The Effective Field Theory Approach to Fluid Dynamics

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Abstract

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In this thesis we initiate a systematic study of fluid dynamics using the effective field theory (EFT) program. We consider the canonical quantization of an ordinary fluid in an attempt to discover if there is some kind of quantum mechanical inconsistency with ordinary fluids at zero temperature. The system exhibits a number of peculiarities associated with the vortex degrees of freedom. We also study the dynamics of a nearly incompressible fluid via (classical) effective field theory. In the kinematical regime corresponding to near incompressibility (small fluid velocities and accelerations), compressional modes are, by definition, difficult to excite, and can be dealt with perturbatively. We systematically outline the corresponding perturbative expansion, which can be thought of as an expansion in the ratio of fluid velocity and speed of sound. This perturbation theory allows us to compute many interesting quantities associated with sound-flow interactions. Additionally, we also improve on the so-called vortex filament model, by providing a local field theory describing the dynamics of vortex-line systems and their interaction with sound, to all orders in perturbation theory. Next, we develop a cosmological model where primordial inflation is driven by a 'solid'. The low energy EFT describing such a system is just a less symmetric version of the action of a fluid—it lacks the volume preserving diffeomorphism. The symmetry breaking pattern of this system differs drastically from that of standard inflationary models: time translations are unbroken. This prevents our model from fitting into the standard effective field theory description of adiabatic perturbations, with crucial consequences for the dynamics of cosmological perturbations. And finally, we introduce dissipative effects in the effective field theory of hydrodynamics. We do this in a model-independent fashion by coupling the long-distance

degrees of freedom explicitly kept in the effective field theory to a generic sector that "lives in the fluid", which corresponds physically to the microscopic constituents of the fluid. At linear order in perturbations, the symmetries, the derivative expansion, and the assumption that this microscopic sector is thermalized, allow us to characterize the leading dissipative effects at low frequencies via three parameters only, which correspond to bulk viscosity, shear viscosity, and—in the presence of a conserved charge—heat conduction. Using our methods we re-derive the Kubo relations for these transport coefficients.

Contents

List of Figures

Acknowledgments

There are many people, both scientifically and personally, to whom I owe so so much. It is funny to say, but in numerious ways this thesis represents my life's work up to this point. As a stack of printed paper it may not merit the calories and effort of 27 years, but aside from its literal impact on the world, it really *is* what I have been working towards for so many years. As far as the actual work contained in this thesis is concerned, I have been thinking about it in some degree for almost 5 years. Before that, I spent 4 years getting ready for those 5 years. Before that, 4 years getting ready for those 4, and so on. All the way back to when I was a little boy obsessed with snakes. I feel privileged that I get to do science on a daily basis, and my fortune is based on the amazing support that I have received for all these years.

If you really go way back it all starts with my dad taking me camping, my brother giving me science fiction comics, and my mom reading to me on the front porch. Then there is Mr. Lee and Mr. G who showed me that the physical sciences were just as cool as the life sciences. Then Karen Frindell made me fall in love with chemistry (and quantum mechanics). It is sort of odd, but her class was one of the best times of my life. I remember walking around Healdsburg as a senior in high school reading Feynman's *QED* and wondering if I would ever understand how to do those crazy integrals he was talking about. I may not have known it at the time, but I guess that was when I became hooked on quantum field theory. My physics buddies in college: Geoff Fudenberg, Oleg Polyakov, and Thomas Dumitrescu. You guys made it feel like a team sport. Philip Kim for being a great role model and taking me on as a useless undergraduate—especially after the B in quantum mechanics. Throughout graduate school the incredible support, physics knowledge, and most importantly friendship of Andrej Ficnar (for coffee), AAron Veicht (for taking my crazy ideas seriously, sometimes), Kurt Hinterbichler (for deep knowledge), and Junpu Wang (for providing the rigor for both of us). The good postdocs of ISCAP: Rachel Rosen, Bart Horn and Riccardo Penco. Don't

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Introduction

Hydrodynamics is notoriously hard to solve. This is usually blamed on the non-linear nature of its equations of motion, and we will not quarrel with that. It is instructive however to ponder what makes hydrodynamics stand out with respect to *other* non-linear field theories (general relativity, for instance) where perturbative techniques exist that can be applied efficiently to a variety of physical situations. Our position is that there are certain configurations—the vortices—that in general cannot be dealt with consistently in perturbation theory. Struggling with (or in some sense admitting defeat to) these degrees of freedom introduces a constant source of tension throughout the following thesis.

From the perspective of a naive high energy physicist, fluid dynamics (especially classically) could be viewed as solved. Certainly the non-relativistic fundamental equations are well known; most are easily derived countless times in undergraduate textbooks. Yes fluid dynamics is hard and non-linear. But that is not a fundamental problem. We can obtain solutions by simply throwing the system on a good enough computer. What could there possibly be left for the pen-and-paper theorist to investigate? It is a subject over a 100 of years old that is (oh the humanity!) mostly taught in engineering departments.

Surely this arrogant viewpoint is wrong. The first piece of evidence to the contrary brought forth is turbulence, the "most important unsolved problem of classical physics" [1]. If we need any more evidence that fluids are an interesting system we can look no further

[1] Richard Feynman

than the money: you get a million dollars from the Clay Institute if you can prove something interesting about the Navier-Stokes equations.

The narrative arc of this thesis is formed around how to handle these vorticose degrees of freedom. Throughout our investigation we will highlight the effective field theory (EFT) viewpoint. The author and (if he may speak for them) his collaborators feel that this is a valuable formalism to attack a host of both quantum *and* classical questions. Not only does the EFT formalism allow us to utilize the full power of the mathematics developed in quantum field theory (QFT) which is particularly powerful for handling perturbation theory and quantum mechanical effects [2]. But possibly even more importantly, working in such a framework allows us to make *model independent* predictions. That is, in contrast to kinetic theory or any other explicit model of the microphysics. For instance, the AdS/CFT correspondence has been used to motivate the so-called "viscosity/entropy bound" [23]. This conjectured bound is statement claiming that in all fluid systems $\eta/s \geq 1/4\pi$. Moreover, in the limit of infinitely strong coupling (like the strongly coupled large N gauge theory AdS/CFT studies) $\eta/s \sim 1/4\pi$. Now while the AdS/CFT construction motivates this bound it certainly can not be used to *prove* the bound as it is just a particular realization of the microphysics. The model independence of the EFT construction will allow us to prove—or at least attempt to prove—statements about *all* fluid systems. Precisely the kinds of systems the bound is supposed to apply to. Additionally, different perspectives often suggest different kinds of questions. We will see an example of this in Chapter 3.

This thesis is organized as follows: In Chapter 1 we construct a relativistic perfect fluid [3] in the effective field theory framework. That is, we identify the degrees of freedom and

[2] We probably don't need to remind the reader that QFT as it stands has been developed over the last half century by some of the most brilliant minds in physics and mathematics. We are very much "standing on the shoulder's of giants" by utilizing these techniques.

[3] We will find the that keeping the theory relativistic will make its construction and treatment oftentimes *simpler*; the non-relativistic limit can then be easily taken at any moment. Given that this is the case, we

symmetries of our system and construct the most general action consisting of the former and compatible with the latter. This so called "effective action" will be organized in a derivative expansion. The leading order term in this action will correspond precisely to usual perfect fluid dynamics. We extend our formalism to include conserved charges, reproduce a relativistic extension of Kelvin's circulation theorem, and discuss the symmetry breaking pattern and associated goldstone modes.

Chapter 2 asks the question: why do we not see any standard (as opposed to super, etc.) fluids at zero temperature? In other words, do we not see any such systems in nature because there is something inconsistent quantum mechanically with the theory? With the EFT language described in Chapter 1 we are in a position to ask these questions and attempt to perform a sharp quantitative analysis. In engaging these issues we generate almost as many questions as we answer, but one thing is clear: the vortex degrees of freedom are at the heart of all the puzzling issues that we raise.

Ok, so the zero temperature quantum theory is confusing and most possibly inconsistent at any energy scale. Is there any hope of justifying our, seemingly overzealous, formalism? Indeed we can. Though in the previous chapters we saw that perturbatively the vortex degrees of freedom are ill-behaved, in Chapter 3 we can side-step this whole issue by applying perturbation theory *only* to degrees of freedom that are under control. By definition, in a nearly incompressible fluid sound waves are difficult to excite [4]. As a result, athough we have to treat the vorticose flow at the full non-linear level we can deal perturbatively in the dynamics of the compressional modes where the small expansion parameter is the background (vorticose) flow velocity over the speed of sound. In Chapter 3 we systematically develop this construction using our EFT techniques, which as advertised, are ideal for perturbation theory. Subsequently, we calculate many interesting quantities: the sound generated by

find no reason not to deal with the more general relativistic theory.

[4]Incompressibility is really a kinematical statement; provided the fluid flow velocity is much smaller than the speed of sound, $v \ll c_s$, any kind of fluid behaves as incompressible.

turbulence, ultra-sound scattering off turbulence, and the non-local potential induced by phonon exchange between vortex sources [5]. In order to correctly interpret this last calculation we develop an action for vortex lines. In doing so we improve on the so-called vortex filament model by providing a local field theory that describes the dynamics of vortex-line systems and their interactions with sound—to all orders in perturbation theory. Of course, it should be noted that expanding around the incompressible flow limit is not a completely new idea, and in fact was pioneered over 60 years ago to give birth to the field of aeroacoustics [25] [26]. Naturally, however, we feel that our formalism is more systematic, intuitive, and in the end, more powerful. We will try to defend these claims in the main body of the text.

Chapter 4 further justifies our effective field theory framework by applying it to early universe cosmology, in particular, inflation. As mentioned, one of the benefits of the EFT framework is that generic coupling to other systems (in this case, gravity) is straightforward. One of the frustrating features of inflationary models is that there are too many of them. This was remedied in some sense with the introduction of the so-called effective field theory of inflation (EFTI) [47]. The logic behind it can be briefly summarized by the following: In order for inflation to end there has to be some kind of "clock", therefore the background matter field has some kind of time dependence [6]. The Goldstone mode associated with this spontaneous breaking of time-translations can be shown to be the adiabatic perturbations of interest in inflationary cosmology. And so, this symmetry breaking pattern puts non-trivial constraints on the possible interactions between the adiabatic modes. Strikingly, it seems that the EFTI encompasses the core dynamics of all inflationary models. This logic is, in fact, not watertight. Indeed, as we will show in the main text, we can instead spontaneously

[5] This last quantity we actually calculate to next-to-leading-order. Although surely challenging to confirm experimentally, this "in principle" calculation would be *extremely* difficult in the standard formulation of the theory. Thus we use this as a demonstration of the perturbative power of our methods.

[6] For instance, in the case of standard slow-roll inflation this is just given by the value of the field as it slowly "rolls" down the nearly flat potential.

break *spacial*-translations, and use the metric scale factor as a clock. Incorporating the necessary symmetries in order to have a homogeneous and isotropic universe we recover the effective field theory of a solid [7]. In our language a solid is simply a less symmetric fluid: both solids and fluids are described by the same degrees of freedom but solids lack the internal volume preserving diffeomorphism which defines the fluid.[8] Throughout the chapter we will compute quantities of interest in observational cosmology whose behavior is, to our knowledge, strikingly different from the standard paradigm.

Finally, we will approach a topic which we will have danced around for some time now but neglected to address head on: how to incorporate dissipative effects in our effective field theory framework? Though it is true that we can often neglect these effects as they enter as higher derivative corrections, in order to approach many other subjects of great interest in the field of fluid dynamics viscosity plays a central role. For instance, boundary effects, shock waves and, most importantly, turbulence. As a more personal motivation we feel that our formalism's model independence and easy inclusion of quantum mechanical effects gives it the potential of playing a pivotal roll in settling the so-called entropy viscosity bound [74] [23]. In Chapter 5 we attempt to perform this construction and are only partially successful. We are able to incorporate in the EFT language dissipative effects at first order in the Goldstones but not to the full non-linear order. We accomplish this by coupling our fluid degrees of freedom to some hidden sector which "lives in the fluid". By being agnostic about the exact details of these hidden degrees of freedom except for the symmetries imposed by our physical system we will find that at low frequencies and momenta (i.e. to leading order in the derivative expansion) these degrees of freedom will introduce effects characterized by

[7] Many friends recommended that we should maybe use the word "jelly" instead of solid to emphasize its homogeneous and isotropic nature. We are dealing with a solid without any sort of lattice structure; it has $SO(3)$ rotational invariance, as opposed to some discrete subgroup. Nevertheless we will stick with the word "solid" for not better reason than because "jelly inflation" sounds weird.

[8] In fact, we will show that one *can't* use a fluid to generate inflation.

three parameters only: corresponding to bulk and shear viscosity, and in the presence of a conserved charge, heat conduction. In essence, we re-*derive* the famous Kubo relations for these quantities. So while our motivating questions regarding dissipation and fluids remain unsolved, we view this construction and the non-trivial constraints it has on our system as a solid step in the right direction.

We take a step back and examine our progress (or lack thereof) in the Outlook and include in the Appendix many laborious technical results necessary to support the claims throughout the main text.

But before we delve into the successful formulation of perturbative techniques for fluid dynamics (or the understanding of precisely why some fail), it is worth making a few comments on the literature. The study of fluid dynamics is vast and we would be foolish to claim the ability to present any comprehensive report, or for that matter, understanding, of what a great many brilliant minds have contributed and are contributing to the subject. However, there are a few names that are worth mentioning, even if only to betray the seeds of interest and motivation of this author and his collaborators.

People have thought about fluid dynamics for some serious time. Although a Lagrangian description of relativistic fluids has been around for quite a while [8] [9] as far as the author's knowledge is concerned the first people to highlight the EFT point of view in fluid dynamics were [10]. In some sense this thesis is owed to them. There is also a constellation of work which not only motivates the investigation of fluids more seriously but also demonstrates (even for classical quantities) the power of EFT theory methods. There really are three distinct camps here. There are the high energy and nuclear theorists thinking about strongly coupled systems, AdS/CFT, and cosmological issues with influential papers like [23, 74], and [47]. Then there are the high energy theorists using EFT methods to compute gravitational wave obervables in the black hole inspiral problem to great accuracy [30] and the whole industry that has arisen out of this work. Lastly, there are the classic (sometimes relativistic)

fluid dynamicists, for instance [42], [40], [41], [11], [34], and [25]. All have left their imprint on our work, and we would be nowhere without their guidance.

The results contained in this thesis are, unless otherwise noted explicitly, reported in several papers written by the author and coauthors Alberto Nicolis, Rafael A. Porto, Riccardo Rattazzi, and Junpu Wang. Chapter 2 follows [1], Chapter 3 follows [2] and [3], Chapter 4 follows [4] and Chapter 5 follows [5]. Figures 3.1, 3.2, and 3.6 by Max Cohen, with acknowledgement and thanks to Gene H. LePere.

Other work by the author: For unity of the concepts and formalism reported in this thesis, the author has omitted his other work published during his time as a PhD candidate under the wing of Professor Alberto Nicolis. The following is a brief outline of this work. The author has spent some time considering modified gravity theories, culminated in two papers. The first paper was work done in collaboration with K. Hinterbichler, L. Hui, A. Nicolis, and J. Wang [6] . The main thrust of the work is an investigation into the possibility of stable solitonic solutions in scalar theories with *derivative* interactions [9]. Of particular focus are the so-called Galileon theories. The author and his colleagues show that these theories do not possess such solutions and as a by-product extend the proof to a different class of derivatively coupled theories including our fluid effective theory. The second paper, done in collaboration with J. Wang, considers the classical stability of the Galileon field in the presence of non-relativistic sources [7]. It introduces the concept of absolute stability of a theory: if one can show that a field at a single point—infinity, for instance—in spacetime is stable, then stability of the field over the rest of spacetime is guaranteed for any positive energy source configuration. Amazingly the Dvali-Gabadadze-Porrati (DGP) model and previously studied spherically symmetric solutions of the Galileon are stable in this manner.

[9] By stable solitonic solutions we mean a static non-trivial solution of the field equations with finite total energy.

These results suggest that the whole Galileon model is stable in this way. The paper finds, however, that this is not the case. In fact, the DGP model is a unique point in the parameter space of Galileon theories. This implies that general solutions for general Galileon models may not be stable. Further investigation into these unstable solutions may prove fruitful.

Notations and Conventions: Unless otherwise stated, we work in natural units where $\hbar = c = k_B = 1$. The signature of the metric tensor is taken to be $(-, +, +, +)$.

Chapter 1

Fluid Dynamics in a Field Theory Language

1.1 The classical theory

To begin with, let us review how classical hydrodynamics can be cast into a field theoretical language. Our first task will be identifying the relevant low energy degrees of freedom associated with our system. With these in hand, we will next determine the symmetries imposed on the action consisting of these same degrees of freedom. We can now construct the most general action consisting of the former invariant under the later. Generically, an action that is constructed this way will contain an infinite tower of terms organized by powers of derivatives. However, at low enough frequencies (small time derivatives) and momenta (small spacial derivatives) there is a leading order contribution to this derivative expansion. For instance, from the perspective of effective field theory general relativity is precisely such a construction. When considering a metric $g_{\mu\nu}$ (the degrees of freedom) whose action is invariant under general covariance the leading order low energy term would just be the usual Einstein-Hilbert action while higher order terms in the derivative expansion would

correspond to a tower of gauge invariant quantities like R^2, $R_{\mu\nu}R^{\mu\nu}$, etc.

The construction of an action that generates the correct dynamics for relativistic fluids is not particularly new, and in fact is reviewed in the text book [8] and was in essence written down a little over a hundred years ago by [9]. The action as constructed then was more of a concise packaging to then generate the relevant equations of motion. However, in the path integral formulation of quantum field theory the action plays a crucial role and once we have such an action for the system we are interested in studying we can bring to bear the mathematical power of quantum field theory, even when computing *classical* observables.

For remainder of this document we will adopt the notation of [10] which is consistent with that of the articles written by the author and his peers.

1.1.1 The degrees of freedom

For a barotropic fluid we can parameterize the fluid's configuration space by giving at time t the comoving ('Lagrangian') coordinates ϕ^I of each fluid element, as a function of the physical ('Eulerian') position $\vec{x}$ occupied by that fluid element:

$$\phi^I = \phi^I(t, \vec{x}) \ , \qquad I = 1, 2, 3 \ . \tag{1.1}$$

In simpler language, in order to keep track of our fluid each "droplets" making up the fluid is label by three scalars. And for a given time t and a particular position $\vec{x}$ the three scalar functions $\phi^1(t, \vec{x})$, $\phi^2(t, \vec{x})$, and $\phi^3(t, \vec{x})$ will give us the particular label of the fluid element at that point in space-time.

Of course this description is completely equivalent to the inverse one, whereby one gives $\vec{x}$ as a function of the comoving coordinates (the labels ϕ^I) and of time, and at some point throughout this work we will in fact use this parametrization. However, we find working with the comoving coordinates to be a more convenient starting point to construct the theory. It is particularly convenient because it identifies the fluid's macroscopic degrees of freedom with three scalar functions of spacetime coordinates, i.e. with three scalar fields: this way

keeping track of Poincaré invariance is straigthforward, and so is coupling the fluid to other systems, like gravity for instance. Moreover, as we will see in a moment, hydrodynamics follows straightforwardly via standard effective field theory (EFT) logic once we identify the correct internal symmetries.

1.1.2 The symmetries

Before proceeding, it is worth stressing that we will be dealing with a fully relativistic theory, even though most laboratory fluids are highly non-relativistic. For these one could impose Galilean invariance rather than Poincaré invariance, but this would not simplify the analysis we are going to carry out, conceptually or algebraically. We thus see no reason why not to keep track of relativistic effects and just neglect them when appropriate.

We now come to the symmetries. The spacetime ones are of course the Poincaré group, under which our ϕ^I's transform as scalars. As for the internal ones, we have a huge redundancy in choosing the fluid's comoving coordinates. This is not a symmetry—it is the standard arbitrariness one has in parameterizing a Lagrangian system's configuration space. To make any progress, we should make an explicit choice. A particularly convenient one is the following: At some given reference pressure we demand that for the homogeneous and static fluid configuration—the fluid's 'ground state'—the comoving coordinates be aligned with the physical ones:

$$\phi^I = x^I \ . \tag{1.2}$$

It is then clear that homogeneity and isotropy for the physical properties of such a state cannot emerge unless the dynamics are invariant under *internal* translations and rotations:

$$\phi^I \ \rightarrow \ \phi^I + a^I \tag{1.3}$$

$$\phi^I \ \rightarrow \ O^I{}_J \, \phi^J \ , \tag{1.4}$$

where O is an $SO(3)$ matrix. So far, we have not specified what distinguishes a fluid from

an isotropic solid (a 'jelly'). It is an additional symmetry—the invariance under volume-preserving diffeomorphisms—

$$\phi^I \to \xi^I(\phi^J) \,, \qquad \det \frac{\partial \xi^I}{\partial \phi^J} = 1 \,. \tag{1.5}$$

This should not be confused with a trivial relabeling of the comoving coordinates, which we already got rid of. Rather, it corresponds to an invariance of the dynamics under physically moving fluid elements around without compressing or dilating the fluid anywhere. If we were to do so in a solid, we would feel transverse stresses trying to pull all volume elements back to their rest position. In a fluid, on the other hand, we only feel reaction forces against compression or dilation.

Invariance under shifts (eq. (1.3)) forces each field ϕ^I to be acted upon by at least one-derivative. At low momenta/low frequencies, the most relevant terms are those with the fewest derivatives. Therefore, the lowest order low-energy Lagrangian will involve exactly one derivative acting on each ϕ^I. Poincaré invariance then forces the Lagrangian to depend on the matrix

$$B^{IJ} = \partial_\mu \phi^I \, \partial^\mu \phi^J \tag{1.6}$$

only. Internal rotations (eq. (1.4)) force us to focus on $SO(3)$ invariant functions of B^{IJ}, and the volume preserving diffeomorphisms (eq. (1.5)) select the determinant among these. We thus have that the most generic low-energy Lagrangian compatible with all the symmetries is [10] [1]

$$S = \int d^4x \, F(b) \,, \qquad b \equiv \sqrt{\det B^{IJ}} \,, \tag{1.7}$$

where F is a generic function. Why we choose to use the square root of the determinant and not just the determinant itself will become clear in a moment.

[1] It is interesting to consider what effects higher order terms in the derivative expansion correspond to. In particular, it is worth noting that in the next-to-leading-order terms must be be completely dissipative as discussed nicely in [11], and [12].

It is straightforward to check that the action (1.7) describes the dynamics of a perfect fluid. The stress energy tensor is

$$T_{\mu\nu} = -F'(b)\, b\, B^{-1}_{IJ}\, \partial_\mu \phi^I \partial_\nu \phi^J + \eta_{\mu\nu} F(b) \tag{1.8}$$

which matches the standard form $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, \eta_{\mu\nu}$ upon the identifications [10]

$$\rho = -F(b)\ , \qquad p = F(b) - F'(b)\, b\ , \qquad u^\mu = \frac{1}{6\, b} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK}\, \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K\ . \tag{1.9}$$

In particular, we see that both ρ and p depend just on the degree of compression b, or equivalently, p depends on ρ only—we confirm that our fluid is 'barotropic'. Different choices for $F(b)$ thus correspond to different equations of state $p(\rho)$, and once the equation of state is given, $F(b)$ is uniquely determined.

Another fundamental object is the vector

$$J^\mu \equiv \tfrac{1}{6} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K\ . \tag{1.10}$$

It is *identically* conserved,

$$\partial_\mu J^\mu = 0 \tag{1.11}$$

as a result of its ϵ-tensor structure, and is related to b and the fluid's four-velocity u^μ via

$$b^2 = -J_\mu J^\mu\ , \qquad u^\mu = \frac{1}{b} J^\mu\ . \tag{1.12}$$

Notice that ρ, p, J^μ and u^μ are all invariant under our internal symmetries, eqs. (1.3–1.5), and so is $T_{\mu\nu}$. In fact, u^μ is invariant under *generic* internal diffs, with no volume-preserving restriction. What matters for characterizing the fluid flow is just that comoving coordinates do not change along it. Such a requirement is clearly preserved by generic diffeomorphisms of the comoving coordinates.

Since we have the correct stress-energy tensor for a fluid, we also have the correct hydrodynamical equations, which follow from stress-energy conservation.

1.2 Kelvin's Theorem

As is well known symmetries correspond to conserved quantities. The question is, what conserved quantities do the *huge* internal symmetry group of volume preserving diffeomorphisms given by (1.5) correspond to? As it turns out this symmetry generates and infinite set of conserved charges, each one corresponding to a line integral around a loop moving through the fluid. In the non-relativistic limit we will recover the usual

$$\Gamma = \oint_C d\vec{x} \cdot \vec{v} = \text{constant} \ , \tag{1.13}$$

where $\vec{v}$ is the velocity of the flow and the $\oint_C dx^i$ indicates an integral over a closed material contour, that is, an integral over closed loop traveling with the fluid flow. In non-relativistic fluid dynamics the "charge" above is usually called the circulation. This is Kelvin's circulation theorem. We reproduce the analysis found in [8] in what follows; the reader interested in a more geometrical proof can find one in [10].

Following Noether's theorem type logic let's look at infinitesimal expressions of our symmetry. A general diffeormorphism can be expressed as

$$\phi^I \to \phi^I + \epsilon \, \xi^I(\phi^J) \tag{1.14}$$

where ϵ is infinitesimal and ξ^I is suitably well behaved, however, in order that this be a volume preserving diffeomorphisms to first order in ϵ we have the further requirement that

$$\frac{\partial \xi^I}{\partial \phi^J} = 0 \ . \tag{1.15}$$

Now, as our Lagrangian is invariant under such an internal transformation we have, to first order in ϵ,

$$\mathcal{L}(\partial_\mu \phi^I + \epsilon \partial_\mu \xi^I) - \mathcal{L}(\partial_\mu \phi^I) = \epsilon \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^I\right)} \partial_\mu \xi^I = 0 \ , \tag{1.16}$$

where $\mathcal{L}(\partial_\mu \phi^I) = F\left(b\left(\partial_\mu \phi^I\right)\right)$ is simply our fluid Lagrangian. Now, note that the equations of motion for our fluid system (as the Lagrangian is only a function of $\partial_\mu \phi^I$) is given simply

by

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^I \right)} \right) = 0 \ . \tag{1.17}$$

We can rewrite (1.16) in such a way that we can utilize these equations of motion. Pulling out the derivative (1.16) becomes

$$\epsilon \partial_\mu J^\mu_\xi - \epsilon \xi_a \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^I \right)} \right) \right) = 0 \ , \tag{1.18}$$

where

$$J^\mu_\xi = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^I \right)} \xi^I \ . \tag{1.19}$$

We can see immediately that when the equations of motion are satisfied there is a conserved current J^μ_ξ and in particular, there is a conserved charge

$$Q_\xi = \int d^3x \, J^0_\xi(t, \vec{x}) \tag{1.20}$$

for each possible choice of $\xi^I(\phi^J)$ which satisfies (1.15). This is perfectly fine as it is, but there is a particular set of choices of ξ^I that will take the above expression and put it into a more aesthetically appealing (and familiar form).

Our symmetry tells us that we can take our comoving elements and reorganize them with no penalty provided we do this in a volume preserving fashion. The simplest example of such a rearranging would be to take a closed loop in the comoving space of the fluid and shift all the fluid elements along this loop while leaving the rest of the fluid untouched. Let's construct a ξ^I that does exactly this and see what charge it corresponds to.

Let a particular loop in comoving space (a loop which moves with the fluid flow in Eulerian space) be given by

$$\phi^I = \Omega^I(\tau) \quad \text{for} \quad 0 \leq \tau \leq 1 \quad \text{with} \quad \Omega^I(0) = \Omega^I(1) \ . \tag{1.21}$$

As promised can now choose $\xi^I(\phi^J)$ in order to move the comoving coordinates on the loop along the loop

$$\xi^I(\phi^J) = - \int_0^1 d\tau \, \frac{d\Omega^I}{d\tau} \delta^3(\phi^J - \Omega^J(\tau)) \ . \tag{1.22}$$

First of all, we can directly confirm that this function is divergence free, i.e. obeys (1.15) as:

$$\begin{aligned}\frac{\partial \xi^I}{\partial \phi^I} &= -\int_0^1 d\tau\, \frac{d\Omega^I}{d\tau}\frac{\partial}{\partial \phi^I}\delta^3(\phi^J - \Omega^J(\tau)) && (1.23)\\ &= -\int_0^1 d\tau\, \frac{d}{d\tau}\delta^3(\phi^J - \Omega^J(\tau)) && (1.24)\\ &= -(\delta^3(\phi^I - \Omega^I(1)) - \delta^3(\phi^I - \Omega^I(0)) = 0\ . && (1.25)\end{aligned}$$

And so, for this particular choice of ξ^I, the conserved quantity associated with it is given by

$$Q_{\rm loop} = -\int_0^1 d\tau \int d^3x\, \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi^I\right)}\frac{d\Omega^I}{d\tau}\delta^3(\phi^J - \Omega^J(\tau))\ . \tag{1.26}$$

We can massage this quantity by expressing the identity matrix δ^{IJ} as $(\partial\phi^I/\partial x^i)(\partial x^i/\partial\phi^J)$ and by changing the variables of integration to $d^3\phi$ at the price of a Jacobian; we can then integrate over the delta function. After some simple reorganization, our conserved quantity thus becomes

$$Q_{\rm loop} = -\int dx^i \left|\frac{\partial x}{\partial \phi}\right| (\partial_i \phi^I)\frac{\partial \mathcal{L}}{\partial\left(\partial_0\phi^I\right)}\bigg|_{\vec{x}(\Omega(\tau))}\ , \tag{1.27}$$

where the $|\partial x/\partial\phi|$ denotes the determinant of the Jacobian. Note that we have now manipulated the charge such that the integral over dx^i is over the material curve moving with the fluid flow. Inserting $\mathcal{L} = F(b)$ into the above expression we can take the derivative directly. After some algebra one can show that

$$-(\partial_\mu \phi^I)\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu\phi^I\right)} = -F'(b)\, b\, \left(u^\nu u_\mu + \delta^\nu_\mu\right)\ . \tag{1.28}$$

As for the determinant, it is easily evaluated as $(bu^0)^{-1}$. Putting everything together we have

$$Q_{\rm loop} = \oint dx^i\, (-F'(b))\, u^i\ . \tag{1.29}$$

This is the relativistic version of Kelvin's circulation theorem. In the non-relativistic limit the density is much much greater than the pressure, and therefore $F(b) \sim b$ and we have, as promised, the expression given in (1.13).

1.3 Inclusion of a conserved charge and thermodynamic relations

This formalism can be straightforwardly extended to accommodate a conserved charge carried by the fluid, like e.g. baryon number [12, 13]. One first introduces a new scalar field $\psi(\vec{x}, t)$ that shifts under the $U(1)$ symmetry transformation associated with the charge:

$$\psi \to \psi + a \ , \qquad a = \text{const} \ . \tag{1.30}$$

Then, in order to describe an *ordinary* charge-carrying fluid—as opposed to a superfluid[2]—one promotes such a symmetry to a ϕ^I-dependent shift symmetry:

$$\psi \to \psi + a\big(\phi^I\big) \ . \tag{1.31}$$

This guarantees that the Noether current associated with the $U(1)$ transformation (1.30) aligns with the fluid's four velocity,

$$j^\mu = n\, u^\mu \ , \tag{1.32}$$

as befits an ordinary fluid at lowest order in the derivative expansion. The lowest order (in the derivative expansion) action should now read

$$S = \int d^4x \, F(b, y) \ , \tag{1.33}$$

where b is the same as above, and y is defined as

$$y \equiv u^\mu \, \partial_\mu \psi \ . \tag{1.34}$$

The stress-energy tensor and the charge current follow straightforwardly from the action. They are

$$T_{\mu\nu} = \big(F_y y - F_b b\big) B^{-1}_{IJ} \, \partial_\mu \phi^I \, \partial_\nu \phi^J + \big(F - F_y y\big) \eta_{\mu\nu} \tag{1.35}$$

$$j^\mu = F_y u^\mu \ , \tag{1.36}$$

[2] See [14] for the effective field theory construction of a finite temperature superfluid.

where F's subscripts denote differentiation.

Equating $T^{\mu\nu}$ and j^μ above with the standard forms they take for a perfect fluid, and using the thermodynamic identities

$$\rho + p = T\,s + \mu\,n \quad \text{and} \quad d\rho = T\,ds + \mu\,dn\ , \tag{1.37}$$

yields the following "dictionary" between field theory variables and hydrodynamical ones [12]:

$$\rho = F_y y - F \tag{1.38}$$

$$p = F - F_b b \tag{1.39}$$

$$s = b\ , \qquad T = -F_b \tag{1.40}$$

$$n = F_y\ , \qquad \mu = y \tag{1.41}$$

and u^μ is still given by (1.9). Once again, we see that the function $F(b, y)$ is related to the equation of state: for instance, from the first, third, and fourth lines of the above we have

$$F(s, \mu) = n(s, \mu)\,\mu - \rho(s, \mu)\ . \tag{1.42}$$

It is interesting to note that the field theory language selects entropy density and chemical potential as the natural pair of thermodynamic variables to work with.

1.4 Expanding around an equilibrium background

The classical ground state of the fluid—the equilibrium configuration at a given pressure or density—is given by eq. (1.2).[3] This spontaneously breaks all of our spacetime and internal symmetries, except for the diagonal combinations of internal shifts and spacial translations, and of internal rotations and spacial ones. As a result, there are gapless Goldstone bosons—the phonons—of which only the longitudinal one propagates. Indeed, we can study the

[3]See Appendix 6.1 for an analysis of the symmetry breaking pattern, Goldstone bosons, and propagating degrees of freedom associated with the fluctuations about the equilibrium configuration of a perfect fluid with a conserved charge.

propagation of small perturbations of the ground state by splitting $\phi^I = x^I + \pi^I$ and expanding the action at second order in the π's. We get [10]

$$S_2 = \int d^4x \, (-F_b b) \left[\tfrac{1}{2} \dot{\vec{\pi}}^{\,2} - \tfrac{1}{2} c_s^2 \left(\vec{\nabla} \cdot \vec{\pi} \right)^2 \right] \tag{1.43}$$

where we defined the coefficient c_s^2 as

$$c_s^2 = \frac{F_{bb} b}{F_b} \tag{1.44}$$

where the function F, its derivatives and b are all evaluated at the background value of $b = 1$. We have also stopped differentiating between internal indices and spacial ones, since they transform in the same way under the unbroken combination of internal rotations and spacial ones. In other words, from now on we should think of $\vec{\pi}$ as a spacial vector field. We see from the quadratic action for $\vec{\pi}$ that only its longitudinal component has a gradient energy. The corresponding free solutions are plane waves propagating with speed c_s—the speed of sound. From the expressions for ρ and p as a function of b, eq. (1.9), one realizes that $c_s^2 = dp/d\rho\big|_{b=b_0}$, thus making contact with the usual expression for the sound speed a fluid. For a non-relativistc fluid $c_s^2 \ll 1$, whereas for an ultra-relativistic one $c_s^2 \simeq 1/3$. We will not commit to either case, but instead leave c_s^2 as a generic parameter.

The transverse excitations do not have a gradient energy and as a consequence obey a free particle-like equation of motion, whose general solution is linear in time:

$$\vec{\pi}_T = \vec{\nabla} \times \left(\vec{a}(\vec{x}) + \vec{b}(\vec{x}) \cdot t \right) , \tag{1.45}$$

where $\vec{a}$ and $\vec{b}$ are arbitrary vector functions. This is the linearized limit of a vortex in constant rotation. For this reason we will refer to the transverse excitations as 'vortices'. Their lack of gradient energy is, of course, a direct consequence of the volume-preserving internal diff invariance, eq. (1.5), and is at the origin of all the peculiarities we are going to unveil in the following chapters. Notice, however, that our diff-invariance is not a local symmetry, and as a consequence the configurations spanned by it—the vortices—are not

gauge-modes, but real dynamical degrees of freedom. For instance, they have non-vanishing conjugate momenta; they just do not feature wave solutions.

Chapter 2

The Quantum Mechanics of Perfect Fluids

2.1 Motivation

Empirically, all fluids we know of undergo a phase transition when we lower the temperature. Either they freeze, or they transform into more exotic systems, like super-fluids or Fermi liquids. Why are there no *ordinary* fluids at zero temperature? For weakly coupled systems we have a microscopic understanding of the low-temperature macroscopic behavior at finite density [15] (we also understand why such systems at high temperatures exhibit hydrodynamic behavior.) However for strongly coupled ones, such as the so-called non-Fermi liquids, we don't. Of course a strongly coupled system could look like anything at long-distances—there is no in-principle preference for the ordinary fluid dynamics. Nevertheless, classical hydrodynamics is so common in nature at high temperatures, and as we have seen in the previous chapter, it can be defined purely in terms of low-energy degrees of freedom and symmetries like an ordinary QFT, that it is natural to ask whether there exist strongly-coupled quantum systems that at zero temperature and finite density do behave like ordinary fluids.

It is tempting to conjecture that we know of no such systems simply because the corresponding quantum effective field theory would be inconsistent. As usual, the advantage of this viewpoint is that, as long as we allow for the most generic local dynamics involving all long-distance degrees of freedom and compatible with the symmetries, we are allowed to be completely agnostic about the microphysics yielding such macroscopic dynamics. Further motivation to investigate the consistency of the ordinary-fluid effective theory comes from the recent interest in the so-called holographic liquids at low temperatures. There, one deals with strongly coupled systems at finite density via a dual description in terms of classical gravity, which can be thought of as providing *the* microphysics behind these liquid-like states. Characterizing the long-distance dynamics of such systems is non-trivial however (see e.g. [16] and references therein). Our approach allows us to make progress on the (perhaps modest) question: can some of these low-temperature, finite-density systems behave like ordinary fluids? Partial indication that the answer may be 'yes' comes from the results of [17], but [16] argues that these cannot be consistently interpreted as a sign of hydrodynamic behavior.

Without committing to any models for the microphysics, we will argue that: *(i)* the effective theory of an ordinary fluid is not consistent; *(ii)* the effective theory of an ordinary fluid might be consistent after all; *(iii)* there is no guarantee that such effective theory be unitary at low energies. Clearly, the matter deserves further study.

Before starting our quantitative analysis which substantiates these claims, we conclude this introductory section with a few qualifications. First, we will only consider fluids at zero temperature. Thus, our results cannot be readily exported to the finite-temperature case. Indeed we know that at high temperatures ordinary fluids abound in the real world (hence the 'ordinary'), and they exhibit no funny quantum effects at long distances like those we are going to discuss. In these cases we expect quantum effects to be overwhelmed by thermal ones. It would be interesting to validate this expectation quantitatively—a task which we

leave for future work.

Second, we are going to neglect dissipative effects throughout this chapter. We feel we are justified in considering *perfect* fluids only because, as discussed in Chapter 5, dissipative effects in hydrodynamics, like for instance those parameterized by viscosity and heat conduction, are associated with higher derivative corrections to the perfect fluid dynamics. Therefore in the far infrared, that is for processes taking place on sufficiently long length scales, they can be safely ignored. Moreover, one may expect that the actual coefficients weighing these higher derivative corrections approach zero when the temperature is taken to zero. For instance in [16] it is argued that a finite viscosity at zero temperature is incompatible with hydrodynamic behavior. So, it is conceivable that by working at low enough temperatures and at long enough distances, one can make dissipation doubly negligible.

Third, a crucial role in our analysis will be played by vortices. Precisely the existence of 'light' vortices is what distinguishes an ordinary fluid from a superfluid at the classical level. As long as one concentrates on the compressional modes—the sound waves—both systems obey hydrodynamics [15], and this holds at the non-linear, relativistic level as well [10]. However in a superfluid the velocity field is irrotational, which implies that any vortex-like configuration will be singular at the center of the vortex, along a line, with the curl of the velocity field behaving like a delta-function peaked on this line. This means that from the viewpoint of the long distance/low energy effective field theory, the vortices are really UV-degrees of freedom, with finite energy per unit length (which is in fact mildly IR-divergent.) So, for instance, one cannot form vortices by scattering phonons of very low energies—there is a gap, and as long as one works below the gap, the vortex degrees of freedom can be ignored [1]. On the other hand, in ordinary fluids, one can build vortex configurations that are arbitrarily mild, that is, that involve arbitrarily low momenta only. As a consequence, there is no gap in the energy one can store in a vortex. Vortices in an ordinary fluid are

[1] A close relative of the superfluid vortex is the roton excitation, which is also gapped, and which can then also be neglected in the far infrared.

low-energy degrees of freedom, and they belong in the low-energy/long distance effective field theory together with the sound waves. In fact, we will see that, in a sense to be made precise below, the vanishing of the vortex gap is stronger than that of the sound wave gap—vortices are 'more massless' than sound waves. This will be the origin of all the quantum-mechanical peculiarities we will discuss, which are therefore absent for a superfluid.

2.2 The naive effective theory

The structure of the quadratic Lagrangian (1.43) already signals that, upon canonical quantization, we might be facing a strong-coupling problem for the vortices. The reason is the following: Consider first as a toy model a quantum-mechanical oscillator with some anharmonic corrections to the potential. In perturbation theory, one first solves the harmonic problem, thus getting the standard oscillator spectrum, and then treats the anharmonicities as small corrections. The approximation is justified for those states whose wavefunctions are localized in a region where the potential is dominated by its quadratic approximation. So, for perturbation theory to be applicable in this case, one needs at least the ground state to have a localized enough wave-function (highly excited states will always be outside the regime of validity of perturbation theory.) Of course, what localizes the ground state is the curvature of the harmonic potential—the oscillator's frequency. For the system to be 'weakly coupled', one thus needs a steep enough quadratic potential. If we now move on to field theory, the role of the quadratic potential is usually played—in the absence of mass terms—by the gradient energy. For a given spatial momentum $\vec{k}$, the gradient energy gives a potential $\propto k^2|\varphi_k|^2$. The vacuum wavefunction is thus localized about $\varphi_k = 0$, and cannot probe large field values where interactions may become important. In the absence of a gradient energy, on the other hand, each mode's vacuum wavefunction is totally delocalized in the quadratic approximation, and its dynamics are completely determined by the interactions. We thus reach the conclusion that a (massless) field theory without gradient energies is prone to

strong coupling, at all scales.

There is a number of caveats in applying the above logic to our case. The first is that the absence of gradient energy may be an accidental feature of the lowest order in the derivative expansion. This is the case, for instance, for the ghost condensate [18], where gradient energy starts at the four-derivative order, $E_{\rm grad} \propto \left(\nabla^2\pi\right)^2$. In the absence of quadratic terms with fewer spatial derivatives, such a term cannot be relegated to the class of higher-dimension operators, because it is marginal by definition—together with the kinetic energy $E_{\rm kin} \propto \dot{\pi}^2$ it determines how things behave under rescalings. In this case then, there is a well defined perturbative expansion. But this way out is not available to our vortices: the absence of gradient energy for them is enforced by a symmetry, which also forbids higher spatial-derivative quadratic terms. In the absence of time-dependence, exciting vortices costs nothing: we can deform the ground state $\phi^I = x^I$ in the 'transverse' direction via eq. (1.5) and pay no energy price, and this extends to non-linear order as well. The second caveat, more relevant for us, is that the above quantum oscillator toy model assumes that the anharmonic interactions are of the potential form—only in this case delocalization of the wavefunction *necessarily* leads to strong coupling, because having access to large values of q entails having access to large interactions. But in our case, by construction, we only have derivative interactions, and moreover the very same symmetry that forbids the vortex gradient energy is also going to forbid many interactions involving vortices. In particular, as we will see more concretely in the following, all vortex interactions that do not involve at least two time derivatives are forbidden. Therefore the connection between wavefunction delocalization and strong-coupling is less obvious in our case.

To settle the question, we should probe the theory by computing some physical quantity and check whether the perturbative expansion holds. The ideal candidates are usually S-matrix elements, but here we face a complication. The longitudinal phonon has standard wave solutions, which upon canonical quantization, get mapped onto standard free-particle

states. The transverse phonons, in contrast, do not behave as waves, and as a consequence there are no quantum asymptotic states associated with them. The classical field $\vec{\pi}_T$ behaves like a collection of infinitely many free particles rather than infinitely many oscillators. Upon quantization, its Hilbert space is not made up of standard Fock states. Without asymptotic states there is no S-matrix.

A possible alternative, is to compute instead local n-point functions in real space, and to check whether perturbation theory holds for them. They may be as physical as the S-matrix: they characterize the physical interaction among local sources that couple to our fluid. We do not need asymptotic states to set up such a question. For instance, we can define the theory and the associated correlation functions via the path-integral formulation. Another possibility, which we will choose, is to give the theory asymptotic states for the vortex degrees of freedom by deforming it in the IR. We can add to the classical action a term that is compatible with all the symmetries except for the volume-preserving diffs,

$$\Delta S = F_b(1) \int d^4x \, \tfrac{1}{2} c_T^2 \, B^{II} \,, \qquad c_T^2 \ll c_s^2 \tag{2.1}$$

and whose only effect, once expanded about the ground state, is to introduce a small gradient energy for the transverse Goldstones [2] :

$$S_2 \to \int d^4x \, (-F_b(1)) \left[\tfrac{1}{2} \dot{\vec{\pi}}^{\,2} - \tfrac{1}{2} c_s^2 \left(\vec{\nabla} \cdot \vec{\pi}_L \right)^2 - \tfrac{1}{2} c_T^2 \left(\nabla_i \pi_T^j \, \nabla_i \pi_T^j \right) \right] . \tag{2.3}$$

We thus have wave solutions, propagating with speed c_T, for the 'vortices' in the deformed theory—we promoted the vortices to real transverse phonons. Essentially, we are deforming the fluid into a solid/jelly that is stiff under compressional stresses but very soft under

[2] More precisely, the expansion of B^{II} is

$$B^{II} = -\dot{\vec{\pi}}^{\,2} + 2 \, \vec{\nabla} \cdot \vec{\pi} + \left(\nabla_i \pi^j \, \nabla_i \pi^j \right) . \tag{2.2}$$

The linear term is a total derivative, and can thus be neglected. The other terms, on top of giving the transverse phonons a gradient energy, correct the kinetic and gradient energies already present in (1.43). However in the limit $c_T^2 \ll c_s^2 < 1$ these corrections are also negligible.

transverse ones. The original theory is recovered in the $c_T \to 0$ limit, with a qualification of course. With this c_T deformation we are perturbing drastically the far infrared of the theory. We are adding asymptotic states, and we are going from not having an S-matrix to having one. So from this viewpoint the fluid limit is obviously discontinuous. However, we expect more local quantities like n-point functions to be continuous in this limit. The situation should be similar to having a fairly narrow unstable particle: strictly speaking it is not an asymptotic state, yet for processes happening at time- and distance-scales much shorter than the particle's lifetime, we can treat it as an asymptotic state and associate an S-matrix to it. For instance, for scattering processes faster than ~ 10 minutes, neutrons behave like asymptotic states.

So, concretely, here is our program. We will consider scattering and decay processes in the c_T-deformed theory. In particular, for simplicity we will stick to processes that involve at most four external legs. Thus to carry out calculations at tree-level, we need to expand the action up to quartic order in the $\vec{\pi}$ field; this is done in the Appendix, and the result is reported below. By construction, interactions involve one derivative per field. For finite c_T, the theory is a standard derivatively coupled theory, and thus strongly-coupled in the UV. The strong coupling scale will depend on all parameters of the theory; however, we are interested in the c_T-dependence, since eventually we will be taking c_T to zero while keeping everything else fixed. If the strong coupling energy scale slides to zero in this limit, or equivalently, if cross sections and decay rates at fixed momentum or energy blow up in this limit, the theory is strongly-coupled at all scales, and thus inconsistent. Notice that we are trying to ascertain the consistency of the theory by computing something—the S-matrix—that loses its meaning in the limit we are interested in. Still at finite c_T we expect that the strong-coupling scale for the S-matrix be related to a similar strong-coupling scale for n-point functions—that is, that for the latter perturbation theory break down at a distance-scale given by the strong-coupling scale inferred from the S-matrix. Thus, even though the

S-matrix does not exist in the $c_T \to 0$ limit, the formal fact that it appears to be strongly coupled at all scales is probably signaling that real-space n-point functions cannot be reliably computed in the fluid theory, at any distance scales.

A final point before we begin the program outlined above. The analysis throughout this chapter representing the work performed in the [1] was completed before we had a thermodynamic understanding of our fluid system as given by section 1.3. With our new understanding we should be concerned about the action presented in (1.43) and the associated deformed action (2.3). As stated, we are explicitly looking at the T=0 field theory but, from section 1.3, we know that $T = 0 \Rightarrow F_b = 0$ and so our action is trivial! In order to have a non-trivial quadratic Lagrangian we need a conserved charge. This is, naturally, obvious in hindsight. When we dial the temperature down on a fluid without a conserved charge (like a fluid of photons) the fluid actually vanishes! So despite the fact that the following discussion centers around the $F(b)$ Lagrangian we will argue in the conclusions of this chapter that we can in fact trust, at least qualitatively, the following analysis for the more appropriate $F(b, y)$ case as well.

2.2.1 Expanding the Lagrangian—Interactions

It is convenient to rewrite the original Lagrangian (1.7) as

$$\mathcal{L} = -w_0 f\big(b\big) \tag{2.4}$$

where $w_0 = -F_b(1) = (\rho + p)_{b=1}$ is the ground-state's enthalpy density, and f is normalized accordingly, so that $f'(1) = 1$. With this new notation the speed of sound (1.44) is simply $c_s^2 = f''(1)/f'(1) = f''(1)$. Note that the derivatives here are with respect to b. Also, we will use $\partial\pi$ to denote the matrix with entries $(\partial\pi)_{ij} = \partial_i \pi_j$, and the brackets $[\dots]$ to denote the

trace of the matrix within. Then, up to fourth order the action is (see Appendix 6.2)

$$\begin{aligned}\mathcal{L} \;\to\; & w_0\Big\{\tfrac{1}{2}\dot{\vec{\pi}}^2 - \tfrac{1}{2}c_s^2[\partial\pi]^2 - \tfrac{1}{2}c_T^2[\partial\pi^T\partial\pi] \\ & + \tfrac{1}{2}c_s^2[\partial\pi][\partial\pi^2] - \tfrac{1}{6}\big(3c_s^2+f_3\big)[\partial\pi]^3 + \tfrac{1}{2}(1+c_s^2)\,[\partial\pi]\dot{\vec{\pi}}^2 - \dot{\vec{\pi}}\cdot\partial\pi\cdot\dot{\vec{\pi}} \\ & - c_s^2[\partial\pi]\det\partial\pi - \tfrac{1}{8}c_s^2[\partial\pi^2]^2 + \tfrac{1}{4}\big(c_s^2+f_3\big)[\partial\pi^2][\partial\pi]^2 - \tfrac{1}{24}\big(3c_s^2+6f_3+f_4\big)[\partial\pi]^4 \\ & + \dot{\vec{\pi}}\cdot\partial\pi^2\cdot\dot{\vec{\pi}} - (1+c_s^2)[\partial\pi]\,\dot{\vec{\pi}}\cdot\partial\pi\cdot\dot{\vec{\pi}} + \tfrac{1}{2}|\partial\pi^T\cdot\dot{\vec{\pi}}|^2 \\ & + \tfrac{1}{4}\big((1+3c_s^2+f_3)\,[\partial\pi]^2 - (1+c_s^2)\,[\partial\pi^2]\big)\dot{\vec{\pi}}^2 + \tfrac{1}{8}(1-c_s^2)\,\dot{\vec{\pi}}^4\Big\}\,. \end{aligned} \tag{2.5}$$

The first line is the free part of the Lagrangian, including the c_T-deformation. The second line collects the trilinear interactions, whereas the third and fourth lines collect the quartic ones. f_3 and f_4 stand for $f'''(1)$ and $f''''(1)$, respectively. Finally, notice that via the suffix T we indicate the transpose of a matrix, rather than the transverse part of $\vec{\pi}$ as we did above.

At this order we have four free parameters: c_s^2, c_T^2, f_3, and f_4. The dimensionful overall factor of w_0 just gives us some reference units—we could use units in which it is one. For c_T^2, we know that we want $c_T^2 \ll c_s^2$. As to c_s^2, it will be much smaller than one for a non-relativistic fluid, and of order one (1/3) for an ultra-relativistic one. In the former case we expect f_3 and f_4 to be naturally of order c_s^2, or smaller. If they were larger, $b = 1$ would be a special point for the shape of $f(b)$, since by going to, say, $b = 2$, the second derivative of f, which controls c_s^2, would undergo a relative change of more than order one. Likewise, in the ultra-relativistic case, for the same reason we probably want f_3 and f_4 of order one, or smaller. That is, if we assume that $b = 1$ is a fairly generic point for f, f_3 and f_4 have to be at most of order c_s^2. [3] On the other hand it may be possible to have a fluid with some

[3] This is what happens for instance for the constant (i.e., b-independent) c_s^2 Lagrangian

$$\mathcal{L} = -w_0\big(b\big)^{1+c_s^2}\,, \tag{2.6}$$

which corresponds to the simple equation of state $p = c_s^2\rho$. Notice however that such a simple case, besides being extensively considered by cosmologists, is not preferred in any sense over more generic equations of state—with the exception of the ultra-relativistic case, where the linear equation of state $p = 1/3\,\rho$ follows

feature in the equation of state, where f'' is small but higher derivatives of f are large. In the following we will make no assumptions about these couplings, since even carefully chosen values for them do not lead to drastic simplifications for our computations. Similarly, the non-relativistic case $c_s^2 \ll 1$ is only slightly simpler that the fully relativistic one, and we thus see no reason why not to investigate the latter.

2.2.2 Sound-wave strong-coupling scale

As a warmup, we estimate the strong-coupling scale for longitudinal phonon scatterings by ignoring the vortices, both as external states as well as internal lines. We assume for simplicity that f_3 and f_4 are of their 'natural' size, c_s^2. We also assume that there are no cancellations among the various interactions. So, schematically the structure of the action is

$$S_{\rm sound} \sim \int d^3x\, dt\, w_0 \left[(\dot\pi^2 - c_s^2\, \partial^2\pi^2) + c_s^2\, \partial^3\pi^3 + c_s^2\, \partial^4\pi^4 \right] , \tag{2.7}$$

where ∂ stands for a typical spatial derivative, and for the interactions we used $\dot\pi \sim c_s\, \partial\pi$, valid for not terribly off-shell phonons. Now, to estimate the size of the amplitude at a given energy or momentum, we can proceed as follows. First, we redefine the time variable

$$t \to t/c_s , \tag{2.8}$$

to get a relativistic-looking kinetic term:

$$\begin{aligned} S_{\rm sound} &\sim \int d^3x\, dt/c_s\, w_0 \left[c_s^2(\dot\pi^2 - \partial^2\pi^2) + c_s^2\, \partial^3\pi^3 + c_s^2\, \partial^4\pi^4 \right] & (2.9) \\ &\sim w_0 c_s \int d^4x \left[(\dot\pi^2 - \partial^2\pi^2) + \partial^3\pi^3 + \partial^4\pi^4 \right] . & (2.10) \end{aligned}$$

Then, we notice that c_s has factored out of the action, and that combined with w_0 it gives the only energy/momentum scale in the action: $M^4 \equiv w_0 c_s$. The rest has standard relativistic scaling (without Lorentz-invariant contractions though), which means that we can

from scale-invariance.

apply standard relativistic amplitude estimates. All interactions inside the integral have unit coefficient; the typical $2 \to 2$ dimensionless amplitude is thus k^4, combined with the appropriate powers of M to match dimensionality

$$\text{interaction strength} \sim \frac{k^4}{w_0 c_s} \sim \frac{E^4}{w_0 c_s^5} \,, \tag{2.11}$$

where E is the typical energy in units of the original time variable (notice that spatial coordinates are untouched, so there is no such ambiguity for k.) The strong-coupling momentum and energy are thus

$$k_* = (w_0 c_s)^{1/4} \,, \qquad E_* = c_s \, k_* \,, \tag{2.12}$$

respectively.

The above estimate yields the correct strong-coupling scale for longitudinal phonons. We could do the same for the vortex sector, at finite c_T. However as we will see, for the vortex interactions there are cancellations that are not manifest in eq. (2.5) and that would impair this simple estimate.

2.2.3 Hunting for all factors of c

When we start computing amplitudes and physical quantities like cross sections and decay rates, we have to be careful about extra factors of c_s and of c_T besides those appearing explicitly in the various interaction terms. For instance, we just saw that the longitudinal phonon interaction strength is of order $k^4/(w_0 c_s)$, whereas the Lagrangian interaction terms are proportional to c_s^2. In hindsight, this result just follows from dimensional analysis, once we keep separate units for space and time (we can still set $\hbar = 1$ though). The quartic interactions involve four powers of momentum. The interaction strength is thus k^4 divided by whatever combination of w_0 and c_s has the same units as k^4. Of course, we have an ambiguity as to the units of w_0—is it a mass- or energy-density? We can easily resolve this ambiguity by looking at the kinetic term. By construction our π^I have units of length; the

action is dimensionless (for $\hbar = 1$); w_0 is thus a mass density, and $w_0 c_s^2$ an energy density: $[\rho_0 c_s^2] = Ek^3 = c_s k^4$. The dimensionless combination therefore is $c_s k^4/(\rho_0 c_s^2) = k^4/(\rho_0 c_s)$, as expected.

So, a possible strategy to get all the factors of c_s right in amplitudes, cross sections, and rates, is to use the standard relativistic formulae, and then insert suitable powers of c_s to match dimensions. Essentially this is equivalent to redefining the time variable as we did above, to end up with a relativistic kinetic term with $c_s = 1$. However this strategy is going to fail once we include vortices/transverse phonons in our processes: with two different propagation speeds c_s and c_T for longitudinal and transverse polarizations, dimensional analysis does not suffice. Equivalently, by redefining the time variable we can cast only one of the two kinetic terms in relativistic form.

In Appendix 6.3 and 6.4 we will therefore briefly review the standard relativistic formulae and derive the modifications needed to apply them to our case with $c \neq 1$. We adopt this somewhat cumbersome action plan, rather than going through some standard condensed matter textbook and trying to dig up the relevant non-relativistc formulae, for no other reason than we are more familiar with the relativistic Feynman rules and related formulae—and we assume that the reader is also. The bottom line is pleasantly surprising: We can use the standard relativistic Feynman rules and formulae for infinitesimal cross-sections and rates, with no modifications, even when we start considering different fields with different speeds. By 'standard relativistic rules and formulae' we mean those associated with the so-called relativistic normalization of single-particle states, as derived for instance in Peskin-Shroeder [19].

As to the overall factor of w_0 in (2.5), it is straightforwardly kept track of. Either by inserting for each Feynman diagram a w_0 for each vertex, a $1/w_0$ for each internal line, and a $1/\sqrt{w_0}$ for each external line, or most simply by setting it to one and retrieving it at the end of the computation via dimensional analysis.

2.3 Simple processes – the vortex strong coupling

We compute, to tree level, a number of simple processes in order of increasing number of vortices on the external legs. For scattering processes, for simplicity we will only consider initial states with zero total momentum. Given that Lorentz boosts are spontaneously broken and that we have a preferred reference frame, this is a non-trivial choice—we are setting some kinematic invariants to zero. With an abuse of language, we will refer to this choice as "working in the center of mass (CM) frame." For the decay of a single finite energy excitation, on the other hand, this choice is not an option, of course. We will use the formulae for amplitudes, cross sections and rates found in the Appendix. But, as commented on before, the good news is that these formulae look just like the usual relativistic ones that we are used to dealing with. So, except for the additional factors of c_s and c_T coming from the external states' dispersion relations and from the internal lines' propagators, everything goes through just as usual: each external line carries a polarization-vector factor (times $1/\sqrt{w_0}$), each incoming or outgoing time-derivative contributes a $\mp i\omega$, each incoming or outgoing spacial-gradient contributes a $\pm i\vec{k}$, and so on. From eq. (2.5), we immediately get the Feynman propagator:

$$\langle T\pi^I(x)\pi^J(y)\rangle \to \frac{1}{w_0}\cdot\frac{iP_L^{IJ}}{\omega^2-c_s^2p^2+i\epsilon}+\frac{1}{w_0}\cdot\frac{iP_T^{IJ}}{\omega^2-c_T^2p^2+i\epsilon}\,, \tag{2.13}$$

where P_L and P_T are the longitudinal and transverse projectors, respectively.

We will not content ourselves with amplitudes. Rather, we will compute physical, measurable quantities like cross sections and decay rates. The reason is that amplitudes depend crucially on the normalization chosen for the single-particle states. For instance going from the so-called relativistic normalization to the non-relativistic one, would move some factors of c_s and c_T from the amplitudes to the phase-space elements, in such a way as to keep cross-sections and rates unaffected. Ascertaining the strong-coupling of the theory in the $c_T \to 0$ limit at the level of amplitudes requires a derivation of partial waves, a la Jacob-Wick, being

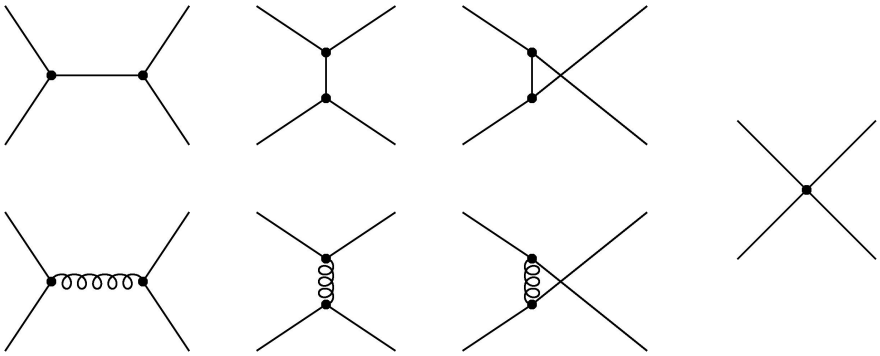
careful about the factors of c_s and c_T. Although we have also derived our results using that method, we found it simpler to present them by focussing on cross-sections and decay rates.

A final remark about external vortices. When we take the $c_T \to 0$ limit we have to decide whether we are going to keep their momenta or their energies fixed. The first choice is the more conservative, since it corresponds to taking their energies to zero, thus weakening any possible strong-coupling phenomenon we are going to encounter. It is also the only consistent one, since the alternative one would send the vortex momenta to infinity, outside the regime of validity of any effective theory. In the following we parameterize everything in terms of momenta rather than energies, so that taking the $c_T \to 0$ limit is straightforward. Notice also that only if we keep the vortex momenta fixed is our deformed theory with small c_T close to the fluid one with $c_T = 0$: in the Lagrangian c_T^2 weighs the gradient energy, so that by sending c_T to zero while keeping the momenta fixed one is in fact sending the magnitude of that Lagrangian term to zero. Related to this, it is somewhat tricky to deal with processes that include longitudinal phonons in the initial state but no longitudinal phonon in the final state: the initial longitudinal phonons' finite energy should be divided among the final vortices, thus making their momenta diverge for $c_T \to 0$. In other words, one cannot send c_T to zero and keep all momenta fixed. We will see an example of this below, in the decay of a longitudinal phonon into two transverse vortices. We postpone a discussion of the related subtleties until then.

We will use $\vec{p}$'s to denote the momenta of the longitudinal modes, and $\vec{k}$'s and $\hat{\epsilon}$'s to denote the momenta and polarizations of the transverse modes. Our $\hat{\epsilon}$'s are real, thus corresponding to linear polarizations, and normalized to one (hence the 'hat'.) For longitudinal phonons the polarization vector is $\hat{p}$, of course. For all the processes we will just compute the leading contribution in the limit $c_T/c_s \ll 1$, for which we hope to learn something about the original fluid ($c_T = 0$).

