \phi^{(i-1)}, \phi^{(i+1)}, ..., \phi^{(n)})$.

Due to the anonymity of CDS transactions, each dealer does not know other dealers’ positions. This lack of information may be more difficult for each dealer
to compute its own optimal position. Hence, we impose the following standing assumption.

**Assumption 4.2.** Each dealer \( i \) has full knowledge of the values of \( GF^{(i)}(\phi) \) and \( \frac{\partial GF^{(i)}}{\partial \phi^{(i)}}(\phi) \).

Assumption 4.2 means that each dealer has the sufficient information its guaranty fund contribution, even though the dealers do not know other dealers’ positions. This information can be supplied by the CCP without revealing the identities of dealers.

Next, we provide the solutions to the dealer’s mean-variance optimization problems. To facilitate the notation, we denote \( GF^{(i)}(\phi^{(i)}) \equiv GF^{(i)}(\phi) \).

**Proposition 4.3.** Dealer \( i \)’s mean-variance optimal position is given as follows.

1. The optimal clearing position is given by \( \phi^{\ast(i)} = 0 \) if and only if

\[
\frac{(\mu^{(i)} - \xi - f) N - r c N - (r + \frac{p}{2}) \frac{\partial GF^{(i)}}{\partial \phi^{(i)}}(0)}{\alpha^{(i)} \sigma^2 N^2} \leq \psi^{(i)} \quad \text{and} \quad \psi^{(i)} \leq \frac{(\mu^{(i)} - \xi + f) N + r c N - (r + \frac{p}{2}) \frac{\partial GF^{(i)}}{\partial \phi^{(i)}}(0)}{\alpha^{(i)} \sigma^2 N^2}.
\]

(4.14)

2. The optimal clearing position \( \phi^{\ast(i)} < 0 \) is a unique solution of the following equation

\[
\alpha^{(i)} \sigma^2 N^2 \phi^{\ast(i)} + \left( r + \frac{p}{2} \right) \frac{\partial GF^{(i)}}{\partial \phi^{(i)}}(\phi^{\ast(i)}) + \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha^{(i)} GF^{(i)}(\phi^{\ast(i)}) \cdot \frac{\partial GF^{(i)}}{\partial \phi^{(i)}}(\phi^{\ast(i)})
\]

\[
= (\mu^{(i)} - \xi + f) N + r c N - \alpha^{(i)} \sigma^2 N^2 \psi^{(i)},
\]

(4.15)

if and only if

\[
\psi^{(i)} > \frac{(\mu^{(i)} - \xi + f) N + r c N - (r + \frac{p}{2}) \frac{\partial GF^{(i)}}{\partial \phi^{(i)}}(0)}{\alpha^{(i)} \sigma^2 N^2}.
\]

(4.16)
3. The optimal clearing position \( \phi^{*}(i) > 0 \) is a unique solution of the following equation

\[
\alpha^{(i)} \sigma^2 N^2 \phi^{*}(i) + \left( r + \frac{p}{2} \right) \frac{\partial G^F(i)}{\partial \phi^{(i)}}(\phi^{*}(i)) + \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha^{(i)} G^F(i)(\phi^{*}(i)) \cdot \frac{\partial G^F(i)}{\partial \phi^{(i)}}(\phi^{*}(i)) \\
= (\mu^{(i)} - \xi - f) - r c N - \alpha^{(i)} \sigma^2 N^2 \psi^{(i)},
\]

if and only if

\[
\psi^{(i)} < \frac{(\mu^{(i)} - \xi - f) N - r c N - (r + \frac{p}{2}) \frac{\partial G^F(i)}{\partial \phi^{(i)}}(0)}{\alpha^{(i)} \sigma^2 N^2}.
\]

Corollary 4.4. The optimal clearing position of each dealer \( i \), \( \phi^{*}(i) \) is increasing in \( \mu^{(i)} \), \( 1 \leq i \leq n_0 \).

In particular, if the market spread \( \xi \) increases, then every expected spread return \( \mu^{(i)} - \xi \) decreases. This in turn reduces the dealer’s position \( \phi^{*}(i) \).

4.1.3 Inter-Dealer Market Equilibrium

We now discuss the equilibrium in terms of the dealers’ clearing positions. Given an arbitrary CDS spread \( \xi \), we denote the inter-dealer market position of dealer \( i \) by \( \phi^{*}(i)(\xi, \phi^{\ast\ast}(i)) \). There are two requirements for the equilibrium of dealers’ clearing positions. The equilibrium is obtained by solving the following problem.

