Homogenization Theory for Partial Differential Equations with Large, Random Potential

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ABSTRACT

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Partial differential equations with highly oscillatory, random coefficients describe many applications in applied science and engineering such as porous media and composite materials. Homogenization of PDE states that the solution of the initial model converges to the solution to a macro model, which is characterized by the PDE with homogenized coefficients. Particularly, we study PDEs with a large potential, a class of PDEs with a potential properly scaled such that the limiting equation has a non-trivial (non-zero) potential.

This thesis consists of the investigation of three issues. The first issue is the convergence of Schödinger equation to a deterministic homogenized PDE in high dimension. The second issue is the convergence of the same equation to a stochastic PDE in low dimension. The third issue is the convergence of elliptic equation with an imaginary potential.
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Chapter 1

Motivations and Overview

1.1 Motivation

The modeling of the physical processes in strongly heterogeneous medium motivates the study of partial differential equations with oscillating coefficients. The basic problem is this: a family of physical processes are assumed to solve

\[ A_\varepsilon u_\varepsilon = f_\varepsilon \]  

with appropriate initial and/or boundary conditions. Here, \( A_\varepsilon \) is a differential operator, and the parameter \( \varepsilon \) stands for the period of oscillation.

If the knowledge of small scale variation of the heterogeneous medium is only known to statistical extent, the medium is modeled as a random field. We are interested in deriving a homogenized equation that captures the effective properties of the heterogeneous medium, as the computational cost of solving the multi-scaled equation (1.1) is prohibitive. Specifically, we would like to understand the following issues:
1.2. EXAMPLE

(1) Does the solution $u_\varepsilon$ converge? If the limiting solution $u$ solve the equation

$$\mathcal{A}u = f,$$  \hspace{1cm} (1.2)

we call (1.2) a homogenized equation.

(2) Which type of convergence do we have in (1)? Is it a $L^2$ convergence to a deterministic PDE, or is it a weak convergence to a stochastic PDE?

(3) What is the rate of convergence? In other words, can we prove

$$\|u_\varepsilon - u\| \leq C\varepsilon^\gamma$$  \hspace{1cm} (1.3)

for some $\gamma > 0$.

1.2 Example

Let us see an example of the random homogenization problem:

$$\frac{\partial}{\partial t} u_\varepsilon(t, x) - \Delta u_\varepsilon(t, x) + \frac{1}{\varepsilon^\alpha} V\left(\frac{x}{\varepsilon}\right) u_\varepsilon(t, x) = 0$$

$$u_\varepsilon(0, x) = u_0(x). \hspace{1cm} (1.4)$$

This equation can be seen as a continuous version of the parabolic Anderson model. The asymptotic behavior of this equation depends on the dimension $d$. For simplicity, we assume $V$ to be a Gaussian field.

For $d = 1$ and $\alpha = 1/2$, $u_\varepsilon$ converges weakly to the stochastic PDE

$$\frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) + \sigma u(t, x) \circ \dot{W} = 0,$$

$$u(0, x) = u_0(x). \hspace{1cm} (1.5)$$
where $\dot{W}$ denotes spatial white noise, $\circ$ denotes Stratonovich product, and

$$\sigma^2 := \int_{\mathbb{R}^d} \mathbb{E}\{V(0)V(x)\} dx. \quad (1.6)$$

For $d = 2$ and $\varepsilon^\alpha := \varepsilon|\log \varepsilon|$, or $d > 2$ and $\alpha = 1$, $u_\varepsilon$ converges in $L^2(\Omega \times \mathbb{R}^d)$ to the deterministic PDE

$$\frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) - \rho u(t, x) = 0,$$

$$u_\varepsilon(0, x) = u_0(x), \quad (1.7)$$

where

$$\rho := \begin{cases} \frac{c_d \hat{R}(0)}{d}, & d = 2 \\ \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi, & d > 2. \end{cases} \quad (1.8)$$

Here, $c_d$ is the volume of unit sphere on $(d - 1)$-dimensional hyperplane. $\hat{R}(\xi)$ is the Fourier transform of the covariance function $R(x) = \mathbb{E}\{V(y)V(x + y)\}$.

### 1.3 Overview

The rest of the thesis is structured as follows.

Chapter 2 reviews a few topics of random fields that are related to the random homogenization problems discussed in this thesis. As noted above, there is no universal approach to prove homogenization and to find correctors. Methods and results are often associated with the properties of the random fields, such as mixing conditions. Therefore, it makes sense to get an understanding of the random fields before we delve into homogenization. Section 2.1 introduces the concept of stationarity and different types of mixing conditions. Section 2.2 narrows down to a special class of random fields called stationary Gaussian fields, and discusses the techniques tailored to them.

Chapter 3 and 4 investigate the homogenization of a Schrödinger equation with highly
oscillatory potential modeled as Gaussian field. Similar to the heat equation discussed in Section 1.2, Schrödinger equation also converges to a deterministic PDE in high dimension in the $L^2$ sense, and to a stochastic PDE in low dimension in the weak sense. As it is pointed out in [4], the reason is that the random solution to visit the whole space of stochasticity more easily in high dimension than in low dimension.

Chapter 5 studies the homogenization of an elliptic equation with an imaginary potential modeled as a mixing random filed, and provides an estimate for the size of the error. It is shown that even if the random potential has an imaginary high frequency part, the potential of the limiting equation is a real constant.
Chapter 2

Preliminaries

In this chapter we first introduce the basic notions of random fields that play an essential role in almost all random homogenization problems. We then review some techniques that will be useful in studying homogenization for PDEs with Gaussian potential in Chapter 3 and 4.

2.1 Random fields

The random fields discussed in the context of random homogenization are often assumed to be stationary and ergodic. In the cases when ergodicity is insufficient to yield homogenization, stronger mixing conditions will be needed. In this section we will explain the connection between these properties and homogenization. We begin with the definition of random fields.

**Definition 2.1.** Given a parameter space $X$, a random field $V$ over $X$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables

$$\{V(x) : x \in X\}. \quad (2.1)$$
When $X = \mathbb{R}$, we also call $V$ a random process.

In random homogenization problem, $X$ is taken as the position of the medium. The distribution of $V(x)$ reflects our knowledge about the statistical distribution of the point $x$ in the medium. When the medium appears highly heterogeneous, but the statistics of the medium is believed to be invariant up to a spatial transition, it is modeled as a stationary random field.

**Definition 2.2.** A random field $V(x)$ is called stationary if for every $\{x_1, x_2, \cdots, x_n\} \subset X$ and $y \in X$, the distributions of vectors $(V(x_1), V(x_2), \cdots, V(x_n))$ and $(V(x_1 + y), V(x_2 + y), \cdots, V(x_n + y))$ are the same.

Since the root idea of random homogenization is the law of large numbers, it is not surprising that we require the random field to be ergodic.

**Definition 2.3.** For a stationary random field $V(x)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, the shift transforms $\tau_y$ for $y \in X$ defined on $\mathcal{F}$ are measure preserving. If every Borel set invariant under all shift transforms has measure zero or one, we call the random field $V(x)$ ergodic.

The simplest example of stationary and ergodic random field is the i.i.d random field. However, ergodicity is not easy to check in general. Sufficient conditions of ergodicity includes strong mixing.

**Definition 2.4.** Given a stationary random field $V(x)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and parameterized by $X$, define a function $\alpha(s)$, called $\alpha$-mixing coefficient, as

$$
\alpha(s) := \sup\{|P(S \cap T) - P(S)P(T)| : S \in \mathcal{F}_A, T \in \mathcal{F}_B, A \text{ and } B \text{ are Borel sets in } X, \text{dist}(A, B) \geq s\}.
$$

(2.2)

$V(x)$ is called $\alpha$-mixing if $\lim_{s \to \infty} \alpha(s) = 0$.

$\alpha$-mixing means asymptotic independent. For any two possible states of the random field, when given sufficient distance between these two states, the occurrence of the states
is roughly independent. It is easy to show that $\alpha$-mixing is stronger than ergodicity. An even stronger version of mixing property is called $\rho$-mixing.

**Definition 2.5.** Given a stationary random field $V(x)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and parameterized by $X$, the $\rho$-mixing coefficient is defined as

$$
\rho(s) := \sup \{ \text{corr}(\xi, \eta) : \xi \in \mathcal{F}_A, \eta \in \mathcal{F}_B, A \text{ and } B \text{ are Borel sets in } X, \text{dist}(A, B) \geq s \}.
$$

(2.3)

$V(x)$ is called $\rho$-mixing if $\lim_{s \to \infty} \rho(s) = 0$.

To prove that (2.3) the stronger is than (2.5), we simply replace $\xi$ and $\eta$ with the indicator function $I(A)$ and $I(B)$.

For a stationary random field $V(x)$, the auto-covariance function $R(x) = \mathbb{E}V(y)V(x+y)$ does not depend on $y$. We say $V(x)$ to have short range correlation if $R(x)$ is integrable, and long range correlation otherwise.

### 2.2 Homogenization of PDEs with Gaussian coefficients

Gaussian fields are unique among all random fields in that the first and second order moments (mean and covariance) completely determines the distribution. Without loss of generality, we assume Gaussian fields to have zero mean hereafter. For Gaussian fields, moments of all orders can be calculated in terms of the second order moments.

**Definition 2.6.** A Gaussian field is a random field involving only Gaussian distributed random variables.

**Theorem 2.7.** If $(Z_1, \ldots, Z_n)$ is a zero mean multivariate Gaussian random vector, then

$$
\mathbb{E}[Z_1 \cdots Z_n] = \begin{cases} 
0, & \text{n is odd} \\
\sum \prod \mathbb{E}[Z_i Z_j], & \text{n is even,}
\end{cases}
$$

(2.4)
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where the notation $\sum \prod$ means summing over all distinct ways of partitioning $Z_1, \cdots, Z_n$ into pairs.

2.2.1 Duhamel expansion

Given the convenience of computing the high order moments of Gaussian variables, it is useful to write the solution for a PDE with Gaussian distributed coefficient as an expansion and analyze cross moments of all terms. This technique is called Duhamel expansion. Let us use (1.4) as an example again. By Duhamel principle, $u_\varepsilon(t, x)$ solves the integral equation

$$u_\varepsilon(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)V_\varepsilon(y)u_\varepsilon(s, y)dyds + \int_{\mathbb{R}^d} G(t, x - y)u_0(y)dy,$$

where $G$ denotes the Green’s function of the operator $\partial/\partial t - \Delta$. Iterating step (2.5) gives

$$u_\varepsilon(t, x) = \sum_{n=0}^{\infty} u_{\varepsilon,n}(t, x) \tag{2.6}$$

where

$$u_{\varepsilon,n}(t, x) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \int_{\mathbb{R}^{(n+1)d}} G(t - s_1, x - y_1)V_\varepsilon(y_1) \cdots G(s_{n-1} - s_n, y_{n-1} - y_n)V_\varepsilon(y_n)$$

$$G(s_n, y_n - y_{n+1})u_0(y_{n+1})ds_1 \cdots ds_n dy_1 \cdots dy_{n+1}. \tag{2.7}$$

Remark 2.8. We need to prove the convergence of (2.6) in order to show that it is indeed a solution to (1.4).

Recall the results presented in Section 1.2, in dimension $d \geq 2$, $u_\varepsilon$ converges to a deterministic function $u$ in $L^2(\Omega \times \mathbb{R}^d)$. One way to prove this convergence as well as estimate the error is to compute the second order moments of every pair of the terms in (2.6), namely $\mathbb{E}\{u_{\varepsilon,n}u_{\varepsilon,m}\}$. According to the expression (2.7), computing these moments is just a simple application of Theorem 2.7. In dimension $d < 2$, the convergence of $u_\varepsilon$
is in the weak sense. We shall introduce the approach to prove weak convergence in next subsection.

### 2.2.2 From moment convergence to weak convergence

Assume we want to prove the weak convergence of the stochastic process $Z_{\varepsilon}(t)$ to $Z(t)$. We would like to take the following path:

\[
\text{moment convergence } \rightarrow \text{finite dimensional distribution convergence } \rightarrow \text{weak convergence}
\]

Let us explain this idea step by step. The following theorem provides the condition under which weak convergence is equivalent to convergence of finite dimensional distribution.

**Theorem 2.9.** If $Z_{\varepsilon}(t)$ is tight and all finite dimensional distribution of $Z_{\varepsilon}(t)$ converges to that of $Z(t)$, then $Z_{\varepsilon}(t)$ converges to $Z(t)$ weakly.

The following theorem, called Kolmogorov criterion, provides an easy way to verify tightness.

**Theorem 2.10.** The sequence of processes $Z_{\varepsilon}(t)$ in $C(0,1)$ is tight if there exists $\delta, \beta, C > 0$, such that

\[
\mathbb{E}\{|Z_{\varepsilon}(s) - Z_{\varepsilon}(t)|^\beta\} \leq C|t - s|^{1+\delta}
\]

uniformly in $\varepsilon, t, s \in (0,1)$.

On the other hand, the convergence of finite dimensional distribution is linked to moment convergence by the following two theorems.

**Theorem 2.11.** Let $\mu$ be a measure on $\mathbb{R}$ such that all the moments

\[
m_n = \int_{-\infty}^{+\infty} x^n d\mu(x), \quad n = 0, 1, 2, \ldots
\]
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are finite. If
\[ \sum_{n=1}^{\infty} m_{2n}^{-\frac{1}{2n}} = +\infty, \]  
then \( \mu \) is determined by its moments \( m_n \); that is, \( \mu \) is the only measure on \( \mathbb{R} \) having \( m_n \) as its sequence of moments.

**Theorem 2.12.** Suppose that the distribution of \( X \) is determined by its moments, that the \( X_n \) have moments of all orders, and that \( \lim_{n} E[X_n^r] = E[X^r] \) for \( r = 1, 2, \ldots \). Then \( X_n \Rightarrow X \).

Theorem 2.11 provides a condition for a distribution to be determined by its moments. Theorem 2.12 states that moment convergence can imply finite dimensional distribution if the limiting distribution is determined by its moments. Theorem 2.9, 2.11, and 2.12 combine to create a route from moment convergence to weak convergence.

We would like to point out that the method for proving weak convergence discussed above is applicable to Gaussian fields because Theorem 2.7 allows us to compute moments of all orders.
Chapter 3

Homogenization of Schrödinger equation

3.1 Introduction

In this chapter, we investigate the homogenization limit for the (time-dependent) Schrödinger equation. The results of [1, 16] apply to parabolic equations of the form of a heat equation with large, mean-zero, highly oscillatory, random potentials (zeroth-order terms). In [3], it was shown that for large dimensions, the random solution converged to a deterministic solution, which is consistent with the homogenization framework. However, such results could only be obtained for short times, and it is unclear whether they hold for larger times. Because the solution operator is unitary in the Schrödinger case, we expect to be able to control the long-time asymptotic behavior of the solution. The method of proof, as in [1, 3], is based on a Duhamel expansion of the random solution in terms involving increasing numbers of scattering events. As the number of terms grows exponentially with the number of scattering events, we need to assume that the potential is Gaussian in order to control such a growth. For non-Gaussian random potential, we expect the homogenization results
to hold, but our method of proof does not adapt to this setting. Readers are referred to [16] for discussion on homogenization of one dimensional parabolic equation with mixing potential. The summation technique used in [3] cannot extend to long time controls. A similar difficulty occurs in the derivation of radiative transfer equations for the energy density of high frequency waves propagating in highly oscillatory media [7, 9, 17]. The method of partial time integration for estimating infinite terms in Duhamel expansion using unitarity property of the differential operator was introduced in [9] to allow for long-time expansions. We adapt this technique to the asymptotic analysis of Schrödinger equations with large potentials.

We now present in more detail the model considered and the main convergence result. Let \( m > 0 \). We consider the following Schrödinger equation:

\[
\begin{align*}
\left( i \frac{\partial}{\partial t} + (P(D) - \frac{1}{\varepsilon^{m/2}} q(\frac{x}{\varepsilon})) \right) u_\varepsilon(t, x) &= 0, \quad t > 0, \; x \in \mathbb{R}^d \\
u_\varepsilon(0, x) &= u_0(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]

where the dimension \( d \) satisfies both \( d > m \) and \( d \geq 3 \). Here, \( P(D) \) is the pseudo-differential operator with symbol \( \hat{p}(\xi) = |\xi|^m \). We assume that \( q(x) \) is a real valued mean zero stationary Gaussian process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with correlation function \( R(x) = \mathbb{E}\{q(y)q(x+y)\} \), and the non-negative power spectrum \( \hat{R}(\xi) \) is radially symmetric, smooth, and decays fast. For simplicity, we assume \( \hat{R} \in \mathcal{S}(\mathbb{R}^d) \). In fact, \( \|R\|_{12d, 12d} < +\infty \), where

\[
\|f\|_{d_1, d_2} := \|\langle x \rangle^{d_1} \langle \nabla x \rangle^{d_2} f(x)\|_2, \quad \langle x \rangle := (1 + x^2)^{1/2},
\]

is enough. We choose the initial condition \( u_0(x) \) to be smooth such that \( \hat{u}_0(\xi) \langle \xi \rangle^{6d} \in L^2(\mathbb{R}^d) \). For any finite time \( T > 0 \), the existence of a weak solution \( u_\varepsilon(t, x) \in L^2(\Omega \times \mathbb{R}^d) \) uniformly in time \( t \in (0, T) \) and \( 0 < \varepsilon < \varepsilon_0 \) can be proved by using a method based on Duhamel expansion.
Equation (3.1) may be seen as the model of quantum dynamics for a regime of wavelengths of the initial condition that are much larger than the scale of oscillations of the random potential. The case $m = 1$ corresponds to the relativistic model of quantum dynamics while $m = 2$ corresponds to the classical quantum mechanical model. The general choice of the parameter $m$ for instance in the super-diffusive case $m > 2$ allows us to precisely display the interactions between the elliptic operator $P(D)$ and the random process $q(x)$ in the limit of homogenization.

As $\varepsilon \to 0$, we show that the solution $u_\varepsilon(t)$ to (1.1) converges strongly in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in $t \in (0, T)$ to its limit $u(t)$ solution of the following homogenized equation

$$
\begin{cases}
(i \frac{\partial}{\partial t} + P(D) - \rho)u(t, x) = 0, & t > 0, \ x \in \mathbb{R}^d \\
\phantom{\rho}u(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
$$

(3.3)

where the potential is given by

$$
\rho = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^m} d\xi.
$$

(3.4)

The main result of this chapter is the following convergence result:

**Theorem 3.1.** There exists a solution to (1.1) $u_\varepsilon(t) \in L^2(\Omega \times \mathbb{R}^d)$ uniformly in $0 < \varepsilon < \varepsilon_0$ for $t > 0$. Moreover, we have the convergence results for all $t \in (0, T)$

$$
\lim_{\varepsilon \to 0} \mathbb{E}\| (u_\varepsilon - u)(t) \|_2^2 = 0.
$$

(3.5)

The rest of the chapter is organized as follows. Section 2 recasts the solution to (3.1) as a Duhamel series expansion in the frequency domain. We estimate the $L^2$ norm of the first $n_0$ terms by calculating the contributions of graphs in three categories similar to those defined in [3]. Section 3 estimates the $L^2$ norm of the error term by first subdividing the time integration into time intervals of smaller sizes, and then using Duhamel formula in each time interval. This method, introduced in [9], significantly improves the error estimates.
compared to the direct estimates of infinite Duhamel terms and enables the elimination of the restriction to short times. The estimates given in these sections are used in section 4 to characterize the limit of the solution $u_\varepsilon(t, x)$. Section 5 provides the proofs for the inequalities used for justifying the estimates in the previous sections.

The analysis of a parabolic equation of the type of the heat equation (with $i\partial_t$ replaced by $\partial_t$) is performed in [3] for $d \geq m$. Up to a logarithmic correction, we expect the limit of the solution to (3.1) to be deterministic also for the critical dimension $m = d$. In [3], the random fluctuations about the deterministic limit are also analyzed for the heat equation. We expect a similar behavior to occur for the Schrödinger equation (3.1) for short times. We do not know the behavior of the random fluctuations for arbitrary times $t \in (0, T)$.

In lower spatial dimension $d < m$, the limit of the solutions to (3.1) as $\varepsilon \to 0$ remains stochastic. This behavior was analyzed for the heat equation in [1, 14]. The limit of $u_\varepsilon$ is then shown to be the solution of a stochastic partial differential equation with multiplicative noise (written as a Stratonovich product). The analysis of (3.1) for $d < m$ is performed in [19]. Note that several results of convergence may be extended to the case of random potential with long range correlations or random potentials that display both temporal and spatial fluctuations [4].

### 3.2 Duhamel expansion

We denote by $e^{itH}$ the propagator for the equation (1.1). The Duhamel expansion then states that for any $n_0 \geq 1$

$$u_\varepsilon(t) = e^{itH}u_0 = \sum_{n=0}^{n_0-1} u_n,\varepsilon(t) + \Psi_{n_0,\varepsilon}(t), \quad (3.6)$$
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where for $H_0 := (-\Delta)^m$, we have defined

$$u_{n,\varepsilon}(t) := (-i)^n \left( \frac{1}{\varepsilon^{m/2}} \right)^n \int_0^t \cdots \int_0^t (\prod_{k=0}^n ds_k) \delta(t - \sum_{k=0}^n s_k) e^{is_0 H_0 q(x)} \cdots q(x) e^{is_n H_0} u_0, \quad (3.7)$$

and the error term is given by

$$\Psi_{n_0,\varepsilon}(t) = (-i) \left( \frac{1}{\varepsilon^{m/2}} \right) \int_0^t ds e^{i(t-s)H} q(x) u_{n_0-1,\varepsilon}(s). \quad (3.8)$$

We shall choose

$$n_0 = n_0(\varepsilon) := \frac{\gamma |\log \varepsilon|}{\log |\log \varepsilon|}, \quad (3.9)$$

for some fixed $0 < \gamma \ll \lambda$ sufficiently small, where $\lambda$ is defined as

$$\lambda = \begin{cases} 
  d - m & m < d \leq 2m \\
  m & d > 2m.
\end{cases} \quad (3.10)$$

Let us introduce $\hat{q}_\varepsilon(\xi) = \varepsilon^{d-m} \hat{q}(\varepsilon \xi)$, the Fourier transform of $\varepsilon^{-\frac{m}{2}} q(x)$. In the Fourier domain, the equation (3.1) may be recast as

$$\begin{cases} 
  (i \partial_t + \xi^m) \hat{u}_\varepsilon = \hat{q}_\varepsilon \ast \hat{u}_\varepsilon, & t > 0, \xi \in \mathbb{R}^d \\
  \hat{u}_\varepsilon(0, \xi) = \hat{u}_0(\xi), & \xi \in \mathbb{R}^d,
\end{cases} \quad (3.11)$$

where the $\ast$ is the operator of convolution. Hereafter, we use the notation $\xi^m = |\xi|^m$.

For the set of indices $I \subset \mathbb{N}$, we define the kernel for the evolution in the Fourier space as

$$K(t; \xi, I) := (-i)^{|I|-1} \int_0^t \cdots \int_0^t (\prod_{k \in I} ds_k) \delta(t - \sum_{k \in I} s_k) \prod_{k \in I} e^{is_k \xi_k^m}. \quad (3.12)$$

In the special case where $I_n := \{0, \cdots, n\}$, we denote by $K(t; \xi, n) := K(t; \xi, I_n)$ and $\xi_n := \xi_{I_n}$. Define $\xi_{n,0} = \xi_{I_n \setminus \{0\}}$. 

