Discrete Gibbsian line ensembles and weak noise scaling for directed polymers

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ABSTRACT

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In this thesis we investigate three projects within in the field of KPZ universality class and integrable probability.

The first project studies the weak KPZ universality for half-space directed polymers in dimension 1+1. This is the half-space analogue of the full-space polymers studied in [AKQ]. The novelty is the extra random environment introduced at the boundary. The new technical challenges are the estimates for half-space heat kernels which are super-probability measures and accurate estimates on visits to origin (weighted by the boundary randomness) for a simple symmetric walk in dimension 1 and 2 respectively.

The second project introduces a framework to prove tightness of a sequence of discrete Gibbsian line ensembles $\mathcal{L}^N = \{\mathcal{L}^N_k(u), k \in \mathbb{N}, u \in \frac{1}{N}\mathbb{Z}\}$, which is a countable collection of random curves. The sequence of discrete line ensembles $\mathcal{L}^N$ we consider enjoys a resampling invariance property, which we call $(\mathbf{H}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property. We assume that $\mathcal{L}^N$ satisfies technical assumptions A1-A4 on $(\mathbf{H}^N, \mathrm{H}^{\mathrm{RW},N})$ and the assumption that the lowest labeled curve with a parabolic shift, $\mathcal{L}^N_1(u) + \frac{u^2}{2}$, converges weakly to a stationary process in the topology of uniform convergence on compact sets. Under these assumptions, we prove our main result Theorem 3.1.13 that $\mathcal{L}^N$ is tight as a sequence of line ensembles and that the $\mathbf{H}$-Brownian Gibbs property holds for all subsequential limit line ensembles with $\mathbf{H}(x) = e^x$. As an application of Theorem 3.1.13, under the weak noise scaling, we show that the scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ is tight, which is a sequence of discrete line ensembles associated with the log-gamma polymer model via the geometric RSK correspondence. The $\mathbf{H}$-Brownian Gibbs property (with $\mathbf{H}(x) = e^x$) of its subsequential limits also follows.

The third project proves an analogue of the classical Komlós-Major-Tusnády (KMT) embedding theorem for random walk bridges and it serves as a key technical input for the second project. The random bridges we consider are constructed through random walks with i.i.d jumps that are conditioned on the locations of their endpoints. We prove that such bridges can be strongly coupled to Brownian bridges of appropriate variance when the jumps are either continuous or integer valued under some mild technical assumptions on the jump distributions. Our arguments follow a similar dyadic scheme to KMT's original proof, but they require more refined estimates and stronger assumptions necessitated by the endpoint conditioning. In particular, our

result does not follow from the KMT embedding theorem, which we illustrate via a counterexample.

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Acknowledgments

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I dedicate this thesis to July.

Chapter 1

Introduction

The KPZ equation was introduced in 1986 by Kardar, Parisi and Zhang [KPZ] as a model for random interface growth, interacting particle systems, and directed polymers. In one-spatial dimension (sometimes also called (1+1)-dimension to emphasize that time is one dimension too), it describes the evolution of a function $h(t,x)$ recording the height of an interface at time $t > 0$ above position $x \in \mathbb{R}$. The equation for $h(t,x)$ is the following non-linear stochastic partial differential equation (SPDE)

$$\partial_t h(t,x) = \frac{1}{2}\partial_x^2 h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x), \tag{1.1}$$

which is driven by a space-time white noise ξ.

The Kardar-Parisi-Zhang (KPZ) universality class is a large class of models, which covers a wide range of mathematical and physical systems of distinct origins, including interacting particle systems, random matrices, traffic models, directed polymers in random media and non-linear stochastic PDEs. All models in the KPZ universality class can be transformed to a kinetically growing interface reflecting the competition between growth in a direction normal to the surface, a surface tension smoothing force, and a stochastic term which tends to roughen the interface. The KPZ equation is a key member of the KPZ universality class and a model belongs to the KPZ universality class if it bears the same long-time, large-scale behavior as the KPZ equation. Numerical simulations along with some theoretical results have confirmed that in the long time t scaling limit, fluctuations in the height of such evolving interfaces scale like $t^{1/3}$ and display non-trivial spatial correlations in the scale $t^{2/3}$ (known as the 3:2:1 KPZ scaling). There are two main goals of great importance: (1) to show the universality of the scalings, statistics and limit objects (known as the **strong KPZ universality conjecture**), and (2) to find systems with solvability and integrability which enables exact formulas for quantities associated with these systems. These two complementary goals are behind significant amounts of on-going research within probability, mathematical physics and related fields.

How does the KPZ equation arise from microscopic systems? This is a proxy for understanding how discrete models may converge to the KPZ equation in (1.1). The first approach is to look at the scalings for $h(t,x)$ such that the KPZ equation remains invariant. Fixing $(b,z) \in \mathbb{R}$ and letting $h_\varepsilon(t,x) := \varepsilon^b h\left(\varepsilon^{-z}t, \varepsilon^{-1}x\right)$), one could directly compute the equation satisfied by $h_\varepsilon(t,x)$ and it is a scaled version of (1.1). There are no choices for (b,z) besides $(0,0)$ which leave the equation invariant, see for instance [Cor]. One may, however, simultaneously scale coefficients in (1.1) to compensate for the effects of the (b,z)-scaling, when one performs (b, z)-scaling while also scaling model parameters to effectively tune coefficients. This is called **weak scaling**, and significant efforts have sought to show **weak KPZ universality**, meaning that general classes of processes converge to under the KPZ equation in (1.1) under such weak scaling.

In this thesis I will investigate three projects I did, within in the scope of the discussions above.

1.0.1 Half-space directed polymers

The directed polymers were introduced in the statistical physics literature by Huse and Henley [HH] in 1985 and received first rigorous mathematical treatment in 1988 by Imbrie and Spencer [IS]. The monograph [Com] is a great resource for the foundational work in this area. The polymer measure in dimension 1+1 is a random probability measure on paths which favors higher weighted paths. It is constructed through up / right paths on $\mathbb{Z}^2$ with path measure re-weighted by an i.i.d. random environment ω presented at each lattice points. Energy of a path is defined as the sum of the weights the walker collects from the bulk random environment ω and it is weighted by the so-called inverse temperature β. The highest energy path(which enjoys the largest sum of weights along the path) dominates as β goes to infinity. One general goal for directed polymer model is to study the behavior of the free energy when n goes to infinity and β varies.

In the direction of strong KPZ universality of directed polymers, the first rigorous verification of the 1/3 fluctuation of polymer free energy was proven for a special case [Sep], where the intergrable log-gamma polymers were introduced. Among directed polymers, the log-gamma directed polymer model was first introduced in [Sep], which is special in the same way as the last-passage percolation model with exponential or geometric weights is special among corner growth models, namely, both demonstrate integrable structures and permit explicit computations. Via a Fredholm determinant identity, [BCR] proves the limiting GUE Tracy-Widom fluctuation for the log-gamma polymer free energy. A major break through for the weak KPZ universality is achieved in the work by Alberts, Khanin and Quastel [AKQ], where the convergence under **weak noise scaling** for general polymer free energy to the KPZ equation is established.

It is natural to ask the same question for the half-space polymers. The half-space directed polymers are constructed through up / right paths constrained to stay in the first quadrant with path measure re-weighted by two random environments(X present only at the boundary and ω in the bulk). Boundary environment X penalizes or rewards the path measure every time the walker visits the origin in an i.i.d manner. As was treated in [AKQ] for the full space case, they showed by scaling inverse temperature like $\beta n^{-1/4}$ and scale random walk diffusively, the point-to-point partition function converges in distribution to the chaos series for the solution to SHE with delta initial data, (the Hopf-Cole solution to the KPZ equation), which suggests the connection between polymer models and the KPZ universality. We adopt the same scaling regime for the half-space polymer model and we show that the point-to-point partition function now converges to chaos series for the solution to SHE with Robin boundary condition. Moreover, we deal with boundary condition by tuning the reflection rate $\gamma = 1 - \mu/\sqrt{n}$ of the boundary environment. The discrete heat kernel estimates for $\mu < 0$ introduces extra technical difficulties as the the polymer measure are super probability measures and one needs accurate estimates on visits to origin (weighted by the boundary randomness) for a simple symmetric walk in dimension 1 and 2 respectively. We summarize the setup of half-space polymers in the following Table 1.1.

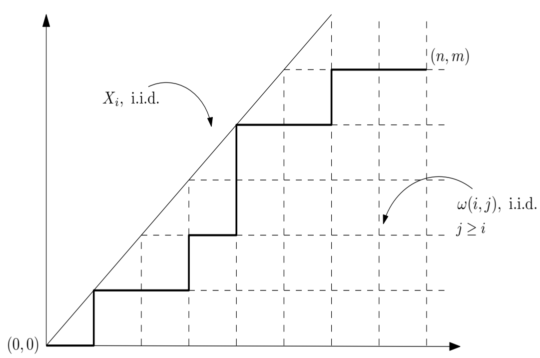
Half-quadrant polymers with ω and X	Definition of partition functions
	Let $\beta > 0$, the point-to-point partition function is defined as: $Z_n^{\omega,X}(x;\beta) := \mathbb{E}_R\left[e^{\beta H_n^\omega(S)}\left(\prod_{\substack{i:S_i=0\\0\le i<n}} X_i\right)\cdot \mathbb{1}\{S_n = x\}\right]$, where the expectation is taken with respect to probability measure $\mathbb{P}_R$ and preserves randomness from ω and X. $H_n^\omega(S) := \sum_{i=0}^{n-1}\omega(i,S_i)$ is the energy of an n-step nearest neighbor walk S.

Table 1.1: Summary of the half-space polymer model.

The limiting behavior of the point-to-point partition function $Z_n^{\omega,X}$ under weak noise scaling is the main focus of this project. Our main result below shows that by taking β_n to zero ($\beta_n := \beta n^{-1/4}$), scaling the reflected random walk diffusively and X weakly, $Z_n^{\omega,X}$ converges in distribution to the solution of stochastic heat equation (SHE) on a half-line with Robin boundary condition in the topology of uniform convergence on compact sets. The precise statement is

Theorem 1.0.1.

$$e^{-n\lambda(\beta n^{-1/4})}\frac{\sqrt{n}}{2}\cdot Z_n^{\omega,X}\left(x\sqrt{n};\beta n^{-1/4}\right) \xrightarrow{(d)} \mathcal{Z}_{\sqrt{2}\beta}(1,x).$$

Here $\mathcal{Z}_{\sqrt{2}\beta}(1,x)$ *is the time 1 solution to half-space SHE equation*

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_{xx}\mathcal{Z} + \sqrt{2}\beta\mathcal{Z}\cdot\xi$$

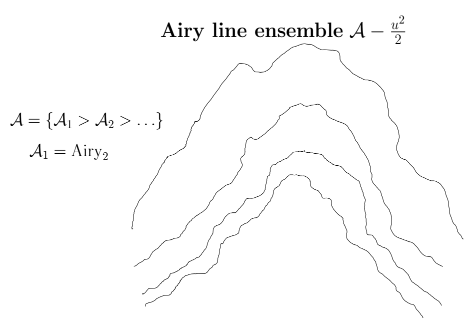
with delta initial data and Robin boundary condition:

$$\mathcal{Z}(0, \cdot) = \delta(0),$$
$$\partial_x \mathcal{Z}(\cdot, x)|_{x=0} = \mu \cdot \mathcal{Z}(\cdot, 0).$$

By a slight modification of the arguments of Theorem 1.0.1, a particularly interesting application is the weak convergence of the point-to-point partition function of the integrable half-space inverse gamma polymer. This convergence result has been applied in [Pa19b] to prove an equality (in one-point distribution) for the solutions to SHE on the half space with different boundary conditions. Prior to my study, in the direction of integrable probability, [BBC] obtains integral formulas for the Laplace transform of the partition function. There has also been many progresses on the half-space ASEP. [CS] shows that under weak asymmetry scaling, the height functions for open ASEP on the half-line and on a bounded interval converge to the Hopf-Cole solution of the KPZ equation Neumann boundary conditions where the boundary parameters are positive. Later [Pa19a] extends the results in [CS] to negative parameters and for more general initial conditions.

1.0.2 Discrete Gibbsian line ensembles

There is a large class of stochastic integrable models from random matrix theory, last passage percolation, and more generally from the Kardar-Parisi-Zhang (KPZ) universality class that naturally carry the structure of random paths with some Gibbsian resampling invariance. One particularly interesting and central example is the Airy line ensemble [CH14] $\mathcal{A} = \{\mathcal{A}_1 > \mathcal{A}_2 > \cdots\}$, a collection of non-intersecting continuous random curves indexed by $\mathbb{N}$, see Table 1.2 for some basic properties of the Airy line ensemble $\mathcal{A}$.



- Non-intersecting paths, i.e. for any $u \in \mathbb{R}$, $\mathcal{A}_1(u) > \mathcal{A}_2(u) > \cdots$
- Top curve $\mathcal{A}_1$ is Airy_2 process.
- $\mathcal{A}(u)$ is the Airy point process for $u \in \mathbb{R}$ fixed.
- Brownian Gibbs property of Airy line ensemble after subtracting a parabola $\frac{u^2}{2}$.

Table 1.2: Illustration of Airy line ensemble $\mathcal{A}$, with four curves drawn.

The Airy line ensemble $\mathcal{A}$ has been proven to be a universal edge scaling limit of a wide range of models, e.g. Gaussian unitary ensemble, Dyson Brownian motion, Brownian last passage percolation, polynuclear growth model, see [OY, PS, DNV]. Another beautiful aspect about Airy line ensemble $\mathcal{A}$ is that the Airy$_2$ process and the Airy point process are embedded together into $\mathcal{A}$. Moreover, after the subtraction of a parabola, $\widetilde{\mathcal{A}} := \mathcal{A} - \frac{u^2}{2}$ enjoys the Brownian Gibbs property (introduced in [CH14]), which is a spatial Markov property and also a global resampling invariance property under the following resampling process. Taking any $a < b \in \mathbb{R}, k \in \mathbb{N}$, first remove the trajectory of the k-th curve $\widetilde{\mathcal{A}}_k$ between $[a, b]$ and then resample a trajectory according to the law of a Brownian bridge which avoids the upper curve $\widetilde{\mathcal{A}}_{k-1}$ and the lower curve $\widetilde{\mathcal{A}}_{k+1}$ (note that we could run the same process for finite adjacent curves by resampling non-intersecting Brownian bridges). In other words, conditioned on the values of $\widetilde{\mathcal{A}}$ outside a compact set $C = \{k_1, k_1 + 1, \cdots, k_2\} \times [a, b]$, the law of $\widetilde{\mathcal{A}}$ inside C only depends on the boundary data (i.e. independent of values of $\mathcal{L}$ outside C). Furthermore, this conditional law of $\mathcal{A}$ on C is equivalent to the law of Brownian bridges with endpoints to be $\vec{x} = (\mathcal{A}_{k_1}(a), \cdots, \mathcal{A}_{k_2}(a))$ and $\vec{y} = (\mathcal{A}_{k_1}(b), \cdots, \mathcal{A}_{k_2}(b))$ conditioned not to intersect (including not to touch upper and lower boundaries $\mathcal{A}_{k_1-1}$ and $\mathcal{A}_{k_2+1}$).

