Essays on Market Design and Auction Theory

Youngwoo Koh

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ABSTRACT

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This dissertation consists of three essays on market design and auction theory. In the first chapter, we develop a model of decentralized college admissions in which students’ preferences for colleges are uncertain, and colleges must incur costs when their enrollments exceed their capacities. Colleges’ admission decisions then become a tool for strategic yield management, because the enrollment at a college depends on not only students’ uncertain preferences but also other colleges’ admission decisions. We find that colleges’ equilibrium admission decisions exhibit “strategic targeting”—colleges may forgo admitting (even good) students likely sought after by the others and may admit (not as good) students likely overlooked by the others. Randomization in admissions may also emerge. The resulting assignment fails to be efficient (among students, among colleges and among all parties including colleges and students) and leads to justified envy among students. When the colleges consider multiple dimensions of students’ merits, their evaluations are unlikely to be perfectly correlated. In such a case, colleges may avoid head-on competition by distorting their evaluation to place excessive weight on less correlated dimensions, such as extra curricular activities and non-academic aspects of students’ application portfolios. Restricting the number of applications or allowing for wait-listing might alleviate colleges’ yield management problem, but the resulting assignments are still inefficient and admit justified envy. Centralized matching via Gale and Shapley’s Deferred Acceptance algorithm eliminates colleges’ yield management problem and justified envy among students and attains efficiency. It also attains the outcome that is jointly optimal among colleges, but some colleges may be worse off relative to decentralized matching.
The second chapter studies a keyword auction model where bidders have constrained budgets. In the absence of budget constraints, Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) analyze “locally envy-free equilibrium” or “symmetric Nash equilibrium” bidding strategies in generalized second-price (GSP) auctions. However, bidders often have to set their daily budgets when they participate in an auction; once a bidder’s payment reaches his budget, he drops out of the auction. This raises an important strategic issue that has been overlooked in the previous literature: Bidders may change their bids to inflict higher prices on their competitors because under GSP, the per-click price paid by a bidder is the next highest bid. We provide budget thresholds under which equilibria analyzed in Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) are sustained as “equilibria with budget constraints” in our setting. We then consider a simple environment with one position and two bidders and show that a search engine’s revenue with budget constraints may be larger than its revenue without budget constraints.

In the third chapter, we study the procurement of an innovation in which firms exert effort and create innovations, where the quality of innovation is stochastic. Both the effort level and the quality of innovation are unverifiable, and the procurer cannot extract up-front payment from the firms. Given the uncertainty of quality realization, there is a trade-off regarding the number of participating firms in the procurement process: If many firms participate in the process, they may be discouraged from expending their initial investment because each of them has a small chance of winning (we call this incentive effect). At the same time, as the number of participants increases, the procurer has a growing chance of getting a higher quality because of the randomness of the quality realization (sampling effect). Therefore, the procurer faces a nontrivial problem of how many firms to invite in the procurement process. We consider two prominent contest mechanisms, a first-price auction and a fixed-prize tournament. We show that if the randomness is large enough,
it is optimal for the buyer to invite as many firms as possible in both mechanisms, and the fixed-prize tournament outperforms the first-price auction. In the limit at which the randomness vanishes, inviting only two firms is optimal in both mechanisms, and the first-price auction outperforms the fixed-prize tournament. Under the first-price auction, we show that any equilibrium converges to an equilibrium as the randomness diminishes and provide a characterization of the limit equilibrium. We also provide a constructive example of a mixed-strategy equilibrium with two firms when the randomness is moderate.
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Chapter 1

Decentralized College Admissions

Yeon-Koo Che and Youngwoo Koh
1.1 Introduction

The standard market design research on matching focuses on how best to design a centralized matching mechanism, taking the societal consensus on centralization as a given. While such a consensus exists in a number of markets (e.g., medical residency matching and public school matching), many markets remain decentralized (e.g., college admissions and graduate school admissions). Decentralized markets often exhibit congestion and do not operate efficiently (Roth and Xing, 1997). Although it is widely believed that these markets will benefit from improved coordination or centralization, it is not well understood why they remain decentralized and what welfare benefits would be gained by improving coordination or by centralizing them.

At least part of the problem is the lack of an analytical grasp of decentralized matching markets. Often treated as a black box, the equilibrium and welfare implications of decentralized matching markets have not been understood well in the literature. Indeed, we have yet to develop a workhorse model of decentralized matching that could serve as a useful benchmark for comparison with a centralized system.\footnote{The main exceptions are two excellent works by Chade and Smith (2006) and Chade, Lewis, and Smith (2011). As we discuss more fully later, they focus on the portfolio decisions students face in application and colleges’ inference of students’ abilities based on imperfect signals. By contrast, the current paper focuses on the matching implications of college admissions, paying special attention to the yield management problem arising from (aggregately) uncertain students’ preferences.}

The current paper develops an analytical framework for understanding decentralized matching markets in the context of college admissions. In essence, college admissions are a case of two-sided, many-to-one matching, and much is understood about how best to organize such a market using a central clearinghouse.\footnote{See Abdulkadiroğlu and Sönmez (2012) for an excellent survey.} However, in many countries, such as the US, Korea and Japan, college admissions are organized similarly to decentralized labor markets, with exploding and binding admissions made by schools during a short window of
With limited offers and acceptances to clear the markets, decentralized matching provides only a limited chance for colleges to learn students' preferences and to condition their admission decisions on them. This presents a challenge for colleges in managing its yield. Inability to forecast yield accurately could result in too many or too few students enrolling a college relative to its capacity. Either mistake is costly. For instance, 1,415 freshmen accepted Yale’s invitation to join its incoming class in 1995-96, although the university had aimed for a class of 1,335. At the same year, Princeton also reported 1,100 entering students, the largest in its history. The college sets up mobile homes in fields and built new dorms to accommodate the students (Avery, Fairbanks, and Zeckhauser, 2003).

The yield management problem becomes increasingly important in many countries. In Korea, for example, students apply for departments not for colleges. Since each department has a small quota and there are many potential choices for students, departments try to predict yield rates in order to ensure that they fill their capacities. In the US, most colleges continue to experience increase in the number of applications they receive, and the average yield rate of four-year colleges in the US has declined significantly over the past decade, from 49 percent in 2001 to 38 percent in 2011. Declining rates signal greatly increased uncertainty for colleges.

Importantly, the uncertainty facing a college with respect to a student’s enrollment depends not just on her preference but also on what other set of admissions she receives. This makes a college’s admission policy a strategic yield management decision. We provide a

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3College admissions are centralized in varying degrees in Australia, China, Taiwan, Turkey and the UK.

4The cost may also take the form of an explicit sanction imposed on the admitting unit (e.g., department) by the government (as in Korea) or by the college (as in Australia).

5The application increase in recent years is due partly to the increased number of high school graduate but also to an increase in applications submitted per student. Seventy-nine percent of Fall 2011 freshmen applied to three or more colleges, and twenty-nine percent of them submitted seven or more applications. (Clinedinst, Hurley, and Hawkins, 2012)
simple model of colleges’ strategic yield management problems and characterize the equilibrium outcomes of these strategic decisions. The explicit analysis of equilibrium allows us to evaluate the resulting assignment in terms of welfare and fairness and to compare this with outcomes that arise from other coordinated admissions and centralized matching.

In our baseline model, there are two colleges, each with limited capacity, and a unit mass of students with “scores” that are common for both colleges (e.g., high school GPA or SAT scores). Students apply to colleges at no cost. Colleges prefer students according to their scores, but they do not know students’ preferences toward them. This uncertainty takes an aggregate form: the mass of students preferring one college over the other varies across states that are unknown to the colleges. Over-enrollment is costly for a college in that it incurs a sufficiently high cost for each incremental enrollment in excess of its capacity. Our baseline model involves a simple time line: Initially, students simultaneously apply to colleges. Each college observes only the scores of those students who apply to them. Next, the two colleges simultaneously offer admissions to sets of students. Finally, the students who are admitted by either or both colleges decide on which admission they will accept.

Given that application is costless, students have a (weak) dominant strategy of applying to both colleges. Hence, the main focus of the analysis is the college’s admission decisions. Our main finding in this regard is characterized by “strategic targeting:” Since the students who attract competing admissions from the other college presents a greater enrollment uncertainty and add to a higher capacity cost, a college seeks to systematically avoid such students. Hence, in equilibrium, each college may forgo good students who are sought after by the other college and may admit less attractive students who appear overlooked by the other college. Randomization in admissions for students may also emerge. We then provide the existence of these equilibria. Next, we study the welfare and fairness properties of the equilibrium assignments and show that the assignment is typically unfair, that is, it entails justified envy among students, and fails to achieve efficiency among students, among colleges
and among all parties including colleges and students.

These results can be illustrated via a simple example. Suppose there are only two students, 1 and 2, applying to colleges \( A \) and \( B \). Each college has one seat to fill and faces a prohibitively high cost of having two students. Student \( i \) has score \( v_i, i = 1, 2 \), where \( 0 < v_2 < v_1 < 2v_2 \). Each student has an equal probability of preferring either school, which is private information (unknown to the other student and to the colleges). Each college values having student \( i \) at \( v_i \). The applications are free of cost, and the timing is the same as that explained above.

Given the large cost of over-enrollment, each college admits only one of the students. Their payoffs are described as follows:

<table>
<thead>
<tr>
<th>A’s strategy \ B’s strategy</th>
<th>Admit 1</th>
<th>Admit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Admit 1</td>
<td>( \frac{1}{2}v_1, \frac{1}{2}v_1 )</td>
<td>( v_1, v_2 )</td>
</tr>
<tr>
<td>Admit 2</td>
<td>( v_2, v_1 )</td>
<td>( \frac{1}{2}v_2, \frac{1}{2}v_2 )</td>
</tr>
</tbody>
</table>

This game has a battle of the sexes’ structure (with asymmetric payoffs), so it is not difficult to see that there are two different types of equilibria. First, there are two asymmetric pure-strategy equilibria in which one college admits student 1 and the other admits student 2. There is also a mixed-strategy equilibrium in which each college admits 1 with probability \( \gamma := \frac{2v_1-v_2}{v_1+v_2} > 1/2 \) and admits 2 with probability \( 1 - \gamma \), where \( \gamma \) is chosen such that the other college is indifferent. Both types of equilibria show the pattern of strategic targeting. In the pure-strategy equilibria, colleges manage to avoid competition and thus randomness in enrollment by targeting different students. In the mixed-strategy equilibrium, when a college misses student 1 with probability \( 1 - \gamma \), the other college may admit the student. Thus, it also exhibits targeting as colleges seek to avoid head-to-head competition, while it does not result in perfect coordination.

This example, while extremely simple, suggests problems with decentralized matching
in terms of welfare and fairness. First, the student with high score (student 1) may be assigned to a less preferred school (in both types of equilibria) even though both colleges prefer the high scoring student; that is, justified envy arises. Second, it could be the case that student 1 prefers \text{A} and student 2 prefers \text{B}, but the former is assigned to \text{B} and the latter is assigned to \text{A}, showing that the equilibrium outcome is inefficient among students. Lastly, the mixed-strategy equilibrium is Pareto inefficient because both colleges may admit the same student, in which case one college is unmatched and would rather match with the other student.

We next study admissions problem when students have multidimensional types. Some measures, such as students’ academic performances, are highly correlated among colleges, but others measures, such as students’ extra curricular activities, are less correlated among them. In this case, colleges may evaluate students based not only on academic abilities but also other extra-curricular activities, or they may put different weights on different dimensions of students performance measures. In Japan and Korea, for instance, students take a nationwide exam and also often take essay tests/oral exams for the school of choice. Clinedinst, Hurley, and Hawkins (2012) reports some evidence that selective colleges places more emphasis on many factors, such as essay/writing sample and extracurricular activities, than less selective schools. We show that colleges’ desires to avoid head-on competition, and thus to lessen enrollment uncertainty, lead them to bias their evaluation in favor of less correlated measures by placing excessive weight on theses dimensions.

We also study two common ways for colleges to alleviate their yield management problem. One common way is “self-targeting,” whereby colleges coordinate to restrict the number of applications each student can submit. This form of coordination is observed in many countries; for instance, students in the UK cannot apply to both Cambridge and Oxford, students in Japan can apply to at most one public university, and students in Korea face a similar restriction. Self-targeting reduces the enrollment uncertainty for the colleges, and
thus alleviates their yield management burden. Yet, we show that this method may not completely eliminate the yield management problem and justified envy, and it may also fail to achieve efficiency.

Another way to cope with the enrollment uncertainty is to employ a sequential admissions strategy: Colleges admit some students and place others in the waiting lists in each of multiple rounds and later extend further admissions to those in the waiting list when seats open up from the previous round. This method is also observed in many countries, including France and Korea. Sequential admissions may alleviate colleges’ yield management problem, since colleges may adjust their admission offers based on the students’ acceptance behavior and the information the colleges may learn over the course of the process. We show, however, that colleges may still engage in strategic targeting under this mechanism, and the welfare and fairness problems still remain.

Finally, we consider a centralized matching via Gale and Shapley’s Deferred Acceptance algorithm (DA in short). This eliminates colleges’ yield management problem and justified envy completely and attains efficiency. At the same time, it is possible for one college to be worse off relative to the decentralized matching. For instance, in the above example, suppose a pure-strategy equilibrium in which college $i$ always gets student 1 is played. Then, that college will clearly be worse off from a switch to a centralization via DA because the college will not always attract student 1. This may explain a possible lack of consensus toward centralization and may underscore why college admissions remain decentralized in many countries.

The paper is organized as follows. Section 1.1.1 discusses the related literature. The model is introduced in Section 3.2. Equilibrium is characterized in Section 1.3. Section 1.3.1 establishes existence of equilibrium. Section 1.3.2 discusses welfare and fairness implications of equilibria. Section 1.4 studies admissions problem when students’ types are multidimensional. In Section 1.5, self-targeting via restriction on application is studied, and in
Section 1.6, sequential admissions are studied. Centralized matching via DA is considered in Section 1.7. Section 3.5 concludes the paper. Proofs are provided in the Appendix unless stated otherwise. The Appendix also extends the baseline model to allow for more than two colleges and shows that our analysis in the two-college model carries over.

1.1.1 Related Literature

Several papers in the matching literature have considered decentralized matching markets. Roth and Xing (1997) study the entry-level market for clinical psychologists in which firms make offers to workers sequentially within a day and workers can accept, reject or hold an offer. They find that, mainly based on simulations, such a decentralized (but coordinated) market exhibits congestion, i.e., not enough offers and acceptances could be made to clear the market, and the resulting outcome is unstable. Neiderle and Yariv (2009) also study a decentralized (one-to-one matching) market in which firms make offers sequentially through multiple periods. They provide sufficient conditions under which such decentralized markets generate stable outcomes in equilibrium in the presence of market friction (namely, time discounting) and preference uncertainty. Like these models, our model concerns about the consequence of congestion arising from decentralized matching, but unlike Roth and Xing (1997), we have an analytical model that allows us to characterize both the equilibrium admission decisions and their welfare and fairness properties. In particular, the current framework develops a new theme of strategic targeting. Moreover, the explicit analysis of equilibria permits a clear comparison with the outcome that would arise from a centralized matching.

The college admissions problem has recently received attention in the economics literature. Chade and Smith (2006) study students’ application decision as a portfolio choice problem. Chade, Lewis, and Smith (2011) analyze colleges’ admission decisions together
with the students’ application decisions. In their model, students with heterogeneous abilities make application decisions subject to application costs, and colleges set admission standards based on noisy signals on students’ abilities. Avery and Levin (2010) and Lee (2009) study early admissions. Unlike our model, these models have no aggregate uncertainty with respect to students’ preferences, which means that the colleges in their model do not face any enrollment uncertainty. Hence, colleges do not employ strategic targeting; they instead use cutoff strategies.

Some aspects of our equilibrium are related to political lobbying behavior studied by Lizzeri and Persico (2001, 2005). Just as colleges target students in our model, politicians in these models target voters for distributing their favors. In their models, voters are homogeneous, and a voter votes for the candidate that offers her the largest favor. In our model, however, students have heterogeneous abilities and preferences. Thus, colleges’ admission decisions are more complicated—admission probabilities vary according to students’ scores.

Our model also shares some similarities with directed search models, such as Montgomery (1991) and Burdett, Shi, and Wright (2001). In these studies, each firm (seller) posts a wage (price), and each worker (buyer) decides which job to apply for. Firms have a fixed number of job openings and cannot hire more than the capacity, and workers can only apply to one firm. Workers’ inability to precisely coordinate their search decisions causes a “search friction,” so they randomize on application decisions. Just like the workers in these models, colleges in our model can be seen to engage in “directed searches” on students. The difference is that the colleges in our model offer admissions to many students, which raises a qualitatively novel problem.
1.2 Model

Our model is described as follows. There is a unit mass of students with score \( v \) distributed on \([0, 1]\) according to an absolutely continuous distribution \( G(\cdot) \). There are two colleges, \( A \) and \( B \), each with capacity \( \kappa < \frac{1}{2} \). (Section A.8 will extend the model to include more than two colleges, showing that our main results carry over to that extension.) Each college values a student with score \( v \) at \( v \) and faces a cost \( \lambda \geq 1 \) for each incremental enrollment exceeding the quota. Each student has a preference over the two colleges, which is private information. A state of nature, \( s \in [0, 1] \), determines the fraction of students who prefer \( A \) over \( B \): The state \( s \) is drawn from \([0, 1]\) according to the uniform distribution. In state \( s \), a fraction \( \mu(s) \in [0, 1] \) of students prefers \( A \) to \( B \), where \( \mu(\cdot) \) is strictly increasing and continuous in \( s \).\(^6\) While we shall consider a general environment with respect to \( \mu(\cdot) \), some result will consider a symmetric environment in which \( \mu(s) = 1 - \mu(1-s) \) for all \( s \in [0, 1] \). In a symmetric environment, the measure of students who prefer \( A \) over \( B \) is symmetric around \( s = \frac{1}{2} \).

The timing of the game is as follows. First, Nature draws the (aggregate uncertainty) state \( s \). Next, all students simultaneously apply to colleges. Each college observes the scores of only those students who apply to it. Next, colleges simultaneously decide which applicants to admit. Last, those students who have received at least one admission offer decide on which offer to accept.

We assume that there is no application cost for the students, so it is a weak dominant strategy for each student to apply to both colleges. Throughout this paper, we focus on a perfect Bayesian equilibrium in which students play the weak dominant strategy.\(^7\)

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\(^6\)There is no loss of generality to assume the uniform distribution, because for a distribution \( F(\cdot) \) of \( s \), we can simply relabel \( s \) and the popularity of a college over the other is captured by \( \mu(\cdot) \).

\(^7\)The strategy of applying to both colleges can be made a strictly dominant strategy if students have some uncertainty about their scores, which is realistic in case the scores are either not publicly observable
Colleges distribute admissions based on students’ scores. Let $\alpha : [0, 1] \to [0, 1]$ and $\beta : [0, 1] \to [0, 1]$ be college $A$ and $B$’s admission strategies, respectively, in terms of the probability of offering an admission to each type $v$.

For given $\alpha(\cdot)$ and $\beta(\cdot)$, let $\mathcal{V}_A := \{v \in [0, 1]|\alpha(v) > 0\}$ and $\mathcal{V}_B := \{v \in [0, 1]|\beta(v) > 0\}$ be the set of students to whom colleges $A$ and $B$, respectively, make an admission offer with positive probability. Let $\mathcal{V}_{AB} := \mathcal{V}_A \cap \mathcal{V}_B$. If $\mathcal{V}_{AB}$ has a positive measure in an equilibrium, this means that a positive measure of students has admissions from both colleges. We call such an equilibrium competitive. An equilibrium in which $\mathcal{V}_{AB}$ has zero measure is called non-competitive.

Consider a student with score $v$. The student will attend college $A$ if either she is admitted only by $A$, which happens with probability $\alpha(v)[1 - \beta(v)]$, or admitted by both colleges but prefers $A$ to $B$, which happens with probability $\mu(s)\alpha(v)\beta(v)$ in state $s$. Thus, the mass of students who attend $A$ in state $s$, given strategies $\alpha(\cdot)$ and $\beta(\cdot)$, is

$$m_A(s) := \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] dG(v).$$

Similarly, the mass of students who attend $B$ in state $s$ is

$$m_B(s) := \int_0^1 \beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)] dG(v).$$

Each college realizes the scores of enrolled students as its gross payoff and incurs cost $\lambda$ for each increment beyond its capacity. Thus, college $A$ and $B$’s ex ante payoffs are, respectively,

$$\pi_A := E_s\left[\int_0^1 v\alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] dG(v) - \lambda \max\{m_A(s) - \kappa, 0\}\right]$$

or depend on multiple dimensions of attributes, the weighting of which may be unknown to the students.
\[\pi_B := \mathbb{E}_s \left[ \int_0^1 v\beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)] dG(v) - \lambda \max \{m_B(s) - \kappa, 0\} \right].\]

One immediate observation is that each college’s payoff is concave in its own admission strategy,\(^8\) that is, \(\pi_A(\eta\alpha + (1 - \eta)\alpha') \geq \eta \pi_A(\alpha) + (1 - \eta)\pi_A(\alpha')\) for any feasible strategies \(\alpha\) and \(\alpha'\) and for any \(\eta \in [0, 1]\). Therefore, mixing over \(\alpha\)’s is unprofitable for college \(A\) (similarly \(\beta\)’s for college \(B\)). For this reason, any equilibrium is characterized by a pair \((\alpha, \beta)\). Of course, this does not mean that the equilibrium is in pure-strategies; the values of \(\alpha\) and/or \(\beta\) may be strictly interior, in which case the admission strategies would involve randomization.

In the following sections, we characterize different types of equilibria and establish their existence. We then provide welfare and fairness properties of equilibria.

### 1.3 Characterization of Equilibrium

We analyze colleges’ admission decisions in this section. To this end, we fix any equilibrium \((\alpha, \beta)\) and explore the properties the equilibrium satisfies. Later, we shall establish existence of the equilibria. We begin with the following observations, whose proofs are in Section A.1.

**Lemma 1.1.** In any equilibrium \((\alpha, \beta)\), the follows hold.

(i) \(m_A(0) \leq \kappa \leq m_A(1)\) and \(m_B(1) \leq \kappa \leq m_B(0)\).

(ii) \(\mathcal{V}_A \cup \mathcal{V}_B\) is a connected interval with \(\sup \{\mathcal{V}_A \cup \mathcal{V}_B\} = 1\) and \(\inf \{\mathcal{V}_A \cup \mathcal{V}_B\} > 0\).

(iii) If the equilibrium is competitive (i.e., \(\mathcal{V}_{AB}\) has a positive measure), then there exists a unique \((\hat{s}_A, \hat{s}_B) \in (0, 1)^2\) such that \(m_A(\hat{s}_A) = \kappa\) and \(m_B(\hat{s}_B) = \kappa\).

\(^8\)See Lemma A.2 in Section A.4 for the proof.
(iv) If the equilibrium is non-competitive (i.e., $\mathcal{V}_{AB}$ has zero measure), then $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$. Further, almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives an admission offer from exactly one college. That is, $\alpha(v) = 1$ for almost every $v \in \mathcal{V}_A$ and $\beta(v) = 1$ for almost every $v \in \mathcal{V}_B$.

Part (i) of the lemma states that in equilibrium, colleges cannot have strict over-enrollment and/or strict under-enrollment in all states. This is obvious since if there were over-enrollment in all states for a college, then since $\lambda \geq 1$, it will profitably deviate by rejecting some students with $v < 1$, and if there were under-enrollment in all states, a college will likewise profitably deviate by accepting more students. Part (ii) suggests that if a student with score $v$ is admitted by either college, then all students with scores higher than such $v$ must be admitted by some college at least with positive probability, and there is a positive mass of students in the low tail who are never admitted by either college. Part (iii) suggests that in a competitive equilibrium, the colleges will suffer from under-enrollment in some states and over-enrollment in other states. This is intuitive since given (aggregately) uncertain preferences on the part of students, the presence of students who receive admissions from both colleges presents non-trivial enrollment uncertainty. Each college will deal with uncertainty by optimally trading off the cost of over-enrollment with the loss from under-enrollment, thus entailing both types of mistakes depending on the states. Part (iv) states that in a non-competitive equilibrium colleges avoid the over- and under-enrollment problems, and almost every top $2\kappa$ students receive admissions from only one college. This is, again, intuitive since the colleges in this case face no enrollment uncertainty, so they will fill their capacities exactly in all states with students who have the top $2\kappa$ scores.

In what follows, we shall focus on competitive equilibria. There are several reasons for this. It will be seen that competitive equilibria always exist (see Theorem 1.3). By contrast, non-competitive equilibria can be ruled out if either $\lambda$ is not too large or $\kappa$ is not too small.
(see Section A.2). Finally, even if a noncompetitive equilibrium exists, the characterization provided in Lemma 1.1-(iv) is sufficient for our welfare and fairness statements, as will be seen later.

Therefore, fix any competitive equilibrium \((\alpha, \beta)\). For ease of notation, let \(\mu_+ (s) := \mathbb{E}[\mu(\tilde{s}) | \tilde{s} \geq s]\) and \(\mu_- (s) := \mathbb{E}[\mu(\tilde{s}) | \tilde{s} \leq s]\). It is convenient to rewrite \(A\)'s payoff at the equilibrium as follows:

\[
\pi_A = \int_0^1 v\alpha(v)[1 - \beta(v) + \overline{\mu}\beta(v)]dG(v) - \lambda \mathbb{E}_A[m_A(s) - \kappa | s > \hat{s}_A](1 - \hat{s}_A)
= \int_0^1 \alpha(v)H_\alpha(v, \beta(v))dG(v) + \lambda (1 - \hat{s}_A)\kappa,
\]

where \(\overline{\mu} := \mathbb{E}[\mu(s)]\), \(\hat{s}_A \in (0, 1)\) is such that \(m_A(\hat{s}_A) = \kappa\) (as defined in Lemma 1.1-(iii)), and

\[
H_\alpha(v, \beta(v)) := v[1 - \beta(v) + \overline{\mu}\beta(v)] - \lambda(1 - \hat{s}_A)[1 - \beta(v) + \mu_+(\hat{s}_A)\beta(v)]
= (1 - \beta(v))[v - \lambda(1 - \hat{s}_A)] + \beta(v)\overline{\mu}\left[v - \lambda(1 - \hat{s}_A)\frac{\mu_+(\hat{s}_A)}{\overline{\mu}}\right]
\]

is \(A\)'s marginal payoff from admitting a student with \(v\) for given \(\beta(\cdot)\) and \(\hat{s}_A\) in equilibrium. (We shall suppress its dependence on \(\hat{s}_A\) unless it is important.) This captures \(A\)'s local incentive, that is, what \(A\) gains by admitting \(v\), holding fixed its opponent’s decision and its own decisions for the rest of the students at \(\alpha(\cdot)\).

Notice that the first and the second square brackets in (1.1) are the marginal payoffs of college \(A\) from admitting type-\(v\) student when she does not receive admission from \(B\) and when she does, respectively. Recall that the college incurs capacity cost only when there is over-enrollment. Suppose first that a type-\(v\) student does not receive a competing offer from \(B\). Then, she accepts \(A\)'s admission for sure. Hence, over-enrollment occurs with probability \((1 - \hat{s}_A)\), so the marginal cost of admitting the student is \(\lambda(1 - \hat{s}_A)\), which explains the second term of the first square bracket in (1.1). Suppose next that the student receives
a competing offer form B. Then, she accepts A’s offer only when she prefers A to B. Hence, conditional on acceptance, the over-enrollment arises with probability $1 - \hat{s}_A$ (since the likelihood of over-enrollment (conditional on acceptance) is higher when the student has a competing offer from B than she does not: $(1 - \hat{s}_A) > (1 - \hat{s}_A)$, where the strict inequality follows since $\hat{s}_A \in (0, 1)$ (by Lemma 1.1-(iii) and since $\mu_+ (\hat{s}_A) > \overline{\mu}$ for $\hat{s}_A > 0$). That is, when the student receives an offer from B but accepts A’s offer, the state is more likely to be high (since she is more likely to accept A’s offer when $\mu(s)$ is high than when it is not) comparing to the case that she does not receive a competing offer.

This observation implies that $H_\alpha(v, \beta(v))$ partitions the students’ type space into three intervals, as depicted in Figure 1.1. For a student with $v > \overline{v}_A := \lambda(1 - \hat{s}_A)$, we have $H_\alpha(v, 1) > 0$, so college A admits such a student even if college B admits the student in equilibrium. For a student with $v < \underline{v}_A := \lambda(1 - \hat{s}_A)$, we have $H_\alpha(v, 0) < 0$, so college A has no incentive to admit such a student even if college B does not admits the student. For a student with $v \in (\underline{v}_A, \overline{v}_A)$, we have $H_\alpha(v, 0) > 0$ but $H_\alpha(v, 1) < 0$. Hence, college A has an incentive to admit such a student if B does not admit the student, but not if college B admits that student. Intuitively, each college considers the enrollment uncertainty worth taking on only when the student has a sufficiently high score, and for a student with a lower score (but above the lower cutoff), the college finds admission is worthwhile only when it is
assured of facing no competition and thus no uncertainty in enrollment. As will be seen, the presence of this intermediate range of scores leads to non-cutoff equilibria.

The characterization of $B$’s admission strategy is completely symmetric. As before, $B$’s payoff from admitting a type-$v$ student is expressed as:

$$\pi_B = \int_0^1 \beta(v) H_\beta(v, \alpha(v)) dG(v) + \lambda \hat{s}_B \kappa,$$

where

$$H_\beta(v, \alpha(v)) := v[1 - \alpha(v) + (1 - \mu)\alpha(v)] - \lambda \hat{s}_B[1 - \alpha(v) + (1 - \mu - (\hat{s}_B))\alpha(v)]$$

is $B$’s marginal payoff from admitting a student with score $v$, holding fixed $A$’s admission decision and its own decisions for the remaining students at $\beta(\cdot)$. Just as before, $H_\beta(v, \alpha(v))$ partitions the students’ type space into three intervals separated by two threshold values $\underline{v}_B$ and $\overline{v}_B$, where $\underline{v}_B := \lambda \hat{s}_B < \overline{v}_B := \lambda \hat{s}_B \frac{1 - \mu - (\hat{s}_B)}{1 - \mu}$, such that $B$ admits all students with $v > \overline{v}_B$ and rejects all students with $v < \underline{v}_B$ for sure, and accepts students with $v \in (\underline{v}_B, \overline{v}_B)$ if they are not admitted by $A$ but rejects them when they are admitted by $A$.

Combining the two colleges’ admission decisions leads to the following characterization of equilibria.

**Theorem 1.1.** In any competitive equilibrium, there exist $\underline{v}_i < \overline{v}_i$, $i = A, B$, such that college $i$ admits students with $v > \overline{v}_i$ and students with $v \in [\underline{v}_i, \underline{v}_j]$ and rejects students with $v < \underline{v}_i$ and students $v \in [\overline{v}_j, \overline{v}_i]$, where $j \neq i$. Students with $v \in [\max\{\underline{v}_A, \underline{v}_B\}, \min\{\overline{v}_A, \overline{v}_B\}]$ are admitted by at least one college with positive probability.

Theorem 1.1 describes the structure of any competitive equilibrium. Figure 1.2 depicts a typical pure-strategy equilibrium. Here, top students with $v > \overline{v}_A = \max\{\overline{v}_A, \overline{v}_B\}$ receive offers from both colleges, because their scores are above the high cutoffs for both colleges.
The next tier students with $v \in (\bar{v}_B, \bar{v}_A)$ receive offers only from $B$, since $A$ finds them admission-worthy only if $B$ does not admit them, but in this case, $B$ is interested in admitting them no matter what $A$ does. Each of the students in the intermediate range of scores, i.e., $[\underline{v}_A, \bar{v}_B]$, receives an admission offer from only one college. Obviously, how the two colleges coordinate exactly on these students are indeterminate, and the figure depicts one possible coordination. The students with scores $v \in [\underline{v}_B, \underline{v}_A]$ receive offers only from $B$, since it is the only college that finds them admission-worthy given that they are not admitted by $A$. Finally, the students at the bottom below $\underline{v}_B = \min\{\underline{v}_A, \underline{v}_B\}$ do not receive any offers.

Clearly, strategic targeting occurs in this equilibrium: A college does not admit good students because they are sought after by the other college, and it admits less attractive students because they are not sought after by the other college. This feature stands in stark contrast with the cutoff strategy equilibrium found by the existing literature (see Chade, Lewis, and Smith, 2011).
As noted, there may be many ways for colleges to coordinate their admissions for students with \( v \in [\overline{v}, \tilde{v}] \), where \( \overline{v} := \max \{v_A, v_B\} \) and \( \tilde{v} := \min \{v_A, v_B\} \). The range of different pure-strategy equilibria can be summarized by two extreme types of equilibria. We call a competitive equilibrium an \( A \)-priority equilibrium if \( \alpha(v) = 1 \) for all \( v \in [\overline{v}, \tilde{v}] \), and a \( B \)-priority equilibrium if \( \beta(v) = 1 \) for all \( v \in [\overline{v}, \tilde{v}] \). In words, in an \( i \)-priority equilibrium, the coordination is tilted in favor of college \( i \). Clearly, between these two equilibria, one can construct (infinitely) many equilibria.

In practice, it is implausible for colleges to achieve the kind of precise coordination described in the pure-strategy equilibria. It seems much more plausible for colleges to randomize over students with the intermediate range of scores \( v \in [\overline{v}, \tilde{v}] \).\(^9\) A typical mixed-strategy equilibrium is depicted in Figure 1.3.

\(^9\)It is important to note that the thresholds are not necessarily the same as in the pure-strategies, since different equilibria involve different cutoff states, \((\hat{s}_A, \hat{s}_B)\), which affect the marginal payoff functions \( H_\alpha \) and \( H_\beta \).
Notice that the admission strategies outside the intermediate range is similar to that in the above pure-strategy equilibrium, as this is completely pinned down by Theorem 1.1. For the intermediate range of scores, interior-valued admissions strategies can be structured so as to keep each college indifferent, as follows. For each \( v \in [\underline{v}, \overline{v}] \), let \( \alpha(v) = \alpha_0(v) \) and \( \beta(v) = \beta_0(v) \), where

\[
H_{\alpha}(v, \beta_0(v)) = 0 \quad \text{and} \quad H_{\beta}(v, \alpha_0(v)) = 0,
\]

or equivalently,

\[
\alpha_0(v) := \frac{v - \lambda \hat{s}_B}{\overline{v} - \lambda \hat{s}_B \mu_-(\hat{s}_B)} \quad \text{(1.2)}
\]

and

\[
\beta_0(v) := \frac{v - \lambda(1 - \hat{s}_A)}{v(1 - \overline{v}) - \lambda(1 - \hat{s}_A)\mu_+(\hat{s}_A)}. \quad \text{(1.3)}
\]

One can easily check that \( \alpha_0(v), \beta_0(v) \in [0, 1] \) for \( v \in [\underline{v}, \overline{v}] \). If college \( B \) adopts \( \beta_0(v) \) for a student \( v \), then college \( A \)'s marginal gain from admitting that student is zero, so it is indifferent about admitting that student. Here, it is college \( A \)'s best response to randomize according to \( \alpha_0(\cdot) \). Since \( H_{\beta}(v, \alpha_0(v)) = 0 \), college \( B \) is indifferent, making its randomization a best response. Observe that both \( \alpha_0(\cdot) \) and \( \beta_0(\cdot) \) are increasing in \( v \), which means that colleges admit students with higher scores with higher probabilities. This is intuitive: A higher score student is more valuable all else equal, so a high probability of admission for a high score student is necessary to keep the opponent college indifferent. It is also interesting to observe discrete jumps in this figure — \( \alpha_0(\underline{v}_A) > 0 \) and \( \beta_0(\overline{v}_B) < 1 \). The former follows from the fact that \( \underline{v}_A > \underline{v}_B \), which implies \( H_{\beta}(\underline{v}_A, 0) > 0 \), and the latter follows from \( \overline{v}_A > \overline{v}_B \), which implies \( H_{\alpha}(\overline{v}_B, 1) < 0 \).

There could be many ways for colleges to play mixed-strategies: For instance, colleges could coordinate to use a pure-strategy for some students, say \([\hat{v}, \overline{v}]\) for some \( \hat{v} \in (\underline{v}, \overline{v}) \), and use mixed-strategies for \( v \in [\underline{v}, \hat{v}] \). Consistent with our selection, we focus on the maximally
mixed equilibrium (MME, in short) in which both colleges play mixed-strategies \((\alpha_0, \beta_0)\) for students with \(v \in [\underline{v}, \bar{v}]\) and according to Theorem 1.1 for outside that range.

The characterization of equilibria has so far rested on the necessary conditions for competitive equilibria, particularly the “local” incentive compatibility with respect to each type of students. Whether the preceding characterizations based on MME and \(i\)-priority equilibria admit a well-defined strategy profile and, if so, whether they constitute competitive equilibria are not clear. We shall address these issues in the next subsection.

Before proceeding, though, it is important to recognize that the randomization by colleges results from their attempts to avoid competition for students in the intermediate range of scores. In this sense, as long as a competitive equilibrium admits the intermediate region, i.e., if \(\underline{v} < \bar{v}\), one can say that equilibrium involves strategic targeting, regardless of whether the colleges play a mixed-strategy or a pure-strategy. Formally, we say an competitive equilibrium exhibits strategic targeting if \(v < \bar{v}\).
When do competitive equilibria exhibit strategic targeting and when do not? Certainly, Theorem 1.1 does not preclude a competitive equilibrium in which $\bar{v} = \min \{\overline{v}_A, \overline{v}_B\} < \max \{\underline{v}_A, \underline{v}_B\} = \underline{v}$. Figure 1.4 depicts such a possibility with $\underline{v}_B < \overline{v}_B < \underline{v}_A < \overline{v}_A$. As before, college $i$ admits students with $v > \underline{v}_i$ and rejects those with $v < \underline{v}_i$. Observe that college $A$ does not admit any student with $v \in [\underline{v}_A, \underline{v}_A]$, since college $B$ admits them for sure (because $\underline{v}_B < \underline{v}_A$). Even though colleges have targeting incentives in this example, the resulting equilibrium is indistinguishable from the cutoff equilibria featured in the existing research.