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Denote the contribution from the potential term by
\[ L(\xi, n) := \prod_{k=1}^{n}\hat{q}_\varepsilon(\xi_k - \xi_{k-1}). \quad (3.13) \]

For simplicity of notations, let \( \xi = \xi_0 \) hereafter. We may then rewrite the \( n^{th} \) order wave function as
\[ \hat{u}_{\varepsilon,n}(t, \xi) = \int K(t, \xi, n) L(\xi, n) \hat{u}_0(\xi_0) d\xi_{n,\hat{0}}. \quad (3.14) \]

We need to introduce the following moments
\[ U^{n}_\varepsilon(t, \xi) = \mathbb{E}\{\hat{u}_{\varepsilon,n}\}, \quad (3.15) \]

which are given by
\[ U^{n}_\varepsilon(t, \xi) = (-i)^n \int K(t, \xi, n) \mathbb{E}\{L(\xi, n)\} \hat{u}_0(\xi_0) d\xi_{n,\hat{0}}, \quad (3.16) \]

and
\[ U^{n,m}_\varepsilon(t, \xi, \zeta) = \mathbb{E}\{\hat{u}_{\varepsilon,n}(t, \xi) \overline{\hat{u}_{\varepsilon,m}(t, \zeta)}\}, \quad (3.17) \]

which are given by
\[ U^{n,m}_\varepsilon(t, \xi, \zeta) = (-i)^{n+m} \int K(t, \xi, n) K(t, \zeta, m) \mathbb{E}\{L(\xi, n)L(\zeta, m)\} \hat{u}_0(\xi_0) \overline{\hat{u}_0(\zeta_m)} d\xi_{n,\hat{0}} d\zeta_{m,\hat{0}}. \quad (3.18) \]

We need to estimate moments of the Gaussian process \( \hat{q}_\varepsilon \) using Isserlis' theorem. The expectation in \( U^{n,m}_\varepsilon \) vanishes unless there is \( \tilde{n} \in \mathbb{N} \) such that \( n + m = 2\tilde{n} \) is even. The moments are thus given as a sum of products of the expectation of pairs of terms \( \hat{q}_\varepsilon(\xi_k - \xi_{k+1}) \), where the sum runs over all possible pairings. We define the pair \((\xi_k, \xi_l), 1 \leq k < l, \)
as the contribution in the product given by

\[
E\{\hat{q}_\varepsilon(\xi_{k-1} - \xi_k)\hat{q}_\varepsilon(\xi_{l-1} - \xi_l)\} = \varepsilon^{d-m} \hat{R}(\varepsilon(\xi_k - \xi_{k-1}))\delta(\xi_k - \xi_{k-1} + \xi_l - \xi_{l-1})
\]

\[
= \varepsilon^{d-m} r(\varepsilon(\xi_k - \xi_{k-1}))r(\varepsilon(\xi_l - \xi_{l-1}))\delta(\xi_k - \xi_{k-1} + \xi_l - \xi_{l-1})
\]

(3.19)

with \( r(\xi) := \hat{R}(\xi)^{1/2} \).

In each instance of the pairings, we have \( \bar{n} \) terms \( k \) and \( \bar{n} \) terms \( l \equiv l(k) \). Note that \( l(k) \geq k+1 \). The collection of pairs \((\xi_k, \xi_{l(k)})\) for \( \bar{n} \) values of \( k \) and \( \bar{n} \) values of \( l(k) \) constitutes a graph \( \pi \in \Pi(n, m) \), where \( \Pi(n, m) \) denotes the set of all graphs \( \pi \) with \( n \) copies of \( \hat{q} \) and \( m \) copies of \( \bar{q} \). The graphs are defined similarly in the calculation of \( U^n_\varepsilon(t, \xi_0) \) in (3.16) for \( n = 2\bar{n} \), and we denote by \( \Pi(n) \) the set of graphs with \( n \) copies of \( \hat{q} \). We denote by \( A_0 = A_0(\pi) \) the collection of the \( \bar{n} \) values of \( k \) and by \( B_0 = B_0(\pi) \) the collection of the \( \bar{n} \) values of \( l(k) \).

Define

\[
F(\xi, n) := \prod_{k=1}^{n} r(\varepsilon(\xi_k - \xi_{k-1}))\hat{u}_0(\xi_n).
\]

(3.20)

Denote by \( \Delta_\pi \) the product of delta functions associated with the graph \( \pi \). Our analysis is based on the estimate of

\[
U^n_{\varepsilon}(t, \xi) = \sum_{\pi \in \Pi(n)} I_{\pi}
\]

(3.21)

with

\[
I_{\pi} := (-i)^n \int K(t, \xi, n)\Delta_\pi(\xi)F(\xi, n)d\xi,
\]

(3.22)

and

\[
\int U^{n,m}_{\varepsilon}(t, \xi) d\xi = \sum_{\pi \in \Pi(n,m)} C_\pi
\]

(3.23)

with

\[
C_\pi := (-i)^{n+m} \int K(t, \xi, n)K(t, \xi, m)\Delta_\pi(\xi, \zeta)\delta(\xi_0 - \zeta_0)F(\xi, n)\overline{F(\zeta, m)}d\xi nd\zeta n.
\]

(3.24)
By Lemma 3.10, \( K(t; \xi, I) \) can also be written as

\[
K(t; \xi, I) = ie^{i\eta} \int d\alpha e^{-i\alpha t} \prod_{k \in I} \frac{1}{\alpha + \xi_k + i\eta}.
\]  

(3.25)

We let \( \eta = t^{-1} \) in this section. Therefore, \( I_\pi \) in (3.22) and \( C_\pi \) in (3.24) can be written explicitly as

\[
I_\pi = -i^{n+1} e^{i\eta} \int d\alpha d\beta e^{-i(\alpha - \beta)t} \prod_{k=0}^{n} \frac{r(\varepsilon(\xi_k - \xi_{k-1}))}{\alpha + \xi_k + i\eta} \prod_{l=0}^{m} \frac{1}{\beta + \xi_l - i\eta} \prod_{k=1}^{n} r(\varepsilon(\xi_k - \xi_{k-1}))
\times \prod_{l=1}^{m} r(\varepsilon(\zeta_l - \zeta_{l-1})) \Delta_\pi(\xi, \zeta) \hat{u}_0(\xi_0) \hat{u}(\zeta_0) \delta(\xi_0 - \zeta_0)d\xi_n d\zeta_m. 
\]  

(3.26)

In order to consider the two sets of momenta in a unified way, we introduce the notation

\[
\alpha_k = \begin{cases} 
\alpha & 0 \leq k \leq n \\
\beta & n+1 \leq k \leq n+m+1,
\end{cases}
\quad \eta_k = \begin{cases} 
\eta & 0 \leq k \leq n \\
-\eta & n+1 \leq k \leq n+m+1,
\end{cases}
\]  

(3.28)

and define \( \xi_{n+k+1} = \zeta_{m-k} \) for \( 0 \leq k \leq m \). (3.27) can then be rewritten as

\[
C_\pi = -i^{n+m} \int d\alpha d\beta e^{-i(\alpha - \beta)t} \prod_{k=0}^{n+m+1} \frac{1}{\alpha_k + \xi_k + i\eta_k} \prod_{k=1, k \neq n+1}^{n+m+1} r(\varepsilon(\xi_k - \xi_{k-1}))
\times \hat{u}_0(\xi_n) \hat{u}_0(\xi_m) \delta(\xi_0 - \zeta_0) d\xi_{n+m+1}.
\]  

(3.29)

Now we introduce several classes of graphs for \( C_\pi \). We say that the graph has a crossing if there is a \( k \leq n \) such that \( l(k) \geq n+2 \). We denote by \( \Pi_c(n, m) \subset \Pi(n, m) \) the set of graphs with at least one crossing and by \( \Pi_{nc}(n, m) = \Pi(n, m) \\setminus \Pi_c(n, m) \) the non-crossing graphs. We denote by simple pairs the pairs such that \( l(k) = k+1 \), which thus involve a
delta function of the form \( \delta(\xi_k - 1 - \xi_{k-1}) \). The unique graph with only simple pairs is called the simple graph, which is denoted by \( \Pi_s(n, m) \). \( \Pi_{ncs}(n, m) = \Pi_{nc}(n, m) \setminus \Pi_s(n, m) \) denotes the set of non-crossing, non-simple graphs. We also use the notation \( \Pi_s(n) \) and \( \Pi_{ncs}(n) \) defined for \( I_\pi \), which denote the simple graph and the set of non-crossing, non-simple graphs, respectively.

We shall estimate \( F(\xi, n) \) before we proceed to analyze the graphs. For the initial condition, we define

\[
\Phi(\xi) = \langle \xi \rangle^{6d} |\hat{u}_0(\xi)|.
\]

From our assumption on \( u_0 \) given in Section 1, we have \( \Phi(\xi) \in L^2(\mathbb{R}^d) \). Since \( \|\hat{R}\|_{12d, 12d} < +\infty \), we have

\[
r(\xi) \leq \frac{C}{\langle \xi \rangle^{6d}}.
\]

Hence, we obtain the estimate for \( F(\xi, n) \)

\[
|F(\xi, n)| \leq \prod_{k=1}^{n} \frac{1}{(\varepsilon(\xi_k - \xi_{k-1}))^{6d}} \langle \xi_n \rangle^{6d} \Phi(\xi_n) \leq \prod_{k=1}^{n} \frac{1}{(\varepsilon(\xi_k - \xi_{k-1}))^{6d}} \langle \xi_n \rangle^{2d} \langle \varepsilon \xi_n \rangle^{4d} \Phi(\xi_n) \leq C^n \Phi(\xi_n) \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{i=1}^{2} \frac{1}{\langle \varepsilon \xi_{l_i} \rangle^{2d}} \prod_{k=1}^{n} \frac{1}{(\varepsilon(\xi_k - \xi_{k-1}))^{2d}}.
\]

The last inequality may be obtained by noting the fact that

\[
\frac{1}{(\varepsilon \xi_n)^{2d}} \prod_{k=1}^{n} \frac{1}{(\varepsilon(\xi_k - \xi_{k-1}))^{2d}} \leq \frac{C^n}{\langle \varepsilon \xi_{l_2} \rangle^{2d}}
\]

for any \( 0 \leq l \leq n - 1 \). We thus have the freedom to choose \( l_1 \) and \( l_2 \) between 0 and \( n \).

We analyze the crossing, non-crossing, and simple graphs respectively in the following. As we shall see, the order of the sum of all crossing graphs for \( n, m \leq n_0 \) is \( O(\varepsilon^{-3\gamma}) \), and
only the simple graph contributes an $O(1)$ term in the limit $\varepsilon \to 0$.

### Analysis of the crossing graphs

**Lemma 3.2.** If $\pi \in \Pi_c(n, m)$, we have

$$|C_\pi| \leq (C \log \varepsilon)^{\beta \lambda} \|\Phi(\xi)\|^2_2.$$  \hspace{1cm} (3.34)

**Proof.** Denote by $(\xi_{q_m}, \xi_{l(q_m)})$, $1 \leq m \leq M$, the crossing pairs and define $Q = \max_m \{q_m\}$.

Let us define $A' = A_0 \backslash \{Q\}$. From (3.32), we have

$$|F(\xi, n)F(\xi, m)| \leq |\Phi(\xi_n)||\Phi(\xi_{n+1})| \frac{1}{(\varepsilon \xi_0)^{2d}} \frac{1}{(\varepsilon \xi_n)^{2d}} \prod_{k \in A'} \frac{1}{(\varepsilon (\xi_k - \xi_{k-1}))^{2d}}.$$ \hspace{1cm} (3.35)

The terms $\frac{1}{|\alpha_k + \xi_k^m + it^{-1}|}$ for $k \notin A' \cup \{0, n, n + 1, n + m + 1\}$ are bounded by $C$. This allows us to obtain

$$|C_\pi| \leq C^n \int d\alpha d\beta \frac{1}{|\alpha + \xi_0^m + it^{-1}| |\beta + \xi_0^m + it^{-1}|} \frac{1}{(\varepsilon \xi_0)^{2d}} \prod_{k \in A'} \frac{1}{|\alpha_k + \xi_k^m + it^{-1}| |\beta + \xi_k^m + it^{-1}|} \frac{1}{(\varepsilon (\xi_k - \xi_{k-1}))^{2d}} \delta(\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k-1)})$$ \hspace{1cm} (3.36)

$$\delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1})$$

$$\frac{1}{|\alpha + \xi_n^m + it^{-1}| |\beta + \xi_n^m + it^{-1}|} |\Phi(\xi_n)|^2 d\xi_{n+m}.$$  

For each $k \in A' \cup \{0\}$, we perform the change of variables $\xi_k \to \frac{\xi_k}{\varepsilon}$, and define

$$\xi_k^\varepsilon = \begin{cases} 
\xi_k & k \neq A' \cup \{0\} \\
\frac{\xi_k}{\varepsilon} & k \in A' \cup \{0\}. 
\end{cases}$$ \hspace{1cm} (3.37)

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CHAPTER 3. HOMOGENIZATION OF SCHröDINGER EQUATION

We then find the estimate

\[ |C_n| \leq C^n \int d\alpha d\beta \frac{1}{|\alpha + \xi^m_0/\varepsilon^m + it|} \frac{1}{|\beta + \xi^m_0/\varepsilon^m + it|} |\varepsilon^{-m} \frac{1}{(\xi_0)^2d}| \prod_{k \in A'} \frac{1}{|\alpha + \xi^m_k/\varepsilon^m + it|} \frac{1}{|\beta + \xi^m_k/\varepsilon^m + it|} |\varepsilon^{-m} \frac{1}{(\xi_k)^2d}| \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1} + \xi_{(k)} - \xi_{(k)-1}\right) \left|\frac{1}{|\alpha + \xi^m_n + it|} \frac{1}{|\beta + \xi^m_n + it|} \frac{1}{(\xi_n)^2d}\right| \Phi(\xi_n)^2 d\xi_{n+m}. \]

(3.38)

We now estimate the above product. Assume \( Q < n \) and define \( k_0 \) such that \( n = l(k_0) \). When \( Q = n \) or \( l(Q) = n + m + 1 \), the derivation of the same estimates is simpler and left to the reader. Define \( k_1 \) such that \( l(k_1) = n + m + 1 \). For each \( k \in A'(\pi) \setminus (k_0 \cup k_1) \), we use Lemma 3.16 below to find the estimate

\[ \varepsilon^{-m} \int \frac{1}{(\xi_k - \varepsilon \xi_{k-1})^{2d}} \frac{1}{|\alpha + \xi^m_k/\varepsilon^m + it|} |\varepsilon^{-m} \frac{1}{(\xi_k - \varepsilon \xi_{k-1})^{2d}} d\xi_k d\xi_{(k)} \leq C |\log \varepsilon|. \]

(3.39)

The integration in the \( \xi_{l(Q)} \) variable is estimated by using the above delta function. The delta function for \( k = k_0 \in A'(\pi) \) may be written in the form \( \delta(\xi_Q - \xi_{Q-1} + \xi_0 - \xi_n + \sum_{m=1}^{M-1} \xi_{q_m} - \xi_{q_m-1}) \), and is thus used to integrate in the variable \( \xi_Q \). The term \( \frac{1}{(\xi_k - \varepsilon \xi_{k-1})^{2d}} \) is used to integrate in the variable \( \xi_{k_0} \). The integral in \( \alpha \) and \( \beta \) is estimated using Lemma 3.11 by

\[ \int \frac{1}{|\alpha + \xi^m_0/\varepsilon^m + it|} \frac{1}{|\alpha + \xi^m_n + it|} d\alpha \leq C \frac{1}{|\xi^m_0/\varepsilon^m - \xi^m_n/\varepsilon^m + it|} \left[ 1 + \log_{+} \left| \frac{\varepsilon^m_0/\varepsilon^m - \varepsilon^m_n}{t-1} \right| \right]. \]

(3.40)

The integral in \( \xi_0 \) is estimated using Lemma 3.13 by

\[ \varepsilon^{-m} \int \frac{1}{(\xi_0)^2d} \left[ 1 + \log_{+} \left| \frac{\varepsilon^m_0/\varepsilon^m - \varepsilon^m_n}{t-1} \right| \right]^2 d\xi_0 \leq C \varepsilon^\lambda (\xi^\lambda_0 \vee 1). \]

(3.41)
3.2. DUHAMEL EXPANSION

Following the usual convention, we use \( a \vee b := \max\{a, b\} \) and \( a \wedge b := \min\{a, b\} \). The delta function \( \delta(\xi_{k_1} - \xi_{k_1-1} - \xi_{n+m+1} - \xi_{n+m}) \) is seen to be equivalent to \( \delta(\xi_n - \xi_{n+1}) \), which handles the integration in the variable \( \xi_{n+1} \). Finally we integrate in \( \xi_n \)

\[
\int \frac{1}{(\xi_n)^2} |(\xi_n^{\lambda} \vee 1)\Phi(\xi_n)|^2 d\xi_n \leq \|\Phi(\xi)\|_2^2, \tag{3.42}
\]

and obtain

\[
|C_\pi| \leq (C|\log \varepsilon|)^{\bar{n}} \varepsilon^\lambda \|\Phi(\xi)\|_2^2. \tag{3.43}
\]

Using Stirling’s formula, we find that \(|\Pi_{ncs}(n, m)| \leq \frac{(n-1)!}{2^{n-1}(\frac{n}{2})!} \) is bounded by \((\frac{2n}{e})^{\bar{n}}\). After summation in \( n, m \leq n_0 \), we obtain

\[
\mathbb{E}\| \sum_{n=0}^{n_0} (\hat{u}_{n, \varepsilon} - U^n_{\varepsilon})(t) \|_2^2 \leq \sum_{n=0}^{n_0-1} \sum_{m=0}^{n_0-1} \left( \frac{2\bar{n}}{\varepsilon} \right)^{\bar{n}} (C|\log \varepsilon|)^{\bar{n}} \varepsilon^\lambda \|\Phi(\xi)\|_2^2 \leq n_0 \mathbb{E}|\Pi_{ncs}(n, m)| \|\Phi(\xi)\|_2^2 \lesssim \varepsilon^{\lambda-3\gamma}, \tag{3.44}
\]

where \( a \lesssim b \) means \( a \leq Cb \) for some \( C > 0 \).

**Analysis of the non-crossing graphs**

**Lemma 3.3.** If \( \pi \in \Pi_{ncs}(n) \), we have

\[
|I_\pi| \leq (C|\log \varepsilon|)^{\frac{n}{2}} \varepsilon^{\lambda(1-\delta)} |\Phi(\xi)|, \tag{3.45}
\]

where \( 0 < \delta \ll 1 \). Moreover, if \( \pi \in \Pi_{ncs}(n, m) \),

\[
|C_\pi| \leq (C|\log \varepsilon|)^{\bar{n}} \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2. \tag{3.46}
\]
Proof. In a graph $\pi \in \Pi_{ncs}(n)$, the delta function

$$\delta(\xi_0 - \xi_n)$$

is obtained by adding up all the delta functions in $\Delta_\pi$. We perform the change of variables for all $k \in A_0$, $\xi_k \rightarrow \xi_k/\varepsilon$ and define as before

$$\xi_k^\varepsilon = \begin{cases} 
\xi_k & k \notin A_0 \\
\xi_k/\varepsilon & k \in A_0.
\end{cases}$$

(3.48)

We shall solve the following two cases in different ways.

(i) If there exists $k_2 \in A_0$, such that for all $k$ satisfying $k_2 + 1 \leq k \leq l(k_2) - 2$, $k \in A_0$, $(\xi_k, \xi_{l(k)})$ are simple pairs, then the delta function $\delta(\xi_{k_2}/\varepsilon - \xi_{l(k_2)-1})$ is present. From (3.32), we have

$$F(\xi, n) \leq C^m |\Phi(\xi_n)| \frac{1}{\langle \xi_{k_2} \rangle^{2d}} \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{k \in A_0} \frac{1}{\langle \xi_k - \varepsilon \xi_{k-1} \rangle^{2d}}. \quad (3.49)$$

The estimate of integration in $\xi_{k_2}$ is then obtained by using Lemma 3.13 below:

$$\int \varepsilon^{-m} \frac{1}{\langle \xi_{k_2} \rangle^{2d}} \frac{1}{|\alpha + \xi_{k_2}^m/\varepsilon^m + it^{-1}|^2} d\xi_{k_2} \leq C \varepsilon^{\frac{\lambda(1-\delta)}{m}} (\alpha/\varepsilon^{m} \vee 1). \quad (3.50)$$

The delta function in which $\xi_n$ is involved is equivalent to $\delta(\xi_n - \xi_0)$, which we use to integrate in $\xi_n$:

$$\int \frac{1}{\langle \xi_n \rangle^{2d}} \frac{1}{|\alpha + \xi_n^m + it^{-1}|^2} \delta(\xi_n - \xi_0) d\xi_n \leq \frac{1}{\langle \xi_0 \rangle^{2d}} \frac{1}{|\alpha + \xi_0^m + it^{-1}|^2}. \quad (3.51)$$

For $k \in A_0, k \neq k_2$, we have the estimate

$$\int \varepsilon^{-m} \frac{1}{\langle \xi_k - \varepsilon \xi_{k-1}^\varepsilon \rangle^{2d}} \frac{1}{|\alpha + \xi_k^m/\varepsilon^m + it^{-1}|^2} \delta(\xi_k/\varepsilon - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}) d\xi_k d\xi_{l(k)} \leq C |\log \varepsilon|. \quad (3.52)$$
3.2. DUHAMEL EXPANSION

The estimate of integration in the variable $\alpha$ is then given by

$$
\int \frac{\alpha^{(1-\delta)}}{|\alpha + \xi_0^m + it^{-1}|^2} \leq C \xi_0^{(1-\delta)}. \tag{3.53}
$$

The extra term $\xi_0^{(1-\delta)}$ that arises in the last estimate can be canceled by the term $1/\langle \xi_0 \rangle^{2d}$ in (3.51), which concludes (3.45).

(ii) If there exists no such $k_2 \in A_0$ satisfying the condition in case (i), then we first delete all simple pairs that exist in the graph $\pi$. In fact, the simple pairs can be handled first by using the bound as in (3.52). Therefore without loss of generality, we need only to consider a graph $\pi$ with no simple pair. Let us define $k_4 = \min \{k | k \in A_0, l(k) - 1 \in A_0 \}$, and $k_5 = l(k_4) - 1$. Note that $k_5 \geq k_4 + 1$.

We have from (3.32) that

$$
F(\xi, n) \leq C^n |\Phi(\xi_n)| \frac{1}{\langle \xi_{l(k_4)} \rangle^{2d}} \frac{1}{\langle \xi_{k_5} \rangle^{2d}} \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{k \in A_0} \frac{1}{\langle \xi_k - \varepsilon \xi_{k-1} \rangle^{2d}}. \tag{3.54}
$$

The integration in $\xi_{l(k_4)}$ provides the terms which we will need for integration in $\xi_{k_5}$

$$
\int \frac{1}{\langle \varepsilon \xi_{l(k_4)} \rangle^{2d}} \frac{1}{\langle \alpha + \xi_{l(k_4)}^m + it^{-1} \rangle^{2d}} |\xi_{k_4} - \varepsilon \xi_{k_4-1} + \xi_{l(k_4)} - \xi_{k_5} \varepsilon | d\xi_{l(k_4)} = \frac{1}{\langle \xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1} \rangle^{2d}} \frac{1}{\langle \alpha + \xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1}^m / \varepsilon^m + it^{-1} \rangle}. \tag{3.55}
$$

We can estimate the integration in $\xi_{k_5}$ using Lemma 3.13

$$
\int \varepsilon^{-m} \frac{1}{\langle \xi_{k_5} \rangle^{2d}} \frac{1}{\langle \xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1} \rangle^{2d}} \frac{1}{\langle \alpha + \xi_{k_5}^m / \varepsilon^m + it^{-1} \rangle} \frac{1}{\langle \alpha + \xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1}^m / \varepsilon^m + it^{-1} \rangle} d\xi_{k_5} \leq C \varepsilon^{\lambda(1-\delta)} \langle \alpha \rangle^{\lambda(1-\delta) \lor 1}. \tag{3.56}
$$

All the other integrations are handled the same way as in case (i). In order to make sure that no integration above is affected by other integrands we plan to use for integrating in
other variables, we just need to first integrate in $\xi_{l(k)}$ with index in decreasing order and then integrate in $\xi_k$ with index in decreasing order. This gives (3.45).