The construction of the Airy line ensemble [PS, AM, CH14] is a marriage of integrability and probability, through taking a functional limit of Brownian watermelon under edge scaling limit, see Figure 1.1. In

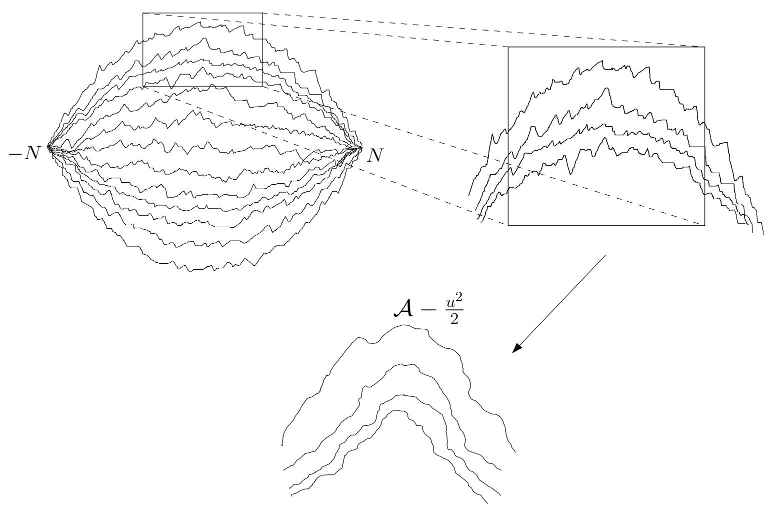


Figure 1.1: Airy line ensemble as the edge scaling limit of Brownian watermelon.

the top left we have a Brownian watermelon $\mathcal{B} := \{B_1, \cdots, B_N\}$, a collection of N Brownian bridges $B_i : [-N, N] \to \mathbb{R}, B_i(-N) = B_i(N) = 0, 1 \le i \le N$, conditioned not to intersect. The edge scaling limit of this system is obtained by taking a weak limit as $N \to \infty$ of the collection of curves scaled so that the point $(0, 2^{1/2}N)$ is fixed and space is squeezed, horizontally by a factor of $N^{2/3}$ and vertically by $N^{1/3}$. Tightness under such edge scaling was established in [CH14] by extensively exploiting the Brownian Gibbs property of Brownian watermelon (which naturally holds by the construction of Brownian watermelon) and showed the tightness. Moreover, it is demonstrated in [CH14] that this Brownian Gibbs property survives under weak convergence of line ensembles, i.e. $\mathcal{A} - \frac{u^2}{2}$ enjoys the Brownian Gibbs property.

The KPZ equation is a central model in the KPZ universality class for random growth processes, interacting particle systems, and directed polymers, see [Cor, QS]. Many discrete growth models have a tunable asymmetry and the KPZ equation appears as a continuum limit in the diffusive time scale (known as the weak scaling) as this parameter is critically tuned close to zero, e.g. ASEP, directed polymers, stochastic vertex models, see [BG, AKQ, CGHT, Lin].

As the Brownian Gibbs property for Airy line ensemble has proven to be a powerful probabilistic resampling tool, it is motivating to embed and study the KPZ equation as the top curve of some Gibbsian line ensemble $\mathcal{H}$, which is constructed and explored in [CH16], called the KPZ line ensemble. The construction of the KPZ line ensemble in[CH16] is based on subsequence extraction through O'Connell-Yor semi-discrete direct polymers and the characterization of the KPZ line ensemble through O'Connell-Warren's [OW] multilayer extension of the solution to the stochastic heat equation with narrow wedge initial data are established in [Nic].

While the non-intersecting property for Airy line ensemble $\mathcal{A}$ being a nature of the zero-temperature models, the curves of KPZ line ensemble $\mathcal{H}$ now could go out of order but subject to an exponential penalization. More specifically, the KPZ line ensemble enjoys the **H**-Brownian Gibbs property, which is a more general type of Gibbs property compared to Brownian Gibbs property. The **H**-Brownian Gibbs property for $\mathcal{H}$ specifies the law of $\mathcal{H}$ inside a compact set $C := [k_1, k_2] \times [a, b]$ conditioned on the values of $\mathcal{H}$ outside C such that the conditional law is equivalent to that of a few independent Brownian bridges reweighted by a penalization factor for being out order, see an illustration in Figure 1.2.

A longstanding conjecture about the KPZ equation is that the solution of the narrow wedge initial data KPZ equation converges to the Airy_2 process (as t goes to infinity) with a parabolic shift, under horizontal scaling by $t^{2/3}$ and vertical scaling by $t^{1/3}$. While the full conjecture is widely open, there has been breakthrough on the one point convergence, see [SS, ACQ]. This conjecture was further strengthen in [CH16] that the the KPZ line ensemble should converge to the Airy line ensemble under the same scaling. [CH16] also provided a plausible route to this conjecture and one of the key steps is to characterize Airy line ensemble without relying on the determinantal formula of its finite dinmensional distributions, thus providing a new method for proving convergence in the KPZ universality class. A very recent work [DM] showed that the Airy line ensemble could be characterized by the finite-dimensional marginals of its top curve and the Brownian Gibbs property.

Upon discovery, the Brownian Gibbs property has served as a powerful probabilistic tool. Recently, one of the authors of [CH14], Hammond, developed a more delicate treatment in [Ham1] for Brownian Gibbs resampling invariance to estimate the modulus of continuity for line ensembles with Brownian Gibbs property

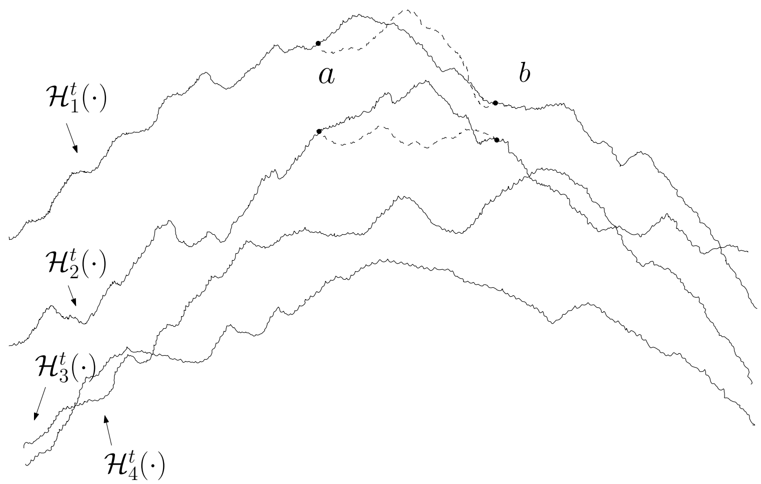


Figure 1.2: An overview of KPZ$_t$ line ensemble $\mathcal{H}^t$ for fixed t. Curves $\mathcal{H}_1^t(\cdot)$ through $\mathcal{H}_4^t(\cdot)$ are sampled. The lowest indexed curve $\mathcal{H}_1^t$ is distributed according to the time t solution to KPZ equation with narrow wedge initial data. The dotted first two curves $\mathcal{H}_1^t$ and $\mathcal{H}_2^t$ between a and b indicate a possible resampling, as a demonstration for the **H**-Brownian Gibbs property.

(e.g. Airy line ensemble and the line ensemble associated with Brownian last passage percolation). Hammond also established L^p-norm bounds (for finite $p > 0$) on Radon-Nikodym derivative of the line ensemble curves (with an affine shift) with respect to Brownian bridges and other refined regularity properties. Furthermore in the subsequent papers [Ham2, Ham3, Ham4], the work in [Ham1] was applied to understanding the geometry of last passage paths in Brownian last passage percolation with more general initial data. Another breakthrough is the construction of the Directed Landscape [DOV], the conjectural central limit of the KPZ universality class [CQR], where they found that the Airy sheet is already embedded in the Airy line ensemble and the Brownian Gibbs property is a key input of their estimates.

While there has been many successes for the study of continuous Gibbs line ensembles in the KPZ universality class, the discrete Gibbsian line ensembles are also worth exploration and will be the focus of this thesis, due to the richness of discrete integrable models in the KPZ universality class. The discrete Gibbsian line ensembles enjoy a discrete analogue resampling invariance of the previous Brownian Gibbs property. We call such resampling invariance random walk Gibbs property to emphasize that for line ensembles of Brownian Gibbs property or more generally, **H**-Brownian Gibbs property, the underlying paths resemble Brownian bridges, while for discrete Gibbsian line ensembles, the underlying paths resemble random walk bridges. To give a few examples, through various versions of the Robinson-Schensted-Knuth (RSK) correspondence [O'Con1], one can link geometric last passage percolation to non-intersecting random walk bridges with geometric jumps, exponential last passage percolation to non-intersecting random walk bridges

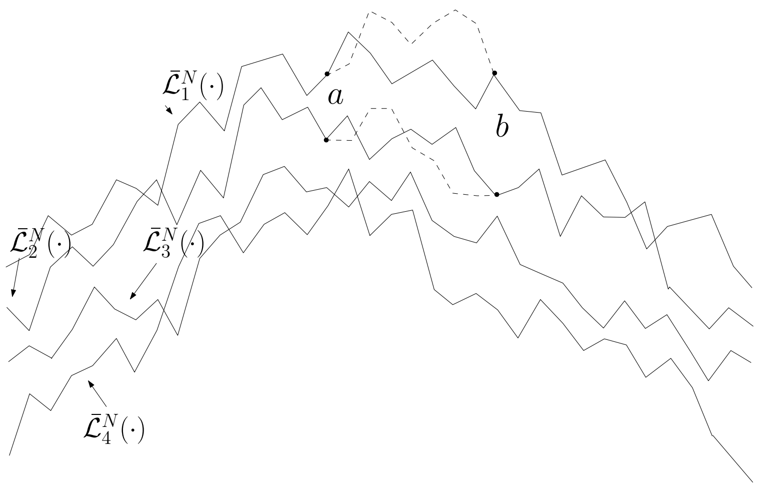


Figure 1.3: An overview of scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$. Discrete curves $\overline{\mathcal{L}}_1^N(\cdot)$ through $\overline{\mathcal{L}}_4^N(\cdot)$ are sampled on $\frac{2}{\sqrt{N}}\mathbb{Z}$ (linked through the linear interpolation between two adjacent points). The lowest indexed curve $\overline{\mathcal{L}}_1^N(\cdot)$ converges weakly to time t solution to the KPZ equation with narrow wedge initial data as N goes to infinity. The dotted first two curves $\overline{\mathcal{L}}_1^N(\cdot)$ and $\overline{\mathcal{L}}_2^N(\cdot)$ between a and b indicate a possible resampling, as a demonstration for the random walk Gibbs property.

with exponential jumps, see [DNV] for a recent study on the uniform convergence to Airy line ensemble for these two line ensembles.

In this thesis, we aim to study a sequence of log-gamma discrete line ensemble $\mathcal{L}$ with a discrete Gibbs resampling invariance property, which we call $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property. The construction of this line ensemble $\mathcal{L}$ and its $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property come from the study in [COSZ] of a geometric RSK correspondence, when applied to the log-gamma directed polymers. We further apply the weak noise scaling to the log-gamma line ensemble $\mathcal{L}$. This scaling regime is known as the intermediate disorder regime in [AKQ], where they showed the convergence of directed polymers with general random environment to the KPZ equation with narrow wedge initial data, hence establishing the weak KPZ universality for directed polymers. Denoting $\overline{\mathcal{L}}^N$ as the scaled log-gamma line ensemble (see Figure 1.3 for an illustration), we want to take a functional limit of $\overline{\mathcal{L}}^N$ as N goes to infinity (in the topology of uniform convergence for continuous functions on compact sets). Moreover we prove that the $\mathbf{H}$-Brownian Gibbs property with $\mathbf{H}(x) = e^x$ (the same Gibbs property as enjoyed by the KPZ line ensemble $\mathcal{H}$) for all subsequential limits of $\overline{\mathcal{L}}^N$.

Instead of studying directly with the sequence of scaled log-gamma line ensembles $\overline{\mathcal{L}}^N$, we introduce a general framework for studying the tightness of a sequence of $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs line ensembles, see our main result Theorem 3.1.13. We propose assumptions A1-A4 that capture the properties enjoyed by $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ that we rely on. To summarize here, consider a sequence of discrete line ensembles

$\mathcal{L}_i^N(u), (i,u) \in \{1, \cdots, K\} \times \frac{1}{N}\mathbb{Z}$, which enjoys $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property such that $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ satisfy assumptions A1-A4, and assume that $\mathcal{L}_1^N(u) + u^2/2$ (defined through linear interpolation) converges weakly to a stationary process, we obtain the following result (Theorem 3.1.13 in the main text),

1. For any $T > 0$ and $1 \leq k \leq K$, the restriction of the line ensemble $\mathcal{L}^N$ to $\{1, \cdots, k\} \times [-T, T]$ is sequentially compact as N varies.

2. Any subsequential limit line ensemble $\mathcal{L}^\infty$ satisfies $\mathbf{H}$-Brownian Gibbs property with $\mathbf{H}(x) = e^x$.

We then verify that the scaled log-gamma line ensembles $\overline{\mathcal{L}}^N$ fall into the category where this theorem applies and obtain the above results for $\overline{\mathcal{L}}^N$. One of the key technical input is achieved in next section.

It is worth mentioning that another particularly successful instance of the discrete Gibbs line ensemble is studied in [CD], where the authors investigated a discrete Gibbsian line ensemble related to the ascending Hall-Littlewood process (a special case of the Macdonald processes [BC14]). By developing discrete analogues of the arguments in [CH16], [CD] were successful in establishing the long-predicted 2/3 critical exponent for the asymmetric simple exclusion process (ASEP).

1.0.3 KMT coupling for random walk bridges

Let me start with briefly investigating the history of the original Komlós, Major and Tusnády (KMT) [KMT75, KMT76] coupling between random walks and Brownian motions. Let X be a random variable with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = 1$. Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence of random variables with the same law as X and let $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$. A classical problem in probability theory, called the *embedding problem*, asks to construct the process $\{S_m\}_{m=1}^n$ and a standard Brownian motion $(B_t)_{t\geq 0}$ on the same probability space so that the uniform distance

$$\max_{1\leq k\leq n} |S_k - B_k|$$

grows as slowly as possible in n. The first major results about the above embedding problem, or *strong approximation/coupling problem*, were obtained in the works of Skorohod [Sko61, Sko65] and Strassen [St67], who showed that if $\mathbb{E}[X^4] < \infty$ then with high probability

$$\max_{1\leq k\leq n} |S_k - B_k| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}).$$

For more than a decade the above rate for strong approximation was the best available result, and the method of obtaining it is now known as the *Skorohod embedding*.

Nearly fifteen years after Skorohod's original work, Komlós, Major and Tusnády showed using completely different techniques that one can achieve $\max\limits_{1\leq k\leq n} |S_k - B_k| = O(\log n)$ for the rate of strong coupling, provided that X has a finite moment generating function in a neighborhood of zero [KMT75, KMT76]. The construction used to achieve this celebrated result is now referred to as the *KMT approximation or coupling.* Unless X is normally distributed, the $\log n$ rate of approximation is optimal. Since its inception, the KMT coupling has become an invaluable tool in probability theory and statistics.

This is a joint project with Evgeni Dimitrov that establishes the KMT coupling in the setting of random walk bridges. We provide general conditions under which we could couple a family of random walk bridges and Brownian bridges in the same probability space to get the optimal logarithmic bound on the uniform distance. The random bridges we consider are constructed through random walks with i.i.d jumps that are conditioned on the locations of their endpoints. We deal with jumps that are either continuous or integer valued under some mild technical assumptions on the jump distributions.