2.3.1 Longitudinal $2 \to 2$ scattering

This is the simplest of the scattering processes. To tree level, the only relevant diagrams are:



We designate, here and for the rest of the chapter, the solid lines as longitudinal excitations and the curly lines as the transverse excitations. Time flows to the right.

When done in the center of mass frame, the only kinematic variables are the momentum of the longitudinal phonons p and the scattering angle θ. To tree level, the total amplitude is given by

$$i\mathcal{M}_{LL\to LL} = -\frac{ip^4c_s^2}{w_0}\left[f_4/c_s^2 - 2f_3^2/c_s^4 + 3c_s^2 + 2f_3 + c_s^4 + 2(1-3c_s^2)\cos^2\theta\right] \tag{2.14}$$

Remarkably, the graphs with transverse propagators do not contribute to the amplitude, even individually. The infinitesimal cross section is

$$d\sigma = \frac{1}{c_s^6}\frac{|\mathcal{M}_{LL\to LL}|^2}{64\pi^2(2p)^2}d\Omega\ , \tag{2.15}$$

where we made use of the phase space element computed in the Appendix (eq. (6.37)). We can easily calculate the total cross section. The final particles are identical, so we over-count when we integrate over all final phase space. To counteract this we simply include a $1/2$ symmetry factor. To all orders in c_s the total cross-section is

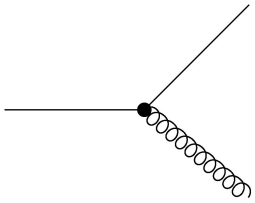
$$\sigma_{LL\to LL} = \frac{1}{256\pi}\frac{1}{p^2}\left(\frac{p^4}{w_0c_s}\right)^2\left[2\alpha^2 + \frac{4\alpha\beta}{3} + \frac{2\beta^2}{5}\right] \sim \frac{1}{p^2}\left(\frac{p^4}{w_0c_s}\right)^2 \tag{2.16}$$

where $\alpha \equiv (f_4/c_s^2 - 2f_3^2/c_s^4 + 3c_s^2 + 2f_3 + c_s^4) = \mathcal{O}(1) + \mathcal{O}(c_s^2) + \mathcal{O}(c_s^4)$ (assuming $f_3, f_4 \sim c_s^2$) and $\beta \equiv 2(1-3c_s^2)$.

This result matches our dimensional estimate of the strong coupling scale for longitudinal phonons in eq. (2.12): the cross-section (2.16) is the geometric cross-sectional area for wave-packets of wavelength $1/p$, times the square of the dimensionless interaction strength we estimated in sect. 2.2.2. We have strong coupling when σ becomes of order $1/p^2$—in such a case the two wave-packets have an $\mathcal{O}(1)$ probability of interacting—in agreement with sect. 2.2.2.

2.3.2 Longitudinal decay and vorticity production

In addition to scattering cross sections, we can also calculate the decay rate of a longitudinal phonon into a longitudinal phonon and a transverse one. This is kinematically allowed thanks to the difference in propagation speeds between longitudinal and transverse excitations. To tree level, we simply have one diagram, the longitudinal-longitudinal-transverse vertex:



Imposing the kinematical constraints coming from momentum and energy conservation, expanding in c_T/c_s, and keeping only to first order in this parameter, we can write the amplitude as

$$i\mathcal{M}_{L\to LT} = \frac{-4c_T c_s\, p^3}{\sqrt{w_0}}(\hat{\epsilon}\cdot\hat{p})\sin\theta/2\,(\cos\theta - c_s^2) \tag{2.17}$$

where θ is the angle between the outgoing longitudinal mode and the decaying one (with $\theta = 0$ corresponding to forward decay.) If the vortex is polarized orthogonally to the scattering plane the decay amplitude vanishes because of parity conservation, whereas for parallel polarization it further simplifies to

$$i\mathcal{M}_{L\to LT_\parallel} = \frac{-2c_T c_s\, p^3}{\sqrt{w_0}}\sin\theta\,(\cos\theta - c_s^2) \tag{2.18}$$

Notice the fairly featureless angular dependence, which combined with a similarly featureless phase space (eq. (6.40)) yields the surprising result that with an order-one relative probability the longitudinal phonon will recoil *backwards*, i.e. with $\theta > \pi/2$, by emitting a high-momentum forward vortex. Moreover, the unusual kinematical constraints associated with the $c_T \ll c_s$ hierarchy force the vortex to be emitted always in the forward half-space, and at an obtuse angle relative to the final longitudinal phonon.

The physical quantity that we want to calculate is the decay rate Γ, given by (6.21). Using the infinitesimal phase space for a two-particle final state with non-zero total momentum and $c_T \ll c_s$, given by (6.40), summing over possible transverse polarizations and integrating over the solid angle we get:

$$\Gamma_{L\to LT} = \frac{c_T}{c_s}\frac{p^5}{w_0}\,\frac{4}{315\pi}(5 - 6c_s^2 + 21c_s^4) \tag{2.19}$$

which, as we can see, smoothly goes to zero as $c_T \to 0$.

In the same way we can study the decay $L \to TT$ of a longitudinal phonon into two vortices. Notice that the decay kinematics in the $c_T/c_s \ll 1$ limit implies $k_1 \simeq -k_2$ and $2|k_1| \simeq |p|c_s/c_T \gg |p|$. That is approximately two back-to-back vortices, carrying each half the energy and with momentum scaled up by a factor $\sim c_s/c_T$ with respect to the initial state phonon. We find the amplitude

$$i\mathcal{M}_{L\to TT} = \frac{c_s^2\,p^3}{4\sqrt{w_0}}\left[\hat{\epsilon}_1\cdot\hat{\epsilon}_2(1+2\cos\theta^2) + \hat{p}\cdot\hat{\epsilon}_1\hat{p}\cdot\hat{\epsilon}_2\right] \tag{2.20}$$

where $\hat{\epsilon}_i$ represent the polarizations of the two vortices and $\cos\theta = \hat{p}\cdot\hat{k}_1$. For the decay rate we have thus roughly

$$\Gamma_{L\to TT} \sim \frac{p^5 c_s^3}{w_0 c_T^3} \tag{2.21}$$

corresponding to a 'quality factor' $\Gamma_{L\to TT}/\omega \sim p^4 c_s^2/w_0 c_T^3$. A phonon with momentum

$$p_* \sim (w_0 c_s)^{1/4}(c_T/c_s)^{3/4} \tag{2.22}$$

has a width comparable to energy. We can thus identify p_* as a strong interaction scale for longitudinal phonons. Notice that this scale is parametrically smaller than the naive

estimate in eq. (2.12). This fact is largely a consequence of the peculiar kinematics of the decay $L \to TT$, where starting from an initial quantum of momentum p, the final state quanta have a much higher momentum scale $k \sim pc_s/c_T$. This higher scale is naturally associated with a stronger interaction strength. The fact that starting with soft quanta one can probe much shorter distances due to the large final state momentum also suggests more care with the use of the notion of effective field theory. We will elaborate briefly on this in sect. 2.5. Notice also that the corresponding vortex momentum strong scale is instead $k_* \sim p_* c_s/c_T \sim (w_0 c_T)^{1/4}(c_s/c_T)^{1/2}$. The computation of $TT \to TT$ scattering in the next section will show that the vortex momentum cut off is actually $\sim (w_0 c_T)^{1/4}$, which is parametrically smaller. Then, within the resulting smaller range of validy of the effective field theory, the process $L \to TT$ remains weakly coupled.

The results just derived display one general property of amplitudes involving vortices: they are accompanied by at least one power of the vortex energy. [4] That property directly follows from invariance under volume preserving diffeormorphisms and can be made evident by chosing a suitable field parametrization. In a neightbourhood of $\vec{\phi} = \vec{x}$ the most general field configuration can indeed be written implicitly as

$$\vec{\phi}(x,t) = \vec{g}(\vec{y},t) \qquad \vec{y} = \vec{x} + \vec{\pi}_L(\vec{x},t) \tag{2.23}$$

where $\vec{\pi}_L \equiv \vec{\nabla}\psi(\vec{x},t)$ is a longitudinal perturbation while $\vec{g}(\vec{y},t)$ is a volume-preserving diffeormorphism generated by "exponentiating" a transverse vector field $\vec{\pi}_T(\vec{y},t)$ ($\vec{\nabla}_y \cdot \vec{\pi}_T = 0$). That is

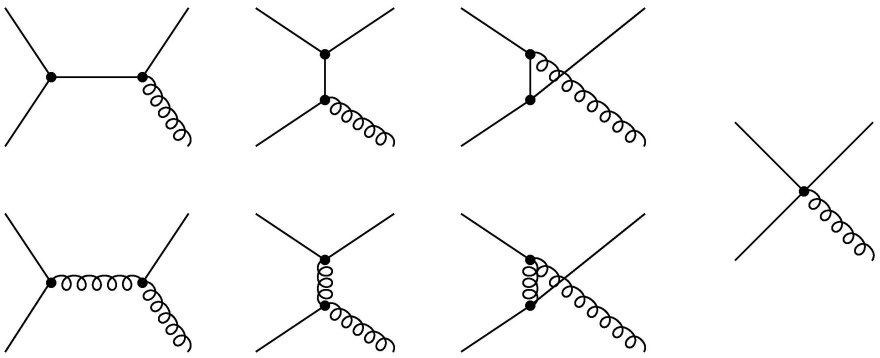
$$\vec{g}(\vec{y},t) = \lim_{N\to\infty}\left[\left(e + \frac{\vec{\pi}_T}{N}\right) \circ \cdots \circ \left(e + \frac{\vec{\pi}_T}{N}\right)\right](\vec{y},t) = \vec{y} + \vec{\pi}_T(\vec{y},t) + O(\pi_T^2) \tag{2.24}$$

where $e(\vec{y}) = \vec{y}$ is the identity function and $\circ$ represent function composition. Our procedure to define a finite transformation $\vec{g}$ starting from the infinitesimal one $\vec{y} + \vec{\pi}_T$ is just the

[4] In the case of the $L \to LT$ decay this also corresponds to a significant suppression of the amplitude, given the parametrically suppressed value of the vortex energy in that process. In the case of $L \to TT$ the energy of L and T modes is instead comparable.

standard exponential map of Lie groups. In the end, eq. (2.23) is a three dimensional (in field space) family of field configurations that for $\pi_{T,L} \to 0$ reduces to the most general one $\vec{x}+\vec{\pi}_L+\vec{\pi}_T$. It follows that in a neighbourhood of the identity, eq. (2.23) is a faithful (one to one) parametrization, acceptable to perform perturbation theory. Now a time independent $\vec{\pi}_T(\vec{y},t) \equiv \vec{\pi}_T(\vec{y},0)$ is just a symmetry tranformation of the action in the limit $c_T = 0$. Therefore, apart from the $\mathcal{O}(c_T^2)$ kinetic perturbation, π_T enters the lagrangian with at least one time derivative, and therefore amplitudes have the corresponding suppression. Notice that in the computation of $L \to LT$, where we used the simple parametrization $\vec{\phi} = \vec{x} + \vec{\pi}$, that result arose via a non trivial cancellation of different terms in the amplitude.

As a further check of the above general property of amplitudes involving transverse modes, consider the process $LL \to LT$. The relevant Feynman diagrams are



The amplitude vanishes if the vortex is polarized orthogonally to the scattering plane, whereas for parallel polarization it is

$$i\mathcal{M}_{LL\to LT_\parallel} = -\frac{4ip^4 c_s c_T}{w_0}\cot\theta\big[3 - f_3/c_s^2 - 2c_s^2 - f_3 + 3c_s^4 - 2(3-c_s^2)\cos^2\theta\big] + \mathcal{O}(c_T^2)\ , \quad (2.25)$$

again vanishing, as expected, as $c_T \to 0$ with p fixed. This result corresponds to a cross-section scaling as c_T ,

$$\sigma_{LL\to LT} \sim \frac{c_T}{c_s}\frac{1}{p^2}\left(\frac{p^4}{w_0 c_s}\right)^2 \ . \quad (2.26)$$

The results just derived deserve one additional comment. For a classical inviscid fluid, we know that if we start with zero vorticity everywhere in space, we cannot produce any. How-

ever this is not because of some global charge conservation, but, more prosaically, because the source of vorticity is proportional to vorticity itself. [5] For a non-relativistic fluid:

$$\dot{\vec{\Omega}} = -\big(\vec{v}\cdot\vec{\nabla}\big)\vec{\Omega} + \big(\vec{\Omega}\cdot\vec{\nabla}\big)\vec{v} - \vec{\Omega}\big(\vec{\nabla}\cdot\vec{v}\big)\ . \tag{2.27}$$

This is similar to, say, a scalar field theory with two fields, ϕ and χ, with mutual interactions of the form $\phi^2\chi^2$. In the χ e.o.m., the source term is proportional to χ itself,

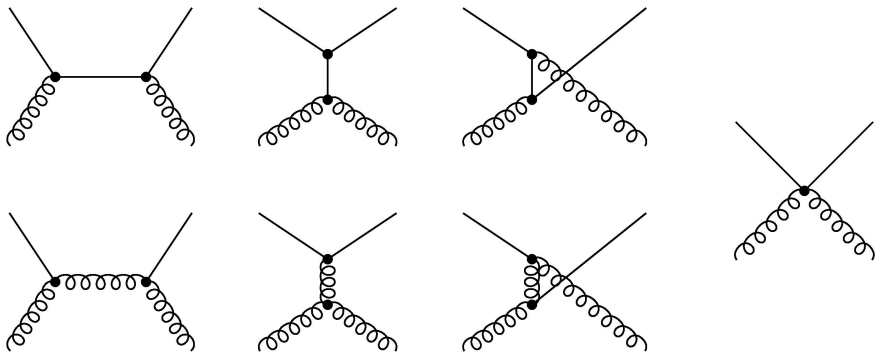
$$\frac{\delta\mathcal{L}}{\delta\chi} \supset \phi^2\chi\ , \tag{2.28}$$

which means that $\chi = 0$ is a perfectly good classical solution no matter what ϕ does. On the other hand, we know that this fact does not survive quantum mechanically. Two ϕ quanta in the initial state can annihilate in a $\phi^2\chi^2$ vertex to yield two χ quanta in the final state. However this way one will never produce a *single* χ quantum if this is not already present in the initial state. Coming back to our fluid, it is suggestive to interpret our results above in light of this analogy. We expect that the classical non-generation of vorticity will not survive at the quantum-mechanical level. Vortex excitations will be generically produced in scattering and decay processes, even if there are no vortices in the initial state. However, the production of a *single* vortex quantum should be prohibited, being immune from the aforementioned '$\phi^2\chi^2$ effect'. Indeed consistently with this expectation, at fixed phonon momentum p, we found the single vortex amplitude Eq. (2.17) vanishes for $c_T \to 0$, while the two vortex amplitude Eq. (2.20) does not.

2.3.3 Longitudinal and transverse scattering

Another interaction we can consider is the scattering of a longitudinal excitation and a transverse excitation. The tree level diagrams are given schematically by:

[5] In fact, associated with the invariance under volume-preserving diffs there are infinitely many conserved local currents and global charges [10]. Eq. (2.27) is a consequence of these infinitely many conservation laws, but it does not take the form of a simple conservation law for vorticity itself.



Besides the usual scattering angle, there are also the additional degrees of freedom associated with the polarization of the transverse modes. The amplitude is

$$i\mathcal{M}_{LT\to LT} = -\frac{i2c_s c_T p^4}{w_0}\left[(\hat{p}\cdot\hat{p}') - c_s^2\right](\hat{\epsilon}_1\cdot\hat{p}')(\hat{\epsilon}_2\cdot\hat{p}) + \mathcal{O}(c_T^2) \tag{2.29}$$

Note that $\mathcal{M} \propto c_T$, consistent with the general argument presented in the previous section. It should also be noted that there is explicitly no dependence on f_4 and f_3. This result can be easily understood using the field parametrization discussed in the previous section. The terms proportional to f_4 and f_3 only depend on the longitudinal field π_L. Then f_4 obviously does not contribute to $LT \to LT$ while f_3 can only contribute via the second diagram in the upper line in the figure, which is clearly of order c_T^2. Again, in the standard parametrization all these results follow from a non-trivial set of cancellation, which also represent a check of our computations.

The infinitesimal cross-section is given by

$$d\sigma = \frac{1}{2}\sum_{\text{initial }\epsilon}\sum_{\text{final }\epsilon}\left(\frac{1}{p^2}\right)\left(\frac{p^4}{4\pi w_0(c_T+c_L)}\right)^2\left\{\left[(\hat{p}\cdot\hat{p}') - c_s^2\right](\hat{\epsilon}_1\cdot\hat{p}')(\hat{\epsilon}_2\cdot\hat{p})\right\}^2 d\Omega \tag{2.30}$$

Here we are averaging over the incoming polarizations (hence the 1/2) and summing over the final ones. A good basis to do this in would be parallel to the scattering plane and perpendicular to the scattering plane. As we can see from the form of $\mathcal{M}$, any perpendicular component of the polarization does not contribute to the amplitude. Putting everything together, keeping all powers of c_s, we have the total cross section:

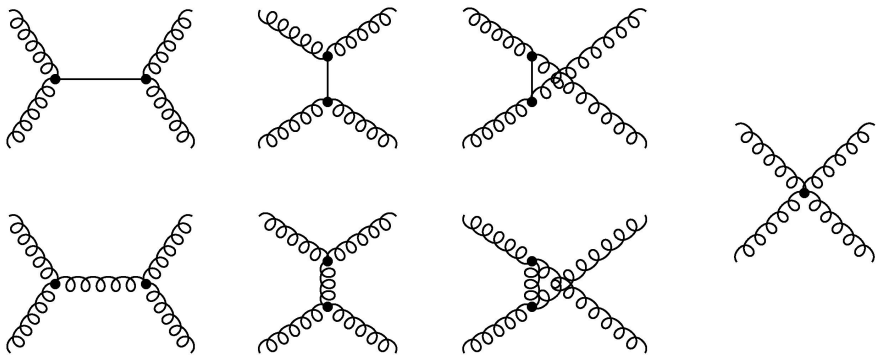
$$\sigma = \frac{1}{105\pi}\frac{1}{p^2}\left(\frac{p^4}{w_0 c_s}\right)^2\left[1+7c_s^4\right] + \mathcal{O}(c_T) \tag{2.31}$$

We now see the importance of $\mathcal{M}_{LT\to LT} \propto c_T$. This dependence on the "transverse speed of sound" is necessary to avoid a divergent physical quantity (here the scattering cross-section) as $c_T \to 0$. Remarkably, the strong-coupling scale for this process is the same as for purely longitudinal scattering—cf. eq. (2.16). Note also that because of the independence of the cross section on f_3 and f_4 this cross section is generic for all fluid types, regardless of the particular functional form of $f(b)$. Its dependence on the particular fluid model only comes through the speed of sound.

2.3.4 Transverse $2 \to 2$ scattering

As we can see from formula (6.18) for the differential scattering cross section, the more transverse incoming and outgoing states the more a rate could possibly diverge as $c_T \to 0$. The results so far agree with that expectation. The only problematic quantity we encoutered is the rate for $L \to TT$, which has however peculiar (singular) kinematics as $c_T \to 0$. The processes with smooth kinematics were instead found to have a well behaved rate. In the $T + L \to T + L$ cross section we picked up a c_T^{-1} from the phase space of the outgoing excitation and we picked up a c_T^{-1} from one of the $1/2E$ normalization factors. It was thus critical that $|\mathcal{M}_{LT\to LT}| \propto c_T^2$ in order that the cross section be well defined.

From eq. (6.18) we can see that for transverse 2 to 2 scattering $\frac{d\sigma}{d\Omega} \propto c_T^{-6}|\mathcal{M}|^2$. In order that our physical process be finite in the $c_T \to 0$ limit we need (at least) $\mathcal{M} \propto c_T^3$. We find that this is not the case. In fact, simple power counting using the parametrization discussed in section 4.2 indicates the amplitude will be suppressed (at least) as c_T^2. This expectation is confirmed working in the standard parametrization: by dramatic cancelations among Feynman diagrams the zeroth and first order terms vanish, and the leading non-trivial term is of order c_T^2. The necessary tree-level Feynman diagrams are:



In the CM frame with $\parallel$ and $\perp$ denoting polarizations parallel and perpendicular to the scattering plane, respectively, at lowest order in c_T the amplitude is

$$i\mathcal{M}_{TT\to TT} = \frac{ik^4 c_T^2}{w_0} \times \begin{cases} \cos 2\theta & \text{for } \parallel\parallel\to\perp\perp\,,\perp\perp\to\parallel\parallel \\ \frac{1}{2}\big(\cos\theta - \cos 2\theta\big) & \text{for } \parallel\perp\to\parallel\perp \end{cases} \tag{2.32}$$

and zero for all other combinations of polarizations. In the $\parallel\perp\to\parallel\perp$ case, θ is the angle between the two $\parallel$-polarized phonons. Note that, as in the previous physical process, there is no dependence on f_3 and f_4. So, once again, this process is generic for all fluid models regardless of the details of $f(b)$.

After squaring the amplitude, we average over incoming polarizations and sum over the final ones. We have:

$$\frac{1}{4}\sum_{\text{initial }\epsilon}\sum_{\text{final }\epsilon}|\mathcal{M}|^2 = \frac{1}{4}\left(\frac{k^4 c_T^2}{w_0}\right)^2\left[2 + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{2}\cos 4\theta\right]\,, \tag{2.33}$$

and so the total cross section, including a $1/2$ symmetry factor, is

$$\sigma_{TT\to TT} = \frac{1}{256\pi}\left(\frac{13}{15}\right)\frac{1}{k^2}\left(\frac{k^4}{w_0 c_T}\right)^2\,. \tag{2.34}$$

As we can see, it blows up as we take $c_T \to 0$ indicating that our theory is breaking down. We emphasize once again the absence of the free parameters f_3 and f_4 in this result, which implies that the pathology just unveiled cannot be avoided by a judicious choice of their values.

2.4 The infrared situation

Our S-matrix analysis indicates that the transverse degrees of freedom are strongly-coupled at arbitrarily low energies. However the strong-coupling phenomenon we unveiled is quite peculiar. We deformed the theory in the far IR by introducing a small deformation parameter c_T. This changes the asymptotic states of the theory, and we discovered that some of these get strongly coupled in the UV, at an energy scale that drops to zero when we recover the original theory—the limit $c_T \to 0$. A vanishing *ultraviolet* strong-coupling energy scale suggests that our problem is probably more properly thought of as an *infrared* one—we may be approaching strong-coupling from the wrong side! That is, in the deformed theory at finite c_T we encounter strong coupling in moving to high energies, but since the deformed theory differs from the original one at low energies, it may be that the strong coupling scale is in fact a divide between the two theories—that there is no regime where the two theories look alike. If we stick to the original description, without ever introducing c_T, we may realize that we have some form of strong coupling in the IR.

The distinction we are putting forward may sound like a matter of definition, but it is not. A theory that becomes strongly coupled in the UV is simply not defined at energies of order of the strong-coupling scale and above—it needs infinitely many parameters for its definition. If this were the case for us, our theory would not be consistent, in any energy range. On the other hand, there are a number of ways in which perturbation theory can break down in the IR without impairing the consistency of a theory. There is for instance real QCD-like strong coupling, where perturbation theory does break down but the non-perturbative theory is perfectly well defined—it is just hard to solve! Or there are QED-like infrared divergences, which can be tamed by focusing on suitable infrared-safe observables, for which the perturbative expansion applies. Or there may be huge quantum IR fluctuations without necessarily implying large interactions, like for instance for would-be Goldstone bosons in $1+1$ dimensions [20]. This would signal a bad identification of the theory's vacuum state.

And in general, to ascertain the consistency of an IR-problematic theory, at least in certain energy and momentum regimes, one can just put the system in a finite-size 'box' and consider its time evolution for a short time, in which case perturbation theory typically does not even break down.

The first signal that our problem with the transverse degrees of freedom may be infrared in nature comes from considering quantum fluctuations about the semiclassical vacuum state with $\langle \phi^I \rangle = x^I$. An order parameter that conveniently quantifies the amount of spontaneous symmetry breaking in a manifestly translationally invariant fashion is

$$\langle \partial_\mu \phi^I \rangle = \delta^I_\mu \sim 1 \ . \tag{2.35}$$

It is straightforward to estimate quantum fluctuations in this quantity. We decompose the fields as $\phi^I = x^I + \pi^I$, and from the π^I propagators,

$$\langle T\pi^I(x)\pi^J(y)\rangle \to \frac{1}{w_0} \cdot \frac{iP_L^{IJ}}{\omega^2 - c_L^2 p^2 + i\epsilon} + \frac{1}{w_0} \cdot \frac{iP_T^{IJ}}{\omega^2 + i\epsilon} \ , \tag{2.36}$$

where P_L and P_T are the longitudinal and transverse projectors, we get

$$\langle \partial_i \pi^I \, \partial_j \pi^J \rangle \ \sim \ \frac{1}{w_0} \frac{p^5}{\omega} \tag{2.37}$$

$$\langle \partial_0 \pi^I \, \partial_j \pi^J \rangle \ \sim \ \frac{1}{w_0} p^4 \tag{2.38}$$

$$\langle \partial_0 \pi^I \, \partial_0 \pi^J \rangle \ \sim \ \frac{1}{w_0} p^3 \omega \tag{2.39}$$

These are real-space correlators, and in the right-hand sides the dimensionless, order-one part of the Fourier transform, $\int d\Omega \, d\log p \, d\log \omega \, e^{i(\cdots)}$, is understood. Also we are considering considerably off-shell (ω, p) pairs, by taking for instance the separation in real space to be space-like with the respect to the sound speed (i.e., by working in Euclidean space.) The correlators (2.38, 2.39) behave in essentially the same way as for standard field theories in dimensions higher than $1+1$: quantum fluctuations in order parameters are damped at low momenta and low energies, and as a consequence in the IR there is spontaneous symmetry

breaking. However the correlator (2.37) ruins this familiar picture: for fixed momentum, it is IR-divergent. Equivalently, in the p-ω plane there is a sector extending all the way to $p = 0$, $\omega = 0$ where quantum fluctuations in our order parameter (2.35) are huge. This reminds us of Coleman's theorem [20] in $1 + 1$ dimensions, and suggests that our strong-coupling problems may stem from an incorrect definition of the vacuum. In other words, we have been assuming all along that doing perturbation theory about the semiclassical vacuum $\phi^I = x^I$ is sensible, but eq. (2.37) is somehow telling us that quantum fluctuations want to dismantle this state. As an aside, notice that we have every reason to believe that Lorentz boosts *are* spontaneously broken: since the correlators (2.38, 2.39) are damped in the IR, it seems that only the breaking of the spatial symmetries (translations, rotations, volume-preserving diffs) is affected by our phenomenon. More concretely, we can compute quantum fluctuations in an order parameter that is invariant under all symmetries but Lorentz boosts, like for instance the fluid velocity u^μ. Classically in the ground state we have $u^\mu = (1, \vec{0})$. At first order in the π field the fluctuation reads $\delta u^\mu = (0, -\dot{\vec{\pi}})$. From eq. (2.39) we thus get the velocity-velocity correlator,

$$\langle \delta u^\mu \, \delta u^\nu \rangle \sim \delta^\mu_i \delta^\nu_j \, \frac{1}{w_0} p^3 \omega \ , \tag{2.40}$$

which is damped in the IR, thus signaling that quantum fluctuations do *not* want to restore Lorentz invariance.

Several natural questions arise: How do we check the above statements more concretely? Can we identify the correct vacuum state? Does it support a well defined perturbative theory? And perhaps the most physically relevant: is there a semiclassical limit where we recover classical hydrodynamics? To start addressing these questions, we step back from our fluid case and consider a much simpler system with somewhat similar features: the free particle in quantum mechanics.

2.4.1 The free quantum particle

Consider a free particle living on a line. Classically, $x = x_i = \text{const}$ is a perfectly good state. However, quantum mechanically we know that if we start from a state localized around x_i, time-evolution will make the wave-function spread out, and at very late times the state will be totally delocalized. Related to this, the ground state of the theory wants to have a constant wave-function throughout the whole line. (This is non-normalizable for an infinite line, but for simplicity we can replace the line by a very large circle.) The spontaneous breaking of translations that we see classically, $x = \text{const}$, quantum mechanically is gone.

Given the system's simplicity, there are many equivalent ways to describe this phenomenon quantitatively. One that will prove readily exportable to the fluid case, is the path integral one. Consider, in the path-integral representation, the amplitude for evolving from x_i at $t = 0$ to x_f at $t = T$:

$$\langle x_f, T \,|x_i, 0\rangle = \int_{x(0)=x_i}^{x(T)=x_f} \mathcal{D}x \, e^{iS[x]} \propto \exp \frac{i\,(x_f - x_i)^2}{2T} \,. \tag{2.41}$$

We are using $\hbar = 1$ units. Also for simplicity we are setting the particle's mass to one. The exponent is of course the classical action for the solution interpolating between x_i and x_f in time T. The prefactor that we omitted is the determinant of the kinetic operator about the classical solution. Since we have a free theory such an object does not depend on the solution under consideration, or on the overall distance $(x_f - x_i)$, and will play no role in our discussion—it is just an overall normalization factor.

We see that, for a given T, at small distances $|x_f - x_i| \lesssim \sqrt{T}$ the amplitude is essentially constant, whereas it starts oscillating rapidly when we move to much larger distances, $|x_f - x_i| \gg \sqrt{T}$. We are tempted to conclude from this that the width of the wave-function grows like $\sqrt{T}$, but this is premature because the amplitude we got is a pure phase at all distances, which means that the probability is uniform in x_f. In other words, as soon as T is bigger than zero, the wave function is completely delocalized. This is clearly not what we expect physically, and is an artifact of having chosen an initial δ-like wave-function: a

$\delta(x)$ has infinite momentum spread, which means that in zero time the particle is going to be everywhere with non-vanishing probability. If we instead choose a more reasonable state, say a gaussian centered at $x_i = 0$ with width σ, the final wave function we get is:

$$\psi(x_f, T) = \int dx_i \, \psi(x_i, 0) \, \langle x_f, T \, | x_i, 0 \rangle \propto \begin{cases} \exp -\frac{x_f^2}{2\sigma^2} \ , & T \ll \sigma^2 \\ \\ \exp -\frac{x_f^2 \, \sigma^2}{T^2} \ , & T \gg \sigma^2 \end{cases} \tag{2.42}$$

where we ignored overall normalization constants, as well as factors that do depend on x_f but are pure phases. We thus see that at early times the wave-function width is—not surprisingly—dominated by the initial width σ, whereas at late times it grows linearly with T.

We can turn the problem around and ask: suppose we want to have a somewhat localized state that looks stationary over a time period of order T. What is the minimum wave-function width we should allow for? Answer: $\sqrt{T}$. As a consequence, in the long-time limit there cannot be spontaneous symmetry breaking. With hindsight, we can go back to our original amplitude eq. (2.41) and learn how to read it properly. Despite being purely imaginary, the exponent really tells us the minimum uncertainty we should allow in the initial position in order for our state to look approximately stationary over a time of order T.

Before moving on to the field theory case, it is worth pointing out that if we have N free particles we can run the above computation independently for each of them, since all amplitudes factorize. This means that, roughly speaking, each particle can afford an $\mathcal{O}(1)$ action, so that the whole system can explore trajectories with an $\mathcal{O}(N)$ action.

2.4.2 Coleman's theorem from the path integral

We can now apply the same logic to a massless free field theory in $d+1$ dimensions,

$$S[\phi] = \int d^{d+1}x \, \tfrac{1}{2} (\partial \phi)^2 \ , \tag{2.43}$$

and recover Coleman's theorem in the path-integral language. We have a shift symmetry $\phi \to \phi + c$, which classically is spontaneously broken by any Poincaré invariant configuration $\phi(x) = \text{const}$. Without loss of generality we can set the constant to zero, and ask whether quantum fluctuations tend to disrupt this configuration. Coleman teaches us that this will be the case for $d = 1$.

We want to compute the path integral

$$\langle f(\vec{x}), T \,|0,0\rangle = \int_{\phi(0)=0}^{\phi(T)=f(\vec{x})} \mathcal{D}\phi \, e^{iS[\phi]} \propto e^{iS[\phi_{\rm cl}]} \,, \tag{2.44}$$

for a generic final configuration $f(\vec{x})$. Given what we learned for the free particle on a line, the result will tell us which are the field configurations that inevitably get populated after a time T if we start at $t = 0$ with a wave-functional for $\phi(\vec{x})$ centered around $\phi = 0$. To compute $S[\phi_{\rm cl}]$, we have to solve the classical equations of motion with the given boundary conditions. In (spatial) Fourier space the immediate solution is:

$$\hat{\phi}_{\rm cl}(\vec{k}, t) = \hat{f}(\vec{k}) \cdot \frac{\sin kt}{\sin kT} \,, \tag{2.45}$$

where $\hat{f}(\vec{k})$ is the final configuration's Fourier transform. The action is

$$S[\phi_{\rm cl}] = \int d^d k \left|\hat{f}(\vec{k})\right|^2 \cdot k \cot kT \,. \tag{2.46}$$

We can for simplicity consider the two regions $k \ll 1/T$ and $k \gg 1/T$ separately, and ignore what happens at intermediate momenta. As to the former region, we can set $\cot kT \to 1/kT$, so that we have

$$S(k \ll 1/T) \simeq \frac{1}{T} \int^{1/T} d^d k \left|\hat{f}(\vec{k})\right|^2 \,. \tag{2.47}$$

The high momentum region, however, contains all the poles of the cotangent. These poles are just enforcing the right periodicity for the corresponding momenta. That is, for a given T all modes with $k = n\pi/T$ should go back to zero after a time T, as enforced by the perfect harmonicity of our system. As a consequence, if the final $\hat{f}(\vec{k})$ does not vanish for these special momenta, the corresponding action is infinite. To have a finite action $\hat{f}(\vec{k})$ has to

have infinitely many zeroes, at the right locations. Apart from this peculiarity, when we integrate over momenta much larger than $1/T$ the cotangent behaves like a number of order one, and to get an estimate for the action we can just ignore it. We thus have

$$S(k \gg 1/T) \sim \int_{1/T} d^d k \left|\hat{f}(\vec{k})\right|^2 \cdot k \ . \tag{2.48}$$

We can now consider a specific final configuration $f(\vec{x})$. We choose it to be localized in a region of size L, with Fourier momenta of order $1/L$ and with typical magnitude $\bar{f}$. As to the final remark of sect. 2.4.1, here we are considering just a few independent degrees of freedom—the Fourier modes with $k \sim 1/L$ in a volume of size L—so that they get spontaneously excited to the desired level $\bar{f}$ only if the corresponding action in (2.44) is of order one, or smaller. At early and late times we get, respectively

$$S(T \ll L) \sim \frac{L^d}{T} \bar{f}^2 \ , \qquad S(T \gg L) \sim L^{d-1} \bar{f}^2 \tag{2.49}$$

Our path-integral formula (2.44) at late times thus yields

$$\langle f(\vec{x}), T \,|0, 0\rangle \sim \exp i \, L^{d-1} \bar{f}^{\,2} \ , \qquad T \gg L \tag{2.50}$$

This is Coleman's theorem in our language. For $d > 1$, the amplitude for finding the system significantly away from $\phi = 0$ (large $\bar{f}$) in bigger and bigger regions becomes smaller and smaller. Put another way, the typical spread of the field decreases with distance scale. This is the standard behavior of a field theory in high dimensionality. On the other hand, for $d = 1$ the L-dependence drops out, and we are left with a finite amplitude of finding an order-one $\bar{f}$ on all scales. In fact we know that a more careful estimate shows that the typical spread of the field *grows* logarithmically with distance scale—as usual one cannot get the logs right by performing Fourier transforms by dimensional analysis. Our analysis for the fluid will be insensitive to such minutiæ.

2.4.3 Back to the fluid

For our fluid, it is easy to convince oneself that the situation is more similar to the free particle case than to a $1+1$ standard field theory. The reason is simply that all the peculiarities associated with the transverse excitations in a perfect fluid stem from their not having a gradient energy, that is, from their behaving like infinitely many free particles. Of course, this is a linearized statement. At the non-linear level, for large vorticous excitations, interactions become important and the free-particle approximation breaks down. But, in fact, it is easy to see that at low fluid velocities we can really describe the dynamics of vortices as a system of free particles constrained to move on a quite non-trivial infinite-dimensional manifold—the group of volume preserving diffeomorphisms.

To see this, we have to take the incompressible fluid limit. Of course, a fluid's compressibility is not a dimensionless quantity. It is a measure of the pressure gradient needed to sustain a given density gradient. Since $dp/d\rho$ is the (squared) speed of sound, incompressibility is not an intrinsic property of the fluid, but rather emerges in the appropriate kinematical regime: *any* fluid behaves as an incompressible one at low enough fluid velocities. In this limit, one can consistently restrict to the vortex sector of the theory, i.e. to configurations $\phi(\vec{x},t)$ that at fixed time are volume-preserving diffs of $\vec{x}$, as we now show.

Integrating out sound

It is easy to see that for low speeds and accelerations (i.e. weak time-dependence of ϕ), integrating out classically the compressional modes introduces new Lagrangian terms for the vortices that are of order ∂_t^4 and higher, whereas the vortex dynamics that we get by simply ignoring the compressional modes starts at order ∂_t^2. Expanding about the incompressible limit following the logic spelled out in the next chapter and Appendix 6.5, and keeping the leading terms in a small ∂_t expansion we have from (6.52)-(6.55):

$$S \simeq S_0 + w_0 \int d^3x dt \left[-\tfrac{1}{2}c_s^2 \left(\nabla^2\Psi\right)^2 - \tfrac{1}{2}c_s^2 \left(\nabla^2\Psi\right)\frac{v^2}{c^2} - \vec{\nabla}\Psi \cdot \frac{D}{Dt}\vec{v}\right] , \qquad (2.51)$$

where S_0 is the compression-independent part of the action, v is the vorticose fluid flow velocity field, $\frac{D}{Dt}$ is the convective derivative $\frac{\partial}{\partial t} + (v \cdot \nabla)$ and Ψ corresponds to the scalar potential representing the compressional modes. Note that the second term in the action is the only one suppressed by c^2, and is clearly associated with relativistic effects. For a non-relativistic fluid with $c_s \ll c$, it can be safely ignored. On the other hand, for relativistic fluids ($c_s \sim c$) with non-relativistic flows ($v \ll c$), such a term should be kept, being of the same order as the other ones.

Varying the action above with respect to Ψ and solving for it, we get

$$\Psi = -\frac{1}{2}\frac{1}{\nabla^2}\frac{v^2}{c^2} + \frac{1}{c_s^2}\frac{1}{\nabla^4}\vec{\nabla}\cdot\frac{D}{Dt}\vec{v}\,. \tag{2.52}$$

Plugging back into the action we get the effective action for the vortex variables,

$$\begin{aligned} S_{\text{eff}} &= S_0 + \Delta S & (2.53)\\ S_0 &= w_0 \int d^3\phi\, dt \left[\frac{1}{2}v^2 + \frac{1}{8}\frac{v^4}{c^2} - \frac{1}{8}\frac{c_s^2 v^4}{c^4}\right] & (2.54)\\ \Delta S &\simeq w_0 \int d^3x dt \left[\frac{1}{8}\frac{c_s^2 v^4}{c^4} - \frac{1}{2}\frac{v^2}{c^2}\Big(\frac{1}{\nabla^2}\vec{\nabla}\cdot\frac{D}{Dt}\vec{v}\Big) + \frac{1}{2}\frac{1}{c_s^2}\Big(\frac{1}{\nabla^2}\vec{\nabla}\cdot\frac{D}{Dt}\vec{v}\Big)^2\right] & (2.55) \end{aligned}$$

where in S_0 we dropped the rest-mass, $\vec{v}$-independent contribution. As advertised, integrating out the compressional modes introduces $\mathcal{O}(\partial_t^4)$ corrections to the vortex action. At small velocities and accelerations we can just restrict our original Lagrangian to the volume-preserving configurations, that is we can just use the lowest order term in S_0. In this approximation, and in comoving coordinates, the dynamics are free, but of course we have a very non-trivial constraint on the configuration space: $\vec{x}(\vec{\phi}, t)$ cannot leave the space of volume preserving diffs.

The absence of spontaneous symmetry breaking

We can now formulate precisely the question of spontaneous symmetry breaking for our fluid. We want to compute the amplitude to propagate from $\vec{x} = \vec{\phi}$ to $\vec{x} = \vec{\xi}(\phi)$, with $\vec{\xi}$ a volume

preserving diff:

$$\left\langle \vec{\xi}(\phi), T \,\middle|\, \vec{\phi}, 0 \right\rangle = \int_{\vec{x}(0)=\vec{\phi}}^{\vec{x}(T)=\vec{\xi}(\phi)} \mathcal{D}\vec{x}\, e^{iS[\vec{x}]} \propto e^{iS[\vec{x}_{\rm cl}]} \tag{2.56}$$

If at late times all volume-preserving diffs get populated with equal probability, there is no spontaneous symmetry breaking and our perturbative analysis in the broken phase is not applicable.

Unfortunately, we are not able to carry out this computation for a completely generic final configuration $\vec{\xi}(\phi)$: given the simple free dynamics of the system, we can confidently say that there will be a classical solution with time-independent velocity field $\vec{v}(\vec{x})$ evolving from $\vec{x} = \vec{\phi}$ to $\vec{x} = \vec{\xi}(\phi)$. However, because of the complexity of the manifold where motion takes place, determining the velocity field that connects the initial and final configuration would be quite hard. And without knowledge of the velocity field, we cannot compute the classical action.

Nevertheless, to assess the question of spontaneous symmetry breaking we do not need to be completely general. We can focus, for instance, on sample vortex configurations with a high degree of symmetry. For example, we can consider $\vec{\xi}(\phi)$ to be a rotation around the z-axis, of an angle $\Delta\varphi$ that depends on the distance r from the axis (dropping to zero above some distance R), localized in a region of length $L \gg R$ in the z direction.[6] The classical solution connecting the initial configuration to this is a vortex in constant rotation with r-dependent angular velocity,

$$\omega(r) = \frac{\Delta\varphi(r)}{T} \tag{2.57}$$

localized in a cylinder of radius R and height L. From eq. (2.54), the classical action for this

[6]Similarly to the sample $f(\vec{x})$ considered in sect. 2.4.2, the final configuration considered here corresponds to exciting an $\mathcal{O}(1)$ number of degrees of freedom. In a region of sizes L, R, and R we are considering essentially just the 'fundamental harmonic', with $k_z \sim 1/L$ and $k_x \sim k_y \sim 1/R$. This makes the comment at the end of sect. 2.4.1 immaterial for our purposes.

solution is

$$S_{\rm cl}[\Delta\varphi] \simeq 2\pi \int dz r dr dt \, \tfrac{1}{2} w_0 r^2 \omega^2(r) = \frac{w_0}{T}\pi \int dz dr \, r^3 \Delta\varphi^2(r) \sim w_0 \frac{R^4 L \bar{\Delta\varphi}^2}{T} \,, \qquad (2.58)$$

where $\bar{\Delta\varphi}$ is the typical overall rotation of the final configuration in the region of interest. The amplitude we are interested in therefore is

$$\left\langle \text{rotation of } \bar{\Delta\varphi} \text{ in } R, L\,; T \left| \vec{\phi}, 0 \right\rangle \sim \exp i \frac{w_0 R^4 L}{T} \bar{\Delta\varphi}^2 \,. \qquad (2.59)$$

Angles of order one get populated in a cylindrical region of any given size R, and L if we wait long enough:

$$T \gtrsim w_0 R^4 L \,. \qquad (2.60)$$

At late times the symmetry is completely restored. Notice how the estimate (2.60) matches precisely what we could have inferred from the linearized statement (2.37).

2.5 Quantum viscosity?

Even if the above arguments alleviate the concerns raised by the perturbative analysis of sect. 2.3, we are still left with indications that the effective theory at hand may not be unitary.

A standard effective field theory can be unitary at low energies thanks to the decoupling of the short distance degrees of freedom. That is, thanks to the fact that to excite the microscopic degrees of freedom one is neglecting/being agnostic about, one needs a non-zero energy. There is a gap, and as long as one works below the gap, the long-distance degrees of freedom are sufficient to parameterize the dynamics.

In our case, we are not in such good shape. To excite very microscopic vorticose deformations of the fluid we need no energy at all. High momenta are not associated with high energies. This is also evident from eq. (2.37), where energy and momentum play strikingly different roles. This suggests that for any given cutoff in momentum space, the effective

theory may not be exactly unitary. No matter how small the energy of the process under consideration, there may be a non-trivial probability flow across the cutoff. From the effective theory viewpoint this should look like dissipation. In fact, for classical turbulence in a viscous fluid this is exactly what happens—viscosity drives vorticity from large scales to smaller and smaller ones, down to the UV cutoff of the fluid description (the mean free path of the underlying microscopic system.) Could it be that in our case we have some sort of quantum contribution to viscosity due to this non-decoupling of micro-vortices? How can we test this conjecture?

2.6 Summary & Outlook

Our findings raise more questions than they answer. The perturbative analysis about the naive, semiclassical vacuum $\phi^I = x^I$ indicates that the ordinary fluid effective field theory is strongly coupled at all scales, and thus inconsistent (sect. 2.3). On the other hand, a more careful non-perturbative study of the theory's quantum-mechanical vacuum shows that this has essentially nothing to do with the semiclassical one, and suggests that the naive perturbative degrees of freedom and their dynamics have no quantum-mechanical counterpart (sect. 2.4). One is thus tempted to welcome the latter conclusion and drop the former, and simply ignore the perturbative results. In particular, the strong coupling problem we isolated in sect. 2.3 might not be there—being associated with excitations that might themselves not be there. Also, in the $\vec{x}(\vec{\phi}, t)$ parameterization of sect. 2.4, the vortex dynamics are essentially free—the only thing resembling an interaction is the volume-preserving constraint. It is not clear what 'strong coupling' would mean in such a description of the system.

However, there is a number of confusing aspects that suggest that this optimistic attitude may be naive. The first is that even for theories where Coleman's theorem applies, one can recover correct information about physical quantities—like the spectrum for instance—by doing perturbative computations about the *wrong* semiclassical vacuum [21] (see also a

related discussion in [22]). We do not know yet whether, and to what extent, the results of [21] apply in our case. If they do, they would demote our Coleman theorem-like result to a somewhat formal statement about the vacuum structure in the far infrared, with less crucial consequences for more local physics.

The second confusing fact—which points in the same general direction as the previous one—is that for tiny but finite c_T we have a well defined effective theory that is perturbative up to some finite energy/momentum scale. For this effective theory there is no funny Coleman theorem-like behavior. All the symmetries that are spontaneously broken classically, remain so quantum-mechanically. The associated Goldstone bosons are described precisely by our perturbative analysis of sect. 2.3. On the other hand, as we stressed above, if $c_T \ll c_S$ we expect this theory to be physically equivalent to the $c_T = 0$ one for local questions, which do not rely crucially on the precise nature of the asymptotic states of the theory. While this kind of logic may be misleading for massive gauge theories or massive gravity, where having or not having the mass really determines the number of *local* (as opposed to asymptotic) physical degrees of freedom, here there is no such subtlety. The transverse degrees of freedom are perfectly physical even for vanishing c_T; they have non-vanishing conjugate momenta and are thus standard Hamiltonian degrees of freedom. Only, they do not feature wave solutions.

It thus seems that for local questions the perturbative analysis of sect. 2.3 should be perfectly fine. Of course the S-matrix is not a local quantity, and it may well be that the pathologies we encountered there are irrelevant for local questions. But we find it unlikely: if an effective theory exhibits strong coupling in scattering processes at some energy and momentum scales, it is probably useless for computing local correlation functions at the corresponding length scales. To settle the question one should compute directly local n-point functions, and see whether the perturbative expansion breaks down there. However, for the S-matrix we have a very powerful property—unitarity—that makes the tree level sufficient for such a question. For local correlation functions instead, one should really

ascertain the validity of the perturbative expansion by computing loop corrections—which, given that Lorentz invariance is spontaneously broken, is certainly doable but somewhat less transparent than usual. We leave this to future work.

Our results invite one to focus on correlators of quantities that are invariant under volume-preserving diffs. For instance, for a free massless scalar in $1+1$ dimensions, even though Coleman theorem applies, correlators of shift-invariant quantities are perfectly well-defined, and match what one would naively expect by doing perturbation theory about the wrong classical vacuum where the shift symmetry is spontaneously broken. Whether this property survives the inclusion of interactions and the generalization to our fluid, we do not know. If all diff-invariant correlators are well defined for our fluid, then one could in principle decide that those are the only observable quantities. This would correspond to *gauging* the problematic volume-preserving symmetry. This is certainly an interesting possibility to consider, and maybe the theory defined this way would be consistent. Yet it is not clear to us what resemblance it would bear with a physical fluid: as we tried to make clear, at the classical level the volume preserving diffs are not a gauge redundancy—they are real symmetries acting on physical and measurable degrees of freedom.

On a different note, even if the effective theory is not strongly coupled, we find it interesting that microscopic vortices may impair its unitarity (sect. 2.5). We plan to make this statement more systematic and quantitative. In particular, it would be interesting to understand to what extent this effect can be parameterized as a new contribution to viscosity, and whether it has any relation to the conjectured viscosity-over-entropy bound [23].

And finally, to make good on our promise discussed in the comment before section 2.2.1 we note again that one may take issue with all the previous analysis because, with our thermodynamic understanding of the EFT, the zero temperature limit corresponds to $F_b = 0$ (or equivalently $w_0 = 0$) and so it appears that our quadratic action is trivial. In order to remedy this situation we need to include a conserved charge and work instead with the

effective field theory described by $F(b, y)$ as defined in section 1.3. So while one may be temped to say that the previous analysis and discussion (which was done before a complete understanding of this subtlety) is entirely moot we believe that this should not be the case. The reason being that the essential features that plagued our system—modes without gradient energies—are still there in the conserved charge case. If we look at perturbations in the conserved charge case (see Appendix 6.1 for more details) in the zero temperature limit the quadratic action is non-trivial

$$\mathcal{L} \simeq \tfrac{1}{2} w_0 \big(\dot{\vec{\pi}}_L^2 - c_s^2 (\vec{\nabla} \cdot \vec{\pi}_L)^2 \big) + \tfrac{1}{2} w_0 \, \dot{\vec{\pi}}_T^2 + \tfrac{1}{2} F_{yy} y_0^2 \, (\dot{\tilde{\pi}}^0)^2 \, , \tag{2.61}$$

where, importantly $w_0 = (F_y y_o - F_b b_0) \to F_y y_0 \neq 0$. Clearly, there are modes without gradient energy terms—in fact, there is now one more of them. The same restrictions apply to this π_0 mode as do the π_T modes in the sense that both are restricted to not having a gradient energy to any order in derivatives by an infinite dimensional symmetry. And in fact, the two infinite dimensional symmetries taken together form a subclass of 4-dimensional volume preserving diffeomorphisms.

And so, while the details still need to be worked out, which we leave for the moment for future work, the qualitative features of the previous analysis should still survive: There will be a similar incompressible limit where we can once again prove, in an identical manner a la section 2.4.3 that we do not have spontaneous symmetry breaking. We believe that set of calculations as to those in section 2.3 can be performed for the charged fluid system and because of the nearly identical starting ingredients the conclusions will be the same. And so while the details will be worked out in the future, we still feel that the analysis throughout this chapter, and all the confusion they generate, is still of value.

Chapter 3

Vortex–Sound Interactions

In the preceding chapter, where the quantization of hydrodynamics was attempted, we discussed at length about how the vortex degrees of freedom are problematic at the quantum level but the trouble is there already at the classical level. Consider, for instance, a non-relativistic incompressible fluid, where the only allowed configurations are vortices, i.e., divergence-free velocity fields. In that limit, the continuity equation is just the constraint $\vec{\nabla} \cdot \vec{v} = 0$, the only degree of freedom is the vorticity field $\vec{\Omega} = \vec{\nabla} \times \vec{v}$, and the Euler equation can be rewritten as

$$\frac{\partial}{\partial t}\vec{\Omega} = \vec{\nabla} \times \left(\vec{v} \times \vec{\Omega}\right) \ . \tag{3.1}$$

If one neglects the non-linearity on the right hand side there is no time-evolution. The dynamics are completely dominated by non-linearities.

An equivalent viewpoint is the effective field theory one put forward in Chapter 1. As discussed in the previous chapters, vortices correspond to excitations that, because of an infinite-dimensional symmetry (volume-preserving diffeomorphisms), have vanishing energy in the limit of vanishing velocities, regardless of the excitations' spatial gradients. It is then easy for the system to excite these zero-energy gradients, and to make them exit the regime of validity of perturbation theory. Indeed, in field theory, gradient energy counts as quadratic

potential energy in the Hamiltonian. In the absence of a quadratic potential, the dynamics are dominated by anharmonicities.

The situation is strikingly different for compressional modes (in a compressible fluid). At the equation of motion level they correspond to perturbations $\delta\rho$ in the density field, obeying a wave equation plus non-linear corrections:

$$\Big(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\Big)\delta\rho + \cdots = 0\ . \tag{3.2}$$

Even neglecting the non-linearities, there are some non-trivial dynamics corresponding to the free propagation of sound waves. Then—for small $\delta\rho$—the non-linearities can be treated as small corrections to this. At the effective field theory level, compressional modes are standard gapless fields, which carry kinetic *and* gradient energy, and for which non-linear terms correspond to standard perturbative interactions .

These considerations indicate that there should exist physical situations in which compressional modes can be dealt with in perturbation theory, even though there might be an underlying vorticose 'background' that cannot. A nearly incompressible fluid provides such a system; there, by definition, compressional modes are difficult to excite.

It should be remembered that near incompressibility is not an intrinsic property of certain fluids, but rather a kinematical regime that exists for *all* fluids: for fluid flows that are much slower than sound, any fluid behaves as nearly incompressible [24], and vice versa, for fluid flows that are as fast as sound, any fluid is quite compressible. The analysis that follows thus applies to any fluid in the appropriate kinematical regime.

We will develop an effective field theory of sound interacting with vortices, and we will outline the systematics of the perturbative expansion that can be carried out close to the incompressible limit (sect. 3.1).[1] Some physical aspects of our analysis are not entirely new.

[1]We will ignore dissipative effects due to viscosity and heat conduction throughout this chapter. This is consistent at low frequencies and long wavelengths, because these effects correspond to higher order terms in the derivative expansion. An initial attempt at incorporating dissipation in our effective field-theory

For instance, the study of sound generation due to fluid flow (as opposed to time-dependent boundary conditions) was pioneered in the 1950's by Lighthill [25, 26]. Since then, its study has had a long history full of many contributions and contributors, the details of which can be found in the relatively recent texts [27, 28].

The basic idea of Lighthill is to identify the non-linear terms in (3.2) (the 'dots') as the source of sound, and to try to solve that equation perturbatively. In a sense our field theory is doing just that. However, by insisting on field theoretical ideas formulated in terms of a local action we can be completely systematic in how we carry out such a perturbative expansion—simply because perturbation theory in quantum field theory has been exhaustively studied. We can organize each computation in terms of Feynman diagrams. By restricting to tree-level diagrams, we will be, in fact, just solving the classical equations of motion perturbatively. The Feynman diagram language is an extremely powerful organizational and computational tool to do that.

It should be noted that Lighthill's approach to eq. (3.2) is the hydrodynamical analog of the so-called post-Netwonian expansion of general relativity (see e.g. [29]), which has been recently recast into an effective field theory language by Goldberger and Rothstein [30]. Our work owes much to their's.

Our first application of our tools will be to reproduce two preexisting results in the literature: the rate at which vorticose motions emit sound (sect. 3.2), and the cross-section they have for scattering it (sect. 3.3). However, since our field theory is relativistic by construction, we will be able to provide the relativistic corrections to these results at no additional cost. To the best of our knowledge, these have never been computed before.

On top of providing powerful computational tools, the effective field theory also offers a novel viewpoint on the dynamics, which can lead to simple predictions of previously overlooked phenomena. For instance, since vortices interact with sound waves, and since these

language can be found in [5].

are gapless, from the field theoretical viewpoint it is *obvious* that sound waves can mediate long-range interactions between vortices. These are in addition to the well known purely kinematical "dragging" interactions, whereby the long-distance tail in each vortex's velocity profile drags all the other vortices with it. We are not aware of any mention of our sound-mediated interactions in the literature. We will compute the leading contribution to the associated potential (sect. 3.4).

Interpreting the effects of such a potential on the vortices themselves will lead us to studying a somewhat different problem. The dynamics of vortices can be extremely counterintuitive. Even in the simplest case of vortex *lines*—which are the only possible vortices in superfluids, but exist in ordinary fluids as well—the equations of motion for the lines' positions are first order in time-derivatives. This means that concepts like that of 'force' on a vortex (which we would naively derive from our potential) do not really apply. We remedy this by constructing an action that describes the vortex lines' dynamics in the purely incompressible limit (sect. 3.5). This action includes the aforementioned "dragging" interactions as long-distance (that is, non-local) Lagrangian terms. Then, our sound-mediated potential energy should just be interpreted as the first-order—in departures from the incompressible limit—correction to this action, whose consequences at the equation of motion level can be derived just by varying the action in the usual way.

Finally, bringing everything together (sect. 3.6), we rewrite the non-local dragging interactions as being mediated by an auxiliary local field, and we reintroduce the local couplings of vortices to sound, thus ending up with a convenient, *local* field theory describing the dynamics of vortex lines and their interactions—among themselves and with sound—to all orders in perturbation theory. We feel this to be substantial technical as well as conceptual improvement over the more standard "vortex filament" model [31].

3.1 Coupling of incompressible vortex flows to sound waves

Our goal here is to systematically incorporate compressional effects by expanding this Lagrangian around slow (with respect to the speed of sound) vorticose background fluid flows. The details of this expansion are given in Appendix 6.5 but we reproduce the key logical steps below simply because this expansion, while pretty technical, is of vital importance for the remainder of the chapter.

The vortex/compressional mode separation is made most clear by working first in the $\vec{x}(\phi, t)$ parameterization of the fluid, i.e. working in comoving coordinates. Additionally, we make the speed of light c explicit in order to better keep track of relativistic effects. In these coordinates the action reads:

$$S = -w_0 c^2 \int d^3\phi dt \, \det J \, f\big((\det J^{-1})\sqrt{1 - v^2/c^2}\big) \, . \tag{3.3}$$

Where J is the Jacobian matrix

$$J^i{}_j = \frac{\partial x^i}{\partial \phi^j} \, , \tag{3.4}$$

$\vec{v}$ is the fluid's velocity field,

$$\vec{v} = \partial_t \vec{x}(\phi, t)|_\phi \, , \tag{3.5}$$

and $w_0 c^2$ is its enthalpy density at the reference point $b = 1$:

$$w_0 c^2 = (\rho + p)_{b=1} = -F_b(1) \, . \tag{3.6}$$

We are giving w_0 units of a mass density, and f is a dimensionless function of its dimensionless argument, defined by

$$F(b) = -w_0 c^2 \, f(b) \, . \tag{3.7}$$

The above expression for the action is derived in Appendix 6.5 and was originally given in reference [10]. The form of equation (3.3) is convenient for taking the non-relativistic limit.

However, we will find that keeping track of all relativistic effects is not particularly difficult and so we will do so.

For incompressible flows we have

$$\vec{x}(\phi,t) = \vec{x}_0(\phi,t) \ , \tag{3.8}$$

where, at each moment in time, $\vec{x}_0(\phi,t)$ is a volume preserving diffeomorphism of the co-moving coordinates $\vec{\phi}$,

$$\det\left(J_0\right) = 1 \ , \qquad J_0{}^i{}_j \equiv \det\left(\frac{\partial x_0^i}{\partial \phi^j}\right) \ . \tag{3.9}$$

This flow will only be a solution to the equations of motion, given by varying (3.3) with respect to $x^i(\phi,t)$, when the time-dependence of the flow vanishes. Consequently, for slow (relative to the speed of sound) vorticose flows, $x_0^i(\phi,t)$ is only an approximate solution with compressional corrections. So for these *nearly* incompressible flows we parametrize the fluid's configuration by

$$\vec{x}(\phi,t) = \vec{x}_0(\phi,t) + \vec{\psi}(\phi,t) \tag{3.10}$$

where $\vec{\psi}(\phi,t)$ vanishes in the limit of negligible time-dependence of $\vec{x}_0(\phi,t)$, and is curl-free *with respect to* $\vec{x}_0$, in the sense that

$$\vec{\psi} = \vec{\nabla}_{x_0}\Psi(\vec{x}_0,t) \ , \tag{3.11}$$

for some function Ψ. It is easy to convince oneself that, at least to lowest order, this is the correct characterization of compressional modes. For instance the determinant of the Jacobian (3.4), which characterizes the compression level of the fluid, at linear order in $\vec{\psi}$ is

$$\det J \simeq 1 + \vec{\nabla}_{x_0}\cdot\vec{\psi} \ , \tag{3.12}$$

where we made use of the volume preserving property of $\vec{x}_0(\phi,t)$. So, a curl component (wrt to $\vec{x}_0$) in $\vec{\psi}$ does not contribute to compression—only a gradient component does. In the following we will denote derivatives wrt to $\vec{x}_0$ simply by $\vec{\partial}_0$—not to be confused with the

usual relativistic notation of denoting ∂_0 as the partial derivative with respect to time, which will be denoted here simply by ∂_t.

We can now expand (3.3) in terms of these small perturbations. To quadratic order in the compressional modes we have

$$\det J = 1 + [\partial_0\psi] + \tfrac{1}{2}\left([\partial_0\psi]^2 - [\partial_0\psi\partial_0\psi]\right) + \dots \ , \tag{3.13}$$

where $[\dots]$ represents the trace [2]. Meanwhile, the velocity field is given simply by

$$\vec{v} = \vec{v}_0 + \partial_t\vec{\psi}(\phi,t) \ , \tag{3.14}$$

where $\vec{v}_0$ is the divergence-free velocity field associated with $\vec{x}_0(\phi,t)$:

$$\vec{v}_0 = \partial_t\vec{x}_0(\phi,t) \ , \qquad \vec{\partial}_0\cdot\vec{v}_0 = 0 \ . \tag{3.15}$$

Taylor-expanding and changing coordinates from the $\vec{\phi}$ to $\vec{x}_0$, and dropping total-derivative terms, the action can be written to lowest order as

$$S = S_0 + w_0\int d^3x_0dt\,\left\{\tfrac{1}{2}\left(\tfrac{D}{Dt}\vec{\psi}\right)^2 - \tfrac{1}{2}c_s^2[\partial_0\psi]^2 + \vec{v}_0\cdot\tfrac{D}{Dt}\vec{\psi} - \tfrac{1}{2}\tfrac{c_s^2}{c^2}\,v_0^2[\partial_0\psi] + \dots\right\} \tag{3.16}$$

where

$$S_0 = w_0\int d^3\phi dt\,\tfrac{1}{2}\vec{v}_0{}^2 + \dots \tag{3.17}$$

has no ψ-dependence. The $\dots$'s in S_0 are relativistic corrections, which are only powers of $\vec{v}_0{}^2$, c and c_s. Since S_0 does not depend on $\vec{\psi}$, it should be thought of as the action for the incompressional flow parameterized by $\vec{x}_0(\vec{\phi},t)$. To use S_0 in this sense, one should supplement it with a constraint that $\vec{x}_0$ be a volume-preserving diff of $\vec{\phi}$. For instance, it is straightforward to check that adding to S_0 the term

$$\int\lambda(\phi,t)\big[\det(\partial x_0^i/\partial\phi^j) - 1\big] \ , \tag{3.18}$$

[2] So $[\partial\psi] = \partial_i\psi^i$ and $[\partial\psi\partial\psi] = \partial_j\psi^i\partial_i\psi^j$ etc.

where λ is a Lagrange multiplier, yields the correct Euler equation for incompressible fluids.[3] However, as we stressed in the Introduction, the incompressional dynamics are very difficult to solve for, whereas, at least for slow fluid flows, we can make substantial progress by systematically applying perturbation theory to the *compressional* degrees of freedom, for a given background incompressible flow. So, in the following we will treat the volume-preserving evolution $\vec{x}_0(\phi, t)$ as given, and we will solve for the small compressional perturbations of such a background. In the quantum field theory/functional integral language, this is equivalent to performing the path integral—at tree level—over the $\vec{\psi}$ field, in the presence of given external sources $\vec{x}_0(\phi, t)$.