If $\pi \in \Pi_{ncs}(n,m)$, we may denote the pairings for $k \leq n$ and for $n + 1 \leq k \leq n + m + 1$ by $\pi_1$ and $\pi_2$ and then $\pi = \pi_1 \cup \pi_2$, since there is no crossing in $\pi$. Hence it follows that

$$|C_\pi| \leq \int |I_{\pi_1}(\xi)I_{\pi_2}(\xi)|d\xi \leq (C \log \varepsilon) \bar{n} \varepsilon^{2\lambda(1-\delta)}\|\Phi(\xi)\|_2^2.$$  \hspace{0.5cm} (3.57)

Similarly to (3.44), we obtain

$$\|\sum_{n=0}^{n_0-1} (U^n_\varepsilon - U^n_{\varepsilon,s})(t)\|_2^2 \leq \sum_{n=0}^{n_0-1} \sum_{m=0}^{n_0-1} \left(\frac{2\bar{n}}{e}\right)^n (C|\log \varepsilon|) \bar{n} \varepsilon^{2\lambda(1-\delta)}\|\Phi(\xi)\|_2^2$$  \hspace{0.5cm} (3.58)

$$\leq n_0^n (C|\log \varepsilon|) n_0 \varepsilon^{2\lambda(1-\delta)}\|\Phi(\xi)\|_2^2 \lesssim \varepsilon^{2\lambda(1-\delta)-3\gamma},$$

where

$$U^n_{\varepsilon,s} := I_\pi$$

(3.59)

for $\pi \in \Pi_s(n)$.

Collecting the results obtained in (3.44) and (3.58), we have shown that

$$\mathbb{E}\|\sum_{n=0}^{n_0-1} (\hat{u}_{\varepsilon,n} - U^n_{\varepsilon,s})(t)\|_2^2 \lesssim \varepsilon^{2\lambda(1-\delta)-3\gamma}.$$  \hspace{0.5cm} (3.60)

Analysis of the simple Graphs

**Lemma 3.4.** If $\pi \in \Pi_s(n)$, we have

$$|I_\pi| \leq \left(\frac{C^n}{(n/2)!} + O(C^n \varepsilon^{m/2})\right)|\hat{u}_0(\xi)|.$$  \hspace{0.5cm} (3.61)
Moreover, if \( \pi \in \Pi_s(n,m) \), we have

\[
|C_\pi| \leq \left( \frac{C^{n+m}}{(n/2)!(m/2)!} + O(C^{n+m} \varepsilon^{m/2}) \right) \| \hat{u}_0(\xi) \|_2^2. \quad (3.62)
\]

**Proof.** In the case of a simple graph \( \pi \), we can explicitly write out the product of delta functions

\[
\Delta_\pi = \prod_{k \in A_0} \delta(\xi_{k-1} - \xi_{k+1}), \quad (3.63)
\]

which is independent of \( \xi_k \) for all \( k \in A_0 \), and forces \( \xi_k = \xi_0 \) for all \( k \notin A_0 \).

Integrating in \( \xi_k \) for all \( k \in B_0 \) using delta functions, we obtain

\[
I_\pi = \int K(t, \xi_0, \cdots, \xi_0, \xi_{k_{a_1}}, \cdots, \xi_{k_{a_{n/2}}}) \prod_{k \in A_0} \varepsilon^{d-m} \hat{R}(\varepsilon(\xi_k - \xi_0)) \hat{u}_0(\xi_0) d\xi_{A_0}, \quad (3.64)
\]

where \( \{k_{a_1}, \cdots, k_{a_{n/2}}\} = A_0 \).

This implies

\[
|I_\pi| \leq \int d\alpha \frac{1}{|\alpha + \xi_0^m + i\eta|^{n/2+1}} \prod_{k \in A_0} \left| \int \varepsilon^{d-m} \frac{\hat{R}(\varepsilon(\xi_k - \xi_0))}{\alpha + \xi_k^m + i\varepsilon^m \eta} d\xi_k \right| |\hat{u}_0(\xi_0)|. \quad (3.65)
\]

Define

\[
\Theta_{\alpha,\eta}(\xi_0) = \int \frac{\varepsilon^{d-m} \hat{R}(\varepsilon(\xi_k - \xi_0))}{\alpha + \xi_k^m + i\varepsilon^m \eta} d\xi_k. \quad (3.66)
\]

Perform the change of variable \( \xi_k \rightarrow \frac{\xi_k}{\varepsilon} \). This gives

\[
\Theta_{\alpha,\eta}(\xi_0) = \int \frac{\hat{R}(\xi_k - \varepsilon \xi_0)}{\varepsilon^m \alpha + \xi_k^m + i\varepsilon^m \eta} d\xi_k. \quad (3.67)
\]

It is clear from Lemma 3.17 that

\[
|\Theta_{\alpha,\eta}(\xi_0)| \leq C. \quad (3.68)
\]
Thus (3.65) already implies

\[ |I_\pi| \leq C^n |\hat{u}_0(\xi)|. \]  

(3.69)

However, this estimate is not sufficient and there is in fact a term \(1/(n/2)!\) missing. We now recover this factor.

Introduce the notation

\[ \Theta(\xi) = \lim_{\eta \to 0} \Theta - \xi_m,\eta(\xi), \]  

(3.70)

From the estimate in Lemma 3.17, we have

\[ |\Theta_{\alpha,\eta}(\xi_0) - \Theta(\xi_0)| \leq \varepsilon^{m/2}(|\alpha + \xi_m^m|/|\eta|^{-1/2} + |\eta|^{1/2}). \]  

(3.71)

We shall show that the leading term of \(I_\pi\) is

\[ K(t, \xi_0, \cdots, \xi_0)\Theta(\xi_0)^{n/2}\hat{u}_0(\xi_0). \]  

(3.72)

In fact, the error term is bounded by

\[ \int d\alpha \frac{1}{|\alpha + \xi^m_0 + i\eta|^{n/2+1}}|\Theta^{n/2}(\xi_0) - \Theta^{n/2}(\xi_0)||\hat{u}_0(\xi_0)|. \]  

(3.73)

Using the uniform bound on \(\Theta\) in (3.68) and the estimate in (3.71), we can bound (3.73) by \(O(\varepsilon^{m/2}C^n |\hat{u}_0(\xi_0)|)\).

We can now use Lemma 3.10 to bound (3.72) by

\[ \frac{C^m}{(n/2)!}|\hat{u}_0(\xi_0)|. \]  

(3.74)

Finally, we obtain (3.62) as an immediate consequence of (3.61). This concludes Lemma 3.4.
3.3 Partial Time Integration

In this section, we estimate the $L^2$ norm of $\Psi_{n_0,\varepsilon}$. The central idea is to subdivide the time integration into smaller time intervals of size $t/\kappa(\varepsilon)$ with

$$\kappa(\varepsilon) := |\log \varepsilon|^{1/\gamma^2}.$$ \hfill (3.75)

We then use the Duhamel formula to estimate the evolution in each time interval. Recall the error term $\Psi_{n_0,\varepsilon} = \sum_{n=n_0}^{+\infty} u_n,\varepsilon$. The Duhamel formula states that

$$\Psi_{n_0,\varepsilon}(t) = (-i) \frac{1}{\varepsilon^{m/2}} \int_0^t ds e^{i(t-s)H} q(x) u_{n_0-1,\varepsilon}(s),$$ \hfill (3.76)

where $e^{itH}$ denotes the propagator of equation (1.1).

Let $\theta_j = j t / \kappa$ for $j = 0, 1, \cdots, \kappa$. Rewrite

$$\Psi_{n_0,\varepsilon}(t) = (-i) \frac{1}{\varepsilon^{m/2}} \sum_{j=0}^{\kappa-1} e^{i(t-\theta_j)H} \int_{\theta_j}^{\theta_{j+1}} e^{i(\theta_{j+1}-s)H} q(x) u_{n_0-1,\varepsilon}(s) ds.$$ \hfill (3.77)

Define the $(n-n_0)$-th term of the Duhamel expansion of $(-i) \frac{1}{\varepsilon^{m/2}} \int_{\theta_j}^{\theta_{j+1}} e^{i(\theta_{j+1}-s)H} q(x) u_{n_0-1,\varepsilon}(s) ds$ as

$$u_{n_0,\theta_j} = (-i)^n u_{n-n_0,\theta_j+1} \frac{1}{\varepsilon^{m/2}} \int_{\theta_j}^{\theta_{j+1}} \int_0^t \cdots \int_0^t \prod_{k=0}^{n-n_0} ds_k \delta(\theta_{j+1} - s - \sum_{k=0}^{n-n_0} s_k) e^{is_0 H_0 q(x) \cdot q(x)} e^{is_{n-n_0+1} H_0 u_{n_0-1,\varepsilon}(s) ds}. \hfill (3.78)$$
We may further obtain the form of $u_{n,n_0,\theta_j}$ in terms of $u_0$ by writing $u_{n_0-1,\varepsilon}(s)$ out explicitly using (4.18)

$$u_{n,n_0,\theta_j} = (-i)^n \left( \frac{1}{\varepsilon^m/2} \right)^n \int_{\theta_j}^{\theta_{j+1}} \int_0^t \cdots \int_0^t (\prod_{k=0}^n ds_k) \delta(\theta_{j+1} - s - \sum_{k=0}^{n-n_0} s_k) \delta(s - \sum_{k=n-n_0+1}^n s_k) e^{i\varepsilon_0 H_0 q_x(s)} \cdots q_x(s) e^{i\varepsilon_0 H_0 u_0 ds}. $$

(3.79)

The amputated versions of these functions are defined as

$$\tilde{u}_{4n_0,n_0,\theta_j}(x) = \frac{1}{\varepsilon^m/2} q_x(s) u_{4n_0-1,n_0,\theta_j}(x). $$

(3.80)

Duhamel formula then gives

$$\Psi_{n_0,\varepsilon} = U_1 + U_2, $$

(3.81)

where

$$U_1(t) = \sum_{n_0 \leq n < 4n_0} \sum_{j=0}^{n-1} e^{i(t-\theta_{j+1})H} u_{n,n_0,\theta_j}(\theta_{j+1}), $$

$$U_2(t) = (-i) \sum_{j=0}^{n-1} e^{i(t-\theta_{j+1})H} \int_{\theta_j}^{\theta_{j+1}} e^{-i(\theta_{j+1} - s)H} \tilde{u}_{4n_0,n_0,\theta_j}(s) ds. $$

(3.82)

From the unitarity of $e^{i(t-\theta_{j+1})H}$ and the triangle inequality, we can bound $U_1$ by

$$\|U_1\|_2^2 \leq C n_0 \kappa \sum_{n_0 \leq n < 4n_0} \sum_{j=0}^{n-1} \|u_{n,n_0,\theta_j}(\theta_{j+1})\|_2^2 \leq C n_0^2 \kappa^2 \sup_{n_0 \leq n < 4n_0} \|u_{n,n_0,\theta_j}\|_2^2. $$

(3.83)

Applying the Cauchy-Schwarz inequality, we can bound $U_2$ by

$$\|U_2\|_2^2 \leq t \sum_{j=0}^{n-1} \int_{\theta_j}^{\theta_{j+1}} \|\tilde{u}_{4n_0,n_0,\theta_j}(s)\|_2^2 ds \leq t^2 \sup_{\theta_j \leq s \leq \theta_{j+1}} \|\tilde{u}_{4n_0,n_0,\theta_j}(s)\|_2^2. $$

(3.84)

Denote

$$I_{n_0+1,n} := \{n - n_0 + 1, \ldots, n\}. $$

(3.85)
3.3. PARTIAL TIME INTEGRATION

Define the free evolution operator with constraint given by the parameters \( n_0 \) and \( \theta \) as

\[
K^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) := \int_{\theta_j}^{\theta_{j+1}} K(\theta_{j+1} - s; \xi, n - n_0)K(s, \xi, I_{n-n_0+1,n})ds. \tag{3.86}
\]

We can write the wave function in Fourier space \( \hat{u}_{n,n_0,\theta_j} \) as

\[
\hat{u}_{n,n_0,\theta_j}(\theta_{j+1}, \xi_0) = \int K^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) L(\xi, n) \hat{u}_0(\xi_0) d\xi_0. \tag{3.87}
\]

We then write

\[
E\|u_{n,n_0,\theta_j}(\theta_{j+1})\|^2 = \sum_{\pi \in \Pi(n,n)} C^\#_{\pi}, \tag{3.88}
\]

where

\[
C^\#_{\pi} := \int d\xi d\zeta K^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) K^\#(\theta_{j+1}, \theta_j; \zeta, n, n_0) \Delta_{s} F(\xi, n) F(\zeta, n). \tag{3.89}
\]

For the amputated function, we have

\[
E\|\tilde{u}_{n,n_0,\theta_j}(\theta_{j+1})\|^2 = \sum_{\pi \in \Pi(n,n)} \tilde{C}^\#_{\pi}, \tag{3.90}
\]

\[
\tilde{C}^\#_{\pi} := \int d\xi d\zeta \tilde{K}^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) \tilde{K}^\#(\theta_{j+1}, \theta_j; \zeta, n, n_0) \Delta_{s} F(\xi, n) F(\zeta, n), \tag{3.91}
\]

where

\[
\tilde{K}^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) := \int_{\theta_j}^{\theta_{j+1}} K(\theta_{j+1} - s; \xi, I_{1,n-n_0})K(s, \xi, I_{n-n_0+1,n})ds. \tag{3.92}
\]
Recall Lemma 3.10. We can extend it to the following identity for $K^\#(\theta_{j+1}, \theta_j; \xi, n, n_0)$:

$$K^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) = - \int_{\theta_j}^{\theta_{j+1}} d\theta e^{-i\theta_j} \int_{-\infty}^{+\infty} d\alpha \alpha e^{-i\alpha \theta_j} e^{i\alpha s} \prod_{k=0}^{n-n_0} \frac{1}{\alpha + \xi_k + i\eta} \prod_{k \in I_{n-n_0+1,n}} \frac{1}{\alpha + \xi_k + i\eta},$$

(3.93)

where we choose $\eta := (t/\kappa)^{-1}, \tilde{\eta} := t^{-1}$. We can integrate in $s$ to have

$$K^\#(\theta_{j+1}, \theta_j; \xi, n, n_0) = i \int_{-\infty}^{+\infty} d\alpha d\tilde{\alpha} e^{-i\theta_j} \prod_{k=0}^{n-n_0} \frac{1}{\alpha + \xi_k + i\eta} \prod_{k \in I_{n-n_0+1,n}} \frac{1}{\alpha + \xi_k + i\eta}.$$

(3.94)

Hence we can bound $C^\#_\pi$ by

$$|C^\#_\pi| \leq \int d\xi \int_{-\infty}^{+\infty} d\alpha d\beta d\tilde{\beta} \prod_{0 \leq k \leq 2n+1} \frac{1}{|\alpha_k + \xi_k + i\eta_k|} \prod_{k=1, k \neq n+1}^{2n+1} r(\varepsilon(\xi_k - \xi_{k-1})),$$

(3.95)

where $\alpha_k$ and $\eta_k$ are defined as

$$\alpha_k := \begin{cases} 
\tilde{\alpha} & \text{if } k \leq n - n_0 \\
\alpha & \text{if } n - n_0 + 1 \leq k \leq n \\
\beta & \text{if } n + 1 \leq k \leq n + n_0 \\
\tilde{\beta} & \text{if } k \geq n + n_0 + 2 
\end{cases}$$

(3.96)

and

$$\eta_k := \begin{cases} 
\tilde{\eta} & \text{if } k \leq n - n_0 \\
\eta & \text{if } n - n_0 + 1 \leq k \leq n \\
\eta & \text{if } n + 1 \leq k \leq n + n_0 \\
\tilde{\eta} & \text{if } k \geq n + n_0 + 2 
\end{cases}$$

(3.97)
We present the following lemmas to show \( \lim_{\varepsilon \to 0} E \| \Psi_{n_0, \varepsilon} \|^2_2 \to 0. \)

**Lemma 3.5.** Let \( n = 4n_0 \). For any \( \pi \in \Pi(n, n) \) we have

\[
|\tilde{C}^\#_{\pi}| \leq \left( \frac{C |\log \varepsilon|}{\kappa^{n_0}} \right)^{4n_0}.
\]

**Proof.** The following bound can be easily obtained using Lemma 3.16

\[
|\tilde{C}^\#_{\pi}| \leq \left( \frac{C |\log \varepsilon|}{\kappa^{n_0}} \right)^{4n_0}.
\]

To recover the denominator in (3.98), notice that among the \( \eta_k \) for \( k \in B_0 \), there are at least \( n - 2n_0 - 2(\geq n_0) \) of them with \( \eta_k = \kappa/t \). Hence

\[
\prod_{k \in B_0} \left| \frac{1}{\alpha_k + \xi_k^m + i\eta_k} \right| \leq \prod_{k \in B_0} |\eta_k|^{-1} \leq t^n \left( \frac{t}{\kappa} \right)^{n_0} \leq \frac{t^n}{\kappa^{n_0}},
\]

which provides the term \( \kappa^{-n_0} \) and hence gives (3.98).

**Lemma 3.6.** If \( \pi \in \Pi_c(n, n) \), then we have

\[
|C^\#_{\pi}| \leq (C \log \varepsilon)^n \varepsilon^\lambda \| \Phi(\xi) \|^2_2.
\]

**Proof.** The proof of Lemma 3.6 is essentially the same as in Lemma 3.2. The only difference is that the integral in \( \alpha \) and \( \tilde{\alpha} \) (\( \beta \) and \( \tilde{\beta} \)) is estimated using Proposition 5.3 by

\[
\int \int \frac{1}{|\alpha - \alpha + i(\eta - \bar{\eta})|} \frac{1}{|\tilde{\alpha} + \xi^m + i\eta|} \frac{1}{|\alpha + \xi^m / \varepsilon + i\tilde{\eta}|} d\alpha d\tilde{\alpha} \leq C \left[ 1 + \log \left( \frac{|\xi^m - \xi^m / \varepsilon|}{|\xi^m - \xi^m / \varepsilon + i\tilde{\eta}|} \right) \right]^2,
\]

(3.102)
CHAPTER 3. HOMOGENIZATION OF SCHröDINGER EQUATION

and the integral in $\xi_0$ is estimated using Proposition 5.6

$$
\varepsilon^{-m} \int \frac{1}{(\xi_0)^{2d}} \left[ 1 + \log_+ \frac{|\xi_0^m/\eta - \xi_m^m|}{|\xi_0^m/\varepsilon \xi_n^m + i\eta|^2} \right]^4 d\xi_0 \leq C\varepsilon^\lambda (\xi_n^\lambda \lor 1). \tag{3.103}
$$

Lemma 3.7. If $\pi \in \Pi_{ncs}(n,n)$, then we have

$$
|C^\#_{\overline{\pi}}| \leq (C \log \varepsilon)^n \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2. \tag{3.104}
$$

Proof. The proof of Lemma 3.7 is similar to that of Lemma 3.3. The only difference is that the integral in $\alpha$ and $\tilde{\alpha}$ ($\beta$ and $\tilde{\beta}$) is estimated by

$$
\begin{align*}
&\int \int (\alpha \phi(1-\delta)/m \lor 1) \frac{1}{|\alpha - \alpha + i(\eta - \tilde{\eta})|} \frac{1}{|\alpha + \xi_0^m + i\eta|} \frac{1}{|\alpha + \xi_0^m + i\eta|} d\alpha d\tilde{\alpha} \\
&\leq C \int (\alpha \phi(1-\delta)/m \lor 1) \frac{1}{|\alpha + \xi_0^m + i\eta|^2} \left[ \log_+ \left| \frac{\alpha + \xi_0^m}{\tilde{\eta}} \right| + 1 \right] d\alpha \leq C\xi_0^{\lambda(1-\delta)}. \tag{3.105}
\end{align*}
$$

Lemma 3.8. If $\pi \in \Pi_s(n,n)$, then we have

$$
|C^\#_{\overline{\pi}}| \leq \frac{C^m}{(n!)^{1/2}} \|\tilde{u}_0(\xi)\|_2^2. \tag{3.106}
$$

Proof. The proof is essentially the same as in Lemma 3.4. We shall not repeat the argument here.

We now apply the above lemmas to estimate $\Psi_{n_0,\varepsilon}$. From Lemma 3.6, 3.7, and 3.8 we have

$$
\|U_1\|_2^2 \leq n_0^2\kappa^2 [(C \log \varepsilon)^{4n_0} \varepsilon^\lambda] (\frac{8n_0}{\varepsilon})^{4n_0} + \frac{C^m n_0^2 \kappa^2}{(n_0!)^{1/2}}. \tag{3.107}
$$
From Lemma 3.5 we have
\[ \|U_2\|_2^2 \leq \left( \frac{C|\log \varepsilon|}{n_0} \right) \left( \frac{8n_0}{e^4} \right)^{4n_0}. \] (3.108)

The $L^2$ estimate of $\Psi_{n_0,\varepsilon}$ is therefore given by
\[ E\| \sum_{n_0} \hat{u}_{\varepsilon,n}(t) \|_2^2 = E\| \Psi_{n_0,\varepsilon} \|_2^2 \leq 2(E\|U_1\|_2^2 + E\|U_2\|_2^2) \to 0 \] (3.109)
as $\varepsilon \to 0$.