For simplicity, we restrict our exposition here to continuous random variables X such that its density function f_X has finite support $[\alpha, \beta]$ (note that this is a special case of the general result in [DW] which deals with random variables without compact support). For all $n \geq 1$ and $z \in L_n = (n\alpha, n\beta)$, let $\{S_m^{(n,z)}\}_{m=1}^n$ denote the random process with law equal to that of the random walk $S_m := X_0 + X_1 \cdots + X_m$ conditioned on $S_0 = 0$ and $S_n = z$. As a natural extension we define $S_t^{(n,z)}$ for non-integer t by linear interpolation, i.e. if $t \in (m, m+1)$ we have

$$S_t^{(n,z)} = (m+1-t) \cdot S_m^{(n,z)} + (t-m) \cdot S_{m+1}^{(n,z)}.$$

Our main goal in this project is to demonstrate that given a reference slope $p \in (\alpha, \beta)$ and an endpoint z, which is close to np, we can construct a probability space that supports the process $\{S_t^{(n,z)}\}_{t \in [0,n]}$ and suitable Brownian bridges $B_t^{(n,z)}$ conditioned on $B_0^{(n,z)} = 0$ and $B_n^{(n,z)} = z$ such that

$$\sup_{0 \leq t \leq n} |S_t^{(n,z)} - B_t^{(n,z)}| = O(\log n)$$

with high probability. More precisely, we obtain

Theorem 1.0.2 (Dimitrov-Wu, [DW])**.** *Fix $p \in (\alpha, \beta)$ and $K > 0$. There exist constants $0 < C, M < \infty$ (depending on p, K and the probability density function $f_X(\cdot)$ of X) such that the following holds. For every positive integer n, there is a probability space on which are defined Brownian bridges $B^{(n,z)}$ (with variances that explicitly depend on p and $f_X(\cdot)$) and a family (in z) of processes $S^{(n,z)}$ for $|z - pn| \leq K\sqrt{n}$ such that*

$$\mathbb{P}\left[\sup_{0 \leq t \leq n} \left|S_t^{(n,z)} - B_t^{(n,z)}\right| \geq M \log n + x\right] \leq Ce^{-x}. \tag{1.2}$$

Our arguments follow a similar dyadic scheme to KMT's original proof. Due to the conditioning on the endpoint, we need stronger technical assumptions than those of the classical KMT embedding theorem for random walks and Brownian motions [KMT75], otherwise we could construct a counterexample X such that (1.2) does not hold. The counterexample X we construct has two-side Gaussian tails but have many spikes and it emphasizes on the point that while the density p_X of each jump X may be an extremely well-behaved distribution, the conditional distribution of a bridge can become quite singular in the presence of spikes in

p_X. This means that one needs better control of the conditional distribution and we provide easy-to-verify sufficient conditions for the extra assumptions on the jump distribution, which still cover a wide range of random variables, e.g. geometric random variables, exponential random variables and log-gamma random variables.

Moreover, we establish quantitative dependence on the density function of X and the endpoints with certain uniformity. These explicit dependence is important in the problem discussed in the previous section for the case of log-gamma random walk bridges with varying parameter. The underlying random walk for log-gamma line ensemble is of a diverging slope, which is equivalent to conditioning the random walk at very atypical endpoint, hence produced a difficulty for such couplings. We resolve this difficulty by a tilting argument. There is a large class of stochastic integrable models that naturally carry the structure of random paths with some Gibbsian resampling invariance. Another application is in [CD], where the authors investigated a discrete Gibbsian line ensemble related to the ascending Hall-Littlewood process.

Chapter 2

Weak noise scaling for half-space polymers

2.1 Definitions of the model and main results

In this section we first introduce notations and definitions needed and state the main results.

2.1.1 Half-space polymers

Considering an n-step simple symmetric random walk on non-negative integers $\mathbb{Z}_{\geq 0}$ with a totally reflecting barrier at the origin. The law of this walk is equal to that of the absolute value of a standard symmetric random walk on $\mathbb{Z}$. Denote the reflecting random walk probability measure by $\mathbb{P}_R$ on paths starting from origin at time 0 and we also denote $\mathbb{P}_R^{m,x}$ as the probability measure on paths starting at $x \geq 0$ at time $m \geq 0$. This measure $\mathbb{P}_R^{m,x}$ will serve as our background probability measure throughout this paper and we omit the superscript when there is no ambiguity about the starting point and time. For a path S, let S_i denote its location at time i and define transition probability for a random walk starting at x at time m and arriving at $y \geq 0$ at time $n \geq m$ by

$$p(m,n,x,y) := \sum_{S:S_n=y} \mathbb{P}_R^{m,x}(S).$$

Note that the sum is over paths of lengths $n-m$.

To add the randomness, let $X = \{X_i\}$ be a sequence of i.i.d. non-negative random variables such that $\mathbb{E}[X_1] = \gamma$ and we refer X as the boundary random environment. Define the random transition kernel

$$p_X(m,n,x,y) := \sum_{S:S_n=y} \left(\prod_{m \leq i < n : S_i = 0} X_i \right) \cdot \mathbb{P}_R^{m,x}(S), \tag{2.1}$$

and the corresponding random measure $\mathbb{P}_X$,

$$\mathbb{P}_X(S) := \left(\prod_{i\geq 0: S_i=0} X_i\right) \mathbb{P}_R(S),$$

which is a measure-valued random variable with randomness inherited from X. Note that in general $\mathbb{P}_X$ is not a probability measure due to the "punishing" or "rewarding" effects caused by the random environment X when paths repeatedly visit the origin.

When the boundary random environment is deterministic such that $X_i \equiv \gamma \geq 0$, where γ is denoted as the reflection rate for the barrier at origin, it follows that the barrier is absorbing if $0 \leq \gamma < 1$, totally reflecting if $\gamma = 1$, and rewarding if $\gamma > 1$. Now the transition kernel $p_\gamma(m, n, x, y)$ becomes also deterministic.

Let $\omega(i, z)$ for $(i, z) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ be an i.i.d. sequence of random variables, and the collection $\omega := \{\omega(i, z)\}$ is called bulk random environment. The energy of an n-step nearest neighbor walk S in the environment ω is defined as:

$$H_n^\omega(S) := \sum_{i=0}^{n-1} \omega(i, S_i).$$

Define the polymer probability measure with randomness inherited from both the bulk random environment ω and the boundary random environment X as:

$$\begin{aligned}
\mathbb{P}_{n,\beta}^{\omega,X}(S) &:= \frac{1}{Z_n^{\omega,X}(\beta)} e^{\beta H_n^\omega(S)} \cdot \mathbb{P}_X(S) \\
&= \frac{1}{Z_n^{\omega,X}(\beta)} e^{\beta H_n^\omega(S)} \cdot \left(\prod_{0\leq i<n: S_i=0} X_i\right) \cdot \mathbb{P}_R(S),
\end{aligned}$$

where β is called inverse temperature and the normalization term $Z_n^{\omega,X}(\beta)$ is a point-to-line partition function, defined as:

$$Z_n^{\omega,X}(\beta) := \mathbb{E}_R\left[e^{\beta H_n^\omega(S)} \left(\prod_{0\leq i<n: S_i=0} X_i\right)\right],$$

where the expectation is taken with respect to probability measure $\mathbb{P}_R$ and preserves randomness from ω and X.

The limiting behavior of the following point-to-point partition function under intermediate disorder scaling $\beta n^{-1/4}$ will be the main focus of this paper, which is defined as:

$$Z_n^{\omega,X}(x;\beta) := \mathbb{E}_R\left[e^{\beta H_n^\omega(S)} \left(\prod_{0\leq i<n: S_i=0} X_i\right) \cdot \mathbb{1}\{S_n = x\}\right], \tag{2.2}$$

where $\mathbb{1}$ is the indicator function.

For technical considerations, we work with a modified point-to-point partition function and then show how this could be viewed as a perturbed version of the above point-to-point partition function $Z_n^{\omega,X}(x;\beta)$.

Denote D_k^n as a discrete integer simplex:

$$D_k^n := \{\mathbf{i} = (i_1 \cdots , i_k) \in \mathbb{Z}_{\geq 0}^k : 0 \leq i_1 < \cdots < i_k \leq n-1\}, \tag{2.3}$$

we define a k-fold transition kernel $p_{X,k|x}(\mathbf{i}, \mathbf{x})$ for $(\mathbf{i}, \mathbf{x}) \in D_k^n \times \mathbb{Z}_{\geq 0}^k$ of a half-space random walk with a barrier at origin and conditional on arriving at x in n steps.

$$p_{X,k|x}(\mathbf{i}, \mathbf{x}) = p_X(i_k, n, x_k, x) \prod_{j=1}^{k} p_X(i_{j-1}, i_j, x_{j-1}, x_j). \tag{2.4}$$

And the modified point-to-point partition with β scaled is defined as:

$$\mathfrak{Z}_n^{\omega,X}(x; \beta n^{-1/4}) := \mathbb{E}_R\left[\prod_{i=0}^{n-1}\left(1 + \beta n^{-1/4}\omega(i, S_i)\right) \cdot \left(\prod_{0 \leq i < n : S_i = 0} X_i\right) \cdot \mathbb{1}\{S_n = x\}\right] \tag{2.5}$$

Expanding the above product in the expectation and by direct computation, $\mathfrak{Z}_n^{\omega,X}(x; \beta n^{-1/4})$ could be written as a discrete sum of weighted chaoses:

$$\mathfrak{Z}_n^{\omega,X}(x; \beta n^{-1/4}) = p_X(0, n, 0, x) + \sum_{k=1}^{n-1} \beta^k n^{-k/4} \sum_{\mathbf{i} \in D_k^n} \sum_{\mathbf{x} \in \mathbb{Z}_{\geq 0}^k} p_{X,k|x}(\mathbf{i}, \mathbf{x}) \omega(\mathbf{i}, \mathbf{x}), \tag{2.6}$$

where $\omega(\mathbf{i}, \mathbf{x}) = \prod_{j=1}^{k} \omega(i_j, S_{i_j})$.

2.1.2 Stochastic Heat Equation and Wiener chaos

In this section we introduce the SHE with Robin boundary condition and provide the expression of the chaos series for its solution, which is a series of multiple stochastic integrals over a Robin heat kernel with respect to a space-time white noise. While the modified partition functions could be expanded as a discrete chaos series and our main result will show how this discrete series in the intermediate disorder regime converges under diffusive scaling and intermediate disorder regime to a continuum chaos series, which is identified with the solution to SHE.

2.1.2.1 1-D heat equation with Robin boundary condition

Definition 2.1.1. *We say $\varrho_\mu(t, x, y)$ is the fundamental solution to 1-D heat equation on $\mathbb{R}_{\geq 0}$ with Robin boundary condition and initial data $\delta(y - x)$ if*

$$\partial_t \varrho_\mu(t, x, y) = \frac{1}{2} \partial_{xx} \varrho_\mu(t, x, y) \tag{2.7}$$
$$\partial_x \varrho_\mu|_{x=0} = \mu \cdot \varrho_\mu|_{x=0},$$

and if for any function $\varphi(x)$,

$$v(t, x) = \int_0^\infty \varrho_\mu(t, x, y) \varphi(y) dy$$

solves heat equation with initial condition

$$v(0,x)=\varphi(x).$$

It is proved in [CS, Lemma 4.4] by a generalized image method that the Robin heat kernel $\varrho_\mu(t,x,y)$ is of the form

$$\begin{aligned}\varrho_\mu(t,x,y)=&\frac{1}{\sqrt{2\pi t}}\cdot\left\{\exp\left(-\frac{(y-x)^2}{2t}\right)-\exp\left(-\frac{(y+x)^2}{2t}\right)\right\} &(2.8)\\ &+\int_0^\infty e^{-\mu s}\cdot(y+x+s)\cdot e^{-\frac{((y+x)+s)^2}{2t}}ds\\ =&\frac{1}{\sqrt{2\pi t}}\cdot\left\{\exp\left(-\frac{(y-x)^2}{2t}\right)+\exp\left(-\frac{(y+x)^2}{2t}\right)\right\} &(2.9)\\ &-\mu\exp\left[\mu(y+x)+\frac{1}{2}\mu^2t\right]\cdot\mathrm{erfc}\left[\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right],\end{aligned}$$

where $\mathrm{erfc}(x)$ is the complementary error function

$$\mathrm{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-s^2}ds.$$

Denote $\varrho(t,x,y):=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-x)^2}{2t}}$ as the standard heat kernel, the following lemma shows that for any $\mu\in\mathbb{R}$, $\varrho_\mu(\cdot,x,\cdot)$ is in $L^2([0,1]\times\mathbb{R}_{\geq 0})$. Note that ϱ_μ is decreasing in μ and ϱ_0 solves heat equation with Neumann boundary condition, which implies that for any $\mu\geq 0$, $\varrho_\mu(\cdot,x,\cdot)\geq 0$ is in $L^2([0,1]\times\mathbb{R}_{\geq 0})$.

Lemma 2.1.2. *For any $\mu\in\mathbb{R}$, there exists a positive constant $C(\mu)$ depending only on μ, such that for any $(t,x,y)\in(0,1]\times\mathbb{R}^2_{\geq 0}$,*

$$\varrho_\mu(t,x,y)\leq C(\mu)\varrho(t,x,y).$$

Proof. If $\mu\geq 0$,

$$\begin{aligned}\varrho_\mu(t,x,y)&\leq\frac{1}{\sqrt{2\pi t}}\cdot\left\{\exp\left(-\frac{(y-x)^2}{2t}\right)+\exp\left(-\frac{(y+x)^2}{2t}\right)\right\}\\ &\leq\frac{2}{\sqrt{2\pi t}}\cdot\exp\left(-\frac{(y-x)^2}{2t}\right).\end{aligned}$$

If $\mu<0$, note that

$$\exp[\mu(y+x)+\frac{1}{2}\mu^2t]=\exp\left[\left(\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right)^2\right]\cdot\exp\left[-\frac{(y+x)^2}{2t}\right],$$

and

$$\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2}\leq\int_x^\infty e^{-s^2}ds\leq\frac{1}{2x}e^{-x^2}.$$

Thus if $(y+x)/\sqrt{2t}+\mu\sqrt{t/2}>1$,

$$\begin{aligned}
&-\mu\exp\left[\mu(y+x)+\frac{1}{2}\mu^2 t\right]\cdot\operatorname{erfc}\left[\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right]\\
&\le\frac{-2\mu}{\sqrt{\pi}}\cdot\exp\left[\mu(y+x)+\frac{1}{2}\mu^2 t\right]\cdot\left(\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right)^{-1}\\
&\quad\cdot\exp\left[-\left(\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right)^2\right]\\
&\le\frac{-2\mu}{\sqrt{\pi}}\exp\left[-\frac{(y+x)^2}{2t}\right]\le C(\mu)\varrho(t,x,y),
\end{aligned}$$

if $(y+x)/\sqrt{2t}+\mu\sqrt{t/2}\le 1$,

$$\begin{aligned}
&-\mu\exp\left[\mu(y+x)+\frac{1}{2}\mu^2 t\right]\cdot\operatorname{erfc}\left[\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right]\\
&\le C(\mu)\exp\left[\left(\frac{y+x}{\sqrt{2t}}+\mu\sqrt{t/2}\right)^2\right]\cdot\exp\left[-\frac{(y+x)^2}{2t}\right]\\
&\le C(\mu)\exp\left[-\frac{(y+x)^2}{2t}\right]\\
&\le C(\mu)\varrho(t,x,y).
\end{aligned}$$