A natural question is when such an equilibrium can be ruled out. The exact condition for its existence appears difficult to find, but we show next that the symmetric environment is sufficient to guarantee strategic targeting behavior.

**Theorem 1.2.** If the environment is symmetric (i.e., $\mu(s) = 1 - \mu(1 - s)$ for all $s$), then every competitive equilibrium exhibits strategic targeting.

**Proof.** See Section A.3.

### 1.3.1 Existence of MME and $i$-Priority Equilibrium

We now show that there exists an equilibrium in which $\alpha(\cdot)$ and $\beta(\cdot)$ involve maximal mixing, or $A$- or $B$-priority.\(^\text{10}\)

**Theorem 1.3.** There exists a competitive equilibrium with maximal mixing, or $A$- or $B$-priority.

**Sketch of Proof.** The proof involves three steps. The first step shows the existence of admission strategies that provide optimal local incentives for each other college. The second

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\(^{10}\)Note that a general equilibrium existence follows from the Glicksberg-Fan theorem, since each college’s strategy space is compact and convex, and each college’s payoff function is concave in its own strategy. That is, if one does not insist on the particular structure of behavior we impose on MME (or $A$- or $B$-priority), it is easy to show the existence of equilibrium admission strategies.
step shows that \( V_{AB} \) has a positive measure in the identified strategy profile. The last step shows that the identified strategies are indeed mutual (global) best responses.

**Step 1:** For the first step, we prove that there exists \( (\alpha, \beta) : [0, 1]^2 \to [0, 1]^2 \) such that for each \( v \in [0, 1], \)

\[
\alpha(v; \hat{s}) = \begin{cases} 
1 & \text{if } H_\alpha(v, 1; \hat{s}) > 0 \\
0 & \text{if } H_\alpha(v, 1; \hat{s}) < 0, \ H_\beta(v, 1; \hat{s}) > 0 \\
\alpha_0(v; \hat{s}) & \text{if } H_\alpha(v, 1; \hat{s}) < 0 < H_\alpha(v, 0; \hat{s}), \ H_\beta(v, 1; \hat{s}) < 0 < H_\beta(v, 0; \hat{s}) \\
1 & \text{if } H_\alpha(v, 0; \hat{s}) > 0, \ H_\beta(v, 0; \hat{s}) < 0 \\
0 & \text{if } H_\alpha(v, 0; \hat{s}) < 0 
\end{cases}
\]

(1.4)

and

\[
\beta(v; \hat{s}) = \begin{cases} 
1 & \text{if } H_\beta(v, 1; \hat{s}) > 0 \\
0 & \text{if } H_\beta(v, 1; \hat{s}) < 0, \ H_\alpha(v, 1) > 0 \\
\beta_0(v; \hat{s}) & \text{if } H_\beta(v, 1; \hat{s}) < 0 < H_\beta(v, 0; \hat{s}), \ H_\alpha(v, 1; \hat{s}) < 0 < H_\alpha(v, 0; \hat{s}) \\
1 & \text{if } H_\beta(v, 0; \hat{s}) > 0, \ H_\alpha(v, 0; \hat{s}) < 0 \\
0 & \text{if } H_\beta(v, 0; \hat{s}) < 0 
\end{cases}
\]

(1.5)

where \( \alpha_0(\cdot) \) satisfies \( H_\beta(v, \alpha_0(v)) = 0 \) for \( v \in [\underline{v}, \tilde{v}] \), as given by (1.2), and \( \beta_0(\cdot) \) satisfies \( H_\alpha(v, \beta_0(v)) = 0 \) for \( v \in [\underline{v}, \tilde{v}] \), as given by (1.3), and \( \hat{s} = (\hat{s}_A, \hat{s}_B) \) satisfies

\[ \hat{s}_A = \inf \{ s \in [0, 1] | m_A(s) - \kappa > 0 \}, \]
if the set in the RHS is nonempty, or else \( \hat{s}_A \equiv 1 \), and

\[
\hat{s}_B = \sup \{ s \in [0, 1] | m_B(s) - \kappa > 0 \},
\]

if the set in the RHS is nonempty, or else \( \hat{s}_B \equiv 0 \).

In other words, the strategy profile is required to satisfy the conditions of MME based on the local incentives \( H_\alpha \) and \( H_\beta \). One can also easily structure the strategy profile to satisfy the requirements of an \( A \)-priority equilibrium by replacing \( \alpha_0(\cdot) \) and \( \beta_0(\cdot) \) with 1 and 0, respectively, and of a \( B \)-priority equilibrium by replacing them with 0 and 1, respectively.

To prove the existence of such a strategy profile, we construct a mapping \( T : S \to S \), where \( S := [0, 1]^2 \), and show that its fixed point exists, and given its fixed point \((\hat{s}_A^*, \hat{s}_B^*)\), the profile \((\alpha(\cdot; \hat{s}_A^*, \hat{s}_B^*), \beta(\cdot; \hat{s}_A^*, \hat{s}_B^*))\) satisfies (1.4) and (1.5).

To begin, fix any \( \hat{s} = (\hat{s}_A, \hat{s}_B) \in S \), and consider the resulting profile \((\alpha(\cdot; \hat{s}), \beta(\cdot; \hat{s}))\). This strategy profile in turn induces the mass of students enrolling in colleges \( A \) and \( B \). Specifically, for college \( A \), we obtain

\[
m_A(s; \hat{s}) = \int_0^1 \alpha(v; \hat{s})[1 - \beta(v; \hat{s}) + \mu(s)\beta(v; \hat{s})]dG(v)
\]

and similarly for college \( B \), we obtain

\[
m_B(s; \hat{s}) = \int_0^1 \beta(v; \hat{s})[1 - \alpha(v; \hat{s}) + (1 - \mu(s))\alpha(v; \hat{s})]dG(v).
\]

Observe that \( m_A(\cdot; \hat{s}) \) and \( m_B(\cdot; \hat{s}) \) in turn yield a new profile of cutoff states:

\[
\tilde{s}_A = \inf \{ s \in [0, 1] | m_A(s; \hat{s}) - \kappa > 0 \},
\]
if the set in the RHS is nonempty, or else $\tilde{s}_A \equiv 1$, and similarly $\tilde{s}_B$ for college $B$.

$$\tilde{s}_B = \sup \{ s \in [0,1] | m_B(s; \hat{s}) - \kappa > 0 \},$$

if the set in the RHS is nonempty, or else $\tilde{s}_B \equiv 0$.

We then define $T$ such that $T(\hat{s}) = \tilde{s}$. We show in Section A.4 that $T$ is a continuous map. Therefore, it has a fixed point by the Brouwer’s fixed point theorem. From the construction of $T$, it is immediate that, given the fixed point $\hat{s}^*$, $\tilde{s} = \hat{s}^*$, the profile $(\alpha(\cdot; \hat{s}^*), \beta(\cdot; \hat{s}^*))$ satisfies (1.4) and (1.5).

**Step 2:** For the second step, we show that $\mathcal{V}_{AB}$ has a positive measure in the strategy profile identified in Step 1. To prove, suppose to the contrary that $\mathcal{V}_{AB}$ has measure zero. Then, $\hat{s}^*_B = 0$ and $\hat{s}^*_A = 1$. But in that case, $H_\alpha(v,1) > 0$ and $H_\beta(v,1) > 0$ for all $v$. Hence, $\overline{v}_A = \overline{v}_B = 0$. Thus, we cannot have a non-competitive equilibrium.

**Step 3:** Observe that the strategy profile $(\alpha, \beta)$ identified in Step 1 forms best responses but based on the local incentives of the colleges — namely, $(\alpha, \beta)$ entails no incentive for each college to unilaterally deviate in its admission decision on each student, holding constant its own admission strategies with respect to the other students. Hence, it does not rule out profitable deviation in its admission decisions on a mass of students. The third step shows that such deviation is not profitable; that is, the identified strategies are mutual (global) best responses.

To this end, let $\tilde{\alpha}(v) \in [0,1]$ be an arbitrary strategy for $v \in [0,1]$, and consider a variation of $\alpha(\cdot)$ such that for any $t \in [0,1]$,

$$\alpha(v; t) := t\tilde{\alpha}(v) + (1 - t)\alpha(v).$$
Define college A’s payoff function in terms of $\alpha(v; t)$,

\[
V(t) := \int_0^1 v\alpha(v; t)[1-\beta(v)+\bar{\mu}\beta(v)]dG(v)-\lambda \int_{\hat{s}_A(t)}^1 \left[ \int_0^1 \alpha(v; t)[1-\beta(v)+\mu(s)\beta(v)]dG(v)-\kappa \right]ds,
\]

where $\hat{s}_A(t)$ is the threshold state given by $\alpha(v; t)$.

Observe that $\pi_A(\tilde{\alpha}) = V(1)$ and $\pi_A(\alpha) = V(0)$. Therefore, the proof is completed by showing that $V(1) \leq V(0)$. Because $\tilde{\alpha}(\cdot)$ is arbitrary, this will prove that $\alpha(\cdot)$ is a best response for a given $\beta(\cdot)$.

To see this, observe first that $V(\cdot)$ is concave in $t$ since $\pi_A$ is concave in $\alpha$ (which follows from the enrollment uncertainty) and $\alpha(v; t)$ is linear in $t$ (see Lemma A.3). Therefore, we have

\[
\pi_A(\tilde{\alpha}; \beta) = V(1) \leq V(0) + V'(0).
\]

Next, using the local condition, (1.4), one can show that (see Lemma A.4)

\[
V'(0) = \int_0^1 [\tilde{\alpha}(v) - \alpha(v)]H_{\alpha}(v, \beta(v))dG(v) \leq 0,
\]

where the inequality holds since if $H_{\alpha}(v, \beta(v)) > 0$ for some $v$, then $\alpha(v) = 1$ and $\tilde{\alpha}(v) \leq 1$ for such $v$; if $H_{\alpha}(v, \beta(v)) < 0$ for some $v$, then $\alpha(v) = 0$ and $\tilde{\alpha}(v) \geq 0$ for such $v$; and $H_{\alpha}(v, \beta(v)) = 0$ otherwise.

Combining (1.6) and (1.7), we conclude that

\[
\pi_A(\tilde{\alpha}) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_A(\alpha),
\]

and this completes the proof.
1.3.2 Properties of Equilibria

We have seen that the equilibrium outcome involves strategic targeting. We now consider the implications of the equilibria in welfare and fairness.

Let us first define assignment and outcome. For a fixed \( s \), an assignment is a mapping from \( V \times \{A, B\} \) to \( \{A, B\} \cup \{\emptyset\} \), where \( V = [0, 1] \) is the students’ score space. That is, an assignment is an allocation of students to colleges where each student has a score and preference over \( A \) and \( B \) and no student is assigned to more than one college. An outcome is a mapping from a state to an assignment, i.e., the realized allocation in state \( s \).

We say that a student has a justified envy if she prefers a college to the one he enrolls in, even though the former enrolls a student with a lower score. An outcome is said to be fair if for almost every state, the assignment it selects has no justified envy for almost all students. Next, an outcome is Pareto efficient if for almost every state, the assignment it selects is not Pareto dominated, i.e., there is no other assignment in which both colleges and all students are weakly better off and either there is a college that is strictly better off or there is a positive measure of students who are strictly better off, relative to the initial assignment. One may be interested in students’ welfare taking colleges as exogenous resources of the society. We say that an outcome is student efficient if for almost every state, there is no other assignment in which all students are weakly better off and a positive measure of students are strictly better off relative to the initial assignment that the outcome selects. Finally, we may consider college’s welfare only. An outcome is said to be college efficient if for almost every state, no other assignment can make both colleges weakly better off and at least one college strictly better off relative to the assignment that the outcome selects.

The next theorem states properties of equilibria that arise in decentralized matching.

**Theorem 1.4.** (i) Any non-competitive equilibrium is unfair, student inefficient, but college
efficient.

(ii) Any non-competitive equilibrium is Pareto inefficient unless almost every student admitted by one college has higher score than those admitted by the other college.

(iii) Any competitive equilibrium is college inefficient and Pareto inefficient.

(iv) Any MME with $v < \tilde{v}$ is unfair and student inefficient.

(v) Any competitive equilibrium with $\tilde{v} < v$ is fair and student efficient.

Proof. Consider non-competitive equilibrium first.

Proof of (i). Consider any non-competitive equilibrium. For each state $s$ except $\mu(s) = 0$ or 1, the equilibrium must admit a positive measure of students who prefer $A$ but are assigned to $B$, and a positive measure of students who are assigned to $A$ but have scores lower than those of the first group of students; that is, justified envy arises. Since justified envy arises for a positive measure of students for almost every state,\footnote{since $\mu(\cdot)$ is strictly increasing and continuous in $s$, $\mu(s) \in (0,1)$ for almost every state.} the outcome is unfair. Also, for almost every state, there must be a positive measure of students assigned to $A$ but prefer $B$ and a positive measure of students assigned to $B$ but prefer $A$. Thus, the outcome is student inefficient. Next, the equilibrium is college efficient. To see this, observe first that in any non-competitive equilibrium, almost all top $2\kappa$ students are assigned to either college. Suppose now that for a given state, there is another assignment that makes both colleges weakly better off and at least one college strictly better off. Then, it must also admit almost all top $2\kappa$ students, or else at least one college is strictly worse off. Therefore, it is a reallocation of the initial assignment, hence if one college is strictly better off, then the other college must be strictly worse off. Thus, we reach a contraction. □

Proof of (ii). Suppose that almost all top $\kappa$ students are assigned to one college, and the next top $\kappa$ students are assigned to the other college. Then, any change of assignments by
positive measure of students will leave the former college strictly worse off, hence it is Pareto efficient.

Suppose it is not the case in a non-competitive equilibrium. Note that for a fixed \( s \), there are some \( V'_i, V''_i \subset V_i \) and \( V'_j \subset V_j, i \neq j \), all with positive measures, such that \( v' < \hat{v} < v'' \) whenever \( v' \in V'_i, v'' \in V''_i \) and \( \hat{v} \in V'_j \). Let \( i = A \) and \( j = B \) without loss of generality. We can choose \( V'_A, V''_A \) and \( V'_B \) that satisfy

\[
\frac{\int_{V'_A \cup V''_A} v \, dG(v)}{\int_{V'_A \cup V''_A} 1 \, dG(v)} = \frac{\int_{V'_B} v \, dG(v)}{\int_{V'_B} 1 \, dG(v)}
\]  

(1.8)

and

\[
(1 - \mu(s)) \int_{V'_A \cup V''_A} 1 \, dG(v) = \mu(s) \int_{V'_B} 1 \, dG(v). 
\]  

(1.9)

(If either (1.8) or (1.9) is violated, we can adjust \( V'_A, V''_A \) and/or \( V'_B \) by adding or subtracting a positive mass of students.) Note that the LHS (resp. RHS) of (1.9) is the measure of students who prefer \( B \) (resp. \( A \)) in \( V'_A \cup V''_A \) (resp. \( V'_B \)). From (1.8), we have

\[
\frac{\int_{V'_A \cup V''_A} v \, dG(v)}{(1 - \mu(s)) \int_{V'_A \cup V''_A} 1 \, dG(v)} = \frac{\int_{V'_B} v \, dG(v)}{(1 - \mu(s)) \int_{V'_B} 1 \, dG(v)}
\]

\[
\Leftrightarrow \frac{\int_{V'_A \cup V''_A} v \, dG(v)}{\mu(s) \int_{V'_B} 1 \, dG(v)} = \frac{\int_{V'_B} v \, dG(v)}{(1 - \mu(s)) \int_{V'_B} 1 \, dG(v)}
\]

\[
\Leftrightarrow (1 - \mu(s)) \int_{V'_A \cup V''_A} v \, dG(v) = \mu(s) \int_{V'_B} v \, dG(v),
\]

where the first equivalence follows from (1.9). The last equivalence shows that the average value of students who prefer \( B \) in \( V'_A \cup V''_A \) is the same as that of students who prefer \( A \) in \( V'_B \). Thus, in state \( s \), a fraction \( 1 - \mu(s) \) of students in \( V'_A \cup V''_A \) who prefer \( B \) to \( A \) can be swapped with a fraction of \( \mu(s) \) of students in \( V'_B \) who prefer \( A \) to \( B \). This reassignment leaves both colleges the same in welfare and makes all students weakly better off and some positive measure of students strictly better off. Since this argument holds for all \( s \) except
\( \mu(s) = 0 \text{ or } 1 \), the outcome is Pareto inefficient. \( \square \)

Consider now competitive equilibrium.

**Proof of (iii).** Recall that there are cutoff states \((\hat{s}_A, \hat{s}_B)\) such that colleges have a mass of unfilled seats in a positive measure of states, \([0, \hat{s}_A)\) for A and \((\hat{s}_B, 1]\) for B, despite the fact that there are unmatched and acceptable students (\(\inf \{V_A \cup V_B\} > 0\) in Lemma 1.1-(ii)). Assigning those unmatched students to a college with excess capacity improves the social welfare. Thus, it is college inefficient and Pareto inefficient. \( \square \)

**Proof of (iv).** Consider a MME with \(v < \tilde{v}\). Fix a state \(s\) such that \(\mu(s) \neq 0, 1\). For those students in \([v, \tilde{v}]\), there is a positive measure of students who are assigned to a college, say B, but prefer A, and their scores are higher than a positive measure of students who are assigned to A, even though both colleges prefer the high-score students. Moreover, students in \([v, \tilde{v}]\) get zero admissions with positive probabilities even when their scores are high. Thus, it entails justified envy for a positive measure of states for almost every state. Student inefficiency follows from that for almost every state, there are two groups of positive measure of students in \([v, \tilde{v}]\), one preferring A but assigned to B and the other preferring B but assigned to A. \( \square \)

**Proof of (v).** Consider a competitive equilibrium with \(\tilde{v} < v\). Let \(v_B < v_A\), as depicted in Figure 1.4, without loss of generality, so college B alone admits students with scores in \([v_B, v_A]\) and those with \(v > v_A\) are admitted by both colleges.

Only the students who are not admitted by either college or admitted only by college B may have envies. However, the students whom they envy have higher scores. So, no justified envy arises in every state \(s\), making the outcome fair. For student efficiency, observe that those students who are admitted by both colleges choose their preferred college. Hence, they cannot be better off from swapping their assignments with others. Next, those who are
admitted only by $B$ may prefer $A$, but there are no students assigned to $A$ who prefer $B$ to $A$. Thus, the outcome is student efficient. □

1.4 Evaluation Distortion in the Presence of Multidimensional Performance Measures

The common performance measure in the baseline model is special. In practice, colleges value multiple dimensions of students’ qualities and performance. Often colleges evaluate students based not just on academic performance measure but also on other non-academic measure. Some performance dimensions are more common to colleges than others. For instance, the SAT scores or grade points average of students are commonly observed and interpreted virtually the same by colleges. Non-academic performance measures are often rich and not summarized by objectively agreed indices; and colleges may consider different aspects and may use their information differently. For instance, some colleges may pay attention to students’ community service or leadership activities. Others may pay more attention to extra curricular activities such as musical or athletic talents. So colleges’ evaluation of students on these dimensions are likely to be less correlated. We show that strategic targeting entails evaluation distortion placing excessive weight on non-common performances.

To this end, we extend our model as follows. A student’s type is described as a triple $(v, e, e') \in V \times E \times E' \equiv [0, 1]^3$, where $v$ is distributed according to $G(\cdot)$ with density $g(\cdot)$, and $e$ and $e'$ are conditionally independent on $v$ and are distributed according to $X(\cdot|v)$ and $Y(\cdot|v)$, respectively, which admit densities $x(\cdot|v)$ and $y(\cdot|v)$. We also assume that $X_v(e|v) < 0$ and $Y_v(e'|v) < 0$ for all $e, e' \in [0, 1]$. That is, a student with higher $v$ has a higher probability to have higher $e$ and $e'$. We also assume full support of $G$, $X$, $Y$. College $A$ only values $(v, e)$ and college $B$ only cares about $(v, e')$. Specifically, we assume college $A$ derives payoff $U(v, e)$ from
matriculating student with type \((v, e, e')\), where \(U\) is strictly increasing and differentiable in both arguments. Likewise, college \(B\) realizes payoff \(V(v, e')\) from matriculating the same type of student, where \(V\) is strictly increasing and differentiable in \((v, e')\).

One interpretation is that \(v\) is an academic performance measure observed commonly to both colleges, and \(e\) and \(e'\) correspond to different dimensions of extra curricular activities that the two colleges focus on. Alternatively, \(v\) is a student’s test scores of the nationwide exam, and \(e\) and \(e'\) may represent a student’s performance on college-specific tests or interviews.\(^{12}\)

College \(A\)’s strategy is now described as a mapping \(\alpha : \mathcal{V} \times \mathcal{E} \rightarrow [0, 1]\) with the interpretation that the college admits a student with type \((v, e)\) with probability \(\alpha(v, e)\). Likewise, college \(B\)’s strategy is described by a mapping \(\beta : \mathcal{V} \times \mathcal{E}' \rightarrow [0, 1]\). The enrollment uncertainty facing college \(A\) with regard to a student type \((v, e)\) depends on whether that student receives admission from college \(B\). But since \(e'\) is conditionally uncorrelated with \(e\), the probability that student type \((v, e)\) receives admission from \(B\) is \(\overline{\beta}(v) := \mathbb{E}[\beta(v, e')|v]\). Likewise \(\overline{\pi}(v) := \mathbb{E}e[\alpha(v, e)|v]\) is relevant for college \(B\) to assess its enrollment uncertainty.

For given \(\overline{\pi}(\cdot)\) and \(\overline{\beta}(\cdot)\), the mass of students enrolling to college \(A\) in state \(s\) is

\[
m_A(s) = \int_0^1 \int_0^1 \alpha(v, e)[1 - \overline{\beta}(v) + \mu(s)\overline{\beta}(v)]dX(e|v)dG(v).
\]

Hence, college \(A\)’s payoff is described as follow:

\[
\pi_A = \int_0^1 \int_0^1 U(v, e)\alpha(v, e)[1 - \overline{\beta}(v) + \mu(s)\overline{\beta}(v)]dX(e|v)dG(v) - \lambda \mathbb{E}_s[m_A(s) - \kappa|s > \hat{s}_A](1 - \hat{s}_A) \\
= \int_0^1 \int_0^1 H_\alpha(v, e, \overline{\beta}(v))dX(e|v)dG(v) + \lambda (1 - \hat{s}_A)\kappa,
\]

\(^{12}\)In Korea, for instance, students take a nationwide exam and each college has its own essay tests and/or oral interviews. In Japan, there is a nationwide exam called National Center Test (NCT). Public universities use both NCT and their own exams, and private universities often use their own exam only.
where $\bar{\mu} := \mathbb{E}[\mu(s)]$, $\hat{s}_A \in (0,1)$ is such that $m_A(\hat{s}_A) = \kappa$, $\mu_+(\hat{s}_A) := \mathbb{E}[\mu(s) | s > \hat{s}_A]$ and

$$H_\alpha(v,e,\overline{\beta}(v)) := U(v,e) [1 - \overline{\beta}(v) + \mu_+ \overline{\beta}(v)] - \lambda (1 - \hat{s}_A) \left(1 - \overline{\beta}(v) + \mu_+(\hat{s}_A) \overline{\beta}(v)\right). \quad (1.10)$$

We focus on a cutoff strategy equilibrium in which college $A$ admits student type $(v,e)$ if and only if $e \geq \eta(v)$ for some $\eta$ nonincreasing in $v$ and college $B$ admits student type $(v,e')$ if and only if $e' \geq \xi(v)$ for some $\xi$ nonincreasing in $v$. For instance, the shaded area in Figure 1.5 depicts the types of students college $A$ may admit under a cutoff strategy. Such an equilibrium is quite plausible here since the use of non-common performance measure by the colleges lessens their head-on competition and the associated enrollment uncertainty. Section A.9 provides a condition under which cutoff equilibrium exists.

The question we focus here is whether the colleges may further reduce the head-on competition and the enrollment uncertainty by placing more weight on the non-common performance measures relative to their common preferences. Consider college $A$. (College $B$’s incentive will be analogous.) Inspection of college $A$’s preference makes it clear that under the cutoff equilibrium college $A$ must accept student types $(v,e)$ if and only if $H_\alpha(v,e,\overline{\beta}(v)) \geq 0$. In particular, the cutoff locus $e = \eta(v)$ must satisfy $H_\alpha(v,\eta(v),\overline{\beta}(v)) = 0$.
whenever $\eta(v) \in (0, 1)$. Its slope $-\eta'(v)$ shows the “relative worth” of a student’s common performance $v$ in $A$’s evaluation of the student, as measured in the units of students’ non-common performance college is willing to give up to obtain a unit increase in her common performance. The higher this value is, the higher weight college $A$ places on the common performance. In particular, we shall say that the college under-weights a student’s common performance $v$ and over-weights her non-common performance $e$ if for all $v$,

$$-\eta'(v) \leq \frac{U_v(v, \eta(v))}{U_e(v, \eta(v))}$$

and the inequality is strict for a positive measure of $v$. Suppose for instance $U(v, e) = (1 - \rho)v + \rho e$, then the condition means that $-\eta'(v) \leq \frac{1-\rho}{\rho}$, so the college places a weight less than $1 - \rho$ to common performance $v$ and the weight of more than $\rho$ to non-common performance $e$.

**Theorem 1.5.** In a cutoff equilibrium, each college under-weights a student’s common performance and over-weights her non-common performance.

*Proof.* Suppose there is a cutoff equilibrium with strategy profiles $(\alpha, \beta)$ where $\alpha(v, e) = \mathbf{1}_{\{e \geq \eta(v)\}}$ and $\beta(v, e') = \mathbf{1}_{\{e' \geq \xi(v)\}}$, for some $\eta(\cdot)$ and $\xi(\cdot)$ which are nonincreasing.

Here, we focus on college $A$, since college $B$’s behavior is analogous. Since $U_e > 0$, by the Implicit Function Theorem, $H_\alpha(v, e, \bar{\beta}(v)) = 0$ implicitly defines $\eta(v)$. Since $\mu_+(\hat{s}_A) > \overline{\mu}$, we must have

$$1 - \bar{\beta}(v) + \mu_+(\hat{s}_A)\bar{\beta}(v) > 1 - \bar{\beta}(v) + \overline{\mu}\bar{\beta}(v).$$

Then, $H_\alpha(v, \eta(v), \bar{\beta}(v)) = 0$ implies that

$$U(v, \eta(v)) > \lambda(1 - \hat{s}_A).$$

(1.11)
Next, totally differentiate $H_a$ to obtain:

$$U_v(v, \eta(v)) + U_e(v, \eta(v))\eta'(v)$$

$$= \frac{1}{1 - \beta(v) + \mu \beta(v)}(U(v, \eta(v))(1 - \beta) - \lambda(1 - s_A)(1 - \mu_+(s_A)))\beta'(v).$$

(1.12)

Since college B adopts a cutoff strategy, $\beta(v) = 1 - Y(\xi(v)|v)$, we have that

$$\beta'(v) = -y(\xi(v)|v)\xi'(v) - Y_v(\xi(v)|v) > 0,$$

(1.13)

where the inequality holds since $\xi'(v) \leq 0$ and $Y_v(e|v) < 0$.\(^{13}\)

Further, $\beta < \mu_+(s_A) \leq 1$, so it follows from (1.11) that the RHS of (1.12) is strictly positive for any $v$ such that $\eta(v) \in (0,1)$. Hence, for all $v$,

$$-\eta'(v) \leq \frac{U_v(v, \eta(v))}{U_e(v, \eta(v))},$$

(1.14)

and the inequality is strict for a positive measure of $v$.

\[\Box\]

### 1.5 Coordinated Matching: Self-Targeting

So far, we have characterized the pattern of colleges’ strategic targeting and provided existence and welfare and fairness properties of such equilibria. And we also show that when students’ types are multidimensional, strategic targeting entails evaluation distortion, that is, colleges place more weights on non-common performance than on common performance.

In the current and the following sections, we study two common ways for colleges to alle-

\(^{13}\)When $v$ and $e$ are independent, $\beta'(v) = -y(\xi(v))\xi'(v) \geq 0$. This implies that each college under-weights a students’ common performance and over-weights her non-common performance at least weakly and one college does so strictly. Further, together with college B’s condition (total differentiation of $H_\beta$), one can show that $\beta'(v) > 0$ for a positive measure of $v$, generically.
violate their yield management burden in decentralized matching. We consider the case that students’ type is single dimensional as in the baseline model.

We begin with students’ self-targeting: Colleges coordinate to limit the set of schools to which students can apply, thereby forcing students to “self-target” colleges. For instance, students cannot apply to both Cambridge and Oxford in the UK, and applicants in Japan can only apply to one public university.\textsuperscript{14} In Korea, all schools (more precisely, college-department pairs) are partitioned into three groups, and students are allowed to apply to only one in each group.

This method alleviates colleges’ yield management problem by improving the odds of enrollment for the colleges, since students apply only to those colleges that they are mostly likely to accept when admitted.\textsuperscript{15} In our model with two colleges, if the number of applications is restricted to one, colleges face no enrollment uncertainty because no student admitted by a college will turn down its offer. However, students’ application behavior will be strategic; thus, the overall welfare effects are not clear a priori.

We now provide a simple model showing students’ application behavior when the students can apply to only one of the two colleges. To this end, we introduce students’ cardinal preferences for colleges.\textsuperscript{16} Each student has a taste $y \in [0, 1]$, which is independent of score $v \in [0, 1]$. A student with taste $y$ obtains payoff $y$ from attending college $A$ and $1 - y$ from attending college $B$. Thus, students with $y \in [0, \frac{1}{2}]$ prefer $B$ to $A$, and those with $y \in [\frac{1}{2}, 1]\textsuperscript{16}$

\textsuperscript{14}More precisely, public schools may hold three exams. The first one is called “zenki(former period)-exam” and the last one is called “koki(later-period)-exam”. There are very small number of schools that have exam between these two exams. Students can apply to at most one public school at each exam date but the deadline for registering to the school that a student is admitted at zenki-exam is earlier than the date for applying the koki-exam.

\textsuperscript{15}Although there is no such restriction in the US, high application fees may serve this role. See Chade and Smith (2006) and Chade, Lewis, and Smith (2011) for students application decisions subject to application costs, without aggregate uncertainty.

\textsuperscript{16}Note that this does not alter the previous analyses, because even if students have cardinal preferences, it is still a weak dominant strategy for students to apply to both colleges in the previous model.
prefer $A$ to $B$. To facilitate the analysis, we assume that colleges observe an applicant’s score $v$ but not her preference $y$, while each student knows her preference $y$ but not her score $v$. In reality, even though students submit their records to colleges, they do not know precisely how they are ranked by colleges. See Avery and Levin (2010) for a similar treatment.

A student’s taste $y$ is drawn according to a distribution that depends on the underlying state. For a given $s$, let $K(y|s)$ be the distribution of $y$ with a density function $k(y|s)$. Then, $\mu(s) \equiv 1 - K(\frac{1}{2}|s)$ is the mass of students who prefer $A$ to $B$ in state $s$. We assume that $k(y|s)$ is continuous and obeys (strict) monotone likelihood ratio property (MLRP). That is, for any $y' > y$ and $s' > s$,

$$\frac{k(y'|s')}{k(y|s')} > \frac{k(y'|s)}{k(y|s)},$$

meaning that a student’s taste is more likely to be high in a high state. We further assume that there is $\delta$ such that $\left| \frac{k(y|s)}{k(y|s')} \right| < \delta$ for any $y \in [0,1]$ and $s \in [0,1]$, which means that students’ tastes changes moderately according to states. Each student with taste $y$ forms a posterior belief about the states,

$$l(s|y) := \frac{k(y|s)}{\int_0^1 k(y|s)ds}.$$

Before proceeding, we make the following observations: First, for the students, applying to a school dominates not applying at all. Second, since students do not know their scores and their preferences are independent of the scores, students’ applications depend only on their preferences. Third, since students’ preferences depend on states, the mass of students applying to each college varies across states. Let $n_i(s)$ be the mass of students who apply to
Consider colleges’ admissions decisions. Since a college faces no enrollment uncertainty, a cutoff strategy is optimal. If \( n_i(s) \geq \kappa \) in state \( s \), then college \( i \) will set its cutoff so as to admit students up to its capacity. Otherwise, it will admit all applicants. More precisely, the cutoff of college \( i \) in state \( s \), denoted by \( c_i(s) \), is given by

\[
c_i(s) := \inf \{ c \in [0, 1] | n_i(s)[1 - G(c)] \leq \kappa \}.
\]

Consider now students’ application decisions. Fix any \( \sigma : [0, 1] \to [0, 1] \) which maps from taste to a probability of applying to \( A \). This induces the mass of students applying to \( A \) in each state \( s \),

\[
n_A(s) := \int_0^1 \sigma(y)k(y|s) dy.
\]

Clearly, \( n_B(s) = 1 - n_A(s) \). A student with taste \( y \) has a probability of being admitted by \( i \)

\[
P_i(y|\sigma) = \mathbb{E}_s[1 - G(c_i(s)) | y, \sigma] = \int_0^1 q_i(s|\sigma)l(s|y) ds,
\]

where \( q_i(s|\sigma) \equiv \min \{ \kappa/n_i(s|\sigma), 1 \} \) for \( i = A, B \). Note that a student with taste \( y \) will apply to \( A \) if and only if

\[
yP_A(y|\sigma) \geq (1 - y)P_B(y|\sigma).
\]

or equivalently,

\[
T(y|\sigma) := yP_A(y|\sigma) - (1 - y)P_B(y|\sigma) \geq 0.
\]

**Lemma 1.2.** Suppose \( \delta \leq \frac{1}{2} \). Any equilibrium involves cutoff strategy where students with \( y \geq \hat{y} \) apply to \( A \) and those with \( y < \hat{y} \) apply to \( B \). And such an equilibrium exists.

We defer the proof to Section A.5. By the lemma, we can focus on cutoff equilibrium. Let \( \hat{y} \) be the cutoff. Since all students with \( y \geq \hat{y} \) apply to \( A \), the mass of students applying
to $A$ is $n_A(s) = \int_{\hat{y}}^{1} k(y|s)dy = 1 - K(\hat{y}|s)$, and similarly $n_B(s) = K(\hat{y}|s)$.

**Theorem 1.6.** Suppose $\mu(s) \geq \frac{1}{2}$ for all $s$ Then, $\hat{y} \in [\frac{1}{2}, 1)$, where $\hat{y}$ is the equilibrium cutoff.

The proof is found in Section A.5. Theorem 1.6 shows students’ strategic applications when college $A$ is more popular than the other for all states. Consider a student with taste $y$ who expects that $P_B(y) > P_A(y)$ since $A$ is more popular than $B$. If she prefers $B$ ($y < \frac{1}{2}$), then it is optimal for her to apply to $B$. If the student prefers $A$ ($y \geq \frac{1}{2}$), then there is a trade-off since her payoff is higher if she can attend $A$ over $B$, but she believes that she has a higher chance of admission to $B$. Thus, if she only mildly prefers $A$, then she may apply to $B$ instead of $A$. We provide a simple example with two states to illustrate the results. Figure 1.6 depicts the equilibrium assignments of the example.

**Example 1.1.** Suppose that there are two states $a$ and $b$ with equal probability. Let $K(y|a) = y^2$, $K(y|b) = y$ and $\kappa = 0.4$. Then, we have

<table>
<thead>
<tr>
<th>$\hat{y}$</th>
<th>$n_A(a)$</th>
<th>$n_B(a)$</th>
<th>$c_A(a)$</th>
<th>$c_B(a)$</th>
<th>$n_A(b)$</th>
<th>$n_B(b)$</th>
<th>$c_A(b)$</th>
<th>$c_B(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.547</td>
<td>0.701</td>
<td>0.299</td>
<td>0.429</td>
<td>0</td>
<td>0.453</td>
<td>0.547</td>
<td>0.116</td>
<td>0.269</td>
</tr>
</tbody>
</table>
Observe that if \( n_i(s) \geq \kappa \) for all \( s \) and all \( i = A, B \), then the self-targeting eliminates colleges’ yield management problem, since each college fills its capacity with the best students among those who applied to it. However, it does not hold in general because there can be under-subscription to a college in some state. In the above example, for instance, the mass of applicants to college \( B \) in state \( a \) is smaller than its capacity \( (n_B(a) = 0.299 < \kappa = 0.4) \).

Let us now consider welfare and fairness properties of the equilibrium outcome. First, the equilibrium is unfair. That is, justified envy arises in that (i) students who happen to have applied to a more popular school for a given state may be unassigned even though their scores could have been good enough for the other school (see Figure 1.6(a)); and (ii) students who mildly prefer the popular school to the less popular one may be assigned to the latter college even though they could have been assigned to the popular one (see Figure 1.6(b)).

Second, there can be under-subscription to a college in equilibrium so that its capacity is not filled even though there are unassigned, acceptable students. By assigning those students to unfilled seats of a college, both the students and college will be better off. Thus, the equilibrium outcome is still Pareto, student and college inefficient.

In the next theorem, we provide conditions under which justified envy among students and/or under-subscription to a college arise. If there is a college that suffers from under-subscription, then the equilibrium outcome is not Pareto and student efficient as discussed above.