Problem 4.5. (Inter-Dealer Market Equilibrium) The equilibrium of inter-dealer market positions \( (\phi^{*}(1)(\xi^*), \ldots, \phi^{*}(n_0)(\xi^*)) \) and the equilibrium CDS spread \( \xi^* \) together satisfy

\[
\phi^{*}(i)(\xi^*, \phi^{\ast\ast}(i)) \text{ is a solution to Problem 4.3 for } 1 \leq i \leq n_0,
\]

where

\[
\sum_{i=1}^{n_0} \phi^{*}(i)(\xi^*, \phi^{\ast\ast}(i)) = 0.
\]
Let us discuss the intuition behind this equilibrium problem. Given an arbitrary initial spread $\xi$, the market clearing condition may be violated. For instance, if
\[ \sum_{i=1}^{n_0} \phi^*(i)(\xi, \phi^*(i)) > 0, \]
this implies that there is a higher aggregate demand for long protection than for short positions. Dealers will a demand for positive positions will offer to pay at a higher CDS spread, which in turns triggers a higher demand for the opposite positions. This will continue until the demand for long and short positions are equal.

### 4.2 Two Heterogeneous Groups

In search of explicit solutions, we now assume that there exist only two heterogeneous groups representing protection buyers and protection sellers in the inter-dealer market. Suppose there are $n = n_0/2$ number of dealers in each group. Within each group, the members are homogeneous with identical parameters. Across the two groups, dealers have the same risk aversion parameter $\alpha$ but all other parameters can differ. We define $(\phi^{*(1)}, \phi^{*(2)})$ as an inter-dealer market equilibrium where $\phi^{*(i)}$ is the equilibrium position for group $i \in \{1, 2\}$. With only two groups, the market clearing condition is simplified to $\phi^{*(1)} = -\phi^{*(2)}$.

#### 4.2.1 Inter-Dealer Market Equilibrium

The required initial margin for every dealer in both groups is $IM^{(i)} = |\phi^*| c N$, where $\phi^* := \phi^{*(1)}$. The total amount of guaranty fund is given by

\[ GF(\phi^*) = 2 \max_{1 \leq i, j \leq 2} \{ |\phi^{*(1)}| + |\phi^{*(2)}| \} (b - c) N = 2 |\phi^*|(b - c) N. \]
The guaranty fund contribution of a dealer in group \( i \in \{1, 2\} \) is

\[
GF^{(i)}(\phi^*) = \frac{GF(\phi)}{2n} = \frac{(b - c) N}{n} |\phi^*|.
\]

We are interested in the inter-dealer market equilibrium positions \((\phi^*, -\phi^*)\). In particular, we want to identify the condition under which a non-zero equilibrium exists. Without loss of generality, we assume that

\[
\mu^{(1)} - \alpha \sigma^2 \psi^{(1)} > \mu^{(2)} - \alpha \sigma^2 \psi^{(2)},
\]

where the superscripts indicate the group number.

**Theorem 4.6. (Two-Group Inter-dealer Market Equilibrium)** The sufficient and necessary condition for the existence of the non-trivial equilibrium is

\[
\frac{1}{2} \left( \mu^{(1)} - \mu^{(2)} + \alpha N \sigma^2 (\psi^{(1)} + \psi^{(2)}) \right) - f - r c - \left( r + \frac{p}{2} \right) \left( \frac{b - c}{n} \right) > 0.
\]

(4.21)

Under this condition, the unique inter-dealer market equilibrium position is given by

\[
\phi^* = \frac{1}{2} \left( \mu^{(1)} - \mu^{(2)} + \alpha N \sigma^2 (\psi^{(1)} + \psi^{(2)}) \right) - f - r c - \left( r + \frac{p}{2} \right) \left( \frac{b - c}{n} \right) > 0.
\]

(4.22)

The market equilibrium CDS spread \( \xi^* \) is given by

\[
\xi^* = \frac{\mu^{(1)} + \mu^{(2)} + \alpha N \sigma^2 (\psi^{(1)} + \psi^{(2)})}{2}.
\]

Otherwise, the equilibrium position is trivial, i.e. \( \phi^* = 0 \), for every dealer in both groups.

Alternatively, we can express the condition for the existence of a non-trivial equilibrium in terms of the group size \( n \).
Corollary 4.7. There exists a non-trivial equilibrium if and only if

\[ n > \frac{(r + \frac{b}{2}) (b - c)}{\bar{\mu} - f + \alpha N \sigma^2 \bar{\psi} - r c}, \]  

(4.23)

and

\[ \bar{\mu} + \alpha N \sigma^2 \bar{\psi} > r c + f, \]  

(4.24)

where \( \bar{\mu} := (\mu^{(1)} - \mu^{(2)})/2 \) and \( \bar{\psi} := (\psi^{(1)} + \psi^{(2)})/2 \).