□

For $\Delta_k=\{\mathbf{t}\in\mathbb{R}^k_{\ge 0}:0=t_0<t_1<\cdots<t_k\le 1\}$, we denote the k-fold standard heat kernel and k-fold Robin heat kernel as:

$$\varrho_{k|x}(\mathbf{t},\mathbf{x})=\varrho(1-t_k,x_k,x)\prod_{j=1}^{k}\varrho(t_j-t_{j-1},x_{j-1},x_j),(\mathbf{t},\mathbf{x})\in\Delta_k\times\mathbb{R}^k;\tag{2.10}$$

$$\varrho_{\mu,k|x}(\mathbf{t},\mathbf{x})=\varrho_\mu(1-t_k,x_k,x)\prod_{j=1}^{k}\varrho_\mu(t_j-t_{j-1},x_{j-1},x_j),(\mathbf{t},\mathbf{x})\in\Delta_k\times\mathbb{R}^k_{\ge 0}.$$

And we have

$$\begin{aligned}
\int_{\Delta_k}\int_{\mathbb{R}^k}\varrho_{k|x}(\mathbf{t},\mathbf{x})^2 d\mathbf{x}d\mathbf{t}&=(2\pi)^{-(k+1)/2}\int_{\Delta_k}\int_{\mathbb{R}^k}\varrho_{k|\sqrt{2}x}(\mathbf{t},\sqrt{2}\mathbf{x})\frac{1}{\sqrt{1-t_k}}\prod_{j=1}^{k}\frac{1}{\sqrt{t_j-t_{j-1}}}d\mathbf{x}d\mathbf{t} \\
&=(2\pi)^{-(k+1)/2}\cdot 2^{-k/2}\frac{1}{\sqrt{2\pi}}e^{-x^2}\int_{\Delta_k}\frac{1}{\sqrt{1-t_k}}\prod_{j=1}^{k}\frac{1}{\sqrt{t_j-t_{j-1}}}d\mathbf{t}\\
&=\frac{e^{-x^2}}{2^{k+1/2}\Gamma((k+1/2))}.
\end{aligned}\tag{2.11}$$

2.1.2.2 Stochastic Heat equation with Robin boundary condition

Consider the stochastic heat equation with multiplicative noise

$$\partial_t z_\beta=\frac{1}{2}\partial_{xx}z_\beta+\beta z_\beta\cdot\xi\tag{2.12}$$

with delta initial data and Robin boundary condition:

$$z_\beta(0,\cdot) = \delta(0)$$
$$\partial_x z_\beta(\cdot,x)|_{x=0} = \mu \cdot z_\beta(\cdot,0).$$

Here $\xi(t,x)$ is a white noise on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with covariance structure

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

With the help of the Robin heat kernel, the mild solution is given by

$$z_\beta(t,x) = \sum_{k=0}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k_{\geq 0}} \varrho_\mu(t-t_k, x_k, x) \cdot \beta^k \prod_{i=1}^{k} \varrho_\mu(t_i - t_{i-1}, x_{i-1}, x_i) d\xi^{\otimes k}(\mathbf{t}, \mathbf{x}), \tag{2.13}$$

where $\Delta_k(t) = \{0 = t_0 < t_1 < \cdots < t_k \leq t\}$ and $x_0 = 0$.

For details about white noise and Wiener chaos, we refer to [AKQ, Section 3]. Further discussions can be found in [Ja].

Our main result below shows under intermediate disorder scaling($\beta n^{-1/4}$), the convergence in distribution of the point-to-point partition function to $z_{\sqrt{2}\beta}(1,x)$, time 1 solution to SHE in the topology of supremum norm on bounded continuous functions, denoted as weak convergence $\xrightarrow{(d)}$ for process. Denote $\mathbb{E}[X] := \gamma$ and $\lambda(\beta) = \log \mathbb{E}[e^{\beta\omega}]$, our main theorem is as follows:

Theorem 2.1.3. *Let ω be i.i.d random environment with mean zero and variance one. Weakly scale the random variables X such that $\gamma = 1 - \frac{\mu}{\sqrt{n}}$, $\mathrm{Var}[X] = o(n^{-\varepsilon})$ for some positive ε and $\mathbb{E}\left|X - \mathbb{E}[X]\right|^3 = o\left(n^{-\varepsilon}\right)$. Under the assumption that the ω satisfy $\lambda(\beta) < \infty$ for β sufficiently small, we have*

$$e^{-n\lambda(\beta n^{-1/4})} \frac{\sqrt{n}}{2} Z_n^{\omega,X}\left(x\sqrt{n}; \beta n^{-1/4}\right) \xrightarrow{(d)} z_{\sqrt{2}\beta}(1,x).$$

Remark 2.1.4. *Here we require the third moment assumption in order to prove tightness and we don't believe this is the optimal case.*

2.2 U-statistics

The strategy for proving Theorem 2.1.3 is to first prove the convergence for the modified partition function $\mathfrak{Z}_n^\omega$ and then rewrite the unmodified partition function Z_n^ω in the same form as $\mathfrak{Z}_n^\omega$ with a perturbed environment $\widetilde{\omega}_n$, depending on n, still of mean zero but with variance only asymptotically one. In addition, the same the strategy will be applied in the log-gamma polymer model, where we will need to deal with the issue that the random environment will only be i.i.d on the diagonal and bulk respectively.

First we recall the definition of U-statistics and quote some technical lemmas, which will be needed in the proof of our main theorem. Proofs of those lemmas can be found in [AKQ, Section 4]. See [CSZ2] for a more general treatment for discrete chaos expansion with more general random environment. We also state Lemma 2.2.5, which give sufficient conditions on the perturbed random environment $\widetilde{\omega}$ so that the convergence result for U-statistics $S_k^n(g;\widetilde{\omega}_n)$(with ω replaced by $\widetilde{\omega}_n$) in Lemma 2.2.2 and Lemma 2.2.3 still hold and could be applied to both of the partition function Z_n^ω and in the log-gamma case.

For a L^2 function g on $[0,1]^k \times \mathbb{R}_{\geq 0}^k$, the corresponding U-statistics of g could be viewed as the sum of the sequence of $\omega(\mathbf{i},\mathbf{x})$ weighted by a discretized version of g defined on a collection of rectangles indexed by the lattice points $(\mathbf{i},\mathbf{x})$.

Due to the periodicity of symmetric simple random walk, i and x have the same parity, which we denote as $i \leftrightarrow x$. More generally, $\mathbf{i} \leftrightarrow \mathbf{x}$ means the all corresponding entries share the same parity. Let $\mathcal{R}_k^n$ be a collection of rectangles R, defined by

$$\mathcal{R}_k^n := \left\{ \left[\frac{\mathbf{i}}{n}, \frac{\mathbf{i}+\mathbf{1}}{n}\right) \times \left[\frac{\mathbf{x}}{\sqrt{n}}, \frac{\mathbf{x}+\mathbf{2}}{\sqrt{n}}\right) : \mathbf{i} \in D_k^n, \mathbf{x} \in \mathbb{Z}_{\geq 0}^k, \mathbf{i} \leftrightarrow \mathbf{x} \right\},$$

see D_k^n in (2.3) and with $\mathbf{1}$ being the vector of all ones and

$$\left[\frac{\mathbf{i}}{n}, \frac{\mathbf{i}+\mathbf{1}}{n}\right) := \left[\frac{i_1}{n}, \frac{i_1+1}{n}\right) \times \cdots \times \left[\frac{i_k}{n}, \frac{i_k+1}{n}\right),$$

similarly

$$\left[\frac{\mathbf{x}}{\sqrt{n}}, \frac{\mathbf{x}+\mathbf{2}}{\sqrt{n}}\right) := \left[\frac{x_1}{\sqrt{n}}, \frac{x_1+2}{\sqrt{n}}\right) \times \cdots \times \left[\frac{x_k}{\sqrt{n}}, \frac{x_k+2}{\sqrt{n}}\right).$$

With the collection $\mathcal{R}_k^n$, we now discretize L^2 functions by replacing their values with integral mean on rectangles in $\mathcal{R}_k^n$. Consider $g \in L^2([0,1]^k \times \mathbb{R}_{\geq 0}^k)$ and define $\overline{g}_n$ by

$$\overline{g}_n|_R := \frac{1}{|R|}\int_R g.$$

where $|R| = 2^k n^{-3K/2}$. Note that $\overline{g}$ is constant on every single R and for each n,k fixed, each pair $(\mathbf{i},\mathbf{x})$ corresponds to a unique $R \in \mathcal{R}_k^n$ for $(\mathbf{i},\mathbf{x}) \in D_k^n \times \mathbb{Z}_{\geq 0}^k$ and $\mathbf{i} \leftrightarrow \mathbf{x}$.

For the convenience of applying U-statistics results we consider sums over unordered

$$E_k^n = \{\mathbf{i} \in [n]^k : \mathbf{i}_j \neq \mathbf{i}_l \text{ for } j \neq l\},$$

where $[n] = \{1,2,\cdots,n\}$ and we define the corresponding U-statistics as

$$\mathcal{S}_k^n(g) := 2^{k/2} \sum_{\mathbf{i}\in E_k^n} \sum_{\mathbf{x}\in\mathbb{Z}_{\geq 0}^k} \overline{g}_n\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}\right) \mathbb{1}\{\mathbf{i} \leftrightarrow \mathbf{x}\} \cdot \omega(\mathbf{i},\mathbf{x}). \tag{2.14}$$

The following lemma, , proved as [AKQ, Lemma 4.1], bounds the second moment from above.

Lemma 2.2.1. *Let $\mathcal{S}_k^n(g)$ be a U-statistics as in (2.14) and $\alpha_1,\cdots,\alpha_k \in \mathbb{R}$, then for any linear combinations of functions $g_1,\cdots,g_m \in L^2([0,1]^k \times \mathbb{R}_{\geq 0}^k)$, we have*

$$\sum_{l=1}^m \alpha_l \mathcal{S}_k^n(g_l) = \mathcal{S}_k^n\left(\sum_{l=1}^m \alpha_l g_l\right)$$

with probability one.

Moreover, if random environment variables satisfy moment conditions $\mathbb{E}[\omega(i,x)] = 0$ *and* $\mathrm{Var}[\omega(i,x)] = 1$*, then the second moment of* $\mathcal{S}_k^n(g)$ *is bounded from above,*

$$\mathbb{E}\big[\mathcal{S}_k^n(g)^2\big] \le n^{3k/2} \, \|g\|^2_{L^2([0,1]^k \times \mathbb{R}^k_{\ge 0})}.$$

Note that the U-statistics is invariant under permutation for $(\mathbf{i}, \mathbf{x})$ and we denote

$$\mathrm{Sym}\, g(\mathbf{i}, \mathbf{x}) = \frac{1}{k!} \sum_{\pi \in \sigma_k} g(\pi \mathbf{i}, \pi \mathbf{x}),$$

where σ_k is the symmetric group of degree k.

The following lemma shows that U-statistics converges to multiple stochastic integrals in distribution under proper scaling.

Lemma 2.2.2. *Let* $g \in L^2([0,1]^k \times \mathbb{R}^k_{\ge 0})$*. Then as* $n \to \infty$*,*

$$n^{-3k/4} \mathcal{S}_k^n(g) \xrightarrow{(d)} I_k(g) := \int_{[0,1]^k} \int_{\mathbb{R}^k_{\ge 0}} \mathrm{Sym}\, g(\mathbf{t}, \mathbf{x}) \xi^{\otimes k}(d\mathbf{t}\, d\mathbf{x}).$$

Moreover for any finite collection of $k_1, \dots, k_m \in \mathbb{Z}_{\ge 0}$ *and* $g_1, \dots, g_m$ *with* $g_i \in L^2([0,1]^{k_i} \times \mathbb{R}^{k_i}_{\ge 0})$*, the joint convergence holds,*

$$\big(n^{-3k_1/4} \mathcal{S}^n_{k_1}(g_1), \dots, n^{-3k_m/4} \mathcal{S}^n_{k_m}(g_m)\big) \xrightarrow{(d)} \big(I_{k_1}(\mathrm{Sym}\, g_1), \dots, I_{k_m}(\mathrm{Sym}\, g_m)\big).$$

The following lemma shows that the discrete chaos expansion converges to the continuum chaos series in distribution.

Lemma 2.2.3. *Let* $G = (g_0, g_1, g_2, \dots) \in \bigoplus_{k \ge 0} L^2([0,1]^k \times \mathbb{R}^k_{\ge 0})$ *with* $\sum_{k=0}^\infty \|g_k\|_{L^2} < \infty$*, then as* $n \to \infty$*,*

$$\mathcal{S}^n(G) := \sum_{k=0}^\infty n^{-3k/4} \mathcal{S}_k^n(g_k) \xrightarrow{(d)} I(G) := \sum_{k=0}^\infty \int_{[0,1]^k} \int_{\mathbb{R}^k} \mathrm{Sym}\, g_k(\mathbf{t}, \mathbf{x}) \xi^{\otimes k}(d\mathbf{t}\, d\mathbf{x}).$$

Moreover if $G_1, \dots, G_m \in \bigoplus_{k \ge 0} L^2([0,1]^k \times \mathbb{R}^k_{\ge 0})$*, then as* $n \to \infty$*, we have the joint convergence:*

$$\big(\mathcal{S}^n(G_1), \dots, \mathcal{S}^n(G_m)\big) \xrightarrow{(d)} \big(I(G_1), \dots, I(G_m)\big).$$

For the application to log-gamma poymer model, we need the following lemma for a perturbed random environment $\widetilde{\omega}$. Assume respectively, that the environment variables $\widetilde{\omega}_n(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ are i.i.d random variables and environment random variables $\widetilde{\omega}_n(i,0), i \in \mathbb{Z}_+$ are also i.i.d random variables (possibly with different distributions from $\widetilde{\omega}_n(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}_+$).

Lemma 2.2.4. *Assume* $\mathbb{E}[\widetilde{\omega}_n(i,0)] = \mathbb{E}[\widetilde{\omega}_n(i,x)] = 0$; $\mathbb{E}[\widetilde{\omega}^2_n(i,x)] = 1 + o(1), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ *and* $\mathbb{E}[\widetilde{\omega}^2_n(i,0)] \le C$*. Then the convergence result for* $S_k^n(g)$ *in Lemma 2.2.2 still holds when replacing the random environment* ω *by* $\widetilde{\omega}_n$*.*

Proof. The proof is similar to the proof of [AKQ, Theorem 4.5]. Re-examining the proof of Lemma 2.2.2, which is [AKQ, Theorem 4.3] and it suffices to check steps 1 and 5 for [AKQ, Theorem 4.3]. The central limit theorem argument in step 1 stills works for the perturbed environment $\widetilde{\omega}_n$ with two family of mean 0 i.i.d variables and the variance assumptions, $\mathbb{E}[\widetilde{\omega}_n^2(i,x)] = 1 + o(1), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $\mathbb{E}[\widetilde{\omega}_n^2(i,0)] \le C$.

And the step 5 is checked in the same way as in [AKQ, Theorem 4.3]. □

The following lemma is a restatement of [AKQ, Lemma 4.6] under the assumption for $\widetilde{\omega}$ as in Lemma 2.2.4.