### 3.4 Homogenization

We come back to the analysis of $U_{\varepsilon,s}(t,\xi)$. We find that $U_{\varepsilon,s}$ is the solution to the following equation
\[
U_{\varepsilon,s} = e^{it \xi^m} \hat{u}_0(\xi) - \int_0^t e^{is \xi^m} \int_0^{t-s} e^{i s_1 \xi_1^m} \int_0^\varepsilon e^{d-m} \hat{R}(\varepsilon(\xi_1 - \xi)) U_{\varepsilon,s}(t-s-s_1,\xi) d\xi_1 dsds_1 \\
:= e^{it \xi^m} \hat{u}_0(\xi) + A_\varepsilon U_{\varepsilon,s}(t,\xi).
\] (3.110)

**Lemma 3.9.** Let us define $\Upsilon_\varepsilon(t,\xi)$ to be the solution to
\[
(i \frac{\partial}{\partial t} + \xi \mathbf{m}) \Upsilon_\varepsilon(t,\xi) = 0
\]
\[ \Upsilon_\varepsilon(0,\xi) = \hat{u}_0(\xi), \] (3.111)

with $\rho_\varepsilon = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi_1 - \varepsilon \xi)}{\xi_1^m} d\xi_1$. We have the convergence results
\[
|U_{\varepsilon,s} - \Upsilon_\varepsilon(t)| \leq \max\{\varepsilon^m, \varepsilon^{d-m} |\log \varepsilon|\} |\hat{u}_0(\xi)|.
\] (3.112)
Proof. (1) We obtain from Duhamel’s principle that

\[ \Phi_\varepsilon(t, \xi) = e^{i t \xi^m} \hat{u}_0(\xi) - i \int_0^t e^{i \varepsilon \xi^m \nu} \int_{\mathbb{R}^d} \frac{\hat{R}(\xi_1 - \varepsilon \xi)}{\xi_1^m} d\xi_1 \Phi_\varepsilon(t - \nu, \xi) d\nu \]

\[ := e^{i t \xi^m} \hat{u}_0(\xi) + B\varepsilon \Phi_\varepsilon(t, \xi) \]

\[ := e^{i t \xi^m} \hat{u}_0(\xi) + A\varepsilon \Phi_\varepsilon(t, \xi) + E\varepsilon \Phi_\varepsilon(t, \xi), \]

where the operator \( A\varepsilon \) is defined in (3.110) and may be recast as

\[ A\varepsilon \Phi_\varepsilon = - \int_0^t e^{i \varepsilon \xi^m \nu} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i s_1 (\xi_1^m - \varepsilon \xi^m)} ds_1 \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 \Phi_\varepsilon(t - \nu, \xi) d\nu \]

\[ = - \int_0^t e^{i \varepsilon \xi^m \nu} \int_{\mathbb{R}^d} \frac{1}{i (\xi_1^m - \varepsilon \xi^m)} (e^{i \varepsilon \xi^m \nu} - 1) \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 \Phi_\varepsilon(t - \nu, \xi) d\nu, \]

and the remainder \( E\varepsilon \) is then given by

\[ E\varepsilon \Phi_\varepsilon = i \int_0^t \int_{\mathbb{R}^d} \frac{1}{\xi_1^m - \varepsilon \xi^m} e^{i \varepsilon \xi^m \nu} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 \Phi_\varepsilon(t - \nu, \xi) d\nu \]

\[ + i \int_0^t \int_{\mathbb{R}^d} \frac{\varepsilon \xi^m \nu}{\xi_1^m (\xi_1^m - \varepsilon \xi^m)} (e^{i \varepsilon \xi^m \nu} - e^{i \varepsilon \xi^m \nu}) \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 \Phi_\varepsilon(t - \nu, \varepsilon) d\nu \]

\[ := I_1 + I_2. \]

For the calculation of \( I_1 \), note that equation (3.111) has the explicit solution:

\[ \Phi(t, \xi) = e^{i t (\xi^m - \rho(\xi))} \hat{u}_0(\xi). \]

On one hand, we may obtain the following expression of integral in \( \nu \) using the method of
3.4. HOMOGENIZATION

separation of variables:

\[
\int_0^t e^{i \frac{m}{\varepsilon} (\xi_1^m - \varepsilon^m \xi^m)} \mathcal{U}_\varepsilon(t - v, \xi) dv = \frac{\varepsilon^m}{\xi_1^m - \varepsilon^m \xi^m} \int_0^t \mathcal{U}_\varepsilon(t - v, \xi) d\left(e^{i \frac{m}{\varepsilon} (\xi_1^m - \varepsilon^m \xi^m)} \hat{u}_0(\xi) - e^{i(t-v)(\xi^m - \rho_\varepsilon(\xi))} \hat{u}_0(\xi)\right) + i \int_0^t e^{i \frac{m}{\varepsilon} (\xi_1^m - \varepsilon^m \xi^m)} e^{i(t-v)(\xi^m - \rho_\varepsilon(\xi))} \hat{u}_0(\xi) dv. \tag{3.117}
\]

On the other hand, we have the following simple estimate of this integral:

\[
\left| \int_0^t e^{i \frac{m}{\varepsilon} (\xi_1^m - \varepsilon^m \xi^m)} \mathcal{U}_\varepsilon(t - v, \xi) dv \right| \leq \int_0^t \left| e^{i \frac{m}{\varepsilon} (\xi_1^m - \varepsilon^m \xi^m)} \mathcal{U}_\varepsilon(t - v, \xi) \right| dv \leq C |\hat{u}_0(\xi)|. \tag{3.118}
\]

This gives that

\[
\left| \int_0^t e^{i \frac{m}{\varepsilon} (\xi_1^m - \varepsilon^m \xi^m)} \mathcal{U}_\varepsilon(t - v, \xi) dv \right| \leq C \varepsilon^m \left( \frac{1}{\xi_1^m - \varepsilon^m \xi^m} \wedge \frac{1}{\varepsilon^m}\right) \max\{|\hat{u}_0(\xi)|, |\xi^m \hat{u}_0(\xi)|\}. \tag{3.119}
\]

We therefore find the estimate of $|I_1|$ using Lemma 3.18:

\[
|I_1| \leq C \varepsilon^m \int_{\mathbb{R}^d} \left( \frac{1}{\xi_1^m - \varepsilon^m \xi^m} \wedge \frac{1}{\varepsilon^m}\right) \max\{|\hat{u}_0(\xi)|, |\xi^m \hat{u}_0(\xi)|\} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 \leq C \max\{1, \xi^m\} \max\{\varepsilon^m, \varepsilon^{d-m}\} \log \varepsilon (\xi^m + 1)^{d-2}, \varepsilon^{d-m}(\xi^m + 1)^{d-2}\} |\hat{u}_0(\xi)|. \tag{3.120}
\]

For the calculation of $I_2$, we first estimate the integral in $v$:

\[
\varepsilon^m \left| \int_0^t e^{i \frac{m}{\varepsilon} v} - e^{i \frac{m}{\varepsilon^m} v} \mathcal{U}_\varepsilon(t - v, \xi) dv \right| \leq C \left( \frac{1}{\xi_1^m - \varepsilon^m \xi^m} \wedge \frac{1}{\varepsilon^m}\right) \max\{|\hat{u}_0(\xi)|, |\xi^m \hat{u}_0(\xi)|\}. \tag{3.121}
\]

Hence

\[
|I_2| \leq C \int_{\mathbb{R}^d} \left( \frac{\xi^m}{\xi_1^m} \wedge \frac{1}{\xi_1^m - \varepsilon^m \xi^m} \wedge \frac{1}{\varepsilon^m}\right) \hat{R}(\xi_1 - \varepsilon \xi) \max\{|\hat{u}_0(\xi)|, |\xi^m \hat{u}_0(\xi)|\} d\xi_1 \leq C \max\{1, \xi^m\} \max\{\varepsilon^m, \varepsilon^{d-m}\} \log \varepsilon (\xi^m + 1)^{d-2}, \varepsilon^{d-m}(\xi^m + 1)^{d-2}\} |\xi^m \hat{u}_0(\xi)|. \tag{3.122}
\]

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Note that
\[ |A_\varepsilon U(t, \xi)| \leq C \int_0^t |U(s, \xi)| ds, \]  
(3.123)
over a bounded interval in time. The equation
\[ (I - A_\varepsilon)U(t, \xi) = S(t, \xi) \]  
(3.124)
therefore admits a unique solution by Gronwall's Lemma, which is bounded by
\[ |U(t, \xi)| \leq \|S\|_\infty e^{Ct}. \]  
(3.125)
We verify that the solution to
\[ (I - B_\varepsilon)\mathcal{U}_\varepsilon = e^{it\xi} \hat{u}_0(\xi), \]  
(3.126)
is given by
\[ \mathcal{U}_\varepsilon(t, \xi) = e^{it(\xi^m - \rho_\varepsilon(\xi))} \hat{u}_0(\xi). \]  
(3.127)
The error \( V_\varepsilon(t, \xi) = (U_{\varepsilon,s}(t, \xi) - \mathcal{U}_\varepsilon(t, \xi)) \) is a solution to
\[ (I - A_\varepsilon)V_\varepsilon = E_\varepsilon \mathcal{U}_\varepsilon(t, \xi), \]  
(3.128)
so that over bounded intervals in time, we find that
\[ |U_{\varepsilon,s}(t, \xi) - \mathcal{U}_\varepsilon(t, \xi)| = |V_\varepsilon(t, \xi)| \leq C \max\{\varepsilon^m, \varepsilon^{d-m}\log \varepsilon\}|\hat{u}_0(\xi)|. \]  
(3.129)
From our assumption that $\hat{R}(\xi) \in C^2(\mathbb{R}^d)$, we find that

$$|e^{it(\xi^m - \rho_\varepsilon(\xi))} - e^{it(\xi^m - \rho)}| \leq t|\rho_\varepsilon(\xi) - \rho| \leq C\varepsilon^2 \xi^2. \quad (3.130)$$

The reason for the second-order accuracy is that $\hat{R}(-\xi) = \hat{R}(\xi)$ and $\nabla \hat{R}(0) = 0$ so that first-order terms in the Taylor expansion vanish.

In terms of the solutions of PDE we defined in (3.3), we may recast the above result as

$$|(U_{\varepsilon,s} - \hat{u})(t)| \leq C \max\{\varepsilon^2, \varepsilon^m, \varepsilon^{d-m}|\log \varepsilon|\}|\hat{u}_0(\xi)| \quad \hat{u}(t, \xi) = e^{i(\xi^m - \rho)t}\hat{u}_0(\xi), \quad (3.131)$$

We now prove Theorem 3.1. By the triangle inequality, we have the estimate

$$\|\hat{u}_\varepsilon(t, \xi) - \hat{u}(t, \xi)\|_2$$

$$\leq \sum_{n=0}^{n_0(\varepsilon)-1} (u_{\varepsilon,n} - U_{\varepsilon,s}^n)(t)\|_2 + \sum_{n_0(\varepsilon)}^{+\infty} \hat{u}_{\varepsilon,n}(t)\|_2 + \|(U_{\varepsilon,s} - \hat{u})(t)\|_2 + \sum_{n_0(\varepsilon)}^{+\infty} U_{\varepsilon,s}^n(t)\|_2. \quad (3.132)$$

The vanishing of the first three terms on right hand side of this inequality when $\varepsilon$ goes to zero follows from (3.60), (3.109) and (3.131) respectively. The fourth term also vanishes because of the $L^2$ convergence of $U_{\varepsilon,s}$.

### 3.5 Inequalities and Proofs

In this section, we present and prove several inequalities used in earlier sections. There are similar versions of Lemma 3.11 and 3.16 in [9]. The proofs are given below for the convenience of the reader. The proofs of similar versions of Lemma 3.17 can be found in [9].
Lemma 3.10. We have the following identity for $\eta > 0$:

$$K(t, \xi, I) = i e^{t \eta} \int d\alpha e^{-i \alpha t} \prod_{k \in I} \frac{1}{\alpha + \xi_k^m + i \eta}. \quad (3.133)$$

We also claim the following estimate with $n := |I| - 1$:

$$|K(t, \xi, I)| \leq \frac{t^n}{n!}. \quad (3.134)$$

Proof. We can replace the upper integration bound $t$ in (3.12) with $\infty$. Hence

$$K(t; \xi, I) = (-i)^{|I|-1} \int_0^\infty \cdots \int_0^\infty (\prod_{k \in I} ds_k) \delta(t - \sum_{k \in I} s_k) e^{(t - \sum_{k \in I} s_k)\eta} \prod_{k \in I} e^{is_k \xi_k^m}. \quad (3.135)$$

Use the formula $\delta(t) = \int e^{-i \alpha t} d\alpha$. We obtain

$$K(t; \xi, I) = e^{t \eta} \int_0^t \cdots \int_0^t (\prod_{k \in I} ds_k) e^{-i \alpha t} \prod_{k \in I} e^{is_k (\alpha + \xi_k^m + i \eta)} d\alpha$$

$$= i e^{t \eta} \int e^{-i \alpha t} \prod_{k \in I} \frac{1}{\alpha + \xi_k^m + i \eta} d\alpha. \quad (3.136)$$

We need to introduce $\eta > 0$ in (3.135) for the integration in $s_k$ as $\lim_{s \to \infty} e^{is (\alpha + \xi_k^m + i \eta)} = 0$. □

Lemma 3.11. Assume $\eta > 0$. We have the following inequality:

$$\int_{-\infty}^{\infty} \frac{d\alpha}{|\alpha + A + i \eta||\alpha + B + i \eta|} \leq \frac{C}{|A - B + i \eta|} \left[ 1 + \log_+ \left( \frac{|A - B|}{\eta} \right) \right], \quad (3.137)$$

where $\log_+ x := \max\{0, \log x\}$ for $x > 0$ and $\log_+ 0 := 0$.

Proof. Without loss of generality, we assume $B > A$. We split the integration over
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$(-\infty, -\frac{A+B}{2})$ as follows:

\[
\int_{-\infty}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} = \int_{-\infty}^{-B} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} + \int_{-B-(B-A)}^{-B} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} + \int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|}.
\]

The first term is estimated as

\[
\int_{-B}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \frac{1}{-\frac{A+B}{2} + A + i\eta} \int_{-B}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + B + i\eta|} \leq \frac{1}{|A-B+i\eta|} \int_{0}^{\frac{B-A}{\eta}} \frac{d\alpha}{\sqrt{\alpha^2 + 1}} \leq \frac{2}{C} \int_{0}^{\frac{B-A}{\eta}} \frac{d\alpha}{\sqrt{\alpha^2 + 1}} \leq \frac{1}{|A-B+i\eta|} \left[ 1 + \log + \left| \frac{A-B}{\eta} \right| \right].
\]

Likewise, the second term is estimated as

\[
\int_{-B-(B-A)}^{-B} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \frac{1}{|A-B+i\eta|} \int_{-B-(B-A)}^{-B} \frac{d\alpha}{|\alpha + B + i\eta|} \leq \frac{1}{|A-B+i\eta|} \int_{0}^{\frac{B-A}{\eta}} \frac{d\alpha}{\sqrt{\alpha^2 + 1}} \leq \frac{1}{|A-B+i\eta|} \left[ 1 + \log + \left| \frac{A-B}{\eta} \right| \right].
\]

We obtain the bound for the third term by using the inequality $|\alpha + A + i\eta| \leq |\alpha + B + i\eta|$ on $(-\infty, -B-(B-A))$.

\[
\int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \int_{-\infty}^{\frac{B-A}{\eta}} \frac{d\alpha}{|\alpha + B + i\eta|^2} = \frac{1}{\eta} \int_{\frac{B-A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1}.
\]
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If $B - A \geq \eta$, we have

$$\int_{\frac{B - A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} \leq \int_{\frac{B - A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} \leq \frac{\eta}{B - A},$$

(3.142)
in which case

$$\int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \frac{1}{B - A} \leq \frac{\sqrt{2}}{|A - B + i\eta|}.$$  

(3.143)

If $B - A < \eta$, we have

$$\int_{\frac{B - A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} \leq \int_{1}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} + \int_{\frac{B - A}{\eta}}^{1} \frac{d\alpha}{\alpha^2 + 1} \leq \frac{\pi}{4} + (1 - \frac{B - A}{\eta}),$$

(3.144)
in which case

$$\int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \frac{1}{\eta} \left(\frac{\pi}{4} + 1\right) \leq \frac{\sqrt{2}(\pi/4 + 1)}{|A - B + i\eta|}.$$  

(3.145)

By symmetry, the integration over $(-\infty, -\frac{A + B}{2})$ admits an identical bound.

Proposition 3.12. Assume $\eta, \tilde{\eta} > 0$, $\tilde{\eta} > 2\eta$, and $\eta^{-1}$ bounded. We have the following inequality:

$$\int \int \frac{1}{|\tilde{\alpha} - \alpha + i(\eta - \tilde{\eta})||\tilde{\alpha} + A + i\eta|} \frac{1}{|\alpha + B + i\eta|} \frac{d\tilde{\alpha} d\alpha}{|\alpha - \tilde{\alpha} + i(\eta - \tilde{\eta})||\alpha + A + i\eta|} \leq C \left[1 + \log \left|\frac{A + B}{\eta}\right|\right]^2.$$  

(3.146)

Proof. Without loss of generality, we assume $B > A$.

We first integrate in $\tilde{\alpha}$ by using Lemma 3.11:

$$\int \frac{1}{|\tilde{\alpha} - \alpha + i(\eta - \tilde{\eta})||\tilde{\alpha} + A + i\eta|} \frac{1}{|\alpha + B + i\eta|} d\tilde{\alpha} \leq \frac{1}{|\alpha + A + i\eta|} \left[1 + \log \left|\frac{\alpha + A}{\eta}\right|\right].$$  

(3.147)

The integration over $(-\infty, -\frac{A + B}{2})$ is split as in (3.138) into three pieces: $(-B, -\frac{A + B}{2})$, $(-B - (B - A), -B)$ and $(-\infty, -B -(B - A))$.  

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The term \(1 + \log_+ \left| \frac{\alpha + A}{\eta} \right|\) is bounded by \(1 + \log_+ \left| \frac{A - B}{\eta} \right|\) on the interval \((-B, -\frac{A+B}{2})\). The integral over \((-B, -\frac{A+B}{2})\) is then bounded by

\[
\int_{-\frac{A+B}{2}}^{-B} \frac{1 + \log_+ \left| \frac{\alpha + A}{\eta} \right|}{\left| \alpha + A + i\eta \right| \left| \alpha + B + i\eta \right|} d\alpha \leq \frac{C}{\left| A - B + i\eta \right|} \left[ 1 + \log_+ \left| \frac{A - B}{\eta} \right| \right]^2.
\] (3.148)

The same bound holds for the integral over \((-B - (B - A), -B)\). It remains to estimate the integral over \((-\infty, -B - (B - A))\). The domain of integration can be divided into two parts. On the set \(|\alpha + A| < \tilde{\eta}\), we have \(\log_+ \left| \frac{\alpha + A}{\eta} \right| = 0\), so that applying Lemma 3.11 immediately gives the desired result. On the set \(|\alpha + A| \geq \tilde{\eta}\), we use the simple bound on this interval

\[
1 + \log_+ \left| \frac{\alpha + A}{\eta} \right| \leq \left| \frac{\alpha + A}{\eta} + i \right|^{1/2}.
\] (3.149)

It then follows that

\[
\int_{-\infty}^{-B-(B-A)} \frac{1 + \log_+ \left| \frac{\alpha + A}{\eta} \right|}{\left| \alpha + A + i\eta \right| \left| \alpha + B + i\eta \right|} d\alpha \leq C \int_{-\infty}^{-B-(B-A)} \frac{1}{\left| \alpha + B + i\eta \right|^{3/2}} d\alpha \leq C \int_{-\infty}^{+\infty} \frac{d\alpha}{\left( \alpha^2 + 1 \right)^{3/2}}.
\] (3.150)

After performing an analysis similar to that from (3.142) to (3.145) we find the estimate:

\[
\int_{-\infty}^{-B-(B-A)} \frac{1 + \log_+ \left| \frac{\alpha + A}{\eta} \right|}{\left| \alpha + A + i\eta \right| \left| \alpha + B + i\eta \right|} d\alpha \leq \frac{C}{\left| A - B + i\eta \right|} \left[ 1 + \log_+ \left| \frac{A - B}{\eta} \right| \right],
\] (3.151)

which concludes the proof.

**Lemma 3.13.** Assume \(\eta > 0\) and \(\eta^{-1}\) is bounded. We have the following inequality:

\[
\int \varepsilon^m \frac{1}{\langle \xi \rangle^{2d}} \frac{1}{\varepsilon^m \alpha + \varepsilon^m \eta_{\varepsilon} + i \varepsilon^m \eta} \xi d\xi \leq C\varepsilon^\lambda (\alpha_{\varepsilon} \wedge 1).
\] (3.152)
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**Proof.** The domain of integration may be split into two parts. On the set \(|\varepsilon^m \alpha + \xi^m| \geq 1\), the integral is bounded by \(\varepsilon^m \|1/\langle \xi \rangle^{2d}\|_1\). On the set \(|\varepsilon^m \alpha + \xi^m| < 1\), we change to polar coordinates

\[
\int_{|\varepsilon^m \alpha + \xi^m| < 1} \varepsilon^m \frac{1}{\langle \xi \rangle^{2d}} \frac{1}{|\varepsilon^m \alpha + \xi^m + i\varepsilon^m \eta|^2} d\xi \leq C \int_{|\varepsilon^m \alpha + \xi^m| < 1} \varepsilon^m \frac{1}{\langle \xi \rangle^{2d}} \frac{\xi^{d-1}}{|\varepsilon^m \alpha + \xi^m + i\varepsilon^m \eta|^2} d|\xi|. \tag{3.153}
\]

When \(m < d < 2m\), we use the inequality \(|1/\langle \xi \rangle^{2d}| \leq 1\) and change variables to \(Q = \varepsilon^m \alpha + \xi^m\)

\[
\int_{|\varepsilon^m \alpha + \xi^m| < 1} \varepsilon^m \frac{\xi^{d-1}}{|\varepsilon^m \alpha + \xi^m + i\varepsilon^m \eta|^2} d|\xi| = \int_0^1 \varepsilon^m (Q - \varepsilon^m \alpha) \frac{d}{m} - 1 \frac{d}{Q + i\varepsilon^m \eta} dQ. \tag{3.154}
\]

On the set \([0, \varepsilon^m \eta]\), we bound \(|Q + i\varepsilon^m \eta|\) from below by its imaginary part and estimate the integral

\[
\int_0^{\varepsilon^m \eta} \varepsilon^m (Q - \varepsilon^m \alpha) \frac{d}{m} - 1 \frac{d}{Q + i\varepsilon^m \eta} dQ \leq \int_0^{\varepsilon^m \eta} \varepsilon^m Q \frac{d}{m} - 1 + (\varepsilon^m \alpha) \frac{d}{m} - 1 dQ \leq C \varepsilon^{d-m} (\alpha \lor 1) \frac{d}{m} - 1. \tag{3.155}
\]

On the set \([\varepsilon^m \eta, 1]\), we bound \(|Q + i\varepsilon^m \eta|\) from below by its real part and estimate the integral

\[
\int_0^{\varepsilon^m \eta} \varepsilon^m (Q - \varepsilon^m \alpha) \frac{d}{m} - 1 \frac{d}{Q} dQ \leq \int_0^{\varepsilon^m \eta} \varepsilon^m Q \frac{d}{m} - 1 + (\varepsilon^m \alpha) \frac{d}{m} - 1 \frac{d}{Q} dQ \leq C \varepsilon^{d-m} (\alpha \lor 1) \frac{d}{m} - 1. \tag{3.156}
\]

This concludes the proof for \(m < d < 2m\). When \(d \geq 2m\), we use the inequality \(|\xi^{d-2m}/\langle \xi \rangle^{2d}| \leq 1\) instead of \(|1/\langle \xi \rangle^{2d}| \leq 1\) and the rest of the proof is similar. \(\square\)

**Proposition 3.14.** Assume \(\eta > 0\) and \(\eta^{-1}\) is bounded. We have the following inequality:

\[
\int \varepsilon^m \frac{1}{\langle \xi \rangle^{2d}} \frac{1}{|\varepsilon^m \alpha + \xi^m + i\varepsilon^m \eta|^2} d\xi \leq C \varepsilon^{\lambda(1-\delta)} (\alpha^{-\lambda(1-\delta)} \lor 1). \tag{3.157}
\]

**Proof.** To prove (3.157), we just need to use \(1/\langle \xi \rangle^{2d}\) to bound an additional \(\xi^{(d-m)\delta}\) and the rest of the proof is the same as in Lemma 3.13. As a result, we lose a contribution \(\varepsilon^{(d-m)\delta}\) for lowering the order of \(\alpha\) by \(\frac{\beta \delta}{m}\). \(\square\)
Proposition 3.15. Assume $\eta > 0$, $\eta^{-1}$ bounded, and $k \in \mathbb{N}$. We have the following inequality:

$$
\int \varepsilon^m \frac{1}{(\xi)^{2d}} \left[ 1 + \log \left(\frac{\alpha + \varepsilon^m / \varepsilon}{\eta} \right) \right]^k \frac{1}{(\varepsilon^m \alpha + \xi^m + i \varepsilon^m \eta)^2} d\xi \leq C \varepsilon^{\lambda} (\alpha \lor 1)^{\frac{\lambda}{m}}.
$$

(3.158)

Proof. The integral on the set $|\alpha + \xi^m / \eta^m| < 1$ is immediately bounded by $C \varepsilon^{\lambda} \alpha^{\frac{\lambda}{m}}$ using Lemma 3.13. The integral on the set $|\alpha + \xi^m / \eta^m| \geq 1$ is bounded by $C \varepsilon^{m} \|1 / (\xi)^{2d}\|_1$. \hfill \Box

Lemma 3.16. Assume $\eta > 0$. We have the following inequalities:

$$
\sup \omega \int \frac{1}{|\alpha + \xi^m + i \eta|} \frac{1}{(\xi - \omega)^{2d}} d\xi \leq C \frac{|\log \eta|}{\langle \alpha \rangle},
$$

(3.159)

where the $\langle \alpha \rangle$ in the denominator again has to be dropped if we remove the factor $(\xi)^{2d}$.