**Theorem 1.7.** Suppose \( \mu(s) \geq \frac{1}{2} \) for all \( s \). Then, there exists a positive mass of students who have justified envy. Suppose \( K(\hat{y}|s) < \kappa \) for a positive measure of states \( s \), where \( \hat{y} \) is the cutoff defined in Theorem 1.6. Then, college \( B \) suffers from under-subscription, and the assignment is Pareto, student and college inefficient.

The first part of the theorem provides a sufficient condition under which justified envy arises. Observe that justified envy arises whenever \( c_A(s) \neq c_B(s) \). Suppose \( c_A(s) > c_B(s) \).
Then, the students with scores in \([c_B(s), c_A(s)]\) and tastes in \([\hat{y}, 1]\) are not assigned to any school, even though their scores are good enough to be assigned to college \(B\). Suppose \(c_A(s) < c_B(s)\). Then, the students with scores in \([c_B(s), 1]\) and tastes in \([\frac{1}{2}, \hat{y}]\) (whenever \(\hat{y} > \frac{1}{2}\)) are assigned to \(B\), even though their scores are good enough to be assigned to \(A\); and those with scores in \([c_A(s), c_B(s)]\) and tastes in \([0, \hat{y}]\) are not assigned to any school, even though their scores are good enough for college \(A\). We show in Section A.5 that there is a positive measure of states in which \(c_A(s) \neq c_B(s)\).

To see the second part of the theorem, recall that for given \(\hat{y}\) in equilibrium, the mass of students applying to \(B\) is \(K(\hat{y}|s)\). Thus, if there is a positive measure of states in which \(K(\hat{y}|s) < \kappa\), college \(B\) faces under-subscription in such states (as state \(a\) in the above example). Therefore, the equilibrium outcome is inefficient.

### 1.6 Sequential Admissions

In this section, we consider sequential admissions using waiting lists, another way of alleviating enrollment uncertainty that colleges use in decentralized matching. In France and Korea, for instance, colleges initially make admission offers to students up to their capacity limits, filling any declined seats with offers to students on the waiting list.

Even though this method may generate more admission offers and acceptances than the baseline model or self-targeting, we show that it is not enough to eliminate congestion altogether, and colleges may still want to engage in strategic targeting.

The main intuition is as follows. Suppose a college, say \(A\), wishes to make admission offers to the most preferred candidates up to its capacity, who are also sought after by other colleges, planning to approach the next best students in the case that some of those first group of students turn its offer down. Often, college \(A\) is uncertain if the next best students are willing to wait for it when a less popular college approaches to them in the first round.
Each of those students may also be uncertain about the likelihood that college A finds her acceptable, and hence she may accept the less popular college’s admission offer immediately. This uncertainty means that when college A is turned down by some of the first group of students, it may not have the second best group of students available. In fact, the students who are available at that point are far worse than the second best group of students, hence college A may wish to directly offer admissions to some of those second group of students instead of some of the first group of students.

We provide a simple model to formalize the intuition. There are three colleges, A, B and C, each with a mass $\kappa < \frac{1}{3}$ capacity. There is a unit mass of students with score $v$, where $v$ is distributed over $[0, 1]$ according to $G(\cdot)$ as before. All students like A and B better than C, but C is sufficiently better than not attending any school. Colleges’ preferences are given by students’ scores, but for each student, there is a probability $\varepsilon$ that colleges A and B find that the student is unacceptable. College C simply likes students according to $v$’s.

There are two states, $a$ and $b$. In state $i, i = a, b$, a fraction $s_i$ of students gets utility $u$ from A and $u'$ from B, and the remaining $1 - s_i$ students have the opposite preference, where $s_a = 1 - s_b > \frac{1}{2}$. In either state, students get utility $u''$ from C, where $u > u' > u''$ and $u'' > (1 - \varepsilon)u$. The latter assumption means that even for a small uncertainty about the students’ acceptability by the better school, the certain utility from college C of a student is greater than the uncertain utility from the better school. Note that in state $a$, the mass of students who prefer A to B is larger than that of those who prefer B to A ($s_a > \frac{1}{2} > 1 - s_a$), and in state $b$, the former is smaller than the latter ($s_b < \frac{1}{2} < 1 - s_b$). Note that A and B are ex ante symmetric.

Suppose the capacity cost is prohibitively high so that at each time a college makes admission decisions, it must be sure that the capacity will never be violated. The sequential admissions game has the following feature. In each round, the colleges make admission offers to a set of students and wait-list the remaining. The students who received offer(s) from
college(s) must decide to accept or reject the offer immediately. After the first round, colleges A and B learn the state, so the game effectively ends in two rounds.

We show that there is no symmetric equilibrium in which both colleges A and B use a cutoff strategy (i.e., admit the top $\kappa$ students among those who are acceptable) in the first round. We then consider a simpler example and analyze equilibrium admission strategies, which shows that colleges still engage in strategic targeting. The equilibrium outcome of the example shows that the sequential admissions may entail justified envy and inefficiency.

**Theorem 1.8.** There is no symmetric equilibrium in which A and B offer admissions to the top $\kappa$ students (excluding those whom they find unacceptable) in the first round.

**Sketch of Proof.** Suppose there is such an equilibrium to the contrary. Then, colleges A and B will admit all acceptable students with $v > \hat{v}$, where $\hat{v}$ is such that each of A and B fills its capacity in the popular state, i.e., $s_a(1 - \varepsilon)[1 - G(\hat{v})] = \kappa$ (or equivalently, $(1 - s_b)(1 - \varepsilon)[1 - G(\hat{v})] = \kappa$), and wait-lists the remaining students. College C will offer admissions to all of these students (i.e., those whose values are above $\hat{v}$), knowing that exactly measure $\varepsilon^2$ of them will accept its offer (since those are not acceptable for both A and B). It will also offer $\kappa - \varepsilon^2$ admissions to all students with $v \in [\tilde{v}, \hat{v}]$, where $\tilde{v}$ is such that $G(\tilde{v}) - G(\hat{v}) = \kappa - \varepsilon^2$.

The students in $[\tilde{v}, \hat{v}]$ now have a choice to make. If a student accepts C, then she will get $u''$ for sure, but if she turns down C’s offer, then with probability $1 - \varepsilon$ the less popular one between A and B will offer an admission to her (assuming all other students admitted by C have accepted that offer), and the student will earn the payoff $u$ if she happens to like the college, or $u'$ otherwise. Since $u'' > (1 - \varepsilon)u$, she will accept C immediately.

Given this, consider now the incentive for deviation of A. If it does not deviate, there will be seats left, equal to $\kappa - s_b(1 - \varepsilon)[1 - G(\hat{v})]$, in the less popular state. Thus, A will fill them with students whose scores are below $\tilde{v}$ (since those with scores in $[\tilde{v}, \hat{v}]$ are taken
by \( C \). Suppose now that college \( A \) admits a small fraction, say \( \delta' \), of (acceptable) students just below \( \hat{v} \) instead of admitting those who are acceptable and slightly above \( \hat{v} \), say \([\hat{v}, \hat{v} + \delta] \), where \( \delta \) and \( \delta' \) are such that

\[
G(\hat{v} + \delta) - G(\hat{v}) = G(\hat{v}) - G(\hat{v} - \delta').
\] (1.16)

Then, college \( A \) benefits from this deviation for sufficiently small \( \delta \). The reason is as follows. The loss from this deviation is that when \( A \) is popular, it will get a worse group; and even in the unpopular state, \( s_b \) fraction of these students are replaced by the worse group. But, the gain is that when \( A \) is unpopular, it will get a discretely better group of students for the \( s_a - s_b \) fraction of the vacant seats. Thus, for sufficiently small \( \delta \), the order of magnitude for the gain is greater than that for the loss. We relegate a formal proof for this to Section A.6.

We now consider equilibrium admission strategies in a simpler example.

**Example 1.2.** There are three students, 1, 2 and 3, with scores \( v_1 > v_2 > v_3 \), where \( v_2 > \frac{1}{2}(v_1 + v_3) \). There are three colleges, \( A, B \) and \( C \), and each of them has one seat to fill. Colleges value students with score \( v_i \) at \( v_i \), but student 2 has an \( \varepsilon \) chance of being unacceptable by either \( A \) or \( B \), where \( \varepsilon < \frac{v_1 - v_2}{v_2 - v_3} \). All students like \( A \) and \( B \) better than \( C \), as in the above model. Students have a uniform preference (i.e., their preference orderings for colleges are the same). They receive utility \( u \) from \( A \) and \( u' \) from \( B \), or \( u' \) from \( A \) and \( u \) from \( B \) with equal probability, and \( u'' \) from \( C \) for sure, where \( u > u' > u'' (> 0) \) and \( u'' > (1 - \varepsilon)(\frac{1}{2}u + \frac{1}{2}u') \). The timing of the game is the same as before.

Before proceeding on what equilibrium may arise, it is useful to begin with a few observations. First, colleges never make an admission offer to student 3 before they offer admissions to 1 and 2 (when 2 is acceptable), since the worst case for them is to have 3, which is always
possible. Second, the result of Theorem 1.8 still works in this example. That is, there is no equilibrium in which both colleges $A$ and $B$ make admission offers to student 1 for sure (i.e., they use a cutoff strategy).\footnote{Suppose both $A$ and $B$ seek student 1 in the first round. Then, college $C$ will seek student 2 and will be immediately accepted. (Student 2 gets $u''$ if she accepts $C$. When she declines $C$, her expected utility is $(1-\varepsilon)(\frac{1}{2}u + \frac{1}{2}u'')$ because the college that was rejected by 1 will offer admission to her in the second round only when she is acceptable. Since $u'' > (1-\varepsilon)(\frac{1}{2}u + \frac{1}{2}u'')$, it is optimal for her to accept $C$ immediately.) Given this, if $A$ and $B$ offer admissions to student 1, then each of them gets the payoff $\frac{1}{2}v_1 + \frac{1}{2}v_3$. This is optimal for each of $A$ and $B$ only when 2 is unacceptable. However, if 2 is acceptable for a college, then the college can deviate to admit her instead and get $v_2$, since $v_2 > \frac{1}{2}(v_1 + v_3).\phantom{1}$}

Consider now the following equilibrium. In the first round, college $C$ offers admission to student 2 for sure, and both $A$ and $B$ admit student 1 with probability $p = \frac{v_1 - v_2 - \varepsilon(v_2 - v_3)}{(1-\varepsilon)(v_2 - v_3)}$ and student 2 with probability $1 - p$, whenever they find that 2 is acceptable. Each college then places the other students whom they do not admit on the waiting list.

Note that $p$ makes the other college indifferent. That is, when a college, say $B$, offers admission to student 1, its payoff is

$$
[(1-p)(1-\varepsilon) + \frac{1}{2}(1-(1-p)(1-\varepsilon))]v_1 + \frac{1}{2}(1-(1-p)(1-\varepsilon))v_3.
$$

(1.17)

Here, $(1-p)(1-\varepsilon)$ is the probability that $A$ offers admission to student 2 when she is acceptable. In this case, student 1 accepts $B$ immediately, or else she will get an offer from $C$ in the second round (because $A$ offers admission to student 2, and 2 immediately accepts it). $1 - (1-p)(1-\varepsilon)$ is the probability that $A$ offers admission to student 1 (either when 2 is unacceptable or when 2 is acceptable but $A$ offers admission to 1), in which case $B$ is accepted by student 1 with probability $\frac{1}{2}$. If $B$ is rejected, then it seeks student 3 in the second round because student 2 is not available (both $A$ and $B$ happen to make admission offers to student 1, and $C$ offers an admission to student 2 and is immediately accepted, see footnote 18).
Similarly, when $B$ offers admission to student 2 (when she is acceptable), its payoff is

$$\left[\frac{1}{2}(1-p)(1-\varepsilon) + (1 - (1-p)(1-\varepsilon))\right]v_2 + \frac{1}{2}(1-p)(1-\varepsilon)v_1. \quad (1.18)$$

If $A$ offers admission to student 1, which happens with probability $(1 - (1-p)(1-\varepsilon))$, then $B$ is immediately accepted by 2. If $A$ also offers admission to student 2, which happens with probability $(1-p)(1-\varepsilon)$, then $B$ will be accepted by 2 with probability $1/2$. If it is rejected, then it will make an admission offer to student 1 in the next round and will be accepted for sure (both $A$ and $B$ happen to make admission offers to student 2, and $C$ also offers admission to 2; thus, student 1 is available in the second round, and she prefers $B$ to $C$).

Equating (1.17) and (1.18), we have

$$p = \frac{v_1 - v_2 - \varepsilon (v_2 - v_3)}{(1-\varepsilon)(v_2 - v_3)}.$$ 

Note that $p \in (0,1)$ because $\varepsilon < \frac{v_1 - v_2}{v_2 - v_3}$ and $v_2 > \frac{1}{2}(v_1 + v_3)$.

Given this, it is clear that $C$ will not make an admission offer to student 1 in the first round. Suppose that college $C$ does so. Then, even when both $A$ and $B$ happen to make admission offers to student 2, $C$ will be rejected by student 1, since she will be admitted by either $A$ or $B$ in the second round and those are preferred than $C$. Therefore, it is optimal for $C$ to offer an admission to student 2 with probability 1 in the first round.

This example shows that colleges still engage in strategic targeting in sequential admissions: $A$ and $B$ compete for a better student, and $C$ admits student 2, who may be overlooked by the competing colleges. The equilibrium assignment may entail justified envy, since it may be the case that student 1 attends $B$ but she likes $A$, or that student 2 attends $C$ and 3 attends $B$. The assignment is also student inefficient, since it is possible that student 1 likes $A$ and 2 likes $B$, but they are assigned to $B$ and $A$, respectively.
1.7 Centralized Matching via Deferred Acceptance

In the last two sections, we have considered two common ways that colleges use to alleviate their yield management problem in the decentralized matching. In this section, we consider a centralized matching with a Gale and Shapley’s Deferred Acceptance algorithm (henceforth DA). Many markets, such as public school admissions and medical residency assignments, are centralized via such an algorithm. College admissions are also centralized in some countries, although it is organized in varying degrees.\(^{19}\) We consider the equilibrium allocation under DA and compare this with the outcome that arises from the decentralized matching.

Suppose that the matching is organized by a clearinghouse that applies Gale and Shapley’s student-proposing DA.\(^{20}\) The algorithm works as follows. Initially, students and colleges report their preference orderings to the clearinghouse. In the first round, students propose their favorite college (according to the reported order). If a college has more applicants than its quota, it tentatively admits a set of students that maximizes its payoff staying within the quota, rejecting the rest (according to the reported order). In each round, unassigned students propose their favorite college among those that has not rejected them. Each college selects a set of students that maximizes its payoff within the quota among those who are tentatively admitted in the previous round and those who propose it in the current round. The process ends until no further proposals are made, in which case each student is assigned to a college that holds her proposal.\(^{21}\)

To see this more precisely, consider a state \(s\) such that \(\mu(s) \geq 1 - \mu(s)\). In the first round, a fraction \(\mu(s)\) of students proposes college \(A\), and the remaining students propose

\(^{19}\)See Chen and Kesten (2011) for Shanghai mechanism and Westkamp (forthcoming) for Germany medical school matchings.

\(^{20}\)The outcome of college-proposing DA is the same as that of student-proposing DA in our model, since colleges have a uniform rank on students.

\(^{21}\)Abdulkadiroğlu, Che, and Yasuda (2012) and Azevedo and Leshno (2012) provide a model of DA in which a continuum mass of students is matched to a finite number of schools.
college $B$. Each college tentatively admits the top $\kappa$ students among the applicants. Thus, colleges’ cutoffs in this round, denoted by $\hat{\nu}_i(s), i = A, B$, satisfy $\mu(s)[1 - G(\hat{\nu}_A(s))] = \kappa$ and $(1 - \mu(s))[1 - G(\hat{\nu}_B(s))] = \kappa$. Unassigned students then propose another college at the second round, and again, each college admits the top $\kappa$ students among those who are admitted in the first round and those who propose it at the current round. Thus, colleges’ cutoffs in this round satisfy $\mu(s)[1 - G(\hat{\nu}_A(s))] = \kappa$ and $1 - G(\hat{\nu}_B(s)) = 2\kappa$. Since there are no more colleges to which unassigned students can apply, the assignment is finalized in the second round in our model. The process is depicted in Figure 1.7.

Consider now the equilibrium properties of the DA outcome. Under DA, the matching is strategy proof for the students, so the students have a dominant strategy of reporting their preferences truthfully (Dubins and Freedman, 1981; Roth, 1982). In addition, colleges in our model also report their rankings and capacities truthfully.

**Lemma 1.3.** *Given the common college preferences, it is an ex post equilibrium for colleges to report their rankings and capacities truthfully.*

The proof in Section A.7 shows that if one college, say $B$, truthfully reports its capacity and preference, it is a best response for $A$ to do the same. The reason is that when $A$ manip-
ulates its capacity or ranking, the possible gain comes from generating a “rejection chain” (Kojima and Pathak, 2009), i.e., $A$ rejects a positive mass of students by the manipulation, and those students apply to $B$, causing $B$ to reject some other positive mass of students. Then, those student (who are rejected by $B$) will then apply to $A$. If those second group of students are preferred by $A$ over the first group of students, then $A$ could be better off. However, the common preference of $A$ and $B$ implies that the second group of students are worse than the first group of students, since $B$ would not otherwise reject the second group of students.

The matching in the equilibrium involves no justified envy (Gale and Shapley, 1962; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), and is efficient among students (because colleges’ preferences are acyclic in the sense of Ergin (2002)) and Pareto efficient (an implication of stability). It also eliminates colleges’ yield management problem completely. Colleges never exceed their quotas (because it is never allowed by the algorithm), and have no seats left unfilled in the presence of acceptable unmatched students (a consequence of stability).

In fact, given the homogeneous preferences of the colleges, there exists a single cutoff such that a student is assigned to a college under DA if and only if her score exceeds that cutoff. In order words, only those with the top $2\kappa$ scores are assigned. This outcome is jointly optimal for the two colleges. In contrast, recall that competitive equilibrium in decentralized matching entails unfilled seats for colleges in low-demand states and exceeded quotas in high-demand states, so the assignment is far from jointly optimal. This observation suggests that at least one college must be strictly better off from a shift from decentralized matching to centralized matching via the deferred acceptance algorithm. Despite the overall benefit from switching centralization via DA, it is possible for one college to be worse off. To see this, consider the following example.

Example 1.3. Let $v \sim U[0, 1]$, $\lambda = 1$, $\kappa = 0.45$ and $\mu(s) = \frac{1}{2} \sqrt{s} + \frac{1}{2}$. Then, in a decentralized
admission, colleges’ payoffs in equilibrium are $\pi_A = 0.304$ and $\pi_B = 0.158$. Suppose now that the DA is in use. Then, their payoffs are $\pi_{DA}^A = 0.352$ and $\pi_{DA}^B = 0.143$. Notice that $\pi_{DA}^A + \pi_{DA}^B = 0.495 > \pi_A + \pi_B = 0.462$ (overall benefit for the two colleges), $\pi_{DA}^A > \pi_A$ (college A is strictly better off), but $\pi_{DA}^B < \pi_B$ (college B is worse off).

In this example, college A is more popular than B for any state. Under decentralized matching, college B has some chance to have high score students (because of colleges’ strategic targeting as we have seen), while it does not under DA. This may explain why college admissions are remained decentralized in many countries unlike public school matchings. In the former, colleges have their own interests and preferences over students, whereas in the latter, schools are treated as “resources” of a society, in which case schools’ preferences are interpreted as school-priorities that students claim for the schools.

Equilibrium properties of the assignment under DA are summarized in the follow.

**Theorem 1.9.** Under DA, the equilibrium outcome is fair, Pareto and student efficient, and jointly optimal among the colleges. However, some college may be worse off relative to decentralized matching.

### 1.8 Conclusion

The current paper has introduced and analyzed a new model of decentralized college admissions. In the model, colleges make admission decisions subject to aggregate uncertainty about students’ preferences and linear costs for any enrollment exceeding the capacity. We find that colleges’ admission decisions become a tool for strategic yield management, and in equilibrium, colleges try to reduce their enrollment uncertainty by strategically targeting students.

We also obtain the welfare and fairness implications of the equilibrium outcomes. We show that the equilibrium outcome under decentralized matching entails justified envy and
fails to attain efficiency. We also show that when colleges consider students’ non-academic performance or extra-curricular activities, the use of these aspects may lessen head-on competition among colleges. However, strategic targeting still entails as colleges placing overweights on those non-common performance measure. Our analytical model permits a clear comparison of the outcomes that would arise (i) when students are forced to self-target (by the limited set of schools they can apply to), (ii) when admissions are made sequentially, and (iii) when the market is centralized via DA. Both self-targeting and sequential admissions may alleviate colleges’ yield management problems but may not eliminate them completely, and inefficiency and justified envy are still entailed. Centralized matching via DA completely eliminates the yield management problem and justified envy, and it also achieves Pareto and student efficiency in our model. We show, however, that it is possible for one college to be worse off using centralized matching relative to decentralized matching. This observation may explain why college admissions remain decentralized in many countries.

The model reveals several aspects to be further investigated. As assumed in our model, it is often the case that colleges have largely homogeneous preferences/evaluations based on students’ high school GPA and their scores from a nationwide test, such as those in Australia, Japan or Korea. However, colleges may prefer students who have enthusiasm or loyalty towards them, because these qualities may predict future donations and alumni activities (see Avery and Levin, 2010), or colleges may seek diversity in the student body. Interestingly, students’ idiosyncratic preferences may be one way to coordinate strategic targeting. If colleges know that “legacy” admits (who have a family history with the school) are more likely to accept admission offers, they can coordinate on admitting different students by each admitting their own legacy students, even if these students are objectively lower quality.

As noted in the paper, colleges under decentralized matching find ways to alleviate their yield management burden. Although we have considered two common ways, self-targeting
and sequential admissions, used for this purpose, early admissions may also alleviate the yield management problem. By admitting a fraction of students early, a college can reduce uncertainty in the enrollment. Avery, Fairbanks, and Zeckhauser (2003) provide empirical evidence that matriculation rates in early admissions are much higher than those in regular admissions (especially in the case of early decisions). Moreover, students are forced to reveal their preferences during the early admission process, and thus, colleges may have better information about students’ preferences. This may help colleges to manage the final class size in regular admissions and may alleviate their yield management problems.

Finally, colleges may encourage students to build specialized human capital or merit/life experience tailored for their specific requirements. For example, some colleges in Korea fill a portion of their capacities with students who have a high test score for foreign language abilities (e.g., TOEFL or TOEIC) or win a mathematics/physics olympiad or other contests. Since those requirements are usually far beyond standard high school level, this will cause students who made the investments to be more qualified for the colleges for whom targeted investments were made but not for other. This may make students lock in even from the early stage of their high school, so it may help ease the yield management problem.
Chapter 2

Keyword Auctions with
Budget-Constrained Bidders

Youngwoo Koh
2.1 Introduction

Keyword advertising is a form of targeted online advertising. When an Internet user enters a keyword (query) into a search engine, she receives sponsored links along with search results. When the user clicks on a sponsored link, she is directed to the advertiser’s web site and the advertiser pays the search engine for that click. Search engines such as Google, Yahoo!, and Bing generate substantial revenue by auctioning off their ad spaces. The commonly used auction format in the industry is the *generalized second-price* (GSP) auction and its variants.

In general, ads that appear in upper positions on a page garner more clicks than those appearing in lower positions. This feature is captured by different *click-through rates* (CTRs) at each position. Under GSP auctions, each advertiser bids on the per-click price, and his total payment is the per-click price multiplied by the CTR. The advertiser who has submitted the highest bid is assigned to the highest position and pays the second highest bid. The second highest advertiser is assigned to the second highest position and pays the third highest bid, and so on.\(^1\)

Two seminal papers addressing GSP auctions are Edelman, Ostrovsky, and Schwarz (2007, EOS hereafter) and Varian (2007). Both analyze a static model in which advertisers with complete information bid simultaneously. They focus on a refinement of Nash equilibria and identify a class of equilibria, called *locally envy-free* and *symmetric Nash equilibria* (SNEs henceforth), respectively. They assume that advertisers have no budget constraints. In practice, however, search engines require advertisers to set their daily budgets. Google, for example, states how the daily budget influences the placement of the advertisers’ ads as

\(^1\)An assumption here is that CTRs depend only on the position but not on the advertisers’ identity: the rankings based on a per-impression bid—per-click bids multiplied by CTRs—are the same as the rankings based on a per-click bid. In practice, however, the allocation schemes vary across search engines. For example, Google ranks advertisers according to “quality scores,” which are based on bids, past CTRs, and the qualities of the advertisers’ landing pages.
follow:

Your daily budget is the amount that you set for each ad campaign to specify how much, on average, you’d like to spend each day. Once you set a daily budget for your campaign, the system will aim to show your ads as much as possible until your budget is met. When your budget is reached, your ads will typically stop showing for that day. (emphasis added)²

This statement addresses some important aspects that are overlooked by EOS (2007) and Varian (2007). First, one’s payment, say bidder i’s, is determined by the per-click price at his position and exposure time. Once the payment reaches his budget, his ad will be removed from the page and the next highest bidder, say j, takes this position. Second, under GSP, bidder i’s per-click price is determined by bidder j’s bid, the next highest one. Thus, bidder j may benefit by raising his bid to deplete bidder i’s budget. On the other hand, bidder i may also lower his bid to be slightly less than j’s bid to influence bidder j’s exposure time. In sum, budget constraints may create an incentive for bidders to change their bids to inflict higher prices on competitors.

Let us elaborate on this with an example. Suppose that there are two positions and three bidders and that the auction takes place during one day (24 hours). Let the CTR of position 1 be 20 and of position 2 be 18. Suppose bidder 1’s per-click bid is 6, bidder 2’s is 5, and bidder 3’s is 4 so that bidder 1 is assigned to the highest position, bidder 2 to the second highest position, and bidder 3 fails to win any position. If there are no budget constraints, both bidders 1 and 2 can stay at their respective positions until the end of the day. Now suppose that bidders have the same budget of 60. Suppose further that bidder 2 does not wish to outbid bidder 1. Then, bidder 2 can benefit by bidding (slightly less than)
6, because bidder 1 exhausts his budget at time $\frac{60}{20 \times 6} \times 24 (= 12 \text{ hours})$, and then bidder 2 takes the position but still pays the third highest per-click bid. That is, bidder 2 makes bidder 1 *consume* only one-half unit of position 1 and takes the remaining *fraction* of the position. Similarly, bidder 1 may benefit by lowering his bid to (slightly less than) 5. By doing so, bidder 1 makes bidder 2 win position 1 initially. However, after bidder 2 leaves the market at time $\frac{60}{18 \times 5} \times 24 (= 16 \text{ hours})$, bidder 1 moves up to position 1 but pays the third highest per-click bid.

In this paper, we raise two questions pertaining to the issues mentioned above:

- **Question 1**: How large should the budgets be in order for SNEs (without budget constraints)\(^3\) to be sustained as equilibria with budget constraints?

- **Question 2**: How do budget constraints affect bidders’ bidding behavior and the search engine’s revenue?

To answer these questions, we consider a model in which the auction is conducted during a unit time, that represents one “day.” All bidders are endowed with daily budgets and submit their bids when the day begins. We assume that after placing bids, bidders do not adjust their bids within that day.\(^4\) Once one’s payment reaches his budget, the bidder’s ad disappears from the page and the next highest bidder takes that position. We assume that bidders have complete information.

Our first result is the answer to the first question. We identify a profile of budget thresholds \(\left(\hat{B}^1_j, \hat{B}^2_j\right)_{j \in \mathcal{N}}\), where \(\mathcal{N}\) is a set of bidders, such that \(\hat{B}^1_j < \hat{B}^2_j\) for all \(j \in \mathcal{N}\). Let \(B_j\) be bidder \(j\)’s budget. We first show that for any SNE bidding profile without budget constraints, if there is a bidder \(j\) with \(B_j < \hat{B}^1_j\), then the next highest bidder can benefit

\(^3\)With a slight abuse of the terminology, let us refer to “SNE without budget constraints” as SNE.

\(^4\)Sodomka, Lahaie, and Hillard (2012) and Pin and Key (2011) provide empirical evidence showing that bidders update their bids infrequently. Particularly, Sodomka, Lahaie, and Hillard (2012) show that it is very rare for bidders to change their bids more than once per day.
by bidding (slightly less than) \( j \)'s bid. Therefore, any SNE bidding profile cannot be an equilibrium with budget constraints. We then show that \( \tilde{B}_{j}^2 \leq B_{j} \) for all \( j \in N \) if and only if all SNEs without budget constraints are sustainable as equilibria with budget constraints; that is, bidders consume the entire unit of their assigned positions.

For the second question, it is difficult to solve equilibrium bidding strategies analytically in a general environment with multiple positions and budget constraints. We consider a simple environment that has one position and two bidders with the same budget. In fact, without budget constraints, the one-position GSP coincides with the standard second-price auction. In the presence of budget, however, it may be the case that the losing bidder overbids (i.e., bids above his own value) to exhaust the winning bidder’s budgets, so the search engine’s revenue may be larger than its revenue without budget constraints. We show that unless the budget is too small, the search engine earns at least as much revenue as from the standard second-price auction.

There are some papers that study keyword auctions with budget constraints. Zhou and Lukose (2006) note that a lower placed bidder has an incentive to bid slightly less than a higher placed bidder’s bid (called “vindictive bidding”). They show that if there are three or more bidders who are vindictive against each other, then a pure-strategy Nash equilibrium may not exist. However, they only consider an environment with indivisible positions, whereas in our model, positions are treated as divisible goods. Moreover, we do not assume a priori that the difference between two adjacent bids is arbitrarily small. Our first question is under what conditions an SNE can or cannot be sustained as an equilibrium with budget constraints, and we show that if each bidder \( j \)'s budget is at least as large as \( \tilde{B}_{j}^2 \), then any SNE is an equilibrium with budget constraints.

Bidders' overbidding behavior has also been studied in the auction literature. Pitchik and Schotter (1988) and Benoît and Krishna (2001) study sequential auctions with two goods and budget-constrained bidders. In the first auction, a bidder may overbid to exhaust the
other bidder’s budget expecting him to be in a stronger position in the second auction. Thus, the auctioneer may enjoy a larger revenue when bidders are budget constrained than when they are not.\footnote{See Example 8 (p.168) in Benoît and Krishna (2001).} Pagnozzi (2006) considers sequential common-value ascending auctions with two goods and two bidders. He shows that a bidder bids above his budget in the first auction to deplete his rival’s budget, and the seller benefits from this overbidding.\footnote{In Pitchik and Schotter (1988) and Benoît and Krishna (2001), overbidding means that a bidder bids above his own value. Pagnozzi (2006) allows bidders to bid above their budgets, and overbidding means bidding above one’s own budget.} Although our model has a single bidding stage, a similar overbidding arises because of the divisibility of the position.

Other researches that consider multi-unit auctions with budget constraints include Borgs et al. (2005) and Hafalir, Ravi, and Sayedi (2012b). They consider mechanisms that sell multiple copies of a single item. Borgs et al. (2005) show that there is no truthful mechanism that sells all units to distinct bidders, but they provide an asymptotically revenue-maximizing truthful mechanism that leaves some units unsold.\footnote{The mechanism randomly divides bidders into two groups and uses the market clearing price for each group as the price offering to the other group.} In the same setting, Hafalir, Ravi, and Sayedi (2012b) provide a new mechanism, called Sort-Cut, to sell all units.\footnote{The algorithm sorts bidders in decreasing order of their (reported) values, say $v_1 \geq \cdots \geq v_n$, and finds a cutoff bidder $k$ such that bidders $i < k$ win goods until their budgets are exhausted, the cutoff bidder $k$ wins the remaining goods, and bidders $i > k$ do not win any good. The cutoff bidder and the amount of his leftover budget are set to clear the market.} They show that in the sort-cut mechanism, bidders cannot benefit from lying about their budgets or understating their values, but may benefit by overstating their values. Although these works provide mechanisms that can be used when bidders have budget constraints, those mechanisms are applicable when there is a single position to be sold, in which the number of units (copies of the single item) each bidder wins can be understood as a fraction of the single position. In this regard, Hafalir, Ravi, and Sayedi (2012a) study a model of selling a
divisible good to budget-constrained bidders. We also consider a single position GSP with budget constraints and provide a characterization of equilibrium bidding strategies.

The rest of the paper is organized as follows. Section 2.2 presents the structure of keyword auctions with and without budget constraints. Section 2.3 characterizes the budget thresholds when an SNE can be sustained as an equilibrium with budget constraints. Section 2.4 discusses bidders’ equilibrium bidding behavior and the search engine’s revenue in a simple setting, and Section 3.5 concludes.

2.2 Model

In this section, we first briefly review the model in EOS (2007) and Varian (2007) and then introduce budget constraints.

2.2.1 Model without Budget Constraints

Suppose there is a set of bidders, \( N = \{1, \ldots, n\} \), and a set of positions, \( S = \{1, \ldots, s\} \), for a keyword of interest. Throughout this paper, we assume that \( n \geq s \) and the reservation price is zero. Let \( V = \{v_1, \ldots, v_n\} \) be the set of values, where \( v_i \) is bidder \( i \)'s per-click value, and \( C = \{c_1, \ldots, c_s\} \) be the set of CTRs, where \( c_j \) is the CTR of position \( j \). Order the positions such that \( c_1 > \cdots > c_s \) and let \( c_{s+1} = \cdots = c_n = 0 \).\(^9\) \( V \) and \( C \) are given exogenously and known to all bidders.

Let \( \beta = (b_1, \ldots, b_n) \) be a bidding profile. Under GSP, the bidder who submits the \( j \)th highest bid wins the \( j \)th highest position and pays the \((j+1)\)th highest bid per click. Suppose bidder \( j \) makes the \( j \)th highest bid. Then, his payoff is \( c_j(v_j - b_{j+1}) \). Varian (2007)

\(^9\)In practice, bidders may have different values for different positions (Yenmez, 2013) and CTRs may depend on which ads are shown in other positions (Jezierski and Segal, 2012). For simplicity, however, we assume throughout this paper that each bidders’ per-click value is independent of positions and CTRs depend only on positions, following EOS (2007) and Varian (2007).
defines a Nash equilibrium and a symmetric Nash equilibrium. A bidding profile $\beta$ is a *Nash equilibrium* (NE) if

$$
c_j(v_j - b_{j+1}) \geq c_l(v_j - b_{l+1}) \quad \forall \ l > j,
$$

$$
c_j(v_j - b_{j+1}) \geq c_l(v_j - b_l) \quad \forall \ l < j.
$$

(2.1)

We must also have $v_j \geq b_{j+1}$, or else bidder $j$ would lose money, and $b_j \geq b_{j+1}$ by the definition of GSP. A bidding profile $\beta$ is called a *symmetric Nash equilibrium* (SNE) if

$$
c_j(v_j - b_{j+1}) \geq c_l(v_l - b_{l+1}) \forall \ l,j.
$$

(2.2)

Denote the set of SNEs (without budget constraints) by $\mathfrak{B}^{SNE}$. A SNE has several attractive properties. First, any $\beta \in \mathfrak{B}^{SNE}$ is also a Nash equilibrium. Second, under SNE, the matching is assortative, that is, $v_1 \geq v_2 \geq \cdots \geq v_s$. Third, $(b_j)_{j \in \mathcal{N}}$ is a SNE bidding profile if and only if $(c_{j-1}b_j)_{j \in \mathcal{N}}$ is a Walrasian price profile. However, there is a continuum of SNE bids within upper and lower bounds. That is, for any $\beta \in \mathfrak{B}^{SNE}$, $b_j$ satisfies that for $2 \leq j \leq s + 1$,

$$
b_j^L \equiv \frac{(c_{j-1} - c_j)}{c_{j-1}} v_j + \frac{c_j}{c_{j-1}} b_{j+1} \leq b_j \leq \frac{(c_{j-1} - c_j)}{c_{j-1}} v_{j-1} + \frac{c_j}{c_{j-1}} b_{j+1} \equiv b_j^U,
$$

(2.3)

$$
b_1 > b_2, \text{ and } b_i = v_i \text{ for all } i > s + 1.
$$

Observe that $b_j^U$ and $b_j^L$ satisfy $c_{j-1}(v_{j-1} - b_j^U) = c_j(v_j - b_{j+1})$ and $c_j(v_j - b_{j+1}) = c_{j-1}(v_j - b_j^L)$, respectively. That is, $b_j^U$ is the bid at which bidder $j-1$ is indifferent between staying at position $j-1$ and moving down one position, and $b_j^L$ is the bid at which bidder $j$ is indifferent between staying at his current position and moving up one position. When we

$^{10}$Since $b_i \geq b_{i+1}$, we have $c_j(v_j - b_{i+1}) \geq c_j(v_j - b_i)$. Thus, the second line in (2.1) is satisfied.
recursively substitute lower bounds of the SNE, that is, substituting \( b_{j+1}^L \) into \( b_j^L \), \( b_{j+2}^L \) into \( b_{j+1}^L \) and so on, we have the smallest SNE bid, denoted by \( b_j^{LL} \), where

\[
b_j^{LL} = \frac{1}{c_{j-1}} \sum_{k \geq j} (c_{k-1} - c_k)v_k.
\]

Similarly, we also have the largest SNE bid, denoted by \( b_j^{UU} \), where

\[
b_j^{UU} = \frac{1}{c_{j-1}} \sum_{k \geq j} (c_{k-1} - c_k)v_{k-1}.
\]

It is not hard to show that \( b_j^{LL} \geq b_{j+1}^{LL} \) and \( b_j^{UU} \geq b_{j+1}^{UU} \) for all \( j \). The smallest SNE bids are the same as the locally envy-free equilibrium bids analyzed by EOS (2007), and it is well known that the smallest price for each position, \( c_{j-1} b_j^{LL} \), is the same as the payment for Vickery-Clarke-Groves mechanism.