Dealers use inter-dealer market contracts which have to be cleared in the CCP only if the clearing benefit is greater than the other two factors, the funding cost and counterparty risk.

This result provides a number of insights about the equilibrium. There are three major factors affecting the dealers’ clearing positions: (1) the clearing benefit, (2) the initial margin and guaranty fund contribution cost, and (3) the counterparty risk. The condition (4.24) implies that dealers clear CDSs only if the benefit of clearing (left-hand side) is greater than the funding cost of the initial margin plus the clearing fee (right-hand side). The benefit perceived by each dealer is increasing in the volatility of the CDS spread, the risk aversion parameter, and the average client-dealer position in the market. However, this condition alone may not be sufficient to trigger an inter-dealer market trade. A minimum number of dealers is required to make the clearing benefit outweighs the counterparty risk and funding cost generated by the dealers’ contribution to the guaranty fund (see 4.23). Consequentially, this guarantees a non-trivial inter-dealer market equilibrium. To summarize,

- Dealers will clear only if the benefit of clearing outweighs the costs (see 4.24).

- If condition (4.24) holds, then dealers will clear only if a sufficient number of members are present in the market (see 4.23).
4.2.2 Impact of Initial Margin Level Change on Equilibrium

We now examine the effect of changing the initial margin level \( c \) on the equilibrium of inter-dealer market positions \((\phi^*, -\phi^*)\). To this end, we compute from (4.22) that

\[
\frac{\partial \phi^*}{\partial c} = \frac{\frac{3}{2} \alpha N \phi^* (\frac{b-c}{n}) + \frac{1}{2} p - (n-1)r}{n \alpha \left( \sigma + \left( \frac{p}{3} - \frac{p^2}{4} \right) \left( \frac{b-c}{n} \right)^2 \right)} \quad \text{for} \quad c \in [0, b].
\]

(4.25)

This result shows that the sign of the sensitivity is determined by the two factors: the guaranty fund default probability \( p \) and funding rate \( r \). On one hand, an increase of the initial margin level decreases the exposure of clearing members to defaults of other dealers. This counterparty risk effect pushes the equilibrium demand up. Since the exposure is the amount of guaranty fund contribution, this effect depends crucially on the guaranty fund default probability \( p \). On the other hand, an increase in the initial margin level raises the total amount of required collateral, and thus, the cost of clearing. This cost effect pushes the equilibrium demand down, and depends on the funding rate \( r \).

If the counterparty risk effect dominates the cost effect, the equilibrium position increases as the initial margin level increases. Otherwise, the equilibrium position is reduced as the initial margin level decreases. We provide two sufficient conditions that determine the direction of the impact.

Proposition 4.8. The equilibrium clearing position \( \phi^* \) is increasing in the initial margin level \( c \) if

\[
n \leq 1 + \frac{p}{2r}.
\]

(4.26)

In contrast, \( \phi^* \) is decreasing in the initial margin level \( c \) if

\[
n \geq \frac{1}{2} \left( \left( 1 + \frac{p}{2r} \right) + \sqrt{\left( 1 + \frac{p}{2r} \right)^2 + \left( \frac{p}{3} - \frac{p^2}{4} \right) \frac{8b(\bar{\mu} + \alpha N \bar{\psi} \sigma^2)}{r \sigma^2}} \right).
\]

(4.27)
The first condition (4.26) tends to hold if the guaranty fund default probability $p$ is significantly higher than the funding rate $r$. This means that dealers the counterparty risk outweighs the cost effect. When the CCP decreases the initial margin level $c$, the total funding cost of a dealer decreases while the guaranty fund contribution and the corresponding counterparty risk exposure of the dealer increases. As a consequence, the dealers seek to reduce counterparty risk by reducing their inter-dealer market positions. The second condition (4.27) corresponds to the case with sufficiently many dealers in the market and low guaranty fund default probability $p$. In this scenario, when the CCP increases the initial margin level $c$, the increase in total funding cost is the driving factor that triggers the dealers to reduce their inter-dealer market positions.

### 4.3 Numerical Results

In this section, we present some numerical results to illustrate our model. We continue to assume the market with two heterogeneous groups with $n = 5$ dealers each. Following the statistics in Table 4.3, we assume that every dealer in group 1 has $8.56B notional of client-dealer short positions and every dealer in group 2 has $8.56B notional of client-dealer long positions. If we set the notional of a CDS contract $N = \$10M$, $\psi^{(1)} = -\psi^{(2)} = -856$. Drawing from data of major clearing houses such as the Inter-continental Exchange (ICE), and Chicago Mercantile Exchange (CME), we set the liquidation cost level $b = 6\%$ and the volatility of CDS price during the period, $\sigma = 15\%$.