Lemma 2.2.5. *Let $\widetilde{\omega}$ satisfy the assumption of Lemma 2.2.4 and assume $G = (g_0, g_1, g_2, \ldots) \in \bigoplus_{k\ge 0} L^2([0,1]^k \times \mathbb{R}^k_{\ge 0})$ with $\sum_{k=0}^{\infty} \|g_k\|_{L^2} < \infty$ satisfies,*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{k=N}^{\infty} \mathbb{E}[\widetilde{\omega}_n^2]^k \, \|g_k\|_{L^2} = 0, \tag{2.15}$$

then the follwing convergence result stills holds for $\widetilde{\omega}$, as $n \to \infty$,

$$\sum_{k=0}^{\infty} n^{-3k/4} S_k^n(g_k; \widetilde{\omega}) \xrightarrow{(d)} I(g).$$

Moreover if $G_1, \ldots, G_m \in \bigoplus_{k\ge 0} L^2([0,1]^k \times \mathbb{R}^k_{\ge 0})$ individually satisfy (2.15), then as $n \to \infty$, we have the joint convergence:

$$\big(\mathcal{S}^n(G_1; \widetilde{\omega}), \ldots, \mathcal{S}^n(G_m; \widetilde{\omega})\big) \xrightarrow{(d)} \big(I(G_1), \ldots, I(G_m)\big).$$

2.3 Estimates on discrete transition kernel and continuous Robin heat kernel

In order to show the convergence of the partition functions, it is necessary to derive the local limit theorem, L^2 boundedness and L^2 convergence of the transition kernel $p_X(m,n,x,y)$ for random walks with a barrier at origin. When $\mathbb{E}[X] > 1$, the total mass of the random path measure grows expectedly and we need more accurate estimates on the number of visits to origin. The control of variance of $p_X(m,n,x,y)$ is reduced to bound the local time of a 2-D symmetric simple random walk.

Let $T(n,z)$ be the probability that a standard simple random walk on $\mathbb{Z}$ arrives at $x = z$ after n jumps starting at origin. Thus

$$T(n,z) = 2^{-n} \binom{n}{\frac{1}{2}(n+z)}.$$

The following lemma gives an explicit expression for $p_\gamma(m, m+n, x, y)$, see the proof as in [G, (23)]. Another approach for the transition kernel can also be found in [CS, Lemma 4.5].

Lemma 2.3.1. *For any fixed $m \geq 0$,*

$$p_\gamma(m, m+n, x, y) = T(n, y-x) - T(n, y+x) \\ + 2\gamma \sum_{j=0}^{n} \gamma^j \frac{y+x+j}{n-j} T(n-j, y+x+j). \tag{2.16}$$

Note that the j-th term in the second line represents the probability of the event that the walk visits origin $j+1$ times and arrives at y in n steps and we only sum over j for which $\frac{y+x+j}{n-j}T(n-j, y+x+j)$ makes sense.

An local limit theorem and L^2 estimates for $p_\gamma(m, m+n, x, y)$ are needed in the proof of Theorem 2.1.3. Recall that γ is the reflection rate, when $\gamma \leq 1$, $p_\gamma(m, m+n, x, y) \leq p(m, m+n, x, y)$, i.e the totally reflecting case, but when $\gamma > 1$ the system will have mass coming in. Therefore we need to estimate how frequently the walker goes to the barrier in order to estimate the discrete transition kernel. Denote N_n as the number of times that a walker goes to origin within n step starting at origin.

We state the following two lemmas on $T(n, z)$ without proofs, which directly follow from a lengthy computation using Stirling formula.

Lemma 2.3.2. *For any $n \in \mathbb{N}$, $z \in \mathbb{Z}$, $z \leftrightarrow n$ and $|z| \leq n$, we have*

$$T(n, z) = 2(2\pi n)^{-1/2} \exp\left(-\frac{z^2}{2n}\right) \exp\left(O\left(E(n, z)\right)\right), \tag{2.17}$$

where,

$$E(n, z) := \frac{|z|^3}{n^2} + \frac{1}{n - |z| + 1}.$$

In other words, there exists a universal constant $C > 0$ such that

$$\exp\left(-CE(n, z)\right) \leq \frac{1}{2}(2\pi n)^{1/2} \exp\left(\frac{z^2}{2n}\right) T(n, z) \leq \exp\left(CE(n, z)\right). \tag{2.18}$$

Lemma 2.3.3. *There exists a universal constant $C > 0$ such that for any $n \in \mathbb{N}$, $z \in \mathbb{Z}$ and $z \leftrightarrow n$ we have*

$$T(n, z) \leq \frac{C}{\sqrt{n}} \exp\left(-\frac{z^2}{Cn}\right). \tag{2.19}$$

Lemma 2.3.4. *There exists a universal constant $C > 0$ such that for any $n \geq 1$, $x, y \in \mathbb{N}_0$ with $x - y \leftrightarrow n$ and $k \geq 0$,*

$$\mathbb{P}_R^{0,x}(N_n \geq k | S_n = y) \leq C \exp\left(-\frac{k^2}{Cn}\right). \tag{2.20}$$

Proof. We first discuss the case that n is even and $x = y = 0$. Note the we have the following formula from Lemma 2.3.1.

$$\mathbb{P}_R^{0,0}(N_n = j, S_n = 0) = \frac{2j}{n-j} T(n-j, j).$$

Therefore for any $k \geq 0$

$$\begin{aligned}
\mathbb{P}_R^{0,0}(N_n \geq k, S_n = 0) =& \sum_{j \geq k}^{n/2} \frac{2j}{n-j} T(n-j, j) \\
\leq& \sum_{j \geq k}^{n/2} \frac{j}{n} \cdot \frac{C}{\sqrt{n}} \exp\left(-\frac{j^2}{Cn}\right) \\
=& \frac{C}{\sqrt{n}} \sum_{j \geq k}^{n/2} \frac{j}{\sqrt{n}} \exp\left(-\frac{j^2}{Cn}\right) \cdot \frac{1}{\sqrt{n}}
\end{aligned}$$

We have used Lemma 2.3.3 in the first inequality. Let $M_0 > 0$ be the number that $s\exp(-\frac{s^2}{C})$ is decreasing for $s \geq M_0$. For $k < M_0\sqrt{n}$, (2.20) holds by choosing C large enough. In the following we assume $k \geq M_0\sqrt{n}$. Then we have

$$\begin{aligned}
\mathbb{P}_R^{0,0}(N_n \geq k, S_n = 0) \leq& \frac{C}{\sqrt{n}} \sum_{j \geq k}^{n/2} \frac{j}{\sqrt{n}} \exp\left(-\frac{j^2}{Cn}\right) \cdot \frac{1}{\sqrt{n}} \\
\leq& \frac{C}{\sqrt{n}} \int_{k/\sqrt{n}} s \exp\left(-\frac{s^2}{C}\right) ds \\
\leq& \frac{C}{\sqrt{n}} \exp\left(-\frac{k^2}{Cn}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}_R^{0,0}(N_n \geq k | S_n = 0) =& \frac{\mathbb{P}_R^{0,0}(N_n \geq k, S_n = 0)}{\mathbb{P}_R^{0,0}(S_n = 0)} \\
&\leq C \exp\left(-\frac{k^2}{Cn}\right).
\end{aligned}$$

For the case with general x and y, by conditioned on the first time the random walk bridge returns to the origin we have

$$\mathbb{P}_R^{0,x}(N_n \geq k | S_n = y) \leq \max_{0 \leq j \leq n} \mathbb{P}_R^{0,0}(N_{n-j} \geq k | S_{n-j} = y).$$

By reversing the random walk bridge and applying the same argument we have

$$\begin{aligned}\max_{0\le j\le n}\mathbb{P}_R^{0,0}(N_j\ge k|S_j=y)=&\max_{0\le j\le n}\mathbb{P}_R^{0,y}(N_j\ge k|S_j=0)\\ \le&\max_{0\le j\le n}\mathbb{P}_R^{0,0}(N_j\ge k|S_j=0)\\ \le& C\exp\left(-\frac{k^2}{Cn}\right).\end{aligned}$$

□

Lemma 2.3.5. *There exists a constants $C_1=C_1(\mu)$ and a uniform constant C such that for any $M\ge 0$, $1\le m\le n$ and any $x,y\in\mathbb{N}_0$,*

$$\sum_{k\ge M\sqrt{n}}\gamma^k\mathbb{P}_R^{0,x}(N_m(S)=k|S_m=y)\le C_1(\mu)\exp\left(-\frac{M^2}{C}\right).$$

Proof. By (2.20), we obtain

$$\begin{aligned}&\sum_{k\ge M\sqrt{n}}\gamma^k\mathbb{P}_R^{0,x}(N_m=k|S_m=y)\\ =&(1-\gamma^{-1})\sum_{k\ge M\sqrt{n}+1}\gamma^k\mathbb{P}_R^{0,x}(N_m\ge k|S_m=y)\\ &+\gamma^{M\sqrt{n}}\mathbb{P}_R^{0,x}(N_m\ge M\sqrt{n}|S_m=y)\\ \le& C\frac{|\mu|}{\sqrt{n}}\sum_{k\ge M\sqrt{n}}\exp\left(-\frac{k^2}{Cm}-\frac{\mu k}{\sqrt{n}}\right)+C\exp\left(-\frac{M^2n}{Cm}-\mu M\right)\\ \le& C(\mu)\sum_{k\ge M\sqrt{n}}\exp\left(-\frac{k^2}{2Cn}\right)\cdot\frac{1}{\sqrt{n}}+C(\mu)\exp\left(-\frac{M^2}{2C}\right)\\ \le& C_1(\mu)\exp\left(-\frac{M^2}{C}\right)\end{aligned}$$

□

Corollary 2.3.6. *There exists a constants $C_1=C_1(\mu)$ and C such that for any $M\ge 0$, $1\le m\le n$ and any $x\in\mathbb{N}_0$,*

$$\sum_{k\ge M\sqrt{n}}\gamma^k\mathbb{P}_R^{0,x}(N_m(S)=k)\le C_1(\mu)\exp\left(-\frac{M^2}{C}\right).$$

By taking $M=0$, we get a L^1 bound on $p_\gamma(0,m,x,y)$.

Corollary 2.3.7. *For any $1\le m\le n$ and any $x\in\mathbb{N}_0$,*

$$\sum_y p_\gamma(0,m,x,y)\le C_1(\mu).$$

In the following lemma, we prove the local limit theorem for $p_\gamma(m, m+n, x, y)$ as tuning the reflection rate by $\gamma = 1 - \mu/\sqrt{n}$. Note that $p_\gamma(m, m+n, x, y)$ is indeed time-homogeneous and we assume $m = 0$ without loss of generality.

Lemma 2.3.8. *Assume the reflection rate $\gamma = 1 - \frac{\mu}{\sqrt{n}}$ for $\mu \in \mathbb{R}$. Fix $0 < \varepsilon < 1$ and $M > 0$. Assume that $n \geq 1$, $t \in [\varepsilon, 1]$ and $x, y \in [0, M]$ with $nt \in \mathbb{N}$, $\sqrt{n}x, \sqrt{n}y \in \mathbb{Z}$ and $nt \leftrightarrow \sqrt{n}(y - x)$, we have*

$$\frac{\sqrt{n}}{2} p_\gamma(0, nt, \sqrt{n}x, \sqrt{n}y) = \varrho_\mu(t, x, y) + o(1),$$

where the $o(1)$ term depends only on ε, M and n and converges to 0 and $n \to \infty$. See the expression for ϱ_μ in (2.8).

Proof. For any $n \in \mathbb{N}$, $z \in \mathbb{Z}$, $z \leftrightarrow n$ and $|z| \leq n$, by Lemma 2.3.2 we have

$$T(n, z) = 2(2\pi n)^{-1/2} \exp\left(-\frac{z^2}{2n}\right) \exp\left(O\left(E(n, z)\right)\right), \tag{2.21}$$

where

$$E(n, z) := \frac{|z|^3}{n^2} + \frac{1}{n - |z| + 1}.$$

Applying Lemma 2.3.1, we have

$$\begin{aligned} p_\gamma\left(0, nt, \sqrt{n}x, \sqrt{n}y\right) = & T\left(nt, (y - x)\sqrt{n}\right) - T\left(nt, (y + x)\sqrt{n}\right) \\ & + 2\gamma \sum_{j=0}^{nt} \gamma^j \frac{(y + x)\sqrt{n} + j}{nt - j} T(nt - j, (y + x)\sqrt{n} + j). \end{aligned}$$

For the first two terms in above we have,

$$\begin{aligned} T(nt, (y \pm x)\sqrt{n}) = & \frac{2}{\sqrt{2\pi nt}} \exp\left(\frac{-(y \pm x)^2}{2t}\right) \exp\left(O\left(\frac{1}{\sqrt{n}}\right)\right) \\ = & \frac{2}{\sqrt{2\pi nt}} \exp\left(\frac{-(y \pm x)^2}{2t}\right) + O\left(\frac{1}{n}\right). \end{aligned} \tag{2.22}$$

Denote the rest summation as

$$\mathcal{T}_n := 2\gamma \sum_{j=0}^{nt} \gamma^j \cdot \frac{(y + x)\sqrt{n} + j}{nt - j} T\left(nt - j, (y + x)\sqrt{n} + j\right).$$

Since $\gamma = 1 - \frac{\mu}{\sqrt{n}}$,

$$\begin{aligned} \gamma^j = \left(1 - \frac{\mu}{\sqrt{n}}\right)^j = & \left(e^{-1} \exp\left(O\left(\frac{\mu}{\sqrt{n}}\right)\right)\right)^{\frac{\mu j}{\sqrt{n}}} \\ = & \exp\left(-\frac{\mu j}{\sqrt{n}}\right) \exp\left(O\left(\frac{\mu^2 j}{n}\right)\right). \end{aligned}$$

Fix $0 < \delta << 1$. Consider the summation over $j \in [0, (nt)^{\frac{2}{3}-\delta}]$. In this range, $nt - j \approx nt$ and $(y+x)\sqrt{n} + j \lesssim n^{\frac{2}{3}-\delta}$. By (2.21), we have

$$T(nt-j, (y+x)\sqrt{n}+j) = \frac{2}{\sqrt{2\pi nt}} \exp\left(-\frac{((y+x)\sqrt{n}+j)^2}{2nt}\right) \exp\left(O(n)^{-3\delta}\right).$$

Together with

$$\frac{(y+x)\sqrt{n}+j}{nt-j} = \frac{(y+x)\sqrt{n}+j}{nt} \exp\left(O(n^{-\frac{2}{3}-2\delta})\right),$$

we obtain

$$\begin{aligned}
&2\gamma \sum_{j=0}^{(nt)^{\frac{2}{3}-\delta}} \gamma^j \cdot \frac{(y+x)\sqrt{n}+j}{nt-j} T\left(nt-j, (y+x)\sqrt{n}+j\right) \\
=&2\gamma \sum_{j=0}^{(nt)^{\frac{2}{3}-\delta}} \exp\left(-\frac{\mu j}{\sqrt{n}}\right) \left(\frac{y+x+j/\sqrt{n}}{\sqrt{n}t}\right) \frac{2}{\sqrt{2\pi nt}} \exp\left(-\frac{(y+x+j/\sqrt{n})^2}{2t}\right) \exp\left(O(n^{-3\delta})\right) \\
=&\frac{4\gamma}{\sqrt{2\pi nt^3}} \sum_{j=0}^{(nt)^{\frac{2}{3}-\delta}} \left(\frac{1}{\sqrt{n}} \exp\left(-\frac{\mu j}{\sqrt{n}}\right) (y+x+j/\sqrt{n}) \exp\left(-\frac{(y+x+j/\sqrt{n})^2}{2t}\right)\right) \exp\left(O(n^{-3\delta})\right) \\
=&\frac{4}{\sqrt{2\pi nt^3}} \int_0^\infty e^{-\mu s} \cdot (y+x+s) \cdot \exp\left(-\frac{(y+x+s)^2}{2t}\right) ds \cdot \exp\left(o(1)\right) \\
=&\frac{4}{\sqrt{2\pi nt}} \exp\left(-\frac{(y+x)^2}{2t}\right) - 2\mu \exp\left(\mu(y+x) + \frac{1}{2}\mu^2 t\right) \cdot \operatorname{erfc}\left(\frac{y+x}{\sqrt{2t}} + \mu\sqrt{t/2}\right) + o\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