### 2.2.2 Model with Budget Constraints

We now provide a simple model with budget-constrained bidders. Similar to the previous section, there is a set of bidders, \( \mathcal{N} \), and their valuations, \( \mathcal{V} \); a set of positions, \( \mathcal{S} \), and CTRs, \( \mathcal{C} \). Let \( \mathcal{B} = \{B_1, \ldots, B_n\} \) be the set of budgets each bidder possesses. The auction is conducted during the unit time. All bidders submit their bids only once, when the game starts at \( t = 0 \). For the given \( \mathcal{C} = \{c_1, \ldots, c_s\} \), there is a flow of clicks on each position, so the number of clicks on position \( j \) up to time \( t \) is \( c_j t \). We also assume that \( \mathcal{V}, \mathcal{B} \) and \( \mathcal{C} \) are given exogenously and known to all bidders.

Figure 2.1 illustrates the game structure via an example with three positions and four bidders. Suppose \( b_1 > b_2 > b_3 > b_4 \) so that bidders 1, 2 and 3 initially win positions 1, 2 and 3, respectively, and bidder 4 fails to win any position. In Figure 2.1(a), bidder 1 drops out at time \( t_1 \), and bidders 2 and 3 move up one position each. However, their per-click payments
are still $b_3$ and $b_4$, respectively. Bidder 2’s budget is so small that he drops out at $t_2$, and bidder 3 moves up one position again and stays there until $t_3$. Figure 2.1(b) illustrates the case that all bidders have large enough budgets so that they stay at their respective positions until the game ends. Note that the allocation in this case is the same as the case without budget constraints. That is, if each bidder has a sufficiently large budget, then the allocation in our model coincides with that in EOS (2007) and Varian (2007).

### 2.3 Budget Thresholds

In this section, we provide an answer to the question concerning under what budget level, a bidding profile $\beta \in \mathcal{B}^{SNE}$ satisfying (2.2) can be sustained as an equilibrium. In order to avoid a trivial case, we restrict our attention, throughout this section, to $\beta \in \mathcal{B}^{SNE}$ with $c_j b_{j+1} \leq B_j$ for all $j$. That is, we focus on the case that if no bidders wish to change their bids from the given $\beta$, then all bidders can stay at their assigned positions until the game ends. As described in the motivating example in Section 2.1, a bidder may deviate from $\beta$ by either increasing or decreasing his bid. In the former (latter) case, we call it an upward (downward) deviation.
Lemma 2.1. Consider any $\beta \in \mathcal{B}_{SNE}$ such that $c_j b_{j+1} \leq B_j$ for all $j \in \mathcal{N}$. There is no profitable upward deviation if and only if $B_j \geq c_j b_j$ for all $j \in \mathcal{N}$.

Proof. “If” part. Fix any $\beta \in \mathcal{B}_{SNE}$. Suppose $B_j \geq c_j b_j$ for all $j$. Bidder $j$’s current payoff is $c_j(v_j - b_j)$. Suppose bidder $j$ increases his bid from $b_j$ to some $b'_j$ such that

$$ b_1 > \cdots > b_{j-k} > b'_j > b_{j-k+1} > \cdots > b_{j-1} > b_{j+1} $$

so as to be assigned to position $j - k + 1$. Note that bidder $(j - k)$’s per-click payment becomes $b'_j$ and $B_{j-k} \geq c_{j-k}b_{j-k} \geq c_{j-k}b'_j$, hence bidder $j - k$ will stay at position $j - k$ until the game ends. Bidder $j$’s payoff from the deviation is at most $c_{j-k+1}(v_j - b_{j-k+1})$, if he stays at position $j - k + 1$ to the end of the game. Observe that

$$ c_j(v_j - b_{j+1}) \geq c_{j-k+1}(v_j - b_{j-k+1}) \geq c_{j-k+1}(v_j - b_{j-k+2}), $$

where the first inequality follows from (2.2) and the second inequality holds since $b_{j-k+1} \geq b_{j-k+2}$. Therefore, any upward deviation is not profitable for bidder $j$.

We defer the proof of the “only if” part to Appendix B.1. However, Figure 2.2 depicts how
an upward deviation can be profitable if there is a bidder $j$ such that $B_j < c_j b_j$. Suppose there are two positions and three bidders. Let $\beta \in \mathfrak{S}^{SNE}$ be a bidding profile such that $b_1 > b_2 > b_3$. Suppose $B_1 < c_1 b_1$ and consider bidders 1 and 2. Figure 2.2(a) depicts the allocation under the original bidding strategy, and Figure 2.2(b) depicts the allocation when bidder 2 increases his bid from $b_2$ to $b'_2 = b_1 - \varepsilon$. In the latter case, bidder 1 will drop out at time $t = \frac{B_1}{c_1 b_2} < 1$ (as $\varepsilon$ vanishes), and bidder 2 will then move up. Suppose further that bidder 2 can stay at position 1 until the end of the game after he moves up to that position. The shaded areas in the left- and right-hand panels represent the number of clicks that bidder 2 enjoys from the original bidding strategy and the upward deviation, respectively. Observe that bidder 2’s payoff is increased from $c_2(v_2 - b_3)$ to $[c_2 t + c_1(1 - t)](v_2 - b_3)$.

**Theorem 2.1.** If there is a bidder $j$ with $B_j < \hat{B}_j := c_j b_j^{LL}$, then no $\beta \in \mathfrak{S}^{SNE}$ is an equilibrium with budget constraints.

**Proof.** It follows immediately from Lemma 2.1 since for any $\beta \in \mathfrak{S}^{SNE}$, $b_j^{LL} \leq b_j$ for all $j \in \mathcal{N}$. ■

For a given bidding profile, Lemma 2.1 shows that in order to deter an upward deviation, each bidder’s budget must be larger than his bid times the CTR of his current position. Thus, if $B_j < c_j b_j^{LL}$ for some $j$, then no SNE sustains as an equilibrium with budget constraints. However, both Lemma 2.1 and Theorem 2.1 consider only upward deviations but not downward deviations. In the following example, we show that a bidder may find a profitable downward deviation even if $B_j \geq c_j b_j$ for all $j$.

**Example 2.1.** Let $c_1 = 10$, $c_2 = 5$, and $v_1 = 6$, $v_2 = 4$, $v_3 = 2$. The upper and lower bounds of SNE for bidders 2 and 3 are $(b_2^{UU}, b_3^{UU}) = (5, 4)$ and $(b_2^{LL}, b_3^{LL}) = (3, 2)$.

Suppose $B_2 = 27$, and $B_1$ and $B_3$ are arbitrarily large. Fix any $\beta = (b_1, b_2, b_3) \in \mathfrak{S}^{SNE}$. Note that $B_2 > c_2 b_2^{UU} = 25 \geq c_2 b_2$, so the condition in Lemma 2.1 is satisfied. Now consider
bidder 1. His current payoff at position 1 is $c_1(v_1 - b_2)$ (see Figure 2.3(a)). Suppose he lowers his bid from $b_1$ to $b_2 - \varepsilon$ so that bidder 1 is assigned to position 2 and bidder 2 is assigned to position 1. Bidder 1, however, moves up to position 1 at time $t = \frac{B_2}{c_1 b_2} < 1$ (as $\varepsilon$ vanishes) since $B_2 < c_1 b_2^{LL} = 30 \leq c_1 b_2$. His payoff from this deviation is $[c_2 t + c_1 (1 - t)](v_1 - b_3)$ (see Figure 2.3(b)). Unlike the upward deviation, there is a trade-off: When bidder 1 moves down to position 2, he incurs a loss due to the smaller number of clicks ($c_2 t + c_1 (1 - t) < c_1$); when he moves up to position 1, however, he pays less per click ($v_1 - b_3 > v_1 - b_2$). One can show that if all bidders use $b_j^{LL}$, then bidder 1’s payoff decreases from 30 to 22 by the deviation, but if all bidders use $b_j^{UU}$, then his payoff increases from 10 to 14.6. The reason is that in the former case, bidder 1 cannot recoup the initial loss after moving up to position 1 at time $t = \frac{B_2}{c_1 b_2^{LL}} = 0.9$. In the latter case, however, bidder 2’s drop-out time from position 1, $t = \frac{B_3}{c_2 b_3^{UU}} = 0.54$, is earlier than before, so bidder 1 has enough time to make up for his initial loss.

**Lemma 2.2.** Let $\hat{B}_j^2 := c_{j-1} b_j^{UU}$. Suppose there is a bidder $j$ such that $B_j < \hat{B}_j^2$, then $\beta = (b_j^{UU})_{j \in \mathcal{N}}$ is not an equilibrium with budget constraints.

**Proof.** Consider a bidding profile $\beta = (b_j^{UU})_{j \in \mathcal{N}}$. Suppose bidder $j - 1$ lowers his bid from
$b_{j-1}^{UU}$ to $b_{j-1}^{'} = b_{j}^{UU} - \varepsilon$. Then, bidder $j - 1$ is assigned to position $j$, and bidder $j$ is assigned to position $j - 1$. Since $B_j < \hat{B}_j^2$, bidder $j$ drops out from position $j - 1$ at time $t = \frac{B_j}{c_{j-1}b_{j-1}^{'}},$ and then bidder $j - 1$ moves up to position $j - 1$. Thus, bidder $(j - 1)$'s net payoff from the deviation is

$$[c_j t + c_{j-1}(1 - t)](v_{j-1} - b_{j+1}^{UU}) - c_{j-1}(v_{j-1} - b_j^{UU}) > c_j(v_{j-1} - b_{j+1}^{UU}) - c_{j-1}(v_{j-1} - b_j^{UU}) = 0,$$

where the inequality holds since $c_j t + c_{j-1}(1 - t) > c_j$ and the equality follows from (2.3). ■

We now provide budget thresholds under which any $\beta \in \mathcal{B}^{SNE}$ can be sustained as an equilibrium with budget constraints.

**Theorem 2.2.** Any $\beta \in \mathcal{B}^{SNE}$ is sustainable as an equilibrium with budget constraints if and only if $B_j \geq \hat{B}_j^2$ for all $j \in \mathcal{N}$.

**Proof.** “If” part. Consider any $\beta \in \mathcal{B}^{SNE}$. By Lemma 2.1, there is no profitable upward deviation, since $B_j \geq c_{j-1}b_j^{UU} > c_jb_j$ for all $j$. Now consider a downward deviation. Suppose bidder $j$ decreases his bid from $b_j$ to some $b_j^{'}$ such that

$$b_1 > \cdots > b_{j-1} > b_{j+1} > \cdots > b_{j+k} > b_j^{'} > b_{j+k+1},$$

so as to be assigned to position $j + k$. Then, bidder $j + k$ is assigned to position $j + k - 1$, bidder $j + k - 1$ is assigned to position $j + k - 2$ and so on. Note that bidder $(j + k)$'s per-click payment is $b_j^{'}$ and $B_{j+k} \geq c_{j+k-1}b_{j+k}^{UU} \geq c_{j+k-1}b_j^{'}.$. Thus, bidder $j + k$ will stay at position $j + k - 1$ until the game ends, and bidder $j$ will also stay at position $j + k$. Bidder $j$'s payoff from this deviation is $c_{j+k}(v_j - b_{j+k+1})$, which is smaller than the original payoff from position $j$ by the definition of SNE, (2.2). Therefore, any downward deviation is not profitable.
“Only if” part. Suppose, on the contrary, that there is a bidder \( j \) such that \( B_j < c_{j-1} b_j^{LU} \).

By Lemma 2.2, the bidding profile \((b_j^{LU})_{j \in N}\) is not an equilibrium with budget constraints, which contradicts the fact that any \( \beta \in B^{SNE} \) is an equilibrium with budget constraints. ■

Theorem 2.2 provides the necessary and sufficient conditions under which any \( \beta \in B^{SNE} \) can be an equilibrium with budget constraints. It would be trivial that if all bidders have sufficiently large amount of budgets, then no deviations are profitable. However, the proposition says that the budgets need not be extremely large as long as it is greater than \( \hat{B}^j_2 \), which is the same as the payment of one position above bidder \( j \)’s current position if the highest SNE bids were played.

2.4 Search Engine’s Revenue

In this section, we consider a simple environment in which two bidders with the same budget compete for a position. We study bidders’ equilibrium bidding strategies and the search engine’s revenue. Let \( v_i, i = 1, 2, \) be bidder \( i \)’s per-click value, where \( v_1 > v_2 \), and \( B \) be the bidders’ common budget. We normalize the CTR of the position by 1 for simplicity. We assume that bidders have complete information and that ties are broken in favor of a bidder with higher per-click value.\(^{11}\) We look for a Nash equilibrium in undominated strategies.

Note that without budget constraints, the one-position GSP coincides with the standard second-price auction. Thus, the search engine’s revenue is \( v_2 \) in this case. With budget constraints, however, the following example shows that the search engine’s revenue may be larger than \( v_2 \).

Example 2.2. Let \( v_1 = 8, v_2 = 2, \) and \( B = 4 \). Then, \( b_1 = b_2 = 5 \) is an equilibrium. The search engine’s revenue is 4.

\(^{11}\)Our tie-breaking rule can be justified as it produces a limit equilibrium of a game in which there is a minimum bid increment and a random tie-breaking rule is used.
In the example, both bidders submit the same bid, and bidder 1 is initially assigned to the position by the tie-breaking rule. Notice that bidder 2’s bid is higher than his value. That is, bidder 2 overbids to exhaust bidder 1’s budget so that bidder 1 drops out at time $t = \frac{B}{b_2} = 0.8$. Thus, bidder 1’s payoff is $t(v_1 - b_2) = 2.4$ and bidder 2’s payoff is $(1 - t)v_2 = 0.4$.\footnote{We assume that there is no reservation price, hence bidder 2 pays nothing after moving up to the position.} Observe that no bidder has an incentive to deviate. Clearly, bidder 2 has no incentive to change his bid since $b_1 > v_2$. For bidder 1, if he lowers his bid to some $b'_1 \in (B, b_2)$, then bidder 2 will drop out at time $t' = \frac{B}{b'_1} < t = \frac{B}{b_2}$. Hence bidder 1’s payoff from the deviation is $(1 - t')v_1$. Since $(1 - t')v_1 < (1 - t)v_1 = 1.6$, bidder 1 does not benefit from this deviation. The search engine’s revenue is the amount of bidder 1’s payment up to time $t$; that is, $b_2 t = B$.

Now we provide a formal result generalizing the above example. We show that in any Nash equilibrium with undominated strategies, the search engine’s revenue is at least $v_2$ unless $v_2 > B$. Therefore, the search engine may benefit from the presence of budget constraints.

Before proceeding, we make a couple of observations. First, any bidding profile $(b_i, b_j)$ such that $b_i > B \geq b_j$, $i, j = 1, 2$, is not an equilibrium because bidder $j$ can benefit by bidding $b'_j \in (B, b_i)$.\footnote{When $b_i > B \geq b_j$, bidder $j$ gets zero payoff. However, bidder $j$ can earn $(1 - \frac{B}{b'_j})v_j > 0$ by bidding $b'_j \in (B, b_i)$ because bidder $i$ will drop out at time $\frac{B}{b'_j} < 1$.} This implies that $b_1$ and $b_2$ must be greater or smaller than $B$ at the same time in equilibrium. Second, if $b_i > b_j \geq B$ in an equilibrium, we must have $b_j = b_i - \varepsilon$ for an arbitrarily small $\varepsilon > 0$ because this expedites bidder $i$’s dropping out time. By the tie-breaking rule, this implies that both bids are bunching to the same bid, denoted by $b$, and bidder 1 is initially assigned to the position. Thus, if $b \geq B$ is an equilibrium, it satisfies the follows:

$$\frac{B}{b}(v_1 - b) \geq \left(1 - \frac{B}{b}\right)v_1 \quad \text{and} \quad \left(1 - \frac{B}{b}\right)v_2 \geq \frac{B}{b}(v_2 - b).$$

The next proposition provides a characterization of equilibrium bidding strategies.
Theorem 2.3. Consider equilibrium in undominated strategies.

(i) Suppose \( B \geq v_1 > v_2 \). \((b_1, b_2) = (v_1, v_2)\) is a unique Nash equilibrium.

(ii) Suppose \( v_1 > B \geq v_2 \). \((b_1, b_2)\) is a Nash equilibrium if and only if either \( b_1 = b_2 \in [B, \frac{2v_1B}{v_1+B}] \) or \( B \geq b_1 \geq b_2 \geq v_2 \).

(iii) Suppose \( v_1 > v_2 > B \). \((b_1, b_2)\) is a Nash equilibrium if and only if \( b_1 = b_2 \in [\frac{2v_2B}{v_2+B}, \frac{2v_1B}{v_1+B}] \).

We defer the proof to Appendix B.2. In the first case, the budget constraints are not binding, so bidders’ bidding behavior is the same as that under the standard second-price auction. Hence, the search engine’s revenue is \( v_2 \). In the last case, the range of equilibrium bids follows from (2.4). Note that equilibrium bids are greater than \( B \), or else bidder 2 can benefit by bidding \( b_2' \in (B, v_2) \) so as to win the position. Thus, bidder 1 drops out at time \( t = \frac{B}{b_2} \), and so the search engine earns \( B < v_2 \).

The most interesting case is the second. Here, both bidders bid either above or below the budget at the same time. In the former case, the range of equilibrium bids follows from (2.4), again. Since \( b_2 \geq B \), bidder 1 drops out at time \( t = \frac{B}{b_2} \), and the search engine’s revenue is \( B \geq v_2 \). In the latter case, bidder 1 does not drop out during the game because \( B \geq b_2 \), hence the search engine’s revenue is simply \( b_2 \) which is greater than \( v_2 \) in undominated strategies. In sum, if \( v_1 > B \geq v_2 \), the search engine earns at least \( v_2 \).

The search engine’s revenue in each of three cases is summarized in the follow.

Theorem 2.4. In equilibrium with undominated strategies, the search engine’s revenue, denoted by \( R \), is

\[
R = \begin{cases} 
  v_2 & \text{if } B \geq v_1 > v_2, \\
  [v_2, B] & \text{if } v_1 > B \geq v_2, \\
  B & \text{if } v_1 > v_2 > B. 
\end{cases}
\]
2.5 Concluding Remarks

A generalized second-price auction is a standard format for selling online advertising positions on search engines. Previous literature has found a set of equilibria—symmetric Nash equilibria (SNEs)—in a model that does not consider budget constraints. In the current paper, we provide a simple model with budget constraints and show that bidders may have an incentive to deviate from an SNE. We characterize budget thresholds under which any SNE can or cannot be sustained as an equilibrium when there are budget constraints. We also show that in a simple setting, the search engine’s revenue with budget constraints may be larger than its revenue without such constraints.

There are several issues driven by budget constraints that are not addressed in the current paper. First, we have assumed that bidders’ budgets are exogenously given. However, the budget amount can be an important decision variable for the bidders. For example, a bidder may want to get a high position for a short time period for a special fire-sale. In that case, the bidder may submit a high bid and place a small budget amount. Second, a bidder may click competitors’ ads to deplete their budgets without any interest on their web sites, which is known as click fraud (Wilbur and Zhu, 2009). Last, in the presence of bidders’ budgets, a natural question is how to allocate bidders to positions. Goel et al. (2010) consider throttling algorithms that select a set of bidders who participate in the auction upon the arrival of a query.14 Their analysis, however, takes the bids and budgets as given and does not consider bidders’ strategic behavior. The results in Section 2.4 can be understood as an equilibrium analysis in a restrictive setting when the search engine does not use any throttling algorithm, and the analyses for other throttling algorithms are needed.

14 They consider three algorithms. The first is the non-throttling model, in which all bidders participate in the auction and a bidder is removed when his budget is exhausted, as in our model. The second is the strict model, in which the search engine may exclude some bidders with positive remaining budgets. Last, in the non-strict model, the search engine selects a set of bidders, whose budgets may be exhausted or not, and gives some clicks for free to those bidders who exhausted their budgets.
Chapter 3

Incentive and Sample Effects in Procurement of Innovation

Youngwoo Koh
3.1 Introduction

Firms and governments increasingly rely on procuring goods and services from outside sources. In particular, they often seek to procure innovations, and the innovative activities require firms to undertake investments. For instance, when the Department of Defense procures a weapon system, defense contractors often make R&D investments to produce prototypes and then participate in the procurement process. In the healthcare system, a procurer requires suppliers to undertake investments for developing medicines. In general, the quality of the innovation depends on firms’ investment effort level at the R&D stage. However, it is often the case that the realized quality of new research is not deterministic of the exerted effort, hence the realized quality is ex ante uncertain.

In the face of these problems, contests have served as a procurement scheme. Under a contest, the buyer has an ex post incentive to choose the supplier who offered her the highest net surplus, and this provides the firms with incentive to exert investment effort. Two popular contest mechanisms are first-price auction and fixed-prize tournament. In the auction, the buyer procures the innovation from a firm who offers the most favorable price-quality combination (called “score”). In the tournament, the prize is fixed by the buyer, and a supplier who develops the highest quality wins the prize. Intuitively, under an auction mechanism, firms have means of competition, since a low-quality firm can offer a high net surplus to the buyer by lowering its price. Hence, the buyer may prefer holding an auction over a tournament, since the former promotes more competition among the firms than the latter. However, if the randomness of quality realization is sufficiently large, then an auction may leave a high rent to firms, so a tournament may be preferred by the buyer.

Besides the choice of a procurement mechanism, the buyer also faces a nontrivial problem of selecting the number of participants. Since the buyer procures from only one supplier, if too many firms participate in the procurement process, then they may be discouraged from
expending their sunk investments, because each of them has a small chance of winning. We call this an *incentive effect*.\(^1\) This suggests that shortlisting the number of participants—often to two—can be optimal (Taylor, 1995; Fullerton and McAfee, 1999; Che and Gale, 2003). On the other hand, due to the randomness of quality realization, the buyer may benefit by inviting many firms, since the chance of having a higher quality increases as the number of participants increases. We call this a *sampling effect*. Intuitively, if the randomness is negligible, inviting only two firms is optimal, since it is enough to make them compete with each other, and thus firms have incentive to put investment efforts. However, for sufficiently large randomness, the sampling effect may dominate the incentive effect, so the buyer may wish to invite as many firms as possible.

In this paper, we consider two contest mechanisms, a first-price auction and a fixed-prize tournament, and investigate how the incentive and sampling effects operate under each mechanism. To isolate the trade-off between the two effects, we further restrict our attention to the environment in which (i) firms invest nonmonetary efforts, (ii) the level of effort and the resulting quality of the innovation are unverifiable, (iii) the buyer cannot extract up-front payments from the firms, and (iv) there is no outside market except selling to the buyer.

When the quality is verifiable, the terms of contracts can be made contingent on the realized quality of innovation. Any optimal mechanism selects the firm who can deliver the highest quality at a minimal cost (Laffont and Tirole, 1986; McAfee and McMillan, 1987; Riordan and Sappington, 1987). If the cost associated with innovative activities is observable, one could imagine various reimbursement schemes (Rogerson, 2003; Chu and Sappington, 2007). However, these prevailing theories may not work well for innovations. First of all, if investments take the form of nonmonetary effort or opportunity costs, the investment costs may not be observable. It is also often the case that the quality of innovation provided by

\(^1\)A number of papers also examine firms’ incentives to invest in cost reduction (Tan, 1992; Piccione and Tan, 1996; Arozamena and Cantillon, 2004).
the firms is unverifiable. A third party, such as a court, may not be able to discern the quality through an audit. With the unverifiable quality, the buyer may use a simple option contract that requires suppliers to pay an up-front fee (Taylor, 1993). However, this may not be feasible if the firms have limited liability or liquidity constraints, so the buyer cannot charge substantial entry fees. Patent systems may not be helpful when the innovation results do not have an immediate commercial value so that there is no outside market except selling it to the buyer.

Under such an environment, we develop a simple model. There is a set of potential suppliers, who are ex ante identical. The buyer announces a procurement mechanism, either first-price auction or fixed-prized tournament, and selects a set of participants (i.e., the number of participants) in the procurement process. Any participating firm exerts investment effort and pays the cost of it. And then it draws quality from a distribution which depends on the effort level. After observing its own quality (which is also observable by the buyer, but not by the other firms), each firm asks a price if the auction is in use. In the fixed-prize tournament, each firm asks the same price, namely the prize fixed by the buyer. Finally, the buyer selects the firm who offers the highest net surplus.

We first consider the first-price auction. We show that when the randomness is large enough, there is a symmetric pure-strategy equilibrium in which all firms put the same level of investment. Each firm’s price bidding behavior resembles that of the standard first-price auction: Each firm’s realized quality serves as a “type” of the firm, and its net surplus offer to the buyer is equivalent to the firm’s “bid.” We also show that it is optimal for the buyer to invite as many firms as possible in this case, even though individual firm’s investment level is decreasing in the number of participants. That is, the sample effect dominates the incentive effect.

However, when the randomness is small, there is no symmetric pure-strategy equilibrium. Unfortunately, we are not able to fully characterize (mixed-strategy) equilibria in this case.
Instead, we show that the distribution of a mixed-strategy equilibrium converges to some distribution as the randomness vanishes, and provide a characterization of the limit distribution. Based on it, we show that inviting only two firms is optimal for the buyer at the limit. In Section C.1, we provide an example of a symmetric mixed-strategy equilibrium with two firms when the randomness is moderate. This connects the pure-strategy equilibrium and the limit case.

Next, we consider the fixed-prize tournament. We show that similar to the auction case, when the randomness is large, it is optimal for the buyer to invite many firms. More interestingly, the tournament outperforms the auction in this case. In the tournament, the prize is optimally set by the buyer, which is shown to be zero so that the buyer entirely relies on the sampling effect. We also consider the limit case of the tournament and show that inviting only two firms is optimal for the buyer. However, the auction outperforms the tournament in this case, since the auction promotes competition more than the tournament.

Two prominent works on innovation contests are Fullerton et al. (2002) and Che and Gale (2003). They suggest that the buyer prefers the first-price auction to the fixed-prize tournament. Fullerton et al. (2002) compare the auction and the tournament in a stochastic innovation technology by adopting Taylor (1995)’s model. In Taylor (1995), each firm decides the number of periods in which it draws a quality from a distribution and pays a fixed cost. At the end of the last period, the firm who gets the highest quality wins the prize.

Che and Gale (2003) consider a deterministic innovation technology. Under a general framework, they show that the first-price auction with two firms is optimal among various contest mechanisms (including the tournament). A crucial difference between our model and theirs is the randomness on the quality realization. In our model, when the randomness is large, it is beneficial for the buyer to invite many firms and to use the tournament. Our analysis of the convergence of the mixed strategies and its characterization at the limit also shows that the auction with two firms outperforms the auction with many firms and the
tournament. In this sense, our research complements their work.

Another notable paper that compares the auction and the tournament is by Schöttner (2008). She considers a situation in which two firms compete with each other, and a firm’s quality stochastically depends on its investment. She shows that if the innovation technology involves large randomness, the buyer prefers the tournament to the auction. Thus, our work is on the same line with hers. However, we do not restrict the number of firms; rather, it is endogenously determined, and hence we are able to capture the incentive effect, which is not in Schöttner (2008).

The remainder of the paper is laid out as follows. Section 3.2 introduces the model. Section 3.3 considers the first-price auction. Section 3.3.1 analyzes the symmetric pure-strategy equilibrium when the randomness is large, and Section 3.3.2 analyzes mixed-strategy equilibria when the randomness is small and their convergence. Section 3.4 considers the fixed-prized tournament. The comparison of the buyer’s payoffs under two mechanisms is also discussed there. Section 3.5 concludes the paper. Proofs are provided in the Appendix unless stated otherwise. The Appendix also provides a constructive example of a mixed-strategy equilibrium with two firms when the randomness is moderate.

3.2 Model

A buyer wishes to procure an innovation from a set of (potential) suppliers, \( \mathcal{M} = \{1, \ldots, N\} \). The innovation requires an (nonmonetary) investment \( x \in [0, \infty) \) by the firms with a sunk cost \( \psi(x) = \frac{1}{2}x^2 \). The quality of an innovation is summarized by \( q \), which depends on the firms’ investment level. More specifically, if a firm invests \( x \), then its quality \( q \) is drawn from a uniform distribution \( U[x + a - \delta, x + a + \delta] \), denoted by \( F(q|x) \), for a given \( \delta \) and for some fixed \( a \geq \delta \). Notice that \( \delta \) captures the randomness of quality, and \( x \) moves the support of

---

\(^2\)Since \( a \geq \delta \), \( q \) is nonnegative regardless of the investment level \( x \geq 0 \).
the distribution. Thus, for a given $\delta$, if a firm puts a higher $x$, then it has a higher probability of getting a high quality. The assumption on functional forms is somewhat restrictive, but it enables us to derive a closed-form solution both in the auction and the tournament, and therefore we can clearly compare when one of the sample and incentive effects prevails.

In order to procure an innovation, the buyer first selects a set of participating firms in the procurement process, denoted by $\mathcal{N} = \{1, \ldots, n\} \subseteq \mathcal{M}$, where $n \geq 2$. After each firm gets its quality $q$, it is observed also by the buyer (but not by the other firms). The buyer solicits prices from the firms in $\mathcal{N}$. Each quality–price combination is called score $s := q - p$.

The buyer selects at most one firm, and if a firm wins, then it is paid the price it has submitted. Thus, the winning firm’s payoff is $p - \psi(x)$, and losing firms earn $-\psi(x)$. Given the unverifiable nature of the innovation, the buyer chooses the firm that offers the highest score. That is, firm $i$ wins if

$$q_i - p_i > \max \left\{ \max_{j \in \mathcal{N}\setminus\{i\}} \{q_j - p_j\}, 0 \right\}.$$ 

In the first-price auction (also called first-score acution, Che, 1993), firm $i$ chooses its price $p_i$. In the fixed-prize tournament, $p_i$ is given by some $P \geq 0$ for all $i \in \mathcal{N}$, which is determined by the buyer, so $s_i = q_i - P$. Therefore, when the tournament is used, there is no bidding stage and a firm that produces the highest quality wins.

### 3.3 First-Price Auction

Given a set of participants, $\mathcal{N}$, and a fixed $\delta > 0$, any equilibrium consists of a pair of investment level and score offering $(x_i, s_i)_{i \in \mathcal{N}}$. Let $\mathcal{X}_i^\delta$ be the support of firm $i$’s investment, where $\underline{x}_i := \inf \{\mathcal{X}_i^\delta\}$ and $\bar{x}_i := \sup \{\mathcal{X}_i^\delta\}$. When $\underline{x}_i = \bar{x}_i$, firm $i$ adopts a pure strategy in investment. Let $G_i^\delta$ denote the cumulative distribution function of firm $i$’s net surplus offer,
s_i = q_i - p_i, and $S^\delta_i$ be the support of it. The aggregate support is denoted by $S^\delta = \cup_{i \in N} S^\delta_i$.

Firm $i$’s expected payoff when it chooses investment level $x \in \mathcal{X}^\delta_i$ is given by

$$
\Pi_i(x) = \int_{x+a}^{x+a+\delta} (q_i - s_i) G^\delta_{-i}(s_i) \, dF(q|x) - \psi(x),
$$

where $G^\delta_{-i}(\cdot) = \prod_{j \in N \setminus \{i\}} G^\delta_j(\cdot)$ is the probability that firm $i$ wins if it offers $s_i$ to the buyer, and $q_i - s_i = p_i$ is the firm’s revenue conditional on winning.

### 3.3.1 Symmetric Pure-Strategy Equilibrium

We first consider a symmetric pure-strategy equilibrium and conditions under which such an equilibrium exists. We then derive the optimal number of participants for the buyer.

Let $n$ be the number of participating firms. Suppose there is a symmetric investment level $x$; that is, $x_i = \bar{x} \equiv x$ for all $i$. Suppose further that there is a symmetric price offering strategy $p(q)$ such that $s(q) = q - p(q)$ is strictly increasing.

A firm’s optimal price bidding can be found by

$$
p \in \arg \max_p p \, \text{Prob}(q_i - p(q_i) \geq q_j - p(q_j) \, \forall j \in N \setminus \{i\}).
$$

Note that the probability of winning is $\prod_{j \in N \setminus \{i\}} \text{Prob}(q_j \leq s^{-1}(s_i)) = F^{n-1}(s^{-1}(s_i)|x)$. That is, the probability of winning, $G^\delta_{-i}(s_i)$, is represented by the quality distribution, $F^{n-1}(q|x)$, by the symmetry.

Since $p = q - s$, the firm’s problem is reduced to finding an optimal score offering strategy.

$$
q \in \arg \max_{\tilde{q}} \left( q - s(\tilde{q}) \right) F(\tilde{q}|x)^{n-1}.
$$
Maximizing the objective function yields

\[ s(q) = q - \int_{x+a-\delta}^{q} \left[ \frac{F(t|x)}{F(q|x)} \right]^{n-1} dt. \] (3.1)

It is immediately seen that \( s \) is strictly increasing in \( q \), and the price bidding strategy is

\[ p(q) \equiv q - s(q) = \int_{x+a-\delta}^{q} \left[ \frac{F(t|x)}{F(q|x)} \right]^{n-1} dt. \] (3.2)

From (3.1), it can be interpreted that \( q \) is a firm’s “type,” \( s(q) \) is its “bid” in the usual first-price auction, and \( p(q) \) is the degree of “shading,” which depends on the number of firms. If \( x \) is fixed independently of \( n \), then the usual interpretation of “shading” works: As \( n \) increases, it approaches to 0, hence \( s(q) \) approaches \( q \). However, as it will be shown below, the investment level depends on the number of firms.

Now, consider the investment strategy. Suppose all firms except \( i \) invest the same level \( x \). Then, firm \( i \)'s expected payoff from choosing \( x_i \) is

\[
\Pi_i(x_i) = \int_{x_i+a-\delta}^{x_i+a+\delta} p(q)F(q|x)^{n-1} dF(q|x) - \psi(x_i)
= \int_{x_i+a-\delta}^{x_i+a+\delta} F(q|x)^{n-1}(1 - F(q|x_i)) \, dq - \psi(x_i),
\]

where the last equality follows from (3.2) and the integration by parts. Note that \( F(q|x)^{n-1} \) is the probability that other firms except firm \( i \) have qualities smaller than \( q \), and \( 1 - F(q|x_i) \) is the probability that firm \( i \) has a quality greater than \( q \).

Maximizing \( \Pi_i(x_i) \) and using the fact that \( F(\cdot|x) \) is uniform and \( \psi(\cdot) \) is quadratic, we

\[ p(q) = q - \int_{x+a-\delta}^{q} \left[ \frac{F(t|x)}{F(q|x)} \right]^{n-1} dt. \] (3.2)

\[\text{If the qualities are observable, then the equilibrium bidding strategy is } p(q_i) = q_i - q_j \text{ if } q_i > q_j \geq \max_{k \in \mathcal{N} \setminus \{i, j\}} \{q_k\}. \] This resembles quality competition in a Bertrand game (Riordan, 2010).
\[
x^* = \frac{1}{n} \quad \text{and} \quad \Pi^* = \frac{2\delta}{n(n+1)} - \frac{1}{2n^2}.
\]

In order to see whether this is indeed an equilibrium, we check firm \( i \)'s deviations from \( x_i \). Let all firms in \( \mathcal{N} \) except \( i \) employ the same investment strategy \( x^* \). Suppose firm \( i \) slightly decreases its investment level such that \( x^* - 2\delta < x_i < x^* \) so as to \( x^* + a - \delta < x_i + a + \delta \) (see 3.1(c)), called local downward deviation (ldd). It is clear that if \( q_i < x^* + a - \delta \), then firm \( i \) would be defeated by other firms. Thus, firm \( i \)'s payoff from the deviation is

\[
\Pi^{ldd}(x_i; x^*) = \int_{x^* + a - \delta}^{x_i + a + \delta} F(q|x^*)^{n-1}(1 - F(q|x_i)) \, dq - \psi(x_i).
\]

Consider now local upward deviation (lud) in which firm \( i \) slightly increases its investment level such that \( x^* < x_i < x^* + 2\delta \) so as to \( x_i + a - \delta < x^* + a + \delta \) (see 3.1(d)). Suppose \( q_i > x^* + a + \delta \). Then, firm \( i \) knows that its quality is the highest one. The firm would like to reduce the net surplus offer to the buyer, because there is no loss of probability of winning by doing so but it increases the price, \( p_i = q_i - s_i \). Thus, firm \( i \) would offer the score as if the other firms had attained the highest possible quality, that is, \( s(q_i) = s(x^* + a + \delta) \). Inserting \( x^* + a + \delta \) into (3.1) and after some rearrangement, we have firm \( i \)'s payoff from the deviation, which is given by

\[
\Pi^{lud}(x_i; x^*) = \int_{x_i + a - \delta}^{x_i + a + \delta} \left( \int_{x^* + a - \delta}^{q} F(t|q)^{n-1} \, dt \right) \, dF(q|x_i) + \int_{x^* + a + \delta}^{x_i + a + \delta} p(q) \, dF(q|x_i) - \psi(x_i),
\]
Lemma 3.1. Suppose $\delta \geq \frac{1}{2}$. Then, $\Pi^{ldd}_i$ and $\Pi^{lud}_i$ are strictly increasing and strictly decreasing in $x_i$, respectively. Both $\Pi^{ldd}_i$ and $\Pi^{lud}_i$ attain the unique maximum value $\Pi^*$ at $x_i = x_i^*$. 