Corollary 4.7 gives the sufficient and necessary condition of the existence of a non-zero equilibrium. As observed in the left panel of Figure 4.1 the threshold of the number of clearing members in the CCP is less than one for all risk aversion parameters. This means that for all 10 dealers will initiate


<table>
<thead>
<tr>
<th>Clearing Member Firm</th>
<th>Short Positions</th>
<th>Long Positions</th>
<th>Net</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clearing Member 1</td>
<td>16,917</td>
<td>1,275</td>
<td>(15,642)</td>
</tr>
<tr>
<td>Clearing Member 2</td>
<td>14,497</td>
<td>1,605</td>
<td>(12,892)</td>
</tr>
<tr>
<td>Clearing Member 3</td>
<td>13,267</td>
<td>1,861</td>
<td>(11,406)</td>
</tr>
<tr>
<td>Clearing Member 4</td>
<td>923</td>
<td>17,228</td>
<td>16,305</td>
</tr>
<tr>
<td>Clearing Member 5</td>
<td>705</td>
<td>6,472</td>
<td>5,767</td>
</tr>
<tr>
<td>Clearing Member 6</td>
<td>2,527</td>
<td>8,194</td>
<td>5,667</td>
</tr>
<tr>
<td>Clearing Member 7</td>
<td>12,148</td>
<td>13,755</td>
<td>1,607</td>
</tr>
<tr>
<td>Clearing Member 8</td>
<td>4,911</td>
<td>2,065</td>
<td>(2,846)</td>
</tr>
<tr>
<td>Clearing Member 9</td>
<td>3,596</td>
<td>7,455</td>
<td>3,859</td>
</tr>
<tr>
<td>Clearing Member 10</td>
<td>6,999</td>
<td>16,580</td>
<td>9,581</td>
</tr>
<tr>
<td>Absolute Total</td>
<td>76,490</td>
<td>76,490</td>
<td>85,572</td>
</tr>
</tbody>
</table>

Table 4.1: The CDS gross notionals of clearing members (in US$ million) on September 9th, 2009

inter-dealer trades and clear via the CCP. Moreover, as seen in the right panel of Figure 4.1, as the dealer’s risk aversion increases, the equilibrium position moves closer to the average number of client-dealer contracts $\bar{\psi} = 856$.

As shown in in Figure 4.2, the inter-dealer market equilibrium decreases both in opportunity cost $r$ and default probability $p$. Also note that in the range of funding rate and guaranty fund default probability, the equilibrium position decreases in initial margin level. The expected future CDS spreads of two groups, $\mu^{(1)}$ and $\mu^{(2)}$, have the opposite impact on the equilibrium position. As we observe from Figure 4.3, the equilibrium position is increasing in $\mu^{(1)}$ and decreasing in $\mu^{(2)}$. Figure 4.4 shows that the the minimum number of dealers in each group that guarantees the monotonicity of the equilibrium position with respect to the initial margin level $c$ (see (4.27)). In both panels, we see that as long as there are two or more dealers in each group, the equilibrium
Figure 4.1: The minimum number of dealers for a non-trivial equilibrium
The minimum number of dealers in each group (left) and the equilibrium position (right) as a function of risk aversion. Other parameters: \( \{\mu_0^{(1)}, \mu_0^{(2)}, r, p\} = \{10\text{bps}, -10\text{bps}, 1\%, 1\%\} \).

positions are decreasing in the initial margin level.

4.4 Optimal Design of CCPs

We now consider from the CCP’s perspective and determine the optimal policy that maximizes its revenue in the two-group setting. The CCP’s revenue is the fee collected from all the clearing transactions of its clearing members, that is, \( 2n|\phi^*(i)|f \). Recall that the design of a CCP is described the two control variables \( c \) and \( f \). They in turn affect the equilibrium positions of the dealers, denoted by \( \phi^*(i)(c, f) \), whose formula is given by (4.22). Under the non-triviality condition (4.21), the CCP’s revenue maximization problem is

\[
\max_{c, f} \left( \frac{1}{2} (\mu^{(1)} - \mu^{(2)} + \alpha N \sigma^2 (-\psi^{(1)} + \psi^{(2)})) f - r c - (r + \frac{p}{2}) \left( \frac{b-c}{n} \right) \right)
\]

subject to \( c_{\text{min}} \leq c \leq b \) and \( f \geq 0 \). (4.28)

In addition, if (4.26) holds, then the monotonicity result suggests that the
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Figure 4.2: Equilibrium positions for different funding rates and guaranty fund default probabilities

Left: The equilibrium positions for different funding rates $r$, with $p$ fixed at 1%.