Proof. We will first show that

$$
\sup \omega \int \frac{1}{|\alpha + \xi^m + i \eta|} \frac{1}{(\xi - \omega)^{2d}} d\xi \leq C |\log \eta|.
$$

(3.160)

If $\eta \geq 1$, the integral is bounded by

$$
\int \frac{1}{|\alpha + \xi^m + i \eta|} \frac{1}{(\xi - \omega)^{2d}} d\xi \leq \frac{1}{\eta} \int \frac{1}{(\xi - \omega)^{2d}} d\xi \leq \frac{C}{\eta} \leq C |\log \eta|.
$$

(3.161)

Therefore, we only need to look at the case for $\eta < 1$. The integral over $|\alpha + \xi^m| \geq 1$ is bounded by

$$
\int_{|\alpha + \xi^m| \geq 1} \frac{1}{|\alpha + \xi^m + i \eta|} \frac{1}{(\xi - \omega)^{2d}} d\xi \leq C \int \frac{1}{(\xi - \omega)^{2d}} |\xi| d\xi \leq C \int |\xi| \leq C |\log \eta|.
$$

(3.162)

The integral over $|\alpha + \xi^m| < 1$ can be estimated by splitting the integration domain according
to the size \( k \leq |\xi - \omega| \leq k + 1 \) for \( k = 0, 1, \ldots \)

\[
\int_{|\alpha + \xi^m| < 1} \left| \frac{1}{\alpha + \xi^m + i\eta} \right| \frac{1}{(\xi - \omega)^{2d}} d\xi
\]

\[
= \sum_{k=0}^{+\infty} \int_{\{ |\alpha + \xi^m| < 1 \} \cap \{ k \leq |\xi - \omega| \leq k + 1 \}} \left| \frac{1}{\alpha + \xi^m + i\eta} \right| \frac{1}{(\xi - \omega)^{2d}} d\xi
\]

\[
= \sum_{k=0}^{+\infty} \int_{|\alpha + \xi^m| < 1} \frac{d|\xi|}{|\alpha + \xi^m + i\eta|} \int_{k \leq |\xi - \omega| \leq k + 1} \frac{J(|\xi|, \theta_1, \ldots, \theta_{d-1})}{(\xi - \omega)^{2d}} d\theta_1 \cdots d\theta_{d-1},
\]

where \( J(|\xi|, \theta_1, \ldots, \theta_{d-1}) \) denotes the Jacobian to change to polar coordinates. For fixed \(|\xi|\), we have the inequality

\[
\int_{k \leq |\xi - \omega| \leq k + 1} \frac{J(|\xi|, \theta_1, \ldots, \theta_{d-1})}{(\xi - \omega)^{2d}} d\theta_1 \cdots d\theta_{d-1} \leq \int_{|\xi| \geq 1} \frac{J(|\xi|, \theta_1, \ldots, \theta_{d-1})}{(1 + k^2)^{d-1} |\xi|^{m-1}} d\theta_1 \cdots d\theta_{d-1}
\]

\[
\leq \left\{ \begin{array}{ll}
C \frac{(k+1)^{d-1}}{(1+k^2)^{d-1} |\xi|^{m-1}}, & |\xi| \geq 1 \\
C \frac{1}{|\xi|^{m-1}} |\xi|^{m-1}, & |\xi| < 1.
\end{array} \right.
\]

(3.164)

Summing up the terms over \( k = 0, 1, \ldots \) gives

\[
\int_{|\alpha + \xi^m| < 1} \left| \frac{1}{\alpha + \xi^m + i\eta} \right| \frac{1}{(\xi - \omega)^{2d}} d\xi \leq C \int_{|\alpha + \xi^m| < 1} \frac{|\xi|^{m-1} d|\xi|}{|\alpha + \xi^m + i\eta|} \leq C |\log \eta|.
\]

(3.165)

We now prove (3.159). Without loss of generality, we may assume \(|\alpha| \geq 2\).

The integral over the domain \(|\alpha + \xi^m| \geq \alpha/2\) is easily bounded by

\[
\int_{|\alpha + \xi^m| \geq \alpha/2} \frac{1}{|\alpha + \xi^m + i\eta|} \frac{1}{(\xi - \omega)^{2d}} d\xi \frac{1}{(\xi - \omega)^{2d}} d\xi \leq \frac{C}{|\alpha|} \int_{|\alpha + \xi^m| \leq \alpha/2} \frac{1}{|\alpha + \xi^m + i\eta|} \frac{1}{(\xi - \omega)^{2d}} d\xi \frac{1}{(\xi - \omega)^{2d}} d\xi
\]

\[
\leq \frac{C}{|\alpha|} \leq \frac{C |\log \eta|}{\langle \alpha \rangle}.
\]

(3.166)

On the domain \(|\alpha + \xi^m| \leq \alpha/2\), we have \( \xi^m \geq |\alpha|/2 \). Note also that \( \langle \xi \rangle^{2d} \geq \langle \xi^m \rangle \). The
integral over this domain is therefore bounded by

\[ \int_{|\alpha + \xi^m| \leq |\eta|} \frac{1}{\alpha + \xi^m + i\eta} \frac{1}{\langle \xi - \omega \rangle^{2d}} d\xi \leq \frac{C}{\langle \alpha \rangle} \int_{|\alpha + \xi^m| \leq |\eta|} \frac{1}{\alpha + \xi^m + i\eta} \frac{1}{\langle \xi - \omega \rangle^{2d}} d\xi \leq \frac{C|\log \eta|}{\langle \alpha \rangle}. \] (3.167)

\[ \square \]

**Lemma 3.17.** Let

\[ (Y_z\hat{R})(\xi) := \int \frac{\hat{R}(\xi - y)}{z - \xi^m} d\xi \] (3.168)

be a family of linear operators parametrized by a complex parameter $z = \alpha + i\eta$ with $\eta > 0$.

(i) For $d \geq 3$ we have

\[ |Y_z\hat{R}| \leq C\|\hat{R}\|_{2d,2d}. \] (3.169)

(ii) If $z' = \alpha' + i\eta'$ and $\eta \geq \eta'$, then for $d \geq 3$

\[ |Y_z\hat{R} - Y_{z'}\hat{R}| \leq C|z - z'|\eta^{-(1 - \frac{1}{m})}\|\hat{R}\|_{2d,2d}. \] (3.170)

**Lemma 3.18.** Let $\hat{R}$ satisfy the smoothness condition defined in Section 1. We have the following inequality:

\[ \epsilon^m \int \left( \frac{1}{\epsilon^m \xi^m - \epsilon^m \xi^m} \right) \hat{R}(\xi_1 - \epsilon \xi^m) \frac{1}{\xi^m_1} d\xi_1 \leq C \max\{\epsilon^m, \epsilon^{d-m}\} |\log \epsilon| (\epsilon^m + 1)^{\frac{d}{m}-2}, \epsilon^{d-m}(\epsilon^m + 1)^{\frac{d}{m}-1}. \] (3.171)

**Proof.** We decompose the integral into three integrals on $\{\xi^m_1 \geq \epsilon^m \xi^m + 1\}, \{\epsilon^m (\xi^m + 1) \leq \xi^m_1 \leq \epsilon^m \xi^m + 1\}$ and $\{0 \leq \xi^m_1 \leq \epsilon^m (\xi^m + 1)\}$. Clearly, the first integral in bounded by $\epsilon^m \int \frac{\hat{R}(\xi_1 - \epsilon \xi^m)}{\epsilon^m \xi^m} d\xi_1$. For the second integral, note that $\hat{R}$ is bounded, and use $\frac{1}{\xi^m_1 - \epsilon^m \xi^m}$ to bound the term in the bracket on the left hand side of (3.171). We then change variable to
polar coordinates and let $Q = \xi_1^m$ to obtain

$$
e^m \int_{\{\varepsilon^m(\xi^m + 1) \leq \xi_1^m \leq \varepsilon^m(\xi^m + 1)\}} \left( \frac{1}{\xi_1^m - \varepsilon^m \xi_1^m} \land \frac{1}{\varepsilon^m} \right) \frac{\tilde{R}(\xi_1 - \varepsilon \xi)}{\xi_1^m} \, d\xi_1 \leq C e^m \int_{\varepsilon^m(\xi^m + 1)} e^m \frac{Q_d^{\frac{d}{m} - 2}}{Q - \varepsilon^m \xi_1^m} \, dQ$$

$$
\leq C e^{d - m} \log \varepsilon (\xi_1^m + 1)^{\frac{d}{m} - 2}.
$$

(3.172)

For the third part, we use $1/e^m$ to bound the term in the bracket on the left hand side of (3.171) and apply the same type of change of variable to obtain

$$
e^m \int_{\{0 \leq \xi_1^m \leq \varepsilon^m(\xi^m + 1)\}} \left( \frac{1}{\xi_1^m - \varepsilon^m \xi_1^m} \land \frac{t}{\varepsilon^m} \right) \frac{\tilde{R}(\xi_1 - \varepsilon \xi)}{\xi_1^m} \, d\xi_1 \leq t \int_0 e^m(\xi^m + 1) \frac{Q_d^{\frac{d}{m} - 2}}{Q_m^{\frac{d}{m} - 2}} \, dQ$$

$$
\leq C e^{d - m} (\xi_1^m + 1)^{\frac{d}{m} - 1}.
$$

(3.173)
Chapter 4

Convergence of Schrödinger equation to SPDE

4.1 Introduction

In this chapter, we consider the following Schrödinger equation in dimension $d < m$:

$$\begin{cases}
\left( i \frac{\partial}{\partial t} + (P(D) - \frac{1}{\varepsilon^{d/2}} q(\frac{x}{\varepsilon})) \right) u_\varepsilon(t, x) = 0, & t > 0, \ x \in \mathbb{R}^d \\
u_\varepsilon(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}$$

(4.1)

where $P(D)$ is the pseudo-differential operator with symbol $\hat{p}(\xi) = |\xi|^m$. Taking Fourier transform of both sides of (4.1), we obtain

$$\begin{cases}
(i \frac{\partial}{\partial t} + \xi^m) \hat{u}_\varepsilon = \varepsilon^{-\frac{d}{2}} \int \hat{q}(\zeta) \hat{u}_\varepsilon(t, \xi - \varepsilon^{-1} \zeta) d\zeta, \\
\hat{u}_\varepsilon(0, \xi) = \hat{u}_0(\xi).
\end{cases}$$

(4.2)

We assume the Fourier transform of the covariance of the potential $\hat{R}(\xi)$ is bounded and continuous at 0, and the initial condition satisfies $(1 + |\xi|^{2m})|\hat{u}_0(\xi)| \leq C$ uniformly in $\xi \in \mathbb{R}^d$. 
Our main objective is to construct a solution to the above equation in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time on bounded intervals and to show that the solution converges in distribution as $\epsilon \to 0$ to the unique solution of the following stochastic partial differential equation (SPDE)

\[
\begin{cases}
i \frac{\partial u}{\partial t} + P(D)u - \sigma u \circ \dot{W} = 0, & t > 0, \ x \in \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
\] (4.3)

where $\dot{W}$ denotes spatial white noise, $\circ$ denotes the Stratonovich product and $\sigma$ is defined as

\[
\sigma^2 := (2\pi)^d \hat{R}(0) = (2\pi)^d \int_{\mathbb{R}^d} R(x)dx.
\] (4.4)

To study this solution, we may take the Fourier transform of the equation:

\[
(i \frac{\partial}{\partial t} + \xi^m)\hat{u}(t, \xi) = \sigma(2\pi)^{-d} \int e^{-i\xi x} u(s, x) \circ dW(x)
\]

\[
= \sigma(2\pi)^{-d} \int e^{-i\xi x} \int e^{i\xi_1 x} \hat{u}(s, \xi_1) d\xi_1 \circ dW(x),
\] (4.5)

with initial condition $\hat{u}(0, \xi) = \hat{u}_0(\xi)$. To look for a mild solution, we recast (4.5) as

\[
\hat{u} = i\sigma(2\pi)^{-d} \int_0^t e^{i\xi^m(t-s)} e^{-i\xi x} \int e^{i\xi_1 x} \hat{u}(s, \xi_1) d\xi_1 ds \circ dW(x) + e^{it\xi^m} \hat{u}_0(\xi).
\] (4.6)

Define formally the stochastic integral

\[
\mathcal{H}\hat{u}(t, \xi) = (-i\sigma)(2\pi)^{-d} \int_0^t e^{i(t-s)\xi^m} \int e^{-i\xi x} \int e^{i\xi_1 x} \hat{u}(s, \xi_1) d\xi_1 ds \circ dW(x).
\] (4.7)

We may rewrite (4.6) as

\[
\hat{u}(t, \xi) = e^{it\xi^m} \hat{u}_0(\xi) + \mathcal{H}\hat{u}(t, \xi).
\] (4.8)

The mild solution to (4.3) is thus defined as $u(t, x) = \mathcal{F}^{-1}\{\hat{u}(t, \xi)\}$, where $\mathcal{F}^{-1}$ denotes
inverse Fourier transform. Suppose that $d < m$. The following result holds for the solution of (4.5).

**Theorem 4.1.** Suppose $d < m$. The series

$$\hat{u}(t, \xi) := \sum_{n \geq 0} \hat{u}^{(n)}(t, \xi)$$

(4.9)

converges in the $L^2(\Omega \times \mathbb{R}^d)$ sense for each $t \geq 0$ and $\xi \in \mathbb{R}^d$ and is the unique solution to (4.5) in the space $M$ dense in $L^2(\Omega \times \mathbb{R}^d)$, which is defined in Section 4.5, where

$$\hat{u}^{(n)} = (-i\sigma)^n (2\pi)^{-nd} e^{\int_0^{\infty} \cdots \int_0^{\infty} e^{i\beta t} d\beta} \int \prod_{k=1}^n d\xi_k \left\{ \prod_{k=0}^n \left[ 1 - i \left( \xi - \sum_{j=1}^k \xi_j \right)^m \right] \right\}^{-1} \prod_{j=1}^n e^{-i\xi_j x} \hat{u}_0(\xi - \sum_{j=1}^n \xi_j) \circ \prod_{j=1}^n dW(\xi_j).$$

(4.10)

The following theorem shows the weak convergence of $u_\varepsilon(t, x)$ to $u(t, x)$ for any $t > 0$ and $x \in \mathbb{R}^d$.

**Theorem 4.2.** Suppose that $d < m$. For any integers $r \geq 1$, $m_1, \ldots, m_r \geq 0$ and $\xi^{(1)}, \ldots, \xi^{(r)} \in \mathbb{R}^d$, $t_1, \ldots, t_r \geq 0$ we have the convergence of moments

$$\lim_{\varepsilon \to 0_+} \mathbb{E}\{[\hat{u}_\varepsilon(t_1, \xi^{(1)})]^{m_1} \cdots [\hat{u}_\varepsilon(t_r, \xi^{(r)})]^{m_r}\} = \mathbb{E}\{[\hat{u}(t_1, \xi^{(1)})]^{m_1} \cdots [\hat{u}(t_r, \xi^{(r)})]^{m_r}\}. \quad (4.11)$$

The finite dimensional distribution of $\hat{u}(t, \xi)$ is uniquely determined by its moments of all orders. Moreover, the family of processes $\{\hat{u}_\varepsilon(t, \cdot), t \geq 0\}$ is tight, as $\varepsilon \to 0_+$, over $C([0, +\infty); \mathcal{S}'(\mathbb{R}^d))$. The process $\{\hat{u}_\varepsilon(t, \cdot), t \geq 0\}$ converges in law over $C([0, +\infty); \mathcal{S}'(\mathbb{R}^d))$, as $\varepsilon \to 0_+$, to $\{\hat{u}(t, \cdot), t \geq 0\}$. Also, we have that in the spatial domain, the process $\{u_\varepsilon(t, \cdot), t \geq 0\}$ converges in law to $\{u(t, \cdot), t \geq 0\}$.

The rest of the chapter is structured as follows. Section 2 gives the formal Duhamel
solutions for both the multi-scale Schrödinger equation and the limiting SPDE in the Fourier domain. Section 3 demonstrates the first order moment convergence of Duhamel solutions \( \hat{u}_\varepsilon \). Section 4 proves that \( \hat{u} \) as the Duhamel expansion of the limiting equation is well defined in the space of in \( L^2(\Omega \times \mathbb{R}^d) \). Section 5 generalizes the first order moment convergence proved in Section 3 to arbitrary orders. Section 6 follows the approach as in [14] to show that the weak convergence of \( \{ \hat{u}_\varepsilon(t, \xi) \} \) in \( C([0, +\infty), \mathcal{S}') \) to \( \hat{u}(t, \xi) \) follows from tightness and convergence in finite dimensional distribution.

The asymptotic theory of solution to parabolic equation with large potential in dimension \( d < m \) is presented in [1]. In [14], analysis is provided to heat equation with long range correlated potential in \( d = 3 \). The treatment of Schrödinger equation (4.1) with the right scaling of potential \( (O(\varepsilon^{-\frac{m}{2}})) \), in dimension \( d > m \) is presented in [20]. It is then shown that \( u_\varepsilon \) converges in \( L^2(\Omega \times \mathbb{R}^d) \) to the solution of a homogenized equation. For the case \( d = m \) a logarithmic correction to the scaling of potential shows up, for the solution to (4.1) to have a deterministic solution to a homogenized equation. Although this critical dimension case which separates homogenization from stochastic limit is not discussed in [20], it is analyzed in [3] for parabolic equation, to which we refer the readers for more details.

### 4.2 Duhamel Expansion

Iteratively using Duhamel’s formula we obtain

\[
\hat{u}_\varepsilon(t, \xi) = e^{i\xi^m t} \hat{u}_0(\xi) - i\varepsilon^{-\frac{d}{2}} \int_0^t \int e^{i\xi^m (t-s)} \hat{q}(\zeta) \hat{u}_\varepsilon(s, \xi - \varepsilon^{-1} \zeta) ds d\zeta \\
= \sum_{n=0}^{+\infty} \hat{u}_\varepsilon^{(n)}(t, \xi),
\]

(4.12)
where
\[
\hat{u}_\varepsilon^{(n)}(t, \xi) = (-i)^n \varepsilon^{-\frac{nd}{2}} \int \cdots \int_{\Delta_n(t)} \hat{q}(\xi_k) d\xi_k \hat{u}_0(\xi - \varepsilon^{-1} \sum_{j=1}^{n-1} \xi_j).
\]
(4.13)

Here, we introduce the notation \(\sum_{j=1}^{0} \xi_j := 0\), and \(\Delta_n(t) := [t \geq s_1 \geq \cdots \geq s_n \geq 0]\).

Let \(\tilde{\Delta}_n(t) := [\sum_{j=1}^{n} \tau_j \leq t, \tau_j \geq 0]\). Changing variables \(s_j := \sum_{i=j}^{n} \tau_i\) and denoting \(\tau_0 := t - \sum_{i=1}^{n} \tau_i\) we can rewrite (4.13) in the form
\[
\hat{u}_\varepsilon^{(n)}(t, \xi) = (-i)^n \varepsilon^{-\frac{nd}{2}} \int \cdots \int_{\tilde{\Delta}_n(t)} d\tau_1 \cdots d\tau_n \int \cdots \int \prod_{k=1}^{n} \hat{q}(\xi_k) d\xi_k
\]
\[
\times \prod_{k=1}^{n+1} e^{i\tau_{k-1} \cdot |\xi - \varepsilon^{-1} \sum_{j=1}^{k-1} \xi_j|^m} \hat{u}_0(\xi - \varepsilon^{-1} \sum_{j=1}^{n} \xi_j).
\]
(4.14)

Using \(\delta(t) = \int e^{i\beta t} d\beta\), we obtain for any \(\eta > 0\)
\[
\hat{u}_\varepsilon^{(n)}(t, \xi) = (-i)^n \varepsilon^{-\frac{nd}{2}} e^{\eta t} \int_0^{+\infty} \cdots \int_0^{+\infty} \int \cdots \int \prod_{k=1}^{n} \hat{q}(\xi_k) d\xi_k
\]
\[
\times e^{i\beta(t - \sum_{j=0}^{n} \tau_j)} \prod_{k=1}^{n+1} e^{i\tau_{k-1} \cdot |\xi - \varepsilon^{-1} \sum_{j=1}^{k-1} \xi_j|^m} \hat{u}_0(\xi - \varepsilon^{-1} \sum_{j=1}^{n} \xi_j).
\]
(4.15)
Integrating out all $\tau_j$ and choosing $\eta = 1$ we get

$$\hat{u}^{(n)}(t, \xi) = (-i)^n \varepsilon^{-nd} e^{ni \beta t} \int_{\mathbb{R}} e^{i \beta \xi d \beta} \int \cdots \int \prod_{k=1}^{n} \hat{q}(\xi_k) d\xi_k$$

$$\times \left\{ \prod_{k=0}^{n} \left[ 1 - i \left( \left| \xi - \varepsilon^{-1} \sum_{j=1}^{k} \xi_j \right| - \beta \right) \right] \right\}^{-1} \hat{u}_0(\xi - \varepsilon^{-1} \sum_{j=1}^{n} \xi_j).$$

(4.16)

We now come to the analysis of the limiting equation. By Duhamel’s principle, the solution to (4.5) formally satisfies the equation

$$\hat{u}(t, \xi) = e^{i \xi m t} \hat{u}_0(\xi) + \left( -i \sigma \right) (2\pi)^{-d} \int_{0}^{t} \int e^{-i \xi x} u(s, x) ds \circ dW(x)$$

$$= e^{i \xi m t} \hat{u}_0(\xi) + \left( -i \sigma \right) (2\pi)^{-d} \int_{0}^{t} \int e^{-i \xi x} \int e^{i \xi x} \int u(s, \xi_1) d\xi_1 ds \circ dW(x).$$

(4.17)

Integrating (4.17) iteratively, we obtain formally the Duhamel expansion for (4.5)

$$\hat{u}(t, \xi) = \sum_{n=0}^{\infty} \hat{u}^{(n)},$$

(4.18)

where

$$\hat{u}^{(0)} = e^{i \xi m} \hat{u}_0(\xi), \quad \text{and} \quad \hat{u}^{(n)} = \mathcal{H}^n \hat{u}^{(0)},$$

(4.19)

for $n = 0, 1, \cdots$, or more explicitly,

$$\hat{u}^{(n)} = (-i \sigma)^n (2\pi)^{-nd} \int \cdots \int \Delta_n(t) \int \frac{\prod_{k=1}^{n+1} e^{i(s_{k-1} - s_k) \xi_k^m}}{\prod_{j=1}^{n} e^{-i(\xi_j - \xi_{j-1}) x_j} \hat{u}_0(\xi_n) \prod_{k=1}^{n} d\xi_k \prod_{k=1}^{n} ds_k \circ \prod_{j=1}^{n} dW(x_j)}.$$

(4.20)

Here, we define $s_0 = t$, $s_{n+1} = 0$, and $\xi_0 := \xi$. At this point neither the iterative Stratonovich integral of $\hat{u}^{(n)}$ for $n = 1, 2, \cdots$ in (4.20) nor the sum of (4.18) are well defined. We give a justification for these expressions in section 4.
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Using the change of variables \( \xi_k \to \xi_{k-1} - \xi_k \), and applying the same type of transform as in (4.16), we obtain

\[
\hat{u}^{(n)} = (-i\sigma)^n (2\pi)^{-nd} e^{t} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i\beta t} d\beta \int \prod_{k=1}^{n} d\xi_k \left\{ \prod_{k=0}^{n} \left[ 1 - i \left( \xi - \sum_{j=1}^{k} \xi_j \right)^m - \beta \right] \right\}^{-1} \prod_{j=1}^{n} e^{-i\xi_j x_j} \hat{u}_0(\xi - \sum_{j=1}^{n} \xi_j) \circ \prod_{j=1}^{n} dW(x_j),
\]

which is the same as (4.10).