In (3.16), the $\frac{D}{Dt}$'s remind us that the partial time derivatives were taken at fixed *comoving* coordinates, i.e. fixed ϕ. These are the usual convective derivatives of fluid dynamics. In the x_0 coordinates, they simply act as

$$\tfrac{D}{Dt} g(x_0, t) = \partial_t g(x_0, t) + (\vec{v}_0 \cdot \vec{\partial}_0) g(x_0, t) \ . \tag{3.19}$$

This expression is exact, to all orders in $\vec{\psi}$: it is a mathematical identity, which follows simply from the chain rule; the fact that $\vec{x}_0(\phi, t)$ only reproduces part of the full, physical flow, is irrelevant. Finally, we have used that

$$f'(1) = 1 \ , \qquad f''(1) = c_s^2/c^2 \tag{3.20}$$

(c_s is the sound speed), which follow, respectively, from the definitions (3.6), (3.7), and from a straightforward analysis of the perturbation spectrum about the background (1.2).

And so, keeping the free quadratic terms in ψ (those not coupled to v_0), and the linear coupling of ψ to the background vortex velocity, we finally write the lowest order action

[3] In the Euler equation, λ plays the role of pressure.

containing the compressional modes as

$$\Delta S = S_{\text{free}} + S_{\text{int}} \tag{3.21}$$

$$S_{\text{free}} = w_0 \int d^3x_0 dt \; \tfrac{1}{2}\Big((\partial_t \psi^i)^2 - c_s^2[\partial_0 \psi]^2\Big) \tag{3.22}$$

$$S_{\text{int}} = w_0 \int d^3x_0 dt \Big(v_0^i(\vec{v}_0 \cdot \vec{\partial}_0)\psi_i - \tfrac{1}{2}\tfrac{c_s^2}{c^2}\, v_0^2[\partial_0 \psi] + \dots\Big) \,. \tag{3.23}$$

We have discarded a $\vec{v}_0 \cdot \partial_t \vec{\psi}$ term after noting that, as $\vec{\psi} = \vec{\partial}_0 \Psi$, we can place this spacial derivative on $\vec{v}_0$ by a simple integration by parts, and the term will vanish as $\vec{\partial}_0 \cdot \vec{v}_0 = 0$.

This simple action is enough to generate the famous "analogy" of Lighthill [26], with an additional relativistic correction. When we vary (3.21) with respect to ψ^i we recover Lighthill's limit of (3.2). Of course, for the fluids Lighthill was implicitly considering the relativistic effects would be extremely subleading, however for ultra-relativistic fluids where c_s^2 can be as large as $c^2/3$ this correction term can become a leading-order effect.

3.2 Sound emitted by a vorticose source

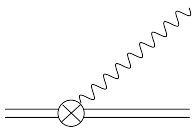
As an illustration of the utility of the effective field theory techniques, we reproduce Lighthill's result with additional relativistic correction terms. Instead of manipulating the lowest order equations of motion we can work directly from the action given by (3.21) utilizing the standard tools of quantum field theory to calculate physical observables. In particular, using the language of Feynman diagrams and restricting to tree level amplitudes allows us to isolate classical quantities [32]. For comparison, a derivation of Lighthill's result using more standard techniques can also be found in Landau and Lifshitz's classic Fluid Mechanics textbook [24].[4]

Say that we have a vorticose, possibly turbulent, configuration $\vec{v}(\vec{x}, t)$ with typical fluid flow of $v \ll c_s$ and some characteristic size ℓ. The velocity field acts as a source for sound waves. A cartoon depiction of this process is shown in Figure 3.1. We can formally deal

[4] It should be noted that there is a small typographical error in their final result.

Figure 3.1: *Sound radiating from a vorticose source.*

with this emission process via QFT tools, by computing the amplitude for emission of a single "phonon" from our compact vorticose source. The linear (in $\vec{\psi}$) terms of S_{int} in (3.21) contribute to the "tadpole" diagram



$$= i\mathcal{M}(\vec{p}) \ , \tag{3.24}$$

where $\mathcal{M}(\vec{p})$ denotes the probability amplitude to emit a single phonon of momentum $\vec{p}$. Given the probability amplitude one can calculate the emission rate with the standard formula:

$$d\Gamma(\vec{p}) = \frac{1}{T}\frac{d^3p}{(2\pi)^3 2c_s p}\left|\mathcal{M}(\vec{p})\right|^2 \ , \tag{3.25}$$

where $T \to \infty$ represents the total integration time, and will drop out in time averaged quantities (which is what we are after). The standard relativistic formula has also been modified with $c \to c_s$ as the energy of a phonon with momentum $\vec{p}$ is simply $c_s p$ (Appendix 6.3 and 6.6 for details).

Proceeding in the usual manner one finds:

$$i\mathcal{M}(\vec{p}) = w_0^{1/2} \frac{p^i p^j}{p} \, \tilde{\kappa}^*_{ij}(c_s p, \vec{p}) \tag{3.26}$$

where $\kappa_{ij}(\vec{x}, t)$ is the "kinetic tensor"

$$\kappa_{ij}(\vec{x}, t) \equiv v_i v_j - \tfrac{1}{2}\delta_{ij} \, \tfrac{c_s^2}{c^2} v^2 \ , \tag{3.27}$$

and $\widetilde{\kappa}_{ij}$ its Fourier transform,

$$\tilde{\kappa}_{ij}(\omega, \vec{p}) = \int d^3x dt \, \kappa_{ij}(\vec{x}, t) \, e^{i\omega t} e^{-i\vec{p}\cdot\vec{x}} \ . \tag{3.28}$$

Before inputting this amplitude into equation (3.25), we can make use of the multipole expansion: The typical frequency of the vorticose source is $\omega \sim v/\ell$. The emitted sound waves inherit such a frequency, and have therefore a typical wavelength $\lambda \sim c_s/\omega \sim \ell \cdot (c_s/v)$, which is, in our approximation, much bigger than the vorticose source's size ℓ. This means that we can treat the source as point-like, or the sound waves' wavelength as infinite. In the monopole approximation we have

$$\tilde{\kappa}_{ij}(c_s p, \vec{p}) \simeq \tilde{\kappa}_{ij}(c_s p, 0) \ . \tag{3.29}$$

In this approximation, integrating over the solid angle we have

$$d\Gamma(\omega) = \frac{\omega^3 \, d\omega}{T} \frac{w_0}{60\pi^2 c_s^5} \left[|\tilde{\kappa}_{ii}|^2 + 2|\tilde{\kappa}_{ij}|^2 \right] \ , \tag{3.30}$$

where the $\tilde{\kappa}$'s are evaluated at frequency ω and vanishing $\vec{p}$, and we have used $\overline{n_i n_k} = \frac{1}{3}\delta_{ik}$ and $\overline{n_i n_k n_l n_m} = \frac{1}{15}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl})$ (the 'overline' denotes the average over the solid angle.)

The emission rate is not the most natural quantity to consider when calculating classical wave emission, a more typical one would be the power radiated. It is easy enough to augment (3.30) in order to get this more standard quantity. If we have $\Gamma = \int d\Gamma$ we can write the power as simply $P = \int E \cdot d\Gamma$. In our case, the energy of a single phonon is simply ω (we

are working in $\hbar = 1$ units). Extending for convenience the range of integration to negative frequencies (and dividing by 2), the total radiated power is

$$P = \frac{1}{T}\frac{w_0}{120\pi^2 c_s^5}\int_{-\infty}^{\infty} d\omega\, \omega^4 \Big[|\tilde{\kappa}_{ii}|^2 + 2|\tilde{\kappa}_{ij}|^2\Big] \,. \tag{3.31}$$

Noting that, for real $f(t)$ and $g(t)$,

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{f}(\omega)\,\tilde{g}^*(\omega)\,\omega^{2n} = \int_{-\infty}^{\infty} dt \frac{d^n f(t)}{dt^n}\,\frac{d^n g(t)}{dt^n}\,, \tag{3.32}$$

we see that the inclusion of the $1/T$ factor, where $T \to \infty$, gives us time averages. Meanwhile, the restriction to vanishing $\vec{p}$ gives us volume integrals of the form

$$K_{ij}(t) \equiv \int d^3x\, \kappa_{ij}(\vec{x}, t)\,. \tag{3.33}$$

Putting everything together, for the radiated power we finally get

$$P = \frac{w_0}{60\pi c_s^5}\Big[\langle \ddot{K}_{ii}^2\rangle + 2\langle \ddot{K}_{ij}{}^2\rangle\Big]\,. \tag{3.34}$$

where $\langle ...\rangle$ denotes the time average, and the over-dots time derivatives.

To compare this result to the standard one—see e.g. [24]—, we define the traceless and pure-trace quantities

$$Q_{ij}(t) \equiv \int d^3x \left(v_i v_j - \tfrac{1}{3}v^2\delta_{ij}\right), \qquad Q(t) \equiv \int d^3x\, \tfrac{1}{3}v^2\,, \tag{3.35}$$

which we can use a basis for our tensor K_{ij}. In terms of these we get

$$P = \frac{w_0}{\pi c_s^5}\Big[\tfrac{1}{4}\langle \ddot{Q}^{\,2}\rangle\left(1 - 3\tfrac{c_s^2}{c^2} + \tfrac{9}{4}\tfrac{c_s^4}{c^4}\right) + \tfrac{1}{30}\langle \ddot{Q}_{ij}{}^2\rangle\Big]\,. \tag{3.36}$$

This matches the result contained [24] with the addition of relativistic corrections. Note that these are much smaller than the leading terms only if c_s is much smaller than c, regardless of how small v/c is. For relativistic equations of state with $c_s^2 \sim 1/3\,c^2$, the relativistic corrections are unsuppressed. The above expression is correct to lowest order in v/c_s—which is our small expansion parameter, associated with near incompressibility—but to *all* orders in c_s/c, and thus applies to those cases as well.

Figure 3.2: *Sound scattering off a vorticose source.*

By going to higher orders in the diagrammatic expansion of eq. (3.24), one could systematically compute higher order (in v/c_s) corrections to this result—for instance "radiative" corrections, as was done in the case of gravitational wave emission by binary systems in [33]. The advantage of the EFT techniques for performing these higher-order computations is manifest: for instance, the UV divergences associated with the point-like approximation for the source can be handled in the standard and well understood ways of renormalization theory.

3.3 Scattering sound waves off vorticose sources

If we include higher order terms in (5.3) we can calculate other interesting quantities. For example, the scattering of sound waves by a nearly incompressible flow. A cartoon illustration depicts this process in Figure 3.2. As emphasized in [34] this can be a powerful probe of the

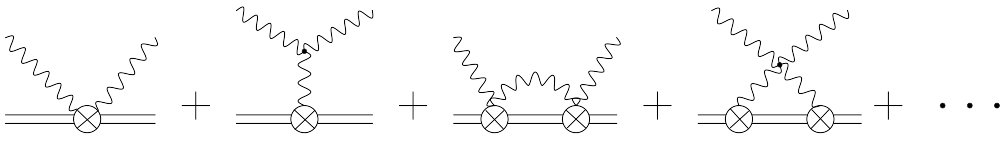


Figure 3.3: *Feynman diagrams contributing to the scattering of sound by a background fluid flow. The circled crosses denote ψ-v_0 interaction terms. There is no propagator associated with the double lines, which just depict the external source v_0.*

details of the fluid flow, e.g. to address questions about turbulence. The simplest scattering diagram will come from terms in the action quadratic in ψ—one ψ for the incoming wave, one for the outgoing one—and coupled to v_0. In principle there are other diagrams contributing to the same physical process, depicted in fig. 3.3. However, it is easy to convince oneself by simple power counting that, for small v_0/c_s, the quadratic-in-ψ vertex is the leading contribution: as derived in Appendix 6.5, it involves a single power of v_0, whereas all the other diagrams start at order v_0^2. The 'cross' in the second diagram of the figure yields directly a v_0^2 factor, while the crosses in the other diagrams yield a factor of v_0 each, but there are two such vertices per diagram.

Continuing the expansion that led to (5.3), and which is summarized in detail in Appendix 6.5, we obtain for this interaction term in the action:

$$S_{\rm int} \supset S_{\psi\, v^n} = w_0 \int d^3x dt \, \left\{ (\partial_t \psi_i)(v\cdot\nabla\psi^i) - \frac{c_s^2}{c^2}(v_i \partial_t \psi^i)[\partial\psi] + \mathcal{O}(v^2) \right\} . \tag{3.37}$$

The second term is a relativistic correction, which is important if the speed of sound is not much smaller than that of light. With an abuse of notation, we have stopped differentiating between x and x_0, and between v and v_0: all x's and v's should be thought of as being x_0's and v_0's.

When calculating the power emitted by a vorticose source in the previous section we only made one assumption about the flow (in addition to its having a finite size): its flow velocity was very subsonic, with $v/c_s \ll 1$. Now however, we have another dial we can control, that of

the frequency of the injected sound waves. In particular, we take the same limit as [34] and take the frequency of the incoming sound to be much larger than the natural frequency of the vorticose fluid flow: $\omega_\psi \gg \omega_v$. This is an experimentally convenient limit, as it distinguishes the scattered radiation from the naturally emitted one discussed in the previous section.

For given velocity profile $\vec{v}(\vec{x},t)$ of the source, the $(\omega_1,\vec{p}_1)\to(\omega_2,\vec{p}_2)$ scattering amplitude associated with the interaction terms above is simply

$$i\mathcal{M} = -i\Big\{(\hat{p}_1\cdot\hat{p}_2)\left[\omega_1 p_2^i + \omega_2 p_1^i\right] - \frac{c_s^2}{c^2}\left[\omega_1\hat{p}_1^i\, p_2 + \omega_2\hat{p}_2^i\, p_1\right]\Big\}\tilde{v}^i(\Delta\omega,\Delta\vec{p})\ , \tag{3.38}$$

where $\Delta\omega$ and $\Delta\vec{p}$ are the energy and momentum transfers:

$$\Delta\omega \equiv \omega_1 - \omega_2\ , \qquad \Delta\vec{p}\equiv\vec{p}_1-\vec{p}_2\ . \tag{3.39}$$

According to our assumptions, the typical frequency and wave-number of $\tilde{v}^i$ are

$$\Delta\omega\sim\omega_v\ll\omega_1\ , \qquad \Delta p\sim\frac{1}{\ell}\sim\frac{\omega_v}{v}\ , \tag{3.40}$$

where ℓ is the typical size of the source. We thus see that from the sound waves' viewpoint, the energy transfer is negligible, both in absolute terms and relative to the momentum transfer:

$$\frac{\Delta\omega}{\omega_1}\ll 1\ , \qquad \frac{\Delta\omega}{\omega_1}\sim\frac{v}{c_s}\cdot\frac{\Delta p}{p_1}\ll\frac{\Delta p}{p_1}\ . \tag{3.41}$$

To simplify the amplitude we can thus set

$$\omega_{1,2}\to\omega\ , \qquad |\vec{p}_{1,2}|\to\omega/c_s\ . \tag{3.42}$$

On the other hand, the momentum transfer—and thus the scattering angle—can be sizable, as long as ω_v/ω is not much smaller than v/c_s:

$$\frac{\Delta p}{p_1}\sim\frac{c_s}{v}\cdot\frac{\omega_v}{\omega}\ . \tag{3.43}$$

We are thus left with

$$i\mathcal{M}\simeq -i\,\frac{\omega^2}{c_s}\Big[(\hat{p}_1\cdot\hat{p}_2)-\tfrac{c_s^2}{c^2}\Big](\hat{p}_1+\hat{p}_2)\cdot\tilde{\vec{v}}(\Delta\omega,\Delta\vec{p})\ . \tag{3.44}$$

Inserting this into the standard formulae for the scattering cross section, which, for the convenience of the reader, are also quickly derived in Appendix 6.6, we have

$$\frac{d\sigma}{d\Omega\, d(\Delta\omega)} = \frac{1}{4c_s^6}\cdot\ \frac{\omega^4}{(2\pi)^3}\cdot\frac{1}{T}\left[(\hat{p}_1\cdot\hat{p}_2)^2 - \tfrac{2c_s^2}{c^2}\,(\hat{p}_1\cdot\hat{p}_2) + \tfrac{c_s^4}{c^4}\right]\left|(\hat{p}_1+\hat{p}_2)\cdot\tilde{\vec{v}}\right|^2, \qquad (3.45)$$

where $T\to\infty$ is a long observation time—long enough to observe the whole scattering process (the same T should be used to define the Fourier transform $\tilde{\vec{v}}$).

If the source's velocity field is nearly stationary, $\tilde{\vec{v}}\sim(2\pi)\,\delta(\Delta\omega)$, or more generally if one does not have the experimental resolution necessary to detect the small frequency transfer $\Delta\omega$, one can integrate in $\Delta\omega$ and end up with a time-averaged expression:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4c_s^6}\cdot\ \frac{\omega^4}{(2\pi)^2}\left[(\hat{p}_1\cdot\hat{p}_2)^2 - \tfrac{2c_s^2}{c^2}\,(\hat{p}_1\cdot\hat{p}_2) + \tfrac{c_s^4}{c^4}\right](\hat{p}_1+\hat{p}_2)^i(\hat{p}_1+\hat{p}_2)^j\left\langle\tilde{v}^{i*}\tilde{v}^j\right\rangle, \qquad (3.46)$$

where now $\tilde{v}=\tilde{v}(\Delta\vec{p},t)$ denotes a purely spatial Fourier transform of the source's velocity field, and $\langle ...\rangle$ denotes a time-average. Needless to say, the second and third terms in brackets are relativistic corrections, and—like in the previous section—they are exact to all orders in c_s/c.

The result expressed in (3.45) can be recast in a form that can be compared to previous results obtained by more traditional (and much more laborious) methods [34]. However, the comparison takes some non-trivial translating and so it is contained in Appendix 6.7.1. When the smoke clears, we see that our answer disagrees with that given in [34] by an overall factor of 2. To test the self-consistency of our computations, we check whether they obey the optical theorem. Since the optical theorem relates something quadratic in the scattering amplitude (the total cross-section) to something linear in it (its imaginary part in the forward limit), this is a non-trivial check of the overall normalization of our final result. According to standard cutting rules, at this order in perturbation theory the optical theorem should relate the cross section we have just computed to the imaginary part of the the third diagram in fig. 3.3. As we show in Appendix 6.7.2, the check works out, which gives us confidence in our results.

3.4 Sound-mediated long distance interactions between vortices

In addition to the physical processes considered above, which have been previously treated by perturbation theory performed at the level of the equations of motion (at least to lowest order for non-relativistic fluids), our formalism naturally motivates the calculation of the long-range interaction due to the exchange of compressional modes between two physically separated vortex configurations. The effective field theory approach *invites* such a question, and gives it a clear-cut qualitative answer: our vortices interact with compressional modes; these are gapless, and, as a consequence, can be exchanged between arbitrarily distant vortices; at some order in perturbation theory, this exchange will yield a non-trivial long range interaction between vortices. There is no question about it. For instance, this is how the classical $1/r$ Coulomb interaction arises in QED—via the tree-level exchange of virtual photons. On the other hand, such a question is not a natural one to ask in the standard approach to fluid dynamics, and in fact, to our knowledge, these long-range interactions have never been postulated before—let alone computed.

Notice that there are other long-range interactions between vortices, which survive even in the strict incompressible limit ($c_s \to \infty$), and are well understood in purely kinematical terms. As we will describe at some length below, each vortex typically carries a long distance "tail" in its velocity profile, scaling for instance as $1/r$ for infinitely long vortices. Then, if one is given several well-separated vortices, each will be dragged by all the others' velocity tails. As will see, our sound-mediated interaction will contribute a small correction to this, suppressed by $(v/c_s)^2$, but by no extra powers of r.

In Chapter 2 we initiated the analysis of our long-range interactions, by solving the linearized equation of motion for the sound modes and plugging the solution back into the action. Here we will use a more systematic technique—that of the effective action.[5] Even

[5] An excellent pedagogical introduction to the effective action as utilized in a very similar context is given

though at the classical level using the effective action *is* equivalent to solving the (generally non-linear) equations of motion for some of the fields and plugging their solutions back into the action, and even though in the end (at least at first) we will restrict to a lowest order computation thus reproducing the result of the previous chapter, we find it useful to set up a more systematic framework anyway, to pave the way for higher-order computations or for more general ones.

For our system, the sources will be given by vortices characterized for simplicity by the same typical length scale l and the same typical velocity $v \ll c_s$, separated by a much bigger length scale $r \gg l$, so that they appear as point-like sources to each other. These vortices interact via the exchange of compressional modes—virtual phonons, in our QFT-inspired language. We can compute the potential energy due to this exchange, and standard thinking indicates that its gradient will give the "force" that the sources exert on each other. As we will see, this intuition is fundamentally *in*correct, but the potential energy is still a well-defined physical quantity worth computing, and we can worry about its dynamical implications later.

The relevant quantity is the effective action $S_{\rm eff}$ one is left with after integrating out the compressional modes in the path integral. Schematically

$$e^{iS_{\rm eff}[\vec{x}_0]} \equiv \int D\vec{\psi}\, e^{iS[\vec{x}_0,\vec{\psi}]} \ . \tag{3.47}$$

Since $\vec{\psi}$ is gapless, $S_{\rm eff}$ will contain *non-local* interaction terms—precisely what we are after. As it turns out, the simple lowest order action given by (3.21),

$$S[\vec{x}_0,\vec{\psi}\,] \simeq S_0[\vec{x}_0\,] + S_{\rm free}[\vec{\psi}\,] + S_{\rm int}[\vec{x}_0,\vec{\psi}\,] \ , \tag{3.48}$$

is all we need to perform this calculation to leading order. S_0 is the lowest order action for $\vec{x}_0(\vec{\phi},t)$, given in (3.17). $S_{\rm free}$ is the quadratic action for $\vec{\psi}$, and $S_{\rm int}$ contains its linear

by [35].

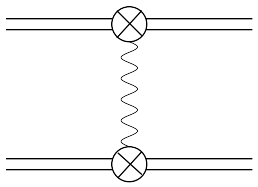


Figure 3.4: *Lowest order diagram contributing to the potential energy mediated by virtual sound between two vorticose sources.*

couplings to v_0:

$$S_{\rm int} = \int d^4x\, \vec{\psi}\cdot\vec{J}\,, \qquad \vec{J}(x) \equiv w_0\big(-\dot{\vec{v}}_0 - (v_0\cdot\nabla)\vec{v}_0 + \tfrac{1}{2}\tfrac{c_s^2}{c^2}\vec{\nabla}(\vec{v}_0{}^2)\big)\,. \tag{3.49}$$

Therefore, to lowest order we just have

$$S_{eff}[\vec{x}_0] = S_0 - i\ln Z[\vec{J}\,]\,, \qquad Z[\vec{J}\,] \equiv \int \mathcal{D}\psi^i e^{iS_{\rm free}+i\int d^4x\,\vec{\psi}\cdot\vec{J}+\epsilon\,\text{terms}}\,, \tag{3.50}$$

and we can compute the Gaussian functional integral via the standard completion of the square method. Up to an irrelevant constant we get simply

$$S_{\rm eff} = S_0 + \frac{1}{2}\int d^4x d^4y\, J_i(x)\Delta^{ij}(x-y)J_j(y)\,, \tag{3.51}$$

where Δ^{ij} is the usual Feynman propagator defined by

$$\Delta^{ij}(x-y) \equiv i\left\langle 0\middle|T\left\{\psi^l(y)\psi^m(x)\right\}\middle|0\right\rangle = \frac{1}{w_0}\int\frac{d^4k}{(2\pi)^4}\frac{\hat{k}^i\hat{k}^j e^{ik(x-y)}}{-k_0^2+c_s^2k^2-i\epsilon}\,. \tag{3.52}$$

Diagrammatically, this leading order correction to the effective action is given by the simple diagram denoted in fig. 3.4.

The compressional modes that mediate a long-distance interaction between slowly-moving sources are going to be (very) off-shell, i.e. $-k_0^2+c_s^2\vec{k}^2\neq 0$, and so we can take $\epsilon\to 0$ straight

away. In particular, we expect the relevant frequencies and momenta to be [6]

$$k^0 \sim \frac{v}{l}, \qquad \vec{k} \sim \frac{1}{r}\,, \tag{3.53}$$

and so if we take $v/c_s \ll l/r$ we can expand the denominator of the propagator as a power series in $(k^0/c_s k)$:

$$\frac{1}{-k_0^2 + c_s^2 k^2} = \frac{1}{c_s^2 k^2}\Big(1 + \frac{k_0^2}{c_s^2 \vec{k}^2} + ...\Big)\,. \tag{3.54}$$

This is analogous to the low-energy expansion of a massive propagator in relativistic QFTs, and for us it means that our interactions, despite being genuinely non-local in space, can be expressed as a series of interaction terms that are at least local in time.

So, to lowest order in this expansion and isolating the interesting interaction term (that is, the cross term when we write the total vortex velocity as the sum of the two separate sources $\vec{v} = \vec{v}_1 + \vec{v}_2$) we have simply,

$$S_{\text{eff}} = S_0 + \Delta S = S_0 + \frac{1}{w_0}\int d^4x d^4y \frac{d^4k}{(2\pi)^4} J_1^i(x) J_2^j(y) \frac{k^i k^j}{c_s^2 k^4} e^{ik(x-y)} \tag{3.55}$$

$$= S_0 + w_0 \int_{x,y} \frac{d^4k}{(2\pi)^4}\, \kappa_1^{ik}(x)\kappa_2^{jl}(y)\, \frac{k^i k^k k^j k^l}{c_s^2 k^4} e^{ik(x-y)}\,, \tag{3.56}$$

where we have freely integrated by parts and utilized the divergence free nature of $\vec{v}$, and the κ's are the two sources' kinetic tensors, as defined in (3.27).

Now, as these sources are supposed to be localized (that is, their velocity field falls off sufficiently fast away from their respective centers) we can multipole expand each source, keep the lowest order term in (l/r) (the monopole), and perform the k^0 integral, which generates a $(2\pi)\delta(t_1 - t_2)$. We may therefore write the integral of interest as:

$$\Delta S \simeq \frac{w_0}{c_s^2}\int dt\; K_1^{ik}(t) K_2^{jl}(t) \int \frac{d^3k}{(2\pi)^3}\; \frac{k^i k^j k^k k^l}{k^4} e^{i\vec{k}\cdot\vec{r}}\,, \tag{3.57}$$

[6] We are assuming that $\dot{v} \sim v^2/l$, so that the typical frequency of the sources is v/l. However, in many cases, like for instance those we will discuss below, one has nearly stationary, incompressible flows, in which case the typical frequencies are zero in first approximation, and the associated "virtual" compressional modes are maximally off-shell.

where $\vec{r}$ is the vortices' relative position vector, and, like in sect. 3.2, each K is defined simply as the monopole moment of the corresponding κ,

$$K^{ij} \equiv \int d^3x\, \kappa^{ij} = \int d^3x \left(v^i v^j - \tfrac{1}{2} \tfrac{c_s^2}{c^2}\, \delta^{ij}\, v^2 \right) . \tag{3.58}$$

The $\vec{k}$ integral is straightforward to perform by first rewriting the k's at the numerator as gradients w.r.t. $\vec{r}$, and then noticing that

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^4} e^{i\vec{k}\cdot\vec{r}} = -\frac{1}{8\pi}\, r + \text{IR-divergent, } \vec{r}\text{-independent pieces,} \tag{3.59}$$

as can be checked by introducing any IR-regulator, like e.g. $1/k^4 \to 1/(k^2+m^2)^2$.

Recalling that the potential enters the action with an overall minus sign, we can finally write the effective vortex-vortex potential mediated by sound in a compact form as

$$\Delta V = \frac{w_0}{8\pi c_s^2}\, K_1^{ik} K_2^{jl}\, \partial_i \partial_j \partial_k \partial_l\, r \ . \tag{3.60}$$

The derivatives, once expanded, yield

$$\partial_i \partial_j \partial_k \partial_l\, r = \frac{1}{r^3} \big[-\big(\delta_{ij}\delta_{kl} + 2 \text{ perms.}\big) - 15 \cdot \hat{r}^i \hat{r}^j \hat{r}^k \hat{r}^l + 3\big(\hat{r}^i \hat{r}^j \delta_{kl} + 5 \text{ perms.}\big) \big] \ . \tag{3.61}$$

Notice that the monopole moment (3.58) obeys positivity properties that forbids its vanishing as soon as one has *some* v—no matter how inventive one is in devising such a velocity profile. For instance, its trace is

$$K^{ii} = \int d^3x\, v^2 \big(1 - \tfrac{3}{2} \tfrac{c_s^2}{c^2}\big) \ , \tag{3.62}$$

which, using $c_s^2/c^2 \leq 1/3$, is always bigger than $\int \frac{1}{2} v^2$—the kinetic energy per unit enthalpy. As a result, the monopole-monopole interaction that we have computed will always be the most important long-distance interaction mediated by sound

For more general or higher order computations it is probably more convenient to perform the multipole expansion directly at the level of the action, *before* integrating out ψ, along the lines of [30]. In this way, on top of organizing the perturbative expansion in a more

systematic fashion, one is able to handle the UV divergences that unavoidably will show up at some order in perturbation theory using the standard tools of renormalization theory. For instance, 'self-energy' diagrams—diagrams in which a vortex exchanges a sound mode with itself—are UV divergent. Their divergent contribution to S_{eff} can be reabsorbed via renormalization into the coefficients of terms that are already present in the (multipole-expanded) Lagrangian. A more systematic treatment of these issues are performed in [3], but we will report the next-to-leading order computations in the next subsection as a proof of principle.

Eq. (3.60) is our final result for the long-distance, monopole-monopole interaction between localized vortex configurations. It is easy to estimate its size. The individual kinetic monopole moments are roughly $K^{ij} \sim v^2 l^3$, the derivatives structure acting on r scales as r^{-3}, and so the effective potential scales like

$$\Delta V \sim \frac{l^3}{r^3} \frac{v^2}{c_s^2} \cdot E_{\text{kin}} \ , \tag{3.63}$$

where $E_{\text{kin}} \sim w_0 v^2 l^3$ it the typical kinetic energy of the vortex configuration. We will see in sect. 3.5 that the aforementioned kinematical dragging phenomenon can be modeled via a long-range potential energy that, for localized vortices, scales as $(l/r)^3 \cdot E_{\text{kin}}$. Our effect is thus suppressed with respect to this one by an extra $(v/c_s)^2$ factor.

3.4.1 Next-to-leading-order in v/c_s contributions

As a demonstration of the ability of our formalism to compute higher order effects in the v/c_s expansion we present a preliminary result of a systematic power counting construction following the lines of [30]. In particular, the following are the (preliminary) results of the next-to-leading-order correction to the vortex-vortex effective action due to virtual sound exchange. This is an "in principle" type of calculation. So while we understand it may not be the most important quantity of interest to our experimental friends at this point [36] we do it because we *can*. In our EFT/Feynman diagram language the calculational program is

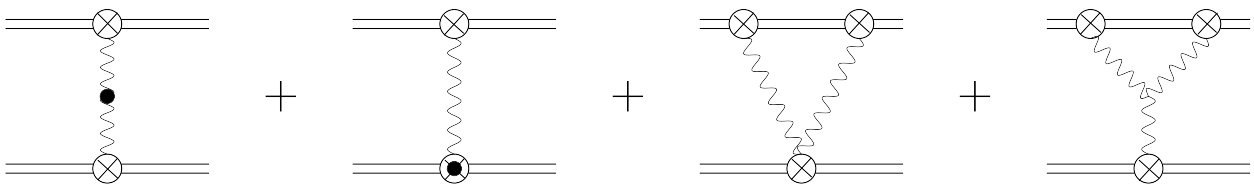


Figure 3.5: *Higher order in v/c_s diagrams contributing to the potential energy mediated by virtual sound between two vorticose sources. Dots denote first order corrections in v/c_s to the coupling and propagator.*

straightforward and relatively simple; we can't imagine doing the same from the equations of motions of perfect fluid dynamics. So while this particular calculation may not be necessary to such high order, surely there are others that may need such accuracy and therefore our methods.

As promised, reading off the interactions terms from (6.52)- (6.55) we can then multipole expand directly at the level of the action and write down the Feynman rules. Treating all our compressional modes as potential, i.e. off-shell, which is appropriate for this calculation and using the scaling that can be extracted from the previous section's results we can calculate the diagrams that contribute at the next order in v/c_s (and lowest order in l/r depending on the character of the sources for this order of v/c_s). The details of the calculation and the systematics of the power counting and can be found in the soon to appear [3].

The necessary diagrams are given by Figure 3.5. There are 4 diagrams (plus their symmetric counterparts) to this order in v/c_s. As commented on previously, the effective action that we are constructing here by integrating out these sound modes is not organized simply by a single expansion parameter as is the case in the simple compact, spin-less, binary inspiral problem [30]. This makes for a more complicated, but also interesting power counting structure. Not only does the relatives size of l/r and v/c_s matter but, as we will see, because we have not completely specified the character of the sources there is some freedom associated with the scaling of particular quantities. For instance, time derivatives on the monopole

moment of kinetic tensor (K^{ij}) for generic sources will scale in one particular way while for simpler sources, like vortex rings[7], they will scale in a completely different way. The only time dependence (other than its self induced constant velocity) that the vortex ring solution exhibits is induced by the other vortex source. So, for instance, in the presence of another vortex ring source (of similar dimensions)

$$\partial_t K_{\text{generic}} \sim \left(\frac{v}{l}\right) K \sim \left(\frac{v}{l}\right)\left(l^3 v^2\right) \tag{3.64}$$

$$\partial_t K_{\text{vortex ring}} \sim \left(\frac{l}{r}\right)^4 \times \left(\frac{v}{l}\right) K \sim \left(\frac{l}{r}\right)^4 \times \partial_t K_{\text{generic}} \,. \tag{3.65}$$

Correction to propagator:

The first diagram in Figure 3.5 represents an insertion of the $\left(\partial_t \phi^i\right)^2$ in the propagator. It corresponds simply to the first term in the series expressed in (3.54) which, we must remember, is valid only for $v/c_s \ll l/r$ for generic sources. Its calculation is straightforward, and we deal with the integral over momentum space in the same way as in did for the leading order diagram. We treat the k^i's in the numerators as derivatives and for this particular diagram must deal with a $1/k^6$ integral. Once again we deal with it by introducing an IR-regulator; the details are contained in Appendix 6.8. When the dust clears, the final correction to the effective potential given by this diagram is:

$$\Delta V_{\text{Prop Corr}} = \frac{w_0}{c_s^4}\frac{1}{4\pi}\frac{1}{12}\left[\dot{K}_1^{ij} + K_1^{ij} V_1^a \partial_a\right]\left[\dot{K}_2^{nm} + K_2^{nm} V_2^b \partial_b\right]\, \partial_i \partial_j \partial_n \partial_m \, r^3 \,, \tag{3.66}$$

where $\vec{V}_n$ are the velocities of the centers (about which we multipole expand) of the vorticose sources. Note that we are performing this calculation to lowest order in (l/r) at the level of the multipole expansion but depending on the character of the sources the $\dot{K}_1^{ij}$ and the $K_1^{ij} V_1^a \partial_a$ terms will scale differently in powers of (l/r). This apparently indicates that one needs to go a little further in the explicit organization of the interaction Lagrangian, using

[7] We will explore vortex rings and offer some references relating to them in the next section. For the moment the reader will just have to trust us on their scaling.

information about the particular sources in question, in order to give an exactly scaling, and therefore hierarchy, of corrections to the lowest order dynamics. We will discuss this scaling, and how they compare to contributions from other diagrams of the same order in (v/c_s) in just a moment.

Higher order coupling:

The second diagram in Figure 3.5 has one higher order coupling, given in (6.54), of the compressional mode to the velocity source. Its computation is straightforward using the same bag of tricks (IR-regulator, etc.) and in the end it contributes

$$\begin{aligned}\Delta V_{\text{H.O. Coupling}} &= \frac{w_0}{c_s^2 c^2}\frac{1}{8\pi}K_1^{ij}\left[\frac{(c^2-c_s^2)}{2c^2}M_2^{kl}\,\partial_l\right.\\ &\quad\left.+\frac{f_3}{8}M_2\,\partial^k-\frac{(c^2-c_s^2)}{2c^2}\left(\dot{P}_2^k+P_2^kV_2^l\partial_l\right)\right]\,\partial_i\partial_j\partial_k\,r\\ &\quad+(1\leftrightarrow 2)\,,\end{aligned}\tag{3.67}$$

where we have defined the the new quantities $M_1^{ij}=\int d^3x\,v_1^2v_1^iv_1^j$ and $P_1^i=\int d^3x\,v_1^2v_1^i$.

Seagull diagram:

The third diagram in Figure 3.5 has a quadratic coupling of the sound to the vortex sources. This diagram is more complicated (but fundamentally no more difficult) than the previous ones. Examining the possible coupling in (6.55) and multipole expanding we see that all the terms that are linear in v^i are going to vanish. To lowest order in (v/c_s) the couplings all take the form of

$$T_{ijnmab}\,v^iv^j\left(\partial^n\psi^a\right)\left(\partial^m\psi^b\right)\tag{3.68}$$

where T_{ijnmab} is just some contractions (δs) scaled by factors. Explicitly, it is given by

$$T_{ijnmab}=\frac{1}{2}\,\delta^{in}\delta^{jm}\delta^{ab}-\frac{c_s^2}{c^2}\,\delta^{ia}\delta^{jn}\delta^{mb}+\left(\frac{c_s^2}{4c^2}+\frac{f_3}{4}\right)\,\delta^{ij}\delta^{na}\delta^{mb}+\frac{c_s^2}{4c^2}\,\delta^{ij}\delta^{nb}\delta^{ma}\,.\tag{3.69}$$

This diagram now has two $1/k^4$ momentum integrals. Up to symmetry factors (which we

will parametrize by α) we have

$$\begin{aligned}\Delta V_{\rm Seagull} &= \frac{w_0}{c_s^4}\frac{\alpha}{(8\pi)^2}K_1'^{ij}\,K_2^{cd}\,K_2^{ef}\,T_{ij\,nm\,ab}\,(\partial_c\partial_d\partial_m\partial_a\,r)\,(\partial_e\partial_f\partial_n\partial_b\,r)\\ &\quad +(1\leftrightarrow 2)\,,\end{aligned} \tag{3.70}$$

where $K_1'^{ij} = \int d^3x\, v_1^i v_1^j$ (the kinetic tensor monopole without the relativistic correction term).

Cubic Interaction:

The final diagram in Figure 3.5 contains the cubic coupling of the sound modes to themselves. There are two distinct structures in (6.53). Let's focus on the simpler

$$w_0\int d^4x\,\frac{c^2 f_3}{6}[\partial\psi]^3 \tag{3.71}$$

piece for the moment. The $S_{\psi^3}\propto[\partial\psi][(\partial\psi)^2]$ piece will follow in a similar fashion but with a little more algebra. In fact, we will not report it here simply because the details are not of any use to the reader for how algebraically (there are many different contractions) complicated the result is. Those interested can find it in [3]. This diagram is more interesting than the others because we will actually have to use some of the tools familiar to quantum field theory in order to compute its effect, thus legitimizing our hard work in casting the problem in that language. It is worth delaying the presentation of the final result to see how this comes about. Before integrating over momenta the (3.71) interaction term gives rise to an effective potential of

$$\Delta V_{[\partial\psi]^3{\rm Cubic}} = \frac{w_0\,c^2 f_3}{c_s^6}K_1^{ab}K_1^{cd}K_2^{ef}\int_{\vec{k},\vec{q},\vec{p}}(2\pi)^3\delta^3(\vec{q}+\vec{p}+\vec{k})\frac{k^a k^b q^c q^d p^e p^f}{k^2q^2p^2}\,e^{i\vec{p}\cdot\vec{r}}\,, \tag{3.72}$$

plus the (1↔2) contribution for the symmetric diagram. When we integrate over d^3q, for instance, we are left with a *loop* integral in d^3k. That is, integrals of the same structure appear in loop corrections/quantum corrections to scattering amplitudes in quantum field theories [19]. In order to do this loop integral it is convenient to employ Feynman parameters

to combine the $\vec{k}^2(\vec{k}+\vec{p})^2$ denominator. Reorganizing and shifting the integral in the standard way we can write the above as

$$\begin{aligned}\Delta V_{[\partial\psi]^3\text{Cubic}} &= \frac{w_0\, c^2 f_3}{c_s^6} K_1^{ab} K_1^{cd} K_2^{ef} \int_{\vec{p}} \frac{p^e p^f}{p^2}\, e^{i\vec{p}\cdot\vec{r}} \int_0^1 dx \\ &\quad \times \int_{\vec{l}} \frac{(l+(x-1)p)^a (l+(x-1)p)^b (l+x\,p)^c (l+x\,p)^d}{(l^2+\Delta)^2}\,, \end{aligned} \tag{3.73}$$

where $\Delta = x(1-x)p^2$.

When we expand the numerator out we can see that there are going to be three different structures:

$$\int_{\vec{l}} \frac{l^4}{(l^2+\Delta)^2}\,, \quad \int_{\vec{l}} \frac{l^2}{(l^2+\Delta)^2} \quad \text{and} \quad \int_{\vec{l}} \frac{1}{(l^2+\Delta)^2}\,. \tag{3.74}$$

As we can see the first two of these are UV-divergent. If we work in dimensional regularization power law divergences are automatically set to zero. In any other consistent renormalization scheme such power law divergences could be simply absorbed into the bare couplings and would not contribute to any physical effects. A modern quantum field theorist would not even bat an eye at these kinds of integrals and dealing with such divergences but to our knowledge such procedures are less well defined at the equations of motion level and would presumably pose a serious barrier to the analysis of effects higher order in v/c_s effects like this.

After using dimensional regularization to perform all the integrals over $\vec{l}$, performing the integral over x and then organize our results we arrive at a sequence of terms of the form

$$\int_{\vec{p}} \frac{p^i p^j p^k p^l p^m p^n}{p^3} e^{i\vec{p}\cdot\vec{r}}\,. \tag{3.75}$$

Just as before we can treat the p's in the numerator as derivatives and use an IR-regulator to handle the $1/p^3$ integral; the result can be found in Appendix 6.8. When the dust clears

we are left with

$$\begin{aligned}\Delta V_{[\partial\psi]^3\text{Cubic}} &= \frac{w_0\, c^2 f_3}{c_s^6}\frac{1}{2048\,\pi^2}K_1^{ab}K_1^{cd}K_2^{ef} \\ &\times\left[\left(\delta^{ab}\delta^{cd}+2\delta^{ac}\delta^{bd}\right)\nabla^2\nabla^2+3\,\partial_a\partial_b\partial_c\partial_d\right. \\ &\left.-10\,\delta_{ab}\partial_c\partial_d\nabla^2-12\,\delta^{ac}\partial_b\partial_d\nabla^2\right]\partial_e\partial_f\,\log r+(1\leftrightarrow 2)\,. \end{aligned} \tag{3.76}$$

The $S_{\psi^3}\propto[\partial\psi][(\partial\psi)^2]$ piece has a very similar structure but with slightly more complicated index contractions.

Relative Scalings:

After all this work we can take stock of our results. As discussed previously we have to be more precise about the nature of the vorticose sources that we are dealing with in order to have a precise scaling. In one limit, if we take the most generic case, some sort of turbulent sources for instance, the leading term comes from the first diagram and scales as

$$\begin{aligned}\Delta V_{\text{Prop Corr}}(\text{generic}) &\sim \frac{l}{r}\,\frac{v^2}{c_s^2}\cdot E_{\text{kin}}\times\left(\frac{v}{c_s}\right)^2 \\ &\sim \Delta V_{\text{Lowest Order}}\times\left(\frac{v}{c_s}\right)^2\left(\frac{r}{l}\right)^2\,. \end{aligned} \tag{3.77}$$

Note that while, as promised, this contribution to the effective potential is subleading in (v/c_s) it is actually *enhanced* by (r/l)! But, in order to expand the propagator in the first place and treat the $(\partial_t\phi^i)$ as an interaction term we require $(v/c_s)(r/l)\ll 1$ and so this term is still subleading in comparison to the effective potential given by 3.60.

For simpler configurations, like the vortex rings we will study below, the first diagram in Figure 3.5 scales at a much higher order in (l/r), so high in fact that it is now subleading with respect to the other diagrams in the figure.

$$\Delta V_{\text{Prop Corr}}(\text{ring})\sim\frac{l^9}{r^9}\,\frac{v^2}{c_s^2}\cdot E_{\text{kin}}\times\left(\frac{v}{c_s}\right)^2\,. \tag{3.78}$$

The next diagram in Figure 3.5 contributes

$$\begin{aligned}\Delta V_{\text{H.O. Coupling}}(\text{ring}) &\sim \frac{l^3}{r^3}\frac{v^2}{c_s^2}\cdot E_{\text{kin}} \times \left(\frac{v}{c_s}\right)^2\left(\frac{c_s}{c}\right)^2 \\ &\sim \Delta V_{\text{Lowest Order}} \times \left(\frac{v}{c_s}\right)^2\left(\frac{c_s}{c}\right)^2 . \end{aligned} \tag{3.79}$$

Meanwhile, the seagull and cubic vertex diagrams scale as

$$\begin{aligned}\Delta V_{\text{Seagull}} \sim \Delta V_{\text{Cubic}} &\sim \frac{l^6}{r^6}\frac{v^2}{c_s^2}\cdot E_{\text{kin}} \times \left(\frac{v}{c_s}\right)^2 \\ &\sim \Delta V_{\text{Lowest Order}} \times \left(\frac{v}{c_s}\right)^2\left(\frac{l}{r}\right)^3 . \end{aligned} \tag{3.80}$$

We see some interesting tension here. If the fluid is relativistic then $c_s \sim c$ and $\Delta V_{\text{H.O. Coupling}}$ dominates while if $(l/r)^3 \gg (c_s/c)^2$, the fluid is non-relativistic to a certain degree, then $\Delta V_{\text{Seagull}}$ and ΔV_{Cubic} dominate.

3.4.2 Potential between two vortex rings

In order to get a better physical sense of eq. (3.60), we can compute it for the simple configuration of two interacting circular vortex loops—so-called vortex rings. A cartoon depiction of this interaction is given in Figure 3.6. Vortex rings are not only the simplest localized vortex-line configuration; they are also beautiful, fascinating objects of intense experimental interest and study, for humans [31,36] as well as other species [37] (see fig. 3.7). For the following we will ignore the relativistic correction term in K^{ij}, but its inclusion is straightforward. We refer the reader to [31] for an extensive review of the vortex rings' physical properties.

Say we have a vortex ring lying in some plane with normal vector $\hat{n}$. The radius of the ring is given by R, the circulation by Γ, and the vortex line thickness by some cutoff a (the so-called core radius). By symmetry K^{ij} will take the form:

$$K^{ij} = A\,\delta^{ij} + B\,\hat{n}^i\hat{n}^j . \tag{3.81}$$

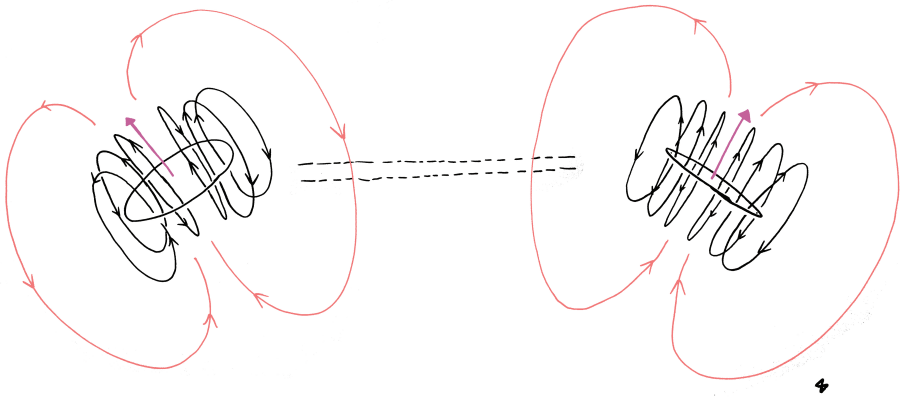


Figure 3.6: *Two vortex rings exchanging virtual sound.*

The parameters A and B are given by the scalar integrals

$$A = \tfrac{1}{2}\int d^3x \left(v^2 - (\vec{v}\cdot\hat{n})^2\right) \tag{3.82}$$

$$B = \tfrac{3}{2}\int d^3x \left((\vec{v}\cdot\hat{n})^2 - \tfrac{1}{3}v^2\right) \tag{3.83}$$

For distances much larger than R the velocity flow is given by that of a dipole, that is, the velocity falls off like $1/r^3$, and so, as anticipated and desired, the main contribution to K^{ij} comes from the region near the vortex ring itself. However, at very close distances to the vortex line the same integral will diverge logarithmically (if the core radius a is taken to zero). For $a \ll R$, we can thus get a good estimate of the integrals above by isolating the coefficient of $\log a$, which we can compute in the straight vortex line limit. Then, by dimensional analysis, $\log a$ has to be accompanied by $-\log R$. Geometric factors coming from precisely integrating over scales of order R (which we can do, at least numerically) and from the details of how the UV cutoff is actually implemented (which is model-dependent), will show up as finite, order-one additions to this large universal $\log R/a$ contribution, and are therefore sub-leading. Implementing this trick we find:

$$K \simeq \tfrac{1}{4}\Gamma^2 R \log(R/a)\,(\mathbb{I} + \hat{n}\otimes\hat{n}) \ . \tag{3.84}$$

Figure 3.7: *Dolphins can set up very clean vortex ring configurations, and play with them. They blow air into them, which, given the pressure gradients associated with the fluid flow, "floats" to the core of the ring, thus serving beautifully as a tracer. Pictures taken from [38, 39].*

However for realistic vortex rings in normal fluids (as opposed to superfluids), R is not that much bigger than a, and the "divergent" log is so weak that one cannot neglect the allegedly sub-leading finite pieces. Therefore, in order to be completely general we can write

$$K = \Gamma^2 R \left(\alpha \, \mathbb{I} + \beta \, \hat{n} \otimes \hat{n}\right) \ . \tag{3.85}$$

where α and β are order-one (or log) coefficients that in general depend on the exact core structure of the vortex rings.

Inserting this into (3.60) and assuming for simplicity the same core structure for the two rings, we can write the effective potential as

$$\Delta V = \frac{w_0}{8\pi c_s^2} \left(\Gamma_1^2 R_1\right) \left(\Gamma_2^2 R_2\right) \cdot \frac{1}{r^3} \cdot f(\hat{n}_1, \hat{n}_2, \hat{r}) \ , \tag{3.86}$$

where f parameterizes the angular structure:

$$\begin{aligned} f(\hat{n}_1, \hat{n}_2, \hat{r}) = &-(\beta^2 + 4\alpha\beta) - 2\beta^2 (\hat{n}_1 \cdot \hat{n}_2)^2 - 15\beta^2 (\hat{n}_1 \cdot \hat{r})^2 (\hat{n}_2 \cdot \hat{r})^2 \\ &+ (6\alpha\beta + 3\beta^2)\left[(\hat{n}_1 \cdot \hat{r})^2 + (\hat{n}_2 \cdot \hat{r})^2\right] + 12\beta^2 (\hat{n}_1 \cdot \hat{r})(\hat{n}_2 \cdot \hat{r})(\hat{n}_1 \cdot \hat{n}_2) \, . \end{aligned} \tag{3.87}$$

Notice that f can be either positive or negative. For instance, if we take $\alpha = \beta = 1$ for simplicity, then for a configuration with $\hat{n}_1 = \hat{n}_2 = \hat{r}$ one gets $f = 8$, while for $\hat{n}_1 = \hat{n}_2 \perp \hat{r}$

one gets $f = -7$. That is, depending on the geometric configuration, this potential can be both "attractive" and "repulsive". As we will see in a moment, the quotes are in order, because these terms are extremely misleading for vortex dynamics.

3.4.3 How (not) to interpret this $1/r^3$ potential

For standard systems, one usually thinks of a potential like that given by (3.60) as a function of $\vec{r}$ that will generate a force in the usual way, $\vec{F} = -\frac{\partial V}{\partial \vec{r}}$. The problem with this interpretation for us is that the notion of "force" does not really apply to vortex dynamics. To see why this is the case, let us restrict ourselves to the study of vortex lines, which exist both in normal fluids and in superfluids, and of which the vortex rings we just discussed are an example.

The way that we visualize a vortex line is by thinking of it as exactly that, a *line*, or more generally a curve. But really such a curve is a placeholder for a very special extended field configuration: the curve is the locus where the *vorticity* is nonzero, but the *velocity* field extends to large distances from the curve, typically in a $1/r$ fashion for very long lines. As a consequence, you cannot do to a vortex line the usual kinds of things that you would to a string-like object in empty space, like, for instance, boost it. Doing so will change the boundary conditions at infinity of the velocity field, thus in effect boosting the whole fluid. This does not mean that vortex lines cannot move, however, it just simply means that for the velocity field to go to zero at infinity, the vortex line will move in only *one* particular way; there is no free initial condition for their velocity so that, at any given time, the local velocity of any vortex-line infinitesimal element is *completely* determined by the dynamics. For instance, isolated, perfect vortex rings always move at a constant speed determined by their radius and circulation (see for instance [31]). Why do they do this?

In the incompressible limit, as ρ is no longer a dynamical variable, the hydrodynamical equations of motion are for $\vec{v}$ only and are given by the simplified continuity equation and

the curl of the Euler equation:

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{3.88}$$

$$\frac{\partial}{\partial t}\vec{\Omega} = \vec{\nabla} \times \left(\vec{v} \times \vec{\Omega}\right) \tag{3.89}$$

where $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ is the vorticity field. The linearity of (3.88) implies that if we have multiple isolated vortex lines (lines where the vorticity is non-zero), then the velocity field outside these lines obeys linear superposition. Furthermore, Kelvin's theorem (which follows from (3.89)) applied to infinitesimal comoving loops surrounding the vortex lines, implies that each line will move with the fluid flow generated by itself and every other vortex line. Since the relation between $\vec{\omega}$ and $\vec{v}$ is formally identical to that between current density and magnetic field in magnetostatics, and since for vortex lines $\vec{\omega}$ is given by a sum of circulations times delta functions supported on the lines, with the circulations playing the role of total electric currents flowing along these lines, the local velocity of the flow at $\vec{x}$ generated by all the vortex lines can be written in a Biot-Savart fashion as

$$\vec{v}(\vec{x}) = -\sum_n \frac{\Gamma_n}{4\pi} \int_n \frac{(\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} \times d\vec{x}\,' \tag{3.90}$$

where Γ_n is the circulation associated with the n-th vortex line.[8]

As we can see, given that a vortex line element placed at $\vec{x}$ will move with this velocity, the vortex lines' positions obey *first order* equations of motion, in contrast to the usual second order equations of motions of any string-like object in empty space. This is the root of their peculiarity: with first-order equations of motion, there is no room for "forces", and the effective potential we have computed above must then be interpreted more carefully. Our strategy in the next section will be to derive a Lagrangian formulation that reproduces precisely these first-order equations motion. Then, our sound-mediated effective potential

[8]There is a well-defined circulation $\Gamma_n = \int_{S_n} \vec{\omega} \cdot d\vec{S}$ for each line (the integral is taken over the cross-section of the "line"), which is in particular constant along that line, because of Stokes theorem, and constant in time, because of Kelvin's.

will just be a correction to such a Lagrangian, which will yield a corresponding correction to the equations of motion via the variational principle.

The model that we briefly sketched above is called the vortex-filament model and was pioneered in [40] and [41]. A modern review of such topics is discussed nicely in [31]. In the next sections we will find a reformulation of the same model that we believe is likely to lead to substantial progress in solving systems involving vortex lines.

Before we proceed, note that the integral over the "self-interacting" line in (3.90), is going to diverge logarithmically as $\vec{x}\,' \to \vec{x}$. Introducing a finite UV cutoff a, say the line's thickness, one gets that the leading contribution to the self-interacting integral is

$$\int_{\text{self}} \frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3} \times d\vec{x}\,' \simeq \hat{z}\frac{1}{R}\log\big(R/a\big) + \dots \ , \tag{3.91}$$

where R is the line's local radius of curvature at $\vec{x}$, $\hat{z}$ is the direction orthogonal to the plane defined by the local curvature, and the dots stand for subleading pieces, which come from the integral over regions that are far from $\vec{x}$, and which are finite for $a \to 0$. This UV divergence tells us that there is a mild dependence of the velocity of vortex lines on the details of their core structure. Changing these details, or equivalently changing the definition of a, will only affect the subleading, finite pieces. As usual, the coefficient of the log divergence is universal. If one is in the large $\log(R/a)$ limit, the velocity is dominated by this universal piece. In the vortex filament model this approximation is called the local induction approximation, because the local velocity is mostly induced by the local curvature of the vortex line.

3.5 An action for vortex-lines and their interactions

Given that this chapter focuses on the Lagrangian formalism, it is natural to ask if there is an action that will generate (3.90) upon variation. As mentioned above, a byproduct of finding such an action is that it would provide us with a straightforward interpretation of our sound-mediated potential (3.60): this should simply be interpreted as a correction to

the action, with obvious implications for the equations of motion.

Strictly speaking, the lowest order incompressible action (3.17), $S_0 = w_0 \int \frac{1}{2}\vec{v}_0{}^2$, contains the dynamics we want to reproduce. However, we want to restrict the velocity field to configurations with a vorticity field localized on lines, and we would like to use the positions of the lines themselves as degrees of freedom. It is not trivial to rewrite S_0 as an action for these degrees of freedom. So, for the moment we will just guess what the right action is, and we will comment later on what its relation to S_0 is.

Consider for the moment just one vortex line, parameterized by $\vec{X}(t, \lambda)$, where λ is some parameter running along the line, over which we are going to integrate. The action that we are going to write down should produce the equation of motion (3.90) upon variation w.r.t. to $\vec{X}$. Let's focus on the left hand side first. To get the velocity, which is just $\vec{v} = \partial_t \vec{X}$, one needs a term in the action with *one* time derivative only. A nontrivial, natural candidate in 2D (for vortex *dots*) would be

$$\int dt\, \tfrac{1}{2}(X^1 \partial_t X^2 - X^2 \partial_t X^1) = \int dt\, \tfrac{1}{2}\epsilon^{ij}\, X^i \partial_t X^j \ . \tag{3.92}$$

Varying w.r.t. to X^1 yields $\partial_t X^2$, and varying w.r.t. to X^2 yields $-\partial_t X^1$. The antisymmetric structure is crucial in order not to end up with a total derivative. Then, a natural generalization to 3D seems to be

$$\int dt\, d\lambda\, \tfrac{1}{3}\epsilon^{ijk} X^i \partial_t X^j \partial_\lambda X^k \ . \tag{3.93}$$

The vector $\partial_\lambda \vec{X}$ is tangent to the curve, and so the combination $\epsilon^{ijk}\partial_\lambda X^k$ behaves like the two-dimensional ϵ-tensor for the plane locally orthogonal the curve, thus giving us back the 2D expression above. The only apparent downside is that when we vary w.r.t. $\vec{X}$ to get the equations of motion, we now end up with an extra $\partial_\lambda X^k$ multiplying (via a cross product) the velocity on the left hand side.

Let's ignore this for the moment and let's move to the right hand side of (3.90). It contains a line integral over all vortex lines, including the one we are trying to understand

the motion of. Then, the corresponding term in the action should be represented by a double line integral, over all possible pairs of lines, including those pairs made up of the same line taken twice. Notice that the line element in (3.90) can be written as $d\vec{x}\,' \to d\lambda\,\partial_\lambda \vec{X}$. Notice also that eq. (3.90) involves a cross-product. Given the extra cross product with $\partial_\lambda \vec{X}$ we expect to get from (3.93), and given that, schematically $\epsilon\epsilon \sim \delta$, we end up with the following inspired guess for the action of a system of vortex lines:

$$S = C \int dt \left[\sum_n \int d\lambda\, \tfrac{1}{3} \Gamma_n \epsilon^{ijk}\, X_n^i\, \partial_t\, X_n^j\, \partial_\lambda X_n^k + \frac{1}{8\pi} \sum_{n,m} \int d\lambda d\lambda' \Gamma_n \Gamma_m \frac{\partial_\lambda \vec{X}_n \cdot \partial_{\lambda'} \vec{X}_m}{|\vec{X}_n - \vec{X}_m|} \right] , \tag{3.94}$$

where the sums run over all the vortex lines, C is an overall—yet to be determined—constant (whose value does not affect the equations of motion), while the relative $1/8\pi$ is needed to get the correct equations of motion, as we will show in a moment. In this expression for the action it is understood that, in the double-integral term, $\vec{X}_n$ is parameterized by λ and $\vec{X}_m$ by λ'. As we stressed above, the double sum also includes $n = m$, in which case the same line is traced out twice, independently by λ and λ'.

Varying the curves in the usual way, $\vec{X}_n(\lambda, t) \longrightarrow \vec{X}_n(\lambda, t) + \delta\vec{X}_n(\lambda, t)$, and ignoring boundary terms, we get the equations of motion for the n-th line:

$$\begin{aligned} &\epsilon^{ijk}\, \partial_t X_n^j\, \partial_\lambda X_n^k \\ &\quad = -\sum_m \frac{\Gamma_m}{4\pi} \int d\lambda' \left[\frac{\left(\vec{X}_n - \vec{X}_m\right)^j \partial_\lambda X_n^j\, \partial_{\lambda'} X_m^i}{|\vec{X}_n - \vec{X}_m|^3} - \frac{\left(\vec{X}_n - \vec{X}_m\right)^i \partial_\lambda X_n^j\, \partial_{\lambda'} X_m^j}{|\vec{X}_n - \vec{X}_m|^3} \right] , \end{aligned} \tag{3.95}$$

or, in vector notation (using the identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$ on the right hand side):

$$\partial_t \vec{X}_n \times \partial_\lambda \vec{X}_n = -\left[\sum_m \frac{\Gamma_m}{4\pi} \int d\lambda' \frac{(\vec{X}_n - \vec{X}_m) \times \partial_{\lambda'} \vec{X}_m}{|\vec{X}_n - \vec{X}_m|^3} \right] \times \partial_\lambda \vec{X}_n \, . \tag{3.96}$$

Now, it seems that our action (3.94) has not successfully reproduced the desired eom (3.90); there is an additional cross product with $\partial_\lambda \vec{X}_n$ which we cannot simply remove from both sides of the equation. As a result, the equations of motion generated by our action are

a little less stringent then the ones generated by "physical reasoning". Should we be worried about this?