For the remaining summation over $j \in [(nt)^{\frac{2}{3}-\delta}, nt]$,

$$2\frac{(y+x)\sqrt{n}+j}{nt-j} T(nt-j, (y+x)\sqrt{n}+j) = \mathbb{P}_R^{0,x}(S_{nt} = y, N_{nt}(S) = j+1)$$

Therefore we have

$$\begin{aligned}
&\sum_{j \geq (nt)^{2/3-\delta}} \gamma^j \cdot \frac{(y+x)\sqrt{n}+j}{nt-j} T\left(nt-j, (y+x)\sqrt{n}+j\right) \\
=&\sum_{j \geq (nt)^{2/3-\delta}} \gamma^{j+1} \mathbb{P}_R^{0,x}(S_{nt} = y, N_{nt}(S) = j+1) \\
\leq&\sum_{j \geq (nt)^{2/3-\delta}} \gamma^{j+1} \mathbb{P}_R^{0,x}(N_{nt}(S) = j+1) \\
\leq&C \exp\left(-\frac{1}{C}(nt)^{1/6-\delta}\right) \\
=&o(\frac{1}{\sqrt{n}}),
\end{aligned}$$

where we have used Corollary 2.3.6 in the above inequality. Adding the above estimates, the desired result follows. □

By using Lemma 2.3.3 instead of Lemma 2.3.2 in estimating $\mathcal{T}_n$ we obtain the following upper bound of p_γ,

Lemma 2.3.9. *Let $\gamma = 1 - \frac{\mu}{\sqrt{n}}$. There exists a constant $C = C(\mu)$ such that for any $0 < t \le 1$ and $x\sqrt{n}, y\sqrt{n} \in \mathbb{N}_0$ such that*

$$p_\gamma(0, nt, x\sqrt{n}, y\sqrt{n}) \le \frac{C}{\sqrt{nt}} \exp\left(-\frac{(x-y)^2}{Ct}\right).$$

For 2-D SSRW (S^1, S^2), denote by $\mathcal{N}_n$ to be the number of times the random walk visits the origin before step $n-1$. In other words

$$\mathcal{N}_n := \#\left\{j \mid 0 \le j \le n-1,\ (S^1_j, S^2_j) = (0,0)\right\}.$$

The following lemma can be found in [Re].

Lemma 2.3.10. *There exists a universal constant $C > 0$ such that for any $n \ge 2$ and any $k \ge 1$ we have*

$$\mathbb{P}^{0,(0,0)}(\mathcal{N}_n \ge k) \le \exp\left(-\frac{k}{C \log n}\right)$$

We also derive the conditional version of it.

Lemma 2.3.11. *There exists a universal constant $C > 0$ such that for any $n \ge 2$, any $k \ge 1$ and any $(x_1, x_2), (y_1, y_2) \in \mathbb{Z}^2$ we have*

$$\mathbb{P}^{0,(x_1,x_2)}(\mathcal{N}_n \ge k | (S^1_n, S^2_n) = (y_1, y_2)) \le \max_{1 \le j \le n} C \cdot \exp\left(-\frac{k}{C \log j} + C \log j\right)$$

Proof. We first consider the case $(x_1, x_2) = (y_1, y_2) = (0, 0)$. In this case since $4^{-2}\binom{n}{n/2}^2 \approx \frac{2\pi}{n}$, we have

$$\mathbb{P}^{0,(0,0)}(\mathcal{N}_n \ge k | (S^1_n, S^2_n) = (0,0)) \le C \cdot \exp\left(-\frac{k}{C \log n} + \log n\right).$$

For general (x_1, x_2) and (y_1, y_2), by conditioning on the first time the random walk bridge arrives at the origin we have

$$\begin{aligned}
&\mathbb{P}^{0,(x_1,x_2)}(\mathcal{N}_n \ge k | (S^1_n, S^2_n) = (y_1, y_2)) \\
&\le \max_{1 \le j \le n} \mathbb{P}^{0,(0,0)}(\mathcal{N}_j \ge k | (S^1_j, S^2_j) = (y_1, y_2)) \\
&= \max_{1 \le j \le n} \mathbb{P}^{0,(y_1,y_2)}(\mathcal{N}_j \ge k | (S^1_j, S^2_j) = (0,0)).
\end{aligned}$$

We have reversed the random walk bridge in the first equality. By conditioning on the first time the random walk bridge arrives at the origin again, we obtain

$$\begin{aligned}
&\max_{1\le j\le n} \mathbb{P}^{0,(y_1,y_2)}(\mathcal{N}_j \ge k|(S_j^1,S_j^2)=(0,0)) \\
&\le \max_{1\le j\le n} \mathbb{P}^{0,(0,0)}(\mathcal{N}_j \ge k|(S_j^1,S_j^2)=(0,0)) \\
&\le \max_{1\le j\le n} C\cdot\exp\left(-\frac{k}{C\log j}+\log j\right).
\end{aligned}$$

□

The following lemma bounds random transition kernel under the condition such that the variance of the boundary random variable decays.

Lemma 2.3.12. *Assume there exists $\varepsilon>0$ such that $\mathrm{Var}[X_i]=\sigma^2=o(n^{-\varepsilon})$ and $\mathbb{E}[X]=\gamma=1-\frac{\mu}{\sqrt{n}}$, thus for any integers $1\le m\le n$ and $x,y\ge 0$,*

$$\mathrm{Var}[p_X(0,m,x,y)]=o(1)\cdot\max\{p^2(0,m,x,y),p^2_{\gamma^2}(0,m,x,y)\}.$$

Here the $o(1)$ term converges to zero as n goes to infinity and is uniform over all such m,x and y.

Proof. Since

$$\mathrm{Var}\,p_X(0,m,x,y)]=\mathbb{E}[p_X^2(0,m,x,y)]-(\mathbb{E}[p_X(0,m,x,y)])^2,$$

we compute

$$\begin{aligned}
\mathbb{E}[p_X^2(0,m,x,y)] &= \mathbb{E}\left[\sum_{S_m=y}\left(\prod_{i:S_i=0}X_i\right)\mathbb{P}_R^{0,x}(S)\cdot\sum_{\widetilde{S}_m=y}\left(\prod_{j:\widetilde{S}_j=0}X_i\right)\mathbb{P}_R^{0,x}(\widetilde{S})\right] \\
&= \mathbb{E}\left[\sum_{S_m=\widetilde{S}_m=y}\left(\prod_{i,j:S_i=\widetilde{S}_j=0}X_iX_j\right)\mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S})\right] \\
&= \sum_{S,\widetilde{S}}\left(\prod_{i=j:S_i=\widetilde{S}_i=0}\mathbb{E}[X_i^2]\right)\left(\prod_{\substack{i:S_i=0\ne\widetilde{S}_i\\ j:\widetilde{S}_j=0\ne S_j}}\mathbb{E}[X_i]\mathbb{E}[X_j]\right)\mathbb{P}_R(S)\mathbb{P}_R(\widetilde{S}).
\end{aligned}$$

By the independence of X,

$$\mathbb{E}[p_X(0,m,x,y)]^2 = \left[\sum_{S_m=y}\left(\prod_{i:S_i=0}\mathbb{E}[X_i]\right)\mathbb{P}_R(S)\cdot\sum_{\widetilde{S}_m=y}\left(\prod_{j:\widetilde{S}_j=0}\mathbb{E}[X_j]\right)\mathbb{P}_R(\widetilde{S})\right]$$

$$= \sum_{S_m=\widetilde{S}_m=y}\left(\prod_{i=j:S_i=\widetilde{S}_i=0}(\mathbb{E}[X_i])^2\right)\left(\prod_{\substack{i:S_i=0\neq\widetilde{S}_i\\ j:\widetilde{S}_j=0\neq S_j}}\mathbb{E}[X_i]\mathbb{E}[X_j]\right)\mathbb{P}_R(S)\mathbb{P}_R(\widetilde{S}).$$

Recall that $\mathbb{E}[X_i]=\gamma=1-\frac{\mu}{\sqrt{n}}$. For two paths $S_i,\widetilde{S}_j$, if we denote $\mathcal{N}_m$ as the number of indices i such that $S_i=\widetilde{S}_i=0$, then

$$\mathrm{Var}[p_X(0,m,x,y)] = \sum_{S_m=\widetilde{S}_m=y}\left(\mathbb{E}[X_i^2]^{\mathcal{N}_m}-\mathbb{E}[X_i]^{2\mathcal{N}_m}\right)\left(\prod_{\substack{i:S_i=0\neq\widetilde{S}_i\\ j:\widetilde{S}_j=0\neq S_j}}\gamma\right)\mathbb{P}_R(S)\mathbb{P}_R(\widetilde{S})$$

$$= \sum_{S_m=\widetilde{S}_m=y}\left((\sigma^2+\gamma^2)^{\mathcal{N}_m}-\gamma^{2\mathcal{N}_m}\right)\left(\prod_{\substack{i:S_i=0\neq\widetilde{S}_i\\ j:\widetilde{S}_j=0\neq S_j}}\gamma\right)\mathbb{P}_R(S)\mathbb{P}_R(\widetilde{S}).$$

We can decompose $\mathcal{N}_m$ into two cases:

$$0\le\mathcal{N}_m\le(\log n)^3=:L \quad\text{and}\quad \mathcal{N}_m>L.$$

For the case $0\le\mathcal{N}_m\le L$, we estimate

$$\begin{aligned}
(\sigma^2+\gamma^2)^{\mathcal{N}_m}-\gamma^{2\mathcal{N}_m} &= \left((1-\frac{\mu}{\sqrt{n}})^2+o(n^{-\varepsilon})\right)^{\mathcal{N}_m}-\left(1-\frac{\mu}{\sqrt{n}}\right)^{2\mathcal{N}_m}\\
&= \left(1-\frac{2\mu}{\sqrt{n}}+o(\frac{1}{n^\varepsilon})+O(\frac{1}{n})\right)^{\mathcal{N}_m}-\left(1-\frac{2\mu}{\sqrt{n}}+O(\frac{1}{n})\right)^{\mathcal{N}_m}\\
&= \left(1-\frac{2\mu}{\sqrt{n}}\right)^{\mathcal{N}_m}+o\left(\frac{\mathcal{N}_m}{n^\varepsilon}\right)-\left(1-\frac{2\mu}{\sqrt{n}}\right)^{\mathcal{N}_m}-O\left(\frac{\mathcal{N}_m}{n}\right)\\
&= o\left(\frac{\mathcal{N}_m}{n^\varepsilon}\right)=o(1).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{\mathcal{N}_m\le L, S_m=\widetilde{S}_m=y}\left((\sigma^2+\gamma^2)^{\mathcal{N}_m}-\gamma^{2\mathcal{N}_m}\right)\left(\prod_{\substack{i:S_i=0\neq\widetilde{S}_i\\ j:\widetilde{S}_j=0\neq S_j}}\gamma\right)\mathbb{P}_R(S)\mathbb{P}_R(\widetilde{S})\\
\le& o(1)\sum_{S_m=\widetilde{S}_m=y}\gamma^{N_m(S)+N_m(\widetilde{S})}\mathbb{P}_R(S)\mathbb{P}_R(\widetilde{S})\\
=& o(1)p_\gamma^2(0,m,x,y).
\end{aligned}$$

Now we discuss the case $\mathcal{N}_m > L$ and $\mu \geq 0$. Denote $\xi = \sigma^2 + \gamma^2$. For $\mathcal{N}_m > L$, we compute

$$\begin{aligned}
&\sum_{\mathcal{N}_m \geq L, S_m = \widetilde{S}_m = y} \left((\sigma^2+\gamma^2)^{\mathcal{N}_m} - \gamma^{2\mathcal{N}_m}\right) \left(\prod_{\substack{i: S_i = 0 \neq \widetilde{S}_i \\ j: \widetilde{S}_j = 0 \neq S_j}} \gamma \right) \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}) \\
\leq &\mathbb{E}^{0,(x,x)} \left[\xi^{\mathcal{N}_m} \mathbb{1}(\mathcal{N}_m \geq L) | (S_m, \widetilde{S}_m) = (y,y) \right] p(0,m,x,y)^2.
\end{aligned}$$

We would like to show that $\mathbb{E}^{0,(x,x)} \left[\xi^{\mathcal{N}_m} \mathbb{1}(\mathcal{N}_m \geq L) | (S_m, \widetilde{S}_m) = (y,y) \right] = o(1)$.

$$\begin{aligned}
&\mathbb{E}^{0,(x,x)} \left[\xi^{\mathcal{N}_m} \mathbb{1}(\mathcal{N}_m \geq L) | (S_m, \widetilde{S}_m) = (y,y) \right] \\
=&\sum_{k \geq L} \xi^k \mathbb{P}^{0,(x,x)}(\mathcal{N}_m = k | (S_m^1, S_m^2) = (y,y)) \\
=&(1-\xi^{-1}) \sum_{k \geq L+1} \xi^k \mathbb{P}^{0,(x,x)}(\mathcal{N}_m \geq k | (S_m^1, S_m^2) = (y,y)) \\
&+ \xi^L \mathbb{P}^{0,(x,x)}(\mathcal{N}_m \geq L | (S_m^1, S_m^2) = (y,y))
\end{aligned}$$

We choose n large such that $C^2 << \log n$, where $C > 0$ comes from Lemma 2.3.11. Then we get

$$\begin{aligned}
&\mathbb{P}^{0,(x,x)}(\mathcal{N}_m \geq k | (S_m^1, S_m^2) = (y,y)) \\
\leq &\max_{1 \leq j \leq m} \exp\left(-\frac{k}{C \log j} + C \log j \right) \\
= &\exp\left(-\frac{k}{C \log m} + C \log m \right). \\
\leq &\exp\left(-\frac{k}{2C \log m} \right).
\end{aligned}$$

Now we can estimate

$$\begin{aligned}
&\xi^L \mathbb{P}^{0,(x,x)}(\mathcal{N}_m \geq L | (S_m^1, S_m^2) = (y,y)) \\
\leq &\exp\left(\left(\frac{C}{n^{\varepsilon}} - \frac{1}{2C \log m} \right) L \right) = o(1).
\end{aligned}$$