We now consider global deviations in which $|x_i - x^*| \geq 2\delta$. Suppose that firm $i$ decreases its investment level such that $x_i \leq x^* - 2\delta$ (global downward deviation, gdd). Then, the highest surplus that the firm can offer to the buyer is $s_i = x_i + a + \delta$ by submitting $p_i = 0$, but the other firms always can offer $s_j = x^* + a - \delta \geq s_i$. Therefore, firm $i$ would be defeated with probability one, hence it would get a zero payoff; that is, $\Pi^{gdd}(x_i, x^*) = 0$.

If firm $i$ increases its investment level such that $x_i \geq x^* + 2\delta$ so as to $x^* + a + \delta \leq x_i + a - \delta$ (global upward deviation, gud), then it would win with probability one. Thus, firm $i$ offers a surplus to the buyer as if all other firms obtain the best possible quality, i.e., $s_i = s(x^* + a + \delta)$. Hence, the firm’s payoff is

$$\Pi^{gud}(x_i, x^*) = \int_{x_i + a - \delta}^{x_i + a + \delta} q dF(q|x_i) - (x^* + a + \delta) + \int_{x^* + a - \delta}^{x^* + a + \delta} F(t|x^*)^{n-1} dt - \psi(x_i).$$

Lemma 3.2. Suppose $\delta \geq \frac{1}{2}$. Then, $\Pi^* > \Pi^{gud}_i(x_i; x^*)$ for any $x_i \geq 0$.

The proofs of Lemmas 3.1 and 3.2 are found in ???. From the two lemmas, it is clear that there exists a symmetric pure-strategy equilibrium if $\delta \geq \frac{1}{2}$, and furthermore, it is the unique symmetric equilibrium employing a pure strategy. Figure 3.3 depicts a firm’s payoff function.
in the symmetric pure-strategy equilibrium when $\delta = 1$. In the left panel, the number of participants is set to be 2, and in the right panel, it is 5.\(^4\) In each panel, the graph of the payoff function is divided into three regions: The solid line in the left-hand side is the payoff from the local downward deviation ($\Pi^{ldd}$), the dashed line in the middle is from the local upward deviation ($\Pi^{lud}$), and the solid line in the right-hand side is from the global upward deviation ($\Pi^{gud}$).

In the equilibrium, the expected surplus offer to the buyer by a firm with quality $q$ is

$$m(q) := \text{Prob}(\text{win}) \times s(q) = F(q|x^*)^{n-1} \times \frac{1}{F(q|x^*)^{n-1}} \int_{x^*+a-\delta}^{q} t \, dF(t|x^*)^{n-1}.$$  

Hence, the buyer’s payoff for a given $n$ and $\delta \geq \frac{1}{2}$ is

$$R(n, \delta) = n \int_{x^*+a-\delta}^{x^*+a+\delta} m(q) \, dF(q|x^*) = (a + \delta) - \frac{4\delta}{n+1} + x^*. \quad (3.3)$$

Observe that

$$R(n + 1, \delta) - R(n, \delta) = \frac{4n\delta - n - 2}{n^3 + 3n^2 + 2n} \geq 0$$

---

\(^4\)In the figures, we set $a = 0$ for convenience.
if and only if $\delta \geq \frac{1}{4} + \frac{1}{2n}$, which always holds since $n \geq 2$, and the inequality is strict unless $n = 2$ and $\delta = \frac{1}{2}$. Therefore, the buyer benefits by inviting all firms from the set of potential suppliers. We summarize these observations in the following proposition.

**Theorem 3.1.** Suppose $\delta \geq \frac{1}{2}$. For a given $n$, the unique symmetric pure-strategy equilibrium is such that $x^* = \frac{1}{n}$ and $s(q)$ is given by (3.1). It is optimal for the buyer to set $N = M$.

The above result shows that with a sufficient amount of uncertainty, inviting many firms is optimal for the buyer; that is, the sampling effect dominates the incentive effect. This feature stands in a sharp contrast to Che and Gale (2003). In their model, there is no randomness about quality realization ($\delta = 0$). So, no pure-strategy equilibrium exists and inviting only two firms is optimal.

Note that if the investment level does not depend on $n$, then inviting more firms increases the buyer’s payoff, clearly. However, since $x^*$ is decreasing with $n$ in equilibrium, the overall effect is less clear a priori. The assumptions of uniform distribution and quadratic cost function help us disentangle the sampling and the incentive effects on the buyer’s payoff. From (3.3), $R(n, \delta)$ can be decomposed by $R^S(n) := \delta - \frac{4\delta}{n+1}$, which comes from the sampling effect, and $R^I(n) := x^*$, which comes from the incentive effect. Observe that

$$\Delta R^S(n) := R^S(n + 1) - R^S(n) = \frac{4\delta}{(n + 1)(n + 2)}$$

is the “marginal benefit” inviting one more firm, and

$$\Delta R^I(n) := \left|R^I(n + 1) - R^I(n)\right| = \frac{1}{n(n + 1)}$$

---

5Since $\delta = 0$, $q_i = x_i + a$. If firm $i$ offers $s_i > 0$ with probability 1, then firm $i$ must win with positive probability in an equilibrium because of the sunk investment. Thus, there is no $j \in N \setminus \{i\}$ such that $s_j$ is slightly below $s_i$, because this is dominated by offering slightly more than $s_i$. Then, $s_i$ is dominated by $s'_i = s_i - \epsilon$. 
is the “marginal cost” of it. When $\delta = \frac{1}{2}$ and $n = 2$, those two are the same, but if either $\delta > \frac{1}{2}$ or $n > 2$, then the former exceeds the latter. Therefore, even though the investment level decreases in $n$, the buyer is better off by inviting many firms.

### 3.3.2 Mixed-Strategy Equilibria

In this section, we consider the case that $\delta < \frac{1}{2}$. As shown in the previous section, there is no symmetric pure-strategy equilibrium in this case. Although we are not able to fully characterize mixed-strategy equilibria in this case, we derive some necessarily conditions and show the convergence of equilibria as $\delta$ vanishes, using those conditions. We also provide a characterization of equilibrium at the limit.

Consider first the bidding stage for a given $q$ and $x$. After getting a quality, firm $i$’s problem is to choose its price, $p_i$, or equivalently its score, $s_i(q_i) = q_i - p_i(q_i)$. Thus, the firm’s problem in the bidding stage is

$$\max_{s_i} (q_i - s_i) G_{-i}^\delta(s_i).$$

Note that we must have $q_i \geq s_i$, or else the firm would risk winning at negative payoff. Recall that $S_i^\delta$ is the support of $s_i$ and $S^\delta = \cup_{j \in \mathcal{N}} S_j^\delta$ for a given $\delta > 0$. We begin with the following observations, whose proofs are in ??.

**Lemma 3.3.** (i) For any $i \in \mathcal{N}$ and any $0 < s \in S_i^\delta$, there exists $j \in \mathcal{N} \setminus \{i\}$ such that $s \in S_j^\delta$.

(ii) $S^\delta$ is a connected interval.

(iii) For any $i \in \mathcal{N}$, $G_{-i}^\delta(\cdot)$ is continuous and strictly increasing in $s \in S_i^\delta$.

(iv) For any $s_i \in \text{int}(S_i^\delta)$, $s_i$ is strictly increasing in $q_i$.

The first part of the lemma states that at least two firms offer the same score in $S^\delta$, and
the second part shows that there is no gap in $S^\delta$. The third part states that the winning probability of a firm in $N$ is continuous and strictly increasing in its score offering.\footnote{Since the investment is sunk, the properties are similar to equilibria in all-pay auctions (Baye, Kovenock, and de Vries, 1996).} Finally, the last part of the lemma shows that the score function is strictly increasing in the quality.\footnote{Since $\pi_i(s_i; q_i)$ satisfies strict single-crossing property, by the Monotone Selection Theorem (Migrom and Shannon, 1994), $s_i$ is nondecreasing. Furthermore, the differentiability of $G^\delta_{-i}$ implies that $s_i$ is strictly increasing in $q_i$ (Edlin and Shannon, 1998).}

We next show that each firm’s score function is drawn from an interval which is bounded at most of order $\delta$. Recall that for any $x \in X_i^\delta$, $q_i \in [x + a - \delta, x + a + \delta]$. It is convenient to parameterize $q$ and $s$ in terms of $x$. Denote $q(x) := x + a - \delta$ and $\overline{q}(x) := x + a + \delta$. Likely, let $\underline{s}(x) := s(q(x))$ and $\overline{s}(x) := s(\overline{q}(x))$.

**Lemma 3.4.** For any $x \in X_i^\delta$, there is an interval of score offering, $S^\delta_i(x) = [\underline{s}(x), \overline{s}(x)]$. Moreover, $\overline{s}(x) - \underline{s}(x) \leq \delta M$ for some $M > 0$.

The proof of the lemma is found in ???. We now turn our attention to limit properties of equilibria. To get the limit distribution of $G^\delta_{-i}$, we derive upper and lower bounds of it and take the limit of those bounds. Let $S^0$ be the support of score offering at the limit of $\delta = 0$. (The characterization of $S^0$ will be established below.) For a given $s \in S^0$, let $I(s) := \{i \in N \mid s_i = s\}$ be the set of firms whose score offering is the same as $s$.

**Theorem 3.2.** Fix any $s \in S^0$. For any sequence of equilibria along $s$ for each $\delta > 0$, $G^\delta_i(s)$ converges to the same distribution $G^0_i(s)$ as $\delta$ approaches to 0 for all $i \in I(s)$.

**Sketch of Proof.** The proof involves several steps. We first characterize $S^0$, the limit support of $S^\delta$. We then show that $G^\delta_{-i}$ is bounded above and below. Using this, we derive the limiting distribution of $G^\delta_{-i}$. Finally, we show the convergence of $G^\delta_i$.

**Step 1.** We first show that $S^0 = [a, \frac{1}{2} + a]$. To do this, suppose first that firm $i$ wins with
probability one. Then,
\[ 1 = \arg \max_{x_i} \left\{ \int_{x_i + a - \delta}^{x_i + a + \delta} q \, dF(q | x_i) - \psi(x_i) \right\} \]
is the efficient investment level for the firm. Let \( s^* = 1 + a - \psi(x^*) = \frac{1}{2} + a \) be the associated score offer to the buyer. Clearly, \( s^* \) is the maximum surplus a firm can profitably offer to the buyer if it wins and \( \delta = 0 \). Using this, one can show that for a sufficiently small \( \delta > 0 \), the supremum of \( S^\delta \), denoted by \( \bar{s} \), is in the \( \delta \)-neighborhood of \( \frac{1}{2} + a \), i.e., \( \bar{s} \in [\frac{1}{2} + a - \delta, \frac{1}{2} + a + \delta] \).
Similarly, \( \underline{s} \in [a - \delta, a + \delta] \), where \( \underline{s} \) is the infimum of \( S^\delta \). See Lemma C.2 in ?? for a formal statement and proof. Note that this implies that as \( \delta \) decreases, \( S^\delta \) approaches \( S^0 \equiv [a, a + \frac{1}{2}] \) since \( S^\delta \) is an interval by Lemma 3.3-(ii).

**Step 2.** In this step, we find the limit distribution of \( G_{-i}^\delta \). Observe that for a fixed \( s \in \text{int}(S^0) \), Step 1 implies that there exists \( \hat{\delta} > 0 \) such that \( s \in S^\delta_i \) for all \( \delta < \hat{\delta} \). Thus, for any selection of equilibrium along \( s \) for each \( \delta \), there always exists a subsequence such that the same firm, say \( i \), is repeated in the sequence. For the fixed \( s \in \text{int}(S^0) \), let
\[ \begin{align*} X^\delta_i(s) := \{ x \in X^\delta_i \mid s \in S^\delta_i(x) \} \end{align*} \]
be the set of investment level of firm \( i \) such that \( s \in S^\delta_i(x) \) for given \( \delta > 0 \).

Next, for any \( x \in X^\delta_i(s) \), let \( s(q) \) be the optimally chosen score for each \( \delta > 0 \). Since \( s, s(q) \in S^\delta_i(x) \) and \( \bar{s}(x) - s(x) \leq \delta M \) (by Lemma 3.4), it follows that
\[ [q - s(q)] G^\delta_{-i}(s(q)) \leq [q - (s - M\delta)] G^\delta_{-i}(s + \delta M) \] (3.4)
for all \( q \in [q(x), \bar{q}(x)] \).

We now derive upper and lower bounds of \( G^\delta_{-i} \). For any \( x \in X^\delta_i(s) \), the firm’s payoff is
bounded above by (3.4); that is,
\[
\int_{x+a-\delta}^{x+a+\delta} (q-s(q)) G_{-i}^\delta(s(q)) \, dF(q|x) - \psi(x) \leq \int_{x+a-\delta}^{x+a+\delta} (q+\delta M - s) G_{-i}^\delta(s+\delta M) \, dF(q|x) - \psi(x).
\]

Suppose now that the firm increases its investment level by \( \varepsilon > 0 \). Then, we have
\[
\int_{x+a-\delta}^{x+a+\delta} (q-s(q)) G_{-i}^\delta(s(q)) \, dF(q|x) - \psi(x) \geq \int_{x+\varepsilon+a-\delta}^{x+\varepsilon+a+\delta} (q-s) G_{-i}^\delta(s+\delta M) \, dF(q|x+\varepsilon) - \psi(x+\varepsilon),
\]
where the inequality follows from the optimality of \( s(q) \).

Combining these two and using the fact that \( F(\cdot|x) \) is the uniform distribution, we have
\[
G_{-i}^\delta(s + \delta M)[\varepsilon - \delta M] \leq \psi(x + \varepsilon) - \psi(x), \tag{3.5}
\]
This serves as an upper bound of \( G_{-i}^\delta \). Likely, we also have a lower bound of \( G_{-i}^\delta \), which is given by
\[
G_{-i}^\delta(s + \delta M)[\delta M + \varepsilon] \geq \psi(x) - \psi(x - \varepsilon). \tag{3.6}
\]

Now, by taking limit of (3.5) and (3.6), we have the limit of \( G_{-i}^\delta \). Lemma C.3 in ?? shows that for any \( s \in \text{int}(S^0) \), we have \( G_{-i}^0(s) = \psi'(x^0) \), where \( x^0 \) is the limit of \( x \in X_{-i}^\delta(s) \).

**Step 3.** Finally, we show the convergence of \( G_{i}^\delta \). Note that Lemma 3.3-(i) implies that there are at least two distinct firms in \( I(s) \). Hence, for any \( i, i' \in I(s) \subseteq N \), we have
\[
\frac{G_{-i}^0(s)}{G_{-i'}^0(s)} = \frac{\prod_{j \in I(s) \setminus \{i\}} G_j^0(s)}{\prod_{j \in I(s) \setminus \{i'\}} G_j^0(s)} = \frac{G_{i'}^0(s)}{G_i^0(s)}.
\]
Since \( G_{-i}^0(s) = G_{-i'}^0(s) = \psi'(x^0) \) by Lemma C.3, this shows that \( G_{-i}^\delta(s) \) and \( G_{i'}^\delta(s) \) converge
to the same distribution at the limit, denoted by $G^0(s)$. Therefore,

$$
\psi'(x^0(s)) = G^0_{-1}(s) = \prod_{j \in I(s), j \neq i} G^0_j(s) = G^0(s)^{|I(s)|-1}.
$$

Thus, we have $G^0(s) = \psi'(x^0(s))^{|I(s)|-1}$.  

Observe that if the buyer has invited only two firms, then $|I(s)| = 2$ for any $s \in S^0$ (by Lemma 3.3-$(i)$), hence there is a unique symmetric equilibrium in this case. With more than two firms, however, there may be an asymmetric equilibrium.  

Theorem 3.3. At the limit, inviting only two firms yields (at least weakly) higher payoff to the buyer than inviting more than two firms.

Proof. Observe that for any $s \in S^0$,

$$
G^0(s)^{|I(s)|} = \psi'(x^0(s))^{|I(s)|-1} \geq \psi'(x^0(s))^2 = G^0(s)^2,
$$

where the inequality holds since $\psi'(x^0(s)) = G^0(s)^{|I(s)|-1} \in [0, 1]$ and $|I(s)| \geq 2$. Notice that the LHS of (3.7) is the cumulative distribution function of the first-order statistic of payoff with arbitrary number of firms; and the right-hand of (3.7) is that with two firms. Since the latter first-order stochastically dominates the former, the buyer collects the highest payoff by inviting only two firms at the limit.

The optimality of inviting two firms is intuitive. As the randomness diminishes, the gain from inviting many firms, i.e., the sampling effect, becomes negligible but the incentive effect prevails. Hence, to promote the investment level of the firms, the buyer invites only

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8 $|I(s)|$ can be different from $|I(s')|$ for some $s \neq s'$.  

two firms. Although Theorem 3.3 does not describe firms’ equilibrium investment strategies, they can be derived from the score offering strategies.\textsuperscript{9} Let $K^0_i$ be the limiting distribution of firm $i$’s investment strategy and $\mathcal{X}^0_i$ be the corresponding support. Let $\mathcal{X}^0 = \bigcup_{j \in \mathcal{N}} \mathcal{X}^0_j$. Then,

$$K^0_i(x) = \text{Prob}(x^0_i(s) \leq x) = G^0_i(\sup \{ s : x^0_i(s) \leq x \}),$$

where the second equality holds since $x^0_i(\cdot)$ is increasing in $s$.\textsuperscript{10} Suppose $|\mathcal{N}| = 2$ so that there is a unique symmetric score offering strategy. Since $\mathcal{S}^0 = [a, a + \frac{1}{2}]$, this implies that both firms’ supports of score offering are the same as $\mathcal{S}^0$, which implies $\mathcal{X}^0 = [0, 1]$. Hence, $\mathcal{X}^0_i$ is the same as $[0, 1]$ for both firms. Therefore, we can precisely define the firms’ equilibrium strategies at the limit.

**Corollary 1.** Suppose $|\mathcal{N}| = 2$. Then, the unique symmetric equilibrium when $\delta = 0$ is that for all $i \in \mathcal{N}$, $K^0(x) = U[0, 1]$, $p = \frac{1}{2} x$ and $G^0(s) = U[a, a + \frac{1}{2}]$.

The proof of the corollary is in ???. So far, we have shown that the set of mixed-strategy equilibria converges at the limit and characterized the limit distribution. Characterizing mixed-strategy equilibrium in the intermediate value of $\delta$, i.e., $\delta \in (0, \frac{1}{2})$, is a hard task, and we are only able to construct a symmetric mixed-strategy equilibrium when there are two firms. We provide this in Appendix C.1.

### 3.4 Fixed-Prize Tournament

In this section, we study the case that the buyer uses the fixed-prize tournament instead of the first-price auction. We also compare the buyer’s payoffs under two mechanisms.

\textsuperscript{9}See also Che and Gale (2003).

\textsuperscript{10}Recall that $s_i(q)$ is increasing in $q$. At the limit, $\delta = 0$, $q$ is the same as $x + a$. Thus, $s$ is also increasing in $x$, so $x$ is increasing in $s$. 
Recall that firm $i$ wins if $q_i - p_i > \max \left\{ \max_{j \in \mathcal{N} \setminus \{i\}} \{q_j - p_j\}, 0 \right\}$. The fixed-prize tournament can be understood as a special case in that $p_i$ is restricted to $P$ for all bidders, where $P \geq 0$ is optimally chosen by the buyer. Therefore, there is no bidding stage, and a firm that has produced the highest quality wins the prize. As it will be seen, the absence of bidding stage makes the analysis simpler than that of the first-price auction.

Consider the firms’ investment decision. We find a symmetric pure-strategy equilibrium. For a given $\mathcal{N}$, suppose that there is such an equilibrium. Suppose further that all firms in $\mathcal{N}$ except firm $i$ undertake the same investment level $x$. Then, firm $i$’s payoff is

$$\Pi^T_i(x_i) = P \int_{x_i + a - \delta}^{x_i + a + \delta} F(q|x)^{n-1} dF(q|x_i) - \psi(x_i),$$

where $n$ is the number firms in $\mathcal{N}$. Since $F(\cdot|x)$ is uniform and $\psi(\cdot)$ is quadratic, we have that from the first-order condition,

$$x^T = \frac{P}{2\delta} \quad \text{and} \quad \Pi^T = \frac{P}{n} - \frac{P^2}{8\delta^2}.$$

By the firms’ individual rationality, we must also have that $P \leq \frac{8\delta^2}{n}$ so that $\Pi^T \geq 0$.

In order to see whether this is indeed an equilibrium, we investigate firm $i$’s incentive to deviate as before. Consider local deviations. (The analysis for global deviations is similar as before. Hence, we omit it.)

Suppose firm $i$ slightly decreases its investment level such that $x^T - 2\delta < x_i < x^T$ (local downward deviation). Then, its payoff is

$$\Pi^{idd}(x_i; x^T) = P \int_{x^T + a - \delta}^{x^T + a + \delta} F(q|x^T)^{n-1} dF(q|x_i) - \psi(x_i),$$

\[11\] We use superscript $T$, which denotes “tournament,” to avoid confusion with the auction setting.
since if \( q_i < x^T + a - \delta \), firm \( i \) will be defeated by other firms for sure. Suppose now that firm \( i \)
slightly increases its investment level such that \( x^T < x_i < x^T + 2\delta \) (local upward deviation).
Then, its payoff becomes

\[
\Pi_{\text{ud}}(x_i; x^T) = P \left[ \int_{x_i+a-\delta}^{x^T+a+\delta} F(q|x^T)^{n-1} dF(q|x_i) + \int_{x_i+a+\delta}^{x^T+a+\delta} 1 dF(q|x_i) \right] - \psi(x_i),
\]

since for any \( q_i > x^T + a + \delta \), firm \( i \) will win for sure.

**Lemma 3.5.** Suppose \( P \leq \frac{4\delta^2}{n-1} \) for any given \( \delta > 0 \). Then, there exists a unique symmetric
pure-strategy equilibrium in which all firms invest \( x^T = P/(2\delta) \).

The condition \( P \leq \frac{4\delta^2}{n-1} \) is necessary for the optimality of \( x^T \). We defer a formal proof to ???. Observe that the investment level increases in \( P \), since for a large amount of prize, the
winning firm’s payoff is high, so the firms become more competitive. However, as \( \delta \) increases,
the firms’ investment level decreases. This is also intuitive since even if a firm invests a high \( x \),
the firm may get a low quality but cannot compete with other firms by lowering its price as
in the auction case. At the same time, even with a low level of investment, a firm may draw
a high quality and has a higher probability of winning.

The investment strategy in Lemma 3.5, however, is not well defined when \( \delta = 0 \). We now
show that in this case, there is a symmetric mixed-strategy equilibrium (of choosing \( x = q \)),
denoted by \( H^T(q) \). Note that a firm with quality (i.e., investment level) \( q \) receives an expected
payoff \( PH^T(q)^{n-1} - \psi(q) \). Since the firms receive zero expected payoff in equilibrium, we
have

\[
H^T(q) = \left( \frac{(q-a)^2}{2P} \right)^{\frac{1}{n-1}}.
\]

Let \( q \) and \( \bar{q} \) be respectively the infimum and the supremum of the support of the distribution. Then, we must have \( PH^T(q)^{n-1} = \psi(q) \) and \( PH^T(\bar{q})^{n-1} = \psi(\bar{q}) \), which implies that
the support of \( q \) is \([a, a + \sqrt{2P}]\). The proof for the optimality of \( H^T(\cdot) \) is fairly standard,
3.4.1 Comparison of the Buyer’s Payoffs

We aim to compare the buyer’s payoffs under the first-price auction and the fixed-prize tournament. But, we have not established the buyer’s optimal prize yet. However, since we were only able to derive the buyer’s payoff in the cases of $\delta = 0$ and $\delta \geq \frac{1}{2}$ in the auction, it is sufficient to consider optimal $P$ in those two cases for the purpose of comparison.

Suppose $\delta \geq \frac{1}{2}$. Under the symmetric investment strategy, $x_i = x^T$, the buyer’s payoff is given by

$$R_T = \int_{x^T + a - \delta}^{x^T + a + \delta} q dF(q|x)^n - P = a + \delta - \frac{2\delta}{n + 1} + P\left(\frac{1}{2\delta} - 1\right).$$

Since $\delta \geq \frac{1}{2}$, this implies that the optimal $P$ is zero. Clearly, $P = 0$ satisfies the condition in Lemma 3.5, hence there is an equilibrium in which $P = 0$ and $x^T = 0$. Observe that the buyer’s payoff is increasing in the number of firms. Therefore, for a large randomness, the buyer does not elicit any effort from the firms but invites as many firms as possible. That is, the buyer entirely relies on the sampling effect.

Recall that in the first-price auction, the buyer’s payoff, when $\delta \geq \frac{1}{2}$, is

$$R_A = a + \delta - \frac{4\delta}{n + 1} + \frac{1}{n}.$$  

Observe that

$$R_T - R_A = \frac{2\delta}{n + 1} - \frac{1}{n} \geq 0 \iff \delta \geq \frac{n + 1}{2n}.$$  

Therefore, for any given $n$, there is $\hat{\delta}_n := \frac{n + 1}{2n}$ such that for any $\delta > \hat{\delta}_n$, the fixed-prize tournament with zero prize dominates the first-price auction in terms of the buyer’s payoff. This is intuitive since in the auction, each firm’s price request is increasing in its quality realization, so the buyer leaves a high rents to the firms when the randomness is
large because she cannot discern the effort level. Observe also that \( \hat{\delta}_n \) converges to \( \frac{1}{2} \) as \( n \) increases. Thus, for sufficiently large \( n \), the tournament gives the buyer a higher surplus than the auction. In Figure 3.4, \( \hat{\delta}_n \) is depicted for each \( n = 2, \ldots, 14 \).

Consider now the case \( \delta = 0 \). Given the symmetric mixed strategy derived above, the buyer’s payoff is

\[
R^T = \int_a^{a+\sqrt{2P}} qdH(q) - P = a + \frac{2n\sqrt{2P}}{3n-1} - P
\]

We thus have for a fixed \( n \),

\[
P^T = \frac{2n^2}{(3n-1)^2} \quad \text{and} \quad R^T = a + \frac{2n^2}{(3n-1)^2}.
\]

It is clear that the buyer benefits by inviting only two firms in this case, since \( R^T \) is strictly decreasing in \( n \). Thus, we have \( P^T = \frac{8}{25} \) and \( R^T = a + \frac{8}{25} \) in equilibrium. Optimality of inviting two firms can be understood as the same reason in the auction: The sampling effect disappears at the limit, but the incentive effect prevails.

Recall that in the first-price auction, the two firms offer scores uniformly over \([a, a + \frac{1}{2}]\). Thus, the buyer’s payoff is \( a + \frac{1}{3} \), which is higher than \( R^T = a + \frac{8}{25} \). That is, the auction outperforms the tournament in this case. Note that even though both mechanisms select
the firm that offers the highest surplus, they induce different investment levels. Under the auction, the quality is uniformly distributed over \([a, a + 1]\), whereas under the tournament, it is uniformly distributed over \([a, a + \frac{4}{5}]\). Thus, the buyer earns a higher payoff under the auction than under the tournament as shown in Che and Gale (2003).

We summarize the results in the following proposition.

**Theorem 3.4.** Suppose \(\delta \geq \frac{1}{2}\). There is a symmetric pure-strategy equilibrium such that \(P_T = 0\) and \(x_T = 0\). Furthermore, for a given \(n\), there is \(\delta_n\) such that for any \(\delta > \delta_n\), the fixed-prize tournament outperforms the first-price auction.

Suppose \(\delta = 0\). It is optimal for the buyer to invite only two firms, and there is a symmetric mixed-strategy equilibrium such that \(P_T = \frac{8}{25}\) and \(H_T(q) = \frac{(q-a)^2}{2P_T}\). Furthermore, the first-price auction with two firms outperforms the fixed-prize tournament.

### 3.5 Conclusion

The current paper has studied a procurement problem in which firms undertake investments and the resulting qualities of the innovative activities are ex ante uncertain. In such an environment, there are two opposing tensions—incentive and sampling effects—and the buyer’s payoff depends on the number of participants as well as the degree of the randomness on the quality of innovation.

We consider two prominent mechanisms, a first-price auction and a fixed-prize tournament. In each mechanism, the number of participating firms in the procurement process are determined endogenously. We show that in both mechanisms, when the randomness of quality realization is large, the buyer does not limit the number of participants but rather invites many firms. If the randomness vanishes, however, inviting only two firms is optimal. It is also often argued that holding an auction is advantageous for the buyer over a fixed-prize tournament. However, our results show that this is not true when the randomness is large.
In fact, the tournament outperforms the auction in this case.

Although we have assumed specific functional forms for the analytical solution, we believe that the main intuition the paper carries over more general environment. The current paper also has focused on an auction and a tournament, but designing an optimal mechanism would be valuable.
Appendices
Appendix A

Omitted Proofs in Chapter 1

A.1 Proof of Lemma 1.1

We begin with a couple of claims. We then prove the lemma in the sequence of (i), (iii), (ii), and (iv).

Claim 1. Suppose $\mathcal{V}_{AB}$ has zero measure. Then, $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$.

Proof. Since $\mathcal{V}_{AB}$ is a measure zero set, colleges do not put a positive probability on admitting the same students; that is, $\alpha(v) = 0$ for almost every $v \in \mathcal{V}_B$ and $\beta(v) = 0$ for almost every $v \in \mathcal{V}_A$. Thus, a student will attend a college for sure if she is admitted by that college. Hence, it is optimal for each college to admit the best students up to the capacity among those who are not admitted by the opponent college. Therefore, $\alpha(v) = 1$ for almost every $v \in \mathcal{V}_A$ and $\beta(v) = 1$ for almost every $v \in \mathcal{V}_B$, so we have $m_i(s) = \kappa$ for all $s \in [0, 1]$ and for all $i = A, B$. ■

Claim 2. Suppose $\mathcal{V}_{AB}$ has a positive measure. Then, $m_A(s) < m_A(s')$ and $m_B(s) > m_B(s')$ for any $s < s'$.

Proof. The results follows immediately from the fact that $\mu(\cdot)$ is strictly increasing in $s$. ■
Proof of Part (i). If \( V_{AB} \) has zero measure, the proof is immediate from Claim 1. Suppose now that \( V_{AB} \) has a positive measure. Suppose \( m_A(1) < \kappa \). Then, \( A \) can benefit by admitting a mass \( \kappa - m_A(1) \) of students. Let \( \tilde{m}_A(s) \) be the mass of students attending \( A \) under such deviation. Then, we have that for any \( s < 1 \),

\[
m_A(s) < m_A(s) + \mu(s)[\kappa - m_A(1)] \leq \tilde{m}_A(s) \leq m_A(s) + [\kappa - m_A(1)] < \kappa,
\]

where the first and the last inequalities hold since \( m_A(s) < m_A(1) \) and \( \mu(s) \geq 0 \). The second inequality becomes equality if all of the newly admitted students has been admitted by \( B \), and the third inequality becomes equality if all of them has not been admitted by \( B \).

Observe that \( A \) benefits from the deviation since it admits more students without paying costs for over-enrollment. Therefore, we must have \( m_A(1) \geq \kappa \) in equilibrium. Similarly, if \( m_A(0) > \kappa \), then \( A \) can benefit by rejecting a mass \( m_A(0) - \kappa \) students. Combining these with Claim 2, we have \( m_A(0) \leq \kappa \leq m_A(1) \). The proof for college \( B \) is analogous.

Proof of Part (iii). Suppose \( V_{AB} \) has a positive measure. Since \( m_A(\cdot) \) is continuous in \( s \) (which follows from the continuity of \( \mu(\cdot) \)), there exists a unique \( \hat{s}_A \) such that \( m_A(\hat{s}_A) = \kappa \). Suppose \( \hat{s}_A = 1 \). Then, \( A \) has strict over-enrollment for all states except \( s = 1 \), a measure zero state. Similarly, if \( \hat{s}_A = 0 \), then \( A \) has strict under-enrollment for all states except \( s = 0 \). These contradict to Part (i). Thus, we have \( \hat{s}_A \in (0, 1) \). The proof for \( \hat{s}_B \) is similar, hence omitted.

Proof of (ii). We first show sup \( \{V_A \cup V_B\} = 1 \). We then show that \( V_A \cup V_B \) is a connected interval and inf \( \{V_A \cup V_B\} > 0 \).

Step 1. sup \( \{V_A \cup V_B\} = 1 \).

Proof. Suppose on the contrary \( \bar{c} := \sup \{V_A \cup V_B\} < 1 \). We show that a college can benefit by rejecting some students in favor of those in \( [\bar{c}, 1] \). Suppose that \( V_{AB} \) has zero
measure. Then, a colleges, say $A$, can benefit by rejecting a positive mass of students from the bottom of $\mathcal{V}_A$ and admits the same mass of students from 1.

Suppose now that $\mathcal{V}_{AB}$ has a positive measure. Let $[\bar{\tau} - \varepsilon, \bar{\tau}] \subset \mathcal{V}_{AB}$. Now, let $A$ reject students in $[\bar{\tau} - \varepsilon, \bar{\tau}]$ and admit those in $[\bar{\tau}, \bar{\tau} + \delta]$, where $\varepsilon$ and $\delta$ are such that

$$G(\bar{\tau} + \delta) - G(\bar{\tau}) = \mu(\hat{s}_A)[G(\bar{\tau}) - G(\bar{\tau} - \varepsilon)].$$

(A.1)

Then, the mass of student attending $A$ from the deviation in state $s$, denoted by $\tilde{m}_A(s)$, is

$$\tilde{m}_A(s) = G(\bar{\tau} + \delta) - G(\bar{\tau}) - \mu(s)[G(\bar{\tau}) - G(\bar{\tau} - \varepsilon)] + m_A(s)$$

$$= (\mu(\hat{s}_A) - \mu(s))[G(\bar{\tau}) - G(\bar{\tau} - \varepsilon)] + m_A(s),$$

(A.2)

where the second equality follows from (A.1). Since $\tilde{m}_A(\hat{s}_A) = m_A(\hat{s}_A)$, $A$’s payoff from the deviation is

$$\tilde{\pi}_A = \int_{\tau}^{\tau + \delta} v dG(v) + \int_{0}^{\tau} v \alpha(v)[1 - \beta(v) + \overline{\mu}_1 \beta(v)] dG(v)$$

$$- \int_{\tau - \varepsilon}^{\tau} v \overline{\mu}_1 \alpha(v) \beta(v) dG(v) - \lambda \mathbb{E}_s[\tilde{m}_A(s) - \kappa|s > \hat{s}_A](1 - \hat{s}_A),$$

and hence,

$$\tilde{\pi}_A - \pi_A = \int_{\tau}^{\tau + \delta} v dG(v) - \int_{\tau - \varepsilon}^{\tau} v \overline{\mu}_1 \alpha(v) \beta(v) dG(v) - \lambda \mathbb{E}_s[\tilde{m}_A(s) - m_A(s)|s > \hat{s}_A](1 - \hat{s}_A)$$

$$\geq \int_{\tau}^{\tau + \delta} v dG(v) - \overline{\mu}_1 \int_{\tau - \varepsilon}^{\tau} v dG(v) - \lambda [G(\bar{\tau}) - G(\bar{\tau} - \delta)] \int_{\hat{s}_A}^{1} [\mu(\hat{s}_A) - \mu(s)] ds.$$
Observe that the first two terms in the RHS of the inequality are

\[
\int_{c}^{c+\delta} v dG(v) - \mu \int_{c-\varepsilon}^{c} v dG(v) = \left[ c [G(c + \delta) - G(c)] - \mu c [G(c) - G(c - \varepsilon)] \right] \\
+ \left[ \delta G(c + \delta) - \int_{c}^{c+\delta} G(v) dv \right] - \mu \left[ \varepsilon G(c - \varepsilon) - \int_{c-\varepsilon}^{c} G(v) dv \right] \\
> c [G(c) - G(c - \varepsilon)] - \mu c [G(c) - G(c - \varepsilon)] \\
= c [G(c) - G(c - \varepsilon)] (\mu(\hat{s}) - \mu),
\]

where the first equality follows from the integration by parts and after some arrangement, and the last equality follows from (A.1).

Therefore,

\[
\widetilde{\pi}_A - \pi_A > c [G(\bar{c}) - G(\bar{c} - \varepsilon)] (\mu(\hat{s}_A) - \mu) - \lambda [G(\bar{c}) - G(\bar{c} - \delta)] \int_{\hat{s}_A}^{1} [\mu(\hat{s}_A) - \mu(s)] ds \\
\geq c [G(\bar{c}) - G(\bar{c} - \varepsilon)] \left[ \mu(\hat{s}_A) - \mu + \int_{\hat{s}_A}^{1} [\mu(s) - \mu(\hat{s}_A)] ds \right] \\
> 0,
\]

where the inequality holds since \( \int_{\hat{s}_A}^{1} [\mu(\hat{s}_A) - \mu(s)] ds \leq 0, \lambda \geq 1, \) and

\[
\mu(\hat{s}_A) - \mu + \int_{\hat{s}_A}^{1} [\mu(s) - \mu(\hat{s}_A)] ds = \int_{0}^{\hat{s}_A} [\mu(\hat{s}_A) - \mu(s)] ds \geq 0.
\]

This proves that \( A \) can benefit from such a deviation. \( \square \)

**Step 2.** \( \mathcal{V}_A \cup \mathcal{V}_B \) is a connected interval.

**Proof.** Suppose on the contrary that there is gap in \( \mathcal{V}_A \cup \mathcal{V}_B \). The proof is analogous to Step 1, where \([\bar{c}, 1]\) is now replaces by the gap in \( \mathcal{V}_A \cup \mathcal{V}_B \). We omit the details. \( \square \)

**Step 3.** \( \inf \{ \mathcal{V}_A \cup \mathcal{V}_B \} > 0 \)
Proof. Suppose $\mathcal{V}_{AB}$ has zero measure in equilibrium. Then, we have $\inf \{\mathcal{V}_A \cup \mathcal{V}_B\} = \xi$, where $1 - G(\xi) = 2\kappa$ from Step 1, Step 2 and Claim 1. Since $\kappa < \frac{1}{2}$, we have the desired result.