As it turns out, the little extra freedom in (3.96) makes manifest a particular redundancy of how we describe the physical configurations of our system, which so far we have been reticent about. Given the spatial configuration of the vortex lines in our system at some particular time t, eq. (3.96) will dictate the velocity of a particular point on the vortex line *only in the direction normal to that line*. That is, the component of the velocity *parallel* to the curve is undetermined. But as the fluid velocity flow is completely determined by the positions of the vortex lines, any motion that does not change the locations and shapes of the lines, such as motion along the lines, is totally unphysical. In hindsight, it is obvious that this is related to a *reparameterization invariance* of the action (3.94):

$$\lambda \to \tilde{\lambda}(\lambda, t) \ . \tag{3.97}$$

Both terms in the action are invariant, because the pieces that transform non-trivially are combined in structures that are manifestly invariant: at fixed t one has

$$d\lambda \, \partial_\lambda \vec{X}_n = d\vec{X}_n \ , \qquad d\lambda \, d\lambda' \, \partial_\lambda \vec{X}_n \cdot \partial_{\lambda'} \vec{X}_m = d\vec{X}_n \cdot d\vec{X}_m \ . \tag{3.98}$$

This is a form of gauge invariance, which must be treated in the usual ways; for instance, in order to solve the equations of motion, one should first *choose a gauge*. The traditional form of the eom, (3.90), corresponds to a particular (and not so simple) gauge choice. Perhaps a more "physical" choice in our language would be to choose $\partial_t \vec{X} \cdot \partial_\lambda \vec{X} = 0$, i.e set the parallel component of the velocity to zero, for all time. But depending on the problem at hand, other choices might be considerably more convenient. For instance, to study an infinite vortex line that is roughly aligned with a coordinate axis, say the z axis, one can choose the gauge $\lambda = z \equiv X^3$, thus effectively eliminating the X^3 degree of freedom from the problem.

3.5.1 Fixing the overall coefficient

While we have been able to reproduce the dynamics of the fluid system given the action (3.94), we still have to determine the action's overall scaling. We can fix this constant C by matching a physical quantity for a simple physical configuration. As usual for effective field theory, the matching of Lagrangian coefficients can be done via an observable of an idealized, simple configuration, and the inferred values for the coefficients can then be used for all other configurations as well.

Consider a single vortex line aligned with the z-axis, with circulation Γ. Its velocity field is

$$\vec{v} = \frac{\Gamma}{2\pi}\frac{1}{r}\,\hat{\varphi}\ . \tag{3.99}$$

The original non-relativistic action for an incompressible fluid, eq. (3.17), implies that such a configuration has an energy per unit length

$$\frac{dE}{dz} = w_0 \int \tfrac{1}{2} v_0^2\, d^2x = w_0 \frac{\Gamma^2}{4\pi} \log L/a\ , \tag{3.100}$$

where L is some IR cutoff (e.g. the size of the container) and a some UV cutoff (e.g. the vortex-line thickness).[9]

We can now compute the same physical quantity with our new action, eq. (3.94). The first piece—the 'kinetic' term—does not contribute to the Hamiltonian of the system, since it is linear in $\dot{X}$:

$$E = H = \frac{\partial L}{\partial \dot{X}^i}\,\dot{X}^i - L\ . \tag{3.101}$$

We are thus left with the second piece only, which yields

$$E = -C\frac{\Gamma^2}{8\pi}\int dz dz' \frac{1}{|z-z'|}\ , \tag{3.102}$$

[9] As before, as long as we are interested in the log-divergence only, we need not be specific about how these cutoffs are implemented, since the coefficient of the log is completely independent of these details—only additive finite pieces are sensitive to them.

corresponding to an energy per unit length

$$\frac{dE}{dz} = -C\frac{\Gamma^2}{4\pi}\log L/a \ . \tag{3.103}$$

Notice that, given the different geometric nature of the integrals involved, the cutoffs used here cannot have exactly the same meaning as those used above; the universality of log divergences allows us to be cavalier about this. Comparing the two expressions for the energy per unit length we get simply

$$C = -w_0 \ . \tag{3.104}$$

3.6 The hydrophoton

Eq. (3.94) reproduces the correct dynamics and energetics of generic vortex line configurations, and is thus *the* correct action for them. Nevertheless, it has an obvious annoying feature: due to the double λ-integral in the second term, it is a *non-local* action. To develop an intuition about the (counter-intuitive) dynamics of the system, especially for perturbation theory questions (stability of solutions, properties and interactions of small perturbations, etc.), having a local action would be much more convenient.

Fortunately, given the $1/r$ nature of the non-locality in question, we know exactly how to fix the problem. We 'integrate in' an auxiliary local field $\vec{A}(\vec{x},t)$, coupled to the 'currents' $\Gamma_n\,\partial_\lambda \vec{X}_n$, which are localized on the vortex lines:

$$\begin{aligned} S_{\rm local} = w_0\Big[&-\sum_n \Gamma_n \int dtd\lambda\, \tfrac{1}{3}\epsilon^{ijk}\, X_n^i\, \partial_t\, X_n^j\, \partial_\lambda X_n^k \\ &+ \int d^3x dt\, \tfrac{1}{2}\big(\partial_i A_j\big)^2 - \sum_n \Gamma_n \int dtd\lambda\, \partial_\lambda \vec{X}_n \cdot \vec{A}\big(\vec{X}_n,t\big)\Big] \ . \end{aligned} \tag{3.105}$$

One can solve the equations of motion for $\vec{A}$ deriving from this action,

$$\nabla^2 \vec{A}(\vec{x},t) = -\sum_n \int d\lambda\, \Gamma_n\, \partial_\lambda \vec{X}_n\, \delta^3\big(\vec{x} - \vec{X}_n(\lambda,t)\big) \ , \tag{3.106}$$

Magnetostatics	Incompressible Hydro
current $\vec{J}$	vorticity $\vec{\omega}$
magnetic field $\vec{B}$	velocity field $\vec{v}$
vector potential $\vec{A}$	hydrophoton $\vec{A}$

Table 3.1: *Schematic dictionary between magnetostatics and our system of vortices in an incompressible fluid.*

and plug the solution back into action, thus getting back the non-local action (3.94). Since $\vec{A}$ appears at most quadratically in the action, this equivalence is *exact*, even at the quantum-mechanical level, which might be important for zero-temperature superfluids.

In this new local picture, interactions can be described in standard field theoretical terms: the action at a distance in (3.94) has been replaced by local interactions of each vortex-line with $\vec{A}$. Then, different vortex-lines, or different pieces of the same line, interact by 'exchanging' $\vec{A}$. It is obvious what $\vec{A}$ corresponds to in the magnetostatics analogy of the previous section: it is the vector potential in Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$). Indeed, given the (localized) current density

$$\vec{J}(\vec{x}, t) = \sum_n \Gamma_n \int d\lambda \; \partial_\lambda \vec{X}_n \, \delta^3\big(\vec{x} - \vec{X}_n(\lambda, t)\big) \; , \tag{3.107}$$

the $\int \vec{J} \cdot \vec{A}$ term in (3.105) is precisely the standard interaction between current and vector potential, and eq. (3.106) is precisely the equation of motion for the vector potential in Coulomb gauge, $\nabla^2 \vec{A} = \vec{J}$. We dub $\vec{A}$ the 'hydrophoton' field. For more general vortex configurations in which the vorticity is not localized on lines, the analogy still works in detail—only, one should identify the current density with the vorticity field $\vec{\omega}$. Table 3.1 summarizes the dictionary between magnetostatics and our system.

In conclusion, the interaction between vortex lines in an incompressible fluid can be described *exactly* as magnetic interaction between currents. The only novelty is how the

lines respond to these magnetic-type fields: given the peculiar structure of the lines' kinetic action in (3.105) (the first line), there is no analog of the Lorentz force. Instead, as discussed in the previous section, the instantaneous velocity of a line-element *is* the local value of the 'magnetic' field.

We can now assess very easily the importance of our sound-mediated potential (3.60), relative to the purely kinematical long-distance interactions which, in our new language, are mediated by $\vec{A}$. For instance, two vortex rings of radii $R_{1,2}$, circulations $\Gamma_{1,2}$, and orientations $\hat{n}_{1,2}$, at a large distance from each other have the same interaction potential energy as two magnetic dipoles,

$$V_{\rm dip} = \frac{w_0}{r^3}\left[3(\hat{r}\cdot\vec{\mu}_1)(\hat{r}\cdot\vec{\mu}_2) - \vec{\mu}_1\cdot\vec{\mu}_2\right] , \tag{3.108}$$

with dipole moments

$$\vec{\mu}_n = \pi(\Gamma_n R_n^2)\,\hat{n}_n \ . \tag{3.109}$$

We thus get

$$V_{\rm dip} \sim \frac{w_0}{r^3}\,(\Gamma_1 R_1^2)\,(\Gamma_2 R_2^2) \ , \tag{3.110}$$

which is indeed a factor of $(c_s/v)^2$ bigger that the sound-mediated potential, eq. (3.86). Notice however that for very thin vortex cores, like e.g. in superfluid vortex lines, eq (3.86) gets enhanced by a factor of $(\log R/a)^2$, which can partially compensate the $(v/c_s)^2$ suppression factor.

The reader might be skeptical about the usefulness of our introducing the local field $\vec{A}$ to describe vortex interactions. In fact, aren't we supposed eventually to solve all the equations of motion? If we first solve the equation of motion for $\vec{A}$ and we plug the solution into the others, we are effectively reproducing the equations of motion deriving from (3.94). So, why bother introducing $\vec{A}$ in the first place? Although this is technically a valid viewpoint—no information is added by introducing $\vec{A}$—such a statement is technically and conceptually equivalent to claiming that it is useless to introduce the local fields $\vec{E}$ and $\vec{B}$ (or V and $\vec{A}$) in electrostatics and magnetostatics, since one can do everything at the level of the

Coulomb force between charges and the magnetic force between currents. A position that, in hindsight, few would defend. We believe that, like for electrostatics and magnetostatics, our local rewriting of vortex interactions will prove valuable in the study of vortex line systems.

3.6.1 Example: Kelvin waves

To prove the usefulness of our approach, we now study via standard field-theoretical techniques the low-frequency spectrum of small perturbations of an infinite straight vortex-line. Let's align the (unperturbed) line with the z-axis. The perturbed fields are

$$\vec{X}(\lambda,t) = \vec{X}_0 + \vec{\pi}(\lambda,t)\ , \qquad \vec{X}_0 = \hat{z}\lambda \tag{3.111}$$

$$\vec{A}(\vec{x},t) = \vec{A}_0 + \vec{\psi}(\vec{x},t)\ , \qquad \vec{A}_0 = -\hat{z}\frac{\Gamma}{2\pi}\log(r/a) \tag{3.112}$$

where $\vec{X}_0$ and $\vec{A}_0$ are the unperturbed configurations, $r = \sqrt{x^2+y^2}$ is the distance from the z-axis, and a is a UV cutoff (the line thickness). We can now expand $S_{\rm local}$ in powers of $\vec{\pi}$ and $\vec{\psi}$. The linear term in the fluctuations is, of course, going to vanish as $\vec{X}_0$ and $\vec{A}_0$ are solutions to the equations of motion. We are thus left with, at lowest order in the fluctuations, a quadratic action:

$$\begin{aligned} S \ \simeq\ & -w_0\Big[\int dtd\lambda\, \tfrac{1}{2}\Gamma\epsilon_{ij}\pi^i_\perp\partial_t\pi^j_\perp - \int d^3xdt\, \tfrac{1}{2}\left(\partial^i\psi^j\right)^2 \\ & +\Gamma\int dtd\lambda\left(\vec{\psi}(0)\cdot\partial_\lambda\vec{\pi} + \vec{\nabla}\psi^\parallel(0)\cdot\vec{\pi} + \vec{\nabla}A_0^\parallel(0)\cdot\vec{\pi}\,\partial_\lambda\pi^\parallel + \tfrac{1}{2}\partial_i\partial_j A_0^\parallel(0)\,\pi^i\pi^j\right)\Big]\ , \end{aligned} \tag{3.113}$$

where '$\perp$' and '$\parallel$' are defined w.r.t. the z-axis, and we have freely integrated by parts and thrown out boundary terms. In the second line, the notation $\vec{\psi}(0)$ and $\vec{A}_0(0)$ reminds us that these 'bulk fields' and their derivatives are to be evaluated on the unperturbed line, at $x = y = 0$ (the last three terms come from expanding their arguments in powers of $\vec{\pi}$). One can thus see that the second-to-last term vanishes as $\partial^j A_0^\parallel$ is odd in either x or y (or simply vanishes for $j = 3$). Continuing further the decomposition of $\vec{\psi}$ and $\vec{\pi}$ into parallel

and perpendicular components, and noticing that $\vec{A}_0$ only has derivatives in the orthogonal direction, we finally get

$$\begin{aligned} S \;\simeq\; & -w_0\Big[\int dtd\lambda\, \tfrac{1}{2}\Gamma\epsilon_{ij}\pi^i_\perp\partial_t\pi^j_\perp - \int d^3xdt\, \tfrac{1}{2}\left(\partial^i\psi^j_\perp\right)^2 + \Gamma\int dtd\lambda\, \vec{\psi}_\perp(0)\cdot\partial_\lambda\vec{\pi}_\perp \\ & - \int d^3xdt \tfrac{1}{2}\left(\partial^i\psi_\parallel\right)^2 + \Gamma\int dtd\lambda\left(\vec{\nabla}_\perp\psi_\parallel(0)\cdot\vec{\pi}_\perp + \tfrac{1}{2}\partial^\perp_i\partial^\perp_j A^\parallel_0(0)\,\pi^i_\perp\pi^j_\perp\right)\Big]\,. \end{aligned} \tag{3.114}$$

Recall that we have not fixed the gauge yet. We would like to choose $\lambda = z = X^3$, that is,

$$\pi_\parallel(\lambda,t) = 0\,. \tag{3.115}$$

Since $\pi^\parallel$ has completely disappeared from the action, we can perform this $\lambda \to z$ gauge choice directly at the level of the action without worrying about losing $\pi^\parallel$'s equation of motion.

Varying (3.114) with respect to $\psi_\perp$, $\psi_\parallel$, and $\pi_\perp$ we generate the three equations of motion:

$$0 \;=\; \nabla^2\vec{\psi}_\perp + \Gamma\delta^2(\vec{x}_\perp)\,\partial_z\vec{\pi}_\perp \tag{3.116}$$

$$0 \;=\; \nabla^2\psi_\parallel - \Gamma\vec{\nabla}_\perp\delta^2(\vec{x}_\perp)\cdot\vec{\pi}_\perp \tag{3.117}$$

$$0 \;=\; \partial_t\vec{\pi}_\perp\times\hat{z} - \partial_z\vec{\psi}_\perp(0) + \vec{\nabla}_\perp\psi_\parallel(0) + \vec{\nabla}_\perp\partial^\perp_i A^\parallel_0(0)\,\pi^i_\perp\,. \tag{3.118}$$

These equations can be recast into a slightly more convenient form by insisting on manifest translational invariance: The background configurations $\vec{X}_0(\lambda,t)$, $\vec{A}_0(\vec{x},t)$ (spontaneously) break translations in the plane orthogonal to our line. As a results, these translations are non linearly linearized on the associated perturbations:

$$\vec{\pi}_\perp \to \vec{\pi}_\perp + \vec{\varepsilon}_\perp\,, \qquad \psi_\parallel \to \psi_\parallel - \vec{\varepsilon}_\perp\cdot\vec{\nabla}_\perp A^\parallel_0\,, \tag{3.119}$$

implying in particular—along the lines of Goldstone theorem—the absence of a gap at zero momentum. Although the action and equations of motion above are invariant under these transformations, the absence of a zero-momentum gap is not obvious at all, and will require non-trivial cancellations, since there are terms in which $\vec{\pi}_\perp$ appears without derivatives. It

would be nicer to have a reformulation of these equations (and of the action), in which the broken translations act simply as a shift on $\vec{\pi}$ and nothing else. This is easily accomplished by redefining the $\psi_\parallel$ field as

$$\psi_\parallel = \tilde{\psi} - \vec{\pi}_\perp \cdot \vec{\nabla}_\perp A_0^\parallel \,, \tag{3.120}$$

so that, using $\nabla_\perp^2 A_0^\parallel = -\Gamma\delta^2(\vec{x}_\perp)$, the eom become

$$\begin{aligned} 0 &= \nabla^2\vec{\psi}_\perp + \Gamma\delta^2(\vec{x}_\perp)\,\partial_z\vec{\pi}_\perp & (3.121)\\ 0 &= \nabla^2\tilde{\psi} - \partial_z^2\vec{\pi}_\perp \cdot \vec{\nabla}_\perp A_0^\parallel & (3.122)\\ 0 &= \partial_t\vec{\pi}_\perp \times \hat{z} - \partial_z\vec{\psi}_\perp(0) + \vec{\nabla}_\perp\tilde{\psi}(0)\,. & (3.123) \end{aligned}$$

Now $\vec{\pi}_\perp$ always appears *with* derivatives, and translational invariance is realized simply as $\vec{\pi}_\perp \to \vec{\pi}_\perp + \vec{\varepsilon}_\perp$.[10]

These equations describe waves propagating in the z direction. To see this, let's go to Fourier space for z and t (but not for $\vec{x}_\perp$), upon which the equations of motion become

$$\begin{aligned} 0 &= \big(\nabla_\perp^2 - k^2\big)\vec{\psi}_\perp + \Gamma\,\delta^2(\vec{x}_\perp)\,ik\,\vec{\pi}_\perp & (3.125)\\ 0 &= \big(\nabla_\perp^2 - k^2\big)\tilde{\psi} + k^2\,\vec{\pi}_\perp \cdot \vec{\nabla}_\perp A_0^\parallel & (3.126)\\ 0 &= -i\omega\,\vec{\pi}_\perp \times \hat{z} - ik\,\vec{\psi}_\perp(0) + \vec{\nabla}_\perp\tilde{\psi}(0)\,, & (3.127) \end{aligned}$$

where now $\vec{\pi}_\perp$ depends on k and ω, and $\vec{\psi}_\perp$ and $\tilde{\psi}$ depends on them as well as on $\vec{x}_\perp$.

The shape of the perturbed line is uniquely determined by $\vec{\pi}_\perp$. Therefore, as long as we are interested in just that, we only need the solutions for $\vec{\psi}_\perp$ and $\tilde{\psi}$ close to the line—see the third equation. Treating then $\nabla_\perp^2$ in the first two equations as being of order $1/r^2 \gg k^2$,

[10] The same change of variables at the level of the action, replaces the second line in (3.114) with

$$w_0 \int d^4x \left[\tfrac{1}{2}\big(\partial_i\tilde{\psi}\big)^2 - \partial_z\tilde{\psi}\,\partial_z\vec{\pi}_\perp \cdot \vec{\nabla}_\perp A_0^\parallel + \tfrac{1}{2}\big(\partial_z\vec{\pi}_\perp \cdot \vec{\nabla}_\perp A_0^\parallel\big)^2\right], \tag{3.124}$$

which has manifest shift-invariance for $\vec{\pi}_\perp$. Varying the new action with respect to the fields one gets, after stratighforward manipulations, precisely the new equations of motion.

close to the line we get

$$\nabla_\perp^2 \vec{\psi}_\perp + \Gamma \delta^2(\vec{x}_\perp)\, ik\, \vec{\pi}_\perp \simeq 0 \qquad \Rightarrow \qquad \vec{\psi}_\perp \simeq -\frac{\Gamma}{2\pi}\, ik\, \log(kr)\, \vec{\pi}_\perp \tag{3.128}$$

$$\nabla_\perp^2 \tilde{\psi} - \frac{\Gamma}{2\pi} k^2\, \vec{\pi}_\perp \cdot \frac{\vec{r}_\perp}{r^2} \simeq 0 \qquad \Rightarrow \qquad \tilde{\psi} \simeq \frac{\Gamma}{4\pi} k^2\, \log(kr)\, \vec{r}_\perp \cdot \vec{\pi}_\perp \tag{3.129}$$

(The exact solutions at all distances involve a Bessel function of the second kind, $K_0(kr)$, and have the precisely these asympthotics for $kr \ll 1$.) Plugging these solutions into the third equation, we finally get a wave equation for $\vec{\pi}_\perp$ only:

$$-i\omega \left(\vec{\pi}_\perp \times \hat{z}\right) + \frac{\Gamma}{4\pi} k^2 \log(1/ka)\, \vec{\pi}_\perp = 0 \,, \tag{3.130}$$

where we have stopped the limit $r \to 0$ at some UV cutoff a, on which, like before, the log weakly depends, and we have ignored a piece proportional to $r_\perp^i \partial_\perp^j \log(kr)\big|_a$, which is finite for $a \to 0$, and can thus be thought of as redefining the UV cutoff a inside the log. Notice that we have expressed the log so that it is positive, since $ka \ll 1$.

Such a wave equation describes two transverse, circularly polarized modes of oscillation, with dispersion relations

$$\omega = \pm \frac{\Gamma}{4\pi} k^2 \log(1/ka) \,, \tag{3.131}$$

and polarizations $\frac{1}{\sqrt{2}}(1, \mp i)$. Note that both these modes circulate (at a fixed height along the vortex line) in the opposite direction of the flow circulation. These are the famous Kelvin waves, and our results match their qualitative and quantitative properties as derived via Victorian methods [42].

Despite the (weak) dependence on the UV cutoff, the dispersion relation above has predictive content. We can rewrite it in an 'RG-invariant' fashion by comparing the frequencies associated with two different momenta:

$$\frac{\omega_1}{k_1^2} - \frac{\omega_2}{k_2^2} = -\frac{\Gamma}{4\pi} \log(k_1/k_2) \,, \tag{3.132}$$

or equivalently

$$\frac{\partial(\omega/k^2)}{\partial \log k} = -\frac{\Gamma}{4\pi} \,. \tag{3.133}$$

Such relations are completely independent of the UV details of the vortex line, and are thus a robust prediction of the low-energy effective field theory. For instance, one can use them to *measure* indirectly the circulation Γ.

3.6.2 Reintroducing sound

It is now time to go back to the first part of this chapter, and look again at how compressional modes couple to incompressional fluid flows. Consider first the lowest-order coupling (5.3), and, to be consistent with this section and the previous one, let's neglect the second term, which is a relativistic correction. In place of $\vec{v}_0$, we could plug in the integral expression (3.90), which gives the incompressional velocity field as a function of the vortex lines' geometry. Clearly, the resulting Lagrangian term would correctly describe the interaction between vortex lines and sound, and can obviously be extended to all orders in $(v/c_s)^2$, following the expansion of Appendix 6.5. The downside of this approach is once again the non-local nature of the resulting action: for each power of v_0, we end up with a line integral over all vortex lines.

The hydrophoton provides us with a conceptually much cleaner solution. Following the magnetostatics analogy, which, for the incompressional part of the fluid flow, is *exact*, we simply have

$$\vec{v}_0 = \vec{\nabla} \times \vec{A}\,. \tag{3.134}$$

So, the local action that describes vortex lines, the hydrophoton, sound, and their interactions, is simply

$$\begin{aligned} S =& w_0 \Bigg[- \sum_n \Gamma_n \int dt d\lambda \, \tfrac{1}{3} \epsilon^{ijk} \, X_n^i \, \partial_t \, X_n^j \, \partial_\lambda X_n^k \\ &+ \int d^3x dt \left[\left(\partial_i A_j\right)^2 + \tfrac{1}{2} \dot{\vec{\psi}}^2 - \tfrac{1}{2} c_s^2 (\vec{\nabla} \cdot \vec{\psi})^2 \right] \\ &+ \sum_n \Gamma_n \int dt d\lambda \, \partial_\lambda \vec{X}_n \cdot \vec{A}\big(\vec{X}_n, t\big) + \int d^3x dt \, (\vec{\nabla} \times \vec{A})^i \big((\vec{\nabla} \times \vec{A}) \cdot \vec{\nabla}\big) \psi^i + \dots \Bigg] \,. \end{aligned} \tag{3.135}$$

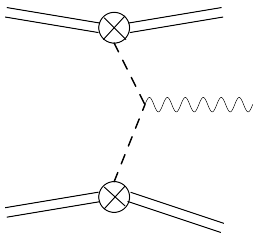


Figure 3.8: *Lowest order Feynman diagram contributing to the emission of sound by the scattering of two vortex rings. The dashed lines represent the hydrophoton and the wavy line the sound.*

The first line contains the vortex lines' peculiar kinetic terms. The second line is the free action for the hydrophoton and sound fields. The third line contains all the interactions: of $\vec{A}$ with the vortex lines, and of $\vec{A}$ with sound. The dots denote all higher order sound interactions of Appendix 6.5, which, given the replacement (3.134), involve more powers of $\vec{\psi}$ and $\vec{A}$, and are straightforward to write down explicitly.

The message is clear: vortex lines interact directly with $\vec{A}$; sound interacts directly with $\vec{A}$; vortex lines and sound interact only indirectly, via $\vec{A}$. For processes that are amenable to a perturbative treatment, one can then use standard field theoretical/diagrammatic techniques to compute observables. Consider for instance a soft scattering process in which two vortex rings pass by each other at large impact parameter (compared to their radii). What is the sound emitted by such a "collision"? The most relevant Feynman diagram is depicted in fig. 3.8: the two vortex rings exchange a long-distance hydrophoton, and this internal line emits a phonon, as dictated by the $AA\psi$ vertex in the action above.

3.7 Summary & Outlook

We have initiated a systematic application of effective field theory techniques to the study of near incompressible hydrodynamical systems. As usual for local field theories, questions that can be dealt with in perturbation theory are reduced, at least conceptually, to "turning

the crank": one can organize each observable as an expansion in Feynman diagrams—of ever increasing complexity as one moves to higher orders in perturbation theory—each of which can be computed in standard ways starting from the action.

The small expansion parameters in our analysis have been the typical fluid-flow speed, which we took by assumption to be much smaller than that of sound, and the typical size of the regions where vorticose flow is localized, which we took to be small relative to the sound waves' wavelengths and to the distance to other vorticose regions. The smallness of the former parameter allows for a perturbative expansion in compressional modes (sound). The smallness of the latter allows for a multipole expansion.

Our insisting on the field theory language has also led us to an entirely *local* action for vortex-line dynamics—which is nontrivial, given that, as is well known, each line carries a long-distance $1/r$ velocity profile with it. We have 'integrated in' such a velocity profile—or better, the associated vector potential (which we have dubbed the "hydrophoton")—thus ending up with a local field theory describing vortex lines, the hydrophoton field, sound modes, and their mutual interactions, to *all* orders in (v/c_s). In this reformulation, vortex lines interact with themselves, with one another, and with sound, by exchanging hydrophotons. We believe this to be substantial technical improvement over the standard "vortex filament" model. Once again, questions amenable to a perturbative expansion are straightforward to analyze—and answer.

From a theoretical viewpoint, there are a number of open conceptual questions about our field-theory formulation of vortex line dynamics. First, we did not construct it bottom-up from standard low-energy effective field theory principles, nor did we derive it top-down from our general fluid action (1.7)—we just guessed the vortex-line action, and checked that it reproduces the correct dynamics as derived from the Euler equation. To be responsible field theorists, we should remedy this. In particular, as we emphasized, the dynamics of vortex lines are strikingly different from those of string-like objects in empty space. Vortex lines

obey first order equations of motion, which means that their local velocity is completely determined by the geometry of the lines' configuration. According to standard effective field theory intuition, all robust properties of the dynamics should follow from symmetry—in order not to be disrupted by renormalization, for instance. So, what is the symmetry protecting such a peculiar behavior? Compared to string-like objects in empty space, we seem to have *less* symmetry—Lorentz or Galileo boosts are gone, being (spontaneously) broken by the surrounding medium. This allows us to write down *more* Lagrangian terms, including our peculiar kinetic term with one time derivative—but the action (3.94), or equivalently (3.105), is hardly the most generic action invariant under translations and rotations. This is all the more frustrating as our original fluid action, eq. (1.7), was constructed precisely as the most generic action invariant under all the appropriate symmetries. Reparameterization invariance under $\lambda \to \tilde{\lambda}(\lambda, t)$—which is symmetry of (3.94) and (3.105)—will play of course a crucial role in reducing the number of allowed Lagrangian terms. We plan to address these symmetry questions in the near future.

Second, for small vortex rings we would like to go one step further, and treat them as point-like objects with multipole moments (starting with a 'magnetic' dipole), and a number of degrees of freedom: their position, radius, orientation, and an infinite tower of excited states (Kelvin waves). Once again, given the first-order nature of the equations of motion, the dynamics of these objects are quite peculiar. For instance, from a preliminary analysis it seems to us that the orientation is *not* a low-energy degree of freedom: one can certainly choose any orientation as an initial condition, but then one cannot change it via low-frequency processes. In a sense, unlike for ordinary string loops in empty space, the orientation degree of freedom seems to belong in the excited Kelvin wave spectrum. Again, all the vortex ring properties should follow from systematically applying symmetry considerations to the effective theory of point-like objects in the fluid, along the lines of [30].

Third, what is this hydrophoton anyway? We have a local field $\vec{A}$, whose equation of

motion in the incompressible limit,

$$\nabla^2 \vec{A} = \vec{\omega} \ , \tag{3.136}$$

implies instantaneous signal propagation at arbitrarily large distances. This is clearly an artifact of the incompressible approximation, and once the speed of sound is brought down to finite values, we expect $\vec{A}$ to propagate at that speed—no small disturbances can propagate faster. This sounds very much like *sound*. Is $\vec{A}$ in some sense sound itself?

But, while we responsibly address all the conceptual points raised above, nothing prevents us from looking for relevant physical applications of our methods, which can range for instance from classical vortex ring systems in ordinary fluids like water [36], to quantized vortex line interactions in laboratory superfluids or in pulsars, where dense arrays of vortex lines in the neutron superfluid are expected to interact strongly with much denser arrays of magnetic flux tubes in the proton superconductor, with crucial consequences for the dynamics of the star as a whole [43].

Chapter 4

Applications to Cosmology: Solid Inflation

4.1 Introduction

There is certainly no shortage of models for primordial inflation. We regret to inform the reader that—as anticipated in the Abstract and Introduction—we are going to add our own to this list. However, we feel that the inflationary model we introduce here presents conceptually novel features that make it stand out as a radical alternative to the standard inflationary scenario. The main reason for this is that our model does not conform to the standard symmetry breaking pattern of inflationary models, and this has far-reaching implications.

In the usual cases, the matter fields ψ_m feature time-dependent cosmological background solutions $\bar{\psi}_m(t)$, which spontaneously break time translations. As a result, there is one fluctuation mode $\pi(x)$ that can be identified with the associated Goldstone excitation. Roughly speaking, it can be thought of as an in-sync perturbation of all the matter fields, of the form

$$\psi(x) = \bar{\psi}_m(t + \pi(x)) \simeq \bar{\psi}_m(t) + \partial_t \bar{\psi}_m(t) \cdot \pi(x) \ . \tag{4.1}$$

When coupling to gravity is taken into account, such a mode describes adiabatic perturbations. As usual for Goldstone bosons, the spontaneously broken symmetry puts completely general, non-trivial constraints on these perturbations' dynamics [46]. This property is at the basis of the model-indepedent approach that goes under the name of "effective field theory of inflation" [47], whose tenets are particularly compelling since they encompass—at first glance—*all* cosmological models: cosmology *is* about time-dependent, homogeneous, and isotropic field configurations.

However, as we will see, there are other possibilities. In our case, we will be dealing with matter fields featuring *time-independent*, $\vec{x}$*-dependent* background solutions. Apparently, this contradicts two facts about inflationary cosmology:

1. The universe is homogeneous and isotropic;

2. In an expanding universe physical quantities depend on time and, more to the point, that one needs a physical 'clock'—a time-dependent observable—to tell the universe when to stop inflating.

As for item 1: $\vec{x}$-dependent solutions *can* be compatible with the homogeneity and isotropy we want for cosmological solutions and for the dynamics of their perturbations, provided extra symmetries are imposed. For instance, to get an Friedmann-Robertson-Walker (FRW) solution for the gravitational field, we need an homogeneous and isotropic background stress-energy tensor. This can arise from matter fields that are not homogeneous nor isotropic, provided there are internal symmetries acting on the fields that can reabsorb the variations one gets by performing translations and rotations. The simplest example is that of a scalar field with a vacuum expectation value

$$\langle \phi \rangle = \alpha x \ . \tag{4.2}$$

Such a configuration breaks translations along x. However, if one postulates an internal shift symmetry $\phi \to \phi + a$, then the configuration above is invariant under a combined

spacial translation/internal shift transformation. As we will see, this is enough to make the stress-energy tensor and the action for small perturbations invariant under translations. To recover isotropy as well, one needs more fields, and more symmetries. For instance—in fact, this is the case that we will consider in this chapter—one can use three scalar fields $\phi^I(x)$ $(I = 1, 2, 3)$, with internal shift and rotational symmetries

$$\phi^I \to \phi^I + a^I \ , \qquad a^I = \text{const} \ , \tag{4.3}$$

$$\phi^I \to O^I{}_J \phi^J \ , \qquad O^I{}_J \in SO(3) \ , \tag{4.4}$$

so that the background configurations

$$\langle \phi^I \rangle = \alpha x^I \tag{4.5}$$

are invariant under combined spacial translation/internal shift transformations, and under combined spacial/internal rotations. As we discussed Chapter 1, such a system has the same dynamics as those of the mechanical deformations of a solid—the phonons. In this sense, the cosmological model that we are putting forward corresponds to having a solid driving inflation. If this interpretation causes the reader discomfort—in particular, if having a solid that can be stretched by a factor of $\sim e^{60}$ without breaking sounds implausible—one should think of our model just as a certain scalar field theory. As we will see, the structure of such a theory is the most general one compatible with the postulated symmetries—and the impressive stretchability we need can also be motivated by an approximate symmetry—so that from an effective field theory standpoint, ours is a perfectly sensible inflationary model. From this viewpoint, the fact that the solids we are used to in everyday life behave quite differently—quantitively, not qualitatively—seems to be an accident: they lack the 'stretchability symmetry'.

As for apparent contradiction number 2: In our model the role of the physical clock will be played by the metric. More precisely, it will be played by (gauge invariant) observables, made up of our scalars and of the metric, like for instance the energy density or the pressure.

These can depend on time even for purely space-dependent scalar backgrounds, because in the presence of a non-trivial stress-energy tensor, the metric will depend on time, in a standard FRW fashion. Doesn't this correspond to a spontaneous breakdown of time translations too? At some level it is a matter of definition, but we will argue in sect. 4.4 that the operationally useful answer is 'no', in the sense that there is no associated Goldstone boson, and that one cannot apply to our case the standard construction of the effective field theory of inflation as given in [47].

Formal considerations aside, our peculiar symmetry-breaking pattern has concrete physical implications, with striking observational consequences. For instance, it predicts a three-point function for adiabatic perturbations with a 'shape' that is not encountered in any other model we are aware of. It is plotted later in fig. 4.1; we will see that it diverges in the squeezed limit, but in a way that depends on the *direction* in which one approaches the limit, with a quadrupolar angular dependence. Additionally, its overall amplitude is also unusually large, corresponding to $f_{\rm NL} \sim \frac{1}{\epsilon}\frac{1}{c_s^2}$.

Before proceeding to spelling out our model in detail, we close this Introduction with two qualifications. The first is that we have been using (and will be using) a somewhat misleading, but fairly standard, language: when spontaneously broken symmetries are gauged, the associated would-be Goldsone bosons are not in the physical spectrum—rather, they are 'eaten' by the longitudinal polarizations of the gauge fields. In fact, there is a gauge, the so-called unitary gauge, in which the Goldstone fields are set to zero. In our case we will be dealing with spontaneously broken translations and rotations, and when gravity is dynamical, these are gauged. In unitary gauge one can set the scalars to their vevs (4.5), and have the corresponding excitations show up in the metric. The Goldstone language is still useful though, in that it captures the correct high energy/short distance dynamics of these excitations. For massive gauge theories, this statement goes under the name of "the equivalence theorem". For cosmological models, it is just the statement that at sub-cosmological

distances and time-scales, in first approximation one can neglect the mixing between matter perturbations and gravitational ones. We hope the Goldstone boson nomenclature will be more useful than misleading.

The second qualification is that our model is not entirely new. A different formulation of essentially the same inflationary model was put forward and briefly analyzed by Gruzinov in [49]. Our emphasis here will be on the peculiar symmetry breaking pattern, on the effective field theory viewpoint, on a systematic analysis of cosmological perturbations—including their non-gaussian features—and in general on the conceptual and technical differences with more standard inflationary models. Cosmological solids have also been used as an exotic model for dark matter [50], and more recently as dynamical media with negative pressure but well-behaved excitations [51].

As a general guide for the reader, the bulk of this (long) chapter can be schematically divided into three parts:

- sects. 4.2–4.4 introduce our model and discuss its field-theoretical and conceptual aspects, including why it cannot fit into the standard EFT of inflation;
- sects. 4.5–4.8 contain a technical analysis of cosmological perturbations, up to their three-point correlation function;
- sects. 4.9 and 4.10 address a number of conceptual subtleties in the analysis of cosmological perturbations, stemming from a very unusual feature of our model: the absence of adiabatic perturbations during inflation.

A brief summary of our results is contained in sect. 4.11, along with a number of possible generalizations of our scenario. Much technical material is contained in the Appendix.

4.2 Effective field theory for solids (and fluids)

To begin with, let us review how one can describe the mechanical degrees of freedom of a solid in modern effective field theory terms. The first systematic approach to this question has probably been that of ref. [52], but here we instead utilize the formalism and language developed in Chapter 1. As a note to avoid future confusion with standard cosmological nomenclature, throughout this chapter we will call the ϕ^I's 'internal' coordinates, and reserve 'comoving' for the standard FRW coordinates for when we introduce gravity.

We want to construct the most general low-energy theory for three scalar fields obeying Poincaré invariance and the internal symmetries (4.3) and (4.4). As discussed in Chapter 1 at lowest order in the derivative expansion, the only Lorentz-scalar, shift-invariant quantity is the matrix

$$B^{IJ} \equiv \partial_\mu \phi^I \, \partial^\mu \phi^J \ . \tag{4.6}$$

We then have to construct $SO(3)$ invariants out of this matrix. For a 3×3 matrix, there are only three independent ones, which we can take for instance to be the traces

$$[B] \ , \quad [B^2] \ , \quad [B^3] \ , \tag{4.7}$$

where the brackets $[\dots]$ are shorthand for the trace of the matrix within. Alternatively, one could take the determinant, and two of the traces above. In the following, we will find it convenient to use one invariant—say $[B]$—to keep track of the overall 'size' of the matrix B, and to choose the other two such that they are insensitive to an overall rescaling of B, e.g.

$$X = [B] \ , \qquad Y = \frac{[B^2]}{[B]^2} \ , \qquad Z = \frac{[B^3]}{[B]^3} \ . \tag{4.8}$$

The most general solid action therefore is

$$S = \int d^4x \, F\big(X, Y, Z\big) + \dots \tag{4.9}$$

where F is a generic function that depends on the physical properties of the solid—e.g. its

equation of state—and the dots stand for higher-derivative terms, which are negligible at low energies and momenta.

As a self-consistency check, notice that the background configuration (4.5) solves the equations of motion for any value of α, which can then be thought of as being determined by the boundary conditions (the external pressure we alluded to above.) The eom for the action above are

$$\partial^\mu \Big(\partial_\mu \phi^J \frac{\partial F}{\partial B^{IJ}} \Big) = 0 \ . \tag{4.10}$$

For a linear configuration like (4.5), all terms in parentheses are constant, since they depend on the ϕ^Is' first derivatives, which are constant. The eom are thus trivially obeyed.

As a side note, in Chapter 1 we continued on to describe a perfect fluid by imposing the infinitely many more internal symmetries described by the volume preserving diff given by (1.5). This expresses the physical fact that for a fluid one can move volume elements around in an adiabatic manner without paying any energy price—only their compression matters. Of the $SO(3)$ invariants above, only one particular combination survives: the determinant of B,

$$\det B = \tfrac{1}{6} \big([B]^3 - 3[B][B^2] + 2[B^3] \big) \ , \tag{4.11}$$

which is of course insensitive to multiplication of B by unit-determinant Jacobians. We can see precisely how, in this formalism, a fluid is just a very symmetric solid.

Back to the solid. The background configurations (4.5) spontaneously break some of our symmetries. There are associated Goldsone bosons, which are nothing but fluctuations of the ϕ^I's about such a background,

$$\phi^I = \alpha(x^I + \pi^I) \ . \tag{4.12}$$

We get these fluctuations' free action by expanding our action (4.9) to second order in π^I. Using

$$B^{IJ} = \alpha^2 \big(\delta^{IJ} + \partial^I \pi^J + \partial^J \pi^I + \partial_\mu \pi^I \partial^\mu \pi^J \big) \ , \tag{4.13}$$

after integrating by parts and neglecting boundary terms we get

$$S \to S_2 = \int d^4x \Big[-\tfrac{1}{3}F_X X \cdot \dot{\vec{\pi}}^{\,2} + \big(\tfrac{1}{3}F_X X + \tfrac{6}{27}(F_Y + F_Z)\big)\,(\partial_i \pi_j)^2 + \big(\tfrac{1}{9}F_{XX}X^2 + \tfrac{2}{27}(F_Y + F_Z)\big)\,(\vec{\nabla}\cdot\vec{\pi})^2 \Big] \,, \tag{4.14}$$

where the subscripts stand for partial derivatives, which are to be evaluated at the background values

$$X \to 3\alpha^2 \,, \qquad Y \to 1/3 \,, \qquad Z \to 1/9 \,. \tag{4.15}$$

These Goldstone excitations are the solid's phonons. For what follows it will be convenient to split the phonon field $\vec{\pi}$ into a longitudinal part and a transverse one,

$$\vec{\pi} = \vec{\pi}_L + \vec{\pi}_T \,, \qquad \vec{\nabla}\times\vec{\pi}_L = 0 \,, \quad \vec{\nabla}\cdot\vec{\pi}_T = 0 \,. \tag{4.16}$$

It is straightforward to extract the longitudinal and transverse propagation speeds from the phonon's action:

$$c_L^2 = 1 + \frac{2}{3}\frac{F_{XX}X^2}{F_X X} + \frac{8}{9}\frac{(F_Y + F_Z)}{F_X X} \,, \qquad c_T^2 = 1 + \frac{2}{3}\frac{(F_Y + F_Z)}{F_X X} \,, \tag{4.17}$$

in terms of which the quadratic action is simply

$$S_2 = \int d^4x\, (-\tfrac{1}{3}F_X X)\left[\dot{\vec{\pi}}^{\,2} - c_T^2\,(\partial_i\pi_j)^2 - (c_L^2 - c_T^2)\,(\vec{\nabla}\cdot\vec{\pi})^2\right] . \tag{4.18}$$

Note that with $c_T \neq 0$ for generic solids the troublesome transverse/vorticose modes are now under control! For these solid systems all the troublesome issues that where discussed in Chapters 2 and 3 are no longer there. Consequently, perturbation theory and the quantum mechanical treatment of these systems should be absolutely straightforward.

If we expand eq. (4.9) to higher orders, we get the interactions among the phonons. The expansion is straightforward, but already at cubic order the result is quite messy, and not particularly illuminating. Below, we will display explicitly the cubic interactions in a particular limit, which yields some simplifications, and which is the physically relevant limit

for an inflationary background. For the moment, it suffices to say that, by construction, the n-th order interaction terms will be schematically of the form $(\partial\pi)^n$, where the derivatives can be spacial or temporal, and the indices are contracted in all possible ways. The coefficients of these interactions terms—the coupling constants—will be given by suitable derivatives of F, evaluated on the background solution. Like for all derivatively-coupled theories, our interactions become strong in the UV, at some energy scale $\Lambda_{\rm strong}$. For our theory to be predictive for cosmological observables, we will need this strong-coupling scale to be above the Hubble rate H, for the whole duration of inflation.

4.3 Inflation

We can now allow for a cosmological spacetime metric and for dynamical gravity, which, operationally, is trivial: the index-contraction in (4.6) should be done via $g^{\mu\nu}$ rather than $\eta^{\mu\nu}$, and the measure in (4.9) should carry a $\sqrt{-g}$. As usual, "minimal coupling" corresponds to the most general coupling one can have between a matter system and gravity at lowest order in the derivative expansion. Then our solid's stress-energy tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = -2\,\frac{\partial F}{\partial B^{IJ}}\,\partial_\mu\phi^I\partial_\nu\phi^J + g_{\mu\nu}\,F\ . \tag{4.19}$$

This "triviality" was one of the advertised advantages of working with the EFT.

As to the scalar fields' background configuration, the x^I in (4.5) should now be interpreted as *comoving* FRW coordinates. The reason is that the FRW metric is invariant under translations and rotations acting on the comoving coordinates, and we want the l.h.s. and the r.h.s. in (4.5) to transform in the same way under the symmetries we are trying to preserve. We can also choose the normalization of the comoving coordinates to set the α parameter to one, so that from now on the background configuration is simply

$$\langle\phi^I\rangle = x^I\ . \tag{4.20}$$

When computed on the background, the stress-energy tensor reduces to the standard $T^{\mu}{}_{\nu} = \mathrm{diag}\big(-\rho, p, p, p\big)$, with

$$\rho = -F \ , \qquad p = F - \tfrac{2}{a^2} F_X \ , \tag{4.21}$$

where the subscript X stands for partial derivative, and F and F_X are evaluated at the background values for our invariants:

$$X \to 3/a^2(t) \ , \qquad Y \to 1/3 \ , \qquad Z \to 1/9 \ . \tag{4.22}$$

Notice that—by construction—X is the only invariant that depends on the scale factor; Y and Z were designed to be insensitive to an overall rescaling of B^{IJ}. This is the reason why only F_X appears in the pressure: for an FRW solution, the pressure is related to the response of the system to changing the scale factor, i.e., the volume. For a more general configuration, the stress-energy tensor (4.19) has a more complicated structure, which depends on F_Y and F_Z as well, which we report here for later use:

$$T_{\mu\nu} = g_{\mu\nu}\, F - 2\, \partial_\mu \phi^I \partial_\nu \phi^J \left(\Big(F_X - \frac{2F_Y Y}{X} - \frac{3F_Z Z}{X} \Big) \delta^{IJ} + \frac{2F_Y B^{IJ}}{X^2} + \frac{3F_Z B^{IK} B^{KJ}}{X^3} \right) . \tag{4.23}$$

Now, in order to have near exponential inflation, we need

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1 \ . \tag{4.24}$$

Via the Friedmann equations [1],

$$H^2 = \frac{1}{3M_{\rm Pl}^2} \rho \ , \qquad \dot{H} = -\frac{1}{2M_{\rm Pl}^2} (\rho + p) \ , \tag{4.25}$$

and eq. (4.21), we can express ϵ directly in terms of our Lagrangian F:

$$\epsilon = 3 \cdot \frac{1}{a^2} \frac{F_X}{F} = \frac{\partial \log F}{\partial \log X} \ , \tag{4.26}$$

[1] We are defining the Planck scale as $M_{\rm Pl}^2 = (8\pi G)^{-1}$

where we used eq. (4.22) for the background value of X. We thus see that if we want our solid to drive near exponential inflation, we need a very weak X-dependence for F. Which is not surprising, since X is the only invariant that is sensitive to the volume of the universe: for inflation to happen, the solid's energy should not change much if we dilate the solid by $\sim e^{60}$; this is only possible if the solid's dynamics do not depend much on X.

This also suggests how to enforce the smallness of ϵ via an approximate symmetry. Consider the scale transformation

$$\phi^I \to \lambda\, \phi^I \ , \qquad \lambda = \text{const} \ . \tag{4.27}$$

The matrix B^{IJ} changes by an overall λ^2 factor, which affects X but not Y nor Z. Therefore, the smallness of F_X can be interpreted as an approximate invariance under (4.27): If F only depended on Y and Z, it would be exactly invariant under (4.27), and this would prevent quantum corrections from generating some X-dependence. If we start with a small F_X at tree level, the symmetry is only approximate, yet all further X-dependence generated at quantum level will be suppressed by the small symmetry-breaking coupling constant—F_X itself.

Notice that in general, *ordinary* scale invariance—that acting on the spacetime coordinates as $x^\mu \to \lambda x^\mu$—cannot be readily used as an approximate symmetry to enforce the smallness of symmetry-breaking parameters. For instance, one cannot solve the Higgs-mass hierarchy problem this way [53]. Moreover, it is generically anomalous, that is, broken by quantum effects, as clearly displayed by the running of coupling constants for interactions that are scale-invariant at tree level. Here instead, we are dealing with a purely internal symmetry—eq. (4.27)—which commutes with all spacetime symmetries. It has nothing to do with spacetime scale-invariance. It is on an equal footing with our other internal symmetries (4.3), (4.4), and, like those, is non-anomalous and *can* be used to constrain the structure of the Lagrangian. To avoid confusion, in the following we will refer to the symmetry (4.27) as 'internal scale invariance'.

Notice in passing that at lowest order in the derivative expansion, imposing internal scale invariance gives us full (internal) conformal invariance as a byproduct. The reason is the following. The invariants we are using in our lowest-order action are combinations of traces of the form $[B^n]$. Under a generic internal diff

$$\phi^I \to \xi^I(\phi) \,, \tag{4.28}$$

these traces transform as

$$[B^n] \to [(J^T J) \cdot B \cdot (J^T J) \cdot B \cdots (J^T J) \cdot B] \,, \tag{4.29}$$

where J is the diff's Jacobian matrix, and we have used the cyclicity of the trace. Now, *by definition*, conformal transformations are the subgroup of diffs that change the flat metric δ^{IJ} only by an overall scalar factor, that is, they have a $(J^T J)$ proportional to the identity. So, under a general conformal transformation, our traces only change by an overall (ϕ-dependent) factor. Our Y and Z combinations are insensitive to such a change, and as long as the Lagrangian only depends on those, it is invariant under all internal conformal transformations. It is interesting that even though our internal scale invariance has in principle nothing to do with ordinary spacetime scale invariance, it shares with it the unavoidable company of special conformal transformations. In both cases, special conformal transformations are not needed to close the symmetry group, yet they are respected by generic scale-invariant dynamics. In the spacetime symmetry case, there are fundamental reasons why that happens [54, 55]. In our case, it appears to be an accidental feature of the lowest order truncation of the derivative expansion.

Back to physics. The smallness of ϵ in our model does not come for free. To see this, notice that by dialing F_X we could go continuously through $\epsilon = 0$ and end up with negative ϵ, that is, positive $\dot{H}$, which violates the null energy condition (NEC). But our EFT conforms to the hypotheses of the general theorem of [10], which links NEC violations to such pathologies as ghost- or gradient-instabilities and superluminality. For small but positive ϵ, we could be

dangerously close to these pathologies [2]. Our quadratic action for the phonons—eq. (4.14)—is quite explicit about this: the phonon's kinetic energy is suppressed by ϵ,

$$S_2 \sim \int d^4x\, \epsilon |F| \cdot \dot{\vec{\pi}}^{\,2} + \dots \tag{4.30}$$

For very small ϵ, this can in principle lead to two problems for our theory:

- Superluminality: the gradient energies in (4.14) are *not* explicitly suppressed by ϵ, and as a consequence the propagation speeds (4.17) are formally of order $1/\epsilon$, unless the numerators are also small. In an effective field theory like ours, with spontaneously broken Lorentz invariance, superluminal signal propagation is not necessarily an inconsistency. However, it prevents the theory from admitting a standard Lorentz-invariant UV-completion [56]. We therefore feel that it should be avoided.

- Strong coupling: unless interactions are also suppressed by suitable powers of ϵ—and it turns that they are not—a smaller kinetic energy means stronger interactions. This is obvious if one goes to canonical normalization for π, by absorbing the prefactor in (4.30) into a redefined phonon field. Then inverse powers of ϵ will show up in the interaction terms, thus signaling that the strong coupling scale of the theory is suppressed by some (positive) power of ϵ. We have to make sure that this strong coupling scale is above H, for the whole duration of inflation.

As for the former issue, notice first of all that the term proportional to F_{XX} in the expression for c_L^2 is forced to be close to $-2/3$. The reason is that not only do we need the 'slow-roll' condition (4.24) for inflation to happen, we also need

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} \ll 1 \tag{4.31}$$

[2]These considerations are irrelevant for standard slow-roll inflation, where the smallness of ϵ is achieved via an approximately flat potential. One cannot play with the potential's slope and end up with positive $\dot{H}$. Positive $\dot{H}$ would require flipping the sign of the inflaton's kinetic energy, which would of course entail ghost-instabilities. On the other hand, in our case the sign of $\dot{H}$ is the same as that of F_X, which is the Lagrangian parameter we are playing with to make ϵ small.

for inflation to last many e-folds.[3] This forces the second derivative F_{XX} also to be small. In particular, given eq. (4.26), and

$$H = \frac{d}{dt}\log a = -\frac{1}{2}\frac{d}{dt}\log X \ , \tag{4.32}$$

we get

$$\frac{F_{XX}X^2}{F_X X} = -1 + \epsilon - \tfrac{1}{2}\eta \ . \tag{4.33}$$

So that, at lowest order in ϵ and η, the propagation speeds (4.17) reduce to

$$c_L^2 \simeq \frac{1}{3} + \frac{8}{9}\frac{(F_Y + F_Z)}{F_X X} \ , \qquad c_T^2 = 1 + \frac{2}{3}\frac{(F_Y + F_Z)}{F_X X} \ . \tag{4.34}$$

It is quite interesting that in this limit they depend on exactly the same $(F_Y + F_Z)$ combination; we see no obvious reason why this should be the case. As a result, the two speeds are not independent: they are related by [4]

$$c_T^2 \simeq \tfrac{3}{4}(1 + c_L^2) \ . \tag{4.36}$$

We thus see that for both speeds to be sub-luminal, we need

$$c_L^2 < \tfrac{1}{3} \ , \tag{4.37}$$

that is, positive $(F_Y + F_Z)$ (recall that $F_X X = \epsilon F$ is negative, because F is). We do not want $(F_Y + F_Z)$ to be too positive though—otherwise, we end up with negative squared speeds, that is, exponentially growing modes. We thus need $(F_Y + F_Z)$ to fit into a small window,

$$0 < (F_Y + F_Z) < \tfrac{3}{8}\epsilon|F| \ . \tag{4.38}$$

[3] In the computations that follow, we will assume all the slow-roll parameters to be of the same order of magnitude.

[4] To all orders in ϵ and η, the exact relation is

$$c_T^2 = \tfrac{3}{4}\big(1 + c_L^2 - \tfrac{2}{3}\epsilon + \tfrac{1}{3}\eta\big) \ . \tag{4.35}$$

We were able to motivate the smallness of F_X,

$$F_X X = \epsilon F \ , \tag{4.39}$$

via an approximate symmetry. It would be desirable to do the same for this new combination of derivatives. One possibility is of course to say that *all* derivatives of F are small, that is

$$\frac{\partial F}{\partial B^{IJ}} B^{KL} \sim \epsilon F \ . \tag{4.40}$$

This formally corresponds to an approximate invariance under *all* internal diffs

$$\phi^I \to \xi^I(\phi) \ , \tag{4.41}$$

that is, to the statement that the value of the Lagrangian does not depend much on the fields. Less formally, and more physically, it corresponds to saying that the bulk of the solid's energy density and pressure are dominated by a cosmological constant, which does not depend on the fields. Although this is of course a technically natural choice—having a *large* cosmological constant was never a problem—it is not particularly interesting. It would be more interesting to find a symmetry that allows large derivatives of F,

$$F_Y, F_Z \sim F \tag{4.42}$$

but that—in the limit of exact symmetry—forces the combination $F_Y + F_Z$ to vanish. Another possibility would be a symmetry that makes $F_Y + F_Z$ saturate the *upper* bound in (4.38), that is, that makes c_L^2 vanish. This is not as unlikely as it sounds: for instance the perfect fluid action (1.7)—whose structure is protected by the volume-preserving diff symmetry—features vanishing propagation speed for the *transverse* phonons, as can be checked explicitly in (4.17), by using expression (4.11) for the determinant. We have not been able to find a symmetry that enforces the condition (4.38) while preserving (4.42). We have to take such a condition as an assumption, which might involve some fine tuning, but which is nonetheless consistent and necessary for the consistency of our inflationary solution.

As for the strong coupling issue, we have to estimate the strong coupling scale $\Lambda_{\rm strong}$ in our small ϵ limit, and make sure that cosmological perturbations are weakly coupled at horizon crossing, that is, at frequencies of order H. Expanding the action (4.9) to all orders in π we get interactions of the form

$$f_n \cdot (\partial\pi)^n \ , \tag{4.43}$$

where f_n is some typical derivative of F. In our case some combinations of derivatives are small,

$$F_X X \sim (F_Y + F_Z) \sim \epsilon F \ , \tag{4.44}$$

but we do not expect this to yield a substantial weakening of interactions. For instance, we will see below that in our approximation the coefficient weighing cubic interactions is F_Y, which, as we argued above, can be as large as the background energy density, $F_Y \sim F$. Assuming that F_Y is a good estimate for the coefficients encountered in interaction terms, and assuming for the moment that both c_L and c_T are of order of the speed of light—so that there is no parametric difference between time- and space-derivatives—we can estimate very easily the strong coupling scale: We can go to canonical normalization for the kinetic term

$$\mathcal{L}_2 \sim \epsilon F \cdot (\partial\pi)^2 \to (\partial\pi_c)^2 \ , \tag{4.45}$$

so that the n-th order interaction becomes

$$\mathcal{L}_n \sim F_Y \cdot (\partial\pi)^n \to \frac{F_Y}{(\epsilon F)^{n/2}} (\partial\pi_c)^n \ . \tag{4.46}$$

This is a dimension-$2n$ interaction, weighed by a scale

$$\Lambda_n \sim \left(\frac{\epsilon^n F^n}{F_Y^2}\right)^{\frac{1}{4n-8}} \tag{4.47}$$

(recall that F and F_Y have mass-dimension four.) If F_Y is of the same order as F, this is simply

$$\Lambda_n \sim F^{1/4} \cdot \epsilon^{\frac{n}{4n-8}} \ , \qquad (F_Y \sim F) \ , \tag{4.48}$$

which, for $n \geq 3$ and $\epsilon \ll 1$, is an increasing function of n. The lowest of all such scales—which defines the strong coupling scale of the theory—is thus that associated with $n = 3$:

$$\Lambda_{\rm strong} = \Lambda_3 \sim F^{1/4}\epsilon^{3/4} \ , \qquad (F_Y \sim F) \ . \tag{4.49}$$

If on the other hand F_Y is much smaller that F, say of order ϵF, we get that all interactions are weighed by the same scale, which then defines strong coupling:

$$\Lambda_{\rm strong} \sim \Lambda_n \sim F^{1/4} \cdot \epsilon^{1/4} \ , \qquad (F_Y \sim \epsilon F) \ . \tag{4.50}$$

Either way, the strong coupling scale is a fractional power of ϵ smaller than the scale associated with the solid's energy density.

If the propagation speeds c_L, c_T are non-relativistic, the estimate of the strong coupling scale depends on the specific structure of the interaction terms, that is, on how many time-derivatives there are. In general, one may expect stronger interactions, i.e., lower strong-coupling scales for non-relativistic excitations (see Chapter 2 for a systematic analysis of this phenomenon in a different limit of our solid action.) Notice first of all that, because of (4.36), the transverse phonon speed c_T is always relativistic,

$$\tfrac{3}{4} < c_T^2 < 1 \ . \tag{4.51}$$

So, our estimates above always work for the transverse phonons' self-interactions. For longitudinal phonons with $c_L \ll 1$, we can repeat the estimate using the cubic interaction, assuming this is still a good indicator of the strong coupling scale of the theory. Expanding (4.9) up to cubic order, and neglecting terms that are proportional to X-derivatives of F or to $(F_Y + F_Z)$, we find

$$S_3 \simeq \int d^4x \, \big(-\tfrac{1}{243} F_Y \big) \cdot \Big\{ 16 \, [\partial\pi]^3 - 36 \, [\partial\pi]^2 \left([\partial\pi \cdot \partial\pi^T] + [(\partial\pi)^2] \right) \\ + 18 \, [(\partial\pi)^3] + 18 \, [(\partial\pi)^2 \cdot \partial\pi^T] \Big\} \ , \tag{4.52}$$

where $(\partial\pi)_{ij} \equiv \partial_i \pi_j$ is the matrix of spacial derivatives of π, $\partial\pi^T$ is its transpose, and the brackets stand for the trace. Notice in particular that there are no time-derivatives. For

$c_L \ll 1$, one can estimate the strong coupling scale via the same trick utilized in Chapter 2. We can redefine the time variable as $t \to t'/c_L$. Now in the kinetic energy term there is no hierarchy between time- and space-derivatives,

$$S_2 \sim \int d^4x\, \epsilon F \cdot \left(\dot{\pi}^2 - c_L^2 (\nabla\pi)^2\right) \to \int d^4x'\, \epsilon F\, c_L \cdot (\partial'\pi)^2 \,, \tag{4.53}$$

and we can apply the usual order-of-magnitude estimates as for relativistic theories. The cubic interaction becomes schematically

$$S_3 \to \int d^4x'\, \frac{F_Y}{c_L} \cdot (\partial'\pi)^3 \,. \tag{4.54}$$

To get the lowest possible value for the strong-coupling scale—that is the most dangerous one—we can take F_Y to be as large as possible, $F_Y \sim F$.[5] Going to canonical normalization and estimating the strong-coupling scale as above we get [6]:

$$p_{\rm strong} \sim F^{1/4} (\epsilon^3 c_L^5)^{1/4} \tag{4.56}$$

This is the strong coupling *momentum* scale, or equivalently, the strong-coupling energy scale in units that are appropriate for our new t' variable. To convert to the original units of energy, we have to multiply by an extra c_L:

$$E_{\rm strong} \sim F^{1/4} (\epsilon^3 c_L^9)^{1/4} \,. \tag{4.57}$$

[5]There is a third possibility where $F_Y \gg F$. While this may not be a problem when analyzing the strong coupling scale of the theory, it is still very unnatural to have the background energy scale much smaller than the other scales in the theory. We already have one cosmological constant problem in cosmology, it would maybe be better if we tried not to introduce a second one.

[6]If one were to repeat the same analysis for a more generic n-th order interaction, also weighed by F_Y like eq. (4.52), and also involving spatial derivatives only, one would get

$$p_{\rm strong} \sim F^{1/4} \epsilon^{\frac{n}{4(n-2)}} c_L^{\frac{n+2}{4(n-2)}} \,, \tag{4.55}$$

which, for small ϵ and c_L, is minimized at $n = 3$.

As we mentioned above, cosmological perturbation theory is under control only if

$$E_{\rm strong} \gg H \ . \tag{4.58}$$

Relating H and F via the Friedmann equation, $H^2 \sim F/M_{\rm Pl}^2$, we get a lower bound on the combination $\epsilon \cdot c_L^3$,

$$\epsilon \cdot c_L^3 \gg \left(H/M_{\rm Pl}\right)^{2/3} \ . \tag{4.59}$$

In principle our H can be several orders of magnitudes smaller than the Planck scale, in which case this bound is not particularly restrictive. Still, it is a nontrivial condition for the self-consistency of the perturbative computations we will perform.

Once the strong-coupling danger is exorcised, large interactions are demoted (or promoted) from a problem to an exciting feature of the model: they imply huge non-gaussianities for our cosmological perturbations. As we will see in sect. 4.5, our non-gaussian signal is peaked on squeezed triangles, with the same size-dependence as the so-called local forms of non-gaussianity, but with a different angular dependence. The corresponding $f_{\rm NL}$ parameter is of order $1/(\epsilon \cdot c_L^2)$, which is a factor of $1/\epsilon$ bigger than what one finds—at the same value of c_L—in single-field models with non-relativistic sound speed.