The remaining terms can be estimated as

$$
\begin{aligned}
&(1-\xi^{-1})\sum_{k\geq L+1}\xi^k\mathbb{P}^{0,(x,x)}(\mathcal{N}_m\geq k|(S_m^1,S_m^2)=(y,y))\\
\lesssim&\frac{1}{n^\varepsilon}\sum_{k\geq L+1}\exp\left(\left(\frac{c_0}{n^\varepsilon}-\frac{1}{2C\log m}\right)k\right)\\
\lesssim&\frac{1}{n^\varepsilon}\sum_{k\geq L+1}\exp\left(\left(-\frac{1}{4C\log m}\right)k\right)\\
\approx&\frac{\log m}{n^\varepsilon}\exp\left(-\frac{L}{4C\log m}\right)\\
=&o(1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sum_{\mathcal{N}_m\geq L, S_m=\widetilde{S}_m=y}\left((\sigma^2+\gamma^2)^{\mathcal{N}_m}-\gamma^{2\mathcal{N}_m}\right)\left(\prod_{\substack{i:S_i=0\neq\widetilde{S}_i\\ j:\widetilde{S}_j=0\neq S_j}}\gamma\right)\mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S})\\
=&o(1)p(0,m,x,y)^2.
\end{aligned}
$$

For the case that $\mu<0$, let $M>0$ be a number to be determined. As we sum over the paths $S,\widetilde{S}$ with $N_m(S),N_m(\widetilde{S})\leq M\sqrt{n}$,

$$
\begin{aligned}
&\sum_{\substack{\mathcal{N}_m\geq L, S_m=\widetilde{S}_m=y\\ N_m(S),N_m(\widetilde{S})\leq M\sqrt{n}}}\left((\sigma^2+\gamma^2)^{\mathcal{N}_m}-\gamma^{2\mathcal{N}_m}\right)\left(\prod_{\substack{i:S_i=0\neq\widetilde{S}_i\\ j:\widetilde{S}_j=0\neq S_j}}\gamma\right)\mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S})\\
\leq&C(M,\mu)\sum_{\mathcal{N}_m\geq L, S_m=\widetilde{S}_m=y}\left((\sigma^2+\gamma^2)^{\mathcal{N}_m}-\gamma^{2\mathcal{N}_m}\right)\mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S})\\
\leq&o(1)C(M,\mu)p(0,m,x,y)^2.
\end{aligned}
$$

The last case is $\max\{N_m(S), N_m(\widetilde{S})\} \geq M\sqrt{n}$. By using Cauchy-Schwarz, we have

$$\begin{aligned}
&\sum_{\substack{\mathcal{N}_m \geq L, S_m=\widetilde{S}_m=y \\ \max\{N_m(S),N_m(\widetilde{S})\}\geq M\sqrt{n}}} \left((\sigma^2+\gamma^2)^{\mathcal{N}_m} - \gamma^{2\mathcal{N}_m}\right) \left(\prod_{\substack{i:S_i=0\neq \widetilde{S}_i \\ j:\widetilde{S}_j=0\neq S_j}} \gamma \right) \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}) \\
\leq & \frac{1}{2} \sum_{\mathcal{N}_m \geq L, S_m=\widetilde{S}_m=y} \left((\sigma^2+\gamma^2)^{\mathcal{N}_m} - \gamma^{2\mathcal{N}_m}\right)^2 \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}) \\
+ & \frac{1}{2} \sum_{S_m=\widetilde{S}_m=y, N_m(S)\geq M\sqrt{n}} \gamma^{2N_m(S)+2N_m(\widetilde{S})} \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}) \\
+ & \frac{1}{2} \sum_{S_m=\widetilde{S}_m=y, N_m(\widetilde{S})\geq M\sqrt{n}} \gamma^{2N_m(S)+2N_m(\widetilde{S})} \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}).
\end{aligned}$$

We have already showed that the first term is $o(1)p(0, m, x, y)^2$. The last two terms are the same and can be estimated as

$$\begin{aligned}
&\sum_{S_m=\widetilde{S}_m=y, N_m(S)\geq M\sqrt{n}} \gamma^{2N_m(S)+2N_m(\widetilde{S})} \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}) \\
=& p_{\gamma^2}(0, m, x, y) \sum_{S_m=y, N_m(S)\geq M\sqrt{n}} \gamma^{2N_m(S)} \mathbb{P}_R^{0,x}(S) \\
=& p_{\gamma^2}(0, m, x, y)p(0, m, x, y) \sum_{k\geq M\sqrt{n}} \gamma^{2k} \mathbb{P}_R^{0,x}(N_m = k | S_m = y) \\
\leq & p_{\gamma^2}(0, m, x, y)p(0, m, x, y)C_2(\mu) \exp\left(-\frac{M^2}{C}\right)
\end{aligned}$$

Because

$$\lim_{n\to\infty} \inf_M \left(o(1)C(M, \mu) + C_2(\mu) \exp\left(-\frac{M^2}{C}\right)\right) = 0,$$

we conclude that

$$\begin{aligned}
&\sum_{\mathcal{N}_m \geq L, S_m=\widetilde{S}_m=y} \left((\sigma^2+\gamma^2)^{\mathcal{N}_m} - \gamma^{2\mathcal{N}_m}\right) \left(\prod_{\substack{i:S_i=0\neq \widetilde{S}_i \\ j:\widetilde{S}_j=0\neq S_j}} \gamma \right) \mathbb{P}_R^{0,x}(S)\mathbb{P}_R^{0,x}(\widetilde{S}) \\
\leq & o(1)p_{\gamma^2}(0, m, x, y)^2.
\end{aligned}$$

□

Remark 2.3.13. *Under the assumption* $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^3] = o(n^{-\varepsilon})$*, we can show that* $\mathbb{E}[p_X^3(0, m, x, y)] = p_\gamma^3(0, m, x, y) + o(1)\max\{p^3(0, m, x, y), p_{\gamma^2}^3(0, m, x, y)\}$ *through a similar argument because the local time of* *higher dimension random walks decays faster. In particular, we have* $\mathbb{E}[(p_X(0,m,x,y) - p_\gamma(0,m,x,y)^\alpha)] = o(1)\max\{p^\alpha(0,m,x,y), p^\alpha_{\gamma^2}(0,m,x,y)\}$ *for any* $0 < \alpha \le 3$.

2.4 Proof for the convergence of the partition functions

We proceed to identify the scaled modified point-to-point partition function $\mathfrak{Z}_n^{\omega,X}(x\sqrt{n}, \beta n^{-1/4})$ as in (2.6). As the transition kernel $p_{X,k|x\sqrt{n}}(\mathbf{i},\mathbf{x})$ as in (2.4) is only defined on lattice points $(\mathbf{i},\mathbf{x}) \in D_k^n \times \mathbb{Z}_{\ge 0}^k$ which verify the parity condition. In order to interpret $\mathfrak{Z}_n^{\omega,X}(x\sqrt{n}, \beta n^{-1/4})$ as a U-statistics, we will interpolate the discrete transition kernel $p_{X,k|x\sqrt{n}}(\mathbf{i},\mathbf{x})$ to be a L^2 function on $[0,1]^k \times \mathbb{R}_{\ge 0}^k$ as follows.

Given $x \in \mathbb{R}_{\ge 0}$ and $i \in \mathbb{Z}_{\ge 0}$, define $[x]_i$ to be the largest integer among the ones that are smaller than x and are of the same parity as i. For a point $\mathbf{x} \in \mathbb{R}_{\ge 0}^k$ and $\mathbf{i} \in D_k^n$, define $[\mathbf{x}]_\mathbf{i} \in \mathbb{Z}_{\ge 0}^k$ by $([\mathbf{x}]_\mathbf{i})_k = [x_k]_{i_k}$. First for $(\mathbf{i},\mathbf{x}) \in D_k^n \times \mathbb{R}_{\ge 0}^k$, define

$$\widetilde{p}_{X,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}) := 2^{-k} p_{X,k|x\sqrt{x}}(\mathbf{i},[\mathbf{x}]_\mathbf{i}),$$

where 2^{-k} deals with the parity condition. Secondly for $(\mathbf{t},\mathbf{x}) \in [0,1]^k \times \mathbb{R}_{\ge 0}^k$, define

$$p^{n,X}_{k|x\sqrt{n}}(\mathbf{t},\mathbf{x}) = \widetilde{p}_{X,k|x\sqrt{n}}(\lfloor n\mathbf{t} \rfloor, \mathbf{x}\sqrt{n}) \cdot \mathbb{1}\{\lfloor n\mathbf{t} \rfloor \in D_k^n\}. \tag{2.23}$$

Under above definition and by the way $\widetilde{p}_{X,k|x\sqrt{n}}$ is extended, $p_k^{n,X|x\sqrt{n}}$ is constant on the rectangles of $\mathcal{R}_k^n$ and it is in $L^2([0,1]^k \times \mathbb{R}_{\ge 0}^k)$. Note that for $\mathbf{i} \in E_k^n, \mathbf{x} \in \mathbb{Z}_{\ge 0}^k$ such that $\mathbf{i} \leftrightarrow \mathbf{x}$,

$$p^{n,X}_{k|x\sqrt{n}}(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}) = \widetilde{p}_{X,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}) \cdot \mathbb{1}\{\mathbf{i} \in D_k^n\} = 2^{-k} p_{X,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}) \cdot \mathbb{1}\{\mathbf{i} \in D_k^n\}.$$

Recall the definition of $\mathcal{S}_k^n$ as in (2.14) and note that $p^{n,X}_{k,x}$ is constant on $D_k^n \times \mathbb{Z}_{\ge 0}^k$ and zero elsewhere, we compute the U-statistics of $p^{n,X}_{k,x}$ as follows,

$$\begin{aligned}\mathcal{S}_k^n(p^{n,X}_{k,x}) &= 2^{k/2} \sum_{\mathbf{i} \in E_k^n} \sum_{\mathbf{x} \in \mathbb{Z}_{\ge 0}^k} p^{n,X}_{k,x}\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}\right) \cdot \omega(\mathbf{i},\mathbf{x}) \cdot \mathbf{i} \leftrightarrow \mathbf{x} \\ &= 2^{-k/2} \sum_{\mathbf{i} \in D_k^n} \sum_{\mathbf{x} \in \mathbb{Z}_{\ge 0}^k} p_{X,k|x}(\mathbf{i},\mathbf{x}) \cdot \omega(\mathbf{i},\mathbf{x}).\end{aligned}$$

where the parity condition is handled by the $p_{X,k|x}$ and summation is over $\mathbf{i} \in D_k^n$ and we could rewrite the modified point-to-point partition function as

$$\mathfrak{Z}_n^{\omega,\gamma}(x\sqrt{n}; \beta n^{-1/4}) = \sum_{k=0}^n 2^{k/2} \beta^k n^{-3k/4} \mathcal{S}_k^n(n^{k/2} p^{n,\gamma}_{k,x\sqrt{n}}). \tag{2.24}$$

The following lemma bounds the L^2 norm of $p^{n,\gamma}_{k,x\sqrt{n}}$ by $\left\|\varrho_{k|x}\right\|^2_{L^2}$ as in (2.10) and shows the L^2 convergence for $p^{n,\gamma}_{k,|x\sqrt{n}}$.

Lemma 2.4.1. *Let $\gamma = 1 - \frac{\mu}{\sqrt{n}}$ for $\mu \in \mathbb{R}$. Thus there exists a constant $C(\mu)$ such that for all $k \geq 0$ and $x \in \mathbb{R}_{\geq 0}$, we have*

$$\sup_n \left\| n^{(k+1)/2} p^{n,\gamma}_{k|x\sqrt{n}} \right\|^2_{L^2} \leq C(\mu)^{k+1} \left\| \varrho_{k|x} \right\|^2_{L^2}, \tag{2.25}$$

$$\lim_{n\to\infty} \left\| \frac{1}{2} n^{(k+1)/2} p^{n,\gamma}_{k|x\sqrt{n}} - \varrho_{\mu,k|x} \right\|_{L^2} = 0. \tag{2.26}$$

Proof. By Lemma 2.3.8 and the fact that $p_\gamma(0,n,0,x)$ decreases in x, there exists a constant $C(\mu)$ such that $\sqrt{n} p_\gamma(0,n,0,x) \leq C(\mu)$. Recall definition for $p_{\gamma,k|x\sqrt{n}}$ in (2.4), it follows that

$$\sup_{\mathbf{x}\in\mathbb{Z}^k_{\geq 0}} p_{\gamma,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}) \leq C(\mu)^{k+1} \frac{1}{\sqrt{n-i_k}} \prod_{j=1}^k \frac{1}{\sqrt{i_j - i_{j-1}}}. \tag{2.27}$$

And we compute

$$\begin{aligned}\left\| n^{(k+1)/2} p^{n,\gamma}_{k|x\sqrt{n}} \right\|^2_{L^2} &= n^{k+1} \sum_{\mathbf{i}\in D^n_k} \sum_{\mathbf{x}\in\mathbb{Z}^k_{\geq 0}} p_{\gamma,k|x\sqrt{n}}(\mathbf{i},\mathbf{x})^2 n^{-3k/2} \\ &\leq n^{-k/2+1} \sum_{\mathbf{i}\in D^n_k} \sup_{\mathbf{x}\in\mathbb{Z}^k_{\geq 0}} p_{\gamma,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}) \sum_{\mathbf{x}\in\mathbb{Z}^k_{\geq 0}} p_{\gamma,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}).\end{aligned}$$

By Corollary 2.3.7 we have

$$\sqrt{n} \sum_{\mathbf{x}\in\mathbb{Z}^k_{\geq 0}} p_{\gamma,k|x\sqrt{n}}(\mathbf{i},\mathbf{x}) \leq C(\mu)^{k+1},$$

combining with (2.27), it follows that

$$\begin{aligned}\left\| n^{k/2} p^{n,\gamma}_k \right\|_{L^2} &\leq C(\mu)^{k+1} n^{(-k+1)/2} \sum_{\mathbf{i}\in D^n_k} \frac{1}{\sqrt{n-i_k}} \prod_{j=1}^k \frac{1}{\sqrt{i_j - i_{j-1}}} \\ &= C(\mu)^{2k+2} n^{-k} \sum_{\mathbf{i}\in D^n_k} \prod_{j=1}^k \left(1 - \frac{i_k}{n}\right)^{-1/2} \left(\frac{i_j}{n} - \frac{i_{j-1}}{n}\right)^{-1/2} \\ &\leq C(\mu)^{2k+2} \int_{\Delta_k} \frac{1}{\sqrt{1-t_k}} \prod_{j=1}^k \frac{1}{\sqrt{t_j - t_{j-1}}} d\mathbf{t} \\ &\leq C(k)^k \left\| \varrho_{k|x} \right\|_{L^2}.\end{aligned}$$

In the last step we abuse the notation of $C(k)$.

For the L^2 convergence (2.26), note that by the way $p^{n,\gamma}_{k|x\sqrt{n}}$ is extended and the local limit theorem 2.3.8, $\frac{1}{2} n^{(k+1)/2} p^{n,\gamma}_{k|x\sqrt{n}}$ converges pointwisely to ϱ_μ. From Lemma 2.3.9, L^2 convergence follows from dominated convergence theorem. □

By identifying $\mathfrak{Z}^{\omega,\gamma}_n(x\sqrt{n}; \beta n^{-1/4})$ with the U-statistics as in (2.24), now we start to prove the main Theorem 2.1.3 in a few steps as follows.