Right: The equilibrium positions for different guaranty fund default probabilities $p$, with $r = 1\%$ fixed. Other parameters: $\{\mu^{(1)}, \mu^{(2)}, \alpha\} = \{10bps, -10bps, 1.5 \times 10^{-5}\}$.

Figure 4.3: The minimum number of dealers for a non-trivial equilibrium

The minimum number of dealers in each group (left) and the equilibrium position (right) as a function of $\mu^{(1)}_0$. Other parameters $\{\alpha, r, p\} = \{1.5 \times 10^{-5}, 1\%, 1\%\}$. 
Chapter 4. Impact of Central Counterparty Design on the Credit Default Swap Market

The minimum number of dealers in each group that guarantees the monotonicity of the equilibrium position with respect to the initial margin level $c$. Left: $\{\mu_0^{(1)}, \mu_0^{(2)}, r, p\} = \{10\text{bps}, -10\text{bps}, 1\%, 1\%\}$. Right: $\{\mu_0^{(1)}, \mu_0^{(2)}, r, p\} = \{-10\text{bps}, 10\text{bps}, 1\%, 1\%\}$.

optimal solution is $c^* = b$. If (4.27) holds, then the optimal initial margin level $c^*$ is simply the lower bound $c_{\text{min}}$. In both cases, the optimal fee is

$$f^* = \frac{1}{2} \left( \mu^{(1)} - \mu^{(2)} + \alpha N \sigma^2 (-\psi^{(1)} + \psi^{(2)}) \right) - r c^* - \left( r + \frac{p}{2} \right) \left( \frac{b - c^*}{n} \right),$$

and the optimal revenue is

$$\left[ \frac{1}{2} \left( \mu^{(1)} - \mu^{(2)} + \alpha N \sigma^2 (-\psi^{(1)} + \psi^{(2)}) \right) - r c^* - \left( r + \frac{p}{2} \right) \left( \frac{b - c^*}{n} \right) \right]^2.$$

Figure 4.5 illustrates the optimal initial margin level $c^*$ as a function of guaranty fund default probability $p$ and funding cost rate $r$. In the range where $p$ is less than $3\%$, the revenue-maximizing initial margin level $c^*$ is the lowest admissible level $c_{\text{min}}$. Otherwise, the CCP will choose the upper bound $b$ as the initial margin level. The bottom panel of Figure 4.5 shows that the optimal revenue is increasing as both $r$ and $p$ decrease.
Figure 4.5: The CCP’s optimal margin level and optimal revenue
Top: The optimal initial margin level $c^*$. Bottom: The optimal revenue of the CCP.
Other parameters: $\{\mu_0^{(1)}, \mu_0^{(2)}, \alpha\} = \{10bps, -10bps, 1.5 \times 10^{-5}\}$. 
Bibliography


Appendix A

Proofs for Chapter 2

A.1 Proof of Theorem 2.5

For \( w \in C^{1,2}_b([0,T] \times \mathcal{D}, \mathbb{R}) \) and the operator \( \mathcal{M} \) in (2.19), we define \( v \equiv v(t,s,x) \) by

\[
v := \mathcal{M} w = \mathbb{E}_{t,s,x} \left[ e^{-\int_t^T \bar{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \bar{r}_v \, dv} f(u, S_u, X_u, w(u, S_u, X_u)) \, du \right].
\] (A.1)

Equation (A.1) admits the same form as (1.2) of Heath and Schweizer [2000]. By Theorem 1 of Heath and Schweizer [2000], \( v \) is a classical solution \( (C^1, 2b([0,T] \times \mathcal{D}, \mathbb{R})) \) of the PDE:

\[
\frac{\partial v(t,s,x)}{\partial t} + \mathcal{L} v(t,s,x) - \bar{r}(t,s,x) v(t,s,x) + f(t,s,x, w(t,s,x)) = 0,
\] (A.2)

for \((t,s,x) \in [0,T] \times \bar{D}_n\). Combined with condition (C7), it implies that the composition \( (t,s,x) \to f(t,s,x, w(t,s,x)) \) is uniformly Hölder-continuous on \([0,T] \times \bar{D}_n \times \mathbb{R}\), thus satisfying \( (A3d') \). Lastly, the boundedness
of \( v \) from Lemma 2.3 corresponds to \((A3e')\). Therefore, we conclude that \( v \) is a bounded classical solution (i.e. \( C^{1,2}_b([0, T) \times \mathcal{D}, \mathbb{R}) \)) of the PDE (A.2) for \((t, s, x) \in [0, T) \times D\).