A clarification is in order: we have been analyzing the viability of our model focusing on the phonons' dynamics, neglecting the background spacetime curvarture and the phonons' mixing with gravitational perturbations. Of course this is not entirely correct. As we mentioned in the Introduction however, at energies much bigger than H, or equivalently, for time-scales much shorter than H^{-1}, curvature and mixing have negligible effects, and in first approximation they can be neglected. Our conditions above, (4.38) and (4.59), should then be thought of as necessary and sufficient for our system to be well-behaved in the UV, at very short distances and time scales. Our detailed analysis of cosmological perturbations in sect. 4.5 will confirm these results.

We should also point out that although we will be using standard 'slow-roll' nomenclature for the conditions (4.24), (4.31) and for the associated perturbative expansion, nothing is

'rolling' in our system, slowly or otherwise: our ϕ^I scalars are *exactly* constant in time. As usual however, the so-called slow-roll expansion really relies on the slowness of certain time-dependent observables like H, $\dot{H}$, etc., which are well defined regardless of the presence of a rolling scalar. We will still use 'slow-roll' to refer to such a weak time-dependence, hoping that this will not cause confusion. As we emphasized, in our case the physical origin of this slowness is the near independence of the dynamics on X, which is, among our invariants, the only one that depends on time. Besides ϵ and η, in the following we will need one more slow-roll parameter,

$$s \equiv \frac{\dot{c}_L}{c_L H} \,, \tag{4.60}$$

which is small, because c_L depends on time only via the Lagrangian's X-dependence.

Finally, we should comment on why we are focusing on a solid rather than on a perfect fluid. First, since eventually we will be interested in quantum mechanical effects—as usual, quantum fluctuations will be the 'seed' for cosmological perturbations—we focus on a solid because, as demonstrated in Chapter 2, we do not know yet how to consistently treat the perfect fluid effective theory as a quantum theory. The problem has to do with the transverse excitations, which appear to be strongly coupled at all scales. Second, even forgetting about the transverse excitations and focusing on the longitudinal ones, we would not be able to keep those weakly coupled for many e-folds. As clear from (4.35), to have vanishing c_T^2 (which is one of the defining features of a fluid) and small ϵ, we need $\eta \sim -1$. But, by definition, $\eta = \dot{\epsilon}/(H\epsilon)$, so that we need $\epsilon = F_X X/F$ to decrease by an order one factor over one Hubble time, i.e. to decrease like some order-one power of $1/a$. F has to be nearly constant over many e-folds, which means that it is actually the numerator $F_X X$ that is tracking $1/a(t)$. But it is precisely combinations like $F_X X$ that control the strong-coupling scale for longitudinal excitations in a fluid, which means that we cannot have $F_X X$ decrease by exponentially large factors without making the system strongly coupled at frequencies of order H at some point during inflation.

4.4 Physical 'clocks' and reheating

Eventually we want our inflation to end and to be followed by a standard hot Big-Bang phase, that is, we want the universe to reheat and to become radiation dominated.[7] In our case, this process can be thought of as a phase-transition from a solid state to a relativistic fluid state. The advantage of our language in dealing with such a transition is that, as we emphasized in sect. 4.2, it describes both solids and fluids in terms of the same long-distance degrees of freedom, our scalars ϕ^I. Only, the fluid action (1.7) enjoys (many) more symmetries. So, regardless of the microscopic dynamics that are actually responsible for the phase transition, at long distances and time scales reheating corresponds to some sort of symmetry enhancement of our action. We will be more specific about this in a moment.

In terms of our infrared degrees of freedom, what triggers reheating? In standard slow-roll inflation, it is the inflaton itself, when its time-dependent background field reaches a critical value. On the other hand, in the absence of perturbations, our ϕ^I's are exactly time-independent: $\langle\phi^I\rangle = x^I$. However the metric is not, and there are gauge-invariant combinations like our

$$X = g^{\mu\nu}\,\partial_\mu\phi^I\partial_\nu\phi^I\ , \tag{4.61}$$

or the energy density and pressure in eq. (4.21), that do depend on time. Usually we are used to solids turning into liquids—that is, melting—when the temperature exceeds a critical value. But we can also envisage a solid that 'melts' at zero temperature, when one of the physical quantities above goes past a critical value. Helium offers an example of such a phenomenon: at zero temperature one can turn liquid helium into a solid by raising the

[7] It should be noted that [49] avoided going through a specific model of reheating by simply evolving the solid until $c_T \to 0$, and demanding that the solid turn into a perfect fluid at that point. Given our analysis above, if we want to keep all the slow roll parameters small, as is reflected in (4.51) c_T^2 cannot become much smaller than 3/4. Via our EFT approach we thus see that the reheating model of [49] necessarily entails a breakdown of the slow-roll expansion before reheating, which is, of course, a consistent possibility.

pressure beyond ~ 25 atm, and melt it back again by lowering the pressure below that value. In our case, we need this zero-temperature melting to be associated with a substantial release of latent heat, so that the fluid we end up with is (very) hot [8]. As far as we can tell, this does not violate any sacred principles of thermodynamics.

Notice that as far as the background solution is concerned, to lowest order in the derivative expansion all choices for what observable triggers our solid's melting are physically equivalent: this is evident in our parameterization of the action (4.9), where all time-dependent observables depend on time only through X, or, equivalently, through $a(t)$. However, when we include fluctuations, we break this equivalence. For instance, the three invariants $[B]$, $[B^2]$, and $[B^3]$ are independent combinations of the fields' derivatives. In the presence of fluctuations, the hypersurface defined by $[B]$ reaching its critical reheating value is different from that defined by $[B^2]$ or $[B^3]$ reaching their critical reheating values. As a result, some of our predictions for cosmological correlation functions might depend on the physical variable chosen to trigger reheating. Notice that *after* reheating, in the hot fluid phase, there is no ambiguity: the lowest-order action (1.7) only depends on one variable, the determinant of B^{IJ}. It is thus natural, although not obviously necessary, to postulate that reheating is triggered by the value of $\det B$.

So, in terms of our action, our assumption is that for large $\det B$ the action has the general structure (4.9), whereas for $\det B$ below a critical value, the action has the more restricted form (1.7). In the space of our X, Y, Z invariants, this corresponds to dividing up the space into two regions, where the action has different symmetries: eqs. (4.3), (4.4) for the former, eq. (1.5) for the latter. Moreover, the slow-roll condition $\dot{H} \ll H^2$ during inflation is protected by another (approximate) symmetry of the solid phase, eq (4.27), which we want to be maximally violated in the post-reheating fluid phase, which has $\dot{H} \sim H^2$. Since renormalization is local in field space, the existence of different regions with different

[8] In the fluid phase, which is described by the action (1.7), the temperature is given by $T = -\frac{dF}{d\sqrt{\det B}}$.

symmetries is protected by precisely those symmetries, and is thus a consistent and natural assumption [9]. For illustrative purposes, consider for instance the following action:

$$\mathcal{L} = F(X, Y, Z) \propto \begin{cases} -(\det B)^{\epsilon/3} \cdot f(Y, Z) & \text{for } \det B > 1 \\ -(\det B)^{2/3} & \text{for } \det B < 1 \,, \end{cases} \tag{4.62}$$

where f is a generic function that evaluates to one for the background values $Y \to 1/3$, $Z \to 1/9$, and we suppressed an overall common factor, which defines the density at reheating. The 'gluing' at $\det B = 1$ can be smoothed at will. In the first regime, $\mathcal{L}$ describes a solid driving an inflationary phase with constant ϵ (for fixed Y and Z, $\det B$ scales like X^3—hence the $\epsilon/3$ power). In the second regime, $\mathcal{L}$ describes an ultra-relativistic fluid, with $p = \rho/3 \propto T^4$. The two regimes have different internal symmetries, as discussed above, but they share the same degrees of freedom. The classical evolution of the background solutions and of perturbations can then be followed smoothly through the transition region, as the long wavelength degrees of freedom are the same all along, and the equations of motion are regular.[10]

Notice that we have been implicitly assuming that reheating is instantaneous, that is, that our solid/fluid phase transition happens in a time interval that is much shorter than the Hubble scale, which is reasonable in principle, but not necessary. One can also consider much slower transitions, which in field space would correspond to replacing the sharp critical values we have been talking about for our observables, with much more continuous transition regions. All our physical considerations above apply unaltered. Since, as we will discuss, some of our predictions are potentially model-dependent, for what follows we need to assume

[9] A similar mechanism is at work in ghost inflation [57], or in the EFT description of finite-temperature superfluids [14].

[10] As mentioned a great deal already, one should refrain from performing quantum computations in the fluid phase. So, eq. (4.62) should not be thought of as a quantum effective theory in the second regime. Still, since at reheating all relevant modes are well outside the horizon and, thanks to the usual reasons, can be treated as classical, we only need (4.62) to be a consistent classical field theory, which it is.

a specific model for reheating. So, we will assume that reheating is fast, much faster than H, and that it is controlled by the value of $\det B$.

4.4.1 Why not the EFT of inflation?

Before turning to a detailed analysis of cosmological perturbations, we close this section by discussing why our model does not conform to the standard EFT of inflation. As we just saw, we do have physical 'clocks', that is time-dependent background observables, so why can't we use the standard results for spontaneously broken time-translations? The reason is that these time-dependent observables depend on time only because the metric does.

To see why this subtlety is important, consider first the dynamics of our system at very short distances—in the so-called decoupling limit—where the matter fluctuations decouple from the gravitational ones and the Goldstone boson language is appropriate. In first approximation, this limit corresponds formally to setting G to zero. But without gravity, our background solution has no time-dependence whatsoever! All observables like density, pressure, etc. are now exactly constant in time, and only *spacial* translations and rotations are broken by the background configuration $\langle\phi\rangle = x^I$. As a result, the Goldstone bosons, whose existence and properties have to be assessed in the decoupling limit—because as recalled in the Introduction, only in this limit does it make sense to talk about them—are those associated with *this* spontaneous symmetry breaking pattern, not with time-translations.

It is not surprising then, that once we re-introduce gravity, the dynamics of cosmological perturbations at all scales are quite different than for the EFT of inflation. This is manifest in the so-called unitary gauge, where one chooses the time-variable according to a physical clock. In the standard case, that clock would be the inflaton, and choosing the equal-time surfaces to be the equal-inflaton surfaces automatically sets to zero the inflaton perturbations and makes the metric the only fluctuating field. In our case, if we choose one of our time-dependent observables—say $\det B$ or $[B]$—to define unitary gauge, we are still left with

matter perturbations, because the background time-dependence that we are using to set the gauge is carried by the metric, not by matter fields. We can either include the matter perturbations explicitly in the Lagrangian terms that we write down in this gauge, or we can set them to zero, by choosing the *spacial* coordinates now so that $\phi^I = x^I$. This is a complete gauge-fixing—all spacetime coordinates have been unambiguously defined—and is of course quite a different gauge choice than the standard unitary gauge. In particular, it is inequivalent to choosing the so-called ζ-gauge for spacial diffs. Either way, the Lagrangian terms one would write down are quite different than for the standard EFT of inflation. We will go into the details of this new unitary gauge in sect. 6.10.

The presence of matter fluctuations in the 'naive' unitary gauge cannot be taken as a sign that we are dealing with what would be called a multi-field model in the standard classification. First, because there is a gauge—our 'improved' unitary gauge—in which all matter fluctuations are set to zero. Second, because our spectrum of cosmological perturbations only includes *one* scalar mode, as clear from the Goldstone quadratic action (4.18). Furthemore, as we will see, this scalar mode is not adiabatic. In other words, in our system there are no adiabatic modes of fluctuation. This is yet another manifestation that we are dealing with a truly unconventional cosmological system.

Notice that at the classical level, a subtle, isolated exception to all of the above is offered by a perfect fluid. On the one hand, in our language a perfect fluid is just a very symmetric solid. In particular, it features the same symmetry breaking pattern. On the other hand however, because of powerful conservation laws for vorticity, classically one can consistently set to zero the transverse excitations—the vortices—and be left with an EFT for the compressional modes only [10]. This admits a description in terms of a single scalar which spontaneously breaks *time*-translations—a $P(X)$ theory—to which the standard EFT-of-inflation construction *is* applicable.[11] In particular, for a perfect fluid scalar cosmological

[11] Here we are using the standard notation of the community: P is a generic function and X here does not refer to the $[B]$ we have been considering throughout the text but rather $\partial_\mu\psi\partial^\mu\psi$, where ψ is some scalar

perturbations are adiabatic. Once quantum effects are taken into account however, transverse excitations cannot be neglected any longer. As we discovered in Chapter 2, for a fluid they are not particularly well behaved quantum mechanically, which is one of the reasons why we have been considering a more generic solid rather than the special perfect-fluid case.

As a technical aside, we should also emphasize that in a gauge where the matter fields are unperturbed, $\phi^I = x^I$, our B^{IJ} matrix reduces simply to g^{IJ}, and our Lagrangian thus becomes the sum of the Einstein-Hilbert action and of a particular function of g^{IJ}, that is, it reduces to a Lorentz-violating theory of massive gravity. Theories like this have been studied in broad generality in [58]. The reader familiar with the EFT of inflation might wonder why we are not writing down directly the action for the perturbations δg^{IJ} in this gauge—the analog of unitary gauge in that case—, instead of going through the (apparently) unnecessary burden of writing an action for the full fields, solving for the background solution, and then expanding the action in small perturbations. The technical reason is that, unlike δg^{00} or δK_{ij} for the EFT of inflation, our δg^{IJ} does not transform covariantly under the residual diffs, which are just time diffs for us. The reason is that g^{IJ} does, but its background value, $\frac{1}{a^2(t)}\delta^{IJ}$ does not. It is then technically more convenient to write an action for the full g^{IJ}, which just amounts to writing an action for B^{IJ}, like we have done.

4.5 Cosmological perturbations

The three sections that follow contain a technical analysis of cosmological perturbations. Before skipping directly to sect. 4.8, the reader uninterested in the details of the derivations should be aware of our results: the scalar tilt (4.115), the tensor-to-scalar ratio (4.116), the tensor tilt (4.100), and the three-point function of scalar perturbations (4.131) (which is analyzed in some detail in sect. 4.8).

As the background stress tensor takes the usual homogeneous and isotropic form repre-

field whose background 'vacuum expectation value' is $\langle\psi\rangle = \bar{\psi}(t)$.

sented by $T^{\mu}{}_{\nu} = \mathrm{diag}\big(-\rho, p, p, p\big)$, all the interesting repercussions of our peculiar symmetry breaking pattern lie in the dynamics of perturbations around the slow roll background. In order to best isolate the dynamical degrees of freedom of the gravitational field it is most convenient to work in the ADM parametrization of the metric:

$$ds^2 = -N^2\, dt^2 + h_{ij}\left(dx^i + N^i\, dt\right)\left(dx^j + N^j\, dt\right)\ . \tag{4.63}$$

It is straightforward to check that the inverse metric $g^{\mu\nu}$ is given by

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = g^{i0} = \frac{N^i}{N^2}, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}\ , \tag{4.64}$$

where h^{ij} is the inverse *spatial* metric: $h^{ik}h_{kj} = \delta^i_j$. For the background FRW metric $N = 1$, $N^i = 0$, and $h_{ij} = a^2(t)\,\delta_{ij}$.

Following [59] we can write the action as

$$S = \int \mathrm{d}^4x\ N\sqrt{h}\ \left\{\tfrac{1}{2}M_{\rm Pl}^2\big[R^{(3)} + N^{-2}(E_{ij}E^{ij} - E^2)\big] +\ F(X,Y,Z)\right\} \tag{4.65}$$

where $R^{(3)}$ is the 3-dimensional Ricci scalar constructed out of h_{ij} and $E_{ij} = N\,K_{ij}$, with K_{ij} denoting the extrinsic curvature of equal-time hypersurfaces. The constraint equations given by varying (4.65) with respect to N and N^i are:

$$0 = \tfrac{1}{2}M_{\rm Pl}^2\left[R^{(3)} - N^{-2}(E_{ij}E^{ij} - E^2)\right] + F(X,Y,Z) + N\frac{\partial F(X,Y,Z)}{\partial N} \tag{4.66}$$

$$0 = \tfrac{1}{2}M_{\rm Pl}^2\nabla_i\left[N^{-1}(E^i_j - \delta^i_j E)\right] + N\frac{\partial F(X,Y,Z)}{\partial N^j}\ . \tag{4.67}$$

The derivatives of F with respect to N and N^j can be calculated easily by noting that our B^{IJ} (and hence X, Y, Z) can be expressed in ADM variables as

$$B^{IJ} = -\frac{1}{N^2}\big(\dot\phi^I - N^k\partial_k\phi^I\big)\big(\dot\phi^J - N^k\partial_k\phi^J\big) + h^{km}\partial_k\phi^I\partial_m\phi^J\ . \tag{4.68}$$

It is usually convenient to work in a gauge where scalar perturbations are removed from the matter fields and appear only in the metric, as $g_{ij} = a^2(t)\,(1+2\zeta)\delta_{ij}$, see e.g. [59]. This possibility is not available to us: By using up the three *spatial* diffs, we can set the matter

field perturbations to zero, $\phi^I = x^I$, but then the spatial metric has an extra scalar mode, proportional to $\partial_i \partial_j \chi$, which we now cannot remove in the usual manner. However, we are still free to use time diffs, but these at best can set the scalar in front of δ_{ij}, not χ, to zero. A more useful choice is to use the time diff to set a physical "clock"—like those we discussed in the last section—to its unperturbed value. If this clock controls reheating, then reheating will happen at the same time for all observers in this gauge. We review this gauge choice, which we call 'unitary', in Appendix 6.10.

For the moment we find it more convenient to work in spatially flat slicing gauge (SFSG)—defined in Appendix 6.10—where we can write the fluctuations about the FRW background as

$$\phi^I = x^I + \pi^I \ , \quad h_{ij} = a(t)^2 \exp(\gamma_{ij}) \ , \quad N = 1 + \delta N \ , \tag{4.69}$$

where γ_{ij} is transverse and traceless, i.e.

$$\partial_i \gamma_{ij} = \gamma_{ii} = 0 \ . \tag{4.70}$$

We can also further split the π^i and N^i fields in terms of their longitudinal scalar and transverse vector components. We therefore write:

$$\pi^i = \frac{\partial_i}{\sqrt{-\nabla^2}} \pi_L + \pi_T^i \ , \quad \text{and} \ \ N^i = \frac{\partial_i}{\sqrt{-\nabla^2}} N_L + N_T^i \ , \tag{4.71}$$

where $\partial_i \pi_T^i = \partial_i N_T^i = 0$. From now on we will stop differentiating between internal $I, J, \ldots$ indices and spacial $i, j, \ldots$ ones. The reason is that of the full original $SO(3)_{\text{spacetime}} \times SO(3)_{\text{internal}}$ symmetry, only the diagonal combination is preserved by the background $\phi^I = x^I$. π^i and N^i both transform as vectors under this unbroken $SO(3)$, and therefore they carry the same kind of index.

For our purposes here we are interested only in the leading non-gaussian behavior. Barring accidental cancellations, this can be captured by keeping terms that are cubic in the fluctuations. In order to reproduce these terms it turns out to be necessary to only know N

and N^i to first order in the fluctuations [12]. From now on, we find it easier to work in spacial Fourier space, with our convention defined for any field $\xi(x)$ by:

$$\xi(t,\vec{x}) = \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\,\tilde{\xi}(t,\vec{k})\ , \qquad \int_{\vec{k}} \equiv \int \frac{d^3k}{(2\pi)^3}\ . \tag{4.72}$$

For convenience, however, we will drop the twiddle as which field variable we intend will be obvious from the arguments. And so, solving the constraint equations (4.66) and (4.67) to first order in fluctuations we have

$$\delta N(t,\vec{k}) = -\frac{a^2\dot{H}}{kH}\,\frac{\dot{\pi}_L - \dot{H}\pi_L/H}{1-3\dot{H}a^2/k^2} \tag{4.73}$$

$$N_L(t,\vec{k}) = \frac{-3a^2\dot{H}\dot{\pi}_L/k^2 + \dot{H}\pi_L/H}{1-3a^2\dot{H}/k^2} \tag{4.74}$$

$$N^i_T(t,\vec{k}) = \frac{\dot{\pi}^i_T}{1-k^2/4a^2\dot{H}} \tag{4.75}$$

where the dot denotes a time-derivative.

Now, plugging these solutions back into (4.65) will give us the correct action for the fluctuations up to cubic order. For instance, the trilinear solid action after mixing with gravity is contained in Appendix 6.12 while the quadratic actions for the tensor, vector and scalar modes are contained in the next section.

Now that we have the correct action for the perturbations in the presence of an inflating background we can compute correlation functions. In the end, we are interested in the post-reheating correlation functions of curvature perturbations, parameterized by either of the gauge invariant (at linear-order) combinations

$$\mathcal{R} = \frac{A}{2} + H\delta u\ , \qquad \zeta = \frac{A}{2} - H\frac{\delta\rho}{\rho} \tag{4.76}$$

[12]This lucky fact is because the higher order terms in N and N^i will be multiplying the constraint equations. In particular: the third order term of N and N^i multiplies the zeroth order constraint equations, and the second order the first order constraint equations [59]. If we were, however, to try and generate the fourth order terms we would need N and N^i to second order.

where we have followed the notation of [29].[13] During our solid inflation phase, in spatially flat slicing gauge these are given by

$$\mathcal{R} = -\frac{k}{3H\epsilon}\frac{\dot{\pi}_L + H\epsilon\,\pi_L}{1 + k^2/3a^2H^2\epsilon}\,, \qquad \zeta = \tfrac{1}{3}\,\vec{\nabla}\cdot\vec{\pi}\,. \tag{4.81}$$

where the non-local piece of $\mathcal{R}$ comes from solving the constraint equation for N_L.

Two peculiarities concerning the behavior of these variables during solid inflation are worth mentioning at this point. First, $\mathcal{R}$ and ζ do not coincide on super-horizon scales. Second, neither of them is conserved. These properties are in sharp contrast with what happens for adiabatic perturbations in standard cosmological models, and stem from the fact that during solid inflation, there are *no* adiabatic modes of fluctuation! We will clarify why this is the case in sect. 4.9.

[13] The general (i.e. before gauge fixing) perturbed metric (to the linear-order) is parametrized by

$$g_{ij} = a(t)^2\left(\delta_{ij}(1+A) + \partial_i\partial_j\chi + \partial_i C_j + \partial_j C_i + D_{ij}\right) \tag{4.77}$$

with $\partial_i C_i = 0$ and $\partial_i D_{ij} = D_{ii} = 0$; furthermore the energy momentum tensor is decomposed into scalar, vector, and tensor modes as

$$\delta T_{00} = -\bar{\rho}\,\delta g_{00} + \delta\rho \tag{4.78}$$

$$\delta T_{i0} = \bar{p}\,\delta g_{i0} - (\bar{\rho}+\bar{p})(\partial_i\delta u + \delta u_i^V) \tag{4.79}$$

$$\delta T_{ij} = \bar{p}\,\delta g_{ij} + a^2(\delta_{ij}\delta p + \partial_i\partial_j\delta\sigma + \partial_i\delta\sigma^j + \partial_j\delta\sigma^i + \delta\sigma_{ij}^T)\,. \tag{4.80}$$

4.6 Two-point functions

Upon plugging the expressions (4.73)–(4.75) back into the action, the quadratic action for tensor, vector, and scalar fluctuations reads:

$$S^{(2)} = S^{(2)}_\gamma + S^{(2)}_T + S^{(2)}_L \tag{4.82}$$

$$S^{(2)}_\gamma = \tfrac{1}{4} M^2_{\rm Pl} \int dt\, d^3x\, a^3 \left[\tfrac{1}{2} \dot\gamma^2_{ij} - \tfrac{1}{2a^2} \big(\partial_m \gamma_{ij}\big)^2 + 2\dot H c_T^2\, \gamma^2_{ij} \right] \tag{4.83}$$

$$S^{(2)}_T = M^2_{\rm Pl} \int dt \int_{\vec k} a^3 \left[\frac{k^2/4}{1 - k^2/4a^2\dot H} \left|\dot{\pi}^i_T\right|^2 + \dot H c_T^2\, k^2 \left|\pi^i_T\right|^2 \right] \tag{4.84}$$

$$S^{(2)}_L = M^2_{\rm Pl} \int dt \int_{\vec k} a^3 \left[\frac{k^2/3}{1 - k^2/3a^2\dot H} \left|\dot\pi_L - (\dot H/H)\pi_L\right|^2 + \dot H c_L^2\, k^2 \left|\pi_L\right|^2 \right] . \tag{4.85}$$

Notice the quite nontrivial k-dependence for $S^{(2)}_T$ and $S^{(2)}_L$ in Fourier space, which would translate into a (spacially) non-local structure in real space.

4.6.1 Tensor perturbations

Using (4.83) we can calculate the two-point function of the tensor perturbations. As usual, it is a simpler calculation than the scalar case and will serve as a warmup. We decompose the tensor modes into their polarizations

$$\gamma_{ij}(\vec k, t) = \sum_{s=\pm} \epsilon^s_{ij}(\vec k) \gamma^s(\vec k, t) \ , \tag{4.86}$$

with $\epsilon^s_{ij}\epsilon^{s'*}_{ij} = 2\delta^{ss'}$. The transverse, traceless conditions on γ_{ij} now simply become $\epsilon_{ii} = k_i \epsilon_{ij} = 0$. We further decompose each $\gamma^s(\vec k, t)$ as

$$\gamma^s(\vec k, t) = \gamma^s_{cl}(\vec k, t)\ a^s(\vec k) + \gamma^s_{cl}(\vec k, t)^*\ a^{s\dagger}(-\vec k) \ . \tag{4.87}$$

where $a^s(\vec k)^\dagger$ and $a^s(\vec k)$ are creation and annihilation operators obeying the usual commutation relation

$$[a^s(\vec k), a^{s'\dagger}(\vec k')] = (2\pi)^3 \delta^3(\vec k - \vec k')\, \delta^{ss'} \ , \tag{4.88}$$

and where the classical solution $\gamma^s_{cl}(\vec{k},t)$ obeys the equations of motion obtained by varying (4.83):

$$\frac{d^2}{d\tau^2}\gamma_{cl} + 2aH\,\frac{d}{d\tau}\gamma_{cl} + \left(k^2 + 4\epsilon a^2 H^2 c_T^2\right)\gamma_{cl} = 0\ . \tag{4.89}$$

In the above we have used conformal time τ, where $d\tau = dt/a$, and the definition of the first slow-roll parameter (4.24). Using the time-dependence of aH, ϵ, and c_T—which is worked out in Appendix 6.9—we can express the e.o.m. for the tensor mode (4.89) up to first order in slow-roll parameters as

$$\frac{d^2}{d\tau^2}\gamma_{cl} - \frac{2+2\epsilon_c}{\tau}\frac{d}{d\tau}\gamma_{cl} + \left(k^2 + \frac{4\epsilon_c c_{T,c}^2}{\tau^2}\right)\gamma_{cl} = 0\ . \tag{4.90}$$

The subscript "c" denotes that the parameters c_T^2, ϵ are evaluated at some reference time τ_c, which is chosen to be the (conformal) time when the *longest* mode of observational relevance today exits the horizon, i.e. τ_c is defined such that

$$|c_{L,c}k_{\rm min}\tau_c| \simeq |c_{L,c}\tau_c H_{\rm today}| = 1 \tag{4.91}$$

where the usual convention for flat spacetime $a_{\rm today} = 1$ is understood.

The most general solution to the above equation takes the form

$$\gamma_{cl}(\vec{k},\tau) = (-\tau)^{3/2+\epsilon_c}\left[\mathcal{A}\,H^{(1)}_{\nu_T}(-k\tau) + \mathcal{B}H^{(2)}_{\nu_T}(-k\tau)\right]\ , \qquad \nu_T \simeq \tfrac{3}{2} + \epsilon_c - \tfrac{4}{3}c^2_{T,c}\epsilon_c\ , \tag{4.92}$$

where $H^{(1,2)}$ are Hankel functions, and $\mathcal{A}$ and $\mathcal{B}$ are constants to be fixed by matching the appropriate initial conditions.

At very early times the *physical* wavelength—k/a—is so small compared to the Hubble scale H that the curvature of spacetime cannot be perceived by such modes; it is therefore expected that the canonically normalized classical solution should match the free wave function in the flat-space vacuum, $\frac{1}{\sqrt{2k}}e^{-ik\tau}$. Note that the canonically normalized field $\gamma^s_{\rm can.}(\vec{k},\tau)$ is related to $\gamma^s(\vec{k},\tau)$ by

$$\gamma^s(\vec{k},\tau) = \frac{\sqrt{2}}{M_{\rm Pl}}\frac{\gamma^s_{\rm can.}(\vec{k},\tau)}{a(\tau)}\ . \tag{4.93}$$

Thus, the initial condition for $\gamma^s_{cl}(\vec{k},\tau)$ is specified by

$$\lim_{\tau\to-\infty}\gamma^s_{cl}(\vec{k},\tau)=\frac{1}{\sqrt{k}M_{\rm Pl}a(\tau)}e^{-ik\tau}\ . \tag{4.94}$$

Comparing this to the general solution given by (4.92) and using the asymptotic form (for large arguments) of Hankel functions,

$$\lim_{x\to+\infty}H^{(1)}_m(x)\to\sqrt{\frac{2}{\pi x}}e^{ix-i\left(m+\frac{1}{2}\right)\frac{\pi}{2}},\quad \lim_{x\to+\infty}H^{(2)}_m(x)\to\sqrt{\frac{2}{\pi x}}e^{-ix+i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} \tag{4.95}$$

it is enforced that

$$\mathcal{A}=\frac{H_c(1-\epsilon_c)}{M_{\rm Pl}}\sqrt{\frac{\pi}{2}}\,(-\tau_c)^{-\epsilon_c}e^{i(\nu_T\pi/2+\pi/4)}+\mathcal{O}(\epsilon^2),\quad \mathcal{B}=0\ . \tag{4.96}$$

Where, once again, $H_c\equiv H(\tau_c)$ and $\epsilon_c=\epsilon(\tau_c)$. This result is valid up to first order in slow roll.

It is interesting to note that even when these tensor modes are well outside the horizon, they are not conserved. A similar story applies to the gauge invariant curvature perturbations ζ and $\mathcal{R}$ defined by (4.81) and the vector perturbation (like π^i_T), as we will see in the following section. This is in opposition to the usual situation in most inflation models, and will be discussed in more detail in Section 4.9. In particular, by utilizing the asymptotic limit (for small argument) of the Hankel function

$$\lim_{x\to0^+}H^{(1)}_m(x)\to(-i)\,\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m\ , \tag{4.97}$$

the mild time-dependence of the tensor mode in late time is given by:

$$\lim_{-k\tau\to0^+}\gamma^s_{cl}(\vec{k},\tau)=k^{-3/2}\left(\frac{\tau}{\tau_e}\right)^{4c^2_{T,e}\epsilon_e/3}(-k\tau_e)^{c^2_{L,e}\epsilon_e}\left(\frac{iH_e}{M_{\rm Pl}}+\mathcal{O}(\epsilon)\right)\ . \tag{4.98}$$

where we have made use of relation (4.36). As we will see soon, the transverse vector modes and scalar modes in our model share this feature as well.

And so finally, we are ready to obtain the two-point function for the tensor perturbations of the metric. In particular, we are interested in its late time behavior, when modes are well

outside the horizon:

$$\left\langle \gamma^{s_1}(\vec{k}_1,\tau)\gamma^{s_2}(\vec{k}_2,\tau)\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\,\delta^{s_1s_2}\left|\gamma_{cl}(\vec{k}_1,\tau)\right|^2 \tag{4.99}$$
$$\stackrel{-k\tau\to 0^+}{\longrightarrow} (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\,\delta^{s_1s_2}\times\frac{H_c^2}{M_{\rm Pl}^2}\,\frac{1}{k_1^3}\,\frac{(\tau/\tau_c)^{8c_{T,c}^2\epsilon_c/3}}{(-k_1\tau_c)^{-2c_{L,c}^2\epsilon_c}}\ .$$

The dependence on k and τ is kept to first order in slow roll while the overall constant is to lowest order.

The advantage of expressing time-dependent quantities in reference to a fixed fiducial time (τ_c), as opposed to the usual convention of using the time at horizon crossing, is that the time- and momentum-dependence are made manifest. We can simply read off the tilt of the spectrum to first order in slow roll from the above expression:

$$n_T - 1 \simeq 2c_{L,c}^2\epsilon_c\ . \tag{4.100}$$

We can see that the two point function for tensor modes is *blue* shifted, which matches the result of [49], and which is a distinctive signature of our scenario, unreproducible by more conventional models of inflation. As to the spectrum's overall amplitude, it is the usual one: $\langle\gamma\gamma\rangle \sim H^2/M_{\rm Pl}^2$.

4.6.2 Scalar Perturbations

We proceed by calculating the scalar two point function in a similar manner as above. As emphasized in section 4.5, the scalar quantity of interest[14] is the gauge invariant quantity ζ, which, in Fourier space, is related to the longitudinal Goldstones π_L simply by $\zeta = -k\pi_L/3$ (see eq. (4.81)).

[14]We find ζ a more interesting quantity than $\mathcal{R}$ for the reason that given our assumption about reheating, ζ evolves continuously from inflation phase to post-inflation phase, while $\mathcal{R}$ does not. See Section 4.10 for details. However, $\mathcal{R}$ does play a vital role as a simplifier in solving for the classical solution for scalar perturbation.

As before, let's decompose the scalar field of interest in terms of creation and annihilation operators:

$$\zeta(\vec{k},t) = \zeta_{cl}(\vec{k},t)\; b(\vec{k}) + \zeta_{cl}(\vec{k},t)^*\; b^\dagger(-\vec{k})\;, \tag{4.101}$$

where the usual commutation relation is obeyed $[b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')$.

The classical equation of motion for ζ_{cl} follows from varying the quadratic π_L action (4.85). The general solution to this equation is quite complicated, however there is a trick that makes its solution much easier. If we re-express the e.o.m. in terms of the other gauge invariant parameter $\mathcal{R}_{cl}$ (see eq. (4.81)) we have

$$-3c_L^2\zeta_{cl}(\vec{k},t) = \frac{1}{H}\dot{\mathcal{R}}_{cl}(\vec{k},t) + (3+\eta(t)-2\epsilon(t))\mathcal{R}_{cl}(\vec{k},t)\;. \tag{4.102}$$

which, together with the definition of $\mathcal{R}$

$$\mathcal{R}_{cl} = \frac{1}{H\epsilon}\,\frac{\dot{\zeta}_{cl} + H\epsilon\,\zeta_{cl}}{1+k^2/3a^2H^2\epsilon}\;, \tag{4.103}$$

forms a system of two first order equations of two variables. Eliminating ζ_{cl} we can generate a second order equation for $\mathcal{R}_{cl}$ which takes the usual form similar to (4.89). Written with respect to conformal time, and up to first order in slow roll, it is given by:

$$\mathcal{R}_{cl}'' + (2+\eta-2s)\,aH\;\mathcal{R}_{cl}' + \left[c_L^2k^2 + (3\epsilon-6s+3c_L^2\epsilon)\,a^2H^2\right]\mathcal{R}_{cl} = 0\;, \tag{4.104}$$

where prime denotes a derivative w.r.t. conformal time, and s is the slow roll parameter defined by (4.60). Once again, using the conformal time dependence of aH, s, η, ϵ, and c_L contained in Appendix 6.9 this equation can be solved in terms of Hankel functions. One finds that the general solution to first order in the slow roll parameters is given by

$$\mathcal{R}_{cl}(\vec{k},\tau) = (-\tau)^{-\alpha}\left[\mathcal{C}H^{(1)}_{\nu_S}\big(-c_L(\tau)\,k\tau(1+s_c)\big) + \mathcal{D}H^{(2)}_{\nu_S}\big(-c_L(\tau)\,k\tau(1+s_c)\big)\right] \tag{4.105}$$

where $\alpha = -\frac{1}{2}(3+2\epsilon_c+\eta_c-2s_c)$ and $\nu_S = \frac{1}{2}(3+5s_c-2c_{L,c}^2\epsilon_c+\eta_c)$. Notice that for this to be a solution, it is important to keep into account—to first order—the time-dependence of c_L in the argument of the Hankel functions.

Once again, in order to match the initial conditions we must canonically normalize π_L. A quick glance at (4.85) reveals that the correct canonically normalized field is

$$\pi_L^{\text{can.}}(\vec{k},\tau) = \sqrt{2}\,\pi_L \left[\frac{M_{\text{Pl}}^2 a^2 k^2}{3\left(1+\frac{k^2}{3a^2H^2\epsilon}\right)}\right]^{1/2} \overset{-k\tau\to\infty}{\longrightarrow} \sqrt{2\epsilon}\,M_{\text{Pl}} H a^2\,\pi_L\ . \tag{4.106}$$

With the usual normalization for the creation and annihilation operators we will recover the Minkowski vacuum for very early times by demanding that

$$\lim_{\tau\to-\infty}\zeta_{cl}(\vec{k},\tau) = -\frac{k\,\pi_{L,cl}^{\text{can.}}}{3\sqrt{2\epsilon}\,M_{\text{Pl}}Ha^2} = -\sqrt{\frac{k}{4\,\epsilon c_L}}\frac{e^{-i(1+s_c)c_L(\tau)k\tau}}{3\,M_{\text{Pl}}Ha^2}\ . \tag{4.107}$$

Or, equivalently that

$$\lim_{\tau\to-\infty}\mathcal{R}_{cl}(\vec{k},\tau) = -\frac{a^2H^2}{k}\frac{d}{d\tau}\left(\frac{\pi_{L,cl}^{\text{can.}}}{H}\right) = i\sqrt{\frac{c_L}{4\,\epsilon k}}\frac{e^{-i(1+s_c)c_L(\tau)k\tau}}{M_{\text{Pl}}a}\ . \tag{4.108}$$

Matching the general solution given by (4.105) to the the initial condition (4.108) will set $\mathcal{D}=0$ and

$$\mathcal{C} = -i\sqrt{\frac{\pi}{8\epsilon_c}}\frac{c_{L,c}H_c}{M_{\text{Pl}}}(-\tau_c)^{s_c-\epsilon_c-\eta_c/2}(1+\tfrac{1}{2}s_c-\epsilon_c)e^{i(\eta_c+5s_c-2c_{L,c}^2\epsilon_c)\pi/4}+\mathcal{O}(\epsilon^{3/2})\ . \tag{4.109}$$

One can now use (4.102) to obtain the full expression for $\zeta_{cl}(\vec{k},\tau)$, which is (as promised) a bit messy and not particularly instructive as for our computation we are only interested in ζ's late time limit. We will not bother to write it out here.

Just like the tensor perturbations, neither $\mathcal{R}$ nor ζ is conserved outside the horizon, though their temporal dependence is mild, i.e., suppressed by slow-roll parameters:

$$\lim_{-k\tau\to0^+}\mathcal{R}_{cl}(\vec{k},\tau) = \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{T,c}^2\epsilon_c-2s_c}(-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c-5s_c/2-\eta_c/2}\left(-\frac{H_c}{\sqrt{4\epsilon_c}M_{\text{Pl}}c_{L,c}^{1/2}\,k^{3/2}}+\mathcal{O}(\epsilon^{1/2})\right) \tag{4.110}$$

$$\lim_{-k\tau\to0^+}\zeta_{cl}(\vec{k},\tau) = \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{T,c}^2\epsilon_c}(-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c-5s_c/2-\eta_c/2}\left(\frac{H_c}{\sqrt{4\epsilon_c}M_{\text{Pl}}c_{L,c}^{5/2}\,k^{3/2}}+\mathcal{O}(\epsilon^{1/2})\right)\ . \tag{4.111}$$

Notice that at this order in slow-roll, on large scales ζ and $\mathcal{R}$ are proportional to each other, with proportionality constant c_L^2, which is in agreement with (4.102):

$$\mathcal{R} \simeq -c_L^2(\tau)\,\zeta \qquad (k\tau \to 0^-)\ . \tag{4.112}$$

Now finally, the two point function of ζ for late times (when the modes are well outside the horizon) is given by

$$\left\langle \zeta(\tau,\vec{k}_1)\zeta(\tau,\vec{k}_2)\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\left|\zeta_{cl}(\tau,\vec{k}_1)\right|^2 \tag{4.113}$$

$$\overset{-k\tau\to 0^+}{\longrightarrow} (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\times\frac{H_c^2}{4\epsilon_c c_{L,c}^5 M_{\rm Pl}^2}\,\frac{1}{k_1^3}\,\frac{(\tau/\tau_c)^{8c_{L,c}^2\epsilon_c/3}}{(-c_{L,c}k_1\tau_c)^{5s_c-2c_{L,c}^2\epsilon_c+\eta_c}} \tag{4.114}$$

where, as before, we have kept the dependence on k and τ to first order in slow roll while the prefactor is expressed only to lowest order in slow roll.

Once again, since all the parameters are evaluated at the global time τ_c as opposed to the time of horizon crossing for each mode, we can simply read off the tilt to first order in slow roll directly from the above expression. It is:

$$n_S - 1 \simeq 2\epsilon_c c_{L,c}^2 - 5s_c - \eta_c\ . \tag{4.115}$$

Notice the overall $1/c_L^5$ factor in front of the spectrum. In a more standard single-field model, this would be replaced by $1/c_L$ (see e.g. [47]). For small c_L, our extra powers of c_L give us a very suppressed tensor-to-scalar ratio:

$$r \sim \epsilon\, c_L^5\ . \tag{4.116}$$

It is crucial however to ascertain whether we should focus on the $\zeta\zeta$ spectrum or the $\mathcal{R}\mathcal{R}$ one. After reheating, when the universe is dominated by a hot fluid, they have to coincide, because of the usual reasons. But during inflation, because of (4.112), they differ by a factor of c_L^4—precisely what suppresses our tensor-to-scalar ratio w.r.t the standard case. In sect. 4.10 we argue that it is the $\zeta\zeta$ spectrum that is continuous at reheating.

4.7 The three-point function

We now compute the $\langle\zeta\zeta\zeta\rangle$ three-point function. Like in single-field models with a small speed of sound, our three-point function will be enhanced by inverse powers of c_L with respect to what one gets in standard slow-roll inflation, for essentially the same reason (see e.g. [47]). However in addition to this, we will find an extra $1/\epsilon$ enhancement, coming from the fact that the quadratic phonon action (4.30) is suppressed by ϵ, whereas the cubic interactions (4.52) are not.

In order to compute the correlation function at a specific time, we need to evolve it from a quantum state we know, that is the early-time flat-space vacuum. Expanding the usual time-evolution operator and working to lowest order in perturbation theory we have the standard result, which is given schematically by:

$$\left\langle \zeta(\tau)^3 \right\rangle = -i \int_{-\infty}^{\tau} d\tau' \left\langle \Omega(-\infty) \left| \left[\zeta(\tau)^3, H_{\rm int}(\tau') \right] \right| \Omega(-\infty) \right\rangle \ . \tag{4.117}$$

For our purposes it is enough to calculate the three-point function to lowest order in slow-roll. As demonstrated in Appendix 6.12, at this order it is enough to work with the phonon cubic action (4.52), which in our FRW curved background takes the form (neglecting boundary terms):

$$\mathcal{L}_3 = M_{\rm Pl}^2\, a(t)^3 H^2\, \frac{F_Y}{F} \left\{ \tfrac{7}{81} (\partial\pi)^3 - \tfrac{1}{9} \partial\pi \partial_j \pi^k \partial_k \pi^j - \tfrac{4}{9} \partial\pi \partial_j \pi^k \partial_j \pi^k + \tfrac{2}{3} \partial_j \pi^i \partial_j \pi^k \partial_k \pi^i \right\} . \tag{4.118}$$

Quite amazingly, this applies both in the decoupling limit ($k \gg aH\epsilon^{1/2}$) and in the opposite limit ($k \ll aH\epsilon^{1/2}$). And so, defining $\zeta_i \equiv \zeta(\tau, \vec{k}_i)$, we have

$$\begin{aligned} \left\langle \zeta_1 \zeta_2 \zeta_3 \right\rangle = i M_{\rm Pl}^2 \frac{F_Y}{F} \frac{k_1 k_2 k_3}{27} \int_{\vec{p}_1, \vec{p}_2, \vec{p}_3} (2\pi)^3 \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\, Q(\vec{p}_1, \vec{p}_2, \vec{p}_3) \times \\ \int_{-\infty}^{\tau} d\tau'\ a^4(\tau') H^2(\tau') \left\langle \left[\pi_L(\tau, \vec{k}_1) \pi_L(\tau, \vec{k}_2) \pi_L(\tau, \vec{k}_3), \pi_L(\tau', \vec{p}_1) \pi_L(\tau', \vec{p}_2) \pi_L(\tau', \vec{p}_3) \right] \right\rangle , \end{aligned} \tag{4.119}$$

where

$$Q(\vec{p}_1,\vec{p}_2,\vec{p}_3) \equiv \frac{7}{81}\,p_1p_2p_3 - \frac{5}{27}\left(p_1\frac{(\vec{p}_2\cdot\vec{p}_3)^2}{p_2p_3} + p_2\frac{(\vec{p}_1\cdot\vec{p}_3)^2}{p_1p_3} + p_3\frac{(\vec{p}_1\cdot\vec{p}_2)^2}{p_1p_2}\right) + \frac{2}{3}\,\frac{(\vec{p}_1\cdot\vec{p}_2)(\vec{p}_2\cdot\vec{p}_3)(\vec{p}_3\cdot\vec{p}_1)}{p_1p_2p_3}\,, \tag{4.120}$$

which is totally symmetric under permutations of $\vec{p}_1,\vec{p}_2,\vec{p}_3$.

Writing π_L in terms of creation and annihilation operators allows us to easily express the integral in terms of the classical solutions. To be precise,

$$\left\langle\zeta_1\zeta_2\zeta_3\right\rangle = -(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\times 6M_{\rm Pl}^2\frac{F_Y}{F}\frac{k_1k_2k_3}{27}\,Q(\vec{k}_1,\vec{k}_2,\vec{k}_3)\,I(\tau;-\infty)\,, \tag{4.121}$$

where the integral $I(\tau_1;\tau_2)$ is defined as

$$I(\tau_1;\tau_2) = J(\tau_1;\tau_2) + J^*(\tau_1;\tau_2) \tag{4.122}$$

$$J(\tau_1;\tau_2) \equiv -i\,\pi_L^{cl}(\tau_1,\vec{k}_1)\pi_L^{cl}(\tau_1,\vec{k}_2)\pi_L^{cl}(\tau_1,\vec{k}_3)\int_{\tau_2}^{\tau_1}d\tau'\,\frac{(\tau'/\tau_c)^{-2\epsilon}}{H_c^2\tau'^4}\pi_L^{cl\,*}(\tau',\vec{k}_1)\pi_L^{cl\,*}(\tau',\vec{k}_2)\pi_L^{cl\,*}(\tau',\vec{k}_3)$$

and we used that Q is an even function of the momenta. Just as in the previous section, we have used the dependence of H and a on conformal time with reference to τ_c to lowest order in slow roll.

Utilizing the classical solution of $\mathcal{R}(\tau,\vec{k})$ given by (4.105) we can immediately recover $\pi_L^{cl}(\tau,\vec{k})$. This general solution, containing Hankel functions, and derivatives of Hankel functions is not particularly useful for attempting to perform the time-integral given by (4.122). We will first simplify $\pi_L^{cl}(\tau,\vec{k})$, which will enable us to express the three-point function in an analytic form.

First, note that at high momenta, above the de-mixing scale the quadratic Lagrangian for π_L takes the simpler form:

$$S_{\rm demix} = \int_{\vec{k}}\int d\tau\,\left(M_{\rm Pl}^2a^4H^2\epsilon\right)\left[\left|\pi_L'\right|^2 - c_L^2k^2\left|\pi_L\right|^2\right]\,, \tag{4.123}$$

which is—apart from the overall time-dependent pre-factor—the standard quadratic action for a scalar with generic propagation speed c_L. Thus, one can solve the equations of motion

generated by varying the above expression, fixing the exact form by requiring the flat-space vacuum in the infinite past (as we have done in the previous sections), without having to go through $\mathcal{R}$. Of course, the two methods are equivalent, but this route makes the method of expansion clear. As for momenta that are deep in the full-mixing limit, where $|c_L k\tau| \ll \epsilon^{1/2}$, $\mathcal{R}$ can be written in a simple form utilizing the asymptotic limit of the Hankel function (4.97), which upon insertion into (4.102) yields a particularly simple $\pi_L = -3\zeta/k$. In fact, even though in principle the two asymptotic expressions we get in this way should not share a common regime of applicability, in practice they are indistinguishable (at lowest order in slow-roll) in a wide range of momenta, $e^{-\mathcal{O}(1/\epsilon)} < |c_L k\tau| \ll 1$. [15] Therefore, as a very good approximation to the full solution, we can use the two asymptotic solutions and 'glue' them together anywhere inside this range. We find it more convenient to glue them at $(c_L k\tau) \sim \epsilon$. In summary, to lowest order in slow roll we can approximate the profile of π_L^{cl} by:

$$\pi_L^{cl}(\tau, \vec{k}) \simeq \begin{cases} \mathcal{B}_k\big(1 + ic_L k\tau - \frac{1}{3}c_L^2 k^2\tau^2\big)e^{-ic_L k\tau} \ , & \text{for } |c_L k\tau| \gtrsim 1 \quad \text{(4.124a)} \\ \mathcal{B}_k\big(-c_{L,c}k\tau\big)^{c_{L,c}^2\epsilon_c+\epsilon_c}\big(-c_{L,c}k\tau_c\big)^{-5s_c/2-\eta_c/2-\epsilon_c} + \mathcal{O}(\epsilon) \ , & \text{for } |c_L k\tau| \lesssim 1 \quad \text{(4.124b)} \end{cases}$$

where

$$\mathcal{B}_k = -\frac{3}{2}\frac{H_c}{M_{\rm Pl}c_{L,c}^{5/2}\epsilon_c^{1/2}}\frac{1}{k^{5/2}} \ . \tag{4.125}$$

Our strategy now is to break up the integral (4.122) into separate regions where one of the functional forms described by (4.124a) and (4.124b) can used. The integral can then be done explicitly.

4.7.1 Analytic Calculation of Integral

To illustrate the point and make the flavor of the analysis transparent, let's look at an almost equilateral configuration of momenta. That is, assume that

$$\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \ , \qquad k_1 \sim k_2 \sim k_3 \sim k \ . \tag{4.126}$$

[15] This stems from the mildness of their time-dependence outside the horizon.

First, notice that given some reference time τ_*, we can split the time-integral as

$$J(\tau;-\infty) = \frac{\pi_L^{cl}(\tau,\vec{k}_1)\pi_L^{cl}(\tau,\vec{k}_2)\pi_L^{cl}(\tau,\vec{k}_3)}{\pi_L^{cl}(\tau_*,\vec{k}_1)\pi_L^{cl}(\tau_*,\vec{k}_2)\pi_L^{cl}(\tau_*,\vec{k}_3)}\, J(\tau_*;-\infty) \tag{4.127}$$
$$- i\,\pi_L^{cl}(\tau,\vec{k}_1)\pi_L^{cl}(\tau,\vec{k}_2)\pi_L^{cl}(\tau,\vec{k}_3)\int_{\tau_*}^{\tau} d\tau'\, \frac{(\tau'/\tau_c)^{-2\epsilon_c}}{H_c^2\tau'^4}\pi_L^{cl\,*}(\tau',\vec{k}_1)\pi_L^{cl\,*}(\tau',\vec{k}_2)\pi_L^{cl\,*}(\tau',\vec{k}_3)\,.$$

Then, choosing τ_* to be precisely the conformal time at which a mode of momentum k transitions from (4.124a) to (4.124b), $-c_L k\tau_* \sim \epsilon$, we find that the real part of the second line vanishes at zeroth order in ϵ, because all the π_L's involved in the expression—inside and outside the integral—are real, and there is an overall i. The remaining piece is all that will contribute to the integral. And so we can write

$$\begin{aligned} J(\tau;-\infty) + J^*(\tau;-\infty) =& \prod_{i=1}^{3}|\mathcal{B}_{k_i}|^2\,(-c_Lk_i\tau)^{c_L^2\epsilon+\epsilon}(-c_Lk_i\tau_c)^{-5s/2-\eta/2-\epsilon}\times \\ & \int_{-\infty}^{\tau_*} d\tau'\,\frac{-i(\tau'/\tau_c)^{+\epsilon_c}}{H_c^2\tau'^4}\prod_{j=1}^{3}\left(1-ic_Lk_j\tau'-\tfrac{1}{3}c_L^2k_j^2\tau'^2\right)e^{+ic_Lk_j\tau'}+\text{c.c.} \\ =& -\frac{1}{27}\frac{c_L^3}{H_c^2}\,k_1k_2k_3\,U(k_1,k_2,k_3)\left(\frac{\tau}{\tau_c}\right)^{4c_T^2\epsilon}\prod_{i=1}^{3}|\mathcal{B}_{k_i}|^2(-c_Lk_i\tau_c)^{c_L^2\epsilon-5s/2-\eta/2}\,, \end{aligned} \tag{4.128}$$

where the scale invariant function $U(k_1,k_2,k_3)$ is given by

$$\begin{aligned} U(k_1,k_2,k_3) =& \frac{2}{k_1k_2k_3(k_1+k_2+k_3)^3}\Big\{3\big(k_1^6+k_2^6+k_3^6\big)+20\,k_1^2k_2^2k_3^2 \\ &+18\big(k_1^4k_2k_3+k_1k_2^4k_3+k_1k_2k_3^4\big)+12\big(k_1^3k_2^3+k_2^3k_3^3+k_3^3k_1^3\big) \\ &+9\big(k_1^5k_2+5\text{ perms}\big)+12\big(k_1^4k_2^2+5\text{ perms}\big)+18\big(k_1^3k_2^2k_3+5\text{ perms}\big)\Big\}\,. \end{aligned} \tag{4.129}$$

In order to ensure convergence of the integral and project onto the right vacuum, the integral is actually computed over a slightly tilted contour, that is $\tau' \to (1-i\varepsilon)\tau'+\tau_*$, with $\varepsilon \to 0^+$, and the limits of integration are from $-\infty$ to 0. Additionally, in the last step, the fact that $1 \gg |c_Lk\tau_*| \sim \epsilon$, and $|\tau_*| > |\tau|$ has been used to collect only the leading order in slow roll contributions.

A more careful analysis of the same flavor applies to more general triangle shapes, see Appendix 6.13. It turns out that the above expression is valid provided that the the triangle formed by the various momenta is not too squeezed, that is, provided

$$k_{\rm long}/k_{\rm short} > \sqrt{\epsilon}\ . \tag{4.130}$$

And so finally, putting everything together, we can express the full three-point function as

$$\begin{aligned}\big\langle\zeta_1\zeta_2\zeta_3\big\rangle(\tau_e) \simeq (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\times \\ \frac{3}{32}\frac{F_Y}{F}\frac{H_c^4}{M_{\rm Pl}^4}\frac{1}{\epsilon^3 c_L^{12}}\left(\frac{\tau_e}{\tau_c}\right)^{4c_T^2\epsilon}\frac{Q(\vec{k}_1,\vec{k}_2,\vec{k}_3)\,U(k_1,k_2,k_3)}{k_1^3\,k_2^3\,k_3^3}\ ,\end{aligned} \tag{4.131}$$

where, we remind the reader, $Q(\vec{k}_1,\vec{k}_2,\vec{k}_3)$ is given by (4.120) and $U(k_1,k_2,k_3)$ is given by (4.129).

The mild time dependence, $(\tau_e/\tau_c)^{4c_T^2\epsilon}$, in the above expression can actually produce an order one correction to the overall magnitude of the three-point function. Indeed, assuming that inflation lasts for $N_e\sim 60$ e-folds after the longest mode of today's relevance exits the horizon, we can see immediately that $(\tau_e/\tau_c)^{\mathcal{O}(\epsilon)}\sim e^{-60\times\mathcal{O}(\epsilon)}$, which, as promised, depending on how small ϵ is, can give an $\mathcal{O}(1)$ correction. On the other hand, the mild momentum dependence $(-c_Lk_i\tau_c)^{c_L^2\epsilon-5s/2-\eta/2}$, appearing in (4.128), is equal to one up to $\mathcal{O}(\epsilon)$ corrections. We thus drop this piece from (4.131), in order to be consistent with the preceding computation of the integral. Our result (4.131) should be understood as the leading order contribution in slow roll.

4.8 The size and shape of non-gaussianities

It is useful to rewrite the three-point function above as an overall amplitude $f_{\rm NL}$ times a *shape* that is a function of the momenta with order-one coefficients [48]. It is customary to do so at the level of correlators of the Newtonian potential Φ during matter-domination,

rather than of ζ. The relation outside the horizon is

$$\Phi = \tfrac{3}{5}\zeta \ . \tag{4.132}$$

Neglecting the tilt, one then writes the two- and three-point functions as

$$\langle \Phi(\vec{k}_1)\Phi(\vec{k}_2)\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\cdot\frac{\Delta_\Phi}{k_1^3} \tag{4.133}$$

$$\langle \Phi(\vec{k}_1)\Phi(\vec{k}_2)\Phi(\vec{k}_3)\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot f(k_1,k_2,k_3) \ , \tag{4.134}$$

and normalizes f on 'equilateral' configurations with $k_1 = k_2 = k_3$ [62],

$$f(k,k,k) = f_{\rm NL}\cdot\frac{6\Delta_\Phi^2}{k^6} \ . \tag{4.135}$$

This defines the parameter $f_{\rm NL}$ in a model-independent fashion, in terms of observable quantities only. In particular, the observed value for the power-spectrum normalization is $\Delta_\Phi \simeq 2\times 10^{-8}$. Notice that, because of momentum conservation, the three momenta $\vec{k}_{1,2,3}$ close into a triangle. As a result, the function f depends on the absolute values $k_{1,2,3}$ only, because a triangle is uniquely defined—up to overall rotations, which are a symmetry of f—by specifying its sides. Notice also that scale-invariance forces f to have overall scaling dimension k^{-6}, and we have used this fact in (4.135). Notice finally that the standard convention would be to call F the function that we call f. Unfortunately we have already been using F for our Lagrangian, so will stick to f for the function defined above. Hopefully this will not lead to confusion.

Applying these definitions to our case we find

$$\Delta_\Phi = \frac{9}{100}\cdot\left(\frac{\tau_e}{\tau_c}\right)^{8c_T^2\epsilon/3}\cdot\frac{H^2}{M_{\rm Pl}^2}\cdot\frac{1}{\epsilon\, c_L^5} \tag{4.136}$$

$$f(k_1,k_2,k_3) = \frac{5}{2}\cdot\frac{F_Y}{F}\frac{1}{\epsilon\, c_L^2}\cdot\left(\frac{\tau_e}{\tau_c}\right)^{-4c_T^2\epsilon/3}\cdot\Delta_\Phi^2\cdot\frac{Q(k_1,k_2,k_3)\cdot U(k_1,k_2,k_3)}{k_1^3k_2^3k_3^3} \tag{4.137}$$

$$f_{\rm NL} = -\frac{19415}{13122}\cdot\left(\frac{\tau_e}{\tau_c}\right)^{-4c_T^2\epsilon/3}\cdot\frac{F_Y}{F}\frac{1}{\epsilon\, c_L^2} \simeq -\mathcal{O}(1)\cdot\frac{F_Y}{F}\frac{1}{\epsilon\, c_L^2} \tag{4.138}$$

The $f_{\rm NL}$ parameter gives us a measure of the absolute size of non-gaussianities. As we argued in sect. 4.3, F_Y is essentially a free parameter, which can be as large as F, in which case our $f_{\rm NL}$ is *huge*, of order $1/(\epsilon\, c_L^2)$. By comparison, single-field inflationary models with small sound speed—whose non-gaussianities are much larger than for standard slow-roll inflation—have, at the same value of the sound speed, an $f_{\rm NL}$ which is a factor of ϵ smaller than ours. Notice that in this case there is a potential tension for our model: the same combination $\epsilon \cdot c_L^2$ appears in the scalar tilt, eq. (4.115). Of course one could have cancellations there, because of the other terms in the expression for the tilt. But assuming that these do not change the overall order of magnitude of the tilt, the tilt is small if non-gaussianities are large, and vice versa. Eventually, one should observe either. If on the other hand our F_Y is of order ϵF, then this $1/\epsilon$ enhancement for non-gaussianities is gone, and our $f_{\rm NL}$ becomes of order $1/c_L^2$, which is the same as for small sound-speed single-field models [16].

But the most interesting feature of our non-gaussian signal is probably its shape, that is, the dependence of f on the shape of the triangle made up by the momenta $\vec{k}_{1,2,3}$. We plot it in fig. 4.1, following the standard conventions of [48]. In particular, it is clear from the plot that our three-point function is peaked on 'squeezed' triangles with $k_3 \ll k_{1,2}$, but its behavior for those configurations depends strongly on the *angle* θ between $\vec{k}_3$ and the other momenta. Quantitatively, focusing on the $\frac{QU}{k^3k^3k^3}$ structure in (4.137) and ignoring the prefactors from now on, we get

$$f(k_1, k_2, k_3 \to 0) \propto -\frac{40}{27}\frac{\left(1 - 3\cos^2\theta\right)}{k_1^3 k_3^3}\ . \tag{4.139}$$

where we used momentum-conservation to rewrite k_2 as $k_2 \simeq k_1 + k_3\cos\theta$. Such an angular dependence is not there in any of the standard inflationary models we are aware of: at least

[16] We remind the reader that cosmological perturbations can still be nearly gaussian, even for huge values of $f_{\rm NL}$, as long as the combination $f_{\rm NL}\sqrt{\langle\zeta^2\rangle}$ is much smaller than one at the relevant scales. For us, in the most strongly coupled case ($F_Y \sim F$), such a combination is of order $H/M_{\rm Pl}\cdot(\epsilon c_L^3)^{-3/2}$, which is much smaller than one if and only if the weak-coupling condition (4.59) is obeyed. As usual, perturbations are nearly gaussian if and only if they are weakly coupled at horizon crossing.

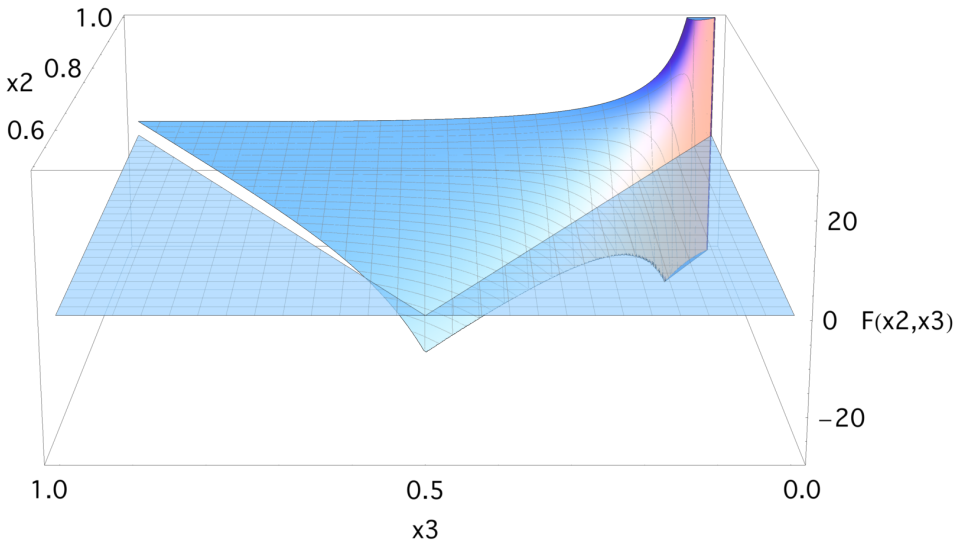


Figure 4.1: *The shape of non-gaussianities for our model, according to the standard conventions and definitions of ref. [48].*

for single-field models, the consistency relations force the behavior of the three-point function in the squeezed limit to be angle-independent—see e.g. [59, 60]. On the other hand, in our case the standard consistency relations are maximally violated, both at the level of angular dependence—as we just mentioned—and at the level of the overall prefactor: usually the squeezed limit of the three-point function is suppressed by the scalar tilt, which is of order ϵ; here instead, there is no suppression like that, and in fact, as we argued above the overall prefactor can be as big as one *over* the tilt. It is easy to see why the consistency relations do not hold in our model: the standard argument of [59,60]—that a long-wavelength background ζ can be traded in for a rescaling of spatial coordinates—does not apply to our case, because in our model there is no gauge in which the curvature perturbation ζ appears as a $\zeta \cdot \delta_{ij}$ correction to the spatial metric.