Suppose now that $\mathcal{V}_{AB}$ has a positive measure in equilibrium. Suppose $\inf \{\mathcal{V}_A \cup \mathcal{V}_B\} = 0$ to the contrary. Then, we have $m_A(s) + m_B(s) = |\mathcal{V}_A \cup \mathcal{V}_B| = 1$ for all $s \in [0, 1]$, where the last equality follows from Steps 1 and 2. Observe that since $m_A(0) \leq \kappa \leq m_A(1)$, we must have $m_B(1) \leq 1 - \kappa \leq m_B(0)$. Adding this to $m_B(1) \leq \kappa \leq m_B(0)$, we have that

$$m_B(1) \leq \frac{1}{2} \leq m_B(0).$$

(A.3)

Similarly, we also have

$$m_A(0) \leq \frac{1}{2} \leq m_A(1).$$

(A.4)

Since $\inf \{\mathcal{V}_A \cup \mathcal{V}_B\} = 0$, at least one of $\nu_A$ and $\nu_B$ is zero. Suppose $\nu_A = \lambda(1 - \hat{s}_A) = 0$. This implies that $\hat{s}_A = 1$, and so $m_A(1) = \kappa$. This contradicts to (A.3) since $\kappa < \frac{1}{2}$. Similarly, if $\nu_B = 0$, then $\hat{s}_B = 0$ and so $m_B(0) = \kappa$, which contradicts to (A.4). □

Proof of Part (iv). The proof follows from Claim 1 and the fact that $\mathcal{V}_A \cup \mathcal{V}_B = [\xi, 1]$ where $\xi = G^{-1}(1 - 2\kappa)$ (from Part (iii)). ■

A.2 Non-Competitive Equilibrium

In this section, we show that when $\kappa < \frac{1}{2}$ is not too small or $\lambda > 1$ is not too large, there does not exist an non-competitive equilibrium. The reason is as follows: Suppose, for instance, $\bar{\mu} = \frac{1}{2}$, $\kappa > \frac{1}{4}$ and $v \sim U[0, 1]$. Suppose further that the worst type college $A$ has is less than $\frac{1}{2}$, and the best type of college $B$ has is 1. Then, regardless of $\lambda$, $A$ can benefit by rejecting sufficiently small mass of students from the bottom and accepting the same mass of students
close to one, because those newly admitted students will accept $A$ with probability close to $\frac{1}{2}$, and the (average) value of those students is discretely better than that of students rejected by $A$. Furthermore, $A$ will never have over-enrollment from this deviation. The first part Lemma B1 generalize this observation and show that for $\kappa$ not too small it is profitable. (In the proof, we do not require the restrictive assumptions made in the above example.)

Of course, this is just one way to deviate. The other way of deviation is to accept more of those close to one than those $A$ rejects at the bottom. But, the profitability of such deviation now depends on $\lambda$. The second part of Lemma B1 shows that for $\lambda$ not too large, it is profitable.

**Lemma B1.** Suppose that $\mathcal{V}_{AB}$ has zero measure. Then, we have the followings:

(i) There is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, one college has an incentive to deviate.

(ii) There is $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, one college has an incentive to deviate.

**Proof.** Suppose $\mathcal{V}_{AB}$ has zero measure. Then, $m_i(s) = \kappa$ for all $s$ and

$$\pi_i = \int_{\mathcal{V}_i} v \, dG(v), \quad i = A, B.$$ 

Now, let $\underline{c}_i = \inf \{\mathcal{V}_i\}$ and $\overline{c}_i = \sup \{\mathcal{V}_i\}$.

**Proof of (i).** Let $\underline{c}_A = \inf \{\mathcal{V}_A \cup \mathcal{V}_B\}$, without loss of generality. Then, $\underline{c}_A = G^{-1}(1 - 2\kappa)$ by Lemma 1.1. We show that college $A$ has an incentive to deviate. Suppose that $A$ rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but accepts those in $[\overline{c}_B - \varepsilon, \overline{c}_B]$, where $\varepsilon$ and $\delta$ are such that

$$G(\overline{c}_B) - G(\overline{c}_B - \varepsilon) = G(\underline{c}_A + \delta) - G(\underline{c}_A). \quad (A.5)$$
Note that the mass of students attending A under this deviation is
\[ \tilde{m}_A(s) = \int_{c_B - \varepsilon}^{c_B} \mu(s) dG(v) + \int_{\mathcal{V}_A \setminus [c_A, c_A + \delta]} 1 dG(v) \]
\[ = \mu(s) [G(c_B) - G(c_B - \varepsilon)] + \kappa - [G(c_A + \delta) - G(c_A)], \]

where the second equality follows from that \( m_A(s) = \kappa \) for all \( s \).

Let \( \hat{s}_A \) be such that \( \tilde{m}_A(\hat{s}_A) = \kappa \) (provided \( \hat{s}_A \in [0, 1] \)), i.e.,
\[ \mu(\hat{s}_A) [G(c_B) - G(c_B - \varepsilon)] = [G(c_A + \delta) - G(c_A)]. \]

Since \( \mu(\cdot) \) is strictly increasing, (A.5) implies A is never over-demanded in a positive measure of states from this deviation. Thus, A’s payoff from the deviation is
\[ \tilde{\pi}_A = \overline{\pi} \int_{c_B - \varepsilon}^{c_B} v dG(v) + \int_{\mathcal{V}_A \setminus [c_A, c_A + \delta]} v dG(v) = \overline{\pi} \int_{c_B - \varepsilon}^{c_B} v dG(v) + \pi_A - \int_{c_B}^{c_B + \delta} v dG(v). \]

Therefore,
\[ \tilde{\pi}_A - \pi_A = \overline{\pi} \int_{c_B - \varepsilon}^{c_B} v dG(v) - \int_{c_B}^{c_B + \delta} v dG(v) \]
\[ = \overline{\pi} \left[ c_B G(c_B) - (c_B - \varepsilon)G(c_B - \varepsilon) - \int_{c_B - \varepsilon}^{c_B} G(v) dv \right] \]
\[ - \left[ (c_A + \delta)G(c_A + \delta) - c_A G(c_A) - \int_{c_B}^{c_A + \delta} G(v) dv \right] \]
\[ > \overline{\pi} \left[ c_B G(c_B) - (c_B - \varepsilon)G(c_B - \varepsilon) - \varepsilon G(c_B) \right] \]
\[ - \left[ (c_A + \delta)G(c_A + \delta) - c_A G(c_A) - \delta G(c_A) \right] \]
\[ = [G(c_B) - G(c_B - \varepsilon)][\overline{\pi} c_B - c_A - \overline{\pi} \varepsilon - \delta], \quad (A.6) \]

where the first equality follows from the integration by parts, and the last equality follows
from (A.5). Observe that if \( \bar{\mu} > \frac{\bar{\epsilon}_A}{\bar{\epsilon}_B} \), then (A.6) is strictly positive for sufficiently small \( \varepsilon \) and \( \delta \), hence \( \bar{\pi}_A > \bar{\pi}_A \). Note that since \( \zeta_A = G^{-1}(1 - 2\kappa) \) and \( m_i(s) = \kappa \) for all \( s \) and \( i \), \( \bar{\epsilon}_B = \sup \{ \mathcal{V}_B \} \) satisfies \( G(\bar{\epsilon}_B) \geq 1 - \kappa \). That is, we have \( \bar{\epsilon}_B \geq G^{-1}(1 - \kappa) \). Therefore,

\[
\frac{\zeta_A}{\bar{\epsilon}_B} \leq \frac{G^{-1}(1 - 2\kappa)}{G^{-1}(1 - \kappa)}.
\quad \text{(A.7)}
\]

Since the RHS of (A.7) is continuous in \( \kappa \) and converges to zero as \( \kappa \) approaches to \( \frac{1}{2} \), there is \( \hat{\kappa} < \frac{1}{2} \) such that for any \( \kappa > \hat{\kappa} \), \( \bar{\mu} > \frac{\zeta_A}{\bar{\epsilon}_B} \) for any given \( \bar{\mu} \). □

**Proof of (ii).** Let \( \bar{\epsilon}_B = \sup \{ \mathcal{V}_A \cup \mathcal{V}_B \} \), without loss of generality. Then, \( \bar{\epsilon}_B = 1 \) by Lemma 1.1. We show that college \( A \) has an incentive to deviate. Suppose \( A \) rejects students in \( [\zeta_A, \zeta_A + \delta] \) but admits students in \( [1 - \varepsilon, 1] \), where \( \varepsilon \) and \( \delta \) are such that

\[
\eta [1 - G(1 - \varepsilon)] = G(\zeta_A + \delta) - G(\zeta_A)
\quad \text{(A.8)}
\]

and \( \eta = \mu(1 - \zeta_A) \). The mass of students who attend \( A \) in state \( s \) under the deviation is

\[
\tilde{m}_A(s) = \int_{1 - \varepsilon}^{1} \mu(s)g(v) \, dv + \int_{\mathcal{V}_A \setminus [\zeta_A, \zeta_A + \delta]} g(v) \, dv = \mu(s)[1 - G(1 - \varepsilon)] + \kappa - [G(\zeta_A + \delta) - G(\zeta_A)].
\]

Let \( \hat{s}_A \) be such that \( \tilde{m}_A(\hat{s}_A) = \kappa \), i.e., \( \mu(\hat{s}_A)[1 - G(1 - \varepsilon)] = [G(\zeta_A + \delta) - G(\zeta_A)] \). Then, \( \mu(\hat{s}_A) = \eta \) by (A.8), i.e., \( \hat{s}_A = \mu^{-1}(\eta) = 1 - \zeta_A \).

Thus, \( A \)'s payoff from the deviation is

\[
\tilde{\pi}_A = \mu \int_{\tau_B - \varepsilon}^{\tilde{\tau}_B} v \, dG(v) + \int_{\mathcal{V}_A \setminus [\zeta_A, \zeta_A + \delta]} v \, dG(v) - \lambda \int_{\hat{s}_A}^{1} [m(s) - \kappa] \, ds
\]
\[
= \mu \int_{\tau_B - \varepsilon}^{\tilde{\tau}_B} v \, dG(v) + \pi_A - \int_{\zeta_A + \delta}^{\zeta_A + \delta} v \, dG(v)
\]
\[
- \lambda \left[ (1 - G(1 - \varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) \, ds - [G(\zeta_A + \delta) - G(\zeta_A)](1 - \hat{s}_A) \right]
\]
and the net payoff from the deviation is

\[ \tilde{\pi}_A - \pi_A = \mu \int_{\tilde{\tau}_B}^{\tau_B} v \, dG(v) - \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds - [G(\xi_A + \delta) - G(\xi_A)](1 - \hat{s}_A) \]

\[ > \mu(1 - \varepsilon)[1 - G(1 - \varepsilon)] - (\xi_A + \delta)[G(\xi_A + \delta) - G(\xi_A)] \]

\[ - \lambda \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds - [G(\xi_A + \delta) - G(\xi_A)](1 - \hat{s}_A) \]

\[ = [1 - G(1 - \varepsilon)] \left( \mu - \eta \xi_A - \mu \varepsilon - \eta \delta - \lambda \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds - \eta(1 - \hat{s}_A) \right), \quad (A.9) \]

where the second inequality follows from the integration by parts of the first two terms of the RHS of the first equality, and the last equality follows from (A.8).

Observe that if \( \mu - \eta \xi_A - \lambda \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds - \eta(1 - \hat{s}_A) > 0 \), then (A.9) is strictly positive for sufficiently small \( \varepsilon \) and \( \delta \). Note that

\[ \mu - \eta \xi_A - \lambda \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds - \eta(1 - \hat{s}_A) = \mu - \lambda \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds + (\lambda - 1) \eta \xi_A, \]

Note that \( \mu = \int_{0}^{1} \mu(s) \, ds > \int_{1 - \hat{s}_A}^{1} \mu(s) \, ds \) (since \( \hat{s}_A < 1 \)). Thus, there exists \( \lambda > 1 \) such that for any \( \lambda < \hat{\lambda} \), \( \tilde{\pi}_A > \pi_A \).  \( \square \)

### A.3 Proofs of Theorem 1.2

Suppose on the contrary that \( \bar{v} \leq v \) in competitive equilibrium. Suppose further that

\[ \underline{v}_B \leq \underline{v}_A \leq \bar{v}_A \leq \bar{v}, \quad (A.10) \]
without loss of generality. For given \( \hat{s}_A \) and \( \hat{s}_B \) in equilibrium, the mass of students attending each college is

\[
m_A(s) = \mu(s)[1 - G(\bar{v}_A)],
\]

\[
m_B(s) = (1 - \mu(s))[1 - G(\bar{v}_A)] + G(\bar{v}_A) - G(\underline{v}_B).
\]

Notice that we must have \( \bar{v}_A \in (0, 1) \) in equilibrium, since if \( \bar{v}_A = 1 \), then \( m_A(s) = 0 \) for any \( s \in [0, 1] \), and if \( \bar{v}_A = 0 \), then \( \underline{v}_B = \bar{v}_B = \bar{v}_A = 0 \). This implies that \( \hat{s}_A < 1 \) (or else, \( \underline{v}_A = \bar{v}_A = 0 \)) and \( \hat{s}_A > 0 \) (or else, \( \bar{v}_A = \lambda \geq 1 \)). Since \( \hat{s}_A < 1 \), (A.10) becomes

\[
\underline{v}_B \leq \bar{v}_B \leq \underline{v}_A < \bar{v}_A,
\]

i.e., the last inequality becomes strict, or equivalently,

\[
\lambda \hat{s}_B \leq \frac{\lambda}{1 - \mu} \int_0^{\hat{s}_B} (1 - \mu(s))ds \leq \lambda(1 - \hat{s}_A) < \frac{\lambda}{\mu} \int_{\hat{s}_A}^1 \mu(s)ds. \quad (A.11)
\]

In equilibrium, we must have that

\[
m_A(\hat{s}_A) = \mu(\hat{s}_A)[1 - G(\bar{v}_A)] = \kappa, \quad (A.12)
\]

\[
m_B(\hat{s}_B) = (1 - \mu(\hat{s}_B))[1 - G(\bar{v}_A)] + G(\bar{v}_A) - G(\underline{v}_B) = \kappa. \quad (A.13)
\]

From (A.12), \( 1 - G(\bar{v}_A) = \frac{\kappa}{\mu(\hat{s}_A)} \). Substituting this into (A.13), we have that

\[
G(\bar{v}_A) - G(\underline{v}_B) = \kappa \left( \frac{\mu(\hat{s}_A) + \mu(\hat{s}_B) - 1}{\mu(\hat{s}_A)} \right).
\]
Since $\overline{v}_A > \overline{v}_B$ by (A.11), we have that

$$\mu(\hat{s}_A) + \mu(\hat{s}_B) > 1 \Leftrightarrow \mu(\hat{s}_B) > 1 - \mu(\hat{s}_A) = \mu(1 - \hat{s}_A),$$

where the last equality follows from the symmetry of $\mu(\cdot)$. Since $\mu(\cdot)$ is strictly increasing, this implies that $\hat{s}_B > 1 - \hat{s}_A$. Therefore, $\lambda \hat{s}_B > \lambda (1 - \hat{s}_A)$ which contradicts to (A.11).

### A.4 Proofs of Theorem 1.3

**Lemma A.1.** $T$ is continuous on $S$.

*Proof.* Fix any $\hat{s} = (\hat{s}_A, \hat{s}_B) \in S$. Then, $\alpha(\cdot; \hat{s})$ and $\beta(\cdot; \hat{s})$ are given by (1.4) and (1.5). Note that $\overline{v}_A$ and $\underline{v}_A$ are continuous in $\hat{s}_A$, and $\overline{v}_B$ and $\underline{v}_B$ are continuous in $\hat{s}_B$. Now let

$$\underline{v} := \min \{ \underline{v}_A, \underline{v}_B \}, \quad \overline{v} := \max \{ \overline{v}_A, \overline{v}_B \}, \quad \overline{v} := \min \{ \overline{v}_A, \overline{v}_B \}, \quad \underline{v} := \max \{ \underline{v}_A, \underline{v}_B \}.$$

For given $\hat{s}$, $T(\hat{s}) = \bar{s}$, where $\bar{s} = (\bar{s}_A, \bar{s}_B) \in S$ satisfies that

$$\bar{s}_A = \inf \left\{ s \in [0, 1] \left| \int_0^1 \alpha(v; \hat{s})[1 - \beta(v; \hat{s}) + \mu(s)\beta(v; \hat{s})]dG(v) - \kappa > 0 \right. \right\},$$

if the set in the RHS is nonempty, or else $\bar{s}_A \equiv 1$, and

$$\bar{s}_B = \sup \left\{ s \in [0, 1] \left| \int_0^1 \beta(v; \hat{s})[1 - \alpha(v; \hat{s}) + (1 - \mu(s))\alpha(v; \hat{s})]dG(v) - \kappa > 0 \right. \right\},$$

if the set in the RHS is nonempty, or else $\bar{s}_B \equiv 0$.

Consider now any $\hat{s}' = (\hat{s}'_A, \hat{s}'_B) \in S$. Then, for such $\hat{s}'$, $\alpha(\cdot; \hat{s}') \equiv \alpha'$ and $\beta(\cdot; \hat{s}') \equiv \beta'$ are
given by (1.4) and (1.5), and

\[ v' := \min \{ v'_A, v'_B \}, \quad v := \max \{ v_A, v_B \}, \quad \bar{v}' := \min \{ \bar{v}'_A, \bar{v}'_B \}, \quad \bar{v} := \max \{ \bar{v}_A, \bar{v}_B \}. \]

Again, \( \tilde{s}' = (\tilde{s}'_A, \tilde{s}'_B) \in S \) is defined by \( T \).

Next, let

\[ v_1 := \min \{ v, v' \}, \quad v_2 := \max \{ v, v' \}, \quad v_3 := \min \{ \bar{v}, \bar{v}' \}, \quad v_4 := \max \{ \bar{v}, \bar{v}' \}, \]

\[ v_5 := \min \{ \tilde{v}, \tilde{v}' \}, \quad v_6 := \max \{ \tilde{v}, \tilde{v}' \}, \quad v_7 := \min \{ v_3, v_4 \}, \quad v_8 := \max \{ v_3, v_4 \}, \]

and consider a partition of \([0, 1]\) such that

\[ V_1 = (\cup_{i=2,4,6,8}[v_{i-1}, v_i]) \cap [0, 1], \quad V_2 = [v_4, v_5] \cap [0, 1], \quad V_3 = [0, 1] \setminus (V_1 \cup V_2). \]

Consider now \( \alpha \) and \( \alpha' \). For any \( v \in [0, 1] \), we have

\[ \int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v) = \sum_{i=1}^3 \int_0^1 |\alpha'(v) - \alpha(v)| \, \mathbf{1}_{V_i}(v) \, dG(v), \]

where \( \mathbf{1}_{V_i}(v) \) is 1 if \( v \in V_i \) or 0 otherwise.

Observe, first, that by the continuity of \( v_i \) and \( \bar{v}_i, i = A, B \), there is a \( \delta_1 > 0 \) such that for any \( \varepsilon > 0 \), if \( \| \hat{s}' - \hat{s} \| < \delta_1 \), then

\[ \int_0^1 \mathbf{1}_{V_1}(v) \, dG(v) < \frac{\varepsilon}{6}. \quad (A.14) \]

Second, for any \( v \in V_2 \), the continuity of \( \alpha_0(\cdot) \), given by (1.2), implies that there is \( \delta_2 \) such
that $\|\hat{s}' - \hat{s}\| < \delta_2$ implies

$$|\alpha'(v) - \alpha(v)| = |\alpha_0'(v) - \alpha_0(v)| < \frac{\varepsilon}{6}, \quad (A.15)$$

Lastly, for any $v \in \mathcal{V}_3$, $\alpha'(v)$ and $\alpha(v)$ are either 0 or 1 at the same time, hence we have that

$$|\alpha(v') - \alpha(v)| = 0. \quad (A.16)$$

Now, let $\delta = \min \{\delta_1, \delta_2\}$ and suppose $\|\hat{s}' - \hat{s}\| < \delta$. Then, we have

$$\int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v) = \int_0^1 |\alpha'(v) - \alpha(v)| \mathbf{1}_{\mathcal{V}_1}(v) \, dG(v) + \int_0^1 |\alpha'_0(v) - \alpha_0(v)| \mathbf{1}_{\mathcal{V}_2}(v) \, dG(v)$$

$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}, \quad (A.17)$$

where the equality follows from (A.16) and the inequality follows from (A.14) and (A.15).

Similarly, we also have that

$$\int_0^1 |\beta'(v) - \beta(v)| \, dG(v) < \frac{\varepsilon}{3}. \quad (A.18)$$

Observe that

$$\left| \int_0^1 \alpha'(v)[1 - \beta'(v) + \mu(s)\beta'(v)] \, dG(v) - \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] \, dG(v) \right|$$

$$= \left| \int_0^1 \left[ \alpha'(v) - \alpha(v) \right] - (1 - \mu(s))[\alpha'(v)\beta'(v) - \alpha(v)\beta(v)] \right| \, dG(v)$$

$$\leq \int_0^1 |\alpha'(v) - \alpha(v)| \, dG(v) + (1 - \mu(s)) \int_0^1 |\alpha'(v)\beta'(v) - \alpha(v)\beta(v)| \, dG(v) \quad (A.19)$$
The first part of (A.19) is smaller than \( \varepsilon/3 \) by (A.17). The second part of (A.19) is

\[
\begin{align*}
&\int_0^1 |\alpha(v)\beta'(v) - \alpha(v)\beta(v)| \; dG(v) \\
&= \int_0^1 |\alpha'(v)\beta'(v) - \alpha'(v)\beta(v) + \alpha'(v)\beta(v) - \alpha(v)\beta(v)| \; dG(v) \\
&\leq \int_0^1 |\beta'(v) - \beta(v)| \; dG(v) + \int_0^1 |\alpha'(v) - \alpha(v)| \; dG(v) \\
&< \frac{2}{3} \varepsilon,
\end{align*}
\]

where the first inequality holds since \( \alpha'(v), \beta(v) \leq 1 \), and the last inequality follows from (A.17) and (A.18). Therefore, if \( \|\hat{s}' - \hat{s}\| < \delta \), then

\[\left| \int_0^1 \alpha'(v)[1 - \beta'(v) + \mu(s)\beta'(v)]dG(v) - \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) \right| < \varepsilon. \quad (A.20)\]

Similarly, we also have

\[\left| \int_0^1 \beta'(v)[1 - \alpha'(v) + (1 - \mu(s))\alpha'(v)]dG(v) - \int_0^1 \beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)]dG(v) \right| < \varepsilon. \quad (A.21)\]

Combining (A.20) and (A.21), we conclude that there is \( \delta > 0 \) such that for any \( \varepsilon > 0 \), if \( \|\hat{s}' - \hat{s}\| < \delta \), then \( \|\hat{s}' - \hat{s}\| < \varepsilon \). Since \( \hat{s} \) is chosen arbitrary, \( T \) is continuous on \( S \).

\section*{Lemma A.2. \( \pi_A \) is concave in \( \alpha \).}

\textit{Proof.} Recall that

\[
\pi_A = \int_0^1 \left[ \int_0^1 v\alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) \right] ds \\
- \lambda \int_0^1 \max \left\{ \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\} ds. \quad (A.22)
\]
Consider any feasible \( \alpha \) and \( \alpha' \). Note that for \( \eta \in [0, 1] \), the first part of (A.22) is linear in \( \alpha \),

\[
\int_0^1 v[\eta \alpha(v) + (1 - \eta)\alpha'(v)][1 - \beta(v) + \mu(s)\beta(v)]dG(v)
= \eta \int_0^1 v\alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) + (1 - \eta) \int_0^1 v\alpha'(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v).
\]

And the second part is convex in \( \alpha \), since

\[
\max \left\{ \int_0^1 [\eta \alpha(v) + (1 - \eta)\alpha'(v)][1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\} \\
= \max \left\{ \eta \left[ \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa \right] + (1 - \eta) \left[ \int_0^1 \alpha'(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa \right] \right\} \\
\leq \eta \max \left\{ \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\} \\
+ (1 - \eta) \max \left\{ \int_0^1 \alpha'(v)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa, 0 \right\}.
\]

Therefore, we have \( \pi_A(\eta \alpha + (1 - \eta)\alpha') \geq \eta \pi_A(\alpha) + (1 - \eta)\pi_A(\alpha') \). That is, \( \pi_A \) is concave in \( \alpha \). \[\blacksquare\]

**Lemma A.3.** \( V(\cdot) \) is concave in \( t \) for any \( t \in [0, 1] \).

**Proof.** Observe that \( \alpha(v; t) \) is linear in \( t \), since for any \( t, t' \in [0, 1] \),

\[
\alpha(v; t + (1 - t) t') = (\eta t + (1 - \eta) t')\tilde{\alpha}(v) + (1 - (\eta t + (1 - \eta) t'))\alpha(v)
= [\eta t \tilde{\alpha}(v) + \eta (1 - t) \alpha(v)] + [(1 - \eta) t'\tilde{\alpha}(v) + (1 - \eta) (1 - t) \alpha(v)]
= \eta \alpha(v; t) + (1 - \eta) \alpha(v; t').
\]
Therefore, we have

\[ V(\eta t + (1 - \eta) t') = \pi_A(\alpha(v; \eta t + (1 - \eta) t')) = \pi_A(\eta \alpha(v; t) + (1 - \eta) \alpha(v; t')) \]

\[ \geq \eta \pi_A(\alpha(v; t)) + (1 - \eta) \pi_A(\alpha(v; t')) \]

\[ = \eta V(t) + (1 - \eta) V(t'), \]

where the second equality follows from Lemma A.2 and the inequality follows from (A.23).

\[ \blacksquare \]

**Lemma A.4.** \( V'(0) \leq 0. \)

**Proof.** Let

\[ W(t, \hat{s}_A(t)) := \int_0^1 \nu(\alpha(v; t)[1 - \beta(v) + \mu]dG(v) \]

\[ - \lambda \int_0^1 \left[ \int_0^1 \alpha(v; t)[1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa \right] ds \]

and denote it by

\[ V(t) := W(t, \hat{s}_A(t)). \]

Observe that

\[ V'(t) = W_1(t, \hat{s}_A(t)) + W_2(t, \hat{s}_A(t))\hat{s}'(t), \]

where

\[ W_1(t, \hat{s}_A(t)) = \int_0^1 (\tilde{\alpha}(v) - \alpha(v))[v[1 - \beta(v) + \mu]dG(v) - \lambda \int_0^1 [1 - \beta(v) + \mu(s)\beta(v)]dG(v) - \kappa] \]

and

\[ W_2(t, \hat{s}_A(t)) = \lambda \int_0^1 \alpha(v; t)[1 - \beta(v) + \mu(\hat{s}_A(t))\beta(v)]dG(v) - \kappa]. \]
Notice that $W_2(0, \hat{s}_A(0)) = 0$ (by definition of $\hat{s}_A$). Therefore, we have

$$V'(0) = W_1(0, \hat{s}(0)) = \int_0^1 (\tilde{a}(v) - \alpha(v))H_\alpha(v, \beta(v)) dG(v) \leq 0,$$

where the inequality follows from local incentives.

A.5 Proofs for Section 1.5

**Proof for Lemma 1.2.** Fix any $\sigma$. To prove the optimality of the cutoff strategy, we show that $T'(y|\sigma) > 0$ for any $y$. Note that

$$T'(y|\sigma) = P_A(y|\sigma) + P_B(y|\sigma) + yP'_A(y|\sigma) - (1 - y)P'_B(y|\sigma)$$

$$\geq y[P_A(y|\sigma) + P'_A(y|\sigma)] + (1 - y)[P_B(y|\sigma) - P'_B(y|\sigma)]$$

$$= y \int_0^1 q_A(s)[l(s|y) + l_y(s|y)]ds + (1 - y) \int_0^1 q_B(s)[l(s|y) - l_y(s|y)]ds.$$ 

Observe that

$$l(s|y) + l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds} \left[ 1 + \frac{k_y(y|s)}{k(y|s)} - \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} \right] > \frac{k(y|s)}{\int_0^1 k(y|s)ds} (1 - 2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} > -\delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s)ds}{\int_0^1 k(y|s)ds} < \delta,$$

because $|\frac{k_y(y|s)}{k(y|s)}| < \delta$. Similarly,

$$l(s|y) - l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds} \left[ 1 - \frac{k_y(y|s)}{k(y|s)} + \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} \right] > \frac{k(y|s)}{\int_0^1 k(y|s)ds} (1 - 2\delta),$$
where the inequality holds since

\[
\frac{k_y(y|s)}{k(y|s)} < \delta \quad \text{and} \quad \int_0^1 k_y(y|s)ds = \int_0^1 \frac{k_y(y|s)}{k(y|s)}k(y|s)ds > -\delta.
\]

Therefore, we have that \(T'(y|\sigma) > 0\) since \(\delta \leq \frac{1}{2}\).

It remains to show that there exists an equilibrium in cutoff strategy. Let \(\hat{y}\) be a cutoff. Then, we have \(n_A(s|\hat{y}) = \int_0^1 k(y|s) = 1 - K(\hat{y}|s)\). Hence,

\[
P_A(y|\hat{y}) = \int_0^1 \min \left\{ \frac{\kappa}{1 - K(\hat{y}|s)}, 1 \right\} l(s|y)ds \quad \text{and} \quad P_B(y|\hat{y}) = \int_0^1 \min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} l(s|y)ds,
\]

Now, let

\[
T(y|\hat{y}) := yP_A(y|\hat{y}) - (1 - y)P_B(y|\hat{y}).
\]

Note that

\[
T(0|\hat{y}) = -P_B(0|\hat{y}) = -\int_0^1 \min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} l(s|0)ds < 0,
\]

where the inequality holds since \(\min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} > 0\) and \(l(s|0) \geq 0\) for all \(s\), and \(l(s|0) > 0\) for a positive measure of states. Similarly, \(T(1|\hat{y}) > 0\). By the continuity of \(T(\cdot|\hat{y})\), there is a \(\hat{y}\) such that \(T(\hat{y}|\hat{y}) = 0\). Moreover, such a \(\hat{y}\) is unique since \(T'(y|\hat{y})\) is nondecreasing and \(P_B(y|\cdot)\) is nonincreasing \(\hat{y}\), \(\tau(\cdot)\) is decreasing. Therefore, there is a fixed point such that \(\tau(\hat{y}) = \hat{y}\), and hence there is \(\hat{y}\) such that \(T(\hat{y}|\hat{y}) = 0\).

**Proof for Theorem 1.6.** Suppose \(\hat{y} < \frac{1}{2}\). Then, \(K(\hat{y}|s) < 1 - K(\hat{y}|s)\) since \(\frac{1}{2} \leq \mu(s) = 1 - K(\frac{1}{2}|s) = 1 - K(\hat{y}|s)\). Therefore, we have that

\[
P_A(y|\hat{y}) - P_B(y|\hat{y}) = \int_0^1 \min \left\{ \frac{\kappa}{1 - K(\hat{y}|s)}, 1 \right\} l(s|y)ds - \int_0^1 \min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} l(s|y)ds \\ \leq 0.
\]
Hence,
\[ T(y|\hat{y}) = \hat{y} P_A(y|\hat{y}) - (1 - \hat{y}) P_B(y|\hat{y}) < \frac{1}{2} [P_A(y|\hat{y}) - P_B(y|\hat{y})] \leq 0, \]
where the first inequality holds since \( \hat{y} < \frac{1}{2} \). Thus, a student with taste \( \hat{y} \) applies to \( B \), which is a contradiction.

Suppose now \( \hat{y} = 1 \). Then, \( n_A(s|1) = 1 - K(1|s) = 0 \), so \( P_A(y|\hat{y}) = 1 \) for any \( y \). Hence, \( T(1|1) = P_A(1|1) = 1 \), which contradicts the fact that \( T(y|\hat{y}) = 0 \). \( \blacksquare \)

**Proof of Theorem 1.7.** We show that there is a positive measure of states in which \( c_A(s) \neq c_B(s) \). Suppose on the contrary \( c_A(s) = c_B(s) \) for almost all \( s \). Recall that equilibrium admission cutoff of each college satisfies
\[ G(c_A(s)) = \max \left\{ 1 - \frac{\kappa}{1 - K(\hat{y}|s)}, 0 \right\} \quad \text{and} \quad G(c_B(s)) = \max \left\{ 1 - \frac{\kappa}{K(\hat{y}|s)}, 0 \right\}. \]

Since \( G(\cdot) \) is strictly increasing, if \( c_A(s) = c_B(s) \), then we must have either \( n_i(s) < \kappa \) for all \( i = A, B \) (so that \( c_A(s) = c_B(s) = 0 \)) or \( n_A(s) = n_B(s) \geq \kappa \).

First, we cannot have \( n_i(s) < \kappa \) for all \( i \) in equilibrium, since this means that all applicants are admitted by either college, and this contradicts to \( 2\kappa < 1 \). Second, suppose \( n_A(s) = n_B(s) \geq \kappa \). This implies that \( K(\hat{y}|s) = \frac{1}{2} \) for all \( s \) (recall that \( n_A(s) = 1 - K(\hat{y}|s) \) and \( n_B(s) = K(\hat{y}|s) \)). However, by (1.15), we have \( K(\hat{y}|s') < K(\hat{y}|s) \) for all \( s' > s \). Therefore, we reach a contradiction again. \( \blacksquare \)
A.6 Proof of Theorem 1.8

Consider a college, say $A$. Under the strategy suggested in the theorem, its payoff is

$$
\pi_A = \frac{1}{2} s_a (1 - \varepsilon) \int_{\hat{v}}^1 v \, dG(v) + \frac{1}{2} \left[ s_b (1 - \varepsilon) \int_{\hat{v}}^1 v \, dG(v) + (1 - \varepsilon) \int_{\tilde{v}} \tilde{v} \, dG(v) \right] 
$$

where $\tilde{v}$ is such that

$$
(1 - \varepsilon) [G(\tilde{v}) - G(\hat{v})] = \kappa - s_b (1 - \varepsilon) [1 - G(\hat{v})].
$$

and the second equality follows from $s_a = 1 - s_b$.

Consider now its payoff under the deviation. Notice first that those students in $[\hat{v} - \delta', \hat{v}]$ accept college $A$, since they prefer it over $C$. Therefore, $A$'s payoff under the deviation is

$$
\pi_A^d = (1 - \varepsilon) \int_{\hat{v} - \delta'}^{\hat{v}} v \, dG(v) + \frac{1}{2} s_a (1 - \varepsilon) \int_{\hat{v} + \delta}^1 v \, dG(v) 
$$

where $\hat{v}$ satisfies

$$
(1 - \varepsilon) [G(\hat{v}) - G(\tilde{v})] = \kappa - (1 - \varepsilon) [G(\hat{v}) - G(\hat{v} - \delta')] - s_b (1 - \varepsilon) [1 - G(\hat{v} + \delta)],
$$

that is, $\hat{v}$ is set to meet the capacity in the less popular state given that the students in
$[\hat{v} - \delta', \hat{v}]$ will attend it for sure in any state. Observe that $\hat{v} > \gamma$, since

$$(1 - \varepsilon)[G(\hat{v}) - G(\hat{v})] = \kappa - s_b(1 - \varepsilon)[1 - G(\hat{v})] - s_a(1 - \varepsilon)[G(\hat{v}) - G(\hat{v} - \delta')]$$

$$= (1 - \varepsilon)[G(\hat{v}) - G(\hat{v})] - s_a(1 - \varepsilon)[G(\hat{v}) - G(\hat{v} - \delta')],$$

(A.25)

where the first equality follows from the fact $s_a = 1 - s_b$ and (1.16), and the last equality follows from (A.24). Thus, we have

$$\frac{2(\pi^d_A - \pi_A)}{1 - \varepsilon} = 2 \int_{\hat{v} - \delta'}^{\hat{v}} v dG(v) - \left[ \int_{\hat{v}}^{\hat{v} + \delta} v dG(v) + \int_{\hat{v} - \delta'}^{\hat{v}} v dG(v) \right]$$

$$= 2 \left[ \hat{v} G(\hat{v}) - (\hat{v} - \delta)G(\hat{v} - \delta') - \int_{\hat{v} - \delta'}^{\hat{v}} G(v) dv \right]$$

$$\quad - \left[ (\hat{v} + \delta)G(\hat{v} + \delta) - \hat{v} G(\hat{v}) - \int_{\hat{v}}^{\hat{v} + \delta} G(v) dv \right]$$

$$\quad - \left[ \hat{v} G(\hat{v}) - \gamma G(\gamma) - \int_{\hat{v} - \delta'}^{\hat{v}} G(v) dv \right]$$

$$\geq (\hat{v} - 2\delta') \left[ G(\hat{v}) - G(\hat{v} - \delta') \right] - \delta \left[ G(\hat{v} + \delta) - G(\hat{v}) \right] - \hat{v} \left[ G(\hat{v}) - G(\hat{v}) \right]$$

where the second equality follows from the integration by parts, and the third equality follows from (1.16). The inequality holds since $\int_{\hat{v} - \delta'}^{\hat{v}} G(v) \leq \delta' G(\hat{v})$, $\int_{\hat{v}}^{\hat{v} + \delta} G(v) dv \geq \delta G(\hat{v})$ and $\int_{\gamma}^{\hat{v}} G(v) \geq (\hat{v} - \gamma) G(\gamma)$. Observe that by rearranging (A.25), we have $G(\hat{v}) - G(\gamma) = s_a \left[ G(\hat{v}) - G(\hat{v} - \delta') \right]$. Hence, using (1.16) again, we get

$$\frac{2(\pi^d_A - \pi_A)}{1 - \varepsilon} \geq \left[ G(\hat{v}) - G(\hat{v} - \delta') \right] (\hat{v} - 2\delta' - \hat{v} s_a) = \left[ G(\hat{v}) - G(\hat{v} - \delta') \right] \left( s_a (\hat{v} - \hat{v}) + s_b \hat{v} - (2\delta' + \delta) \right).$$

Therefore, for sufficiently small $\delta$, we have $\pi^d_A > \pi_A$. 
A.7 Proof of Lemma 1.3

Suppose college $B$ truthfully reports its ranking and capacity. We show that it is a best response for $A$ to do the same.