4.3 Convergence of the first moments

We shall prove the convergence of moments as stated in (4.11). For simplicity, we shall first consider the case when \( n = m = 1 \) and show that the limit

\[
\lim_{\varepsilon \to 0^+} \mathbb{E}\hat{u}_\varepsilon(t, \xi)
\]

exists. The proof of convergence for general moments is given in Section 4.6.

Taking expectation of both sides of (4.12), we obtain

\[
\mathbb{E}\hat{u}_\varepsilon(t, \xi) = \sum_{n \geq 0} \mathbb{E}\hat{u}_\varepsilon^{(2n)}(t, \xi).
\]

This is because expectation of product of odd number of Gaussian random variables is 0. The expectation of product of even number of Gaussian random variables is given as a sum of products of the expectation of pairs of variables, where the summation runs over all possible pairs. The contribution of products of potentials can thus be represented by

\[
\mathbb{E}\{\prod_{k=1}^{2n} \hat{q}(\xi_k)\} = \sum_{\pi} \prod_{(ef) \in \pi} \hat{R}(\xi_e) \delta(\xi_e + \xi_f),
\]

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where \((ef)\) denotes pair of indices, \(\pi\) denotes a pairing of the \(2n\) indices, and the summation is over all possible pairings.

Adding up all the delta functions gives \(\sum_{k=1}^{2n} \xi_k = 0\). We can therefore write

\[
\mathbb{E}\tilde{u}_\varepsilon^{(2n)}(t,\xi) = (-1)^n \varepsilon^{nd} \mathcal{E}t \hat{u}_0(\xi) \int e^{i\beta t} d\beta \int \cdots \int \\
\times \left\{ \prod_{k=0}^{2n} \left[ 1 - i \left( \xi - \varepsilon^{-1} \sum_{j=1}^{k} \xi_j \right) \right] \right\}^{-1} \mathbb{E}\left\{ \prod_{k=1}^{2n} \hat{q}(\xi_k)d\xi_k \right\}
\]

\[
= (-1)^n \varepsilon^{nd} \mathcal{E}t \hat{u}_0(\xi) \sum_{\pi} \int e^{i\beta t} d\beta \int \cdots \int \\
\times \left\{ \prod_{k=0}^{2n} \left[ 1 - i \left( \xi - \varepsilon^{-1} \sum_{j=1}^{k} \xi_j \right) \right] \right\}^{-1} \prod_{(ef)\in\pi} \hat{R}(\xi_e) \delta(\xi_e + \xi_f)d\xi_ed\xi_f.
\]

The above summation extends over all possible pairings \(\pi\) made over vertices \(\{1, \ldots, 2n\}\).

By changing variables \(\xi_k := e^{-1}\xi_k\), we obtain

\[
\mathbb{E}\tilde{u}_\varepsilon^{(2n)}(t,\xi) = (-1)^n e^{t} \hat{u}_0(\xi) \sum_{\pi} \int e^{i\beta t} d\beta \int \cdots \int \\
\times \left\{ \prod_{k=0}^{2n} \left[ 1 - i \left( \xi - \sum_{j=1}^{k} \xi_j \right) \right] \right\}^{-1} \prod_{(ef)\in\pi} \hat{R}(\xi_e) \delta(\xi_e + \xi_f)d\xi_ed\xi_f.
\]

(4.26)

We will show that the exchange of taking limit and expectation:

\[
\lim_{\varepsilon \to 0^+} \mathbb{E}\tilde{u}_\varepsilon(t,\xi) = \sum_{n=0}^{+\infty} \lim_{\varepsilon \to 0^+} \mathbb{E}\tilde{u}_\varepsilon^{(2n)}(t,\xi).
\]

(4.27)

is legitimate based on the fact that the sequence \(\{\sup_{t} |\mathbb{E}\tilde{u}_\varepsilon^{(2n)}(t,\xi)|\}_{n=1}^{\infty}\) is summable. Let \(\mathcal{L}(\pi)\) be the set of all left vertices of a given pairing \(\pi\). Define

\[
A_k := \left| \xi - \sum_{j=1}^{k} \xi_j \right|^m,
\]

(4.28)
and
\[ e_A^{(\rho)} = [1 - i(A - \beta)]^{-1}. \] (4.29)

Using contour integration method, we are able to show that
\[ e_A^{(\rho)} = \begin{cases} e^{iAt}\rho - 1 e^{-t} & t > 0, \\ 0 & t < 0 \end{cases} \] (4.30)

The following estimate is then derived:

**Lemma 4.3.** Suppose that \( \rho > 0 \). There exists a constant \( C > 0 \) such that for an arbitrary \( n \geq 1 \) we have
\[ \left| \int e^{i\beta t} \left\{ \prod_{k=1}^{n} [1 - i(A_k - \beta)] \right\}^{-\rho} d\beta \right| \leq \frac{C^n t^{\rho-1} e^{-t}}{[(n - 1)!]^\rho}. \] (4.31)

Readers are referred to [14] for the proof of this lemma.

With such notation, we may rewrite (4.26) as
\[ E\hat{u}_\xi^{(2n)}(t, \xi) = (-1)^n e^{t\hat{u}_0(\xi)} \sum_{\pi} \int \cdots \int \prod_{k \notin \mathcal{L}(\pi), k \neq 2n} \ast e_{A_k} \ast F(t, \xi; \pi) \times \prod_{(ef) \in \pi} \hat{R}(\varepsilon \xi_e) \delta(\xi_e + \xi_f) d\xi_e d\xi_f, \] (4.32)

where
\[ F(t, \xi; \pi) := \int e^{i\beta t} (1 - i(\xi^m - \beta))^{-2} \prod_{k \in \mathcal{L}(\pi)} \left[ 1 - i \left( \left\| \xi - \sum_{j=1}^{k} \xi_j \right\| - \beta \right) \right]^{-1} d\beta \] (4.33)

and \( \prod_{k \notin \mathcal{L}(\pi), k \neq 2n} \ast e_{A_k} \) denotes the convolution of all \( e_{A_k} \)'s, where \( k \) is the right vertices in the giving pairing \( \pi \), except \( k = 2n \).

The following inequality plays an important role in estimating \( E\hat{u}_\xi^{(2n)}(t, \xi) \).
4.3. CONVERGENCE OF THE FIRST MOMENTS

Lemma 4.4. Suppose $\rho \in (\frac{d}{m}, 1)$. Then

\[
\sup_{\beta \in \mathbb{R}, \omega \in \mathbb{R}^d} \int_{\mathbb{R}^d} | - \beta + |\xi - \omega|^m + i|^{\rho} d\xi < +\infty. \tag{4.34}
\]

Proof. We may first shift $\xi$ to get rid of $\omega$ and perform the spherical change of coordinates

\[
\int_{\mathbb{R}^d} | - \beta + |\xi - \omega|^m + i|^{\rho} \leq \Omega_d \int_0^{+\infty} | - \beta + \xi^m + i|^{\rho} d|\xi|, \tag{4.35}
\]

where $\Omega_d$ is the area of the unit sphere in $\mathbb{R}^d$. Let $Q = \xi^m$. The integral on the right hand side of (4.35) may be rewritten as

\[
\int_0^{+\infty} | - \beta + Q + i|^{\rho} Q^{d/m - 1} dQ. \tag{4.36}
\]

Without loss of generality, we assume $\beta \geq 0$ and the integral above can then be written as the sum of three integrals $I$, $II$, $III$ according to whether $\xi$ belongs to $(0, \beta/2)$, $(\beta/2, 2\beta)$, or $(2\beta, +\infty)$. We have

\[
I \leq [1 + (\frac{\beta}{2})^2]^{-\frac{d}{2}} \int_0^{\beta/2} Q^{d-1} dQ \leq \frac{m}{d} [1 + (\frac{\beta}{2})^2]^{-\frac{d}{2}} (\frac{\beta}{2})^\frac{d}{m} \leq \frac{m}{d}. \tag{4.37}
\]

The second integral can also be estimated as follows

\[
II \leq (2\beta)^{\frac{d}{m} - 1} \int_{\frac{\beta}{2}}^{2\beta} |Q - \beta + i|^{\rho} dQ. \tag{4.38}
\]

If $\beta \leq 1$ we have

\[
II \leq 3 \times 2^{d/2}. \tag{4.39}
\]

If $\beta > 1$, we estimate

\[
II \leq (2\beta)^{\frac{d}{m} - 1} \int_{\frac{\beta}{2}}^{2\beta} |Q - \beta|^{\rho} dQ \leq [1 + (\frac{1}{2})^{1-\rho}]2^{d/2 - 1}. \tag{4.40}
\]
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The third integral is estimated

\[
III \leq \frac{m}{d} (\beta^2 + 1)^{-\frac{d}{2}} \beta \leq \frac{m}{d}. 
\]  
(4.41)

This concludes the proof of the lemma. \(\square\)

Let

\[
F_1(t, \xi; \pi) := \int_{\mathbb{R}} \frac{e^{i\beta t} d\beta}{1 - i(\xi^m - \beta)} \left\{ \prod_{k \in L(\pi)} \left[ 1 - i \left( \left| \xi - \sum_{j=1}^{k} \xi_j \right| - \beta \right) \right]^{-\rho} \right\} 
\]  
(4.42)

and

\[
F_2(t, \xi; \pi) := \int_{\mathbb{R}} e^{i\beta t} d\beta \left\{ \prod_{k \in L(\pi)} \left[ 1 - i \left( \left| \xi - \sum_{j=1}^{k} \xi_j \right| - \beta \right) \right]^{-\rho(1-\rho)} \right\}. 
\]  
(4.43)

Since

\[
F_1(t, \xi; \pi) = e_{B_1(\xi)}^{(\rho)} \cdots e_{B_n+2(\xi)}^{(\rho)}(t) 
\]  
(4.44)

for some \(B_j(\xi)\), we get \(F_1(t, \xi; \pi) = 0\) for \(t < 0\). Likewise, \(F_2(t, \xi; \pi) = 0\) for \(t < 0\). We can therefore write

\[
F(t, \xi; \pi) = \int_0^t F_1(t-s, \xi; \pi) F_2(s, \xi; \pi) ds \geq 0. 
\]  
(4.45)

Observe that by Lemma 3.1

\[
F_2(t, \xi; \pi) \leq \frac{C^n t^{(1-\rho)-1}}{[(n-1)!]^{1-\rho}} 
\]  
(4.46)

and

\[
\prod_{k \in L(\pi), k \neq 2n} *e_{A_k}(t) \leq \frac{C^n t^{n-1}}{(n-1)!} 
\]  
(4.47)
for $t > 0$. On the other hand, we have obviously

$$|F_1(t, \xi; \pi)| \leq C \int_{\mathbb{R}} \frac{d\beta}{1 + |\xi^m - \beta|^2} \left\{ \prod_{k \in \mathcal{L}(\pi)} \left| \frac{1 - i \left( \left| \xi - \sum_{j=1}^k \xi_j \right|^m - \beta \right)}{\left| \xi - \sum_{j=1}^k \xi_j \right|^m - \beta} \right|^{-\rho} \right\}$$

for some constant $C > 0$. As a result, we obtain

$$|\hat{E}_t^{(2n)}(t, \xi)| \leq C e^t |\hat{u}_0(\xi)| \sum_{\pi} \frac{(t - s)^{n-1} ds}{(n - 1)!} \prod_{(kl) \in \pi} \hat{R}(\varepsilon \xi_e) \delta(\xi_e + \xi_f) d\xi_e d\xi_f$$

$$\leq \frac{C^n e^t |\hat{u}_0(\xi)|}{[(n - 1)!]^{2-\rho}} \int_0^t (t - s)^{s^{n(1-\rho)-1} ds} \sum_{\pi} \frac{d\beta}{1 + |\xi^m - \beta|^2} \int \cdots \int \prod_{(ef) \in \pi} \hat{R}(\varepsilon \xi_e) \delta(\xi_e + \xi_f) d\xi_e d\xi_f \left\{ \prod_{k \in \mathcal{L}(\pi)} \left| \frac{1 - i \left( \left| \xi - \sum_{j=1}^k \xi_j \right|^m - \beta \right)}{\left| \xi - \sum_{j=1}^k \xi_j \right|^m - \beta} \right|^{-\rho} \right\}.$$  \hspace{1cm} (4.49)

Here, we just need half of the order of decay of the initial condition $\hat{u}_0(\xi)$ as assumed to have

$$\frac{|\hat{u}_0(\xi)|}{(1 + |\xi^m - \beta|^2)^{1/2}} \leq \frac{C}{(1 + |\xi^m|^2 + |\beta|^2)^{1/2}} \leq \frac{1}{(1 + \beta^2)^{1/2}}.$$  \hspace{1cm} (4.50)

The right hand side of the above inequality is then used for estimating the integration in $\beta$ by Lemma 5.2 in [20]:

$$\int_{-\infty}^{+\infty} \frac{d\beta}{(1 + \beta^2)^{1/2}(1 + |\xi^m - \beta|^2)^{1/2}} \leq C \frac{1 + \log_+ |\xi|}{(1 + \xi^m)^{1/2}},$$  \hspace{1cm} (4.51)

where $\log_+ |\xi| := \max(0, \log |\xi|)$.

Using Lemma 3.2 we conclude therefore that

$$|\hat{E}_t^{(2n)}(t, \xi)| \leq (2n - 1)!! \frac{C^n t^{n(2-\rho)} e^t}{(n!)^{2-\rho}} \frac{1 + \log_+ |\xi|}{(1 + \xi^m)^{1/2}}$$  \hspace{1cm} (4.52)
and (4.63) follows by letting $a_{2n} = (2n - 1)!! \frac{C_n n^{n(2 - \rho)}}{(n!)^{2 - \rho}}$. We obtain

$$\lim_{\varepsilon \to 0^+} \mathbb{E} \hat{u}_\varepsilon(t, \xi) = \sum_{n=0}^{+\infty} \tilde{u}^{(2n)}(t, \xi),$$

where

$$\tilde{u}^{(2n)}(t, \xi) := (-\hat{R}(0))^{n} e^{t} \tilde{u}_0(\xi) \sum_{\pi} \int_{\mathbb{R}} e^{i\beta t} d\beta \int \cdots \int$$

$$\times \left\{ \prod_{k=0}^{2n} 1 - i \left( \left| \xi - \sum_{j=1}^{k} \xi_j \right| - \beta \right) \right\}^{-1} \prod_{(e,f) \in \pi} \delta(\xi_e + \xi_f) d\xi_e d\xi_f.$$  

In what follows, we show that

$$\tilde{u}^{(2n)}(t, \xi) = \mathbb{E} \hat{u}^{(2n)}(t, \xi).$$

(4.55)

Expectation of multiple Stratonovich integrals is defined:

$$\mathbb{E} \{ \prod_{j=1}^{2n} dW(x_j) \} = \sum_{\pi} \prod_{(e,f) \in \pi} \delta(x_e - x_f) dx_e dx_f.$$  

(4.56)

Upon integrating in all variables $x$, the first moment of $\hat{u}^{(2n)}$ can therefore be written as

$$\mathbb{E} \tilde{u}^{(2n)} = (-i \sigma)^{2n} (2\pi)^{-2n} \sum_{\pi} \int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{i\beta t} d\beta \int \prod_{k=1}^{2n} d\xi_k$$

$$\left\{ \prod_{k=0}^{2n} 1 - i \left( \left| \xi - \sum_{j=1}^{k} \xi_j \right| - \beta \right) \right\}^{-1} \tilde{u}_0(\xi) \prod_{(e,f) \in \pi} \delta(\xi_e + \xi_f),$$

(4.57)

which gives exactly (4.55).
4.4 \( L^2 \) CONVERGENCE OF THE SPDE SOLUTION

We now come to the series (4.9), which we claim is the solution to (4.6). The readers are referred to [1] for more complete description of the relationship between Stratonovich and Itô integrals and for additional details on the theory presented in this section.

We first prove that \( \hat{u}(t, \cdot) \) in (4.9) as a series is well-defined in the space of \( L^2(\Omega \times \mathbb{R}^d) \).

Denote \( \hat{u}^{(n)} = I_n(f_n) \), where \( I_n \) denotes the \( n \)-th order iterated Stratonovich integral, and \( f_n \) is a \( n \)-parameter function, i.e.

\[
I_n(f_n) = \int_{\mathbb{R}^nd} f_n(x_1, \ldots, x_n) dW(x_1) \ldots dW(x_n).
\]  

(4.58)

By the definition of the \( L^2 \) norm of multiple Stratonovich integral, we have

\[
\mathbb{E}|\hat{u}^{(n)}|^2 = \mathbb{E}\{I_{2n}(f_n \otimes f_n)\} \\
= \mathbb{E} \int_{\mathbb{R}^{2nd}} f_n(x_1, \ldots, x_n) \bar{f}_n(y_1, \ldots, y_n) dW(x_1) \ldots dW(x_n) dW(y_1) \ldots dW(y_n).
\]  

(4.59)

Taking the same approach as in the calculation of the first moment in (4.57), we obtain

\[
\mathbb{E}|\hat{u}^{(n)}(t, \xi)|^2 = (\sigma^2(2\pi)^{-d}e^t)^2 \sum_{\pi_{2n}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(\beta_1 - \beta_2)t} d\beta_1 d\beta_2 \hat{u}_0(\xi - \sum_{j=1}^{n} \xi_j^{(1)}) \\
\bar{u}_0(\xi + \sum_{j=1}^{n} \xi_j^{(2)}) \times \left\{ \prod_{k=0}^{n} \left[ 1 - i \left( \xi - \sum_{j=1}^{k} \xi_j^{(1)} \right) \right]^{m} + \beta_1 \right\} \prod_{k=0}^{n} \left[ 1 + i \left( \xi + \sum_{j=1}^{k} \xi_j^{(2)} \right) \right]^{m} + \beta_2 \right\}^{-1} \\
\prod_{(ef) \in \pi} \delta(\xi_e + \xi_f) d\xi^{(1)} d\xi^{(2)}.
\]  

(4.60)

We denote \( d\xi^{(l)} = d\xi^{(l)}_1 \ldots d\xi^{(l)}_{n}, \ l = 1, 2 \). \( \xi_e \) and \( \xi_f \) are paired arguments in the graph comprised of \( 2n \) arguments in total with their index \( e \) and \( f \).

The idea that we use to estimate the first moment applies here too. However, note that we have \( \{ k = n \} \in \mathcal{L}(\pi) \) for the crossing graphs defined in [3], and one of the terms [1 –
\[ i(\xi^m - \beta)^{-1} \] in the function \( F_1 \) defined in (4.42) now becomes \([1 - i(\xi - \sum_{j=1}^{n} \xi_j^{(1)} |m - \beta])^{-1}.\]

Therefore, we have to make the following adjustments in the proof. Using the smoothness condition for the initial condition, we have

\[ \left| \hat{u}_0(\xi - \sum_{j=1}^{n} \xi_j^{(1)}) \right| \leq \left| \frac{1}{1 - i(\xi - \sum_{j=1}^{n} \xi_j^{(1)} |m - \beta)} \right| \leq \frac{1}{(1 + |\xi - \sum_{j=1}^{n} \xi_j^{(1)} |2m)^{1/2}} \frac{1}{(1 + \beta_1^2)^{1/2}}. \]  

(4.61)

We use the term \([1 + |\xi - \sum_{j=1}^{n} \xi_j^{(1)} |2m]^{-1/2}\) for the integration in \( \xi_n \), and the term \([1 + \beta_1^2]^{-1/2}\) together with \([1 - i(\xi^m + \beta_1)]^{-1}\) indexed by \( \{k = 0\} \) in the first product in (4.60)

\[ \int_{-\infty}^{+\infty} \frac{d\beta_1}{(1 + \beta_1^2)^{1/2}[1 - i(\xi^m + \beta_1)]} \leq C \frac{1 + \log_+ |\xi|}{(1 + \xi^2m)^{1/2}}. \]  

(4.62)

We also use the smoothness condition for the initial condition and obtain another \((1 + \log_+ |\xi|)/(1 + \xi^2m)^{1/2}\) from the integration in \( \beta_2 \). Finally, we obtain the estimate

\[ \mathbb{E} \left| \hat{u}^{(n)}(t, \xi) \right|^2 \leq (2n - 1)! \frac{C^{n\mu(n^2-\rho)} e^t}{(n!)^{2-\rho}} \frac{1 + \log_+ |\xi|}{(1 + \xi^2m)^{1/2}} \]  

(4.63)

for any \( \rho \in \left( \frac{d}{m}, 1 \right) \). This implies that the iterated Stratonovich integral \( \hat{u}^{(n)} \) is indeed well-defined. Integrating in \( \xi \) and summing the above bound over \( n \) gives

\[ (\mathbb{E} \int \left| \hat{u}(t, \xi) \right|^2 d\xi)^{1/2} \leq \sum_{n \geq 0} (\mathbb{E} \int \left| \hat{u}^{(n)}(t, \xi) \right|^2 d\xi)^{1/2} < \infty \]  

(4.64)

and the \( L^2(\Omega \times \mathbb{R}^d) \) convergence of (4.9) follows. In fact, by first multiplying \( \xi^m \) with (4.63) and performing the integration and summation we can further obtain

\[ \mathbb{E} \int \xi^m \left| \hat{u}(t, \xi) \right|^2 d\xi < \infty. \]  

(4.65)
4.5 Uniqueness of the SPDE solution

Let us now provide the rigorous definition for the operator $H$. Suppose $f(t, \xi)$ is a sum of iterated Stratonovich integrals

$$f(t, \xi) = \sum_{n \geq 0} \mathcal{I}_n(f_n(t, \xi, \cdot)). \quad (4.66)$$

We define

$$Hf(t, \xi) = \sum_{n \geq 1} \mathcal{I}_n((Hf)_n(t, \xi, \cdot)), \quad (4.67)$$

where

$$(Hf)_{n+1}(t, \xi, x, x_1, \cdots, x_n) := (-i\sigma)(2\pi)^{-d} \int_0^t \int e^{i(s-t)\xi^m} e^{i(\xi_0-\xi)x} f_n(s, \xi_0, x_1, \cdots, x_n) d\xi_0 ds. \quad (4.68)$$

It is checked that under this definition $H\hat{u} = \sum_{n \geq 1} \hat{u}^{(n)} = \hat{u} - \hat{u}^{(0)}$. The Duhamel solution (4.9) is therefore a solution to the equation (4.6).

We can also define

$$Jf(t, \xi) = \sum_{n \geq 0} \mathcal{I}_{n+1}((Jf)_{n+1}(t, \xi, \cdot)), \quad (4.69)$$

where

$$(Jf)_{n+1}(t, \xi, x, x_1, \cdots, x_n) := (-i\sigma)(2\pi)^{-d} e^{-i\xi x} \int e^{i\xi_0 x} f_n(s, \xi_0, x_1, \cdots, x_n) d\xi_0. \quad (4.70)$$
For the Duhamel solution (4.9), we have

$$\mathbb{E}|I_{n+1}((\mathcal{J}\hat{u})_{n+1}(t,\xi))|^2 = (\sigma^n(2\pi)^{-d}e^t)^2 \sum_{n=2}^{\infty} \int \int \int_{\mathbb{R}^2} e^{i(\beta_1-\beta_2)t}d\beta_1d\beta_2\hat{u}_0(\xi - \sum_{j=1}^{n+1}\xi_j^{(1)})$$

$$\times \bar{\hat{u}}_0(\xi + \sum_{j=1}^{n+1}\xi_j^{(2)}) \left\{ \prod_{k=1}^{n+1} \left[ 1 - i \left( \left| \xi - \sum_{j=1}^{k}\xi_j^{(1)} \right| \beta_1 \right) \right] \right\}^{-1}$$

$$\prod_{(ef) \in \pi} \delta(\xi_e + \xi_f)d\xi^{(1)}d\xi^{(2)}. \quad (4.71)$$

Although it looks a little different from (4.60), it can be estimated in the same way as

$$(\mathbb{E}|\hat{u}^{(n)}|^2)^{1/2} < (2n - 1)!! \frac{C^n n^{n(2-\rho)} e^t}{(n!)^{2-\rho}} \frac{1 + \log |\xi|}{1 + \xi^{2m}} \left(1/2\right)^{1/2}, \quad (4.72)$$

which upon summation in $n$ implies that $\mathbb{E}|\hat{u}(t,\xi)|^2$ is uniformly bounded for all $\xi \in \mathbb{R}^d$.