Now, let’s select the initial function \( P(0) \in C^{1,2}_b([0, T] \times \mathcal{D}, \mathbb{R}) \), e.g. \( P(0) = 0 \). Then, the subsequent functions \( P(n) = \mathcal{M}P^{(n-1)}, \) \( n = 1, 2, \ldots \), are also \( C^{1,2}_b([0, T] \times \mathcal{D}, \mathbb{R}) \) and satisfy the linear inhomogeneous PDE (2.22). By Proposition 2.4, the contraction mapping \( \mathcal{M} \) ensures the sequence \( (P(n)) \) to converge to a unique fixed point \( P \in C_b([0, T) \times \mathcal{D}, \mathbb{R}) \).

### A.2 Proof of Proposition 2.8

First, we denote \( \tilde{\lambda} := \lambda^{(1)} + \lambda^{(2)} \) and \( \tilde{h}(t, s) := h(t, s) + \lambda^{(0)} l(t, s) \). Applying the positive payoff to the definition of \( P^b = P \) in (2.14), the MtM value with CR provision is given by

\[
P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) du \
+ \int_t^T (\tilde{\lambda} - \alpha) e^{-(r+\lambda)(u-t)} P^b(u, S_u) du \right].
\] (A.3)

To prove (2.33), we substitute it into the RHS of (A.3) and verify that it
indeed reduces to $2.33$. To this end, we get

\[
P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du \right.
\]

\[
+ \int_t^T (\lambda - \alpha) e^{-(r+\lambda)(u-t)} \left\{ e^{-(r+\alpha+\lambda(0))(T-u)} g(S_T) + \int_u^T e^{-(r+\alpha+\lambda(0))(v-u)} \tilde{h}(v, S_v) \, dv \right\} \, du \]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda(0))(T-t)} g(S_T) \right.
\]

\[
+ \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du + \int_t^T (\lambda - \alpha) e^{-(\lambda - \alpha)u - r(v-t) + (-\alpha - \lambda(0))v + \lambda t} \tilde{h}(v, S_v) \, dv \, du \)
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda(0))(T-t)} g(S_T) \right.
\]

\[
+ \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du + \int_t^T (e^{-(r+\alpha+\lambda(0))(u-t)} - e^{-(r+\lambda)(u-t)}) \tilde{h}(u, S_u) \, du \]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda(0))(T-t)} g(S_T) + \int_t^T e^{-(r+\alpha+\lambda(0))(u-t)} \tilde{h}(u, S_u) \, du \right].
\]

Since the last equality resembles $2.33$, we conclude. The same steps will yield the proof for expression $2.34$.

To verify $2.35$, we use the expressions of $\Pi$ in $2.4$ and $\tilde{P}^b = \tilde{P}$ in $2.15$ to get

\[
\tilde{P}^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du \right.
\]

\[
- \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du + \int_t^T \tilde{\lambda} e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right.
\]

\[
+ \int_t^T \tilde{\lambda} e^{-(r+\lambda)(u-t)} \left[ \int_u^T e^{-(r+\lambda(0))(v-u)} \tilde{h}(v, S_v) \, dv + e^{-(r+\lambda(0))(T-u)} g(S_T) \right] \, du \]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\lambda(0))(T-t)} g(S_T) - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right.
\]

\[
+ \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du + \int_t^T (1 - e^{-\tilde{\lambda}(v-t)}) e^{-(r+\lambda(0))(v-t)} \tilde{h}(v, S_v) \, dv \]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\lambda(0))(T-t)} g(S_T) - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du + \int_t^T e^{-(r+\lambda(0))(u-t)} \tilde{h}(u, S_u) \, du \right]
\]

\[
= \Pi(t, s) - \mathbb{E}_{t,s} \left[ \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right].
\]
Applying the same steps to the definition of $\hat{P}^s$ in (2.18), we obtain the equation (2.36).

**A.3 Proof of Proposition 2.10**

From (2.35) and (2.36) and the condition $\alpha, \beta \geq 0$, we obtain the inequalities $\hat{P}^b(t, s) \leq \Pi(t, s)$ and $\hat{P}^s(t, s) \leq \Pi(t, s)$. The price expressions (2.33) and (2.34) imply that

$$P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-\int_t^T (r+\alpha+\lambda(0)) \, du} g(S_T) + \int_t^T e^{-\int_u^T (r+\alpha+\lambda(0)) \, dv} \tilde{h}(u, S_u) \, du \right]$$

$$\leq \mathbb{E}_{t,s} \left[ e^{-\int_t^T (r+\lambda(0)) \, du} g(S_T) + \int_t^T e^{-\int_u^T (r+\lambda(0)) \, dv} \tilde{h}(u, S_u) \, du \right] = \Pi(t, s).$$

(A.4)

Similar arguments give $P^s(t, s) \leq \Pi(t, s)$. Hence, we conclude (2.39).