The fact that our non-gaussianities feature a novel shape, means that the numerical analyses of CMB data that have been carried out so far are quite suboptimal for our case.

Following [48], this can be quantified by computing the overlap—or 'cosine'—between our shape and those used in the analyses. These cosines give a measure of how much one could improve by performing a dedicating analysis, which would entail using directly our shape in the estimator for the $f_{\rm NL}$ parameter. Of all the standard shapes on the market, the *local* one is the only one that is peaked on squeezed triangles, with exactly the same scaling as ours, but with no dependence on the angle at which the squeezed limit is approached. As far as angular dependence goes, this corresponds to a monopole, whereas our squeezed limit (4.139) corresponds to a quadrupole. As a result, the overlap between our shape and the local one vanishes in the squeezed limit, and overall it is very small, [17]

$$\cos(f, f_{\rm local}) \simeq 2\% \ . \tag{4.140}$$

Another popular shape is the *equilateral* one, which is peaked on configurations with $k_1 = k_2 = k_3$. Our shape is fairly suppressed for those configurations, and this results in a smallish overlap:

$$\cos(f, f_{\rm equil}) \simeq -39\% \ . \tag{4.141}$$

Finally, given that our shape is fairly large for degenerate—or 'flattened'—triangles with $k_1 = k_2 + k_3$, one might think that it has a large overlap with the so-called *orthogonal* shape. However, we find a small cosine with the orthogonal template of [63]:

$$\cos(f, f_{\rm orthog}) \simeq -32\% \ . \tag{4.142}$$

[17] According to the nomenclature of [48], we are computing '3D' cosines, which are relevant for analyses of fully 3D data, like e.g. those of large scale structure surveys. For analyses of CMB data, which are projected onto a 2D sphere, one should refer to 2D cosines, which are however much more cumbersome to compute. Even though they are not entirely precise in this case, 3D cosines are still a good indicator of the overlaps between different shapes. We should also mention that for some of these cosines, the computations involve logarithmic divergences in the squeezed limit, which we cutoff at a value of 10^3 for the maximum ratio between momenta in the data set.

All this suggests that *(a)* at the level of the three-point function, our model is very distinguishable from more standard inflationary models, and that *(b)* to test our model, we should perform a dedicated CMB data analysis for our shape of non-gaussianities.

Numerical analyses of three-point correlation functions in the CMB drastically simplify if one can rewrite the theoretical momentum-space three-point function in a *factorizable* form, that is, as a sum of a few terms that are just products of powers of the triangle sides k_1, k_2, k_3 [61]. If one is able to do so, the calculation time decreases by a factor of $N_{\rm pixel}$, which is a huge improvement. We do so in the Appendix for the interested reader to aid in data analysis

4.9 Why Is ζ Not Conserved?

We saw in sect. 4.6 that already at linear level, during our solid inflation phase neither ζ nor $\mathcal{R}$ is conserved on large scales. One might be tempted to attribute this to the presence of isocurvature modes in addition to adiabatic ones. However, in our model there is only one scalar perturbation—parameterized by π_L in the gauge we have been using—and usually isocurvature modes are a luxury that only multi-field models can afford. To sharpen the paradox, our ζ and $\mathcal{R}$ do not coincide on large scales—see eq. (4.112)—whereas usually they do, even for fluctuations that are not purely non-adiabatic, that is, even when they are not conserved. In fact, Weinberg proved a no-go theorem stating that *all* FRW cosmological models—inflationary or not—feature two adiabatic modes of fluctuation, one of which has constant and identical ζ and $\mathcal{R}$ on large scales, while the other has $\zeta = \mathcal{R} = 0$ [32, 64]. This theorem is manifestly violated by our model. But, in Weinberg's own words, "no-go theorems have a way of relying on apparently technical assumptions that later turn out to have exceptions of great physical interest" [65]. We do not know whether our model will ultimately turn out to be of great physical interest, but it certainly offers an exception to an apparently technical assumption of ref. [64]. Before showing this, let us explain in physical

terms why our solid system cannot sustain adiabatic modes.

By definition, an adiabatic mode is a perturbation that for very long wavelengths becomes locally unobservable, being indistinguishable from a slight shift in time of the background solution. An ordinary fluid offers a perfect example of this. Consider a long-wavelength sound wave in a fluid, for the moment in the absence of gravity. For an observer making measurements on scales much shorter than the wavelength, and working in the local rest frame—which is slightly different from the background one—, the only observables are the density and the pressure: neglecting gradients, a fluid is isotropic in its rest frame, and its stress-energy tensor is characterized by ρ and p only, which are related by the equation of state. Then, the only physical effect that is measurable on scales much shorter than the wavelength is the local compression (or dilation) the sound wave induces. When we include gravity into the picture, essentially the same considerations apply for the perturbation, but now the time-evolution of the *unperturbed* FRW background already probes all possible compression levels for the fluid (within some range), that is, all possible values of ρ and p compatible with the equation of state. As a consequence, for wavelengths much longer than the Hubble scale, within any given Hubble patch a sound wave will be indistinguishable from a time-shift of the background solution, that is, it will become physically unobservable. Different Hubble patches will then evolve as separate identical FRW universes, and it can be shown that this translates into a conservation law for $\mathcal{R}$ [66].

For a solid, the situation is very different, already in the absence of gravity. A longitudinal phonon—which is the only scalar fluctuation at our disposal—does not correspond to a purely compressional deformation of the medium. Even for wavelengths that are much longer than the observation scale, the anisotropic stress and the compression associated with the phonon are of the same order of magnitude. This is evident from the form of the stress-energy tensor (4.23), which expanded to first order in $\vec{\pi}$ yields schematically

$$\delta T_{ij} \sim (\vec{\nabla} \cdot \vec{\pi})\, \delta_{ij} + (\partial_i \pi_j + \partial_j \pi_i) \ , \tag{4.143}$$

with similar coefficients in front of the two tensor structures—related to suitable derivatives of F w.r.t. X, Y, Z—whose precise values will not concern us here. For a longitudinal phonon of momentum $\vec{q}$, the anisotropic stress is proportional to $\hat{q}_i\hat{q}_j$, and is of the same order of magnitude as the change in pressure. A local observer can detect this anisotropy if he or she can detect the change in pressure. In other words, unlike a fluid, a solid with small longitudinal deformations is not locally isotropic. Once we include gravity, these scalar fluctuations will be locally distinguishable from the background solution, even for super-horizon wavelengths, since the background *is* isotropic. One could probably apply a 'separate universe'-like argument that includes anisotropic homogeneous backgrounds, to get information about the time-evolution of these modes, even at non-linear order.[18] We leave this for future work. For the moment, we just notice that our scalar modes do not conform to the standard characterization of adiabatic perturbations.

We can now go back to the no-go theorem of ref. [64], and see which assumptions we are violating. The theorem is based on the following ingenious idea. Since an adiabatic mode is, by definition, unobservable once the wavelength is very long, at zero momentum it should reduce to a gauge transformation of the FRW background. Newtonian gauge is a complete gauge-fixing at finite momentum, but it has a residual gauge freedom at zero momentum. By exploiting this gauge freedom one can construct zero-momentum solutions of the linearized perturbation equations. Most of these are pure-gauge, unphysical solutions. To be physical, they have to be the zero-momentum limit of finite momentum solutions, which are physical because there is no residual gauge-freedom at finite momentum. For this to be the case, one needs the zero-momentum solution to obey the finite-momentum version of the (ij) and $(0i)$ linearized Einstein equations,

$$\Phi = \Psi - (8\pi G)\,\delta\sigma \qquad \dot{H}\,\delta u = \dot{\Psi} + H\Phi\ , \tag{4.144}$$

which singles out only two independent modes among the zero-momentum solutions. Here

[18] We thank Matias Zaldarriaga for this remark.

$\delta\sigma$ and δu are the scalar anisotropic stress and the velocity potential of eqs. (4.79), (4.80),

$$\delta T_{ij} \supset a^2(t)\,\partial_i\partial_j\delta\sigma\ , \qquad \delta T_{0i} \supset -(\bar\rho+\bar p)\,\partial_i\delta u\ . \tag{4.145}$$

One has to impose further that all equations of motion—for gravity and for the matter fields—are regular for $\vec q \to 0$, including eq. (4.144). More precisely, if one rewrites all linearized equations of motion in first-order form,

$$\dot y_a(\vec q,t) + C_{ab}(\vec q,t)\cdot y_b(\vec q,t) = 0\ , \tag{4.146}$$

where the y_a's include the fields and their velocities, with constraints for the initial conditions of the form

$$c_b(\vec q)\cdot y_b(\vec q,t_0) = 0\ , \tag{4.147}$$

then one has to demand that all C_{ab} and c_b coefficients be regular for $\vec q \to 0$. This is the only technical assumption of the theorem.[19] If it is obeyed, then the two zero-momentum gauge modes can be promoted to physical, finite-momentum solutions, which are adiabatic by construction, and one of which turns out to have constant ζ and $\mathcal{R}$.

From our physical argument above, we see that the culprit for us is the anisotropic stress, which does not become negligible at long wavelengths. Indeed, comparing (4.145) with (4.143) we get schematically

$$\delta\sigma \sim \frac{\vec q}{q^2}\cdot\vec\pi \tag{4.149}$$

(we are neglecting factors of $a(t)$, and corrections to (4.143) involving the metric, which does not change our conclusion.) Once plugged into eq. (4.144), this gives us an equation

[19] There is also an implicit assumption—used to write down the zero-momentum pure gauge solutions—that all background matter fields only depend on time, which is not obeyed in our case. However, this is easily fixed by performing the correct gauge transformation, which, following the notation of ref. [64], in our case yields the pure gauge solution

$$\Psi = H\varepsilon - \lambda\ , \qquad \Phi = -\dot\varepsilon\ , \qquad \vec\pi = \lambda\,\vec x\ , \qquad \delta\rho = -\dot{\bar\rho}\,\varepsilon\ , \qquad \delta p = -\dot{\bar p}\,\varepsilon\ . \tag{4.148}$$

of motion that is not regular for $\vec{q} \to 0$, thus violating Weinberg's technical assumption. We get a similar singularity in the second equation of (4.144), since from $T_{0i} \sim \dot{\pi}^i$ we get a velocity potential

$$\delta u \sim \frac{\vec{q}}{q^2} \cdot \dot{\vec{\pi}} \tag{4.150}$$

Notice that we cannot reabsorb the annoying q^{-2} factors into a new $\vec{\pi}$ field, thus making $\delta\sigma$ and δu regular for $\vec{q} \to 0$, because one of the equations (4.146) is the equation of motion for $\vec{\pi}$ itself, which is local in real space, thus analytic in $\vec{q}$ in Fourier space. If we were to divide it by q^2, to write an evolution equation for the new $\vec{\pi}/q^2$ field, we would introduce singularities there.

4.10 Matching of correlation functions at reheating, and post-inflationary evolution

Since our scalar perturbations are not adiabatic, our predictions for post-inflationary correlation functions on large scales can in principle be affected by local physical processes happening at reheating. It would be interesting to investigate how much model-independent information about time-evolution across the reheating phase can be obtained by applying an *anisotropic* parallel-universe argument, as mentioned in the last section. For the time being, we adopt what appears to be a reasonable model for reheating. As motivated in sect. 4.4, we postulate that inflation ends when $B \equiv \det B^{IJ}$ reaches some critical value B_e, after which the matter content of the universe turns into a perfect fluid, which can be described by a low-energy effective Lagrangian involving the same set of dynamical d.o.f.'s as our solid, but with different (and more restrictive) symmetry requirements. Furthermore, we assume that the transition (a.k.a. reheating) from solid to fluid occurs "smoothly" (in a sense that will be made explicit) and instantaneously (the transition time being much shorter than the Hubble time).

Since it is the scalar quantity B that plays the role of the "clock" in our model, it is easiest to work in unitary gauge (UG), in which constant time slices correspond to surfaces of uniform B (the properties of this gauge are worked out in Appendix 6.10). Explicitly, our model for reheating can be captured by the following statements:

- Inflation ends at t_e, with $a(t_e)^{-6} = B_e$, where t denotes the time in UG.

- The matter content in the post inflationary era takes the form of a perfect fluid, which can be described by the effective action

$$S_{\text{fluid}} = \int d^4x\sqrt{-g}\,\tilde{F}(B) \tag{4.151}$$

  where $B = \det g^{IJ}$ in UG. The change from one equation of state to the other is effectively instantaneous. For an ultra-relativistic fluid with $p = \frac{1}{3}\rho$, one has

$$\tilde{F}(B) \propto B^{2/3}\ . \tag{4.152}$$

- Energy transfer from the solid phase to the fluid phase during this short reheating period is efficient, and the normalization of $\tilde{F}$ is restricted in such a way that energy conservation is respected, i.e.

$$\rho_{\text{solid}} = -F(X,Y,Z)\big|_{t_e} = -\tilde{F}\left(a(t_e)^{-6}\right) = \rho_{\text{fluid}}\ . \tag{4.153}$$

  However, generally $p_{\text{solid}} \neq p_{\text{fluid}}$, since the equation of state has been changed. Consequently, even though the Hubble parameter H remains continuous, $\dot{H}$ does not.

- Smoothness: the dynamical d.o.f.'s in unitary gauge—A (or χ), C_i, D_{ij}—as well as their first derivatives are continuous across the $t = t_e$ surface [20]. The second derivatives will exhibit discontinuities since the equations of motions are altered due to the instantaneous change in the equation of state.

[20] See Appendix 6.10 for the definition of these fields in terms of fluctuations of the metric.

An immediate consequence following the smoothness requirement is that ζ transits continuously from solid phase to post-inflationary fluid phase, while $\mathcal{R}$ does discontinuously. Indeed notice that in UG, ζ and $\mathcal{R}$ are given by

$$\zeta = \frac{A}{2}, \quad \mathcal{R} = -\frac{H}{2\dot{H}} \frac{\dot{A} - \dot{H}A/H}{1 - k^2/3a^2\dot{H}} \tag{4.154}$$

thus the discontinuity of $\mathcal{R}$ stems from that of $\dot{H}$.

Given that the transition from the solid phase during inflation to the perfect fluid phase during the post-inflationary era occurs effectively instantaneously, we can compute various correlators (of A, C_i and D_{ij}) at t_e and use them as the initial conditions of the post-inflationary evolution. There are two subtleties:

1. How do we relate the correlators of quantities in UG to those in SFSG, which have been computed in Section 4.6 and Section 4.7.

2. Given that the super-horizon modes of A and D_{ij} are not adiabatic during inflation, they start the post-inflationary evolution with a non-vanishing first derivative in time. Although eventually these super-horizon modes become constant (time-independent) before reentering the horizon, a natural question to ask is how much the eventual constants could differ from the initial conditions these modes start with.

In order to address the first issue, let's compute the scalar two- and three-point correlators: $\langle A_1(\bar{t}_e)\, A_2(\bar{t}_e)\rangle$ and $\langle A_1(\bar{t}_e)\, A_2(\bar{t}_e)\, A_3(\bar{t}_e)\rangle$, where the "$\bar{}$" is to remind ourselves that we are using time in UG, and $A_i(\bar{t}\,)$ is shorthand for $A(\vec{k}_i,\, \bar{t}\,)$. Using the transformation rule from SFSG to UG, we have that

$$\bar{t} = t - \frac{1}{3H}\partial_i \pi^i(x) + \mathcal{O}(\pi^2)\ ,$$

and we can write

$$\begin{aligned} &A(x) = A^{(1)} + A^{(2)} + \ldots, \\ &\text{where} \quad A^{(1)}(x) = \tfrac{2}{3}\partial_i \pi^i(x) = 2\zeta, \qquad A^{(2)} \sim \partial\pi\partial\pi\ , \qquad \text{etc.} \end{aligned}$$

It follows immediately that, schematically,

$$\langle A^2\rangle = \langle A^{(1)}A^{(1)}\rangle + 2\langle A^{(1)}A^{(2)}\rangle + ... \sim 4\langle\zeta^2\rangle + \langle(\partial\pi)^3\rangle + ... \ . \tag{4.155}$$

The second term on the right can be neglected, since it is of higher order in the perturbative expansion. Likewise, we have

$$\langle A_1(\bar{t}_e)A_2(\bar{t}_e)\rangle \simeq \langle A_1^{(1)}(\bar{t}_e)A_2^{(1)}(\bar{t}_e)\rangle = 4\langle\zeta_1(\bar{t}_e)\zeta_2(\bar{t}_e)\rangle \simeq 4\langle\zeta_1(t)\zeta_2(t)\rangle \tag{4.156}$$

where the last (approximate) equality is justified as long as the perturbative expansion (in fields) holds, since the difference between t and $\bar{t}$ is of first order in the fields.

We can do the same for the 3-pt correlators:

$$\langle A^3\rangle \simeq \langle A^{(1)}A^{(1)}A^{(1)}\rangle + 3\langle A^{(1)}A^{(1)}A^{(2)}\rangle \sim 8\langle\zeta^3\rangle + \langle\zeta^2\partial\pi\partial\pi\rangle \ . \tag{4.157}$$

Notice that $\langle\zeta^2\partial\pi\partial\pi\rangle \sim \langle\partial\pi\partial\pi\rangle^2 \sim O(\epsilon^{-2})$, while $\langle\zeta^3\rangle \sim O(\epsilon^{-3})$, thus if we restrict ourselves to the leading order in slow-roll, this term can be safely neglected. It follows that

$$\begin{aligned}\langle A_1(\bar{t}_e)A_2(\bar{t}_e)A_3(\bar{t}_e)\rangle &\simeq 8\langle\zeta_1(\bar{t}_e)\zeta_2(\bar{t}_e)\zeta_3(\bar{t}_e)\rangle \\ &\simeq 8\langle\zeta_1(t)\zeta_2(t)\zeta_3(t)\rangle + \frac{8k_1}{3H}\langle\dot{\zeta}_1(t)\,\pi_{L,1}(t)\zeta_2(t)\zeta_3(t)\rangle + \text{perms.} \\ &\simeq 8\langle\zeta_1(t)\zeta_2(t)\zeta_3(t)\rangle + \epsilon\cdot\frac{32c_T^2}{3}\langle\zeta_1(t)\,\zeta_1(t)\zeta_2(t)\zeta_3(t)\rangle + \text{perms.} \ ,\end{aligned} \tag{4.158}$$

where we have used that in the long wavelength limit $\dot{\zeta} \simeq \zeta'/a \simeq \frac{4}{3}c_T^2\epsilon H\,\zeta$. In the last line, the second term and its permutations are negligible at the leading order in slow roll, since $\epsilon\langle\zeta^4\rangle \sim \epsilon\langle\zeta^2\rangle^2 \sim O(\epsilon^{-1})$ while $\langle\zeta^3\rangle \sim O(\epsilon^{-3})$.

Hence, as long as we focus only on the leading contribution in slow roll, the first issue mentioned above can be easily resolved: the 2-pt and 3-pt correlators of scalar perturbations in UG are related to those of (2 times) ζ in SFSG. Not surprisingly, similar relations for tensor perturbations hold if we apply the same logic.

As for the second issue. It can be shown [67,68] that during the post inflationary era, when the matter content of the universe is in the form of a perfect fluid, the scalar perturbation

A is adiabatic in the long wavelength limit, i.e. it is a constant at nonlinear level as long as it stays outside the horizon. However, unlike other inflationary models where there exists a conserved scalar mode in the long wavelength limit *during* inflation, the scalar perturbation A in our model evolves slowly outside the horizon, in the sense that $A \simeq A^{(1)} = 2\zeta \propto (-\tau)^{\frac{4}{3}c_T^2\epsilon}$. Therefore, after the rapid transition from solid phase to perfect fluid phase, rather than staying at its initial value, $A(t_e)$, it approaches its eventual constant value. However the relative difference between the two is only of order ϵ: the slow time-dependence of A during inflation means that right after reheating the initial condition for the velocity is roughly $\dot{A}(t_e) \sim \epsilon \cdot HA(t_e)$. Since then, $\dot{A}$ decreases like $1/a^3$, thus making $A(t)$ approach its asymptotic value in a few Hubble times, during which $A(t)$ moves by $\sim \epsilon A(t_e)$. At the leading order in slow-roll, we can neglect this difference. Notice that this effect cannot change the tilts that we have computed: all modes of interest are outside the horizon during reheating and during the phase when A relaxes to its asymptotic value. As a result, this small correction of order ϵ to the value of A is the same for all modes, i.e., independent of k. By applying the same logic, we can reach the same conclusion for the transverse traceless tensor perturbation D_{ij}, which is not conserved in the long wavelength limit during inflation, but approaches an asymptotic value in the post-inflationary fluid phase in a similar manner as its scalar counterpart.

In conclusion, at the order we are working, we can take the correlation functions for ζ and γ that we have computed during inflation in SFSG, evaluate them right before reheating, and obtain in this way good approximations to the corresponding correlation functions in UG in the post-inflationary phase. In particular, even though our scalar perturbations are not adiabatic during inflation, at reheating they get converted to adiabatic ones, with the same asymptotic constant value of ζ (up to $O(\epsilon)$ corrections) as they had at reheating.

4.11 Summary & Outlook

Our model differs drastically from more standard ones in its symmetry breaking pattern. In particular, time-translations are not broken: there are physical "clocks"—i.e., time-dependent gauge-invariant observables—, but they inherit their time-evolution from the metric, not from the matter fields. As a result, the systematics of the EFT for the associated Goldstone excitations is completely different than the standard effective field theory of inflation. This has far reaching implications, some of which are directly observable.

The observational predictions of our model can be summarized as follows: *(i)* A nearly scale-invariant spectrum of adiabatic scalar perturbations, in agreement with observations. *(ii)* A nearly scale-invariant spectrum of tensor modes, with a slight *blue* tilt; the tensor-to-scalar ratio $r \sim \epsilon c_L^5$ ranges from somewhat smaller than in standard slow-roll inflation, for ultra-relativistic longitudinal phonons with $c_L^2 \simeq 1/3$, to tiny, if they are non-relativistic. *(iii)* A scalar three-point function with a novel shape—peaked in the squeezed limit, with non-trivial angular dependence on how the limit is approached—and a potentially very large amplitude, as big as $f_{\rm NL} \sim \frac{1}{\epsilon}\frac{1}{c_L^2}$.

A fourth prediction, whose full analysis we leave for future work, is the presence of vector modes. Unlike usual inflationary systems, our solid features transverse phonons, which get excited during inflation. In Appendix 6.11 we show that the vector-to-scalar ratio at the end of inflation scales as $(c_L^2/c_T^2)^{5/2}$, which, depending on the value of c_L, ranges from negligible to roughly 6%. After reheating, the universe is dominated by a perfect fluid, and these vector modes should decay in the usual fashion, thus leaving no detectable imprint on the CMB. However, they interact at non-linear order with scalar and tensor perturbations during inflation and at reheating, thus affecting in principle their (higher point) correlation functions.

Also for future work we leave a more thorough understanding of the time-evolution of our perturbations. One should be able to get model-independent information by running a

separate universe-type argument involving anisotropic background solutions. In particular, this approach might elucidate whether and how super-horizon perturbations get affected by local processes at reheating. We showed in sect. 4.10 that a quick phase-transition triggered by $\det B^{IJ}$ conserves ζ and the tensor modes. Moreover, we showed that for the two- and three-point function, there is no difference between using unitary gauge time and SFSG time, which implies that we would get the same post-inflationary correlation functions if reheating were in fact triggered by another "clock", say $[B]$. This gives us confidence that, at least for these correlation functions, our predictions are robust. However it leaves open the possibility that we would get different predictions if the reheating phase lasted longer, for an Hubble time or more.

There are also a number of generalizations of our model that we feel deserve being studied. The first, would be to promote our solid to a 'super-solid'. In physical terms, a super-solid is a solid harboring a superfluid [69]. In our field-theory, symmetry-breaking terms, it is a system of four derivatively coupled scalars, ϕ^0 and ϕ^I, with a shift-symmetry on ϕ^0, our solid symmetries on the ϕ^I's, and a state that spontaneously breaks *all* space-time translations as follows [70]:

$$\langle \phi^0 \rangle = \mu\, t \ , \qquad \langle \phi^I \rangle = \alpha\, x^I \ . \tag{4.159}$$

This system then combines the symmetry-breaking pattern of standard inflationary models with ours, and provides the minimal ingredients—the analog of 'single-field'—for studying the consequences of breaking all space-time translations during inflation, in a way that is consistent with a residual physical homogeneity and isotropy. Cosmological systems of this sort have been considered briefly in [10]. Notice that such a model will have *two* scalar perturbations—roughly speaking, excitations of ϕ^0 and longitudinal excitations of ϕ^I, although in general they will be mixed, like it happens for instance in finite-temperature superfluids [14], where the two resulting modes go under the names of first and second sound. Notice also that, for inflationary purposes, one does not need to assume that the shift-

symmetry on ϕ^0 be exact. One can assume an approximate shift-symmetry, with symmetry-breaking couplings suppressed by slow-roll parameters—in which case, to be consistent with the literature, one should refrain from calling such a system a super-solid.

Another generalization that one should consider, is demoting our solid from isotropic to *crystalline*. We say "demoting", because it would entail lowering the degree of internal symmetries acting on our ϕ^I fields, from the full $SO(3)$ group to one of its discrete subgroups, e.g. the cubic symmetry group. Then, in the background configuration with $\langle\phi^I\rangle = \alpha x^I$, only that particular discrete subgroup will be preserved (now as a linear combination of internal and spatial rotations), and one could have interesting, observable deviations from isotropy. One of course needs to make sure that these deviations are absent from the background evolution and from the spectrum of perturbations—which we know to be isotropic with very good accuracy—but this might be automatic for certain subgroups of $SO(3)$. As an example, consider the background evolution. The potential anisotropy in it is determined by the tensor structure of T_{ij}. Let's assume for definiteness that our discrete subgroup is the cubic group—by which we also mean the individual inversions along the sides—and let's align the sides of the cube with $\hat{x}$, $\hat{y}$, and $\hat{z}$. Then, T_{ij} has to be invariant under permutations of $\hat{x}$, $\hat{y}$, and $\hat{z}$, and under the individual parities $\hat{x} \to -\hat{x}$, etc. The only two-index tensor with these properties is

$$\hat{x}^i\hat{x}^j + \hat{y}^i\hat{y}^j + \hat{z}^i\hat{z}^j \ , \tag{4.160}$$

which is nothing but δ^{ij}. That is, at the two-index level, cubic symmetry accidentally implies full $SO(3)$ invariance. For the spectrum of perturbations to be accidentally isotropic in the same fashion, one needs the phonons' quadratic action to be accidentally isotropic. This involves now four-index background tensors, contracted with the phonons' derivatives. So, the general mathematical question is, what are the discrete subgroups of $SO(3)$ whose invariant two-index and four-index tensors are all isotropic? If one finds one or more subgroups with this property, and if their *six*-index invariant tensors are *not* all isotropic, one has a

model of inflation, that, because of symmetry, has an isotropic background and spectrum of perturbations, and a (potentially) maximally anisotropic three-point function.

Finally, it would be interesting to run a dedicated numerical analysis of CMB data for our specific three-point function template, given its small overlaps with the more standard templates that have been considered so far.

We hope to address all these questions in the near future.

Chapter 5

Dissipation

So far we have neglected a crucial feature of real-world hydrodynamics: the presence of dissipative effects. Dissipation appears in the gradient expansion of hydrodynamics as a first order correction to the perfect fluid equations, which are the continuity and the (relativistic generalization of the) Euler equations. In fact, for fluids that do not carry anomalous charges [71], *all* first order corrections are dissipative. It is then clear that for the EFT program to be useful beyond zeroth order in the derivative expansion, one has to find a way to accommodate dissipative effects.

In the standard parameterization, first order dissipative effects are characterized by three coefficients: bulk viscosity, shear viscosity, and heat conduction. Physically, at least for weakly coupled fluids, dissipation arises because of the fluid's microscopic constituents' diffusion, which tends to erase any gradients the macroscopic quantities like temperature, velocity field, etc., might have.[1] This process effectively converts the mechanical energy carried by long wavelength perturbations, like sound waves for instance, into thermal energy. Since diffusion is essentially unobstructed when the microscopic constituents are weakly interacting, very weakly coupled fluids are, from the viewpoint of their long-distance hydrodynamical description, the *most dissipative* ones. This is the usual counterintuitive property of shear

[1] Bulk viscosity has a different physical origin. See e.g. [72].

viscosity: at lowest order in perturbation theory, it grows linearly with the mean free path (see [73] for a discussion about this point.) One could think that going in the opposite direction—to very *strong* coupling—might make diffusion and thus dissipation completely unimportant. But, apparently, this is not case. It has been conjectured that there exists an absolute lower bound on the ratio of shear viscosity (η) to entropy density (s) [23, 74],

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \,. \tag{5.1}$$

Interestingly, heavy ion collision data indicate that the quark-gluon plasma has an η to s ratio of the same order as the proposed bound.

Systems violating such a bound have been proposed—see [75] for a recent review—but it is still an open question whether there exists a fundamental bound that is just somewhat lower than (5.1). For example, in [76] it has been argued that Eq. (5.1) can be violated for theories with gravitational duals, but still there is a (somewhat weaker) bound: $\frac{\eta}{s} \geq \frac{16}{25}\frac{1}{4\pi}$, which follows after enforcing causality in the bulk or micro-causality in the boundary CFT. In any case, the mere existence of such a bound still defies a purely field-theoretic justification. Therefore, part of our motivation to characterize dissipation in hydrodynamics in an EFT language, is to derive—if it exists—a fundamental bound from sacred properties of relativistic quantum field theory. For instance, it might follow from unitarity, via dispersion relations [77, 78]. It is sad to report that at this stage we make no progress in this particular direction, and keep it open for future investigation, motivation, and inspiration.

Without further ado, we now discuss how to include dissipative effects in the EFT formulation of hydrodynamics. We will paraphrase a method we learned from [79] where absorptive phenomena in black hole physics were dealt with in an EFT fashion. (See also [80–82] for generalizations, [83] for a somewhat different approach, and [84] for an application of the same techniques in a different context.)

5.1 The general idea

Clearly, a field theory with a local action is non-dissipative by construction.[2] But so is Nature: In any physical system, we call 'dissipation' the transfer of energy from the degrees of freedom we are interested in (collectively denoted by ϕ, in the following) to others which we are not keeping track of (collectively denoted by χ), either because we are not concerned about them, or because describing them is too complicated or impractical. So, the best way to approach dissipation from a field theory viewpoint—at least conceptually—is to keep in mind that these additional degrees of freedom should also appear in the action of the system. That is, if we were to write the full action for ϕ and χ, we would have

$$S[\phi, \chi] = S_0[\phi] + S_\chi[\chi] + S_{\rm int}[\phi, \chi] \ . \tag{5.2}$$

S_0 is the action we would write for ϕ alone, if we forgot about χ. S_χ governs the dynamics of χ. $S_{\rm int}$ couples the two sectors, and is responsible for exchanging energy between them. If we now compute observables involving our ϕ only, we can detect 'dissipative' effects—corresponding to exciting the χ degrees of freedom—which cannot be reproduced by using S_0 alone. For instance, the S-matrix restricted to the ϕ-sector is non-unitary whenever producing χ-excitations is energetically allowed.

In the particular case we are interested in, χ stands for the degrees of freedom of the microscopic constituents making up the fluid. For instance, for a weakly coupled, non-relativistic fluid made up of massive point-particles, χ stands for the positions of these particles. On the other hand, ϕ stands for the collective degrees of freedom, like sound waves for instance, which are those explicitly kept by the hydrodynamical description [3].

[2]The formal trick of adding an explicit time-dependence to a Lagrangian to make the energy not conserved—see e.g. [85]—might work to reproduce the desired dissipative equations of motion, but *(i)* is not systematic, i.e. it is not clear what the rules of the game are, and has therefore no predictive power, and more importantly *(ii)* does not correspond to the physical origin of dissipation, which is that there are additional degrees of freedom that have been ignored.

[3]Strictly speaking, to avoid double counting, one should remove from the χ's the combinations of the

Notice that at all times we are dealing with *one and the same* fluid, and its microscopic constituents. The splitting in Eq. (5.2) is one of the key features of the EFT formalism and the (emergent) dynamics in the long-wavelength limit. Hydrodynamics is about the dynamics of ϕ.

To illustrate the general idea, in this section we will not commit to the hydrodynamical case, nor will we go into many details. Rather, we will keep the discussion as general and as schematic as possible. We will only assume that the interaction Lagrangian $S_{\rm int}$ can be treated as a small perturbation. If this is not the case—if the two sectors are strongly coupled to each other—then it is not even clear how to talk separately of the ϕ-sector and of the χ-sector. In other words, we are assuming that as a first approximation, one *can* neglect the χ's when talking about the ϕ's. For hydrodynamics, as will see, this will be guaranteed by the symmetries: at low frequencies and momenta, *all* the interactions of the ϕ's become negligible, including those with the χ's. Notice that we are not assuming anything about interactions *within* the χ sector: they can be arbitrarily strong.

Now, the crucial question is how to make use of expression (5.2), without actually specifying what the χ's and their dynamics really are. The idea is to make the dependence of the interaction piece $S_{\rm int}$ on ϕ explicit, while keeping that on χ implicit. Schematically:

$$S_{\rm int} = \int d^4x \sum_{n,m} \partial^n \phi^m(x)\, \mathcal{O}_{n,m}(x)\ . \tag{5.3}$$

The $\mathcal{O}$'s are 'composite operators' of the χ-sector—local combinations of the χ's and their derivatives. As usual, one expects all couplings allowed by symmetry to appear in the action. So, in particular, the $\mathcal{O}$'s should carry spacetime and possibly internal indices in order to make the combinations appearing in $S_{\rm int}$ invariant under all the symmetries that act on the ϕ's. Apart from symmetry, as usual in EFT, the other organizational principle in the infinite series (5.3) is the derivative expansion: terms with fewer derivatives acting on the long distance/low energy degrees of freedom (ϕ) matter the most at low energies and momenta.

individual particle positions that make up the ϕ's.

Now, in any observable that involves measuring the ϕ's only—like for instance a $\langle\phi\phi\cdots\phi\rangle$ correlation function, or a $\phi\phi \to \phi\phi$ scattering amplitude—all effects due to the presence of the χ's, dissipative or otherwise, are "mediated" by the correlation functions of these $\mathcal{O}$ composite operators. As an example, consider a coupling (linear in ϕ) between the two sectors of the form

$$S_{\rm int} = \lambda \int d^4x \, \phi \, \mathcal{O} \, , \tag{5.4}$$

where λ is a small coupling constant. For instance, suppose that we are interested in computing the T-ordered two-point function of ϕ in the standard vacuum (i.e. the vacuum for both the ϕ sector and the χ sector, for as we will see in a moment, computing this same correlator with a non-vacuum dissipative χ sector will necessarily complicate the story). This two-point function will receive contributions from S_0 and from $S_{\rm int}$. We can compute the latter contribution in perturbation theory for λ. For instance, if ϕ's only interaction is that contained in $S_{\rm int}$ above, this would correspond to the Feynman diagram series of fig. 1. In that case, neglecting combinatoric factors, powers of i, and momentum-conserving delta-functions, we would have schematically

$$\begin{aligned} \langle\phi(p)\phi(-p)\rangle = {} & \langle\phi(p)\phi(-p)\rangle_0 \\ & + \lambda^2 \langle\phi(p)\phi(-p)\rangle_0{}^2 \, \langle\mathcal{O}(p)\mathcal{O}(-p)\rangle_0 \\ & + \lambda^4 \langle\phi(p)\phi(-p)\rangle_0{}^3 \, \langle\mathcal{O}(p)\mathcal{O}(-p)\rangle_0{}^2 + \dots \, , \end{aligned} \tag{5.5}$$

where T-ordering is understood, and the subscript zeroes denote that those two-point functions are to be computed at zeroth order in λ, that is, in the absence of any interactions between ϕ and χ. Once $\langle\phi\phi\rangle_0$ and $\langle\mathcal{O}\mathcal{O}\rangle_0$ are known, the full $\langle\phi\phi\rangle$ can be computed at any order in λ, without any further explicit reference to the χ dynamics. This is analogous to the standard Feynman-diagram expansion for a perturbative QFT, which involves the free propagators only. Here the correlators on the r.h.s. are not the free ones—they are those determined by S_0 (for ϕ) and by S_χ (for $\mathcal{O}$) separately. In a more general case, where ϕ has

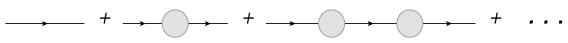


Figure 5.1: *Feynman diagram representation of eq. (5.5): the solid lines represent the ϕ propagators and the gray circles the two-point function of $\mathcal{O}$.*

non-trivial self-interactions and couples to the $\mathcal{O}$'s in a more general way, the right-hand side looks more complicated because it involves higher-point correlation functions of ϕ and $\mathcal{O}$ as well. However, all the correlators are still evaluated at zero coupling (λ) between the two sectors.

As hinted at before, this simple picture gets slightly more complicated for correlation functions in more general states and in particular, thermal states. As we will discuss at some length in the next section, we will be interested in a thermalized χ sector. Its real-time correlation functions and the associated perturbative expansion then have to be handled via the so-called In-In, or Schwinger-Keldysh, formalism (for extensive reviews, see e.g. [86–88]). This entails a doubling of the fields in the path-integral, $\phi \to \phi_\pm$, $\chi \to \chi_\pm$, which complicates somewhat the systematics of the Feynman-diagram expansion. However, for what we are interested in, we can instead consider the effective (linearized) equations of motion for the expectation value of ϕ that we get by "integrating-out" the χ sector via In-In path integrals, which is essentially an In-In generalization of the quantum effective action formalism that is appropriate for systems described by a density matrix.

Following the notation of [86] and utilizing the simple coupling given by (5.4), the In-In generating functional for the correlation functions of ϕ is given schematically by

$$e^{iW[J_+,J_-]} = \text{const} \times \int \mathcal{D}\phi_\pm \mathcal{D}\chi_\pm \tag{5.6}$$

$$e^{i(\pm S_\phi[\phi_\pm] \pm J_\pm \phi_\pm \pm S_\chi[\chi_\pm] \pm \lambda \phi_\pm \mathcal{O}_\pm)} \,, \tag{5.7}$$

where the functional integral over χ_+ and χ_- is understood to include a (thermal) density matrix $\rho(\chi_0^+, \chi_0^-)$ for the initial conditions, which are also integrated over [87,88]. As we will see in a second, we will not need to be explicit about this.

Let's assume that $\langle\mathcal{O}\rangle = 0$ and confine ourselves to quadratic order in the $\phi_\pm$ fields. Noticing that, from the viewpoint of the χ sector, the $\phi_\pm$ fields act as external sources for the operators $\mathcal{O}_\pm$, we can formally perform the functional integration over χ_+ and χ_- and obtain

$$e^{iW[J_+,J_-]} = \text{const} \times \int \mathcal{D}\phi_\pm e^{i(\pm S_2[\phi_\pm]\pm J_\pm\phi_\pm)} e^{\frac{i\lambda^2}{2}\phi^a\mathcal{G}^{ab}_{\mathcal{O}}\phi^b} \ , \tag{5.8}$$

where S_2 is the quadratic action for ϕ, $\phi^a \equiv (\phi_+, -\phi_-)$, and $\mathcal{G}_\mathcal{O}$ is a matrix of $\mathcal{O}\mathcal{O}$ correlators [86]:

$$\mathcal{G}_\mathcal{O}(x_1,x_2) = \begin{pmatrix} \langle T\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle & \langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \\ \langle\mathcal{O}(x_2)\mathcal{O}(x_1)\rangle & \langle\mathcal{O}(x_1)\mathcal{O}(x_2)T\rangle \end{pmatrix} \tag{5.9}$$

(the T to the right of a sequence of operators implies anti-time ordering.) These correlators have to be understood as traces involving the density matrix that is appropriate for the χ sector.

The In-In effective action $\Gamma[\phi_+,\phi_-]$ is then just the Legendre transform of the In-In generating functional $W[J^+,J^-]$, from which the effective equations of motion for $\langle\phi\rangle$ follow simply as [86]

$$\left.\frac{\delta\Gamma}{\delta\phi_+(x)}\right|_{\phi_+=\phi_-=\langle\phi\rangle} = 0 \ , \tag{5.10}$$

However, since we are working at quadratic order in $\phi_\pm$, the effective action Γ is just whatever appears at the exponent in the path integral (5.8) after having set $J^\pm$ to zero:

$$\Gamma_2[\phi_+,\phi_-] = S_2[\phi_+] - S_2[\phi_-] + \tfrac{\lambda^2}{2}\phi^a\mathcal{G}^{ab}_{\mathcal{O}}\phi^b \ , \tag{5.11}$$

where two convolutions are understood for the last term. We thus get that the linear equation of motion for the expectation value of ϕ—which, to keep the notation light, we also call ϕ—is simply

$$\frac{\delta S_2}{\delta\phi} + i\lambda^2\,\langle\mathcal{O}\mathcal{O}\rangle_R * \phi = 0 \ , \tag{5.12}$$

where the second term involves precisely the *retarded* two-point function of $\mathcal{O}$:

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_R \equiv \theta(t_1-t_2)\langle\big[\mathcal{O}(x_1),\mathcal{O}(x_2)\big]\rangle \ . \tag{5.13}$$

Note that the above conforms to the expectations of the usual "linear-response theory" result. What's nice about the In-In formalism is that it allows one to generalize such a result to all orders in perturbation theory in a systematic fashion.

Keeping these qualifications in mind, and coming back to the main message of this section: For generic $S_{\rm int}$, in order to compute observables that involve the ϕ's only—and in particular the time-evolution of $\langle\phi(x)\rangle$—we need not be explicit about the dynamics of the χ's. We 'only' need the n-point correlation functions of the operators the ϕ's couple to. Of course, knowing all such correlators is essentially equivalent to having solved the theory defined by S_χ, which, as we stressed, can be arbitrarily complicated, strongly coupled, or simply unknown. Fortunately, in our particular case of hydrodynamics, we are interested in such correlators at very low frequencies and very long distances only. Moreover, we can assume that the χ sector—whatever it is—is in a state of thermal equilbrium. As we will see, this allows us to parameterize the leading low-frequency, long-distance behavior of the relevant correlators by three coefficients only.

5.2 Low frequency, long distance behavior of correlators

Consider the two-point function for a generic operator in the χ sector, $\langle\mathcal{O}(\vec{x},t)\mathcal{O}(\vec{x}\,',t')\rangle$. If, in the absence of external perturbations—due for instance to our ϕ's—, the χ sector is thermalized, then the average $\langle\dots\rangle$ has to be interpreted as a thermal trace with a density matrix $\rho\propto e^{-\beta H}$, or, in the presence of a conserved charge, $\rho\propto e^{-\beta(H-\mu Q)}$. (We will use a quantum mechanical language, but everything we say applies straightforwardly to classical statistical systems as well.) Now, we will assume that we identified correctly all the degrees of freedom that can propagate at long distances and for long times–we called them ϕ–and that we constructed the most general EFT for them, encoded by $S_0[\phi]$. We will be more

explicit in the next section, but for the moment, it suffices to say that these ϕ's correspond to the degrees of freedom traditionally associated with hydrodynamics: long-wavelength fluctuations in the energy density, in the velocity field, in the charge density, etc. Following the traditional language, we have 'hydrodynamic modes'—i.e. physical variables with non-trivial long-range, late time correlators—for each conserved quantity: energy, momentum, charge. It is usually believed that thermal equilibrium erases all other information that is not associated with conserved charges. In particular, it is usually believed that in a thermal system correlators for quantities that are not densities for conserved charges decay rapidly, faster than any power, at very large distances and at very late times—roughly speaking, at distances and times larger than the mean free path and the mean free time, respectively.

Following this intuition, we will assume that the χ-sector only features such rapidly decaying correlators. As we will see, this does not imply that it does not feature gapless excitations. Indeed: if there were no gapless χ-excitations, it would not be possible for very low frequency ϕ fields to transfer any energy to the χ sector. That is: at frequencies lower than the gap, there would be no dissipation whatsoever. Now, if an $\langle \mathcal{O}\mathcal{O} \rangle$ correlator decays faster than any power at large space- and time-separations, then its Fourier transform

$$G(\omega, \vec{k}\,) \equiv \int d^3x dt\, e^{i(\omega t - \vec{k}\cdot\vec{x})}\, \langle \mathcal{O}(\vec{x}, t)\mathcal{O}(0) \rangle \ , \tag{5.14}$$

is differentiable for real ω and $\vec{k}$—infinitely many times—at $\omega = \vec{k} = 0$. In particular, it admits a Taylor expansion in powers of ω and $\vec{k}$ about the origin. This means that at very low frequencies and momenta, we can parameterize our two-point function by just a few numbers—the coefficients of the leading terms in such a Taylor expansion.

To develop some physical intuition, it is useful to rephrase the above statement in terms of the spectral density for the operator $\mathcal{O}$. So far we have been cavalier about the ordering of operators inside the two-point function. As pointed out in the last section, we will be mostly interested in the retarded two-point function,

$$G_R(\vec{x}, t) \equiv \theta(t) \langle [\mathcal{O}(\vec{x}, t), \mathcal{O}(0)] \rangle \ , \tag{5.15}$$

which describes the causal response of the system to external disturbances, in the sense that adding a term $\int d^3x J\mathcal{O}$ to the Hamiltonian—where $J(\vec{x},t)$ is a given external source—triggers a response in the expectation value of $\mathcal{O}$

$$\langle \mathcal{O}(\vec{x},t)\rangle_J = -i\int_{-\infty}^{\infty} dt' d^3x' G_R(\vec{x}-\vec{x}\,',t-t')\, J(\vec{x}\,',t') + O(J^2) \tag{5.16}$$

(we have assumed that the background expectation value of $\mathcal{O}$ vanishes, i.e $\langle \mathcal{O}(\vec{x},t)\rangle_{J=0} = 0$). Its Fourier transform admits the spectral representation

$$G_R(\omega,\vec{k}) = \int_{-\infty}^{+\infty} \frac{d\omega_0}{\pi} \frac{i}{\omega-\omega_0+i\epsilon}\rho(\omega_0,\vec{k}\,)\ , \tag{5.17}$$

where $\rho(\omega_0,\vec{k}\,)$—the spectral density—is a real, non-negative function (for positive ω_0) that quantifies the density of states the system has at energy ω_0 and momentum $\vec{k}$, weighed by the overlap the operator $\mathcal{O}$ has with them [4].

One is often interested in separating the real and imaginary parts of Fourier-space correlation functions, because they contribute to different phenomena. In particular, the dissipative effects we are after will be associated with the imaginary part of iG_R, which, given the distributional identity

$$\frac{1}{x+i\epsilon} = P\frac{1}{x} - i\pi\,\delta(x)\ , \tag{5.19}$$

is simply the spectral density:

$$\mathrm{Im}\big(iG_R(\omega,\vec{k}\,)\big) = \rho(\omega,\vec{k}\,)\ . \tag{5.20}$$

Our discussion following (5.14) thus implies that the spectral density should be infinitely differentiable for real ω and $\vec{k}$ at $\omega = \vec{k} = 0$, and that it should admit a low-frequency,

[4]The finite-temperature spectral density is given by

$$\rho(\omega,\vec{k}) = \tfrac{1}{2}\left(1-e^{-\beta\omega}\right)\left(\mathrm{Tr}\, e^{-\beta H}\right)^{-1}\sum_{n,m} e^{-\beta E_n} \tag{5.18}$$
$$\times\,(2\pi)^4\delta(\omega+E_n-E_m)\delta^3(\vec{k}+\vec{p}_n-\vec{p}_m)\left|\langle n|\mathcal{O}(0,\vec{0})|m\rangle\right|^2$$

from which the non-negativity (for positive ω) follows immediately.

low-momentum Taylor expansion. Moreover, standard arguments (see e.g. [89]) imply that the imaginary part of iG_R is odd under $\omega \to -\omega$ (while the real part is even), so that in the Taylor expansion of ρ we only have odd powers of ω. The dependence on $\vec{k}$ is constrained by rotational invariance. If $\mathcal{O}$ is a scalar operator, it has to involve 1, $|\vec{k}|^2$, $|\vec{k}|^4$, ...; If $\mathcal{O}$ carries a vector index i, the $\vec{k}$-dependence of the tensor spectral density ρ_{ij} will involve the combinations δ^{ij}, $k^i k^j$, $|\vec{k}|^2\delta^{ij}$, ...; And so on for higher rank tensors. Given these properties, at very low frequencies and momenta, the spectral density of a tensor operator that transforms irreducibly under rotations can be parameterized by just *one* number—the first coefficient in its Taylor expansion:

$$\rho(\omega, \vec{k}\,) \simeq A\,\omega \times \delta \cdots \delta\ , \qquad \omega, k \to 0\ , \tag{5.21}$$

where $\delta \cdots \delta$ stands for the combination of Kronecker-deltas with the right symmetries [5]. Notice that A has to be positive, because ρ is positive for positive ω.

We thus see that the absence of long-range, late-time correlations in the χ sector does *not* forbid the existence of gapless excitations. These can exist, as long as the zero-momentum density of states *(i)* is a regular continuum in a neighborhood of $\omega = 0$, and *(ii)* goes to zero at zero frequency, at least as fast as ω. For instance, a δ-function contribution to the spectral density, peaked at $\omega = 0$, is not allowed. This would correspond to a gapless 'single particle' pole in correlators—i.e. to an excitation with a power-law propagator at very long distances

[5] For any operator $\mathcal{O}$ of given spin s, there is only one possible such combination that can appear in the $\langle \mathcal{O}\mathcal{O} \rangle$ correlator. The reason is that in the tensor product of two spin s representations, the singlet (spin-0) representation appears only once:

$$(2s+1) \otimes (2s+1) = 1 \oplus 3 \oplus \cdots \oplus (4s+1)\ . \tag{5.22}$$

For instance, if $\mathcal{O}_{ij}$ is symmetric and traceless, that is, spin 2, its two point function at zeroth order in $\vec{k}$ has to take the form

$$\langle \mathcal{O}_{ij}\mathcal{O}_{kl} \rangle \propto \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \tfrac{2}{3}\delta^{ij}\delta^{kl}\ . \tag{5.23}$$

and at very late times. According to our assumptions above, this should be included in the ϕ sector.

5.3 The actual couplings: $S_{\rm int}$ to linear order

From Chapter 1 we know the S_0—the fluid action. Let's now be more specific about the structure of $S_{\rm int}$. There is one physical property of the χ's that we have not yet been explicit about: in a sense that we will try to make precise these degrees of freedom "live in the fluid" simply because they "make up" the fluid—they are supposed to describe all the degrees of freedom of the fluid's microscopic constituents that are not explicitly taken into account by the ϕ's. This requirement alone should fix their transformation properties under all the symmetries that act on the ϕ's.

In what follows we will restrict ourselves to the lowest order in the derivative expansion and, more importantly, to linear order in the π fluctuations where, as we shall see shortly, the coupling to the χ sector can be read off from basic properties of Goldstone boson interactions. In order to generalize our results to higher orders in the Goldstone fields, we would need to apply systematically the so-called coset construction to our case. The coset construction allows one to write the most general interactions of Goldstone fields among themselves and with other degrees of freedom (χ, in our case). One only needs to specify the symmetry breaking pattern and the transformation laws of the χ fields under the *unbroken* symmetries, which in our case are suitably redefined translations and rotations. Our physical requirement that the χ's "live in the fluid" should correspond to very specific transformation laws under such symmetries. The general coset construction for internal symmetries has been worked out in [90–92]. It was later generalized to spontaneously broken spacetime symmetries in [93] (see also [94,95] for recent applications). Our case presents yet another twist, in that certain spacetime symmetries mix with internal ones upon spontaneous breaking, in the sense that the unbroken symmetries are specific linear combinations of internal and spacetime ones. We

leave addressing such a systematic construction to future work and content ourselves with the correct linearized description, which we discuss next.

Let us focus first on the case of a fluid without conserved charges. Suppose we start from an equilibrium configuration in which our Goldstones π^I are set zero,

$$\phi_0^I(x) = b_0^{1/3} x^I \ . \tag{5.24}$$

Then, let's turn on a small π^I perturbation,

$$\phi^I(x) = b_0^{1/3} \cdot \big(x^I + \pi^I(x)\big) \ , \tag{5.25}$$

with very mild spatial gradients and time-derivatives. Since π^I appears as an addition to x^I, this is equivalent to performing a small spatial translation of the original equilibrium field configuration (5.24), weakly modulated in space and time:

$$\phi_0^I\big(\vec{x}\big) \to \phi^I(\vec{x}, t) = \phi_0^I\big(\vec{x} + \vec{\pi}(\vec{x}, t)\big) \ . \tag{5.26}$$

We can now be precise about the meaning of "living in the fluid" for the χ sector: if the comoving coordinates ϕ^I are subjected to a weakly modulated spatial translation as in eq. (5.26), the χ degrees of freedom undergo *the same* spatial translation. But, following standard Nöther theorem-type logic, under a modulated spatial translation with parameter $\vec{\pi}(\vec{x}, t)$, the χ action changes by

$$S_\chi[\chi] \to S_\chi[\chi] - \int d^4x \, \partial_\mu \pi^i \, T_\chi^{\mu i} \ , \tag{5.27}$$

where $T_\chi^{\mu i}$ is, *by definition*, the χ sector's contribution to the Nöther current associated with spatial translations, that is, the spatial columns of the χ sector's stress-energy tensor. Therefore we conclude that, at linear order in π^i,

$$S_{\rm int} \simeq - \int d^4x \, \partial_\mu \pi^i \, T_\chi^{\mu i} \qquad \text{(no charges.)} \tag{5.28}$$

Note that $\partial_\mu T_\chi^{\mu i} \neq 0$ and so the above interaction is non-trivial, since we are not including in $T_\chi^{\mu i}$ the π-dependent pieces that are required for conservation of the *total* stress-energy tensor.

A couple of comments about this expression are in order. First, the coupling above, while invariant under spatial translations, rotations, and π-shifts, does not seem to respect the volume-preserving symmetry of eq. (1.5). At linear order this symmetry requires invariance under

$$\vec{\pi}(t,\vec{x}) \to \vec{\pi}(t,\vec{x}) + \vec{\epsilon}\,(\vec{x}) \;, \qquad \vec{\nabla}\cdot\vec{\epsilon} = 0 \;. \tag{5.29}$$

Since the $\vec{\epsilon}$ parameters are time-independent, we note that the 0-component of eq. (5.28) does respect the symmetry, whereas the spatial parts do not. At the moment we have no satisfactory understanding of this issue, but we are confident that (5.28) describes the correct linearized coupling of π to the χ sector, because, as we will see in the next section, it correctly reproduces the first-order dissipative effects of hydrodynamics.

Second, the linear coupling of a Goldstone boson to the associated current—which we motivated via our "living in the fluid" logic—is likely a very general feature of theories with spontaneously broken symmetries [6]. In the Appendix we show that the analog of our coupling holds for a generic theory with a spontaneously broken internal $U(1)$ symmetry, and the logic of that example suggests that analogous results should apply for more generic (internal) symmetry breaking patterns. For spontaneously broken *spacetime* symmetries there will be additional subtleties, but ignoring them for the moment, we are led to postulate that in the case of a fluid *with* conserved charges the leading order interaction Lagrangian will read

$$S_{\rm int} \overset{?}{\simeq} -\int d^4x \left[\partial_\mu \pi^i \, T^{\mu i}_\chi + \, y_0 \, \partial_\mu \pi^0 \, j^\mu_\chi\right] \quad \text{(with charges)}, \tag{5.31}$$

[6] This is *not*—and has no obvious relation with—the usual statement that the current for a spontaneously broken symmetry interpolates the Goldstone particles, in the sense that given a single-Goldstone state $|\vec{p}\rangle$, one has

$$\langle 0|j^\mu|\vec{p}\rangle \neq 0 \;. \tag{5.30}$$

This interpolation property implies that the *full* current has terms that are linear in the Golstone field, e.g. for relativistic theories $j_\mu = f\partial_\mu\pi + \ldots$ Here instead we are focusing on the terms in the current that depend on other fields—our χ's—but not on π, and we are claiming that, in the Lagrangian, the linear coupling of π to this other sector involves precisely this π-independent part of the current.

where π^0 is the Goldstone excitation of ψ, and the associated y_0 factor comes directly from its definition (6.2): j^μ_χ is the current associated with shifts of ψ, and turning on a π^0 field corresponds to shifting ψ by $y_0\pi^0$. As we will see, the second term in (5.31) will turn out not to be the end of the story in this case—hence the question mark—but for the moment, notice that, analogously to the first term, the second term is non-trivial ($\partial_\mu j^\mu_\chi \neq 0$), and *not* invariant under one of the symmetries,

$$\pi^0(t,\vec{x}) \to \pi^0(t,\vec{x}) + a(\vec{x}) \ , \tag{5.32}$$

which is the linearized version of (1.31). We will now check that the first term in (5.31) reproduces correctly the first-order dissipative phenomena associated with bulk and shear viscosity—including the celebrated Kubo relations. On the other hand, we will see that in order to model heat conduction precisely the second term has to be corrected, both in its overall coefficient and in its structure. We leave deriving these corrections from symmetry considerations to future work.

5.4 Rediscovering Kubo relations

As advertised in sect. 5.1, we can now compute observables that involve our Goldstone excitations, and the χ sector will contribute indirectly to these observables only via the correlators of the composite operators that couple to our Goldstones. Since the only couplings that we have so far are linear in the Goldstones, the observables we are able to compute at this point have to do with the free propagation of Goldstone excitations. That is, we are able to compute the Goldstone attenuation rates.

5.4.1 Fluid without charges

Consider first a fluid without conserved charges. Its excitation spectrum—neglecting dissipative effects—is described by the action (6.3), with the $\tilde{\pi}^0$ part omitted. We have a

longitudinal mode $\vec{\pi}_L$ with $\omega = c_s k$, and two transverse modes $\vec{\pi}_T$ with a degenerate dispersion relation $\omega = 0$. Consider now one such excitation propagating in the fluid. Its coupling to the χ sector via the interaction (5.28) will make it slowly decay away, eventually transferring all its initial energy to χ excitations. We can compute the rate at which this decay process takes place at the level of the classical equations of motion for the Goldstones. We could also do the computation at the level of Feynman-diagram perturbation theory, which would be more in line with our field theoretical approach. In particular, since the attenuation rates we are after correspond to imaginary shifts in the excitations' frequencies, we should compute the χ-mediated corrections to the poles of the π^I propagator. However, as reviewed in sect. 5.1, in the in-in formalism each propagator gets replaced by a 2×2 matrix of propagators, which, at least for our simple computation, complicates unneccessarily the systematics of perturbation theory.

Following sect. 5.1, the linearized eom for π^I derived from the Goldstone quadratic action (6.3), augmented by their interaction with the χ sector (5.28), is precisely what one would naively expect from linear response theory [7]:

$$w_0\big(\omega^2\,\pi^i - c_s^2 k^i k^j\,\pi^j\big) + iG_R^{ij}(\omega,\vec{k})\,\pi^j = 0\ , \tag{5.33}$$

where G_R^{ij} is the retarded two-point function of the combination that couples to π^i in (5.28):

$$G_R^{ij}(\omega,\vec{k}) = k_\mu k_\nu \big\langle T_\chi^{\mu i} T_\chi^{\nu j}\big\rangle\ , \tag{5.34}$$

and from now on we will use simply $\langle \dots \rangle$ to denote the Fourier transforms of *retarded* two-point functions, evaluated at ω and $\vec{k}$. Moreover, it will be understood that G_R is evaluated in Fourier space, and its $\omega, \vec{k}$ arguments will be omitted.

In the end we are interested in the imaginary parts of the eigenfrequencies of the system, which—at leading order in perturbation theory—will be related to the imaginary part of iG_R.

[7] We have assumed, as we did in Section 5.1, that $\langle T_\chi^{\mu i}\rangle_{\pi=0}$ vanishes. In our formalism, the equilibrium expectation value for the fluid's *full* stress energy tensor is given by (1.35), evaluated at the equilibrium configuration (6.1), that is, it is fully captured by the ϕ's sector action S_0.

At this point we could parameterize the infrared behavior of $\mathrm{Im}\, i \cdot \langle T^{\mu i}_\chi T^{\nu j}_\chi \rangle$ as described in sect. 5.2, but, before proceeding let us massage this quantity a little in order to rewrite it in a form that the reader familiar with hydrodynamics will recognize. First, notice that according to (5.20), such a quantity is the spectral density of a composite operator ($T^{\mu i}_\chi$) in the χ sector. We argued that all local operators in the χ sector should have very well behaved spectral densities near $\omega = \vec{k} = 0$, at least for real ω and $\vec{k}$, with a Taylor expansion starting as const $\cdot\, \omega$, and continuing with higher powers of ω and $\vec{k}$. At low energies and momenta, we are interested in just that first term, which we can extract formally by taking the nested limit

$$\omega \lim_{\omega\to 0} \left[\tfrac{1}{\omega} \lim_{\vec{k}\to 0} \left(\mathrm{Im}\, i \cdot \langle T^{\mu i}_\chi T^{\nu j}_\chi \rangle \right) \right] . \tag{5.35}$$

Given the regularity of our spectral densities in the infrared, we can take the limits in any order. However, taking the limits in the order we have written them allows us to replace $T^{\mu i}_\chi$ with the *total* $T^{\mu i}$, which includes contributions coming from the Goldstone bosons. The reason is that at lowest order in the χ-π interactions and in the derivative expansion, the Golstones' contribution to *any* spectral density is a Dirac-delta peaked at on-shell values for ω and $\vec{k}$. But the limit in (5.35) carefully dodges such on-shell values, both for longitudinal ($\omega = c_s k$) and for transverse ($\omega = 0$) excitations. At the order we are working we thus have

$$\mathrm{Im}\big(iG^{ij}_R\big) \simeq \omega k_\mu k_\nu \cdot \lim_{\omega\to 0} \tfrac{1}{\omega} \lim_{\vec{k}\to 0} \mathrm{Im}\, i \cdot \langle T^{\mu i} T^{\nu j} \rangle . \tag{5.36}$$

Then, using a standard trick—see e.g. [72]—we can use conservation of the full stress-energy tensor to set to zero terms in the correlator above that have μ or ν equal to zero. The reason is that, because of $T_{\mu\nu}$ conservation, in Fourier space we have the operator equation

$$T^{0\alpha} = \frac{k^k}{\omega} T^{k\alpha} , \tag{5.37}$$

which yields zero if we take $\vec{k}$ to zero first, as we are doing in the limit above [8].