Note that $\mathcal{H}\hat{u}(t,\xi) = (-i\sigma)(2\pi)^{-d} \int_0^t e^{i(t-s)} \xi^m (J\hat{u})(s,\xi)ds$. $\hat{u}(t,\xi)$ is therefore a solution to the equation

$$(i \frac{\partial}{\partial t} + \xi^m)\hat{u}(t,\xi) = \sigma(2\pi)^{-d}J\hat{u}(t,\xi). \quad (4.73)$$

Now we prove that this equation preserves mass. Multiplying this equation by $\hat{u}(t,\xi)$, and integrating in $\xi$ and over the probability space $\Omega$ gives

$$\frac{i}{2} \frac{\partial}{\partial t} \mathbb{E} \int |\hat{u}|^2 d\xi + \mathbb{E} \int \xi^m |\hat{u}|^2 d\xi = \sigma(2\pi)^{-d} \mathbb{E} \int (J\hat{u}) \bar{\hat{u}} d\xi. \quad (4.74)$$
The right hand side of this equation can be written out explicitly as

\[ E \int (\mathcal{J} \hat{u}) \hat{u} d\xi = \int \sum_{n,m} \mathcal{I}_{n+1} \left( e^{-i\xi x} \int e^{i\xi_1 x} f_n(t,\xi_1) d\xi_1 \right) \mathcal{I}_m (\check{f}_m(t,\xi)) d\xi \]

\[ = \int \sum_{n,m} \left( e^{-i\xi x} \int e^{i\xi_1 x} f_n(t,\xi_1) d\xi_1 \sum_{(ef) \in \pi} \delta(x_e - x_f) d\pi \right) d\xi \]

\[ = \sum_{n,m} \sum_{\pi} \left( e^{i\xi_1 x} f_n(t,\xi_1) d\xi_1 \int e^{-i\xi x} \check{f}_m(t,\xi) d\xi \prod_{(ef) \in \pi} \delta(x_e - x_f) d\pi \right) \]

\[ = \sum_{n,m} \sum_{\pi} \int f_n(t,x) \check{f}_m(t,x) \prod_{(ef) \in \pi} \delta(x_e - x_f) d\pi, \]

which is real-valued because of the symmetricity of this summation. Extracting the imaginary part from both sides of (4.74) gives

\[ \frac{\partial}{\partial t} E \int |\hat{u}(t,\xi)|^2 d\xi = 0. \] (4.76)

Finally, we define the space $M$ in which the equation (4.73) admits a unique solution. In light of the equation (4.74), $M$ consists of sum of iterated Stratonovich integrals $f(t,\xi) = \sum_{n \geq 0} \mathcal{I}_n (f_n(t,\xi,\cdot))$ such that

1. $f(t,\xi) \in L^2(\Omega \times \mathbb{R}^d)$,

2. $\mathcal{J} f \in L^2(\Omega)$,

3. $|\xi|^{\mp} f(t,\xi) \in L^2(\Omega \times \mathbb{R}^d)$.

As a reminder, defining the sum of iterated Stratonovich integral $f$ as in (4.66), we have that

\[ f = \sum_{n \geq 0} \mathcal{I}_n (f_n) = \sum_{m \geq 0} \mathcal{I}_m (g_m), \] (4.77)

where

\[ g_m(t,\xi,x) = \sum_{k \geq 0} \frac{(m + 2k)!}{m!k!2^k} \int_{\mathbb{R}^{kd}} f_{m+2k}(t,\xi,x_m,y_k^{\otimes 2}) dy. \] (4.78)
Here, \( y \otimes y \equiv (y, y) \), and

\[
I_m(g_m) := \int_{\mathbb{R}^{md}} g_m(t, \xi, x_1, \ldots, x_m) dW(x_1) \cdots dW(x_m)
\]  

(4.79)
denotes the iterated Itô integral. The \( L^2 \) norm of \( f \) can then be computed using the orthogonality of Wiener Chaos expansion as

\[
\|f\|_{L^2(\Omega)} = \left( \sum_{m \geq 0} m! \left( \sum_{k \geq 0} \frac{(m + 2k)!}{m!k!2^k} \int_{\mathbb{R}^{kd}} |f_{m+2k}(t, \xi, x_m, y_{\otimes 2}^k)| dy \right)^2 dx \right)^{\frac{1}{2}} < \infty,
\]  

(4.80)
The readers are referred to [1] for more details. It is easy to verify that the space defined above is dense in \( L^2(\Omega \times \mathbb{R}^d) \). Denote the space consisting of all functions that satisfy condition (1) and (2) by \( \tilde{M} \). In fact, any function \( f(\xi) \in L^2(\Omega \times \mathbb{R}^d) \) can be written as their Wiener Chaos expansion

\[
f(\xi) = \sum_{m \geq 0} I_m(g_m(\xi, x_m)).
\]  

(4.81)
Each \( g_m \) can be approximated by a function \( f_{m}^k \), which vanishes in a set of measure at most \( k^{-1} \) in the vicinity of measure 0 set of diagonals given by the support of the distributions \( \delta(x_e - x_f) \). By the change of change of coordinates in (4.78), we have \( g_{m}^k = f_{m}^k \) so that the Itô and Stratonovich integrals agree. Define

\[
f^{(k)}(\xi) = \sum_{m \geq 0} I_m(f_{m}^{(k)}).
\]  

(4.82)
Using formula (4.80), we may verify that

\[
\|\mathcal{J} f^{(k)} \|_{L^2(\Omega)} < \infty,
\]  

(4.83)
and
\[
\lim_{k \to \infty} \| f^{(k)}(\xi) - f(\xi) \|_{L^2(\Omega \times \mathbb{R}^d)} = 0.
\] (4.84)

We have shown that \( \tilde{M} \) is dense in \( L^2(\Omega \times \mathbb{R}^d) \). Since \( M \) is dense in \( \tilde{M} \), it is also dense in \( L^2(\Omega \times \mathbb{R}^d) \).

4.6 General moment convergence

We now turn to the general case of (4.11) by considering the limit
\[
\lim_{\varepsilon \to 0} E\{ \hat{u}_\varepsilon(t_1, \xi^{(1)}_1) \cdots \hat{u}_\varepsilon(t_r, \xi^{(r)}_r) \}. \tag{4.85}
\]

It is seen from (4.12) that the above expression before passing to the limit equals
\[
\sum I_\varepsilon(n), \tag{4.86}
\]
where
\[
I_\varepsilon(n) := E\{ \hat{u}_\varepsilon^{(n_1)}(t_1, \xi^{(1)}_1) \cdots \hat{u}_\varepsilon^{(n_r)}(t_r, \xi^{(r)}_r) \}. \tag{4.87}
\]

The summation in (4.86) extends over all non-negative integer multi-indices \( n = (n_1, \cdots, n_r) \).

As expectation of product of Gaussian variables, only the terms for which \( |n| := \sum_{l=1}^r n_l \) are even do not vanish. These term can be written explicitly as
\[
I_\varepsilon(n) := (-i)^{|n|} \exp(\sum_{l=1}^r t_l) \pi \int_{\mathbb{R}^{|n|d}} \exp\{i \sum_{l=1}^r \beta_l t_l\} \prod_{l=1}^r d\beta_l \int \cdots \int \prod_{l=1}^r d\xi^{(l)}
\]
\[
\times \prod_{l=1}^r \hat{u}_0(\xi^{(l)}_1 - \sum_{k=1}^{n_l} \xi^{(l)}_k) \prod_{(ef) \in \pi} \hat{R}(\varepsilon \xi_e) \delta(\xi_e + \xi_f)
\]
\[
\times \left\{ \prod_{l=1}^r \prod_{k=0}^{n_l} \left[ 1 - i \left( \xi^{(l)} - \sum_{j=1}^k \xi^{(l)}_{nj} \right)^m - \beta \right] \right\}^{-1}. \tag{4.88}
\]
We denote $d\xi(l) := d\xi_1(l) \cdots d\xi_n(l)$. The summation extends over all possible pairings $(e,f)$.

Reproducing the work for estimating the first order moment in section 4.3 yields

$$|I_{\varepsilon}(n)| \leq \frac{(CT)^{2}\rho}{(\frac{n}{T})^{2\rho}} e^{eT},$$

assuming $0 \leq t_1, \ldots, t_r \leq T$, where the constant $C$ is independent of $\varepsilon \in (0,1]$, $|n|$ and $r$. The summation of $(N + r - 1)$ non-negative integer valued multi-indices satisfying equation $|n| = N$ is estimated as

$$\left| \sum_{|n|=N} I_{\varepsilon}(n) \right| \leq c_N,$$

where

$$c_N := \left( N + r - 1 \right) (N - 1)!! \frac{(CT)^{2\rho} N e^{eT}}{(\frac{n}{T})^{2\rho}}.$$

The convergence of $\sum_{N=1}^{+\infty} c_N$ is easy to verify, which allows us moving the passage to the limit of $\varepsilon \to 0_+$ inside the summation. As a result, we conclude that

$$\lim_{\varepsilon \to 0_+} E\{\hat{u}_{\varepsilon}(t_1, \xi^{(1)}) \cdots \hat{u}_{\varepsilon}(t_r, \xi^{(r)})\} = \sum_{\varepsilon \to 0_+} I_{\varepsilon}(n) = \sum \tilde{I}(n),$$

where

$$\tilde{I}(n) := (-i\sigma(2\pi)^{-d})^{n} \sum_{\pi} \int_{\mathbb{R}^{n\pi d}} \exp \{ i \sum_{l=1}^{r} \beta_l t_l \} \prod_{l=1}^{r} d\beta_l \int \cdots \int \prod_{l=1}^{r} d\xi(l)$$

$$\times \prod_{l=1}^{r} \hat{u}_0(\xi(l)) - \sum_{k=1}^{n_l} \xi_k^{(j)}(e,f) \prod_{(e,f)\in \pi} \delta(\xi_e + \xi_f)$$

$$\times \left\{ \prod_{l=1}^{r} \prod_{k=0}^{n_l} \left[ 1 - i \left( \left| \xi^{(l)} - \sum_{j=1}^{k} \xi_j^{(l)} \right| - \beta_l \right) \right] \right\}^{-1}$$

$$= E\{\hat{u}^{(n)}(t_1, \xi^{(1)}) \cdots \hat{u}^{(n)}(t_r, \xi^{(r)})\}.$$
4.7 Weak convergence

In order to show the weak convergence of \( \hat{u}_\varepsilon(t, \xi) \), we need to first demonstrate the convergence of finite dimensional distributions, i.e., the convergence of distributions of \( (\hat{u}_\varepsilon(t_1, \xi^{(1)}), \ldots, \hat{u}_\varepsilon(t_r, \xi^{(r)})) \) for an arbitrary \( r \geq 1, t_1, \ldots, t_r \geq 0, \xi^{(1)}, \ldots, \xi^{(r)} \in \mathbb{R}^d \). For simplicity we consider only the case \( r = 1 \), as it is easy to generalize to the case for arbitrary \( r \).

Since we have already proved the moment convergence in the previous sections, it suffices to verify the determinacy of the distributions by their moments. In fact, we point out that the estimate (4.89) still holds if we replace some of the terms in (4.87) with their complex conjugates. Therefore, for \( n \) being even, we have

\[
\mathbb{E}|\hat{u}(t, \xi)|^n \leq \sum_{N=0}^{+\infty} \left( \frac{N+n-1}{n-1} \right) (N-1)!! \left( \frac{(Ct^2-\rho)^N}{(N!)^{2-\rho}} \right) e^{nt}.
\]  

(4.94)

Using Stirling’s formula we can easily obtain that

\[
\left( \frac{N+n-1}{n-1} \right) \leq C(1 + \frac{n-1}{N})^N (1 + \frac{N}{n-1})^{n-1}.
\]  

(4.95)

Plugging this into equation (4.94) therefore gives

\[
\mathbb{E}|\hat{u}(t, \xi)|^n \leq Ce^{nt} \sum_{N=0}^{n-1} \left( 1 + \frac{n-1}{N} \right)^N (N-1)!! \left( \frac{(Ct^2-\rho)^N}{(N!)^{2-\rho}} \right) e^{nt} + Ce^{nt} \sum_{N=n}^{+\infty} \left( 1 + \frac{N}{n-1} \right)^{n-1} (N-1)!! \left( \frac{(Ct^2-\rho)^N}{(N!)^{2-\rho}} \right) e^{nt}.
\]  

(4.96)

The first term can be easily estimated by \( Cn^n \) while the second by a constant \( C \) independent of \( n \). Therefore we have

\[
\sum_{n=1}^{+\infty} \frac{1}{[\mathbb{E}|\hat{u}(t, \xi)|^{2n}]^{1/2n}} \geq C \sum_{n=1}^{+\infty} \frac{1}{n^{1/2}} = +\infty
\]  

(4.97)

and the uniqueness follows from Carleman’s condition.
CHAPTER 4. CONVERGENCE OF SCHröDINGER EQUATION TO SPDE

It remains to prove the tightness of \{\hat{u}_\varepsilon(t, \xi)\} over \(C([0, +\infty), \mathcal{S}'(\mathbb{R}^d))\). By [15], it suffices to prove that \(\{u^\varepsilon_\phi(t) := \langle \hat{u}_\varepsilon(t, \cdot), \phi \rangle_{L^2(\mathbb{R}^d)}, t \geq 0\}\) for an arbitrary \(\phi \in \mathcal{S}(\mathbb{R}^d)\), which, by Kolmogorov’s theorem, follows from:

**Proposition 4.5.** For any \(T > 0\) and \(\phi \in \mathcal{S}(\mathbb{R}^d)\) there exists a constant \(C > 0\) such that

\[
\mathbb{E}|u^\phi_\varepsilon(t) - u^\phi_\varepsilon(s)|^2 \leq C(t - s)^2, \forall \varepsilon \in (0, 1], s, t \in [0, T]. \tag{4.98}
\]

**Proof.** From equation (4.2), we obtain

\[
u^\phi_\varepsilon(t) - u^\phi_\varepsilon(s) = i \int_s^t u^\phi_\varepsilon(\tau)d\tau + \int_s^t v^\phi_\varepsilon(\tau)d\tau, \tag{4.99}
\]

and

\[
v^\phi_\varepsilon(\tau) := \sum_{n \geq 0} v^{n,\phi}_\varepsilon(\tau),
\]

\[
v^{n,\phi}_\varepsilon(\tau) := (-i)^{n+1} \varepsilon^{-\frac{d(n+1)}{2}} e^{\tau} \int_{\mathbb{R}^d} e^{i\beta \tau} \int_{\mathbb{R}^d} \prod_{k=0}^{n} \hat{q}(\xi_k)d\xi_k \int_{\mathbb{R}^d} d\xi \times \left\{ \prod_{k=0}^{n} \left[ 1 - i \left( \left| \xi - \varepsilon^{-1} \sum_{j=0}^{k} \xi_j \right|^{m} - \beta \right) \right] \right\}^{-1} \hat{u}_0(\xi - \varepsilon^{-1} \sum_{j=0}^{n} \xi_j)\phi(\xi), \tag{4.100}
\]

where \(\phi_1(\xi) := \xi^m\phi(\xi)\). Taking the same approach as we did in section 4, we argue that for all \(\tau \in [0, T]\), we have \(\mathbb{E}|u^{\phi_1}(\tau)|^2 \leq C\) and \(\mathbb{E}|v^\phi_\varepsilon(\tau)|^2 \leq C\) for some constant independent of \(\varepsilon \in (0, 1]\). Estimate (4.98) is a consequence of the Cauchy-Schwarz inequality. \(\square\)
4.7. WEAK CONVERGENCE

Using (4.21) for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, we can write

\[
\langle \hat{u}^{(n)}(t, \cdot), \phi \rangle - \langle \hat{u}^{(n)}(s, \cdot), \phi \rangle = \int_s^t e^\tau d\tau \int_{\mathbb{R}} e^{-i\beta \tau} d\beta \int \cdots \int \prod_{k=1}^n d\xi_k \int d\xi
\]

\[
\times \left\{ \prod_{k=0}^n \left[ 1 - i \left( \frac{k}{m} \sum_{j=1}^n \xi_j \right) + \beta \right] \right\}^{-1} \hat{u}_0(\xi) - \sum_{j=1}^n \xi_j \phi(\xi) \prod_{j=1}^n e^{-i\xi_j x_j} \circ \prod_{j=1}^n dW(x_j)
\]

\[
- i \int_s^t e^\tau d\tau \int_{\mathbb{R}} e^{-i\beta \tau} d\beta \int \cdots \int \prod_{k=1}^n d\xi_k \int d\xi
\]

\[
\times \left\{ \prod_{k=0}^n \left[ 1 - i \left( \frac{k}{m} \sum_{j=1}^n \xi_j \right) + \beta \right] \right\}^{-1} \hat{u}_0(\xi) - \sum_{j=1}^n \xi_j \phi(\xi) \prod_{j=1}^n e^{-i\xi_j x_j} \circ \prod_{j=1}^n dW(x_j).
\]

(4.101)

Applying the same technique as in the proof for the $L^2(\mathbb{R}^d \times \Omega)$ setting, we check that for any $T > 0$, we have

\[
\mathbb{E} \left| \langle \hat{u}^{(n)}(t, \cdot), \phi \rangle - \langle \hat{u}^{(n)}(s, \cdot), \phi \rangle \right|^2 \leq (CT)^2(t-s)^2(n!)^{(1-\rho)} \forall n \geq 1, s, t \in [0, T]
\]

(4.102)

for some constant $C > 0$ independent of $n$. This in turn implies that

\[
\mathbb{E} \left| \langle \hat{u}(t, \cdot), \phi \rangle - \langle \hat{u}(s, \cdot), \phi \rangle \right|^2 \leq C(t-s)^2 \forall n \geq 1
\]

(4.103)

on any compact set. By the Kolmogorov-Chentsov Theorem, we have that $\langle \hat{u}(t, \cdot), \phi \rangle$ is continuous almost surely.

Weak convergence of $\{\hat{u}_\varepsilon(t, \xi)\}$ follows from the convergence of finite dimensional distribution and tightness. Back to the spatial space, by the Plancherel theorem, we have $\langle u(t, \cdot), \phi \rangle = \langle \hat{u}(t, \cdot), \hat{\phi} \rangle$. Hence, the process $\{u_\varepsilon(t, x)\}$ converges in law over $C([0, +\infty); \mathcal{S}'(\mathbb{R}^d))$ to $\{u(t, x)\}$. 

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Chapter 5

Homogenization of elliptic equation with an imaginary potential

5.1 Introduction

We study the asymptotic behavior of the solution to the equations parameterized by $\varepsilon$

$$(\Delta - \lambda + \frac{i}{\varepsilon} V(\frac{x}{\varepsilon})) u_\varepsilon(x) = f(x)$$  (5.1)

for $x \in \mathbb{R}^d$ as $\varepsilon \to 0$ in dimension $d \geq 3$. The motivation of studying (5.1) becomes clearer when we rewrite it as a system

$$\begin{pmatrix} \Delta - \lambda & 0 \\ 0 & \Delta - \lambda \end{pmatrix} \begin{pmatrix} u_{1,\varepsilon} \\ u_{2,\varepsilon} \end{pmatrix} + \begin{pmatrix} 0 & -V_\varepsilon \\ V_\varepsilon & 0 \end{pmatrix} \begin{pmatrix} u_{1,\varepsilon} \\ u_{2,\varepsilon} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$  (5.2)

Here, $V_\varepsilon$, a shorthand notation for $i\varepsilon^{-1} V(\varepsilon)$, may be understood as the biological or social interaction between two populations. The potential function $V(\cdot)$ is assumed to be real, stationary, and have bounded and integrable covariance function $R(x) = \mathbb{E}V(y)V(x + y)$. 

Moreover, a further assumption (5.32) is imposed on $V(\cdot)$ such that it has controlled fourth order cross moments. The assumption required for our problem is stronger than that in [6], where the diffusion equation with random and oscillatory Robin boundary condition is studied.

The corrector theory of elliptic equation is provided in [3] and [5]. The logarithmic type corrector of heat equation is analyzed in [12]. For (5.1), we believe the distribution of the corrector is difficult to characterize, and thus only estimate its size. In [18], the Feynman-Kac approach is taken to prove the homogenization of a parabolic equation with a purely imaginary, piecewise constant potential.

Our main result is the following theorem.