Observe first that it is a dominant strategy for $A$ to report its capacity truthfully. Clearly, $A$ has no incentive to over-report its quota, or else it will pay costs for over-enrollment, which is not profitable since $\lambda \geq 1$ and $v \in [0, 1]$; that is, each student beyond the (true) capacity brings about a cost higher than her value. Suppose $A$ under-reports its capacity. Apparently, $A$ will have a positive mass of unfilled seats, a part of loss. The possible gain from this deviation comes from that it causes a “rejection chain,” that is, $A$ rejects some students (who could be assigned to $A$ if $A$ had not underreported its quota), and those students apply to $B$, causing $B$ to reject some other students, who then apply to $A$. If those second group of students are more preferred by $A$ over the first group of students, then $A$ could be better off. However, the common preference of $A$ and $B$ implies that those second group of students are worse than the first group of students, since $B$ would not otherwise reject the second group of students.

Suppose now that $A$ has changed its rankings for students. Then, there exists some value $\hat{v}$ such that a positive mass of students with scores above $\hat{v}$ are reported to be less preferred to some positive mass of students with scores below $\hat{v}$. If the first group of students were not able to be admitted by $A$ for all states when $A$ reports its ranking truthfully, this does not change $A$’s payoff. However, when those first group of students were able to be assigned to $A$ for a positive measure states under truthful report, $A$’s deviation causes a rejection chain. The common preferences of $A$ and $B$, again, implies that this is not profitable for $A.$
A.8 Extension: More than Two Colleges

Our main model in Section 3.2 considers the case with two colleges. In this section, we show that our analysis extends to the case with more than two colleges. While the extension works for any arbitrary number of colleges, we provide the result for the three-college case for expositional simplicity. It will become clear that the method also extends to larger numbers.

Let $\sigma_i : [0, 1] \rightarrow [0, 1]$ be college $i$’s admission strategy, where $i = 1, 2, 3$. In each state $s \in [0, 1]$, let $\mu_{ijk}(s)$, where $i, j, k = 1, 2, 3$, denote the mass of students whose preference ordering is $i \succ j \succ k$. Define the following notations.

- $\mu_{i \succ j}(s) := \mu_{ijk}(s) + \mu_{ikj}(s) + \mu_{kij}(s)$ (the mass of students who prefer $i$ to $j$ in state $s$),
- $\mu_{i \succ j,k}(s) := \mu_{ijk}(s) + \mu_{ikj}(s)$ (the mass of students who prefer $i$ the most strongly in state $s$),

and

$$\bar{\mu}_{i \succ j} := \int_0^1 \mu_{i \succ j}(s) \, ds, \quad \bar{\mu}_{i \succ j,k} := \int_0^1 \mu_{i \succ j,k}(s) \, ds.$$ 

For given $\sigma_i(\cdot)$, $i = 1, 2, 3$, let $n_i(v)$ be the probability that a student with score $v$ attends college $i$ in state $s$ when she is admitted by $i$. That is,

$$n_i(v) := \left[ \prod_{t=j,k} (1-\sigma_t(v)) + \mu_{i \succ j}(s)\sigma_j(v)(1-\sigma_k(v)) + \mu_{i \succ k}(s)\sigma_k(v)(1-\sigma_j(v)) + \mu_{i \succ j,k}(s)\sigma_j(v)\sigma_k(v) \right].$$

The student will attend college $i$ if she is admitted only by $i$, which happens with probability $(1 - \sigma_j(v))(1 - \sigma_k(v))$; or is admitted by college $i$ and one of the less preferred colleges, which happens with probability $\mu_{i \succ j}(s)\sigma_j(v)(1 - \sigma_k(v)) + \mu_{i \succ k}(s)\sigma_k(v)(1 - \sigma_j(v))$ in state $s$; or is admitted by both of the other colleges but prefers $i$ the most, which happens with probability $\mu_{i \succ j,k}(s)\sigma_j(v)\sigma_k(v)$ in state $s$.

Thus, for a given profile of admission strategies, $\sigma = (\sigma_i)_{i=1,2,3}$, in equilibrium, the mass
of students who attend college $i$ in state $s$ is
\[ m_i(s) := \int_0^1 \sigma_i(v) n_i(v) dG(v), \]
and college $i$’s payoff is
\[ \pi_i = \int_0^1 v \sigma_i(v) \pi_i(v) dG(v) - \lambda \int_0^1 \max\{m_i(s) - \kappa, 0\} ds, \tag{A.26} \]
where
\[ \pi_i(v) := \int_0^1 n_i(v) ds. \tag{A.27} \]

Recall that in the two-school case, the monotonicity of $\mu(\cdot)$ yields cutoff states $(\hat{s}_A, \hat{s}_B)$ that trigger over-enrollment for each college, and the set of over-demanded states for each of them is a connected interval, $(\hat{s}_A, 1]$ and $[0, \hat{s}_B)$. To establish the existence of a maximally mixed equilibrium (MME), we have projected the admission strategies to a simpler (state) space, which allows us to use the Brouwer’s fixed point theorem. However, with more than two colleges, we do not know the structure of the set of over-demanded states in general, so we cannot directly define a map from cutoff states to cutoff states. Nonetheless, the main idea of the proof can be carried over, although to do so requires us using a fixed point theorem (Schauder) in a functional space.

Define a subdistribution $F_i : [0, 1] \to [0, 1], i = 1, 2, 3$, such that $F_i(0) = 0$ and
\[ F_i(s) := \text{Prob}(m_i(t) > \kappa \text{ for } t < s). \tag{A.28} \]

The subdistribution of college $i$ places a positive mass only on the states in which college $i$
is over-demanded. Observe that $F_i(\cdot)$ is nondecreasing and

$$0 \leq F_i(s') - F_i(s) \leq s' - s, \quad \forall s' \geq s.$$

Let $\mathcal{F}_i$ be the set of all such subdistributions and $\mathcal{F} := \times_{i=1}^3 \mathcal{F}_i$. (It will become clear that these subdistributions will play a similar role to the cutoff states in the two-school case.)

Using the subdistributions, each college’s payoff is now given by

$$\pi_i = \int_0^1 v \sigma_i(v) \mu_i(v) dG(v) - \lambda \int_0^1 (m_i(s) - \kappa) dF_i(s)$$

$$= \int_0^1 \sigma_i(v) H_i(v, \sigma_j(v), \sigma_k(v)) dG(v) + \lambda \int_0^1 \kappa dF_i(s),$$

where

$$H_i(v, \sigma_j(v), \sigma_k(v)) := v \mu_i(v) - \lambda \int_0^1 n_i(v) dF_i(s)$$

is college $i$’s marginal payoff from admitting a student with score $v$. The first part of the RHS of (A.30) is college $i$’s expected benefit, and the second part is its the expected cost. Note that this marginal payoff depends on the subdistribution $F_i$, as $\mu_i(v)$ is a constant for given admission strategies $(\sigma_i)_{i=1,2,3}$ (by (A.27)) and $n_i(v)$ is evaluated by the subdistribution.

Consider now college $i$’s marginal payoff. First, $H_i(v, 0, 0)$ is its marginal payoff from

---

2The second inequality holds because

$$F_i(s') - F_i(s) = \Pr(m_i(t) > \kappa \text{ for } t < s') - \Pr(m_i(t) > \kappa \text{ for } t < s)$$

$$= \Pr(m_i(t) > \kappa \text{ for } s < t < s')$$

$$\leq \Pr(s < t < s')$$

$$= s' - s.$$

3Note that since $F_i$ is Lipschitz continuous, so it is absolute continuous. Thus, the integration is well defined. Observe also that (A.29) does not involve $\max \{\cdot, \cdot\}$ in the cost (see (A.26) for comparison), as the subdistribution is defined for states where $m_i(s) > \kappa$ by (A.28), and the college’s cost is evaluated by the subdistribution.
admitting a student with score $v$ who is refused by both of the other colleges. Second, $H_i(v, 1, 0)$ and $H_i(v, 0, 1)$ are college $i$’s marginal payoffs from admitting a student with score $v$ who is admitted by college $j$ ($k$) but rejected by $k$ ($j$, respectively). Lastly, $H_i(v, 1, 1)$ is the marginal payoff from a student with $v$ who is admitted by both of the other colleges.

Using what we have so far established, (A.30) is decomposed as follow:

$$H_i(v, \sigma_j(v), \sigma_j(v)) = (1 - \sigma_j(v))(1 - \sigma_k(v))H_i(v, 0, 0) + \sigma_j(v)(1 - \sigma_k(v))H_i(v, 1, 0)$$

$$+ (1 - \sigma_j(v))\sigma_k(v)H_i(v, 0, 1) + \sigma_j(v)\sigma_k(v)H_i(v, 1, 1).$$

Let us now define $v_i^{11}, v_i^{10}, v_i^{01}$ and $v_i^{00}$ such that

$$H_i(v_i^{11}, 1, 1) = 0, \quad H_i(v_i^{10}, 1, 0) = 0, \quad H_i(v_i^{01}, 0, 1) = 0, \quad H_i(v_i^{00}, 0, 0) = 0.$$

That is, $v_i^{11}$ ($v_i^{00}$) is the threshold score that makes college $i$ indifferent from admitting or not a student who was admitted (or refused) by both of the other colleges. Likewise, $v_i^{10}$ and $v_i^{01}$ are the threshold scores that make college $i$ indifferent from admitting or not a student who was admitted by only one of the other colleges.

Similar to the two-school case, $H_i(v, \sigma_j, \sigma_k)$ partitions the students’ type space as depicted in Figure A.1.\(^4\) Each college admits a student with score $v$ such that $H_i(v, 1, 1) > 0$ and

\(^4\)It can be the case that $v_i^{01} < v_i^{10}$. 

<table>
<thead>
<tr>
<th>$H_i(v, 0, 0)$</th>
<th>$H_i(v, 1, 0)$</th>
<th>$H_i(v, 1, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$v_i^{10}$</td>
<td>$v_i^{11}$</td>
</tr>
<tr>
<td>$\sigma_i = 0$</td>
<td>$\Rightarrow \sigma_i = 1$</td>
<td>$\Rightarrow \sigma_i = 1$</td>
</tr>
<tr>
<td>$\sigma_j = 0$</td>
<td>$\Rightarrow \sigma_i = 0$</td>
<td>$\Rightarrow \sigma_i = 0$</td>
</tr>
<tr>
<td>$\sigma_k = 0$</td>
<td>$\Rightarrow \sigma_i = 1$</td>
<td>$\Rightarrow \sigma_i = 1$</td>
</tr>
<tr>
<td>$\sigma_j = 1$</td>
<td>$\Rightarrow \sigma_i = 0$</td>
<td>$\Rightarrow \sigma_i = 0$</td>
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<tr>
<td>$\sigma_k = 1$</td>
<td>$\Rightarrow \sigma_i = 1$</td>
<td>$\Rightarrow \sigma_i = 1$</td>
</tr>
</tbody>
</table>
rejects a student with score \( v \) such that \( H_i(v, 0, 0) < 0 \). That is, each college admits a student whose score is high enough so that its marginal payoff from admitting the student is positive even when she is definitely admitted by both of the other colleges; and each college rejects a student whose score is low enough that its marginal payoffs from admitting the student is negative. Where \( H_i(v, 1, 1) < 0 < H_i(v, 0, 0) \), each college admits a student with \( v \) if \( H_i(v, 1, 0) > 0 \) or \( H_i(v, 0, 1) > 0 \); that is, the college’s marginal payoff from admitting the student is positive only when she is admitted by one of the other colleges. This shows that colleges engage in strategic targeting for those intermediate range of scores.

Randomization may emerge for some students. For students with \( v \) such that

\[
\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\},
\]

all three colleges engage in mixed-strategies, where the mixed-strategies satisfy

\[
H_i(v, \sigma_j(v), \sigma_k(v)) = 0 \quad \forall i, j, k = 1, 2, 3.
\]

For students with \( v \) such that \( H_k(v, 0, 0) < 0 \) and

\[
\max \{H_i(v, 1, 0), H_j(v, 1, 0)\} < 0 < \min \{H_i(v, 0, 0), H_j(v, 0, 0)\},
\]

college \( k \) does not admit such students, but colleges \( i \) and \( j \) engage in mixed-strategies satisfying

\[
H_i(v, \sigma_j, 0) = 0 \quad \text{and} \quad H_j(v, \sigma_i, 0) = 0.
\]

A typical mixed-strategy equilibrium is depicted in Figure A.2 when, for instance,

\[
v_3^{00} < v_2^{00} < v_1^{00} < v_3^{01} < v_2^{01} < v_1^{01} < v_3^{10} < v_2^{10} < v_1^{10} < v_3^{11} < v_2^{11} < v_1^{11}.
\]
Figure A.2: Admission Strategies
Note that, as in the two-school case, there are many ways that colleges could coordinate (even in a mixed-strategy equilibrium). Hence, we consider the maximally mixed-strategy as before and provide the existence of such equilibrium.

For a given profile of subdistributions \((F_i)_{i=1}^3\), let \(\sigma := (\sigma_i)_{i=1}^3\) be the profile of admission strategies that satisfy the local conditions described above. Then, such \(\sigma\) in turn determines a new profile of subdistributions, \((F_i)_{i=1}^3\) via (A.28). Next, we define \(T : \mathcal{F} \to \mathcal{F}\), a self-map from the set of subdistributions to itself, where \(\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i\). The existence of equilibrium is achieved when \(T\) has a fixed point (on the functional space of \(\mathcal{F}\)).

As mentioned earlier, the idea of proving the existence of equilibrium is similar to the idea of Theorem 1.3, projecting the strategy profile into a simpler space. The difference is that in the two-school case, the strategy profiles are projected into the state space, but in the general case, they are projected into the set of subdistributions \(\mathcal{F}\).

**Theorem A.1.** There exists an equilibrium with maximally mixed-strategies.

We first show that \(\mathcal{F}\) is a compact and convex subset of a normed linear space, and \(T : \mathcal{F} \to \mathcal{F}\) is continuous. Then, \(T\) has a fixed point by Schauder’s fixed point theorem. \(^5\) We then show that the identified strategies in the previous step do indeed constitute mutual (global) best responses. We provide a formal proof in the next subsection.

**A.8.1 Proof of Theorem A.1**

For given \((F_i)_{i=1,2,3}\), consider colleges’ strategy profile \((\sigma_i)_{i=1,2,3}\) which satisfies the following local conditions:

- \(\sigma_i(v) = 1\) if \(H_i(v, 1, 1) > 0, i = 1, 2, 3\).

\(^5\)Schauder’s fixed point theorem is a generalization of Brouwer’s theorem on a normed linear space. It guarantees that every continuous self-map on a nonempty, compact, convex subset of a normed linear space has a fixed point (see Ok, 2007). In our framework, \(\text{CB}([0,1])\), the space of the continuous and bounded real maps on \([0,1]\), is a normed linear space, and \(\mathcal{F}\) is a nonempty, compact, convex subset of it.
\( \sigma_1(v) = 0 \) if \( H_1(v, 1, 1) < 0, H_2(v, 1, 1) > 0, H_3(v, 1, 1) > 0. \)

\( \sigma_2(v) = 0 \) if \( H_1(v, 1, 1) > 0, H_2(v, 1, 1) < 0, H_3(v, 1, 1) > 0. \)

\( \sigma_3(v) = 0 \) if \( H_1(v, 1, 1) > 0, H_2(v, 1, 1) > 0, H_3(v, 1, 1) < 0. \)

\[ \sigma_1(v) = 0, \sigma_2(v) = 0, \sigma_3(v) = 1 \] if

\[ \begin{align*}
H_1(v, 1, 1) &< 0 \\
H_2(v, 1, 1) &< 0, H_2(v, 0, 1) > 0 \\
H_3(v, 1, 1) &< 0, H_3(v, 0, 1) > 0
\end{align*} \]

\[ \sigma_1(v) = 1, \sigma_2(v) = 0, \sigma_3(v) = 1 \] if

\[ \begin{align*}
H_1(v, 1, 1) &< 0, H_1(v, 0, 1) > 0 \\
H_2(v, 1, 1) &< 0 \\
H_3(v, 1, 1) &< 0, H_3(v, 1, 0) > 0
\end{align*} \]

\[ \sigma_1(v) = 1, \sigma_2(v) = 1, \sigma_3(v) = 0 \] if

\[ \begin{align*}
H_1(v, 1, 1) &< 0, H_1(v, 1, 0) > 0 \\
H_2(v, 1, 1) &< 0, H_2(v, 1, 0) > 0 \\
H_3(v, 1, 1) &< 0
\end{align*} \]

\[ \sigma_1(v) = 1, \sigma_2(v) = 0, \sigma_3(v) = 0 \] if

\[ \begin{align*}
H_1(v, 1, 1) &< 0, H_1(v, 0, 0) > 0 \\
H_2(v, 1, 1) &< 0, H_2(v, 1, 0) < 0 \\
H_3(v, 1, 1) &< 0, H_3(v, 1, 0) < 0
\end{align*} \]

\[ \sigma_1(v) = 0, \sigma_2(v) = 1, \sigma_3(v) = 0 \] if

\[ \begin{align*}
H_1(v, 1, 1) &< 0, H_1(v, 1, 0) < 0 \\
H_2(v, 1, 1) &< 0, H_2(v, 0, 0) > 0 \\
H_3(v, 1, 1) &< 0, H_3(v, 0, 1) < 0
\end{align*} \]
\[ \sigma_1(v) = 0, \sigma_2(v) = 0, \sigma_3(v) = 1 \text{ if } \begin{cases} H_1(v,1,1) < 0, H_1(v,0,1) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,1) < 0 \\ H_3(v,1,1) < 0, H_3(v,0,0) > 0 \end{cases} \]

\[ \sigma_i(v) = 0 \text{ if } H_1(v,0,0) < 0, i = 1, 2, 3. \]

\[ \sigma_i(v) \text{'s satisfy } H_1(v,\sigma_2(v),\sigma_3(v)) = H_2(v,\sigma_1(v),\sigma_3(v)) = H_3(v,\sigma_1(v),\sigma_2(v)) = 0, \text{ if } \max_{i=1,2,3} \{H_i(v,1,0), H_i(v,0,1)\} < 0 < \min_{i=1,2,3} \{H_i(v,0,0)\} \]

\[ \sigma_i(v) \text{ and } \sigma_j(v) \text{ satisfy } H_i(v,\sigma_j,0) = 0 \text{ and } H_j(v,\sigma_i,0) = 0 \text{ if } h_k(v,0,0) < 0 \text{ and } \max \{H_i(v,1,0), H_j(v,1,0)\} < 0 < \min \{H_i(v,0,0), H_j(v,0,0)\} \]

Now, let \( \mathbf{CB}([0,1]) \) be the space of continuous and bounded real maps on \([0,1]\). Then, \( \mathbf{CB}([0,1]) \) is a normed linear space, with a sup norm \( \| \cdot \| \), i.e., for any \( F, F' \in \mathbf{CB}([0,1]) \),

\[ \| F - F' \| = \sup_{s \in [0,1]} |F(s) - F'(s)|. \]

**Lemma A.1.** \( \mathcal{F} \) is compact and convex.

**Proof.** We first show that \( \mathcal{F}_i, i = 1, 2, 3, \) is closed. To this end, consider any sequence \( \{F^n_i\} \), where \( F^n_i \in \mathcal{F}_i \) for each \( n \), such that \( \|F^n_i - F_i\| \to 0 \) as \( n \to \infty \). We prove that \( F_i \in \mathcal{F}_i \).

Observe first that \( F_i \) is nondecreasing. To see this, note that \( F_i(s') - F_i(s) < 0 \) for some
\( s' > s \) otherwise. But then,

\[
\| F_i^n - F_i \| \geq \max \{ |F_i^n(s') - F_i(s')|, |F_i(s) - F_i^n(s)| \}
\]

\[
\geq \frac{1}{2} (|F_i^n(s') - F_i(s')| + |F_i(s) - F_i^n(s)|)
\]

\[
\geq \frac{1}{2} |F_i^n(s') - F_i(s) + F_i(s) - F_i^n(s)|
\]

\[
\geq \frac{1}{2} |F_i(s) - F_i(s')| > 0
\]

which is a contradiction. Likely, for \( s' > s \), we must have that \( F_i(s') - F_i(s) \leq s' - s \). To see this, suppose on the contrary that \( F_i(s') - F_i(s) > s' - s \). Then,

\[
\| F_i^n - F_i \| \geq \max \{ |F_i(s') - F_i^n(s')|, |F_i^n(s) - F_i(s)| \}
\]

\[
\geq \frac{1}{2} (|F_i(s') - F_i^n(s')| + |F_i^n(s) - F_i(s)|)
\]

\[
\geq \frac{1}{2} |F_i(s') - F_i(s) + F_i^n(s) - F_i^n(s')|
\]

\[
\geq \frac{1}{2} |F_i(s') - F_i(s) - (s' - s)|
\]

\[
> 0,
\]

which is a contradiction again. Combining these, \( F_i \in F_i \), proving that \( F_i \) is closed.

Next, we prove that \( F_i \) is compact. Note that for any \( F_i \in F_i \) and \( s, s' \in [0, 1] \),

\[
|F_i(s') - F_i(s)| \leq |s' - s|,
\]

Hence, \( F_i \) is Lipschitz continuous with Lipschitz constant \( K \), and hence it is equicontinuous and bounded. By the Arzèla-Ascoli theorem,\(^6\) \( F_i \) is compact.

\(^6\)Arzèla-Ascoli theorem gives conditions for a set of \( C(T) \) to be compact, where \( C(T) \) is the space of continuous maps on \( T \) and \( T \) is a compact metric space. A subset of \( C(T) \) is compact if and only if it is
Next, we prove that \( F_i \) is convex. Observe that for any \( F_i, F'_i \in \mathcal{F} \) and \( s, s' \in [0, 1] \), for and \( \eta \in (0, 1) \),

\[
(\eta F_i + (1 - \eta) F'_i)(s') - (\eta F_i + (1 - \eta) F'_i)(s) = \eta(F_i(s') - F_i(s)) + (1 - \eta)(F'_i(s') - F'_i(s))
\leq \eta(s' - s) + (1 - \eta)(s' - s)
= s' - s,
\]

which proves that \( F_i \) is convex.

Since \( \mathcal{F}_i \) is compact and closed, so is its Cartesian product \( \mathcal{F} = \times_{i=1}^3 \mathcal{F}_i \) (with respect to the product topology).

\[\square\]

Lemma A.2. \( T \) is continuous.

Proof. The proof involves several steps:

Step 1. \( v_{ij} \)'s are continuous on \( F_1, F_2, F_3 \).

Proof. We first show that \( v_{ij} \)'s are continuous in \( F_i \). Fix any \( F_i \in \mathcal{F}_i \) and \( \varepsilon > 0 \). Take \( \delta = \varepsilon/2 \). Then, for any \( F_i, F'_i \in \mathcal{F}_i \) such that \( \| F_i - F'_i \| < \delta \), we have that

\[
\left| v_{ij} - v_{ij}' \right| = \left| \frac{\lambda}{\mu} \int_0^1 \mu(s)\nu_{ij,k}(f_i(s) - f'_i(s))ds \right|
= \left| \mu_{\nu_{ij,k}}(1)[F_i(1) - F'_i(1)] - \int_0^1 \mu'_{\nu_{ij,k}}(s)[F_i(s) - F'_i(s)]ds \right|
\leq 2\| F_i(s) - F'_i(s) \|
< \varepsilon,
\]

where the third equality follows from the integration by parts and \( F_i(0) = F'_i(0) = 0 \), and

the first inequality holds since \( \int_0^1 \mu'(s)ds = \mu(1) - \mu(0) \leq 1 \). \( \square \)
Step 2. \( \sigma_i \)'s in mixed-strategies are continuous.

Proof. Consider, at first, students with score \( v \) such that

\[
H_k(v, 0, 0) < 0, \quad \text{(A.31)}
\]

\[
H_i(v, 1, 0) < 0 < H_i(v, 0, 0), \quad \text{(A.32)}
\]

\[
H_j(v, 1, 0) < 0 < H_j(v, 0, 0). \quad \text{(A.33)}
\]

That is, college \( k \) puts zero probability for those students (by (A.31)), and colleges \( i \) and \( j \) use mixed-strategies \( \sigma_i \) and \( \sigma_j \) which satisfy \( H_i(v, \sigma_j, 0) = 0 \) and \( H_j(v, \sigma_i, 0) = 0 \).

Now, let \( J_i : [0, 1]^2 \times [0, 1]^2 \rightarrow [0, 1] \) such that

\[
J_i(F_i, F_j, \sigma_i, \sigma_j) \equiv H_i(v, \sigma_j, 0) = v[(1 - \sigma_j) + \mu_{i \succ j}(\sigma_j(v))] - \lambda \int_0^1 [(1 - \sigma_j) + \mu_{i \succ j}(s)\sigma_j(v)] dF_i(s),
\]

\[
J_j(F_i, F_j, \sigma_i, \sigma_j) \equiv H_j(v, \sigma_i, 0) = v[(1 - \sigma_i) + \mu_{j \succ i}(\sigma_i(v))] - \lambda \int_0^1 [(1 - \sigma_i) + \mu_{j \succ i}(s)\sigma_i(v)] dF_j(s).
\]

Then, \( \sigma_i \) and \( \sigma_j \) are the solution to \( J_i = 0 \) and \( J_j = 0 \) in terms of \( F_i \) and \( F_j \). Observe that

\[
J_i = (1 - \sigma_j)H_i(v, 0, 0) + \sigma_jH_i(v, 1, 0).
\]

Hence,

\[
\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + H_i(v, 1, 0) < 0,
\]

where inequality follows from (A.32). Similarly, we also have by (A.33)

\[
\frac{\partial J_j}{\partial \sigma_i} = -H_j(v, 0, 0) + H_j(v, 1, 0) < 0.
\]
Therefore,

\[
\Delta_{ij} := \begin{vmatrix}
\frac{\partial J_i}{\partial \sigma_i} & \frac{\partial J_i}{\partial \sigma_j} \\
\frac{\partial J_j}{\partial \sigma_i} & \frac{\partial J_j}{\partial \sigma_j}
\end{vmatrix} = \begin{vmatrix}
0 & \frac{\partial J_i}{\partial \sigma_j} \\
\frac{\partial J_j}{\partial \sigma_i} & 0
\end{vmatrix} = \frac{\partial J_i}{\partial \sigma_j} \frac{\partial J_j}{\partial \sigma_i} < 0.
\]

Since \(\Delta_{ji} \neq 0\), the Implicit function theorem implies that there are unique \(\sigma_i\) and \(\sigma_j\) such that

\[J_i(F_i, F_j, \sigma_i, \sigma_j) = 0\quad \text{and} \quad J_j(F_i, F_j, \sigma_i, \sigma_j) = 0.\]

Furthermore, such \(\sigma_i\) and \(\sigma_j\) are continuous.

Consider now the case that \(H_1(v, \sigma_2, \sigma_3) = H_2(v, \sigma_1, \sigma_3) = H_3(v, \sigma_1, \sigma_2) = 0\) when

\[
\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\}. \tag{A.34}
\]

Similar as before, let

\[
J_1(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_1(v, \sigma_2, \sigma_3) = 0,
\]

\[
J_2(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_2(v, \sigma_1, \sigma_3) = 0,
\]

\[
J_3(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_3(v, \sigma_1, \sigma_2) = 0.
\]

Observe that

\[
J_i = (1 - \sigma_j)(1 - \sigma_k)H_i(v, 0, 0) + \sigma_j(1 - \sigma_k)H_i(v, 1, 0) + (1 - \sigma_j)\sigma_kH_i(v, 0, 1) + \sigma_j\sigma_kH_i(v, 1, 1)
\]

\[
= (1 - \sigma_j)H_i(v, 0, 0) + \sigma_j(1 - \sigma_k)H_i(v, 1, 0) - (1 - \sigma_j)\sigma_kH_i(v, 1, 0) + \sigma_j\sigma_kH_i(v, 1, 1).
\]

where the second inequality holds after some rearrangement using the fact that \(1 - \mu_{i \neq k}(s) =\)
Therefore,

\[ \frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + (1 - \sigma_k)H_i(v, 1, 0) + \sigma_k H_k(v, 1, 0) + \sigma_k H_i(v, 1, 1) < 0, \]

where the inequality holds since \( H_i(v, 0, 0) > 0, \) \( H_i(v, 1, 0) < 0, \) \( H_k(v, 1, 0) < 0 \) and \( H_i(v, 1, 1) < 0 \) by (A.34). This implies that

\[
\Delta := \begin{vmatrix}
\frac{\partial J_1}{\partial \sigma_1} & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\
\frac{\partial J_2}{\partial \sigma_1} & \frac{\partial J_2}{\partial \sigma_2} & \frac{\partial J_2}{\partial \sigma_3} \\
\frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & \frac{\partial J_3}{\partial \sigma_3}
\end{vmatrix}
= \begin{vmatrix}
0 & \frac{\partial J_1}{\partial \sigma_3} & \frac{\partial J_1}{\partial \sigma_2} \\
\frac{\partial J_2}{\partial \sigma_3} & 0 & \frac{\partial J_2}{\partial \sigma_2} \\
\frac{\partial J_3}{\partial \sigma_3} & \frac{\partial J_3}{\partial \sigma_2} & 0
\end{vmatrix}
= \frac{\partial J_1}{\partial \sigma_2} \frac{\partial J_2}{\partial \sigma_3} \frac{\partial J_3}{\partial \sigma_1} + \frac{\partial J_1}{\partial \sigma_3} \frac{\partial J_2}{\partial \sigma_1} \frac{\partial J_3}{\partial \sigma_2} < 0
\]

Using the Implicit function theorem again, we conclude that such \( \sigma_1, \sigma_2, \sigma_3 \) exist and they are continuous. \( \square \)

Observe that from Step 1 and Step 2, \( H_i(v, \sigma_j, \sigma_k), \) \( i = 1, 2, 3, \) is continuous on \( (F_i)_{i=1,2,3} \) for a given \( s \) and fixed \( v. \)

**Step 3.** \( m_i(s) \) is continuous.

**Proof.** Consider any \( F_i, F_i' \in F_i \) such that \( \| F_i - F_i' \| < \delta \) for all \( i = 1, 2, 3. \) Let \( \sigma_i \) and \( \sigma_i' \) are admission strategies of college \( i \) which correspond to \( F_i \) and \( F_i' \), respectively. For a given \( s, \) fix any \( v \) and let

\[
n_i(v) = \prod_{t=j,k} (1 - \sigma_t(v)) + \mu_{i\succ j}(s)\sigma_j(v)(1 - \sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1 - \sigma_j(v)) + \mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v)
\]

and

\[
n_i'(v) = \prod_{t=j,k} (1 - \sigma_t'(v)) + \mu_{i\succ j}(s)\sigma_j'(v)(1 - \sigma_k'(v)) + \mu_{i\succ k}(s)\sigma_k'(v)(1 - \sigma_j'(v)) + \mu_{i\succ j,k}(s)\sigma_j'(v)\sigma_k'(v).
\]
Let $X := \{ v \in [0, 1] ||\sigma_i(v) - \sigma'_i(v)|| \geq \varepsilon/2 \}$. Clearly,

$$|\sigma_i(v) - \sigma'_i(v)| = |\sigma_i(v) - \sigma'_i(v)| 1_X(v) + |\sigma_i(v) - \sigma'_i(v)| 1_{X^c}(v),$$

where $1_X(v)$ is the indicator function which is 1 if $v \in X$ or 0 otherwise.

Since $v_i^{jk}$ are continuous by Step 1, we have

$$\int_0^1 1_X(v) dG(v) < \frac{\varepsilon}{2}. \tag{A.35}$$

For $v \in X^c$, it must be the case that either $\sigma_i = \sigma'_i$, or $\sigma_i$ and $\sigma'_i$ are the mixed-strategies. Thus, we have for $v \in X^c$,

$$|\sigma_i(v) - \sigma'_i(v)| < \frac{\varepsilon}{2}. \tag{A.36}$$

Observe that

$$\int_0^1 |\sigma_i(v) - \sigma'_i(v)| dG(v) = \int_0^1 |\sigma_i(v) - \sigma'_i(v)| 1_X(v) dG(v) + \int_0^1 |\sigma_i(v) - \sigma'_i(v)| 1_{X^c}(v) dG(v)
< \int_0^1 1_X(v) dG(v) + \int_0^1 |\sigma_i(v) - \sigma'_i(v)| 1_{X^c}(v) dG(v)
< \varepsilon,$$

where the first inequality holds since $\sigma_i, \sigma'_i \leq 1$, and the last inequality follows from (A.35) and (A.36).

This implies that there exist $\delta_1$ such that $||F_i - F'_i|| < \delta_1$, for all $i, i' = 1, 2, 3$, implies

$$\int_0^1 |\sigma_i(1 - \sigma_j)(1 - \sigma_k) - \sigma'_i(1 - \sigma'_j)(1 - \sigma'_k)| dG(v)
\leq \int_0^1 \left[ |\sigma_i - \sigma'_i| (1 - \sigma_j)(1 - \sigma_k) + |\sigma_j - \sigma'_j| \sigma'_i(1 - \sigma_k) + |\sigma_k - \sigma'_k| \sigma'_i(1 - \sigma'_j) \right] dG(v)
< \frac{\varepsilon}{4}$$
Similarly, there are $\delta_t$, $t = 2, 3, 4$, such that $\|F_i - F'_i\| < \delta_t$ respectively imply that

$$|\sigma_i \sigma_j (1 - \sigma_k) - \sigma'_i \sigma'_j (1 - \sigma'_k)| < \frac{\varepsilon}{4}, \quad |\sigma_i \sigma_k (1 - \sigma_j) - \sigma'_i \sigma'_k (1 - \sigma'_j)| < \frac{\varepsilon}{4}, \quad |\sigma_i \sigma_j \sigma_k - \sigma'_i \sigma'_j \sigma'_k| < \frac{\varepsilon}{4}.$$ 

Now, let $\delta = \min_{t=1,2,3,4} \{\delta_t\}$. We have that $\|F_i - F'_i\| < \delta$ implies

$$|m_i(s) - m'_i(s)| = \left| \int_0^1 \sigma_i(v) n_i(v) dG(v) - \int_0^1 \sigma'_i(v) n'_i(v) dG(v) \right| < \varepsilon.$$

That is, $m_i(s)$ is continuous on $(F_i)_{i=1,2,3}$. □ ■

Lemma A.2 proves the existence admission strategies that satisfy the local conditions. The proof that those strategies are mutual (global) best responses is analogous to that of the two college case. We briefly summarize it below:

Consider a college, say $i$. For given $\sigma_j(\cdot)$ and $\sigma_k(\cdot)$, let $\tilde{\sigma}_i(v) \in [0,1]$ be an arbitrary strategy for $v \in [0,1]$. Let $\tilde{\sigma}_i(v; t)$ be a variation of $\sigma_i(\cdot)$ such that for any $t \in [0,1]$,

$$\sigma_i(v; t) := t \tilde{\sigma}_i(v) + (1 - t) \sigma_i(v).$$

Define $i$'s payoff function in terms of $\sigma_i(v; t)$,

$$V(t) := \int_0^1 v \sigma_i(v; t) \pi_i(v) dG(v) - \lambda \int_0^1 \max \left\{ \int_0^1 \sigma_i(v; t) n_i(v) dG(v) - \kappa, 0 \right\} ds.$$

Observe that $V(\cdot)$ is continuous and concave in $t$. Therefore, we have

$$\pi_i(\tilde{\sigma}_i) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_i(\sigma_i),$$
where the second inequality holds since

\[ V'(0) = \int_0^1 [\tilde{\sigma}_i(v) - \sigma_i(v)] H_i(v, \sigma_j(v), \sigma_k(v)) \, dG(v) \leq 0 \]  \hspace{1cm} (A.37)

because if \( H_i(v, \sigma_j(v), \sigma_k(v)) \geq 0 \) for some \( v \), then \( \sigma_i(v) = 1 \geq \tilde{\sigma}_i(v) \); and if \( H_i(v, \sigma_j(v), \sigma_k(v)) \leq 0 \), then \( \sigma_i(v) = 0 \leq \tilde{\sigma}_i(v) \); and \( H_i(v, \sigma_j(v), \sigma_k(v)) = 0 \) otherwise.

**A.9 The Existence of Cutoff Equilibrium**

**Step 1: Existence of a profile of cutoff strategies for \( A \) and \( B \).**

Define

\[ \delta := \max_{v,e,e'} \left\{ x(e|v) \left( \frac{U_v(v,e)}{U_e(v,e)} \right) - X_v(e|v) \sqrt{y(e'|v) \left( \frac{V_v(v,e')}{V_e(v,e')} \right) - Y_v(v)} \right\}. \]

Let \( \mathcal{M} \) be the set of Lipschitz-continuous nondecreasing functions mapping from \([0,1]\) to \([0,1]\) with Lipschitz bound given by \( \delta \).