[8] The manipulations we just performed may seem dangerous: in fact, in the last section we insisted that

We are thus left with

$$\mathrm{Im}\big(iG_R^{ij}\big) \simeq \omega k_k k_l \cdot \lim_{\omega\to 0} \tfrac{1}{\omega} \lim_{\vec{k}\to 0} \mathrm{Im}\, i \cdot \big\langle T^{ki} T^{lj} \big\rangle \ . \tag{5.38}$$

Following sect. 5.2, we can now split the stress tensor operator T^{ij} into irreducible representations of the (unbroken) rotation group, spin 0 and spin 2,

$$T^{ij} = T_0^{ij} + T_2^{ij} \tag{5.39}$$

$$T_0^{ij} = \tfrac{1}{3}\delta^{ij}\, T^{kk} \ , \qquad T_2^{ij} = T^{ij} - \tfrac{1}{3}\delta^{ij}\, T^{kk} \ , \tag{5.40}$$

and parameterize the low-energy behavior of the associated spectral densities—in the nested limit we are interested in—via two free parameters $A_{0,2}$ as

$$\mathrm{Im}\, i \cdot \big\langle T_0^{ki} T_0^{lj} \big\rangle \simeq A_0\, \omega \cdot \delta^{ki}\delta^{lj} \tag{5.41}$$

$$\mathrm{Im}\, i \cdot \big\langle T_2^{ki} T_2^{lj} \big\rangle \simeq A_2\, \omega \cdot \big(\delta^{kl}\delta^{ij} + \delta^{kj}\delta^{il} - \tfrac{2}{3}\delta^{ki}\delta^{lj}\big) \ .$$

We should also mention that the mixed correlator $\langle T_0 T_2 \rangle$ vanishes at zero momentum, because of rotational invariance.

Plugging these parameterizations into eq. (5.38) we get

$$\mathrm{Im}\big(iG_R^{ij}\big) \simeq \omega\, k^2 \big[(A_0 + \tfrac{4}{3}A_2) P_L^{ij} + A_2\, P_T^{ij}\big] \ , \tag{5.42}$$

where $P_{L,T}^{ij}$ are the longitudinal and transverse projectors

$$P_L^{ij} = \hat{k}^i \hat{k}^j \ , \qquad P_T^{ij} = \delta^{ij} - \hat{k}^i \hat{k}^j \ . \tag{5.43}$$

The reason it's convenient to split this contribution to the π^i eom as a sum of a longitudinal and a transverse part, is that the zeroth-order eom has a similar structure:

$$\omega^2 \pi^i - c_s^2 k^i k^j\, \pi^j \to \big[(\omega^2 - c_s^2 k^2) P_L^{ij} + \omega^2 P_T^{ij}\big] \pi^j \ . \tag{5.44}$$

is important that the π^i does not couple to the full $T^{\mu i}$, but only to a non-conserved part of it, so that the coupling (5.28) is actually non-trivial. There is no contradiction however: the divergence—or the k^μ—one needs to annihilate the full stress-energy tensor does not commute with our nested limit, so that the r.h.s. in eq. (5.36) is actually non-zero.

Then, putting everything back into eq. (5.33) we get immediately the imaginary parts of the (low-momentum) eigenfrequencies:

$$\Delta\omega_L \simeq -i\,\frac{(A_0+\frac{4}{3}A_2)}{2w_0}k^2 \tag{5.45}$$

$$\Delta\omega_T \simeq -i\,\frac{A_2}{w_0}\,k^2\ . \tag{5.46}$$

These are the attenuation rates for, respectively, the longitudinal and transverse modes. We already see two important predictions of our field theoretical approach. First, the *dissipative* nature of the coupling (5.28): these imaginary frequency shifts have the right sign to make the Goldstone excitations decay in time, since the positivity of $A_{0,2}$ is guaranteed by the positivity properties of *any* spectral density, as reviewed in sect. 5.2. Second, the attenuation rates scale as k^2 at low momenta, which agrees with the standard dissipative hydrodynamics results.

But we can go further. Comparing our attenuation rates to the standard ones in the literature—see e.g. [72]—we find that our parameters $A_{0,2}$ correspond to *bulk and shear viscosity*, usually denoted by ζ and η:

$$\zeta = A_0\ , \qquad \eta = A_2\ . \tag{5.47}$$

Then, our *definitions* of $A_{0,2}$ in eq. (5.41), match precisely the famous Kubo relations for bulk and shear viscosity [72]. This is the main result of this chapter: an independent derivation of the Kubo relations via effective field theory techniques.

From now on, we will refer to the nested limit that we used above as 'the Kubo limit', and we will denote retarded correlators in that limit by

$$\langle\cdots\rangle_{\rm K} \equiv \omega \lim_{\omega\to 0}\tfrac{1}{\omega}\big(\lim_{\vec{k}\to 0}\langle\cdots\rangle\big)\ . \tag{5.48}$$

5.4.2 Fluid with charges

We can now extend the same analysis to the case of a fluid with conserved charges. However, as anticipated, the interaction Lagrangian (5.31) now does not work equally well as for

the case without charges. In particular, it yields the correct $\Delta\omega \sim ik^2$ scaling for the attenuation rates, but it does not reproduce the correct numerical factors in the Kubo relations for the corresponding transport coefficients. We blame this on the fact that in order to guess the second term in (5.31), in the absence of a "living in the fluid"-type argument we applied cavalierly the global symmetry lesson of Appendix 6.15 directly to our case, which involves spontaneously broken *space-time* symmetries, and is therefore of a slightly more subtle nature. We are confident that the coset construction will shed light on this issue. For the moment, let's see whether there exists a minimal generalization of (5.31) that reproduces the correct physics of dissipation in this more general case.

Let's assume that the χ sector still couples to our Goldstones only via the currents $T^{\mu\nu}_\chi$ and j^μ_χ. Then, by (unbroken) rotational and shift-invariance, the possible couplings at lowest order in derivatives are

$$\partial_j\pi^i\, T^{ji}_\chi\ , \quad \partial_0\pi^i\, T^{0i}_\chi\ , \quad \partial_i\pi^i\, j^0_\chi\ , \quad \partial_0\pi^i\, j^i_\chi\ , \tag{5.49}$$

$$\partial_j\pi^0\, T^{j0}_\chi\ , \quad \partial_0\pi^0\, T^{00}_\chi\ , \quad \partial_i\pi^0\, j^i_\chi\ , \quad \partial_0\pi^0\, j^0_\chi\ . \tag{5.50}$$

However, given what we learned above by manipulating the $\langle T_\chi T_\chi\rangle$ correlators in the Kubo limit, we notice that correlators involving T_χ or j_χ with a *zero* index will not contribute to the imaginary parts that we are interested in. For the purposes of our computation, we can thus discard the couplings involving those composite operators. Furthermore, among the surviving couplings, the first is the only one involving the transverse Goldstones π^i_T: the only other possibility is $\partial_0\pi^i\, j^i_\chi$, but recall that at lowest order in the derivative expansion π^i_T has a degenerate dispersion relation $\omega = 0$, which means that, for the transverse modes, such a coupling has to be neglected at the order we are working. By rotational invariance, the transverse modes cannot mix with either π^i_L or π^0, and, given our success above in determining their attenuation rate in the absence of conserved charges, we want to keep their couplings to the χ sector unaltered, i.e., we want $\partial_j\pi^i\, T^{ji}_\chi$ to appear with the same coefficient as in (5.28). For the other couplings, we introduce two arbitrary coefficients.

We thus consider the interaction Lagrangian

$$S_{\rm int} \simeq -\int d^4x \left[\partial_j \pi^i \, T_\chi^{ji} + B\, \partial_i \pi^0 \, j_\chi^i + C\, \partial_0 \pi^i \, j_\chi^i\right] , \tag{5.51}$$

and determine the values of B and C by computing the attenuation rates and matching these to known results. We can focus on the π_L-π_0 sector only, since for the transverse modes we have the same couplings as before, and the same analysis applies unaltered. On the other hand, when we consider the π_L and π_0 equations of motion, we are sensitive to new (retarded) correlators,

$$\mathrm{eom}(\pi^\alpha) \supset i G_R^{\alpha\beta} \pi_\beta \, , \tag{5.52}$$

with

$$\mathrm{Im}\big(iG_R^{00}\big) \simeq B^2 \, k^i k^j \, \mathrm{Im}\, i \cdot \langle j^i j^j \rangle_{\rm K} \tag{5.53}$$

$$\mathrm{Im}\big(iG_R^{0L}\big) \simeq -BC \, \omega k^i \hat{k}^j \, \mathrm{Im}\, i \cdot \langle j^i j^j \rangle_{\rm K} \tag{5.54}$$

$$\begin{aligned} \mathrm{Im}\big(iG_R^{LL}\big) \simeq{}& k^k k^l \, \hat{k}^i \hat{k}^j \, \mathrm{Im}\, i \cdot \langle T^{ki} T^{lj} \rangle_{\rm K} \\ &+ C^2 \, \omega^2 \hat{k}^i \hat{k}^j \mathrm{Im}\; i \cdot \langle j^i j^j \rangle_{\rm K} \, . \end{aligned} \tag{5.55}$$

Following the same logic as in the case without conserved charges, we have replaced the χ-sector's current and stress-tensor with the total ones. We have also used that the mixed correlator $\langle j^i T^{jk} \rangle$ vanishes at zero momentum by rotational invariance.

According to the general discussion of sect. 5.2, the imaginary parts of the $\langle jj \rangle_{\rm K}$ and $\langle TT \rangle_{\rm K}$ correlators have to scale as ω. As we already saw, in the TT case this matches the Kubo relations that determine bulk and shear viscosity—see eqs. (5.41), (5.47). In the jj case, there is an analogous Kubo relation [96], which relates the *heat conductivity* χ to the coefficient of ω

$$\mathrm{Im}\, i \cdot \langle j^i j^j \rangle_{\rm K} = \chi T \left(\tfrac{n}{\rho+p}\right)^2 \omega \cdot \delta^{ij} \, . \tag{5.56}$$

Putting everything together, we can rewrite the imaginary parts of the iG_R entries as

$$\mathrm{Im}\big(iG_R^{00}\big) = \chi T\big(\tfrac{n}{\rho+p}\big)^2 B^2\,\omega k^2 \tag{5.57}$$

$$\mathrm{Im}\big(iG_R^{0L}\big) = -\chi T\big(\tfrac{n}{\rho+p}\big)^2 BC\,\omega^2 k \tag{5.58}$$

$$\mathrm{Im}\big(iG_R^{LL}\big) = \big(\zeta + \tfrac{4}{3}\eta\big)\,\omega k^2 + \chi T\big(\tfrac{n}{\rho+p}\big)^2 C^2\,\omega^3\ . \tag{5.59}$$

To find the imaginary shifts of the eigenfrequencies, it is convenient to re-express the iG_R matrix in the $\tilde{\pi}^0$-π_L basis that diagonalizes the lowest-order quadratic Lagrangian (6.3). Given the definition of $\tilde{\pi}^0$, eq. (6.4), and the structure of our interaction Lagrangian, eq. (5.51), this amounts to just replacing

$$C \to C - BA\,\tfrac{k^2}{\omega^2}\ , \qquad A \equiv \tfrac{F_{by}b_0 - F_y}{F_{yy}y_0} \tag{5.60}$$

in the expressions for iG_R above:

$$\mathrm{Im}\big(iG_R^{\tilde{0}\tilde{0}}\big) = \chi T\big(\tfrac{n}{\rho+p}\big)^2 B^2\,\omega k^2 \tag{5.61}$$

$$\mathrm{Im}\big(iG_R^{\tilde{0}L}\big) = -\chi T\big(\tfrac{n}{\rho+p}\big)^2 BC\,\omega^2 k + \chi T\,B^2 A\,k^3 \tag{5.62}$$

$$\mathrm{Im}\big(iG_R^{LL}\big) = \big(\zeta + \tfrac{4}{3}\eta\big)\,\omega k^2 + \chi T\big(\tfrac{n}{\rho+p}\big)^2 \big(C - BA\tfrac{k^2}{\omega^2}\big)^2\,\omega^3\ . \tag{5.63}$$

We thus get that the eigenfrequencies of the system—identified by the vanishing of the determinant of the matrix defining the eom—get shifted by

$$\Delta\omega_{\tilde{0}} \simeq -i\cdot\chi T\big(\tfrac{n}{\rho+p}\big)^2 B^2\,\big(\tfrac{1}{F_{yy}y_0^2} - \tfrac{A^2}{w_0\,c_s^2}\big)\,k^2 \tag{5.64}$$

$$\Delta\omega_L \simeq -i\cdot\Big[(\zeta + \tfrac{4}{3}\eta)\tfrac{1}{2w_0} + \chi T\big(\tfrac{n}{\rho+p}\big)^2 \tfrac{(c_s^2 C - BA)^2}{w_0\,c_s^2}\Big]\,k^2\ , \tag{5.65}$$

where we have kept only the leading order in k.

We are now in a position to match our computations to the classic dissipative fluid results, which, in the case of a fluid with conserved charges, are quite messy—see e.g. [97]. At low

momenta, the attenuation rates for the scalar modes are [9]

$$\Delta\omega_{\text{heat}} \simeq -i \cdot \chi \, \tfrac{n\,(\partial p/\partial n)_T}{(\rho+p)(\partial\rho/\partial T)_n} \, k^2 \tag{5.66}$$

$$\begin{aligned}\Delta\omega_{\text{sound}} \simeq -i \cdot \tfrac{1}{2(\rho+p)} \Big[&(\zeta + \tfrac{4}{3}\eta) + \chi(\partial\rho/\partial T)_n^{-1} \\ &\times \Big((\rho+p) - 2T \tfrac{\partial p}{\partial T}\big|_n + c_s^2 T \tfrac{\partial \rho}{\partial n}\big|_T - \tfrac{n}{c_s^2}\tfrac{\partial p}{\partial n}\big|_T \Big) \Big] \, k^2\end{aligned} \tag{5.67}$$

After a messy computation (see Appendix 6.16) one can see that, despite their complexity, these expressions agree with ours above if we simply choose

$$B = -C = -\frac{y_0 w_0}{b_0 F_b} \, . \tag{5.68}$$

That is, the two coefficients in (5.51) have in fact the same value (up to a sign), which in hydrodynamical/thermodynamical terms is simply the combination $\mu(\rho+p)/Ts$. The emergence of such a simple final result from a long series of fairly messy intermediate steps, gives us confidence in the correctness of (5.51) with this particular choice of coefficients. This completes our matching computation.

5.5 Summary & Outlook

From purely symmetry arguments and the principles of EFT, we were able to derive that the coupling of hydrodynamical modes to a generic thermalized sector that "lives in fluid" yields dissipation, with attenuation rates scaling as k^2. This matches well known features of dissipative effects in hydrodynamics, and is quite independent of the precise structure of the couplings that we would write down, following essentially from the thermal nature of this extra sector. For fluids without conserved charges, the living-in-the-fluid requirement is strong enough to determine—via symmetry considerations—the precise structure of the

[9] In fact, ref. [97] computes only the attenuation rate for the sound mode. However, following that paper's derivation it is easy to spot another scalar mode, which corresponds to our $\tilde{\pi}_0$ and which we call the "heat mode," with an attenuation rate as given below.

interactions, thus allowing us to *re-derive* Kubo relations. For fluids *with* conserved charges, we adopted a "symmetry-inspired" ansatz for the interactions, with two free parameters, which we determined via matching a procedure. The emergence of a remarkably simple value for them, eq. (5.68), from a long series of fairly cumbersome intermediate steps, gives us confidence in the validity of the arguments behind our ansatz.

We feel that these are significant accomplishments. However, for our techniques to be a useful tool rather than simply an alternative derivation of well-know features of hydrodynamics, we need to extend our analysis beyond linear order, and, more importantly, we need to understand the systematics of the symmetry structure of the dissipative couplings. As we already emphasized, we believe that the coset construction will help us in these directions. Until then, we are left with a puzzle: Our couplings to the χ sector do not preserve all the symmetries we started with, in particular the volume preserving diffs (5.29), and the "modulated" shift (5.32). Because dissipation involves time derivatives—the factor of ω determining the infrared behavior of spectral densities—, and the symmetries above are time-independent, this was not an issue on the way to reproduce known results. However, symmetry breaking terms can be generated, for example if we compute the effects associated with the real parts of our correlators, e.g. Re $i\langle T^{ij}_\chi T^{kl}_\chi\rangle$. If these have frequency-independent pieces, they can yield symmetry-violating terms in the Goldstone effective action, like for instance a gradient energy for the transverse modes, in contradiction with standard properties of hydrodynamics. Unfortunately, from standard analytic properties of retarded Green's functions one can derive a dispersion relation of the form (see e.g. [89])

$$\mathrm{Re}\big(iG_R(\omega_0,0)\big) = -\int_{-\infty}^{\infty} \frac{\mathrm{Im}\big(iG_R(\omega,0)\big)}{\omega-\omega_0}\frac{d\omega}{\pi}\ . \tag{5.69}$$

The RHS is *strictly* negative-definite when we take $\omega_0 \to 0$—because of the strict positivity of spectral densities—which leads us to the result

$$\lim_{\omega\to 0}\ \mathrm{Re}\big(iG_R(\omega)\big) < 0\ . \tag{5.70}$$

That is, the unwanted frequency-independent pieces are in fact forced to be there. One could in principle add symmetry-violating local counter-terms to cancel out the undesired, symmetry-violating contributions from the real parts of our Green's functions (these are analytic in frequency and momentum, and therefore local in position space). But this would correspond to fine-tuning certain Lagrangian terms to zero, which would go against the whole point of insisting on symmetries as the guiding principle to construct effective field theories: the only robust properties of physical systems should be those ensured by symmetries.

We conclude with an intriguing application of our results to black-hole physics. Ref. [79]—from which we borrowed the techniques of sect. 5.1—analyzed dissipative effects in the dynamics of black holes, starting from a matching computation involving the absorption of gravitational waves by a black hole in isolation. A classical GR computation for such a process yields—in the language of sects. 5.1 and 5.2—a spectral density for the relevant composite operator scaling as ω at low frequencies [79]. Our arguments of sect. 5.2 show that such a behavior is characteristic of spectral densities for local operators in thermal systems. That is, a black hole absorbs like a thermal system. This is yet another indication of the thermal nature of black holes. What is remarkable is that such an implication here follows from a purely classical computation in GR.

Outlook

We hope the reader feels that in the previous pages some amount of progress has been made in understanding, utilizing, simplifying or calculating observables in fluid (and solid) dynamics. Above all we hope that, if anything, the readers of this thesis view this body of work as proof that effective field theory thinking applied to fluid dynamics can provide us with new possible handles to investigate this ubiquitous and rich system. A detailed Summary & Outlook section relevant to each individual subtopic is included in the ends of Chapters 2-5. For particulars we refer the reader to these sections. By construction, such discussions lacked a more global picture. We try and offer that here.

The arguments and discussions throughout this thesis can be organized into two categories: partial progress and sharp results.

In our mind, the sharp results are clear proof of the usefulness of our techniques. For instance, for the solid inflation model introduced in Chapter 4 we were able to not only to calculate everything, but we were able to confirm theoretical consistency all the way through inflation. Yes we can add more ingredients like spectator fields and yes we can cook up a more complicated phase transition into a hot big bang. But while we can freely complicate the picture (we are pretty much always free to do this, of course) the point remains the same. Our symmetry breaking pattern gives, in general, a completely different picture of the possible dynamics of inflation. Really we have discovered, if that is an appropriate word, an orthogonal EFT of inflation to the usual one. The question remains: does this have

anything to do with nature? The model is restrictive enough that future data will hopefully be able to tell us, but as far as theory is concerned solid inflation in its simplest form is *done*—everything interesting has been computed and the theoretical consistency confirmed. The EFT description was key in doing both of these things easily.

Similarly, our efforts collected in Chapter 3 make precise and systematic the expansion around the incompressible limit. This has allowed us to easily recover some classic results in the theory of vortex-sound dynamics in addition to calculating new ones. In fact, if we can be so bold, we would argue that our construction is *the* way to understand sound-flow interaction. Not only is it conceptually cleaner but it also comes prepackaged with the powerful machinery of QFT. This allows us to deal with divergencies that naturally arise in higher order perturbation theory. Divergencies that would not be easily handled from the equation of motion perspective. Even just the simple fact that we easily extended the vortex filament model to include interactions with sound would be nontrivial progress in its own right. The real unanswered question here is: now that we have all this technology what can we do with it?

What about the partial results? In Chapter 2 we asked a simple question "Why are there no normal fluids at zero temperature?" and were greeted by a host of complications and confusions. In some sense this was a completely natural thing to encounter. Our expectation was that the answer to our question may be that we find no normal fluids at zero temperature simply because something was quantum mechanically *inconsistent* about such a theory. So while we have indications that this is indeed the case, unfortunately for us, we have nothing completely precise. Sometimes we even had convincing arguments that were seemingly contradictory; there clearly is lots more to think about. But aside from confirming that our quantum mechanical confusions remain when we include a conserved charge we don't really offer a concrete next step. Should we be ashamed? We don't think so. This is an important question, and even though we ended our analysis with more questions than the

one we began with, we uncovered previously unknown puzzles—and that is something! If we demonstrated anything at all it is that as a quantum field theory perfect fluid dynamics deserves much more study.

Similarly, there seems to be something deep happening in Chapter 5 when we try and implement the usual program for incorporating dissipation in the EFT language. First, we can't seem to construct the necessary non-linear coupling. But even more interestingly, when we gave up and restricted ourselves to first order couplings we found that we *must* introduce terms that *break* the very symmetry which we were insisting on! Most likely this is not a mistake however because it is precisely these uncomfortable terms which are necessary to beautifully re-derive the Kubo relations. Contrary to the other simple models that we have been considering (like the dissipative degrees of freedom in black holes and our $U(1)$ Goldstone) maybe there is something about the symmetry breaking pattern of the fluid that forces higher order dissipative effects to explicitly break the very symmetries dictating the lowest order dynamics. We do not know yet, but the author feels that this is the most outstanding problem in this EFT description of fluid dynamics. We have made progress in understanding dissipation, but that progress has led us to the position that something even more interesting than we expected must be going on. The resolution of this important puzzle is of great importance and justifies much more effort.

We are enamored by fluids. Their rich complicated structure seemingly holds endless mysteries and beauty. But more importantly, fluids are *real*. The issues that we raise cannot be explained away by concluding that the random theory we cooked up is just mathematically bad. This is not an option for us. Fluids are there and at lowest order they are described by these degrees of freedom and these symmetries. Period. We have to face these issues directly, and we hope that the work described in this thesis represents a brave attempt. There is still so much more to do and we are filled with excitement for the future.

Bibliography

[1] S. Endlich, A. Nicolis, R. Rattazzi and J. Wang, "The Quantum mechanics of perfect fluids," JHEP **1104**, 102 (2011) [arXiv:hep-th/1011.6396].

[2] S. Endlich and A. Nicolis, "The incompressible fluid revisited: vortex-sound interactions ," [arXiv:hep-th/1303.3289].

[3] S. Endlich "Higher order computations in vortex-sound interactions, " in preparation.

[4] S. Endlich, A. Nicolis, and J. Wang, "Solid Inflation," [arXiv:hep-th/1210.0569].

[5] S. Endlich, A. Nicolis, R. A. Porto and J. Wang, "Dissipation in the effective field theory for hydrodynamics: First order effects," [arXiv:hep-th/1211.6461].

[6] S. Endlich, K. Hinterbichler, L. Hui, A. Nicolis, and J. Wang, "Derrick's theorem beyond a potential," JHEP **05**, 73 (2011) [arXiv:hep-th/1002.4873].

[7] S. Endlich and J. Wang, "Classical Stability of the Galileon ," JHEP **1111** 065 (2011) [arXiv:hep-th/1106.1659].

[8] D. Soper "Classical Field Theory," *Wiley (1976) 259p.*

[9] G. Herglotz, *Ann. Phys.* **36**, 493 (1911).

[10] S. Dubovsky, T. Gregoire, A. Nicolis and R. Rattazzi, "Null energy condition and superluminal propagation," JHEP **0603**, 025 (2006) [arXiv:hep-th/0512260].

[11] W. Israel and J. M. Stewart, "Transient relativistic thermodynamics and kinetic theory, Annals Phys. 118, 341 (1979).

[12] S. Dubovsky, L. Hui, A. Nicolis and D. T. Son, "Effective field theory for hydrodynamics: thermodynamics, and the derivative expansion," Phys. Rev. D **85**, 085029 (2012) [arXiv:hep-th/1107.0731].

[13] S. Sibiryakov, unpublished.

[14] A. Nicolis, "Low-energy effective field theory for finite-temperature relativistic superfluids," [arXiv:hep-th/1108.2513].

[15] E. M. Lifshitz and L. P. Pitaevskii, "Statistical physics. Part 2: Theory of the condensed state," *Oxford, UK: Butterworth-Heinemann (1980) 387p.*

[16] D. Nickel and D. T. Son, "Deconstructing holographic liquids," [arXiv:hep-th/1009.3094].

[17] M. Edalati, J. I. Jottar and R. G. Leigh, "Transport coefficients at zero temperature from extremal black holes," JHEP **1001**, 018 (2010) [arXiv:hep-th/0910.0645].

[18] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, "Ghost condensation and a consistent infrared modification of gravity," JHEP **0405**, 074 (2004) [arXiv:hep-th/0312099].

[19] M. E. Peskin and D. V. Schroeder, "An introduction to quantum field theory," *Reading, USA: Addison-Wesley (1995) 842 p.*

[20] S. R. Coleman, "There are no Goldstone bosons in two-dimensions," Commun. Math. Phys. **31**, 259 (1973).

[21] E. Witten, "Chiral symmetry, the $1/N$ expansion, and the $SU(N)$ Thirring model," Nucl. Phys. B **145**, 110 (1978).

[22] S. Dubovsky and S. Sibiryakov, "Superluminal travel made possible (in two dimensions)," JHEP **0812**, 092 (2008) [arXiv:hep-th/0806.1534].

[23] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231].

[24] L. D. Landau and E. M. Lifshitz, "Fluid Mechanics," Pergamon Press (1987) 539p.

[25] M. J. Lighthill, "On sound generated aerodynamically I. General theory," Proc. Roy. Soc. London **A211**, 564-587 (1952).

[26] M. J. Lighthill, "On sound generated aerodynamically II. Turbulence as a source of sound," Proc. Roy. Soc. London **A222**, 1-32 (1954).

[27] Y. Auregan, A. Maurel, V. Pagneux, and J. -F. Pinton (Eds), "Sound-Flow Interactions," Springer-Verlag Berlin Heidelberg (2002) 286p.

[28] M. S. Howe, "Theory of Vortex Sound," Cambridge Texts in Applied Mathematics (No. 33) (2002) 232p.

[29] S. Weinberg, "Gravitation and Cosmology," John Wiley & Sons (1972) 657p.

[30] W. D. Goldberger and I. Z. Rothstein, "An Effective Field Theory of Gravity for Extended Objects," Phys. Rev. D **73**: 104029 (2006) [arXiv:hep-th/0409156v2].

[31] C. F. Barenghi and R. J. Donnelly, "Vortex rings in classical and quantum systems," Fluid Dyn. Res. **41** (2009) 051401.

[32] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," Cambridge, UK: Univ. Pr. (1995) 609 p.

[33] W. D. Goldberger and A. Ross, "Gravitational radiative corrections from effective field theory," Phys. Rev. D **81**:124015 (2010).

[34] F. Lund and C. Rojas, "Ultrasound as a Probe of Turbulence," Physica D **37**: 508-514 (1989).

[35] W. D. Goldberger, "Les Houches Lectures on Effective Field Theories and Gravitational Radiation," (2006) [arXiv:hep-ph/0701129].

[36] D. Kleckner and W. T. M. Irvine, "Creation and dynamics of knotted vortices," Nature Physics (2013) DOI: 10.1038/NPHYS2560.

[37] See video at `http://www.youtube.com/watch?v=TMCf7SNUb-Q`.

[38] `http://travel.aol.co.uk/2012/12/13/dolphins-blowing-bubbles-italy/` .

[39] `http://www.linc.ox.ac.uk/Undergraduates-SubjectsMathematics` .

[40] K. W. Schwarz, "Three-dimensional vortex dynamics in supefluid He4: lineline and lineboundary interaction," Phys. Rev. B **31** 5782-5804 (1985).

[41] K. W. Schwarz, "Three-dimensional vortex dynamics in superfluid He4: homogeneous superfluid turbulence," Phys. Rev. B **38** 2398-2417 (1988).

[42] W. Thomson (Lord Kelvin), Philos Mag. 10, 155 (1880).

[43] M. Ruderman, "Pulsar Spin, Magnetic Fields, and Glitches," in *Neutron Starts and Pulsars*, W. Becker editor, Springer Berlin Heidelberg (2009).

[44] M. Srednicki, "Quantum field theory," Cambridge, UK: Univ. Pr. (2007) 641 p.

[45] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Reading, USA: Addison-Wesley (1995) 842 p.

[46] P. Creminelli, M. A. Luty, A. Nicolis and L. Senatore, "Starting the Universe: Stable Violation of the Null Energy Condition and Non-Standard Cosmologies," JHEP **0612** (2006) 080 [arXiv:hep-th/0606090].

[47] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan and L. Senatore, "The Effective Field Theory of Inflation," JHEP **0803** (2008) 014 [arXiv:hep-th/0709.0293].

[48] D. Babich, P. Creminelli and M. Zaldarriaga, "The Shape of non-Gaussianities," JCAP **0408**, 009 (2004) [arXiv:astro-ph/0405356].

[49] A. Gruzinov, "Elastic Inflation," Phys. Rev. D **70** (2004) 063518 [arXiv:astro-ph/0404548].

[50] M. Bucher and D. N. Spergel, "Is the dark matter a solid?," Phys. Rev. D **60**, 043505 (1999) [arXiv:astro-ph/9812022].

[51] P. W. Graham, B. Horn, S. Kachru, S. Rajendran and G. Torroba, "A Simple Harmonic Universe," [arXiv:hep-th/1109.0282].

[52] H. Leutwyler, "Phonons as goldstone bosons," Helv. Phys. Acta **70**, 275 (1997) [arXiv:hep-ph/9609466].

[53] R. Rattazzi and A. Zaffaroni, "Comments on the holographic picture of the Randall-Sundrum model," JHEP **0104**, 021 (2001) [arXiv:hep-th/0012248].

[54] J. Polchinski, "Scale And Conformal Invariance In Quantum Field Theory," Nucl. Phys. B **303**, 226 (1988).

[55] M. A. Luty, J. Polchinski and R. Rattazzi, "The a-theorem and the Asymptotics of 4D Quantum Field Theory," [arXiv:hep-th/1204.5221].

[56] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, "Causality, analyticity and an IR obstruction to UV completion," JHEP **0610**, 014 (2006) [arXiv:hep-th/0602178].

[57] N. Arkani-Hamed, P. Creminelli, S. Mukohyama and M. Zaldarriaga, "Ghost inflation," JCAP **0404**, 001 (2004) [arXiv:hep-th/0312100].

[58] S. L. Dubovsky, "Phases of massive gravity," JHEP **0410**, 076 (2004) [arXiv:hep-th/0409124].

[59] J. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," JHEP 0305 (2003) 013 [arXiv:astro-ph/0210603v5].

[60] P. Creminelli and M. Zaldarriaga, "Single field consistency relation for the 3-point function," JCAP **0410**, 006 (2004) [arXiv:astro-ph/0407059].

[61] L. -M. Wang and M. Kamionkowski, "The Cosmic microwave background bispectrum and inflation," Phys. Rev. D **61**, 063504 (2000) [arXiv:astro-ph/9907431].

[62] P. Creminelli, A. Nicolis, L. Senatore, M. Tegmark and M. Zaldarriaga, "Limits on non-gaussianities from wmap data," JCAP **0605**, 004 (2006) [arXiv:astro-ph/0509029].

[63] L. Senatore, K. M. Smith and M. Zaldarriaga, "Non-Gaussianities in Single Field Inflation and their Optimal Limits from the WMAP 5-year Data," JCAP **1001**, 028 (2010) [arXiv:astro-ph/0905.3746].

[64] S. Weinberg, "Adiabatic modes in cosmology," Phys.Rev. D67 (2003) 123504 [arXiv:astro-ph/0302326].

[65] S. Weinberg, "The Cosmological Constant Problem," Rev. Mod. Phys. **61**, 1 (1989).

[66] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, "Spontaneous Creation of Almost Scale - Free Density Perturbations in an Inflationary Universe," Phys. Rev. D **28**, 679 (1983).

[67] D. H. Lyth, K. A. Malik, M. Sasaki "A general proof of the conservation of the curvature perturbation" JCAP 0505:004, 2005 [arXiv:astro-ph/0411220v3].

[68] D. Langlois, F. Vernizzi "Conserved nonlinear quantities in cosmology" Phys. Rev. D **72** (2005) 103501 [arXiv:astro-ph/0509078v3].

[69] A. F. Andreev and I. M. Lifshitz, Sov. Phys. JETP **29** (1969) 1107.

[70] D. T. Son, "Effective Lagrangian and topological interactions in supersolids," Phys. Rev. Lett. **94**, 175301 (2005) [arXiv:cond-mat/0501658].

[71] D. T. Son and P. Surowka, "Hydrodynamics with Triangle Anomalies," Phys. Rev. Lett. **103**, 191601 (2009) [arXiv:hep-th/0906.5044].

[72] S. Jeon, "Hydrodynamic Transport Coefficients in Relativistic Scalar Field Theory," Phys. Rev. D **52** (1995) 3591 [arXiv:hep-ph/9409250].

[73] D. T. Son and A. O. Starinets, "Viscosity, Black Holes, and Quantum Field Theory," Ann. Rev. Nucl. Part. Sci. **57** (2007) 95 [arXiv:hep-th/0704.0240].

[74] P. Danielewicz and M. Gyulassy, "Dissipative Phenomena in Quark Gluon Plasmas," Phys. Rev. D **31**, 53 (1985).

[75] S. Cremonini, "The Shear Viscosity to Entropy Ratio: a Status Report," Mod. Phys. Lett. B **25** (2011) 1867 [arXiv:hep-th/1108.0677].

[76] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, "The Viscosity Bound and Causality Violation," Phys. Rev. Lett. **100** (2008) 191601 [arXiv:hep-th/0802.3318].

[77] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, "Causality, analyticity and an IR obstruction to UV completion," JHEP **0610**, 014 (2006) [arXiv:hep-th/0602178].

[78] J. Distler, B. Grinstein, R. A. Porto and I. Z. Rothstein, "Falsifying Models of New Physics via WW Scattering," Phys. Rev. Lett. **98**, 041601 (2007) [arXiv:hep-ph/0604255].

[79] W. D. Goldberger and I. Z. Rothstein, "Dissipative effects in the worldline approach to black hole dynamics," Phys. Rev. D **73**, 104030 (2006) [arXiv:hep-th/0511133].

[80] R. A. Porto, "Absorption effects due to spin in the worldline approach to black hole dynamics," Phys. Rev. D **77**, 064026 (2008) [arXiv:hep-th/0710.5150].

[81] D. Lopez Nacir, R. A. Porto, L. Senatore and M. Zaldarriaga, "Dissipative effects in the Effective Field Theory of Inflation," JHEP **1201**, 075 (2012) [arXiv:hep-th/1109.4192].

[82] D. Lopez Nacir, R. A. Porto and M. Zaldarriaga, "The consistency condition for the three-point function in dissipative single-clock inflation," JCAP **1209**, 004 (2012) [arXiv:hep-th/1206.7083].

[83] C. R. Galley, "The classical mechanics of non-conservative systems," [arXiv:arqc/1210.2745].

[84] P. Meade, N. Seiberg and D. Shih, "General Gauge Mediation," Prog. Theor. Phys. Suppl. **177**, 143 (2009) [arXiv:hep-ph/0801.3278].

[85] H. Goldstein, C. Poole and J. Safko, "Classical mechanics," *San Francisco: Addison Wesley (2002) 638 p.*

[86] R. D. Jordan, "Effective Field Equations for Expectation Values," Phys. Rev. D **33**, 444 (1986).

[87] J. Maciejko, "An Introduction to Nonequilibrium Many-Body Theory," available for download at:

http://www.physics.arizona.edu/∼stafford/Courses/560A/nonequilibrium.pdf.

[88] E. Calzetta and B. L. Hu, "Nonequilibrium Quantum Fields: Closed Time Path Effective Action, Wigner Function and Boltzmann Equation," Phys. Rev. D **37**, 2878 (1988).

[89] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," Class. Quant. Grav. **26**, 224002 (2009) [arXiv:hep-th/0903.3246].

[90] S. Weinberg, "Nonlinear realizations of chiral symmetry," Phys. Rev. **166**, 1568 (1968).

[91] S. R. Coleman, J. Wess and B. Zumino, "Structure of phenomenological Lagrangians. 1.," Phys. Rev. **177**, 2239 (1969).

[92] C. G. Callan, Jr., S. R. Coleman, J. Wess and B. Zumino, "Structure of phenomenological Lagrangians. 2.," Phys. Rev. **177**, 2247 (1969).

[93] D. V. Volkov, "Phenomenological Lagrangians," Fiz. Elem. Chast. Atom. Yadra **4**, 3 (1973).

[94] C. Armendariz-Picon, A. Diez-Tejedor and R. Penco, "Effective Theory Approach to the Spontaneous Breakdown of Lorentz Invariance," JHEP **1010**, 079 (2010) [arXiv:hep-ph/1004.5596].

[95] G. Goon, K. Hinterbichler, A. Joyce and M. Trodden, "Galileons as Wess-Zumino Terms," [arXiv:hep-th/1203.319].

[96] D. T. Son and A. O. Starinets, "Hydrodynamics of r-charged black holes," JHEP **0603**, 052 (2006) [arXiv:hep-th/0601157].

[97] S. Weinberg, "Entropy generation and the survival of protogalaxies in an expanding universe," Astrophys. J. **168**, 175 (1971).

Chapter 6

Appendix

6.1 Perturbations in a fluid with conserved charge

In the following, we will consider small perturbations about a homogeneous equilibrium configuration, which, generalizing (1.2) to arbitrary pressure and to the case with charge, is described in our language by the field configuration

$$\phi_0^I(x) = b_0^{1/3}\, x^I \ , \qquad \psi_0(x) = y_0\, t \ , \tag{6.1}$$

where b_0 and y_0 are the fluid's equilibrium entropy density and chemical potential. Such a configuration spontaneously breaks many of our symmetries: Lorentz boosts, completely; Spatial translations and internal ϕ^I-shifts (eq. (1.3)), down to the diagonal combination; Spatial and internal ϕ^I-rotations (eq. (1.4)), down to the diagonal combination; Internal volume-preserving diffs (eq. (1.5)), completely; Time-translations and internal ψ-shifts (eq. (1.30)), down to the diagonal combination; ϕ^I-dependent ψ-diffs (eq. (1.31)), completely. Associated with this spontaneous symmetry breaking pattern there are Goldstone excitations, which are simply fluctuations of our fields about the equilibrium configuration,

$$\phi^I(x) = b_0^{1/3}\big(x^I + \pi^I\big) \ , \qquad \psi(x) = y_0\big(t + \pi^0\big) \ . \tag{6.2}$$

Not all the π's feature propagating wave solutions. Indeed, after expanding the Lagrangian (1.33) to quadratic order in fluctuations and diagonalizing it, we get [14]

$$\mathcal{L} \simeq \tfrac{1}{2} w_0 \big(\dot{\vec{\pi}}_L^2 - c_s^2 (\vec{\nabla} \cdot \vec{\pi}_L)^2 \big) + \tfrac{1}{2} w_0 \, \dot{\vec{\pi}}_T^2 + \tfrac{1}{2} F_{yy} y_0^2 \, (\dot{\tilde{\pi}}^0)^2 \ , \tag{6.3}$$

where $\vec{\pi}_L$ and $\vec{\pi}_T$ are the longitudinal (curl-free) and transverse (divergence-free) components of π^I, and $\tilde{\pi}^0$ is a suitable linear combinations of π^0 and $\vec{\pi}_L$,

$$\dot{\tilde{\pi}}^0 = \dot{\pi}^0 + \tfrac{F_{by} b_0 - F_y}{F_{yy} y_0} \, \vec{\nabla} \cdot \vec{\pi}_L \ . \tag{6.4}$$

Moreover, $w_0 \equiv (F_y y_0 - F_b b_0)$ is the equilibrium enthalpy density $(\rho + p)_0$, c_s^2 is a somewhat complicated expression involving various derivatives of F—which corresponds precisely to the standard $\big(dp/d\rho\big)_{S,N}$—and all derivatives of F are computed at the equilibrium values y_0 and b_0.

We thus see from (6.3) that only one of our Goldstones—the longitudinal part of π^I—has a standard quadratic Lagrangian for a gapless field, and wave solutions propagating at some finite speed c_s, with $\omega = c_s k$. This Goldstone field corresponds to ordinary sound waves. The other Goldstones—$\vec{\pi}_T$ and $\tilde{\pi}^0$—do not have a gradient energy. As a result, at the order we are working, they have degenerate dispersion laws, $\omega = 0$.

6.2 Expanding the Lagrangian about a homogeneous and isotropic background: $\phi^I = x^I$

We begin with the field theoretical description for a relativistic perfect fluid [8,10]. Its action is given in eq. (1.7), which we reproduce here for completeness:

$$S = \int d^4x \, F(b) \ , \qquad \text{with} \quad b \equiv \sqrt{\det B^{IJ}} \ , \tag{6.5}$$

where $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ and where F is a generic function. A particular functional form of F corresponds to a particular equation of state. However, in order carry out the expansion of

the Lagrangian in fluctuations about a homogeneous and isotropic background (or for that matter, as we will see in Appendix 6.5, a background incompressible flow), there is a slightly more convenient starting point than eq. (6.5). Manipulating b we can pull out the Jacobian determinant for the equal-time $\vec{x} \leftrightarrow \vec{\phi}$ mapping:

$$\begin{aligned} b^2 &= \det \partial_\mu \phi^I \partial^\mu \phi^J \\ &= \det \left(\partial\phi^T \cdot \partial\phi - \dot{\vec{\phi}} \otimes \dot{\vec{\phi}} \right) \\ &= \det \left(\partial\phi^T \cdot \left(1 - ((\partial\phi^T)^{-1} \cdot \dot{\vec{\phi}}) \otimes ((\partial\phi^T)^{-1} \cdot \dot{\vec{\phi}}) \right) \cdot \partial\phi \right) \\ &= (\det \partial\phi)^2 \, \det \left(1 - \vec{v} \otimes \vec{v} \right) , \end{aligned} \tag{6.6}$$

where we defined the matrix $(\partial\phi)_{ij} \equiv \partial_i \phi_j$, and the vector

$$\vec{v} \equiv -(\partial\phi^T)^{-1} \cdot \dot{\vec{\phi}} \, . \tag{6.7}$$

Notice that, by the implicit function theorem, $\vec{v}$ is precisely the usual fluid velocity field—hence the name:

$$\vec{v} = \frac{\partial \vec{x}(\vec{\phi}, t)}{\partial t}\Big|_\phi \, . \tag{6.8}$$

The second determinant in (6.6) is easy to compute by going to a basis where, locally, the x axis is aligned with $\vec{v}$. We get

$$b^2 = (\det \partial\phi)^2 \big(1 - |\vec{v}|^2 \big) \tag{6.9}$$

and therefore we can rewrite the Lagrangian in the form:

$$\mathcal{L} = -w_0 f \big(\det \partial\phi \, \sqrt{1 - |\vec{v}|^2} \, \big) \, . \tag{6.10}$$

Notice that we are essentially reproducing eq. (88) of [10]. We now have to expand f in powers of its argument, and its argument in powers of π. The benefit of doing the expansion this way is that one now only has to expand in π functions of $\partial\phi = 1 + \partial\pi$, rather than of

$\partial_\mu\phi\partial^\mu\phi = 1 + \partial\pi + \partial\pi^T + \partial\pi^T\partial\pi + \dot\pi\dot\pi$. This simplifies the algebra considerably. We just need

$$(\partial\phi)^{-1} \simeq 1 - \partial\pi + \partial\pi^2 \tag{6.11}$$

$$\det\partial\phi = 1 + [\partial\pi] + \tfrac{1}{2}\big([\partial\pi]^2 - [\partial\pi^2]\big) + \tfrac{1}{6}\big([\partial\pi]^3 - 3[\partial\pi][\partial\pi^2] + 2[\partial\pi^3]\big) \tag{6.12}$$

(the determinant of an $n\times n$ matrix stops at n-th order.) A straightforward Taylor-expansion of (6.10) up to fourth order then yields eq. (2.5).

It is worth pointing out two sources of non-trivial cancellations, with important physical consequences. The first forces all Lagrangian terms that do not involve time-derivatives to be weighed by $c_s^2 = f''(1)$ or by higher derivatives of f, as manifest in eq. (2.5). That is, no such term is coming from the expansion of (6.41) at first order in f's argument, i.e. with a coefficient $f'(1) = 1$. The reason is simple: neglecting time-derivatives, the term proportional to $f'(1)$ would be

$$\mathcal{L} \supset -w_0 f'(1)\big(\det\partial\phi - 1\big)\,, \tag{6.13}$$

which is a total derivative:

$$\det\partial\phi = \epsilon\,\epsilon\,\partial\phi\,\partial\phi\,\partial\phi = \partial\big(\epsilon\,\epsilon\,\phi\,\partial\phi\,\partial\phi\big) \tag{6.14}$$

Since one expects higher derivatives of f to be naturally of order c_s^2, for a non-relativistic fluid this cancellation has the effect of weakening the interactions considerably, or equivalently of raising the strong-coupling scale compared to what one may have naively guessed before carrying out the expansion.

The second cancellation involves the transverse phonons only, and has also the effect of weakening some interactions and correspodingly raising the vortex strong-coupling scale. Consider an interaction term with spatial derivatives only, and assume that at least one of the phonons entering the corresponding vertex is transverse. As *not* manifest from the Lagrangian (2.5), such a vertex yields zero. The reason is that we can perform a non-linear field redefinition that makes vortices disappear from all Lagrangian terms without

time-derivatives. The trick is to define $\vec{\pi}$ so that

$$\det \partial\phi = 1 + \vec{\nabla} \cdot \vec{\pi} \tag{6.15}$$

exactly. That this is possible follows from eq. (6.14)—we may as well call the total derivative on the r.h.s. $1 + \vec{\nabla} \cdot \vec{\pi}$. This matches our original definition of $\vec{\pi}$ at linear order, and as a consequence it does not affect the S-matrix. But now it is clear that vortex interactions will only come from the $|\vec{v}|^2$ part of (6.10), and will thus involve at least two time-derivatives. The downside is that in these variables the structure of the Lagrangian will be more complicated than eq. (2.5); in particular we will not have exactly one derivative acting on each field. For this reason we stick to the original definition of the phonon field and to eq. (2.5), but we should expect non-trivial cancellations when computing S-matrix elements involving vortices, as we indeed find in sect. 2.3.

6.3 The S-matrix, cross-sections, and decay rates

Here we briefly review the standard relativistic formulae for the S-matrix and related physical quantities like cross sections and decay rates, and derive the modifications needed for applying them to our $c \neq 1$ case. The rules we will derive are straightforwardly generalizable to the case of different fields with different propagation speeds.

We will borrow the conventions of Peskin-Shroeder [19]. In particular, we use the so-called relativistic normalization for one-particle states:

$$\langle \vec{p} \, | \vec{q} \rangle = (2E) \, (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \ , \tag{6.16}$$

(we are suppressing spin labels—their inclusion is straightforward) and of course the vacuum state $|0\rangle$ is normalized to one. This way, a relativistic canonically normalized scalar field $\phi(x)$ obeys

$$\langle 0 | \phi(x) | \vec{p} \rangle = e^{-i(Et - \vec{p}\cdot\vec{x})} \ , \tag{6.17}$$

and consequently the momentum-space Feynman rules assign a wavefunction one to external spin-0 states. One thus has that for a $2 \to n_f$ scattering process the infinitesimal cross section is

$$d\sigma = \frac{1}{2E_A}\frac{1}{2E_B}\frac{1}{|v_A - v_B|}\left|\mathcal{M}_{AB\to f}\right|^2 d\Pi_{n_f} \ . \tag{6.18}$$

Here $\mathcal{M}_{i\to f}$ is the amplitude computed according to the standard relativistic Feynman rules, and defined by

$$\langle \vec{q}_1 \dots \vec{q}_{n_f} \,|(S-1)|\vec{p}_1 \dots \vec{p}_{n_i}\rangle = (2\pi)^4\,\delta^3(\text{momentum})\,\delta(\text{energy})\cdot i\mathcal{M}_{i\to f} \ , \tag{6.19}$$

and the $d\Pi_{n_f}$ is the relativistic final-state phase-space:

$$d\Pi_{n_f} = (2\pi)^4\,\delta^3(\text{momentum})\,\delta(\text{energy})\cdot\left(\prod_f \frac{d^3q_f}{(2\pi)^3}\frac{1}{2E_f}\right) \tag{6.20}$$

Finally, $|v_A - v_B|$ is the relative velocity between the two colliding beams as measured in the lab frame. Likewise for a $1 \to n_f$ decay process, the infinitesimal rate is

$$d\Gamma = \frac{1}{2E_A}\left|\mathcal{M}_{A\to f}\right|^2 d\Pi_{n_f} \ . \tag{6.21}$$

First, let us check the dimensions of these quantities, by keeping $\hbar$ dimensionless but the speed of light dimensionful. That is, let's give energy = 1/time and momentum = 1/length different units. From their definitions, eqs. (6.19, 6.20), for the amplitude and phase-space element we get

$$\big[\mathcal{M}\big] = Ek^3\big(E/k^3\big)^{\frac{n_i+n_f}{2}} \ , \qquad \big[d\Pi\big] = (Ek^3)^{-1}\big(E/k^3\big)^{-n_f} \ . \tag{6.22}$$

The cross-section and decay rate thus have dimensions

$$\big[d\sigma\big] = k^{-2} = \text{area} \ , \qquad \big[d\Gamma\big] = E = 1/\text{time} \ , \tag{6.23}$$

as they should. This means that the above fomulae are already dimensionally correct with no need of explicit powers of the speed of light.

Next, we notice that nowhere is Lorentz invariance assumed in deriving the Feynman rules and the above expressions for σ and Γ. This is evident e.g. in the derivation of ref. [19], apart from the relative velocity factor in the cross section. However that too is independent of Lorentz invariance, for it arises from the integral of an energy delta-function over the longitudinal (w.r.t. to the collision direction) momenta of the incoming wave-packets:

$$\int dk_A^z\, dk_B^z\, \delta(k_A^z + k_B^z - P_f^z)\, \delta(E_A + E_B - E_f) = \int dk_A^z\, \delta(E_A + E_B - E_f)\big|_{k_B^z = P_f^z - k_A^z} \quad (6.24)$$

$$= \Big|\frac{\partial E_A}{dk_A^z} - \frac{\partial E_B}{\partial k_B^z}\Big|^{-1} . \quad (6.25)$$

For each wave-packet, the derivative of the energy w.r.t. the corresponding momentum is the wave-packet's group-velocity, independently of the actual form of the dispersion law $E(k)$. The above thus yields the factor $1/|v_A - v_B|$ in the cross section, regardless of Lorentz-invariance.

The bottom line is, much ado about nothing. We can use the standard relativistic Feynman rules and formulae for infinitesimal cross-sections and rates for our non-relativistic case as well, with no modifications, even when different fields have different speeds. The only subtlety we should keep in mind is that canonically normalized fields obey eq. (6.17), times possible polarization factors for non-scalar particles. This means that a scalar field ϕ thus normalized should appear in the action as

$$S = \int d^3x dt\, \tfrac{1}{2}\dot{\phi}^2 + \dots\ , \quad (6.26)$$

so that single-particle states are eigenstates of the free Hamiltonian with the right energy:

$$w_0|\vec{p}\rangle = \Big(\int d^3x\, \tfrac{1}{2}\dot{\phi}^2 + \dots\Big)|\vec{p}\rangle = E(\vec{p})|\vec{p}\rangle\ . \quad (6.27)$$

As a check that these conclusions make sense, we estimate the cross section for sound wave-sound wave elastic scattering and show that, indeed, we have strong-coupling at the correct energy. From Feynman rules applied to the Lagrangian (2.5) we have

$$\mathcal{M} \sim c_s^2 \frac{k^4}{w_0} \quad (6.28)$$

where the factor of w_0 comes from the non-canonical normalization of π^I. The final state phase space (6.20) is of order

$$\Pi_f \sim \frac{1}{k^3}\frac{1}{E}\big(k^3/E\big)^2 \,, \tag{6.29}$$

and the relative velocity is of course $2c_s$, so that the cross-section (6.18) is

$$\sigma \sim \frac{1}{k^2}\left(\frac{k^4}{w_0 c_s}\right)^2 . \tag{6.30}$$

This agrees with our estimates of sect. 2.2.2—see the last paragraph of sect. 2.3.1.

6.4 Phase space

We are mostly interested in a two-particle final state, possibly with two independent propagation speeds. The infinitesimal phase space is

$$d\Pi_2 = (2\pi)^4\,\delta^3(\vec{P}-\vec{q}_1-\vec{q}_2)\,\delta(E-E_1-E_2)\cdot\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}\,, \tag{6.31}$$

where E and $\vec{P}$ are the total energy and momentum. The integral in $\vec{q}_2$ eliminates the momentum-conservation delta-function. Then we are left with

$$d\Pi_2 = \frac{d\Omega}{(2\pi)^2}\,q_1^2 dq_1 \frac{1}{2E_1\,2E_2}\delta(E-E_1-E_2)\,, \tag{6.32}$$

with the understanding that E_2 be evaluated at $\vec{q}_2 = \vec{P}-\vec{q}_1$. We have

$$\delta(E-E_1-E_2) = \frac{\delta(q_1-\bar{q}_1)}{\left|\frac{\partial E_1}{\partial q_1}+\frac{\partial E_2}{\partial q_2}\frac{\partial q_2}{\partial q_1}\right|} \tag{6.33}$$

and

$$\frac{\partial q_2}{\partial q_1} \equiv \frac{\partial\big|\vec{P}-\vec{q}_1\big|}{\partial q_1} = \frac{q_1 - P\cos\theta}{q_2}\,, \tag{6.34}$$

where θ is the angle between $\vec{q}_1$ and $\vec{P}$. On the other hand, the derivatives of the energies w.r.t. the corresponding momenta are the particles' group velocities. Integrating over q_1 we thus get

$$d\Pi_2 = \frac{d\Omega}{16\pi^2}\frac{q_1^2 q_2}{E_1 E_2}\frac{1}{\big|c_1 q_2 + c_2 q_1 - c_2 P\cos\theta\big|} \tag{6.35}$$

For a linear dispersion law like in our case, $E_a = c_a q_a$, we finally have

$$d\Pi_2 = \frac{d\Omega}{16\pi^2}\frac{1}{c_1 c_2}\frac{q_1}{\left|c_1 q_2 + c_2 q_1 - c_2 P\cos\theta\right|} \ . \tag{6.36}$$

In special circumstances there are further simplifications:

i) For scattering processes at zero total momentum, we can set $P = 0$ and $q_1 = q_2$. We get

$$d\Pi_2 = \frac{d\Omega}{16\pi^2}\cdot\frac{1}{c_1 c_2(c_1 + c_2)} \qquad (\vec{P} = 0) \ . \tag{6.37}$$

ii) For decay processes at finite total $\vec{P}$, but when one of the final particles is much slower that the other, barring an hierarchy between q_1 and q_2 we have

$$d\Pi_2 \simeq \frac{d\Omega}{16\pi^2}\,\frac{1}{c_1^2 c_2}\,\frac{q_1}{q_2} \qquad (c_2 \ll c_1) \ . \tag{6.38}$$

Of course the ratio q_1/q_2 depends non-trivially on the angle θ we are supposed to integrate over—which we can take to be the angle between $\vec{q}_1$ and $\vec{P}$. We have:

$$\frac{q_1}{q_2} \simeq \frac{1}{2\sin\theta/2} \qquad (c_2 \ll c_1,\ \vec{P} \neq 0) \ . \tag{6.39}$$

Overall we thus get

$$d\Pi_2 \simeq \frac{d\Omega}{32\pi^2}\,\frac{1}{c_1^2 c_2}\frac{1}{\sin\theta/2} \qquad (c_2 \ll c_1,\ \vec{P} \neq 0) \ . \tag{6.40}$$

Notice that this is regular at $\theta = 0$, thus making our 'barring an hierarchy ...' approximation under control. That is, eq. (6.40) is the correct phase-space element at lowest order in c_2/c_1.

6.5 Systematic expansion of the Lagrangian about incompressible flow

Here we make more explicit the steps taken to arrive at the foundation of all of the work in Chapter 3.

Following Appendix 6.2, but including explicitly powers of c in order to better keep track of relativistic effects we begin with

$$\mathcal{L} = -w_0 c^2 f\big(\det \partial\phi \, \sqrt{1 - |\vec{v}|^2/c^2}\,\big) \,, \tag{6.41}$$

where f and w_0 are defined in eqs. (3.6), (3.7). As $\vec{\phi}(x,t)$ is one-to-one with $\vec{x}$ for all t, we can change coordinates to those of the comoving volume elements and we have arrived at equation (3.3):

$$S = -w_0 c^2 \int d^3\phi dt \, \det J \, f\big((\det J^{-1})\sqrt{1 - v^2/c^2}\big) \,. \tag{6.42}$$

$J^i{}_j$ is the Jacobian matrix $\partial x^i/\partial\phi^j$, and $\vec{v} = \partial_t \vec{x}(\phi, t)$.

We now expand this action in powers of v/c_s—which is small by assumption—and of the compressional field $\vec{\psi}$, which, as explained in sect. 3.1, describes the small deviations from a purely volume-preserving time evolution:

$$\vec{x}(\phi,t) = \vec{x}_0(\phi,t) + \vec{\psi}(\phi,t) \,, \qquad \det(J_0) \equiv \det\left(\frac{\partial x_0^i}{\partial\phi^j}\right) = 1 \,, \qquad \vec{\psi} = \vec{\nabla}_0 \Psi \,. \tag{6.43}$$

For $\vec{v}$ and $\det J$ the expansion in $\vec{\psi}$ truncates at finite order (the determinant of a 3×3 matrix is a cubic function of that matrix):

$$\vec{v} = \vec{v}_0 + \partial_t \vec{\psi}\,\big|_\phi \,, \qquad \vec{v}_0 \equiv \partial_t \vec{x}_0\,\big|_\phi \tag{6.44}$$

$$\det J = 1 + [\nabla_0\psi] + \tfrac{1}{2}\left([\nabla_0\psi]^2 - [\nabla_0\psi^2]\right) + \tfrac{1}{6}\left([\nabla_0\psi]^3 - 3[\nabla_0\psi][\nabla_0\psi^2] + 2[\nabla_0\psi^3]\right) \,, \tag{6.45}$$

where [...] means the trace, $\nabla_0\psi^n$ means the n-th power of the matrix $\partial\psi^i/\partial x_0^j$, and we have used the identity

$$J^i{}_j \equiv \frac{\partial x^i}{\partial\phi^j} = \left(\delta^i_k + \frac{\partial\psi^i}{\partial\phi^l}\frac{\partial\phi^l}{\partial x_0^k}\right) J_0{}^k{}_j = \left(\delta^i_k + \frac{\partial\psi^i}{\partial x_0^k}\right) J_0{}^k{}_j \,. \tag{6.46}$$

Of course, for the inverse determinant and the square root entering the action (6.42) the

expansion does not truncate, and goes on to all orders. Up to cubic order in $\vec{\psi}$ we get

$$
\begin{aligned}
S = w_0 \int d^3\phi\, dt \, \Big\{ & + \tfrac{1}{2} v^2 + \tfrac{(c^2-c_s^2)v^4}{8c^4} \\
& + \tfrac{1}{2} \left((\partial_t \psi)^2 - c_s^2 [\nabla_0 \psi]^2 \right) \\
& + v_i \, \partial_t \psi^i - \tfrac{1}{2} \tfrac{c_s^2}{c^2} v^2 \, [\nabla_0 \psi] \\
& + \frac{1}{2c^4} (c^2 - c_s^2) v^2 v_i \, \partial_t \psi^i + \frac{1}{8c^2} f_3 \, v^4 \, [\nabla_0 \psi] \\
& - \frac{c_s^2}{c^2} v_i \partial_t \psi^i [\nabla_0 \psi] + \frac{1}{4c^2} \left(c_s^2 v^2 ([\nabla_0 \psi]^2 + [\nabla_0 \psi^2]) + c^2 \, f_3 \, v^2 [\nabla_0 \psi]^2 \right) \\
& + \frac{1}{6c^2} \left(-3 c_s^2 [\nabla_0 \psi] \left((\partial_t \psi)^2 - c^2 [\nabla_0 \psi^2] \right) + c^4 \, f_3 \, [\nabla_0 \psi]^3 \right) + \dots \Big\} \, .
\end{aligned}
\tag{6.47}
$$

We have used $f'(1) = 1$, $f''(1) = c_s^2/c^2$, $f_3 \equiv f'''(1)$, and for notational simplicity we have dropped the subscript zero in v_0, and so, from now on, $v \equiv v_0$ is the underlying incompressional velocity field. Even though f_3 is a free parameter, as pointed out in [1] one typically expects

$$
f_3 \sim c_s^2/c^2 \, . \tag{6.48}
$$

We have dropped total derivative terms, most notably the term $-w_0 c^2 \, f(1) \, \det J$, since $\det J$ itself is a total derivative—schematically:

$$
\det J = \det \partial_\phi x \propto \epsilon\epsilon \cdot \partial_\phi x \, \partial_\phi x \, \partial_\phi x = \partial_\phi \cdot \left(\epsilon\epsilon \cdot x \, \partial_\phi x \, \partial_\phi x \right) \, . \tag{6.49}
$$

We have also dropped terms that are total derivatives *with respect to* $\vec{x}_0$, even though the integral is in $d^3\phi$, for reasons that will become clear in a moment. Notice that for different powers of ψ, we have stopped the expansion at different orders in v_0. The reason is that the two expansion parameters are not entirely independent, as we will explain below. Notice also that the first line is just the action for the incompressible fluid flow, in the absence of compressional perturbations. To be used in this sense, it should be supplemented by a volume-preserving constraint for $\vec{x}_0(\vec{\phi}, t)$, as explained in the main text. However that is not our goal, since we are interested in solving for the dynamics of $\vec{\psi}$ in the presence of a *given*

background incompressible fluid flow $\vec{v}_0$. We will therefore discard the first line for what follows.