**Theorem 5.1.** In dimension $d \geq 3$, the solution $u_\varepsilon(x)$ to (5.1) converges to the solution to

$$(\Delta - \lambda - \rho)u_0(x) = f(x).$$

Here $\rho = \int \Phi(y)R(y)dy$, where $\Phi = (-\Delta)^{-1}\delta$. Moreover, we have the estimation of the error as

$$
\|u_\varepsilon - u_0\|_{L^2(\Omega \times \mathbb{R}^d)} \sim \begin{cases} 
O(\sqrt{\varepsilon}) & d = 3 \\
O(\varepsilon \sqrt{\log \varepsilon}) & d = 4 \\
O(\varepsilon) & d > 4.
\end{cases}
$$

(5.4)

### 5.2 Estimation of second order moments

Define $\chi_{\varepsilon} = G \ast (\frac{1}{\varepsilon}V(x))$, and $u_1(x) = \chi_{\varepsilon}(x)u_0(x)$, where $G = (-\Delta + \lambda)^{-1}\delta$. Simple calculation leads to

$$(\Delta - 1 + \frac{i}{\varepsilon}V(x))\chi_{\varepsilon}(x)u_0 + \varepsilon u_1 - u_\varepsilon = (\rho + iV(x)\chi_{\varepsilon}(x))u_0 + \varepsilon(\chi_{\varepsilon}\Delta u_0 + 2\nabla \chi_{\varepsilon} \cdot \nabla u_0).
$$

(5.5)
CHAPTER 5. HOMOGENIZATION OF ELLIPTIC EQUATION

The Green’s function $G(x)$ exponentially decays at infinity and of the same size as $|x|^{-(d-2)}$ at zero. Specifically, we have the estimation of $G(x)$ and its gradient as

$$G(x) \leq \frac{\exp(-\nu|x|)}{|x|^{d-2}}, \quad (5.6)$$

and

$$|\nabla G(x)| \leq \frac{\exp(-\nu|x|)}{|x|^{d-1}}, \quad (5.7)$$

for some constant $\nu$ dependent on $\lambda$. The size of $\varepsilon u_1$ is then estimated as

$$\mathbb{E} \int |\varepsilon u_1(x)|^2 dx = \frac{1}{\varepsilon^2} \int G(x - y_1)G(x - y_2)R\left(\frac{|y_1 - y_2|}{\varepsilon}\right)|u_0(x)|^2 dy_1 dy_2 dx$$

$$= \frac{1}{\varepsilon^2} \int G(y_1)G(y_2)R\left(\frac{|y_1 - y_2|}{\varepsilon}\right)|u_0(x)|^2 dy_1 dy_2 dx$$

$$\leq \frac{1}{\varepsilon^2} \int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-2}} \frac{\exp(-\nu|y_2|)}{|y_2|^{d-2}} R\left(\frac{|y_1 - y_2|}{\varepsilon}\right) dy_1 dy_2 dx$$

$$= \frac{1}{\varepsilon^2} \int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-2}} \frac{\exp(-\nu|y_2|)}{|y_1 - y_2|^{d-2}} R\left(\frac{|y_2|}{\varepsilon}\right) dy_1 dy_2 dx$$

$$= \begin{cases} 
\frac{1}{\varepsilon^2} \|u_0\|^2_2 C \exp(-\nu|y_2|) R\left(\frac{|y_2|}{\varepsilon}\right) dy_2, & d = 3 \\
\frac{1}{\varepsilon^2} \|u_0\|^2_2 C \exp(-\nu|y_2|)(\log |y_2| + 1) R\left(\frac{|y_2|}{\varepsilon}\right) dy_2, & d = 4 \\
\frac{1}{\varepsilon^2} \|u_0\|^2_2 C \exp(-\nu|y_2|)(|y_2|^{-(d-4)} + 1) R\left(\frac{|y_2|}{\varepsilon}\right) dy_2, & d > 4 
\end{cases} \quad (5.8)$$

$$\sim \begin{cases} 
O(\varepsilon), & d = 3 \\
O(\varepsilon^2 \log \varepsilon), & d = 4 \\
O(\varepsilon^2), & d > 4. 
\end{cases}$$

By replacing Green’s function with its gradient in (5.8) we find that

$$\mathbb{E} \int |\varepsilon \nabla u_1(x)|^2 dx \sim O(1). \quad (5.9)$$
Define $v_{\varepsilon} = u_0 + \varepsilon u_1 - u_{\varepsilon}$. Multiplying $\bar{v}_{\varepsilon}$, the complex conjugate of $v_{\varepsilon}$ to both sides and taking real part gives

\[
\mathbb{E} \int (\Delta - \lambda) v_{\varepsilon} \bar{v}_{\varepsilon} dx = \Re \left\{ \mathbb{E} \int (\rho + i V(\varepsilon x) \chi_{\varepsilon}(x)) u_0 \bar{v}_{\varepsilon} dx + \varepsilon \mathbb{E} \int (\chi_{\varepsilon} \Delta u_0 + 2 \nabla \chi_{\varepsilon} \cdot \nabla u_0) \bar{v}_{\varepsilon} dx \right\}
\]  
(5.10)

Performing separation of variables, the left hand side of (5.10) can be rewritten as

\[
\left| \mathbb{E} \int (\Delta - \lambda) v_{\varepsilon} \bar{v}_{\varepsilon} dx \right| = \mathbb{E} \int |v_{\varepsilon}|^2 + |\nabla v_{\varepsilon}|^2 dx \leq (\lambda \vee 1) \|v_{\varepsilon}\|_{L^2(\Omega, H^1(\mathbb{R}^d))}^2,
\]  
(5.11)

where $\lambda \vee 1 := \max(\lambda, 1)$.

Now we discuss the estimation of the right hand side of (5.10). The term $\chi_{\varepsilon} \Delta u_0$ can be estimated in the same way as $u_1$, by using Cauchy-Schwartz inequality, we have

\[
\left| \varepsilon \mathbb{E} \int \chi_{\varepsilon} \Delta u_0 \bar{v}_{\varepsilon} dx \right| \leq C \|v_{\varepsilon}\|_{L^2(\Omega, H^1(\mathbb{R}^d))} \times \begin{cases} \sqrt{\varepsilon}, & \text{if } d = 3 \\ \varepsilon \sqrt{\log \varepsilon}, & \text{if } d = 4 \\ \varepsilon, & \text{if } d > 4. \end{cases}
\]  
(5.12)

The integral $\varepsilon \mathbb{E} \int \nabla \chi_{\varepsilon} \cdot \nabla u_0 \bar{v}_{\varepsilon} dx$ is estimated using separation of variables as

\[
\left| \varepsilon \mathbb{E} \int \nabla \chi_{\varepsilon} \cdot \nabla u_0 \bar{v}_{\varepsilon} dx \right| = \left| \varepsilon \mathbb{E} \int (\nabla \cdot (\chi_{\varepsilon} \nabla u_0) - \chi_{\varepsilon} \Delta u_0) \bar{v}_{\varepsilon} dx \right|
\leq \left| \varepsilon \mathbb{E} \int \nabla \bar{v}_{\varepsilon} \cdot \nabla u_0 \chi_{\varepsilon} \bar{v}_{\varepsilon} dx \right| + \left| \varepsilon \mathbb{E} \int \chi_{\varepsilon} \Delta u_0 \bar{v}_{\varepsilon} dx \right|
\leq C \|\nabla v_{\varepsilon}\|_{L^2(\Omega \times \mathbb{R}^d)} \times \begin{cases} \sqrt{\varepsilon}, & \text{if } d = 3 \\ \varepsilon \sqrt{\log \varepsilon}, & \text{if } d = 4 \\ \varepsilon, & \text{if } d > 4. \end{cases}
\]  
(5.13)

Define $f_\varepsilon(x) = G \ast ((\rho + i V(\varepsilon x)) \chi_{\varepsilon}(x)) u_0)$. The integral $\mathbb{E} \int (\rho + i V(\varepsilon x)) \chi_{\varepsilon}(x)) u_0 \bar{v}_{\varepsilon} dx$ is esti-
mated using separation of variables as

\[
\left| E\int (\rho + i V(x/\varepsilon) \chi(x)) u_0 \bar{v}_\varepsilon dx \right| = \left| E\int f_\varepsilon \bar{v}_\varepsilon + \nabla f_\varepsilon \nabla \bar{v}_\varepsilon dx \right| \leq C(E \int |\nabla f_\varepsilon|^2 dx)^{1/2} \|v_\varepsilon\|_{L_2(\Omega, H^1(\mathbb{R}^d))}.
\]

(5.14)

The estimation of \( E \int |\nabla f_\varepsilon|^2 dx \) is more involved and will be discussed in the next section.

### 5.3 Estimation of fourth order moments

In this section we will find the estimation of \( E \int |\nabla f_\varepsilon|^2 dx \). We have the following decomposition for \( ||\nabla f_\varepsilon||_2^2 \)

\[
E \int |\nabla f_\varepsilon|^2 dx = \frac{1}{\varepsilon^4} \int \nabla G(x, y) u_0(y) dy \cdot \frac{1}{\varepsilon^2} \int \nabla G(x, y) V(\frac{y}{\varepsilon}) G(y, z) V(\frac{z}{\varepsilon}) u_0(y) dy dz \] \( dx \)

\[
= \frac{1}{\varepsilon^4} \int \nabla G(x, y_1) G(y_1, z_1) \nabla G(x, y_2) G(y_2, z_2) R(y_1 - y_2) R(z_1 - z_2) u_0(y_1) u_0(y_2) dy_1 dy_2 dz_1 dz_2 dx + \frac{1}{\varepsilon^4} \int \nabla G(x, y_1) G(y_1, z_1) \nabla G(x, y_2) G(y_2, z_2) R(y_1 - z_2) R(z_1 - y_2) u_0(y_1) u_0(y_2) dy_1 dy_2 dz_1 dz_2 dx
\]

\[
+ \frac{1}{\varepsilon} \int \left[ \frac{1}{\varepsilon^2} \int \nabla G(x, y) y R(\frac{y - z}{\varepsilon}) u_0(y) dy dy dz - \rho \int \nabla G(x, y) u_0(y) dy \right]^2 dx.
\]

\[
+ \frac{1}{\varepsilon^4} \int \left( \frac{1}{\varepsilon} q(\frac{y_1}{\varepsilon}) q(\frac{y_2}{\varepsilon}) q(\frac{z_1}{\varepsilon}) q(\frac{z_2}{\varepsilon}) - R(\frac{y_1 - y_2}{\varepsilon}) R(\frac{z_1 - z_2}{\varepsilon}) - R(\frac{y_1 - z_2}{\varepsilon}) R(\frac{y_2 - z_1}{\varepsilon}) \right)
\]

\[
\cdot (I) + (II) + (III) + (IV)
\]

We point out that \( E \int |\nabla f_\varepsilon|^2 dx = (I) + (II) + (III) \) if \( V(\cdot) \) is assume to be Gaussian random field. The gradient of Green’s function is bounded as

\[
|\nabla G(x)| \leq \frac{\exp(-\nu|x|)}{|x|^{d-1}},
\]

(5.16)
5.3. ESTIMATION OF FOURTH ORDER MOMENTS

which will be useful for estimating (I), (II), (III), and (IV).

(1) Estimation of (I) Changing variables \( y_i \) and \( z_i \) to \( x - y_i \) and \( x - y_i - z_i \) for \( i = 1, 2 \) gives

\[
|\langle I \rangle| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1) \nabla G(z_1) \nabla G(y_2) \nabla G(z_2) |R(\frac{y_1 - y_2}{\varepsilon})||R(\frac{y_1 - y_2}{\varepsilon} - \frac{z_1 - z_2}{\varepsilon})|
\]

\[
|u_0(x - y_1)||u_0(x - y_2)|dy_1 dy_2 dz_1 dz_2 dx.
\]

Using \( u_0 \) to integrate in \( x \), we then have

\[
|\langle I \rangle| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1) \nabla G(z_1) \nabla G(y_2) \nabla G(z_2) |R(\frac{y_1 - y_2}{\varepsilon})||R(\frac{y_1 - y_2}{\varepsilon} - \frac{z_1 - z_2}{\varepsilon})|dy_1 dy_2 dz_1 dz_2.
\]

(5.17)

Changing variables \( y_2 \) and \( z_2 \) to \( y_1 - y_2 \) and \( z_1 - z_2 \), and plugging in the bound of the gradient of Green’s function (5.16) yields

\[
|\langle I \rangle| \leq C \frac{1}{\varepsilon^4} \int \exp(-\nu|y_1|) \exp(-\nu|z_1|) \exp(-\nu|y_1 - y_2|) \exp(-\nu|z_1 - z_2|) |R(\frac{y_2}{\varepsilon})||R(\frac{y_2}{\varepsilon} - \frac{z_2}{\varepsilon})|
\]

\[
dy_1 dy_2 dz_1 dz_2.
\]

(5.18)

Now we may apply Lemma 5.3 to integrate in \( y_1 \) and \( z_1 \):

\[
\int \frac{\exp(-\nu|y_1|) \exp(-\nu|y_1 - y_2|)}{|y_1|^{d-1} |y_1 - y_2|^{d-1}} dy_1 \leq C \exp(-\nu|y_2|)(1 + |y_2|^{-(d-2)}), \quad (5.20)
\]

\[
\int \frac{\exp(-\nu|z_1|) \exp(-\nu|z_1 - z_2|)}{|z_1|^{d-2} |z_1 - z_2|^{d-2}} dz_1 \leq \begin{cases} 
C \exp(-\nu z_2), & d = 3 \\
C \exp(-\nu z_2)(1 + \log |z_2|), & d = 4 \\
C \exp(-\nu z_2)(1 + |z_2|^{-(d-4)}), & d > 4.
\end{cases}
\]

(5.21)
The estimation of (I) can be rewritten as

\[
|I| \leq C \frac{1}{\varepsilon^4} \int \int dy_2 dz_2 \exp(-\nu|y_2|)(1 + |y_2|^{-(d-2)}) \exp(-\nu|z_2|)|R\left(\frac{y_2}{\varepsilon}\right)||R\left(\frac{y_2 - z_2}{\varepsilon}\right)|
\]

\[
\times \begin{cases} 
   1, & d = 3 \\
   \log(|z_2|), & d = 4 \\
   (1 + |z_2|^{-(d-4)}), & d > 4.
\end{cases}
\]

(5.22)

It remains to integrate in \(y_2\) and \(z_2\) to obtain

\[
|I| \sim \begin{cases} 
   O(\varepsilon), & d = 3 \\
   O(\varepsilon^2 \log \varepsilon), & d = 4 \\
   O(\varepsilon^2), & d > 4.
\end{cases}
\]

(5.23)

(2) Estimation of (II) After changing variables \(y_1\) and \(z_1\) to \(x - y_i\) and \(x - y_i - z_i\) for \(i = 1, 2\), and integrating in \(x\) using \(u_0\), we have

\[
|I| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1)G(z_1)\nabla G(y_2)G(z_2)|R\left(-\frac{y_1 + y_2 + z_2}{\varepsilon}\right)||R\left(-\frac{y_1 + y_2 + z_1}{\varepsilon}\right)|dy_1 dy_2 dz_1 dz_2.
\]

Changing variable \(y_2\) to \(y_1 - y_2\), plugging in inequality (5.16) gives

\[
|I| \leq C \frac{1}{\varepsilon^4} \int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-1}} \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} \frac{\exp(-\nu|y_2|)}{|y_1 - y_2|^{d-1}} \frac{\exp(-\nu|z_2|)}{|z_2|^{d-2}} |R\left(\frac{z_2 - y_2}{\varepsilon}\right)||R\left(\frac{z_1 + y_2}{\varepsilon}\right)|
\]

\[
dy_1 dy_2 dz_1 dz_2.
\]

(5.24)

(5.25)

We now integrate in \(y_1\) and \(z_1\):

\[
\int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-1}} \frac{\exp(-\nu|y_1 - y_2|)}{|y_1 - y_2|^{d-1}} dy_1 \leq C \exp(-\nu|y_2|)(1 + |y_2|^{-(d-2)}),
\]

\[
\int \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} |R\left(\frac{z_1 + y_2}{\varepsilon}\right)| dz_1 \leq C \varepsilon^2.
\]

(5.26)
5.3. ESTIMATION OF FOURTH ORDER MOMENTS

The estimation is then recast as

\[ |(II)| \leq C \frac{1}{\varepsilon^2} \int \int \exp(-\nu|y_2|)(1 + |y_2|^{-(d-2)}) \frac{\exp(-\nu|z_2|)}{|z_2|^{d-2}} |R \frac{z_2 - y_2}{\varepsilon}| dy_2 dz_2. \quad (5.27) \]

Changing variable \( z_2 \) to \( y_2 - z_2 \), and integrating in \( y_2 \) using Lemma 5.3 yields

\[ |(II)| \leq C \frac{1}{\varepsilon^2} \int dz_2 \exp(-\nu|z_2|) |R \frac{z_2}{\varepsilon}| \times \left\{ \begin{array}{ll}
1, & d = 3 \\
\log(|z_2|), & d = 4 \\
(1 + |z_2|^{-(d-4)}), & d > 4.
\end{array} \right. \quad (5.28) \]

It remains to integrate in \( z_2 \) to obtain

\[ |(II)| \sim \left\{ \begin{array}{ll}
O(\varepsilon), & d = 3 \\
O(\varepsilon^2 \log \varepsilon), & d = 4 \\
O(\varepsilon^2), & d > 4.
\end{array} \right. \quad (5.29) \]

(3) Estimation of (III) Recall that \( \rho = \int \Phi(y) R(y) dy \). The estimation of (III) has the expression as

\[ (III) = \int \left( \frac{1}{\varepsilon^2} \int \int \nabla G(y)(G(z) - \Phi(z)) R \frac{z}{\varepsilon} u_0(x - y) dy dz \right)^2 dx \]
\[ = \left( \frac{1}{\varepsilon^2} \int (G(z) - \Phi(z)) R \frac{z}{\varepsilon} dz \right)^2 \left( \int \nabla G(y) u_0(x - y) dy \right)^2 dx. \quad (5.30) \]

Since \( \int (\int \nabla G(y) u_0(x - y) dy)^2 dx \) is bounded, we have

\[ |(III)| \sim \left\{ \begin{array}{ll}
O(\varepsilon^2), & d = 3 \\
O(\varepsilon^4 \log \varepsilon^2), & d = 4 \\
O(\varepsilon^4), & d > 4.
\end{array} \right. \quad (5.31) \]
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(4) *Estimation of (IV)* Let us first provide an additional assumption regarding the potential $V(\cdot)$ on which we reply to show homogenization. We need to introduce some notations to formulate this assumption.

Consider a set of spatial points $\{x_1, x_2, x_3, x_4\}$ as variables of the potential function $V(\cdot)$. Let $p := (\{p_1, p_2\}, \{p_3, p_4\})$ be two different pairs of indices selected from $\{1, 2, 3, 4\}$. Note that $p_1 \neq p_2$, and $p_3 \neq p_4$, but it is possible for the pairs to have one common index. Define $U_*$ to be the set of two pairs which do not have common index, and $U^*$ to be the set of two pairs which share exactly one common index. The assumption required for our estimation is:

$$\left| \prod_{i=1}^4 V(x_i) - \sum_{q \in U_*} \mathbb{E}\{V(x_{q_1})V(x_{q_2})\} \mathbb{E}\{V(x_{q_3})V(x_{q_4})\} \right| \leq \phi_p(x_{p_1} - x_{p_2}, x_{p_3} - x_{p_4}).$$

(5.32)

for all possible $p \in U^*$, where $\phi_p \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

The motivation of dividing pairs this way comes from the characteristic of the fourth moment of Gaussian random variables. To be specific, if $V(\cdot)$ are mean zero Gaussian random fields, we simply have $\phi_p = 0$. A notable class of potentials that satisfies this condition is Poisson point process, whose cross-moments are derived in [6]. Under condition (5.32), $IV$ has the expression as

$$|IV| \leq C \frac{1}{\varepsilon^4} \int \nabla G(x - y_1)G(y_1 - z_1)\nabla G(x - y_2)G(y_2 - z_2) |\phi_p(\frac{y_1 - z_1}{\varepsilon}, \frac{y_1 - z_2}{\varepsilon})|
\left| u_0(y_1) \right| \left| u_0(y_2) \right| dy_1 dy_2 dz_1 dz_2 dx.$$  

(5.33)

Performing change of variables: $x - y_1 \to y_1$, $x - y_2 \to y_2$, $y_1 - z_1 \to z_1$, $y_2 - z_2 \to -z_2$ yields

$$|IV| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1)G(z_1)\nabla G(y_2)G(z_2) |\phi_p(\frac{z_1}{\varepsilon}, \frac{y_1 - y_2 - z_2}{\varepsilon})|
\left| u_0(x - y_1) \right| \left| u_0(x - y_2) \right| dy_1 dy_2 dz_1 dz_2 dx.$$  

(5.34)
Using $|u_0(x - y_1)||u_0(x - y_2)|$ to integrate in $x$, the bound of (IV) then becomes

$$|(IV)| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1)G(z_1)\nabla G(y_2)G(z_2)|\phi_p\left(\frac{z_1}{\varepsilon}, \frac{y_1 - y_2 - z_2}{\varepsilon}\right)|dy_1dy_2dz_1dz_2. \quad (5.35)$$

Changing variable $y_1 - y_2 \to y_2$ gives

$$|(IV)| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1)G(z_1)\nabla G(y_1 - y_2)G(z_2)|\phi_p\left(\frac{z_1}{\varepsilon}, \frac{y_2 - z_2}{\varepsilon}\right)|dy_1dy_2dz_1dz_2$$

$$\leq C \frac{1}{\varepsilon^4} \int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-1}} \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} \frac{\exp(-\nu|y_1 - y_2|)}{|y_1 - y_2|^{d-1}} \frac{\exp(-\nu|z_2|)}{|z_2|^{d-2}}$$

$$|\phi_p\left(\frac{z_1}{\varepsilon}, \frac{y_2 - z_2}{\varepsilon}\right)|dy_1dy_2dz_1dz_2. \quad (5.36)$$

Using Lemma 5.3 to integrate in $y_1$ we obtain

$$|(IV)| \leq C \frac{1}{\varepsilon^4} \int \int \exp(-\nu|y_2|)(|y_2|^{-(d-2)} + 1) \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} \frac{\exp(-\nu|z_2|)}{|z_2|^{d-2}}$$

$$|\phi_p\left(\frac{z_1}{\varepsilon}, \frac{y_2 - z_2}{\varepsilon}\right)|dy_2dz_1dz_2. \quad (5.37)$$

Changing variable $y_2 - z_2 \to z_2$ and using Lemma 5.3 again to integrate in $y_2$ gives

$$|(IV)| \leq \begin{cases} 
C \frac{1}{\varepsilon^4} \int \int \exp(-\nu|z_2|) \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} |\phi_p\left(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}\right)|dz_1dz_2 & d = 3 \\
C \frac{1}{\varepsilon^4} \int \int \exp(-\nu|z_2|)(|\log|z_2|| + 1) \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} |\phi_p\left(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}\right)|dz_1dz_2 & d = 4 \\
C \frac{1}{\varepsilon^4} \int \int \exp(-\nu|z_2|)(|z_2|^{-(d-4)} + 1) \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} |\phi_p\left(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}\right)|dz_1dz_2 & d > 4 \\
O(\varepsilon) & d = 3 \\
O(\varepsilon^2|\log\varepsilon|) & d = 4 \\
O(\varepsilon^2) & d > 4 
\end{cases} \quad (5.38)$$

Finally, the estimation of $\mathbb{E} \int |\nabla f_\varepsilon|^2 dx$ is given by summing up (5.23), (5.29), (5.31),
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and (5.38) as
\[
E \int |\nabla f_\varepsilon|^2 dx \sim \begin{cases} 
O(\varepsilon) & d = 3 \\
O(\varepsilon^2 |\log \varepsilon|) & d = 4 \\
O(\varepsilon^2) & d > 4
\end{cases}
\tag{5.39}
\]

Plugging (5.11), (5.12), (5.13), and (5.14) into (5.10) gives
\[
\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega \times \mathbb{R}^d)} = \|v_\varepsilon\|_{H^1(\Omega \times \mathbb{R}^d)} \sim \begin{cases} 
O(\sqrt{\varepsilon}) & d = 3 \\
O(\varepsilon \sqrt{|\log \varepsilon|}) & d = 4 \\
O(\varepsilon) & d > 4
\end{cases}
\tag{5.40}
\]

The size of \(\varepsilon u_1\) in \(L^2(\Omega, H^1(\mathbb{R}^d))\) is given by (5.8). By Cauchy-Schwartz inequality, our estimation of the error is
\[
\|u_\varepsilon - u_0\|_{L^2(\Omega \times \mathbb{R}^d)} \sim \begin{cases} 
O(\sqrt{\varepsilon}) & d = 3 \\
O(\varepsilon \sqrt{|\log \varepsilon|}) & d = 4 \\
O(\varepsilon) & d > 4
\end{cases}
\tag{5.41}
\]

The proof of the main theorem is completed.

Remark 5.2. (5.9) implies that \(\varepsilon u_1\) is the leading corrector \(u_\varepsilon - u_0\) in \(L^2(\Omega, H^1(\mathbb{R}^d))\).

5.4 Appendix

The following lemma is proved in [6].

Lemma 5.3. Let us fix two distinct points \(x, y \in \mathbb{R}^d\). Let \(\alpha, \beta\) be positive numbers in \((0, d), \)
and $\lambda$ another positive number. We have the following convolution results.

$$
\int_{\mathbb{R}^d} e^{-\lambda|z-x|} e^{-\lambda|z-y|} \left| z - x \right|^{\alpha} \left| z - y \right|^{\beta} dz \leq \begin{cases} 
C \exp(-\lambda|x - y|)(|x - y|^{d-(\alpha+\beta)} + 1), & \text{if } \alpha + \beta > d; \\
C \exp(-\lambda|x - y|)(|x - y||\log|x - y| + 1), & \text{if } \alpha + \beta = d; \\
C \exp(-\lambda|x - y|), & \text{if } \alpha + \beta < d.
\end{cases}
$$

(5.42)

The above constants depend only on the diam($X$), $\alpha$, $\beta$, $\lambda$, and dimension $d$ but not on $|x - y|$.
Bibliography