Next, applying the definition of $\hat{P}^b \equiv \hat{P}$ in (2.15) along with the inequality (A.4), we get

$$\hat{P}^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(T-t)(r+\lambda)} g(S_T) + \int_t^T e^{-(u-t)(r+\lambda)} \tilde{h}(u, S_u) \, du \right]$$

$$\geq \mathbb{E}_{t,s} \left[ e^{-(T-t)(r+\lambda)} g(S_T) + \int_t^T e^{-(u-t)(r+\lambda)} \tilde{h}(u, S_u) \, du \right]$$

$$+ \int_t^T (\lambda - \alpha) e^{-(u-t)(r+\lambda)} \Pi(u, S_u) \, du$$

$$= P^b(t, s).$$

The last equality follows from the definition of $P^b$ in (2.14). From the definition
of $\hat{P}^s$ in (2.18) and the inequality $0 \leq P^s \leq \Pi$ in (2.39), we obtain

$$\hat{P}^s(t, s) = \mathbb{E}_{t,s} \left[ e^{-(T-t)(r+\lambda)} g(S_T) + \int_t^T e^{-(u-t)(r+\lambda)} \tilde{h}(u, S_u) \, du + \int_t^T (\tilde{\lambda} - \beta) e^{-(u-t)(r+\lambda)} \Pi(u, S_u) \, du \right]$$

$$\geq \mathbb{E}_{t,s} \left[ e^{-(T-t)(r+\lambda)} g(S_T) + \int_t^T e^{-(u-t)(r+\lambda)} \tilde{h}(u, S_u) \, du + \int_t^T (\tilde{\lambda} - \beta) e^{-(u-t)(r+\lambda)} P^s(u, S_u) \, du \right]$$

$$= P^s(t, s).$$

The last equality holds from the definition of $P^s$ in (2.16). Hence, we conclude (2.40).
Appendix B

Proofs for Chapter 4

B.1 Proof of Proposition 4.3

We can solve this problem by treating the two cases, $\phi^{(i)} \leq 0$ and $\phi^{(i)} \geq 0$, separately. Given that $\phi^{(i)} \leq 0$, the maximization problem becomes

$$
\max_{\phi^{(i)}} \phi^{(i)} \mu^{(i)} - r (c \sigma |\phi^{(i)}| + GF^{(i)}) - \frac{p}{2} GF^{(i)}
$$

- $\frac{\alpha^{(i)}}{2} \left( (\psi^{(i)} + \phi^{(i)})^2 \sigma^2 + \left( \frac{p}{3} - \frac{p^2}{4} \right) (GF^{(i)})^2 \right)$

subject to $\phi^{(i)} \leq 0$.

The first order condition (FOC) condition implies that the optimal solution $\phi^{*(i)}$ satisfies

$$
\mu^{(i)} + r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} - \alpha^{(i)} \sigma^2 \phi^{*(i)} - \left( r + \frac{p}{2} \right) \frac{\partial GF^{(i)}}{\partial \phi^{(i)}} (\phi^{*(i)}) - \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha^{(i)} GF^{(i)} (\phi^{*(i)}) \cdot \frac{\partial GF^{(i)}}{\partial \phi^{(i)}} (\phi^{*(i)}) - \nu = 0 \quad \text{(FOC)}
$$

$$
\nu \geq 0 \quad \text{(Dual Feasibility)}
$$

$$
\phi^{*(i)} \leq 0 \quad \text{(Primary Feasibility)}
$$

$$
\nu \cdot (\phi^{*(i)}) = 0 \quad \text{(Complimentary Slackness)}
$$
where $\nu$ is the Lagrangian multiplier. We can restate the FOC as

$$-\nu + \mu^{(i)} + r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} = \alpha^{(i)} \sigma^2 \phi^{* (i)} + \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (\phi^{* (i)})$$

$$+ \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha^{(i)} G F^{(i)} (\phi^{* (i)}) \cdot \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (\phi^{* (i)}), \quad (B.1)$$

The primary feasibility implies that RHS of (B.1) is always non-positive and attains its maximum at $\phi^* = 0$. Therefore, if

$$\mu^{(i)} + r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} > \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (0), \quad (B.2)$$

we have $\phi^{* (i)} = 0$, and the non-zero Lagrangian multiplier satisfies

$$\nu = \mu^{(i)} + r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} - \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (0) > 0.$$

Note that the RHS of (B.1) is strictly increasing in $\phi^{(i)}$. Given that

$$\mu^{(i)} + r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} \leq \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (0), \quad (B.3)$$

we have a unique solution $\phi^{* (i)}$ that satisfies the equation

$$\mu^{(i)} + r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} = \alpha^{(i)} \sigma^2 \phi^{* (i)} + \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (\phi^{* (i)})$$

$$+ \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha^{(i)} G F^{(i)} (\phi^{* (i)}) \cdot \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (\phi^{* (i)}),$$

and the Lagrangian multiplier $\nu = 0$.