We define an operator \( T : [0,1]^2 \times \mathcal{M}^2 \to [0,1]^2 \times \mathcal{M}^2 \) as follows. For any \((\hat{s}_A, \hat{s}_B, \overline{\alpha}, \overline{\beta}) \in [0,1]^2 \times \mathcal{M}^2 \). The third component of \( T(\hat{s}_A, \hat{s}_B, \overline{\alpha}, \overline{\beta}) \) is a function \( \overline{\alpha} \) defined as follows.

First, \( \eta(v) \) is implicitly defined via \( H_\alpha(v, \eta(v), \overline{\beta}(v)) = 0 \) according to the Implicit Function Theorem (since \( U_\alpha > 0 \)). For \( v \) such that \( \eta(v) \in (0,1) \), the same argument as in the proof of Theorem 1.5 implies that

\[ 0 \leq -\eta'(v) \leq \frac{U_v(v, \eta(v))}{U_e(v, \eta(v))}. \]

Hence, \( \eta^{-1}((0,1)) \) forms an interval. For \( v \leq \inf \eta^{-1}((0,1)) \), we extend \( \eta \) such that \( \eta(v) = 1 \) and for \( v \geq \sup \eta^{-1}((0,1)) \), we set \( \eta(v) = 0 \). We now define \( \alpha(v,e) := 1_{\{e \geq \eta(v)\}} \). Let

\[ \overline{\alpha}(v) = \mathbb{E}_v[\alpha(v,e)|v]. \]

Then,

\[ \overline{\alpha}(v) = 1 - X(\eta(v)|v). \]
Since \( \eta \) is nonincreasing, \( a \) is nondecreasing. Further,

\[
\bar{a}'(v) = -x(\eta(v)|v) - X_v(\eta(v)|v)\eta'(v) \leq x(\eta(v)|v) \frac{U_v(v, \eta(v))}{U_v(v, \eta(v))} - X_v(\eta(v)|v) \leq \delta.
\]

It thus follows that \( \bar{a} \in \mathcal{M} \).

The fourth component of \( T(\hat{s}_A, \hat{s}_B, \bar{\alpha}, \bar{\beta}) \), labeled \( \bar{u} \), is analogously constructed via \( e' = \xi(v) \) determined implicitly by \( H_{\beta}(v, \xi(v), \bar{\alpha}) = 0 \), analogously, and belongs to \( \mathcal{M} \). This process also determines \( B \)'s strategy \( \beta \).

The first two components \( (\hat{s}'_A, \hat{s}'_B) \) are determined by the \( m_A(\hat{s}'_A) = m_B(\hat{s}'_B) = \kappa \), much as in the earlier proof, using \( \alpha \) and \( \beta \), along with \( (\hat{s}_A, \hat{s}_B) \) as input.

In sum, the operator \( T \) maps from \( (\hat{s}_A, \hat{s}_B, \bar{\alpha}, \bar{\beta}) \in [0, 1]^2 \times \mathcal{M}^2 \) to \( (\hat{s}'_A, \hat{s}'_B, \bar{\alpha}, \bar{\beta}) \in [0, 1]^2 \times \mathcal{M}^2 \). By Arzela-Ascoli theorem, the set \( \mathcal{M} \) endowed with sup norm topology is compact, bounded and convex. Hence, the same holds for the Cartesian product \([0, 1]^2 \times \mathcal{M}^2 \). Following the approach used earlier in Section A.8, the mapping \( T \) is continuous (with respect to sup norm). Hence, by the Schauder’s theorem, \( T \) has a fixed point. The fixed point then identifies a profile of cutoff strategies \( (\alpha, \beta) \) via \( \alpha(v, e) = 1_{\{e \geq \eta(v)\}} \) and \( \beta(v, e') = 1_{\{e' \geq \xi(v)\}} \).

**Step 2:** The cutoff strategies identified in Step 1 form an equilibrium under a condition.

Consider the following condition:

\[
\left( \frac{U_v(v, e)}{U(v, e)} + Y_v(\xi(v)|v) \right) V_v(v, e') \geq y(\bar{e}|v) \left( \frac{\mu_+(s) - \mu}{\mu_+} \right), \quad \forall v, e, \bar{e}, s,
\]

and

\[
\left( \frac{V_v(v, e')}{V(v, e')} + X_v(\eta(v)|v) \right) U_v(v, e) \geq x(\bar{e}|v) \left( \frac{\bar{\mu} - \mu_-(s)}{(1 - \mu_-(s))(1 - \mu)} \right), \quad \forall v, e', \bar{e}, s,
\]
where $\mu_+(s) := \mathbb{E}[\mu(\tilde{s}) | \tilde{s} \geq s]$ and $\mu_-(s) := \mathbb{E}[\mu(\tilde{s}) | \tilde{s} \leq s]$.

Since the RHS of each inequality is bounded by some constant, the conditions can be interpreted as requiring that each college values the non-common performance sufficiently highly. For instance, if $U(v, e) = (1 - \rho)v + \rho e$ and $V(v, e') = (1 - \rho)v + \rho e'$, then the LHS of each inequality will be no less than $\rho - \gamma$, where $\gamma := \max_{v, e, e'} \{ |X_v(e|v)|, |Y_v(e'|v)| \}$. So the condition will hold if the RHS is less than $\rho - \gamma$.

We now show the cutoff strategies identified by Step 1 form an equilibrium, given this condition. We show this only for college $A$, since the argument for college $B$ is completely analogous. For the proof, note first that $H_\alpha(v, e, \overline{\beta}(v))$ is nondecreasing, so it suffices to show that

$$\frac{\partial H_\alpha(v, e, \overline{\beta}(v))}{\partial v} \geq 0$$

whenever $H_\alpha(v, e, \overline{\beta}(v)) = 0$.

This result holds since

$$\text{sgn} \left( \frac{\partial H_\alpha(v, e, \overline{\beta}(v))}{\partial v} \right)$$

$$= U_v(v, e) - \frac{(U(v, e)(1 - \overline{\beta}) - \lambda(1 - \bar{s}_A)(1 - \mu_+(\bar{s}_A)) - \mu_+(\bar{s}_A))}{1 - \mu_+(\bar{s}_A)} \overline{\beta}(v)$$

$$= U_v(v, e) - U(v, e) \frac{1}{1 - \mu_+(\bar{s}_A)\overline{\beta}(v) + \mu_+(\bar{s}_A)\overline{\beta}(v)} \left( (1 - \mu) - \frac{1 - \overline{\beta}(v) + \mu_+(\bar{s}_A)\overline{\beta}(v)(1 - \mu_+(\bar{s}_A))}{1 - \mu_+(\bar{s}_A))} \right) \overline{\beta}(v)$$

$$= U_v(v, e) - U(v, e) \frac{\mu_+(\bar{s}_A) - \mu}{(1 - \mu_+(\bar{s}_A)\overline{\beta}(v))(1 - \mu_+(\bar{s}_A))\overline{\beta}(v)} \overline{\beta}(v)$$

$$\geq U_v(v, e) - U(v, e) \frac{\mu_+(\bar{s}_A) - \mu}{\mu_+(\bar{s}_A)\overline{\beta}(v)} \overline{\beta}(v)$$

$$= U_v(v, e) + U(v, e) \frac{\mu_+(\bar{s}_A) - \mu}{\mu_+(\bar{s}_A)\overline{\beta}(v)} \left( y(\xi(v)|v)\xi'(v) + Y_v(\xi(v)|v) \right)$$

$$\geq U_v(v, e) - U(v, e) \frac{\mu_+(\bar{s}_A) - \mu}{\mu_+(\bar{s}_A)\overline{\beta}(v)} \left( y(\xi(v)|v) \left( \frac{V_v(v, \xi(v))}{V_e(v, \xi(v))} - Y_v(\xi(v)|v) \right) \right)$$

$$\geq 0,$$

where the second equality is obtained by substituting $H_\alpha(v, e, \overline{\beta}(v)) = 0$, the first inequality
follows since $\bar{\mu}, \mu_+ (\hat{s}_S) \leq 1$, the penultimate equality follows from the fact that $\bar{\beta}(v) = 1 - Y(\xi(v)|v)$, the second inequality follows since the argument in the proof of Theorem 1.5 implies that $-\xi'(v) \leq \frac{V_v(\nu, \xi(v))}{V_{\nu'}(\nu, \xi(v))}$, the last inequality follows from the first part of the above condition.
Appendix B

Omitted Proofs in Chapter 2

B.1 Proof of Lemma 2.1

"Only if" part. Fix a bidding profile $\beta \in \mathcal{B}^{SNE}$ such that $c_j b_{j+1} \leq B_j$ for all $j \in N$. We show that if there is a bidder $j$ such that $B_j < c_j b_j$, then there is a profitable upward deviation. Suppose bidder $j + 1$ increases his bid to $b'_{j+1} = b_j - \varepsilon$. Since $B_j < c_j b_j$, bidder $j$ drops out at time $t \equiv \frac{B_j}{c_j b_j} < 1$ as $\varepsilon$ vanishes, and bidder $j + 1$ moves up one position. Hence, bidder $(j + 1)$’s payment is

$$[c_{j+1} t + c_j (\min\{t', 1\} - t)] b_{j+2},$$

where $t'$ satisfies $[c_{j+1} t + c_j (t' - t)] b_{j+2} = B_{j+1}$. The net payoff from the deviation is

$$[c_{j+1} t + c_j (\min\{t', 1\} - t)](v_{j+1} - b_{j+2}) - c_{j+1}(v_{j+1} - b_{j+2})$$

$$= \begin{cases} 
(B_{j+1} \frac{B_{j+2}}{b_{j+2}} - c_{j+1}) (v_{j+1} - b_{j+2}) \geq 0 & \text{if } t' \leq 1, \\
(c_j - c_{j+1})(1 - t)(v_{j+1} - b_{j+2}) \geq 0 & \text{if } t' > 1.
\end{cases}$$

Therefore, bidder $j + 1$ benefits by increasing his bid from $b_{j+1}$ to $b_j - \varepsilon$. 
B.2 Proof of Theorem 2.3

Proof of (i). The proof is analogous to that of the standard second-price auction. We omit the details. □

Proof of (ii). The proof involves several steps.

Step 1. *Any bidding profile* \( b_2 > b_1 \) *cannot be an equilibrium.*

*Proof.* If \( b_2 > b_1 \geq B \), bidder 2’s payoff is \( (v_2 - b_1) \frac{B}{b_1} < 0 \) because \( v_1 > B \geq v_2 \). Thus, this cannot be an equilibrium. If \( B \geq b_2 > b_1 \), then bidder 1’s payoff is zero. Since \( v_1 > B \geq v_2 \), however, bidder 1 can make a positive payoff \( v_1 - b_2 \) by increasing his bid to \( b'_1 \in (b_2, v_1) \). □

Step 2. A bidding profile \( b_1 \geq b_2 \geq B \) is a Nash equilibrium if and only if \( b_1 = b_2 \in \left[ B, \frac{2v_1B}{v_1+B} \right] \).

*Proof.* “If” part. Let \( b := b_1 = b_2 \). Note that bidder 1 is initially assigned to the position and drops out at time \( t = \frac{B}{b} \leq 1 \). Bidders’ payoffs are \( \pi_1 = t(v_1 - b) > 0 \) and \( \pi_2 = (1 - t)v_2 > 0. \)

Suppose bidder 1 decreases his bid to \( b'_1 < b \). Then, his payoff from this deviation is \( \pi'_1 = (1 - \frac{B}{b'_1})v_1 \). Observe that \( \pi_1 - \pi'_1 > t(v_1 - b) - (1 - t)v_1 = 2vt - (v_1 + B) \geq 0 \), where the first inequality holds since \( \pi'_1 < (1 - t)v \) and the last inequality holds since \( t = \frac{B}{b} \) and \( b \leq \frac{2v_1B}{v_1+B} \). Suppose bidder 2 increases his bid to \( b'_2 > b \). Then, his payoff from this deviation is \( \pi'_2 = \frac{B}{b}(v_2 - b) \). Observe that \( \pi_2 - \pi'_2 = (1 - t)v_2 - t(v_2 - b) = v_2 + B - 2v_2t \geq 0 \), where the last inequality holds that \( b \geq B \geq \frac{2v_2B}{v_2+B} \). Hence, no bidder has an incentive to deviate, and so such a bidding profile is an equilibrium.

“Only if” part. It follows immediately from (2.4) and the fact that \( B \geq \frac{2v_2B}{v_2+B} \). □

Step 3. A bidding profile \( B \geq b_1 \geq b_2 \) is a Nash equilibrium if and only if \( b_1 \in [v_2, B] \) and \( b_1 \geq b_2 \).

\(^1\) Notice that \( v_1 - b \geq v_1 - \frac{2v_1B}{v_1+B} = \frac{v_1(v_1-B)}{v_1+B} > 0 \), hence \( \pi_1 > 0 \).
Proof. “If” part. It is clear that bidder 1 has no incentive for a downward deviation, and bidder 2 cannot benefit by bidding $b_2' > b_1$ since $b_1 \geq v_2$.

“Only if” part. Suppose $B \geq b_1 \geq b_2$ in an equilibrium. Then, $\pi_1 = v_1 - b_2 > 0$ and $\pi_2 = 0$. If $b_1 < v_2$, then bidder 2 will benefit from bidding $b_2' \geq b_1$, since by doing so, he will earn $v_2 - b_1 > 0$. Thus in an equilibrium, we must have $b_1 \geq v_2$. □

Step 4. It is a weakly dominated strategy for bidder 2 to bid $b_2 < v_2$.

Proof. There are three cases to be considered: $b_1 \geq B > v_2$, $B > b_1 \geq v_2$, and $B > v_2 > b_1$.

In the first two cases, $\pi_2 = 0$ for any $b_2 \leq v_2$. Consider the last case, $B > v_2 > b_1$. Suppose $b_2 < v_2$. Then, $\pi_2$ is $v_2 - b_1$ if $b_2 > b_1$ or zero if $b_2 \leq b_1$. However, if $b_2 = v_2$, then $\pi_2 = v_2 - b_1 > 0$. Therefore, bidding $v_2$ weakly dominates $b_2 < v_2$. □

Proof of (iii). Observe first that $B \geq b_i \geq b_j$, $i, j = 1, 2$, cannot be an equilibrium, since bidder $j$ would increase his bid to $b_j' \in (B, v_j)$ and get a positive payoff, whereas his current payoff is zero. The proof for “If” part is the same as that in 2 except $\frac{2v_2 B}{v_2 + B} > B$. The proof for the “only if” part follows from (2.4) and the fact that $\frac{2v_2 B}{v_2 + B} > B$ (since $v_2 > B$). □
Proof of Lemma 3.1. Using the uniform distribution and the quadratic cost function, we get a firm’s payoff from a local downward deviation, where $x^* - 2\delta < x < x^*$:

$$\Pi_{ldd}(x, x^*) = \frac{(2\delta + x - x^*)^{n-1}}{n(n+1)(2\delta)^n} - \frac{1}{2}x^2$$

Maximizing this with respect to $x$ yields the first-order condition

$$-x + \left. \frac{(2\delta + x - x^*)^n}{n(2\delta)^n} \right|_{x=x^*\equiv \frac{1}{n}} = 0$$

and the second-order condition is

$$-1 + \left( \frac{(2\delta + x - x^*)^{n-1}}{(2\delta)^n} \right) < -1 + \frac{1}{2\delta}$$

where the inequality is from $x^* - 2\delta < x < x^*$. Notice that if $\delta \geq \frac{1}{2}$, then the second-order condition is negative. Therefore, $\Pi_{ldd}$ is increasing in $x_i$, and $x = x^*$ is the unique maximizer.
Similarly, the firm’s payoff from a local upward deviation, where \( x^* < x < x^* + 2\delta \), is

\[
\Pi^{lud}(x, x^*) = \frac{2^{-(n+2)}}{n(n+1)\delta} \left\{ 2^{n+3}\delta^2 + 2^n n(n+1)(x-x^*)^2 + 4\delta(x-x^*) \left( 2^n(n+1) - \left( \frac{x-x^*}{\delta} \right)^n \right) \right\} - \frac{1}{2}x^2
\]

The first-order condition is

\[
\frac{\left( x - x^* \right) - 2\delta x}{2\delta} + \frac{1 - \left( \frac{x-x^*}{2\delta} \right)^n}{n} \bigg|_{x = x^* \equiv \frac{1}{n}} = 0
\]

and the second-order condition is

\[
\frac{1}{\delta} \left( \frac{1}{2} - \delta \right) - \frac{(x-x^*)^{n-1}}{(2\delta)^n} < -1 + \frac{1}{2\delta}
\]

where the inequality is from \( x^* < x < x^* + 2\delta \). Notice that if \( \delta \geq \frac{1}{2} \), then the second-order condition is negative. Thus, \( \Pi^{lud} \) is decreasing in \( x \), and \( x = x^* \) is the unique maximizer. It can be easily shown that \( \Pi^{dd} \) and \( \Pi^{lud} \) are the same as \( \Pi^* \) at \( x = x^* \), which proves that \( \Pi^* \geq \max \{ \Pi^{dd}, \Pi^{lud} \} \). \( \blacksquare \)

**Proof of Lemma 3.2.** Using the uniform distribution and the quadratic cost function, we get a firm’s payoff from a global upward deviation, \( x \geq x^* + 2\delta \):

\[
\Pi^{gud}(x, x^*) = (x - x^*) - \delta + \frac{2\delta}{n} - \frac{1}{2}x^2
\]

Notice that \( 1 = \arg \max_x \Pi^{gud} \) and \( \max_x \Pi^{gud} = \frac{1}{2} - \frac{(n-2)\delta + 1}{n} \). Therefore,

\[
\Pi^* - \max_x \Pi^{gud} = \frac{(n-1)(1 + n^2(2\delta - 1))}{2n^2(n+1)} > 0
\]
because $n \geq 2$ and $\delta \geq \frac{1}{2}$.

**Proof of Lemma 3.3.** Proof of (i). Suppose there is no such $j$. Then, there is an $\varepsilon > 0$, independent of $\delta$, such that $B_{\varepsilon}(s) \cap S^\delta_{-i} = \emptyset$, where $B_{\varepsilon}(s)$ is an $\varepsilon$-neighborhood of $s \in S^\delta_i$, and for all $s' \in B_{\varepsilon}(s) \cap S^\delta_i$, $G^\delta_{-i}(s')$ is the same as $G^\delta_{-i}(s)$. Therefore, firm $i$ would reduce its surplus offer from $s$ to $s - \varepsilon$, which gives the same probability of winning but conveys a higher payoff conditional on winning. □

**Proof of (ii).** Suppose that there is a gap in $S^\delta$. Then, there exists a firm $i$ and $s'_i < s''_i$ such that $G^\delta_{-i}(s'_i) = G^\delta_{-i}(s''_i)$. By part (i), there must be no other firms offering $s \in (s'_i, s''_i)$. Thus, firm $i$ would offer a score slightly below $s''_i$, which is a contradiction. □

**Proof of (iii).** Suppose that firm $i$ puts some mass on $s > 0$. Then, $G^\delta_{-i}$ has an upward jump at $s$, which implies that a firm, say $j \neq i$, can be better off by transferring mass from an $\varepsilon$-neighborhood below $s$ to some $\eta$-neighborhood above $s$. Thus, there would be no $\varepsilon$-neighborhood below $s_i$ in which no other firms put mass. Then, it is not an equilibrium strategy for firm $i$ to put mass on $s$. Thus, the continuity of $G^\delta_{-i}$ is established. Since there is no mass point, it is strictly increasing. □

**Proof of (iv).** We first show that $s_i \in \arg \max_{\hat{s}, q_i} \pi_i(\hat{s}; q_i)$ is nondecreasing in $q_i \in Q_i$, where $Q_i$ denotes the support of $q$. To see this, consider $\hat{s}, s \in S^\delta_i$, where $\hat{s} > s$, and $\hat{q}, q \in Q^\delta_i$, where $\hat{q} > q$. Suppose $\pi_i(\hat{s}; q) - \pi_i(s; q) \geq 0$. We want to show that $\pi_i(\hat{s}; \hat{q}) - \pi(s; \hat{q}) > 0$. Notice that

$$\pi_i(\hat{s}; q) - \pi_i(s; q) \geq 0 \iff q[G^\delta_{-i}(\hat{s}) - G^\delta_{-i}(s)] \geq \hat{s}G^\delta_{-i}(\hat{s}) - sG^\delta_{-i}(s)$$

and

$$\pi_i(\hat{s}; \hat{q}) - \pi_i(s; q) > 0 \iff \hat{q}[G^\delta_{-i}(\hat{s}) - G^\delta_{-i}(s)] > \hat{s}G^\delta_{-i}(\hat{s}) - sG^\delta_{-i}(s)$$
Since \( G^\delta_{-i}(\cdot) \) is strictly increasing in \( s \), \([G^\delta_{-i}(\hat{s}) - G^\delta_{-i}(s)] > 0 \). Since \( \hat{q} > q \),

\[
\hat{q}[G^\delta_{-i}(\hat{s}) - G^\delta_{-i}(s)] > q[G^\delta_{-i}(\hat{s}) - G^\delta_{-i}(s)]
\]

which shows that \( \pi_i(\hat{s}; \hat{q}) - \pi_i(s; \hat{q}) > 0 \). Therefore, \( \pi_i(s; q) \) satisfies the strict single crossing property (Migrom and Shannon, 1994). By the Monotone Selection Theorem (Theorem 4'), \( s_i(q_i) \) is nondecreasing in \( q_i \).

Observe now that from part (iii), \( G^\delta_i(\cdot) \) is differentiable almost everywhere. The first-order condition of \( \pi_i \) with respect to \( s_i \) is

\[
-G^\delta_{-i}(s_i) + (q_i - s_i) \left( G^\delta_{-i}(s_i) \right)' = 0 \tag{C.1}
\]

Since \( G^\delta_{-i}(s_i) > 0 \) for any \( s_i \in \text{int}(S^\delta_i) \), it must hold that \((q_i - s_i) \left( G^\delta_{-i}(s_i) \right)' > 0 \). Therefore, \( q_i > s_i \) and \( (G^\delta_{-i}(s_i))' > 0 \). By Edlin and Shannon (1998, Theorem 1), \( s_i \) is strictly increasing. \( \square \)

For the proof of Lemma 3.4, we first establish a useful lemma.

**Lemma C.1.** If \( L : \mathbb{R}^n \to \mathbb{R}^m \) is a linear map (i.e., additive and homogenous of degree 1), then there exists a constant \( M_0 > 0 \) such that \( \|L(x)\| \leq M_0 \|x\| \) for all \( x \in \mathbb{R}^n \).

**Proof.** Let \( M = \sup \{\|L(e_1)\|, \ldots, \|L(e_n)\|\} \), where \( e_1, \ldots, e_n \) is the standard basis for \( \mathbb{R}^n \). Letting \( x = (x_1, \ldots, x_n) \),

\[
\|L(x)\| = \|x_1L(e_1) + \cdots + x_nL(e_n)\| \leq |x_1| \|L(e_1)\| + \cdots + |x_n| \|L(e_n)\| \\
\leq M(|x_1| + \cdots + |x_n|) \leq Mn \|x\|
\]

Taking \( M_0 = nM \), we get the result. \( \square \)
**Proof of Lemma 3.4.** For the first part of the lemma, notice that for any given $x \in \mathcal{X}_i^\delta$, $q$ is drawn from $F(\cdot|x)$, which has the support of $[\underline{q}(x), \overline{q}(x)]$. Since $q$ is in this interval and $s(q)$ is strictly increasing in $q$, $s$ is also in an interval, denoted by $[\underline{s}(x), \overline{s}(x)]$.

Now, we prove the second part of the lemma. Note that $s$ is implicitly defined in (C.1). Since $G_{-1}^\delta$ is differentiable a.e., so is $s$ by the Implicit Function Theorem. Now fix any $x \in \mathcal{X}_i^\delta$. Then for any $q, \tilde{q} \in (\underline{q}(x), \overline{q}(x))$, $|q - q'| < 2\delta$ clearly. By the definition of derivative, this implies that

$$|s(q) - s(\tilde{q}) - s'(\tilde{q})(q - \tilde{q})| \leq |q - \tilde{q}|$$

Since $|s(q) - s(\tilde{q})| - |s'(\tilde{q})(q - \tilde{q})| \leq |s(q) - s(\tilde{q}) - s'(\tilde{q})(q - \tilde{q})|$, we get

$$|s(q) - s(\tilde{q})| \leq |s'(\tilde{q})(q - \tilde{q})| + |q - \tilde{q}|$$

Now set $n = m = 1$ in Lemma C.1 and take $L = s'(\tilde{q})$. Then there is $M_0$ such that $|s'(\tilde{q})(q - \tilde{q})| \leq M_0 |q - \tilde{q}|$. Therefore,

$$|s(q) - s(\tilde{q})| \leq (M_0 + 1) |q - \tilde{q}|$$

Define $M_1 = M_0 + 1$, then $|s(q) - s(\tilde{q})| \leq M_1 2\delta$ for any $q, \tilde{q} \in (\underline{q}(x), \overline{q}(x))$. Now let $M = 2M_1$, which delivers the desired results. ■

**Lemma C.2.** There exists $\hat{\delta} > 0$ such that for any $\delta < \hat{\delta}$, $\overline{s} \in [\frac{1}{2} + a - \delta, \frac{1}{2} + a + \delta]$ and $\underline{s} \in [a - \delta, a + \delta]$.

**Proof.** Since $q \in [x + a - \delta, x + a + \delta]$ and $q \geq s(q)$, it is clear that the maximum surplus offer firm $i$ can make with $\delta > 0$ is $s^* + \delta$. Therefore, for any $x \in \mathcal{X}_i^\delta$, $\overline{s} \equiv \overline{s}(\overline{x}) \leq s^* + \delta$. Suppose now that $\overline{s} < s^* - \delta$. Then, there exists a feasible $(x_i, s_i)$ for some $i$ such that $s_i \in (\overline{s}, s^* - \delta]$ and $q(x_i) - s_i \geq \psi(x_i)$, because by investing $x_i = x^*$, $q(x_i) - s_i \geq x^* + a - \delta - (s^* - \delta) = \psi(x^*)$. Then, firm $i$ wins with probability 1. Therefore, $s^* - \delta \leq \overline{s} \leq s^* + \delta$. 

Suppose that \( s < a - \delta \). If firm \( i \) sets \( s_i = a - \delta \) by investing \( x_i = 0 \), then for any \( s_i \in [a - \delta, a + \delta] \), \( \pi_i(s_i; q_i) \geq 0 \). Suppose \( s > a + \delta \). This implies that \( x > 0 \). Then, firm \( i \)'s payoff from investing \( x \) is

\[
\int_{x+a-\delta}^{x+a+\delta} \left( q - s(q) \right) G_{-i}^\delta(s) \, dF(q|x) - \psi(x)
\]

where the first part shrinks to zero as \( \delta \) goes to zero (notice that the integrand is bounded), while the second part remains positive. Therefore, for sufficiently small \( \delta \), \( x \) should be zero. Since the best possible quality given \( x = 0 \) is \( a + \delta \) and \( q \geq s(q) \), it follows that \( s \leq a + \delta \). ■

**Lemma C.3.** For any \( s \in \text{int}(S^0) \), let \( x \in X^\delta_i(s) \). Then, for all \( i \in \mathcal{N} \), \( G_{-i}^\delta(s) \) converges to \( G_{-i}^0(s) = \psi'(x^0) \) as \( \delta \) vanishes, where \( x^0 \) is the limit of \( x \).

**Proof.** Take the limit superior to (3.5) and the limit inferior to (3.6). Then we have that

\[
\limsup_{\delta \to 0} G_{-i}^\delta(s + \delta M)[\varepsilon - \delta M] \leq \liminf_{\delta \to 0} [\psi(x + \varepsilon) - \psi(x)]
\]

and

\[
\lim_{\delta \to 0} G_{-i}^\delta(s + \delta M)[\delta M + \varepsilon] \geq \lim_{\delta \to 0} [\psi(x) - \psi(x - \varepsilon)],
\]

which respectively imply that

\[
\limsup_{\delta \to 0} G_{-i}^\delta(s + \delta M)\varepsilon \leq \psi(x^0 + \varepsilon) - \psi(x^0)
\]

and

\[
\lim_{\delta \to 0} G_{-i}^\delta(s + \delta M)\varepsilon \geq \psi(x^0) - \psi(x^0 - \varepsilon),
\]
where $x^0$ is the limit of $x$. Therefore, we have

$$\psi'(x^0) = \lim_{\varepsilon \to 0} \frac{\psi(x^0) - \psi(x^0 - \varepsilon)}{\varepsilon} \leq \lim_{\delta \to 0} G^\delta_{-i}(s + \delta M)$$

$$\leq \lim_{\delta \to 0} G^\delta_{-i}(s + \delta M) \leq \lim_{\varepsilon \to 0} \frac{\psi(x^0 + \varepsilon) - \psi(x^0)}{\varepsilon} = \psi'(x^0).$$

Since $s + \delta M$ goes to $s$ as $\delta$ approaches to 0, the limit of $G^\delta_{-i}$ is well defined. Denote the limit distribution by $G^0_{-i}(s)$. Then, $G^0_{-i}(s) = \psi'(x^0)$.

Proof of Corollary 1. Let $|\mathcal{N}| = 2$. Then, for all $i \in \mathcal{N}$ and for any $x \in \mathcal{X}^0$, $K^0(x) = \psi'(x) = x(s)$ for any $s \in S^0$ (the last equality holds since $\psi = \frac{1}{2}x^2$). Since $\mathcal{X}^0 = [0, 1]$, this proves that $K^0$ is the uniform distribution over $[0, 1]$. Now consider the score offer. At the equilibrium, $x$ must satisfy the first-order condition:

$$\frac{\psi(x)(x - s) - \psi'(x)}{(x - s)^2} = 0$$

Invoking the strict monotonicity of $x^0(s)$,

$$p(x^0) = x - x^0^{-1}(x) = \frac{\psi(x)}{\psi'(x)} = \frac{1}{2}x$$

Since $s = q - p = x + a - p$ and $p = \frac{1}{2}x$, it follows that $s = \frac{1}{2}x + a$ and $G^0(s)$ is the uniform distribution over $[a, \frac{1}{2} + a]$.

Proof of Lemma 3.5. For the local deviations, it suffices to show that $\Pi^{lud}$ and $\Pi^{lud}$ attain the unique maximum value $\Pi^T$ at $x = x^T$. The proof for the global deviations is similar, hence we omit it.

Consider local upward deviation at first. From the first-order condition, we have

$$\frac{P}{2\delta} \left[-F(x_i + a - \delta|x|^n - 1) - x_i \right]_{x_i = x^T} = 0$$
Since \( F(x_i + a - \delta | x^t)^{n-1} = \left( \frac{2}{2\delta} x_i - x^t \right)^{n-1} \), the second-order condition yields

\[
-\frac{P}{2\delta} (n - 1) \left( \frac{2}{2\delta} x_i - x^t \right)^{n-2} \frac{1}{2\delta} - 1 \bigg|_{x_i = x^t} = -1 < 0.
\]

Consider now local upward deviation. From the first-order condition, we have

\[
PF(x_i + a + \delta | x^t)^{n-1} \frac{1}{2\delta} - x_i \bigg|_{x_i = x^t} = 0
\]

Since \( F(x_i + a + \delta | x^t)^{n-1} = \left( \frac{2}{2\delta} x_i - x^t + 2\delta \right)^{n-1} \), the second-order condition yields

\[
\frac{P}{2\delta} (n - 1) \left( \frac{2}{2\delta} x_i - x^t + 2\delta \right)^{n-2} \frac{1}{2\delta} - 1 \bigg|_{x_i = x^t} = \frac{P}{4\delta^2} (n - 1) - 1 \leq 0 \Leftrightarrow P \leq \frac{4\delta^2}{n-1},
\]

where the inequality holds since \( P \leq \frac{4\delta^2}{n-1} \).

C.1 First-Price Auction with Two Firms: \( n = 2 \) and \( 0 < \delta < \frac{1}{2} \)

Suppose \( |\mathcal{N}| = 2 \) throughout this section. Consider a discrete mixed strategy such that both firms put mass \( m_1, \ldots, m_k \) on investment level \( x_1, \ldots, x_k \) for some integer \( k \geq 2 \). Let \( H(q) \) be the induced quality distribution, i.e.,

\[
H(q) := \sum_{t=1}^k m_t F(q|x_t),
\]

where \( F(\cdot|x_t) = U[x_t + a - \delta, x_t + a + \delta] \). Let \( h(q) \) be the associated probability density function.
Corollary 1 shows that at the limit, both firms randomize their investment uniformly over \([0, 1]\), thus the quality distribution is the uniform distribution over \([a, a+1]\). In Section 3.3.1, it is shown that if \(\delta = \frac{1}{2}\), then the unique symmetric pure-strategy equilibrium is that the firms put mass on \(x^* = \frac{1}{n} = \frac{1}{2}\), so that the quality distribution is the uniform distribution over \([a, a+1]\). A natural way of constructing an equilibrium from these observations is to make \(H(q)\) as close as possible to the uniform distribution.

Consider the strategy that satisfies the follows. (i) \(k = 2s\) for any \(\frac{1}{2(s+1)} \leq \delta < \frac{1}{2s} , s \in \mathbb{N}\); (ii) mass points are

\[
x_1 = \frac{1}{2}(1 - 2\delta k + \delta^2 k(k + 2)), \quad x_2 = \frac{1}{2}(1 + 4\delta - \delta^2 k(k + 2))
\]

\[
x_t = \begin{cases} 
  x_1 + (t - 1)\delta & \text{if } t \geq 3 \text{ is odd} \\
  x_2 + (t - 2)\delta & \text{if } t \geq 3 \text{ is even}
\end{cases} \tag{C.2}
\]

and (iii) their weights are

\[
m_1 = \frac{1}{s+1}, \quad m_2 = \frac{1}{s(s+1)}, \quad m_3 = \frac{s-1}{s(s+1)}, \quad m_4 = \frac{2}{s(s+1)}, \quad m_5 = \frac{s-2}{s(s+1)} \tag{C.4}
\]

\[
\cdots, \quad m_{2s-2} = \frac{s-1}{s(s+1)}, \quad m_{2s-1} = \frac{1}{s(s+1)}, \quad m_{2s} = \frac{1}{s+1}
\]

Notice that \(x_t\)’s defined (C.2) satisfy

\[
0 \leq x_1 < x_2 < \cdots < x_k
\]

\[
x_1 + \delta = x_3 - \delta, \quad x_2 + \delta = x_4 - \delta, \quad \cdots, \quad x_{k-2} + \delta = x_k - \delta,
\]

\[
x_1 + x_k = 1, \quad x_2 + x_{k-1} = 1, \quad \cdots, \quad x_{s-1} + x_s = 1.
\]

It is immediate to see that \(x_k \leq 1\), and \(x_t\)’s are symmetric around \(\frac{1}{2}\). It is also easy to
see that the weights put symmetrically. That is, $m_i$’s in (C.4) satisfy

$$m_1 = m_k, \ m_2 = m_{k-1}, \ \cdots, \ m_s = m_{s+1}.$$  

For any $t \geq 2$, it holds that $m_t + m_{t+1} = \frac{1}{s+1}$ by construction. Thus, if $s$ is an odd number, then

$$\sum_{t=1}^{s} m_t = m_1 + \sum_{t=2}^{s} m_t = \frac{1}{s+1} + \frac{1}{s+1} \frac{s-1}{2} = \frac{1}{2}.$$  

And if $s$ is an even number, then

$$\sum_{t=1}^{s} m_t = m_1 + \sum_{t=2}^{s-1} m_t + m_s = \frac{1}{s+1} + \frac{1}{s+1} \frac{s-2}{s} + \frac{s/2}{s(s+1)} = \frac{1}{2}.$$  

Therefore, $\sum_{t=1}^{2s} m_t = 1$ for any $s$.

For the given $H(q)$, a firm’s expected payoff from choosing $x_t$ is

$$\Pi(x_t) = \int_{x_t+a-\delta}^{x_t+a+\delta} H(q)[1 - F(q|x_t)] \, dq - \psi(x_t) \quad (C.5)$$

In equilibrium, $\{(x_t; m_t)\}_{t=1}^{k}$, must satisfy

- **Indifference**: $\Pi(x_t) = \Pi(x_{t-1})$ for all $t = 2, \ldots, k$.

- **Optimality**: $\Pi(x_t) \geq \begin{cases} \Pi(x_{td}) & \text{for } x_{td} \leq x_t \\ \Pi(x_{tu}) & \text{for } x_{tu} \geq x_t \end{cases}$.

It is tedious to show that the strategy defined above is indeed an equilibrium. We omit a formal proof for this. Instead, we provide some numerical examples in Figure C.1.\(^1\)

In the figure, there are six examples of $\delta = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$. In each of them, there are three

\(^1\)We set $a$ to be zero in the figure.
subfigures. The first column depicts $h(q)$, the second compares $H(q)$ (the solid line) with the uniform distribution (the dashed line), and the last column depicts a firm’s payoff. In each graph, the horizontal axis is the level of investment, $x$. Observe that $H(q)$ uniformly converges to the uniform distribution and the firms’ payoff shrinks to zero as $\delta$ vanishes.
• $\delta = \frac{1}{2}$.

• $\delta = \frac{1}{3}$.

• $\delta = \frac{1}{4}$.

• $\delta = \frac{1}{5}$.

• $\delta = \frac{1}{6}$.

(a) $h(q)$  
(b) $H(q)$  
(c) Firm Payoff

Figure C.1: $h(q)$, $H(q)$ and Firm Payoff
Bibliography