We are not finished yet. We want to now go into the $\vec{x}_0$ coordinate system. The reason is that in an experiment it is much more convenient to parameterize a velocity field in terms of its dependence on the physical $\vec{x}$ coordinates rather than on the comoving ones. Since the $\vec{x}$ and $\vec{x}_0$ coordinate systems coincide at lowest order in $\vec{\psi}$, using $\vec{x}_0$ will suffice for lowest order computations. This is an easy transformation for most things in the action: The Jacobian J_0 has unit determinant, and all the spacial partial derivatives are with respect to $\vec{x}_0$ already (this is the reason why above we were allowed to neglect total x_0-derivatives). The non-trivial pieces are the partial *time* derivatives as they are taken at constant ϕ values. We can write the time derivatives in a more compatible form:

$$\partial_t \psi^i(x_0(\phi,t),t)|_{\vec{\phi}} = \partial_t \psi^i(x_0,t)|_{x_0} + v_0^j \frac{\partial}{\partial x_0^j}\psi^i(x_0,t) \ . \tag{6.50}$$

So, changing coordinates from $\vec{\phi}$ to $\vec{x_0}$ and inserting the expanded time derivatives in the action above we arrive at an action which can be organized in the following way:

$$S = S_{\psi^2} + S_{\psi^3} + \cdots + S_{\psi\ v^n} + S_{\psi^2\ v^n} + \dots \ , \tag{6.51}$$

where the n's above mean 'positive powers of v' (1, 2, 3, etc.). Dropping for notational convenience all the subscript 0's, and denoting partial time-derivatives by overdots, we have

explicitly:

$$S_{\psi^2} = w_0 \int d^3x dt \, \frac{1}{2}\big(\dot{\vec{\psi}}^2 - c_s^2[\partial\psi]^2\big) \tag{6.52}$$

$$S_{\psi^3} = w_0 \int d^3x dt \, \Big\{ \frac{c_s^2}{2}[\partial\psi][(\partial\psi)^2] + \frac{c^2 f_3}{6}[\partial\psi]^3 - \frac{c_s^2}{2c^2}\dot{\vec{\psi}}^2[\partial\psi] \Big\} \tag{6.53}$$

$$\vdots$$

$$\begin{aligned} S_{\psi\, v^n} = w_0 \int d^3x dt \, \Big\{ & \vec{v}\cdot\dot{\vec{\psi}} + v_i(v\cdot\nabla)\psi^i - \frac{c_s^2}{2c^2}\, v^2[\partial\psi] \\ & + \frac{(c^2 - c_s^2)}{2c^4}\big(v^2\vec{v}\cdot\dot{\vec{\psi}} + v^2 v_i(v\cdot\nabla)\psi^i\big) + \frac{f_3}{8c^2}\, v^4[\partial\psi] + \cdots \Big\} \end{aligned} \tag{6.54}$$

$$\begin{aligned} S_{\psi^2\, v^n} = w_0 \int d^3x dt \, \Big\{ & \dot{\psi}_i(v\cdot\nabla)\psi^i - \frac{c_s^2}{c^2}\big(\vec{v}\cdot\dot{\vec{\psi}}\big)[\partial\psi] + \frac{1}{2}\left((v\cdot\nabla)\psi^i\right)^2 \\ & - \frac{c_s^2}{c^2} v_i(v\cdot\nabla)\psi^i[\partial\psi] + \left(\frac{c_s^2}{4c^2} + \frac{f_3}{4}\right) v^2[\partial\psi]^2 + \frac{c_s^2}{4c^2} v^2[(\partial\psi)^2] \\ & \frac{1}{2c^4}\left(c^2 - c_s^2\right)\left(\vec{v}\cdot\dot{\vec{\psi}}\right)^2 + \frac{1}{4c^4}\left(c^2 - c_s^2\right) v^2\dot{\vec{\psi}}^2 + \cdots \Big\} \end{aligned} \tag{6.55}$$

$$\vdots$$

While all of the above looks like a total mess (and of course there is an infinite tower of terms), it is both easy to generate and easy to interpret perturbatively. In particular:

- S_{ψ^2} describes the free propagation of compressional modes (sound waves); c_s is indeed their propagation speed. The associated (Feynman) propagator is

$$\langle T(\psi^i\psi^j)\rangle = \frac{\hat{p}^i\hat{p}^j}{w_0}\frac{i}{\omega^2 - c_s^2\vec{p}^2 + i\epsilon} \, . \tag{6.56}$$

- S_{ψ^3} describes the sound waves' trilinear self-interactions; in a Feynman diagram these will correspond to a vertex with three lines attached.

- $S_{\psi\, v^n}$ corresponds to 'tadpole' diagrams for ψ—diagrams with a single ψ line attached to the external source $\vec{v}_0$: In the presence of a non-trivial background incompressional velocity field, $\psi = 0$ is not a consistent solution. Terms in the action that are linear in ψ describe, at lowest order in ψ, how a non-trivial ψ field is generated by such a background velocity field.

- $S_{\psi^2\, v^n}$ corresponds to interaction vertices where two ψ lines attach to the external source $\vec{v}_0$, describing for instance the scattering of sound waves by vorticose motion, as in sect. 3.3.

- And so on.

As discussed throughout the text, note that the terms with the additional c_s^2/c^2 factors scale like many of the others in terms of powers of v/c_s, derivatives, and so forth. These terms are the relativistic corrections, and will be necessary to correctly describe ultra-relativistic fluids.

By comparing S_{ψ^2} and $S_{\psi\, v^n}$ we see that, for small v/c_s, the ψ field generated by the background fluid flow scales like $\psi \sim v^2$ (we are implicitly assuming that $\dot{v}$ scales like v^2, keeping the typical length scales fixed.) According to this power-counting scheme, the terms explicitly displayed in (6.52)–(6.55) are all the terms in the action up to order v^6. Things become more complicated when not all of the ψ's appearing in a diagram are "generated" by the background v_0, like for instance in a scattering process. Also, in general there are important power-counting differences between highly off-shell internal lines with $\omega \ll c_s p$, and on-shell external lines with $\omega = c_s p$. These subtleties for power counting in classical perturbation theory are addressed systematically for general relativity in ref. [30]. It would be interesting to derive analogous power counting rules for our case. An attempt at this will appear soon in a paper by the author of this thesis [3].

6.6 Scattering cross section and emission rate from the amplitude, in the presence of sources

For the benefit of the reader we derive the scattering cross section formula for one-to-one scattering. Following Srednicki's [44] quick construction—which avoids the use of wave

packets—we start with the probability of scattering:

$$P = \frac{\left|\left\langle f\middle|i\right\rangle\right|^2}{\left\langle f\middle|f\right\rangle\left\langle i\middle|i\right\rangle} \equiv \frac{\left|\mathcal{M}\right|^2}{\left\langle f\middle|f\right\rangle\left\langle i\middle|i\right\rangle} \ . \tag{6.57}$$

Note that our $\mathcal{M}$ here is different then the usual one because we have not yet removed any delta-functions. Assume that we are performing our experiment in a large box of volume V and over a time T. The norm of a single particle state is given by (with the so-called relativistic normalization, which is in fact still convenient for non-relativistic systems, see Appendix 6.3)

$$\left\langle k\middle|k\right\rangle = 2E\,(2\pi)^3\delta^3(0) = 2EV \ . \tag{6.58}$$

For our $1 \to 1$ scattering off external sources of sect. 3.3, let's take the incoming momentum and energy to be p_1 and E_1, and the outgoing ones to be p_2 and E_2. Additionally, we want to sum over final momentum states We then have

$$P = \sum_{\vec{p}_2} \frac{\left|\mathcal{M}(p_1,p_2)\right|^2}{2E_1 2E_2 V^2} \ , \tag{6.59}$$

which is in the continuous limit becomes

$$P \ \ \to \ \ \frac{V}{(2\pi)^3}\int d^3p_2 \frac{\left|\mathcal{M}(p_1,p_2)\right|^2}{2E_1 2E_2 V^2} \ . \tag{6.60}$$

To get the total cross section we need to divide by the incoming flux (which for an incoming particle moving at speed c_s is simply (c_s/V)) and divide by the total time T. Finally, we arrive at the general expression

$$d\sigma_{(1\to 1)} = \frac{1}{c_s}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{\left|\mathcal{M}(p_1,p_2)\right|^2}{T} \tag{6.61}$$

$\mathcal{M}$ can be computed following the standard *relativistic* Feynman rules [1], with the caveat that whenever an external source $J(x)$ appears in a Lagrangian term, its (four-dimensional) Fourier transform should appear as a factor in $\mathcal{M}$. As a consistency check, notice that for Lagrangian terms with *no* external sources, we formally have $J(x) = 1$, whose Fourier transform is the usual $(2\pi)^4\,\delta(\text{energy})\delta^3(\text{momentum})$.

Now, if our vorticose source is time-independent, then we have an energy-conserving delta function in our $\mathcal{M}$, such that

$$\begin{aligned} \mathcal{M} &\rightarrow \mathcal{M}'(2\pi)\delta(E_1 - E_2) && (6.62) \\ \Rightarrow \quad \left|\mathcal{M}\right|^2 &\rightarrow \left|\mathcal{M}'\right|^2(2\pi)\delta(E_1 - E_2)(2\pi)\delta(0) && (6.63) \\ &= \left|\mathcal{M}'\right|^2(2\pi)\delta(E_1 - E_2)\, T && (6.64) \end{aligned}$$

and thus, in this time-independent limit we have that

$$d\sigma_{(1\rightarrow 1)}(\text{time independent source}) = \frac{1}{c_s}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\left|\mathcal{M}'(p_1,p_2)\right|^2(2\pi)\delta(E_1 - E_2) \quad (6.65)$$

which, as a good check, matches Peskin and Schroeder's Rutherford scattering problem in their fourth chapter [45].

Similar considerations apply to the emission process discussed in sect. 3.2, and yield eq. (3.25).

6.7 Scattering sound waves off sources: Checks

6.7.1 Discrepancy with previous results

The results found in [34] were developed in the context of classical scattering theory with the same set of assumptions as ours: in particular, low Mach number of the source flow, low intensity and high frequency relative to the typical source frequency of the incoming sound waves. Moreover, the computations of ref. [34] are appropriate for non-relativistic fluids only, so for the purposes of this comparison we will set $c_s/c \rightarrow 0$ in our results.

The first step in the comparison is to note that the vorticity field, defined as $\vec{w} = \nabla \times \vec{v}$, satisfies

$$(\hat{p}_1 \times \hat{p}_2) \cdot \tilde{w}(\Delta p^\mu) = i\left[(\hat{p}_1 \cdot \vec{\Delta p})(\hat{p}_2 \cdot \tilde{v}) - (\hat{p}_2 \cdot \vec{\Delta p})(\hat{p}_1 \cdot \tilde{v})\right] \quad (6.66)$$

where $\tilde{w}(\Delta p^\mu)$, $\tilde{v}(\Delta p^\mu)$ are four-dimensional Fourier transforms, and for brevity we are using a relativistic notation, $\Delta p^\mu \equiv (\Delta\omega, \Delta\vec{p})$). Defining the scattering angle through the usual $\hat{p}_1 \cdot \hat{p}_2 = \cos\theta$, we can write

$$(\hat{p}_1 \times \hat{p}_2) \cdot \tilde{w}(\Delta p^\mu) = i\,(1-\cos\theta)\,p_1\,[(\hat{p}_1 + \hat{p}_2)\cdot\tilde{v}] + \mathcal{O}\left(\frac{\Delta\omega}{c_s p_1}\right)\ . \tag{6.67}$$

Here $\Delta\omega/c_s p_1$ is small—see sect. 3.3—and so we will neglect it in the following.

Now, using (6.67) we can re-express (3.45) as

$$\frac{d\sigma}{d\Omega\, d(\Delta\omega)} \simeq \frac{1}{4c_s^4}\cdot\frac{\omega^2}{(2\pi)^3}\cdot\frac{1}{T}\cdot\frac{\cos^2\theta}{(1-\cos\theta)^2}\,|(\hat{p}_1\times\hat{p}_2)\cdot\tilde{w}(\Delta p^\mu)|^2 + \mathcal{O}\left(\frac{\Delta\omega}{c_s p_1}\right) \tag{6.68}$$

Using our notation for the Fourier transform (which is different than that of [34]) we can express

$$\begin{aligned}
\tilde{w}_k(\Delta p^\mu)\tilde{w}_l^*(\Delta p^\mu) &= \int d^4x\, d^4y\; e^{-i\Delta p^\mu x_\mu + i\Delta p^\mu y_\mu}\, w_l(y^\mu) w_k(x^\mu) && (6.69)\\
&= \int d^3R\; d\Delta\; e^{i\left(\Delta\omega\cdot\Delta - \vec{\Delta p}\cdot\vec{R}\right)} && (6.70)\\
&\quad \times \int d^3x \int_{-\infty}^{\infty} d\tau\; w_l(\vec{R}+\vec{x}, \tau+\Delta/2) w_k(\vec{x}, \tau-\Delta/2)
\end{aligned}$$

where we have shifted the integral variables appropriately. Using the notation of [34] we can write our final scattering cross section as

$$\frac{d\sigma}{d\Omega_2 d\nu} = \frac{(2\pi)\nu^2}{4c^4}\cdot\frac{\cos^2\theta}{(1-\cos\theta)^2}\,A_k A_l\,\tilde{S}_{kl}(\vec{q}, \nu-\nu_0) \tag{6.71}$$

where $\nu = \omega_2$, $\nu_0 = \omega_1$, $\vec{q} = \vec{\Delta p}$, $d\Omega_2 = d\Omega$, $A_k = (\hat{p}_1 \times \hat{p}_2)_k$,

$$\tilde{S}_{kl}(\vec{q}, \nu-\nu_0) = \frac{1}{(2\pi)^4}\int d^3r\; d\Delta\; e^{i\left(\Delta E\cdot\Delta - \vec{\Delta p}\cdot\vec{R}\right)} \int d^3x \left\langle w_k(\vec{R}+\vec{x}, \Delta/2) w_l(\vec{x}, -\Delta/2)\right\rangle\ , \tag{6.72}$$

and where

$$\langle f(\Delta/2) f(-\Delta/2)\rangle \equiv \frac{1}{T}\int_{-T/2}^{T/2} d\tau f(\tau+\Delta/2) f(\tau-\Delta/2)\ . \tag{6.73}$$

When we compare (6.71) to that of equation (19) in [34] we see that there is a factor of 2 discrepancy.

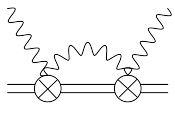
6.7.2 Agreement with optical theorem

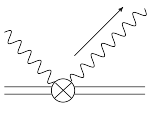
The optical theorem as expressed in quantum field theory is a consequence of the unitarity of the S-matrix. In our case, it relates the imaginary part of the one-to-one amplitude in the forward scattering limit, to the sum over the squares of the one-to-anything amplitudes[1]:

$$2 \text{ Im } \mathcal{M}(i \to i) = \sum_f \int d\Pi_f \ |\mathcal{M}(i \to f)|^2 \ . \tag{6.74}$$

It is a statement about full amplitudes, but perturbatively it can be expressed as a relation between particular graphs. To lowest order (in the coupling) the relationship is simply given by:

$$2 \text{ Im } \left(\right) = \int \frac{d^3q}{(2\pi)^3 2\omega} \left| \right|^2 \ . \tag{6.75}$$





We can rewrite the right hand side of the above equation as the one-to-one scattering cross-section using (6.61). Written this way, we have

$$2 \text{ Im } \mathcal{M}(\vec{p}_1, \vec{p}_1) = 2\omega_1 c_s T \, \sigma_{(1\to 1)} \ . \tag{6.76}$$

Eq. (3.45) provides us with the right hand side of this equation. We need to calculate the left hand side.

The imaginary part of the amplitude comes from the $i\epsilon$ prescription in the internal propagator of sound modes. Using generating functional methods to ensure that all symmetry factors are correct, we have

$$\begin{aligned} \left\langle 0 \left| T\psi^i(x_1)\psi^j(x_2) \right| 0 \right\rangle_v &= \frac{1}{i}\frac{\delta}{\delta J^i(x_1)} \cdot \frac{1}{i}\frac{\delta}{\delta J^j(x_2)} \cdot \frac{1}{2!}\left(i\int d^4y \, \mathcal{L}_{\psi^2 v}\left(\frac{1}{i}\frac{\delta}{\delta \vec{J}(y)} \right) \right)^2 \\ &\quad \times \frac{1}{3!} \cdot \left(\frac{i}{2}\int d^4x d^4x' J^k(x) \Delta^{kl}(x-x') J^l(x') \right)^3 . \end{aligned} \tag{6.77}$$

[1] We use the conventions given in [44] and [45], which should be noted are different than those given in [32].

Upon taking all the functional derivatives it is easy to see that one is left with four terms. Inserting this correlation function into the LSZ formula the amplitude is

$$\begin{aligned} i\mathcal{M}(\vec{p}_1 \to \vec{p}_2) &= i\int \frac{d\omega_q}{2\pi}\frac{d^3q}{(2\pi)^3}\,\frac{1}{-\omega_q^2 + c_s^2\vec{q}^{\,2} - i\epsilon}\int_{y_1 y_2} e^{iy_1(p_1-q)}e^{iy_2(q-p_2)} \\ &\quad\times\left[\omega_1\left(\vec{v}(y_1)\cdot\vec{q}\right) + \omega_q\left(\vec{v}(y_1)\cdot\vec{p}_1\right)\right]\left(\hat{q}\cdot\hat{p}_1\right) \\ &\quad\times\left[\omega_2\left(\vec{v}(y_2)\cdot\vec{q}\right) + \omega_q\left(\vec{v}(y_2)\cdot\vec{p}_2\right)\right]\left(\hat{q}\cdot\hat{p}_2\right), \end{aligned} \tag{6.78}$$

where we have suppressed relativistic corrections for algebraic simplicity; their inclusion is straightforward.

Taking the forward scattering limit, $p_2 \to p_1$, and performing the y integrals which yield Fourier transforms of the velocity fields we arrive at

$$i\mathcal{M}(\vec{p}_1 \to \vec{p}_1) = i\int \frac{d\omega_q}{2\pi}\frac{d^3q}{(2\pi)^3}\,\frac{1}{-\omega_q^2 + c_s^2\vec{q}^{\,2} - i\epsilon}\tilde{v}^i(q-p_1)\tilde{v}^j(p_1-q)V^iV^j, \tag{6.79}$$

where $V^i = \left(\hat{q}\cdot\hat{p}_1\right)\left[\omega_1 q^i + \omega_q p_1^i\right]$. In this form it is easy to see that the only contribution to the Im $\mathcal{M}$ comes from the $i\epsilon$, we have

$$\begin{aligned} \text{Im }\mathcal{M}(p_1,p_1) &= \int \frac{d\omega_q}{2\pi}\frac{d^3q}{(2\pi)^3}\frac{\pi}{2c_s|\vec{q}|}\left[\delta(\omega_q - c_s|\vec{q}|) + \delta(\omega_q + c_s|\vec{q}|)\right] \\ &\times\ \tilde{v}^i(p_1-q)\tilde{v}^j(q-p_1)V^iV^j\,, \end{aligned} \tag{6.80}$$

where we have isolated the imaginary part with the standard formula

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{6.81}$$

where $\mathcal{P}$ is the principal value.

Only the first delta-function in (6.80) is going to overlap with the support of $\tilde{v}^i(p_1-q)$. This is because $\tilde{v}^i(p_1-q)$ offers a narrow width around zero frequency relative to the incoming frequency (as the time scale of the velocity flow is much longer than the incoming frequency). For instance, in the static limit $\tilde{v}^i(p_1-q) \propto \delta(\omega_1-\omega_q)\tilde{v}^i(\vec{p}_1-\vec{q})$. For the same reason, as discussed in Section 3.3, to lowest order in the energy transfer we can take $\omega_1 = \omega_2 = \omega$.

Evaluating this final delta function, relabeling our variables of integration and performing some trivial algebra we have:

$$2 \text{ Im } \mathcal{M}(p_1, p_1) = (2\omega c_s T) \int \frac{d\Omega \, d(\Delta\omega)}{(2\pi)^3} \frac{\omega^4}{4c_s^6 T} (\hat{p}_1 \cdot \hat{p}_2)^2 \left| \tilde{v}^i(\Delta\omega, \vec{\Delta p}) \cdot \left(\hat{p}_1^i + \hat{p}_2^i\right) \right|^2 \tag{6.82}$$

Inserting this into the left-hand side of (6.76) and comparing with our scatting cross section (3.45) calculated in the bulk of the paper we see that indeed the optical theorem is satisfied.

6.8 Collection of momentum integrals

In performing the calculations of section 3.4 we encountered integrals of the form

$$\int_{\vec{k}} \frac{k^{i_1} k^{i_2} .... k^{i_m}}{k^n} e^{i\vec{k}\cdot\vec{x}} \, . \tag{6.83}$$

These integrals will often contribute power law UV divergences. These are trivial divergences that can simply be absorbed into bare couplings in using the standard renormalization program and will not yield any physical consequences. Operationally we regulate these divergences by taking the k^i's out of the numerator by expressing the as derivatives acting on $\vec{x}$. That is

$$\int_{\vec{k}} \frac{k^{i_1} k^{i_2} .... k^{i_m}}{k^n} e^{i\vec{k}\cdot\vec{x}} = (-i)^m \partial_{i_1} \partial_{i_2} ... \partial_{i_m} \int_{\vec{k}} \frac{1}{k^n} e^{i\vec{k}\cdot\vec{x}} \, . \tag{6.84}$$

The particular powers of n that we need are 6, 4, 3, 2, 1, and -1. For $n > 2$ we use an IR regulator and keep the lowest order terms in that regulator. When the derivatives are taken all dependence on the regulator vanishes validating its use. They are collected below:

$$\int_{\vec{k}} \frac{1}{k^2(k^4+a^4)} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi^2}\left(\frac{\pi}{\sqrt{2}a^3} - \frac{\pi\,|\vec{x}|^2}{6\sqrt{2}a} + \frac{\pi\,|\vec{x}|^3}{24} + \mathcal{O}(a)\right) \tag{6.85}$$

$$\int_{\vec{k}} \frac{1}{k^2(k^2+a^2)} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi^2}\left(\frac{\pi}{a} - \frac{\pi\,|\vec{x}|}{2} + \mathcal{O}(a)\right) \tag{6.86}$$

$$\int_{\vec{k}} \frac{1}{k^2(k+a)} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi^2}\big(2 - 2\gamma - 2\log(a) - 2\log(|\vec{x}|) + \mathcal{O}(a)\big) \tag{6.87}$$

$$\int_{\vec{k}} \frac{1}{k^2} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi^2}\left(\frac{\pi}{|\vec{x}|}\right) \tag{6.88}$$

$$\int_{\vec{k}} \frac{1}{k} e^{i\vec{k}\cdot\vec{x}} = -\nabla^2 \cdot \frac{1}{4\pi^2}\big(-2\log(|\vec{x}|)\big) = \frac{1}{4\pi^2}\left(\frac{2}{|\vec{x}|^2}\right) \tag{6.89}$$

$$\int_{\vec{k}} k\; e^{i\vec{k}\cdot\vec{x}} = -\nabla^2 \cdot \frac{1}{4\pi^2}\left(\frac{2}{|\vec{x}|^2}\right) = \frac{1}{4\pi^2}\left(\frac{-4}{|\vec{x}|^4}\right) \tag{6.90}$$

where γ is the Euler-Mascheroni constant.

6.9 Time dependence of background quantities in solid inflation

In order to solve the classical equations of motion for scalar and tensor perturbations, we need know the explicit time dependence of quantities such $a(\tau)$, $H(\tau)$, $\epsilon(\tau)$, ...; the goal of this section is obtaining this time dependence. For the computations we are interested in, it suffices to derive these temporal functions up to the first order in slow roll. To make the notation lighter, we will mostly drop the τ argument: $a(\tau) \to a$, etc. Primes will denote derivatives with respect to τ.

Recall the definition of the first slow roll parameter ϵ, (4.24), and rewrite it as

$$\epsilon = -\frac{H'}{aH^2} = \frac{d}{d\tau}\left(\frac{1}{aH}\right) + 1\ . \tag{6.91}$$

Integrating the above equation once and choosing some suitable additive constant [2], one has

$$\frac{1}{aH} = -(1-\epsilon_c)\,\tau + \mathcal{O}(\epsilon^2) \tag{6.92}$$

[2]The integration constant is chosen by demanding $a(\tau) \gg a(\tau_c)$, for $\tau/\tau_c \to 0$.

where the subscript "c" denotes evaluation at some reference conformal time τ_c, which we find most convenient to choose to be the (conformal) time when the *longest* mode of observational relevance today exists the horizon, i.e. $|c_{L,c}k_{\rm min}\tau_c| \simeq |c_{L,c}\tau_c H_{\rm today}| = 1$.

The reason $\epsilon(\tau)$ is being treated as a constant in integration is that the higher order terms in the Taylor series of $\epsilon(\tau) = \epsilon(\tau_c) + \epsilon'(\tau_c)(\tau - \tau_c) + \dots$ are suppressed by powers of slow roll, for instance $\epsilon'(\tau_c)\tau_c \sim O(\epsilon^2)$. Of course this also depends on the choice of reference time; we don't want the perturbative expansion in slow roll to be contaminated by large values of $\tau/\tau_c - 1$. Since $|\tau(t)|$ is a decreasing function during inflation, and $+\infty > -\tau > 0$, we ought to choose early times (like τ_c) as the reference.

Using the definition of the Hubble parameter, we can extract the time dependence of the scale factor $a(\tau)$ from the above equation (6.92):

$$aH = \frac{a'}{a} = -\frac{1+\epsilon_c}{\tau} \quad \Longrightarrow \quad a(\tau) = a_c \left(\frac{\tau}{\tau_c}\right)^{-1-\epsilon_c} + \mathcal{O}\,.(\epsilon^2) \tag{6.93}$$

Furthermore, we obtain

$$H(\tau) = \frac{a'}{a^2} = -\frac{1+\epsilon_c}{a_c\tau_c}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c} + \mathcal{O}(\epsilon^2)\,. \tag{6.94}$$

Finally, the time dependence of ϵ, c_L and c_T can be revealed by invoking the definitions of other slow roll parameters. For instance,

$$\frac{\epsilon'}{\epsilon} = aH\eta = -\frac{\eta}{\tau} + \mathcal{O}(\epsilon^2) \quad \Longrightarrow \quad \epsilon(\tau) = \epsilon_c\left(\frac{\tau}{\tau_c}\right)^{-\eta_c} + \mathcal{O}(\epsilon^3)\,. \tag{6.95}$$

Similarly we obtain

$$c_L(\tau) = c_{L,c}\left(\frac{\tau}{\tau_c}\right)^{-s_c} + \mathcal{O}(\epsilon^2), \qquad c_L(\tau) = c_{T,c}\left(\frac{\tau}{\tau_c}\right)^{-u_c} + \mathcal{O}(\epsilon^2)\,. \tag{6.96}$$

Notice that, because of the all-order relation between c_T^2 and c_L^2 of footnote 4, $c_{T,c}$ and u_c are not independent parameters—they can be expressed in terms of $c_{L,c}$ and of the slow-roll parameters. The equations (6.93)–(6.96) are frequently used in solving the classical equations of motion.

6.10 Unitary gauge vs. spatially flat slicing gauge in solid inflation

In performing calculations throughout this paper two gauge choices are particularly useful:

- Spatially Flat Slicing Gauge (SFSG) is defined by setting to zero the scalar and vector perturbations in g_{ij}, i.e. by imposing

$$g_{ij} = a(t)^2 \exp \gamma_{ij} \ , \tag{6.97}$$

where γ_{ij} denotes the transverse traceless tensor mode, satisfying

$$\gamma_{ii} = 0 \ , \qquad \partial_i \gamma_{ij} = 0 \ . \tag{6.98}$$

Then the fluctuations in our ϕ^I scalars are unconstrained:

$$\phi^I = x^I + \pi^I \ . \tag{6.99}$$

The three $\pi^I(x)$ fields can be split into a transverse vector and longitudinal scalar as in (4.16), according to their transformation properties under the *residual* rotation group. This gauge choice is of particular convenience for computations of the two- and three-point functions because in the demixing (with gravity) limit the π Lagrangian will contain all the scalar (or longitudinal) and vector (or transverse) degrees of freedom.

- Unitary Gauge (UG) is defined by setting to zero the fluctuations in the ϕ^I fields and in the "clock" field:

$$\phi^I = x^I, \qquad \det(B^{IJ}) = a(t)^{-6} \ . \tag{6.100}$$

Then the spatial metric is unconstrained. And can be parameterized in general as

$$g_{ij} = a(t)^2 \exp(A\,\delta_{ij} + \partial_i\partial_j\chi + \partial_i C_j + \partial_j C_i + D_{ij}) \ , \tag{6.101}$$

where $\partial_i C_i = 0$ and D_{ij} is transverse-traceless.

From the above form of the metric it seems that in UG there are too many degrees of freedom; there is an extra scalar in addition to the scalar, transverse vector, and transverse traceless tensor that we expect. However, when the metric is expressed in terms of the ADM parameters defined by (4.63), the second condition in (6.100) can be rewritten as

$$3A + \nabla^2\chi = \log(1 - N^i N_i/N^2) \, . \tag{6.102}$$

As N^i and N can be expressed in terms of A, χ, C_i, and D_{ij} by solving the constraint equations given by (4.66), (4.67) we can see that (6.102) implies that the two scalar functions $A(x)$ and $\chi(x)$ are not independent in UG. Hence the dynamical d.o.f. in question can be chosen to be $A(x)$ (the scalar mode), $C_i(x)$ (the transverse vector mode), and D_{ij} (the transverse traceless tensor mode). The number of which matches our physical intuition and properly agrees with SFSG. UG is particularly useful in following our degrees of freedom through the reheating surface.

As we find it convenient to calculate correlation functions in SFSG, and yet utilize UG to most easily describe the surface of sudden reheating, we need to develop the transformation rules to go from one gauge to the other. Let's denote by $\{x^\mu\}$ the coordinate system in SFSG and by $\{\bar{x}^\mu\}$ that in UG. A gauge transformation relating SFSG to UG is given by $\bar{x}^\mu = x^\mu + \xi^\mu(x)$, where

$$\xi^0(x) = -\frac{1}{3H}\partial_i\pi^i(x) + \mathcal{O}(\pi^2), \quad \xi^I(x) = \pi^I(x) + \mathcal{O}(\pi^2) \tag{6.103}$$

and the scalar perturbations are related by

$$A = \frac{2}{3}\partial_i\pi^i + \mathcal{O}(\pi^2) \, . \tag{6.104}$$

6.11 Vector perturbations in solid inflation

In this section we will derive the spectrum for the vector modes π_T^i (in SFSG), in the same manner as for scalar perturbations and tensor perturbations. Let's begin by writing the

vector modes as

$$\pi_T^i(\vec{k},t) = \sum_{\lambda=\pm} \epsilon_\lambda^i(\vec{k})\, \pi_{T,\lambda}(\vec{k},t)\ , \tag{6.105}$$

where the polarization vectors satisfy the transverse condition $k_i\epsilon^i(\vec{k}) = 0$ and they form an orthonormal and complete set $\epsilon_\lambda^i(\vec{k})\epsilon_{\lambda'}^i(\vec{k})^* = \delta_{\lambda\lambda'}$. As before, writing

$$\pi_{T,\lambda}(\vec{k},t) = \pi_T^{cl}(\vec{k},t)\ d_\lambda(\vec{k}) + \pi_T^{cl}(\vec{k},t)^*\ d_\lambda^\dagger(-\vec{k}) \tag{6.106}$$

the creation and annihilation operators obey the usual commutation relation $[d_\lambda(\vec{k}), d_{\lambda'}(\vec{k}')^\dagger] = (2\pi)^3\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')$ and the classical equations of motions for π_T^{cl} follow from varying the quadratic π_T action (4.84), which is given by

$$\dot{N}_T^{cl} + 3HN_T^{cl} + 4H^2\epsilon\, c_T^2\pi_T^{cl} = 0 \tag{6.107}$$

$$N_T^{cl} = \frac{\dot{\pi}_T^{cl}}{1 + k^2/4a^2H^2\epsilon} \tag{6.108}$$

where the second one follows from (4.75) (after N_T^i is expressed in terms creation and annihilation operators in the same manner as π_T^i). Eliminating π_T^{cl}, we reach a second order differential equation for N_T^{cl}; using conformal time and keeping terms only up to the first order in slow roll, it reads

$$\frac{d^2}{d\tau^2}N_T^{cl} - \frac{2+4\epsilon_c-\eta_c-2u_c}{\tau}\frac{d}{d\tau}N_T^{cl} + \left[\frac{3\epsilon_c-3\eta_c-6u_c+4c_{T,c}^2\epsilon_c}{\tau^2} + k^2c_T(\tau)^2\right]N_T^{cl} = 0 \tag{6.109}$$

where $u \equiv \dot{c}_T/Hc_T = \frac{3c_{L,c}^2}{4c_{T,c}^2}s + O(\epsilon^2)$. The general solution for the equation above takes the form

$$\begin{aligned} N_T^{cl} &= (-\tau)^\beta\left[\mathcal{E}H_{\nu_V}^{(1)}\left((1+u_c)c_{T,c}\,k|\tau_c|\left(\frac{\tau}{\tau_c}\right)^{1-u_c}\right) + \mathcal{F}H_{\nu_V}^{(2)}\left((1+u_c)c_{T,c}\,k|\tau_c|\left(\frac{\tau}{\tau_c}\right)^{1-u_c}\right)\right] \\ &= (-\tau)^{-\beta}\left[\mathcal{E}H_{\nu_V}^{(1)}\left(-c_T(\tau)k\tau(1+u_c)\right) + \mathcal{F}H_{\nu_V}^{(2)}\left(-c_T(\tau)k\tau(1+u_c)\right)\right] \end{aligned} \tag{6.110}$$

where $\beta = -\frac{3}{2} - 2\epsilon_c + \frac{1}{2}\eta_c + u_c$ and $\nu_V = \frac{3}{2} + \frac{5}{2}u_c - c_{L,c}^2\epsilon_c + \frac{1}{2}\eta_c$. The constants $\mathcal{E}$ and $\mathcal{F}$ are determined via the initial condition, which is specified by

$$\lim_{\tau\to-\infty} N_T^{cl}(\vec{k},\tau) = \lim_{\tau\to-\infty}\frac{4H^2a\epsilon}{k^2}\frac{d}{d\tau}\pi_T^{cl} = -i\frac{\sqrt{4\epsilon c_T}H}{M_{\rm Pl}a}\frac{1}{k^{3/2}}e^{-ic_T(\tau)\,k\tau(1+u_c)} \tag{6.111}$$

where in the second equality we have used the fact that the mode function of the canonically normalized transverse vector field $\pi^{\rm can.}_{T,cl}$ should match that in the flat space vacuum at early times, that is

$$\pi^{cl}_T = \left[\frac{2a^2M_{\rm Pl}^2k^2}{4\left(1+\frac{k^2}{4a^2H^2\epsilon}\right)}\right]^{-1/2}\pi^{\rm can.}_{T,cl} \overset{k\tau\to-\infty}{\longrightarrow} \frac{1}{\sqrt{2\epsilon}\,M_{\rm Pl}Ha^2}\frac{e^{-ic_T(\tau)k\tau(1+u_c)}}{\sqrt{2c_Tk}}\,. \tag{6.112}$$

Therefore, $\mathcal{F}=0$ and

$$\mathcal{E} = -i\sqrt{2\pi\epsilon_c}\,\frac{c_{T,c}H_c^2}{kM_{\rm Pl}}\left(1+\frac{u_c}{2}-\epsilon_c\right)e^{\frac{i\pi}{2}(\nu_V+1/2)}\left(-\tau_c\right)^{-2\epsilon_c+\eta_c/2+u_c}+O(\epsilon^{5/2})\,. \tag{6.113}$$

Then, by (6.107), we can derive the expression for π^{cl}_T; in particular, its asymptotic behavior at very late time when the mode exits the horizon is given by

$$\lim_{k\tau\to0^-}\pi^{cl}_T(\vec{k},\tau) = \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c^2_{T,c}\epsilon_c}\left(-c_{T,c}\,k\,\tau_c\right)^{c^2_{L,c}\epsilon_c-\frac{\eta_c}{2}-\frac{5u_c}{2}}\left(\frac{-3H_c}{\sqrt{4\epsilon_c}M_{\rm Pl}c_{T,c}^{5/2}k^{5/2}}+O(\epsilon^{1/2})\right)\,, \tag{6.114}$$

the time dependence of which is the *same* as the tensor (4.98) and scalar (4.111) perturbations. The vector-to-scalar ratio simply

$$\lim_{k\tau\to0^-}\frac{\left|\pi^{cl}_T(\vec{k},\tau)\right|^2}{\left|\pi^{cl}_L(\vec{k},\tau)\right|^2} = \left(\frac{c^2_{L,c}}{c^2_{T,c}}\right)^{5/2}+\mathcal{O}(\epsilon)\,. \tag{6.115}$$

Given the relation (4.36), this is always smaller than $(1/3)^{5/2}\simeq 6\%$, which is reached only in the ultra-relativistic case with $c_L^2\simeq 1/3$ and $c_T^2\simeq 1$, or equivalently when $|F_Y+F_Z|\sim\mathcal{O}(\epsilon^2)|F|$.

6.12 Full trilinear action in SFSG

Expanding the Lagrangian to third order in fluctuations about the FRW background in SFSG we have after a straightforward but lengthy computation [3]

$$\begin{aligned}
\mathcal{L}^{(3)} = a(t)^3\Big\{ & 3M_{\rm Pl}^2H^2\,\overset{(1)}{\delta N}^3 + 2M_{\rm Pl}^2H\,\overset{(1)}{\partial_i N^i}\overset{(1)}{\delta N}^2 + 2M_{\rm Pl}^2\dot{H}a^2\,\overset{(1)}{N^j}\,\partial_j\pi^i(\dot{\pi}^i - \overset{(1)}{N^i}) \\
& - M_{\rm Pl}^2\,\overset{(1)}{\delta N}\left(\tfrac{1}{4}\,\overset{(1)}{\partial_i N^j}\overset{(1)}{\partial_j N^i} + \tfrac{1}{4}\,\overset{(1)}{\partial_j N^i}\overset{(1)}{\partial_j N^i} - \tfrac{1}{2}(\overset{(1)}{\partial_i N^i})^2\right) \\
& - \overset{(1)}{\delta N}\,M_{\rm Pl}^2\dot{H}\Big(- a^2(\dot{\pi}^i - \overset{(1)}{N^i})^2 - c_T^2[\Pi^T\Pi] + (1-c_T^2)[\Pi^2] - (1+c_L^2-2c_T^2)[\Pi]^2\Big) \\
& + [\Pi]^3\left(\tfrac{4}{3}F_{XXX}a^{-6} - \tfrac{8}{27}(F_{XZ}+F_{XY})a^{-2} + \tfrac{64}{243}F_Z + \tfrac{16}{81}F_Y\right) \\
& + [\Pi][\Pi^2]\left(\tfrac{4}{9}(F_{XZ}+F_{XY})a^{-2} - \tfrac{4}{9}F_Z - \tfrac{8}{27}F_Y\right) \\
& + [\Pi][\Pi^T\Pi]\left(2F_{XX}a^{-4} + \tfrac{4}{9}(F_{XZ}+F_{XY})a^{-2} - \tfrac{16}{27}F_Z - \tfrac{4}{9}F_Y\right) \\
& + \tfrac{2}{27}F_Z[\Pi^3] + \left(\tfrac{2}{3}F_Z + \tfrac{4}{9}F_Y\right)[\Pi^T\Pi\Pi] - \tfrac{4}{9}(F_Y+F_Z)a^2(\dot{\pi}^i - \overset{(1)}{N^i})\partial_i\pi^j(\dot{\pi}^j - \overset{(1)}{N^j}) \\
& + a^2[\Pi](\dot{\pi}^i - \overset{(1)}{N^i})^2\left(-2F_{XX}a^{-4} + \tfrac{4}{27}(F_Y+F_Z)\right)\Big\}\,,
\end{aligned} \tag{6.116}$$

where Π denotes the 3×3 matrix $\partial_i\pi^j$ and $[\cdots]$ indicates the trace; for instance, $[\Pi^T\Pi\Pi] \equiv \partial_j\pi^i\partial_j\pi^k\partial_k\pi^i$.

Now, as discussed in Section 4.5 one only needs to solve the constraint equations $\delta S/\delta N = 0$ and $\delta S/\delta N^i = 0$ to linear order in perturbations. We therefore don't need to worry about terms in N, N^i that are quadratic in the fluctuations contributing to the cubic Lagrangian. The solutions to these equations are given by (4.73), (4.74), and (4.75).

In particular, we are interested in two separate limits for computing the three-point function. The first is the de-mixing regime where $k \gg aH\epsilon^{1/2}$. In this limit, to lowest order in slow roll, we can effectively set δN and N_L to zero. Furthermore, note that all terms that are not of the form Π^3 (like the final $[\Pi]\dot{\pi}^2$ term) are explicitly suppressed by slow roll

[3] Since we are after the three-point function for scalar perturbations, we ignore the interaction between the tensor mode γ and the π fields.

parameters. We are left with

$$\mathcal{L}_3 = a(t)^3 M_{\rm Pl}^2 H^2 \frac{F_Y}{F}\left\{\tfrac{7}{81}(\partial\pi)^3 - \tfrac{1}{9}\partial\pi\partial_j\pi^k\partial_k\pi^j - \tfrac{4}{9}\partial\pi\partial_j\pi^k\partial_j\pi^k + \tfrac{2}{3}\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i\right\}, \quad (6.117)$$

where we neglected boundary terms. We have freely used the Friedmann equations (4.25), various definitions of slow roll parameters, and the the total derivative

$$\det(\partial\pi) = \tfrac{1}{6}\epsilon_{ijk}\epsilon_{lmn}\partial^i\pi^l\partial^j\pi^m\partial^k\pi^n = \tfrac{1}{6}\left([\partial\pi]^3 - 3[\partial\pi][\partial\pi^2] + 2[\partial\pi^3]\right) \quad (6.118)$$

(which is a total derivative because of the ϵ-tensor structure).

The second limit is in the strong mixing (with gravity) regime. This occurs when $k \ll aH\epsilon^{1/2}$. In this limit we can write to lowest order in slow roll

$$\delta N = \frac{k}{a}\frac{d}{d\tau}\left(\frac{\pi_L}{H}\right) \simeq k\epsilon\,\pi_L\,, \quad (6.119)$$

$$N_L = \dot{\pi}_L\,, \quad (6.120)$$

$$N_T^i = \dot{\pi}_T^i\,, \quad (6.121)$$

where we have estimated the time dependence of π via the explicit classical solution to the first order equation of motion given by (4.124a). When we insert the above expressions into the full cubic Lagrangian we see that all the terms involving these auxiliary fields are going to vanish, as the reccurring combination $(\dot{\pi}^i - N^i)$ vanishes to lowest order in slow roll and δN is explicitly of order $\epsilon(\partial\pi)$. So, surprisingly, we see that to lowest order in slow roll we recover the same expression (6.117) for the cubic Lagrangian in the *strong* mixing limit. This is a convenient fact, as it allows us to effectively use the same expression for the cubic interactions during the whole inflationary phase in our calculation of the three-point function in Section 4.7.

6.13 Three point function integral for general triangles

For momenta configurations that are not roughly equilateral, we need to be a little more careful in computing the integral (4.122), as there are many different regions in the integral,

each of which will necessitate the use of different expressions of $\pi_L(\tau, k)$ given by (4.124a) and (4.124b).

Assume for the general case that $k_1 > k_2 > k_3$ and define three separate times τ_1, τ_2, and τ_3 by

$$-c_L k_1 \tau_1 = \epsilon \,, \qquad -c_L k_2 \tau_2 = \epsilon \,, \qquad -c_L k_3 \tau_3 = \epsilon \,, \tag{6.122}$$

and for the observation time τ require $|\tau| < |\tau_1|$—that is, all modes are very long at the time of observation.

Splitting the object in question into regions defined by the times given above we can write schematically:

$$\begin{aligned} I(\tau; -\infty) \;\; = \;\; & \prod_{i=1}^{3} \pi_L(\tau, k_i) \cdot \left(\int_{-\infty}^{\tau_3} d\tau' \quad + \int_{\tau_3}^{\tau_2} d\tau' \quad + \int_{\tau_2}^{\tau_1} d\tau' \quad + \int_{\tau_1}^{\tau} d\tau' \right) \\ & \times \left(\frac{-i}{H_0^2 \tau'^4} \right) \prod_{j=1}^{3} \pi_L^*(\tau', k_j) + \text{c.c.} \end{aligned} \tag{6.123}$$

We can do each integral separately, and will find that provided that the the triangle does not become "too squeezed" (which will become precise in a moment) only the first integral contributes to leading order in slow roll. For each integral there is, of course, going to be an overall $H_0^{-2} \prod_{i=1}^{3} |\mathcal{B}_{k_i}|^2 (-c_L k_i \tau)^{c_L^2 \epsilon + \epsilon}$ as $|\tau| < |\tau_1|$.

6.13.1 Integrating from $-\infty$ to τ_3:

Ignoring for a moment the overall factor, we need to compute

$$\int_{-\infty}^{\tau_3} d\tau' \, \left(\frac{-i}{\tau'^4} \right) \prod_{i=1}^{3} \left(1 - i c_L k_i \tau' - \frac{1}{3} c_L^2 k_i^2 \tau'^2 \right) e^{i c_L k_i \tau'} + \text{c.c} \equiv I_3 \,. \tag{6.124}$$

In order to ensure convergence of the integral and project onto the right vacuum the integral is computed over a slightly tilted contour, that is $\tau' \to (1 - i\varepsilon)(\tau' + \tau_3)$, with $\varepsilon > 0$, and the

limits of integration are from $-\infty$ to 0. Performing the integral and expanding in τ_3 we have

$$\begin{aligned} I_3 &= -\frac{c_L^3 k_1 k_2 k_3 U(k_1,k_2,k_3)}{27} + \frac{c_L^5}{45}\left(k_1^5+k_2^5+k_3^5\right)\cdot\tau_3^2 \\ &\quad + \frac{c_L^7}{3780}\left(-3(k_1^7+k_2^7+k_3^7) + 7(k_1^5k_2^2 + 5 \text{ perms})\right)\cdot\tau_3^4 + \mathcal{O}(\tau_3^6)\ , \end{aligned} \tag{6.125}$$

where $U(k_1,k_2,k_3)$ is the scale invariant function given by (4.129). Now, as $\tau_3 = -\epsilon/c_L k_3$ we can see that provided $k_1/k_3 < (\sqrt{\epsilon})^{-1}$ only the zeroth order term in τ_3 contributes to leading order in slow roll. That is, for a not "too squeezed" triangle the first integral in (6.123) is given by

$$-\frac{c_L^3 k_1 k_2 k_3 U(k_1,k_2,k_3)}{27 H_0^2}\prod_{i=1}^{3}|\mathcal{B}_{k_i}|^2 \times (1+\mathcal{O}(\epsilon))\ , \quad \text{provided } \frac{k_1}{k_3} < \frac{1}{\sqrt{\epsilon}}\ . \tag{6.126}$$

6.13.2 Integrating from τ_3 to τ_2:

Again, dropping the overall factor, first note that

$$\begin{aligned} &\int_{\tau_3}^{\tau_2} d\tau' \left(\frac{-i}{\tau'^4}\right)\left((-c_L k_3\tau')^{\epsilon c_L^2+\epsilon}\right)\prod_{i=1}^{2}\left(1 - ic_L k_i\tau' - \frac{1}{3}c_L^2 k_i^2\tau'^2\right)e^{ic_L k_i\tau'} + \text{c.c} \\ <\ &\left((-c_L k_3\tau_3)^{\epsilon c_L^2+\epsilon}\right)\left[\int_{\tau_3}^{\tau_2} d\tau' \left(\frac{-i}{\tau'^4}\right)\prod_{i=1}^{2}\left(1 - ic_L k_i\tau' - \frac{1}{3}c_L^2 k_i^2\tau'^2\right)e^{ic_L k_i\tau'} + \text{c.c}\right] \\ \simeq\ &\int_{\tau_3}^{\tau_2} d\tau' \left(\frac{-i}{\tau'^4}\right)\prod_{i=1}^{2}\left(1 - ic_L k_i\tau' - \frac{1}{3}c_L^2 k_i^2\tau'^2\right)e^{ic_L k_i\tau'} + \text{c.c} \equiv I_2\ . \end{aligned} \tag{6.127}$$

This final form of the integral is straightforward to compute and yields

$$I_2 = -\frac{c_L^3(k_1^5+k_2^5)(k_2-k_3)(k_2+k_3)}{45 k_2^2 k_3^2}\cdot\epsilon^2 + \mathcal{O}(\epsilon^4) \tag{6.128}$$

which, provided that $k_1/k_3 < (\sqrt{\epsilon})^{-1}$ is first order in slow roll. That is, for a not "too squeezed" triangle the second integral in (6.123) is given by

$$\frac{c_L^3 k^3}{H_0^2}\prod_{i=1}^{3}|\mathcal{B}_{k_i}|^2 \times \mathcal{O}(\epsilon)\ , \quad \text{provided } \frac{k_1}{k_3} < \frac{1}{\sqrt{\epsilon}}\ . \tag{6.129}$$

6.13.3 Integrating from τ_2 to τ_1:

The exact same logic follows for this piece. The meat of the integral,

$$\int_{\tau_2}^{\tau_1} d\tau' \left(\frac{-i}{\tau'^4}\right)\left(1 - ic_L k_1 \tau' - \frac{1}{3}c_L^2 k_1^2 \tau'^2\right) e^{ic_L k_1 \tau'} + \text{c.c} \equiv I_1 \ , \tag{6.130}$$

yields

$$I_1 = \frac{c_L^3}{45} k_1^3 \left(1 - \frac{k_1^2}{k_2^2}\right) \cdot \epsilon^2 + \mathcal{O}(\epsilon^4) \ . \tag{6.131}$$

Which, provided that $k_1/k_2 < (\sqrt{\epsilon})^{-1}$ is first order in slow roll. And so, once again, for a not "too squeezed" triangle the third integral in (6.123) is given by

$$\frac{c_L^3 k^3}{H_0^2} \prod_{i=1}^{3} |\mathcal{B}_{k_i}|^2 \times \mathcal{O}(\epsilon) \ , \quad \text{provided } \frac{k_1}{k_2} < \frac{1}{\sqrt{\epsilon}} \ . \tag{6.132}$$

6.13.4 Integrating from τ_1 to τ:

As done in the main body of the text, we once can immediately see that this integral is going to be of the form:

$$\prod_{i=1}^{3} |\mathcal{B}_{k_i}|^2 \left(-c_L k_i \tau\right)^{c_L^2 \epsilon + \epsilon} \left[\int_{\tau_1}^{\tau} d\tau' \ (-i) g(\tau') + \text{c.c.}\right] \tag{6.133}$$

where $g(\tau)$ is pure real. And thus, the final integral in (6.123) simply vanishes.

6.13.5 Summary:

And so, for not "too squeezed" triangles, $I(\tau; -\infty)$ is completely dominated by the portion of the integral where all the modes are in the de-mixed regime, that is:

$$I(\tau; -\infty) = -\frac{c_L^3 k_1 k_2 k_3 U(k_1, k_2, k_3)}{27 H_0^2} \prod_{i=1}^{3} |\mathcal{B}_{k_i}|^2 \times \left(1 + \mathcal{O}(\epsilon)\right) \ , \quad \text{provided } \frac{k_i}{k_j} > \sqrt{\epsilon} \quad . \tag{6.134}$$

6.14 A factorizable template for non-gaussianities in solid inflation

Apart from isolated examples—most notably the so-called local form of non-gaussianities—factorization of a three-point function is typically possible only approximately.

We present here a factorizable form that is a very good approximation to our three-point function. As for other forms of non-gaussianities, finding an adequate factorizable approximation is a matter of trial and error. The zeroth order desiderata are: *(i)* it should have an overall k^{-6} scale dependence; *(ii)* it should be totally symmetric in k_1, k_2, k_3; *(iii)* it should have the correct squeezed limit. Notice that our squeezed limit, eq. (4.139), *is* factorizable, since

$$\cos\theta = \frac{\vec{k}_1 \cdot \vec{k}_2}{k_1 k_2} \,, \tag{6.135}$$

and the dot product can always be rewritten, using $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$, in terms of absolute values,

$$2\vec{k}_1 \cdot \vec{k}_2 = k_3^2 - k_1^2 - k_2^2 \,. \tag{6.136}$$

So our first (and final) guess, is to take the squeezed limit behavior and rewrite it in a totally symmetric form:

$$f_{\rm factor}(k_1, k_2, k_3) \equiv -\frac{20}{27}\left[\frac{k_1^2 k_3^2 - 3\big(\vec{k}_1 \cdot \vec{k}_3\big)^2}{k_1^5 k_3^5} + 2 \text{ permutations}\right] , \tag{6.137}$$

where the factor of 2 difference with eq. (4.139) takes into account that for $k_3 \ll k_1, k_2$ we are getting the same contribution from the terms explicitly displayed and from one of the permutations.

This ansatz obeys all the properties above, and performs surprising well in approximating our f away from the squeezed limit. Quantitatively, for flattened configurations ($k_1 = k_2 + k_3$)—where our three-point function has quite non-trivial features—the relative difference is very small, reaching a maximum of roughly 3% for 'folded' triangles ($k_2 = k_3 = k_1/2$). For

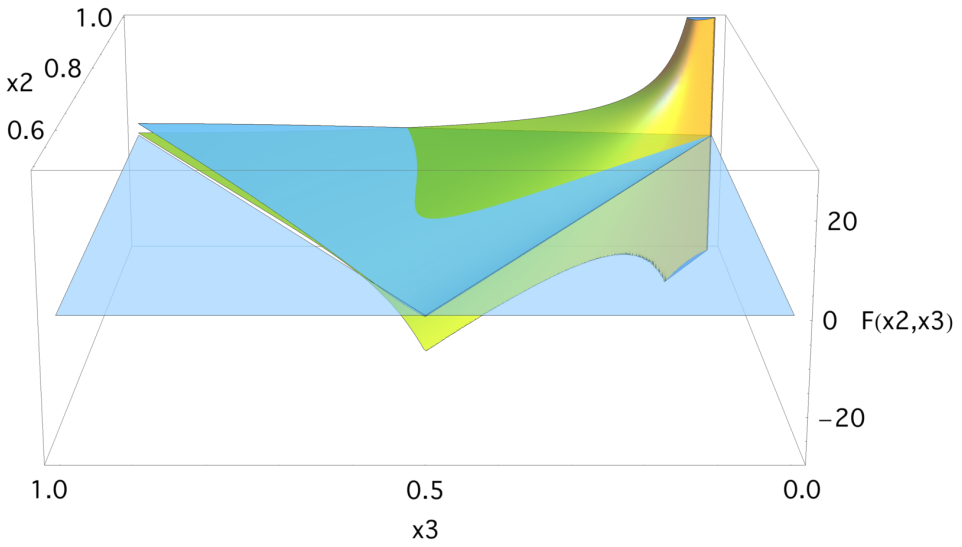


Figure 6.1: *Our factorizable approximation to the three-point function. The flat triangular surface is the difference with our exact three-point function. As clear from the picture, this is maximized for equilateral configurations ($x_2 = x_3 = 1$), and is always quite small.*

equilateral configurations the difference is more substantial, but still quite small in absolute terms, given that our signal is small there. We plot this factorizable form in fig. 6.1, alongside its difference with the exact three-point function. The cosine between the two shapes is

$$\cos(f, f_{\rm factor}) \simeq 97\% \ . \tag{6.138}$$

If this level of precision is not enough, one can improve (6.137) by adding to it the right admixture of the equilateral factorizable form of [62], to make up for the small difference in the equilateral limit:

$$f_{\rm factor} \to f_{\rm factor}(k_1,k_2,k_3) + f_{\rm equil}(k_1,k_2,k_3) \cdot \left. \frac{f - f_{\rm factor}}{f_{\rm equil}} \right|_{k_1=k_2=k_3} \ . \tag{6.139}$$

This way the cosine with f rises to roughly 99.5%.

6.15 Linear couplings of a $U(1)$ Goldstone

Consider a pair of complex scalar fields charged under a $U(1)$ global symmetry. For consistency of notation with our fluid case, let us denote them as ϕ and χ. Then we have a Lagrangian of the sort

$$\mathcal{L} = \mathcal{L}_\phi[\phi] + \mathcal{L}_\chi[\chi, \phi] \,, \tag{6.140}$$

invariant under the $U(1)$ transformation

$$\phi \to e^{i\alpha}\phi \,, \qquad \chi \to e^{iq\alpha}\chi \tag{6.141}$$

(we are allowing for different charges for ϕ and χ.) Notice that for the moment we are using a slightly different notation from the main text: we are including in $\mathcal{L}_\chi$ both the χ-sector's dynamics, and its interactions with the ϕ sector. For instance, $\mathcal{L}_\chi$ might contain interactions of the form

$$\mathcal{L}_\chi \supset \lambda\big(\phi^{2q}\chi^{*\,2} + \text{h.c.}\big) \,. \tag{6.142}$$

As in the case of spatial translations for our fluid, let us imagine now that this $U(1)$ symmetry is spontaneously broken by the vev of ϕ: $\langle\phi\rangle = v$. Then, as it is standard, we can parameterize ϕ as

$$\phi = (v + \rho)e^{i\pi}, \tag{6.143}$$

with π being the Goldstone boson associated with the symmetry breaking, and ρ the (generically) heavy radial excitation, which can be ignored at very low energies. For the sake of argument, let us thus set ρ to zero from now on. Note that the symmetry is now realized non-linearly on π, i.e. $\pi \to \pi + \alpha$.

We can expand the action as

$$\mathcal{L} = \mathcal{L}_\phi[v\, e^{i\pi}] + \mathcal{L}_\chi[\chi, v\, e^{i\pi}] \tag{6.144}$$

the same way we expanded $\phi^I = x^I + \pi^I$ for the fluid. However, in this parameterization of the fields it is not obvious that π is derivatively coupled, as guaranteed from standard

soft-pion theorems for the emission of a single soft π quantum. For instance, from (6.142) we get a coupling

$$\mathcal{L}_\chi \supset \lambda v^2 \big(e^{i\,2q\pi}\chi^{*\,2} + \text{h.c.}\big)\ , \tag{6.145}$$

which does not involve derivatives of π. This stems from a suboptimal choice of the field variables, and is easily remedied via a non-linear redefinition of the χ field, which as usual does not change the S-matrix:

$$\chi = \chi' e^{iq\pi}\ . \tag{6.146}$$

Notice that the new χ' field is invariant under the $U(1)$ symmetry—the transformation of χ is now carried solely by the $e^{iq\pi}$ factor—and the action becomes:

$$\mathcal{L} = \mathcal{L}_\phi[v\,e^{i\pi}] + \mathcal{L}_\chi[\chi' e^{iq\pi}, v\,e^{i\pi}] \tag{6.147}$$

Let's focus on the $\mathcal{L}_\chi$ part. By the $U(1)$ symmetry—which now only acts on π—this action must be invariant under constant π shifts, $\pi \to \pi + \alpha$. Then, interpreting the π in $\mathcal{L}_\chi[\chi' e^{iq\pi}, v\,e^{i\pi}]$ as a weakly spacetime-dependent $U(1)$ transformation parameter, from Nöther's theorem we get

$$\mathcal{L}_\chi[\chi' e^{iq\pi}, v\,e^{i\pi}] = \mathcal{L}_\chi[\chi', v] - \partial_\mu\pi\, J^\mu_\chi + \dots\ , \tag{6.148}$$

where J^μ_χ is the χ-sector's contribution to the $U(1)$ Nöther current, and we omitted terms with more π's or more derivatives. In other words, at linear order in π and at lowest order in the derivative expansion, the interaction between π and the χ sector has to take the form

$$\mathcal{L}_{\text{int}} \simeq -\partial_\mu\pi\, J^\mu_\chi\ , \tag{6.149}$$

which is exactly the $U(1)$ analog of our eq. (5.28). Notice that, at this order, it does not matter whether we evaluate J^μ_χ at χ or χ', since their difference is of first order in π. Notice also that we never really used that χ is scalar. Clearly our proof is completely general and holds for any set of charged fields χ, of any spin.

As an alternative, quicker derivation of the same result, we can go back to eq. (6.144) and use the following trick (a version of Stückelberg's trick): We promote the global symmetry to a gauge symmetry by introducing an auxiliary gauge field $\mathcal{A}_\mu$ which transforms as $\mathcal{A}_\mu \to \mathcal{A}_\mu - \partial_\mu \alpha(x)$, and replace standard derivatives by covariant ones. Expanding the action to linear order in $\mathcal{A}_\mu$ we have

$$\mathcal{L}[\phi, \chi, \mathcal{A}_\mu] = \mathcal{L}[\phi, \chi] + J_\mu \mathcal{A}^\mu + \mathcal{O}(\mathcal{A}^2), \tag{6.150}$$

where $J_\mu[\phi, \chi]$ is the conserved current (in the absence of $\mathcal{A}^\mu$) for the $U(1)$ global charge.[4] Since we promoted this symmetry to a gauge transformation we are guaranteed that π *disappears* from the action, because it can be absorbed into $\mathcal{A}_\mu$ by choosing $\alpha = -\pi$. This means that, to linear order in π, we wind up with the coupling

$$J^\mu_\chi \partial_\mu \pi, \tag{6.151}$$

where J^μ_χ is the χ-dependent component of the full current at zeroth order in π.[5] Hence we conclude that, at leading order in the perturbations, the Goldstone boson couples to the π-independent part of the current associated with the broken symmetry.

The introduction of $\mathcal{A}_\mu$ is equivalent to working in the so called *unitary* gauge, where the Goldstones are set to zero and their interactions are encoded in the gauge field. The previous analysis suggests that one could perform similar manipulations in the case of our fluid, where the Goldstone fields π^I are associated with the breaking of spatial translations. Now, to go to the unitary gauge we must introduce the gauge field associated with spatial translations, namely the metric perturbation $h_{\mu I}$, and the broken generators are the $T^{\mu I}$ components of the stress energy tensor. Hence, the coupling must read: $T^{\mu I}_\chi h_{\mu I}$. To introduce the pions we

[4]Note we included all the gauge-field dependence explicitly (in the $\mathcal{O}(\mathcal{A}^2)$ terms), so that there is no $\mathcal{A}^\mu$ in J^μ, which would be necessary to make it a gauge invariant expression.

[5]The alert reader may have already recognized this is the way longitudinal gauge bosons couple to matter.

do as before, which in our case entails schematically

$$h_{\mu I} = \partial_{(\mu}\alpha_{I)} \to \partial_{(\mu}\pi_{I)}. \tag{6.152}$$

This viewpoint might prove useful in extending our results to non-linear order.

6.16 Matching the attenuation rates

In order to match our computed attenuation rates for the sound and heat modes given by (5.64) and (5.65) with the known values given by (5.66) and (5.67) and fix our free parameters B and C we must express many thermodynamic quantities such as c_s^2, $(\partial\rho/\partial T)_n$, etc. in terms of our effective theoretical variables b, y and various derivatives of F. Explicitly they are as follows:

$$c_s^2 = \frac{\mathrm{d}(F - F_b b_0)}{\mathrm{d}(F_y y_0 - F)}\Big|_{\mathrm{d}(F_y/b)=0} = \frac{(F_y - F_{by}b_0)^2 - b_0^2 F_{yy}F_{bb}}{w_0 F_{yy}}, \tag{6.153}$$

$$\frac{\partial\rho}{\partial T}\Big|_n = \frac{d(-F + F_y y_0)}{d(-F_b)}\Big|_{dF_y=0} = -\frac{F_{yy}F_b}{(F_{by}^2 - F_{bb}F_{yy})}, \tag{6.154}$$

$$\frac{\partial p}{\partial T}\Big|_n = \frac{d(F - F_b b_0)}{d(-F_b)}\Big|_{dF_y=0} = b_0 + \frac{F_y F_{yb}}{F_{bb}F_{yy} - F_{by}^2}, \tag{6.155}$$

$$\frac{\partial p}{\partial n}\Big|_T = \frac{d(F - F_b b_0)}{d(F_y)}\Big|_{dF_b=0} = -\frac{F_y F_{bb}}{F_{by}^2 - F_{bb}F_{yy}}\ . \tag{6.156}$$

(For c_s^2, the $F_y/b = n/s = \text{const}$ constraint is equivalent to the usual one, $S = \text{const}$, $N = \text{const}$ [14].) Along with the relations (1.38)-(1.41) this is all we need to perform the matching. We find that for the two computations to coincide

$$B = -C \quad \text{with } |B| = -\frac{y_0 w_0}{b_0 F_b} = \frac{\mu(\rho + p)}{Ts} \tag{6.157}$$

as quoted in the text.