Similar to the first case, if

$$\mu^{(i)} - r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} < \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (0), \quad (B.4)$$

we have $\phi^{* (i)} = 0$. Otherwise, there is a unique solution $\phi^{* (i)}$ satisfying the equation

$$\mu^{(i)} - r c \sigma - \alpha^{(i)} \sigma^2 \psi^{(i)} = \alpha^{(i)} \sigma^2 \phi^{* (i)} + \left( r + \frac{p}{2} \right) \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (\phi^{* (i)})$$

$$+ \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha^{(i)} G F^{(i)} (\phi^{* (i)}) \cdot \frac{\partial G F^{(i)}}{\partial \phi^{(i)}} (\phi^{* (i)}).$$
B.2 Proof of Theorem 4.6

The market clearing condition \((4.19)\) implies that it is enough to solve the following system of two equations to get the inter-dealer market equilibrium by using \((4.15)\) and \((4.17)\) for \(\phi^* = (\phi^*(1), \phi^*(2))\):

\[
\begin{align*}
(\mu(1) + \xi^* - f - r c) N - \alpha(1) \sigma^2 N^2 \psi(1) &= \alpha(1) \sigma^2 N^2 \phi^* + \left( r + \frac{p}{2} \right) \frac{\partial GF(1)}{\partial \phi(1)}(\phi^*) + \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha(1) \left( GF(1)(\phi^*) \frac{\partial GF(1)}{\partial \phi(1)}(\phi^*) \right), \\
(\mu(2) + \xi^* - f + r c) N - \alpha(2) \sigma^2 N^2 \psi(2) &= \alpha(2) \sigma^2 N^2 \phi^*(2) + \left( r + \frac{p}{2} \right) \frac{\partial GF(2)}{\partial \phi(2)}(\phi^*) + \left( \frac{p}{3} - \frac{p^2}{4} \right) \alpha(2) \left( GF(2)(\phi^*) \frac{\partial GF(2)}{\partial \phi(2)}(\phi^*) \right),
\end{align*}
\]

where

\[ GF(i) = \frac{(\delta - c \sigma)}{\sigma} \phi^*, \quad i = 1, 2. \]

For simplicity, if we assume that \(\alpha(1) = \alpha(2) = \alpha\) then

\[(\mu(1) + \xi^* - r c) N - \alpha \sigma^2 N^2 \psi(1) = - (\mu(2) + \xi^* + r c) N + \alpha \sigma^2 N^2 \psi(2) \]

or equivalently,

\[ \xi^* = \frac{\mu(1) + \mu(2) + \alpha N \sigma^2 (\psi(1) + \psi(2))}{2}. \]

Substituting this into the above system of equations, we get

\[ \phi^* = \frac{1}{2} \left( \frac{\mu(1) - \mu(2) + \alpha \sigma^2 (-\psi(1) + \psi(2))}{\sigma^2} - r c \sigma - (r + \frac{p}{3}) \left( \frac{\delta - c \sigma}{\sigma} \right) \right) > 0. \]

Otherwise, we have a trivial solution, \(\phi^* = 0\).

B.3 Proof of Proposition 4.8

When the condition in \((4.26)\) is satisfied, the partial derivative in \((4.25)\) is positive for all \(c \in [0, b]\). This implies that \(\phi^*\) is increasing in \(c\). Hence, we conclude the first assertion.
Next, the second condition (4.27) implies that

\[ n^2 - \left(1 + \frac{p}{2r}\right) n - \frac{2 \alpha \phi^* (b - c)}{3r} \geq 0, \]

which is equivalent to

\[ n \geq \frac{1}{2} \left( \left(1 + \frac{p}{2r}\right) + \sqrt{\left(1 + \frac{p}{2r}\right)^2 + \left(\frac{p}{3} - \frac{p^2}{4}\right) \frac{8 \alpha \phi^* (b - c)}{r}} \right), \]

when we require \( n \) to be positive. From (4.22), we know that

\[ \phi^* \leq \bar{\psi} + \frac{\mu}{\sigma^2 \alpha N}. \]

The last two inequalities yield the sufficient condition

\[ n \geq \frac{1}{2} \left( \left(1 + \frac{p}{2r}\right) + \sqrt{\left(1 + \frac{p}{2r}\right)^2 + \left(\frac{p}{3} - \frac{p^2}{4}\right) \frac{8 b (\mu + \alpha N \bar{\psi} \sigma^2)}{r \sigma^2}} \right) \]

so that \( \phi^* \) is decreasing in \( c \). Thus, the second part of Proposition 4.8 is proven.