Proof of Theorem 2.1.3. Step 1: We first prove the convergence of $\mathfrak{Z}_n^{\omega,\gamma}(x\sqrt{n};\beta n^{-1/4})$. Since the Robin heat kernel $\varrho_\mu(t,x,y)$ is a L^2 function and so is the k-fold transition kernel $\varrho_{\mu,k|x}$, by Lemma 2.2.3 it follows that for all $\beta > 0$, as $n \to \infty$,

$$\sum_{k=0}^{\infty} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n(\varrho_{\mu,k|x}) \xrightarrow{(d)} z_{\sqrt{2}\beta}(1,x). \tag{2.28}$$

See the chaos expansion of $z_{\sqrt{2}\beta}(1,x)$ (2.13) . Now it suffices to show that the difference

$$J := \sum_{k=0}^{\infty} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n(\varrho_{\mu,k|x}) - \frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,\gamma}(x\sqrt{n};\beta n^{-1/4}).$$

converges to 0 in L^2. By splitting the above series and applying linearity of $\mathcal{S}_k^n$, we have

$$\begin{aligned} J &= \sum_{k=0}^{\infty} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n(\varrho_{\mu,k|x}) - \frac{\sqrt{n}}{2}\sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n(n^{k/2}p_k^{n,\gamma}) \\ &= \sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n\big(\varrho_{\mu,k|x} - \frac{1}{2}n^{k+1/2}p_{k|x\sqrt{n}}^{n,\gamma}\big) + \sum_{k=n+1}^{\infty} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n(\varrho_{\mu,k|x}). \end{aligned}$$

By Lemma 2.2.1 the second term is bounded from above in L^2 by

$$\sum_{k=n+1}^{\infty} 2^k\beta^{2k}\|\varrho_{\mu,k|x}\|^2_{L^2(\Delta_k\times\mathbb{R}^k_{\geq 0})},$$

which converges to zero if the infinite series $\sum_{k\geq 1} 2^k\beta^{2k}\|\varrho_{\mu,k|x}\|^2_{L^2(\Delta_k\times\mathbb{R}^k_{\geq 0})}$ is summable. By (2.11), for any constant C, $C\|\varrho_{k|x}\|^2_{L^2([0,1]^k\times\mathbb{R}^k_{\geq 0})}$ is summable in k. By Lemma 2.1.2, the Robin heat kernel $\varrho_\mu \leq C(\mu)\cdot\varrho$ where $C(\mu)$ is a constant depending only on μ, thus $\sum_{k\geq 1} 2^k\beta^{2k}\|\varrho_{\mu,k}\|^2_{L^2(k\times\mathbb{R}^k_{\geq 0})} < \infty$ since $\sum_{k\geq 1} C^k\|\varrho_{k|x}\|^2_{L^2(k\times\mathbb{R}^k_{\geq 0})} < \infty$ for any C. Thus the second term converges to zero as n goes to infinity.

For the first term, we have

$$\|\sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n\big(\varrho_{\mu,k|x} - \frac{1}{2}n^{k+1/2}p_{k|x}^{n,\gamma}\big)\|^2_{L^2} \leq \sum_{k=0}^{n} 2^k\beta^{2k}\big\|\varrho_{\mu,k|x} - \frac{1}{2}n^{k+1/2}p_{k|x}^{n,\gamma}\big\|^2_{L^2}.$$

Lemma 2.4.1 shows that for any k, as $n \to \infty$,

$$\big\|\varrho_{\mu,k|x} - \frac{1}{2}n^{(k+1)/2}p_{k|x\sqrt{n}}^{n,\gamma}\big\|^2_{L^2} \to 0.$$

and there exists a constant $C(\mu) > 0$ such that

$$\sup_n \|\frac{1}{2}n^{(k+1)/2}p_{k|x\sqrt{n}}^{n,\gamma}\|_{L^2} \leq C(\mu)^k\|\varrho_{\mu,k|x}\|_{L^2}.$$

which implies

$$\lim_{n\to\infty}\sum_{k=0}^{n} 2^k\beta^{2k}\big\|\varrho_{\mu,k|x} - \frac{1}{2}n^{(k+1)/2}p_{k|x\sqrt{n}}^{n,\gamma}\big\|^2_{L^2} = \sum_{k=0}^{\infty} 2^k\beta^{2k}\lim_{n\to\infty}\big\|\varrho_{\mu,k} - \frac{1}{2}n^{(k+1)/2}p_{k|x\sqrt{n}}^{n,\gamma}\big\|^2_{L^2} = 0.$$

Step 2: Having established the convergence for $\mathfrak{Z}_n^{\omega,\gamma}(x\sqrt{n};\beta n^{-1/4})$, we now turn to likewise demonstrating convergence of $\mathfrak{Z}_n^{\omega,X}(x\sqrt{n};\beta n^{-1/4})$ where randomness is also present at the boundary random environment. It suffices to show

$$\frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,X}(x\sqrt{n};\beta n^{-1/4}) - \frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,\gamma}(x\sqrt{n};\beta n^{-1/4}) \xrightarrow{(d)} 0.$$

We have

$$\frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,X}(x\sqrt{n};\beta n^{-1/4}) - \frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,\gamma}(x\sqrt{n};\beta n^{-1/4}) = \sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4}\mathcal{S}_k^n\big(\frac{1}{2}n^{k+1/2}(p_k^{n,X}-p_k^{n,\gamma})\big).$$

As in Step 1, it suffices to show $\frac{1}{2}n^{(k+1)/2}p^{n,X}_{k|x\sqrt{n}} - \frac{1}{2}n^{(k+1)/2}p^{n,\gamma}_{k|x\sqrt{n}}$ converges in L^2, i.e.

$$\int_{[0,1]^k\times\mathbb{R}^k_{\geq 0}} \mathbb{E}\left[n^{k+1/2}(p^{n,X}_{k|x\sqrt{n}} - p^{n,\gamma}_{k|x\sqrt{n}})\right]^2 d\mathbf{t}d\mathbf{x} \to 0.$$

as $n\to\infty$.

Recall the definition for $p^{n,X}_{k,x\sqrt{n}}(\mathbf{t},\mathbf{x})$ as in (2.23) and the definition for the k-fold transition kernel $p_{X,k|x\sqrt{n}}$ as in (2.4), it follows that

$$\begin{aligned}
&\mathbb{E}\left[n^{k+1/2}(p^{n,X}_{k|x\sqrt{n}} - p^{n,\gamma}_{k|x\sqrt{n}})\right]^2\\
&= \mathbb{E}[n\cdot p_X^2(t_k n, n, x_k\sqrt{n}, x\sqrt{n})]\prod_{j=1}^{k}\mathbb{E}[n\cdot p_X^2(x_{j-1}, x_{j-1}\sqrt{n}, x_j\sqrt{n})]\\
&\quad - n\cdot p_\gamma^2(t_k n, n, x_k\sqrt{n}, x)\prod_{j=1}^{k} n\cdot p_\gamma^2(x_{j-1}, x_j x_{j-1}\sqrt{n}, x_j\sqrt{n})
\end{aligned}$$

By Lemma 2.3.12 and Lemma 2.3.9, $n\,\mathrm{Var}[p_X(m,n,x,y)] = o(1)\cdot n\cdot p^2(m,n,x,y) = o(1)$, where $o(1)$ is uniformly in m,x,y. Thus we have $\mathbb{E}[np_X(m,n,x,y)] = p_\gamma(m,n,x,y)+o(1)$ and it follows that

$$\begin{aligned}
&\mathbb{E}\left[n^{k+1/2}(p_{X,k|x\sqrt{n}} - p_{\gamma,k|x\sqrt{n}})\right]^2\\
&= o(1)\cdot n\cdot p_\gamma^2(t_j\cdot n, n, x_j\sqrt{n}, x\sqrt{n})\cdot\prod_{j=1}^{k} n\cdot p_\gamma^2(t_{j-1}\cdot n, t_j\cdot n, x_{j-1}\sqrt{n}, x_j\sqrt{n})\\
&= o(1)\frac{C}{\sqrt{1-t_j}}\exp\left(-\frac{(x-x_j)^2}{C(1-t_j)}\right)\prod_{j=1}^{k}\frac{C}{\sqrt{t_j-t_{j-1}}}\exp\left(-\frac{(x_j-x_{j-1})^2)}{C(t_j-t_{j-1})}\right)
\end{aligned}$$

The last term is L^2 integrable and we have $n^{(k+1)/2}p^{n,X}_{k|x\sqrt{n}} - n^{(k+1)/2}p^{n,\gamma}_{k|x\sqrt{n}}$ converges in L^2 and the desired convergence for $\frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,X}(x\sqrt{n};\beta n^{-1/4}) \xrightarrow{(d)} z_{\sqrt{2}\beta}(1,x)$ follows. □

Step 3: Now we begin the process of extending the convergence to the unmodified point-to-point partition function $\frac{\sqrt{n}}{2}Z_n^{\omega,X}(x\sqrt{n};\beta n^{-1/4})$. Define the environment field $\widetilde{\omega}_n$ by

$$e^{\beta n^{-1/4}\omega(i,x)-\lambda(\beta n^{-1/4})} = 1+\beta n^{-1/4}\widetilde{\omega}_n(i,x), \tag{2.29}$$

so that

$$Z_n^{\omega,X}\left(x\sqrt{n};\beta n^{-1/4}\right) = \frac{\sqrt{n}}{2}\mathbb{E}_R\left[\prod_{i=0}^{n}\left(1+\beta n^{-1/4}\widetilde{\omega}_n(i,S_i)\right)\mathbb{1}\{S_n = x\sqrt{n}\}\right]$$
$$= \frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\widetilde{\omega}_n,X}\left(x\sqrt{n};\beta n^{-1/4}\right).$$

It is straightforward to check that under the exponential moments assumption for ω and the definition of $\lambda(\beta)$ we have $\mathbb{E}(\widetilde{\omega}_n) = 0$ and $\mathrm{Var}(\widetilde{\omega}_n^2) = 1 + O(n^{-1/4})$. Now convergence follows by applying the same arguments in Step 1 and Step 2 and using the convergence result from Lemma 2.2.5 for the perturbed random environment $\widetilde{\omega}$.

Now in order to show the weak convergence as a process, it suffices to show the tightness of the above process, which could be done by a similar argument as in [AKQ, Appendix B]. They first deduced an integral form in terms of the random walk transition kernel for the modified point-to-point partition function $\mathfrak{Z}(x,k)$ from the discrete stochastic heat equation that $\mathfrak{Z}(x,k)$ satisfies and then developed the modulus of continuity for the partition function with estimates for heat kernel. In our case, for deterministic $X_i \equiv \gamma$, we could derive a similar integral form for the point-to-point partition function but in terms of transition kernel for half-line random walk with a barrier at origin and then the similar estimates follow given that Robin heat kernel has similar decay behavior as standard heat kernel as in Lemma 2.1.2. For X_i under the assumption of Theorem 2.1.3, from Remark 2.3.13, we have that $\mathbb{E}[|\sqrt{n}p_1^{n,X} - \sqrt{n}p_1^{n,\gamma}|^\alpha]$ converges to zero in $L^1([0,1]\times\mathbb{R}_{\geq 0})$ for any $1 \leq \alpha < 3$. Here $p_1^{n,X}(t,x)$ and $p_1^{n,\gamma}(t,x)$ are interpolated (random) transition kernel as in (2.23). This allows us to adapt the proof in [AKQ, Appendix B] to the current setting.

2.5 Application to log-gamma polymer models

In this section we consider the half-space log-gamma polymer model and apply the argument for Theorem 2.1.3 to the log-gamma polymer point-to-point partition function. The log-gamma polymer models in dimension $1+1$ are of significant importance among polymer models in the sense that integral formulas are discovered and steepest descent analysis is allowed, see [BBC].

We start with defining the log-gamma polymer model and we follow notations used in the literature. For an endpoint $(m,n), m \geq n$, define the point-to-point partition function by

$$Z_{m,n}^Y := \sum_{S:(0,0)\to(m,n)}\prod_{(i,j)\in S} Y_{i,j},$$

where we sum over the paths S from $(0,0)$ to (m,n) which always stay in the half-quadrant with $m \geq n$. Note that the probabilities of these paths do not sum to one since those paths having crossed boundary $x = y$ are excluded. To match with the general environment setting in the half-space regime with a barrier at origin, we rewrite

$$\begin{aligned} Z^Y_{m,n} &= 2^{m+n} \sum_{S:(0,0)\to(m,n)} 2^{-(m+n)} \cdot 2^{\#_S} \cdot 2^{-\#_S} \cdot \prod_{(i,j)\in S} Y_{i,j} \\ &= 2^{m+n}\mathbb{E}_R\left[\prod_{(i,j)\in S} \widetilde{Y}_{i,j} \cdot \mathbb{1}\{S(m+n)=(m,n)\}\right], \end{aligned} \tag{2.30}$$

where $\#_S$ is the number of times that path S visits the origin and $\mathbb{E}_R$ is the expectation with respected to the reflected random walk measure. And $\widetilde{Y}_{i,j}$ is defined by

$$\widetilde{Y}_{i,i} = \frac{1}{2}Y_{i,i}, \quad \widetilde{Y}_{i,j} = Y_{i,j}, i > j.$$

Starting from now, we fix the parameters of the log-gamma random environment with

$$Y_{i,i} \sim \Gamma^{-1}(\sqrt{n}+\mu+\frac{1}{2}), \quad Y_{i,j} \sim \Gamma^{-1}(2\sqrt{n}),\ i>j, \tag{2.31}$$

as studied in [BBC] with moments formulas derived. Here $\Gamma^{-1}(\alpha)$ is the inverse gamma distribution with shape parameter α and scale parameter 1, and with density

$$\frac{1}{\Gamma(\alpha)}x^{-\alpha-1}e^{-1/x}.$$

The following convergence result holds for log-gamma polymer model.

Theorem 2.5.1. *Let $Y_{i,j}$ be independent random variables distributed as in (2.31) and $x \ge 0$, the following convergence results hold for the half-space log-gamma polymer model as $n \to \infty$,*

$$\frac{\sqrt{n}}{2} \cdot 2^{-n}\mathbb{E}[Y_{i,j}]^{-n} \cdot Z_{\lfloor \frac{1}{2}(n+x\sqrt{n})\rfloor, \lfloor \frac{1}{2}(n-x\sqrt{n})\rfloor} \xrightarrow{(d)} z_1(1,x).$$

Proof. The proof is based on the same argument as in Theorem 2.1.3 and it suffices to rewrite the log-gamma partition function to match with the expression of modified point-to-point partition function as in (2.5) with $\beta = \frac{1}{\sqrt{2}}$.

Note that

$$\begin{aligned} &2^{-n}\mathbb{E}[Y_{i,j}]^{-n} \cdot Z_{\lfloor \frac{1}{2}(n+x\sqrt{n})\rfloor, \lfloor \frac{1}{2}(n-x\sqrt{n})\rfloor} \\ &= \mathbb{E}_R\left[\prod_{(i,j)\in S} \frac{\widetilde{Y}_{i,j}}{\mathbb{E}[Y_{i,j}]}\mathbb{1}\left\{S(n) = \left(\lfloor \frac{1}{2}(n+x\sqrt{n})\rfloor, \lfloor \frac{1}{2}(n-x\sqrt{n})\rfloor\right)\right\}\right]. \end{aligned}$$

Define $\omega_{i,j}$ for $i \ge j$ via

$$\begin{aligned} \frac{\widetilde{Y}_{i,j}}{\mathbb{E}[Y_{i,j}]} &=: 1 + \frac{n^{-1/4}}{\sqrt{2}}\omega_{i,j}, i > j; \\ \frac{\widetilde{Y}_{i,i}}{\mathbb{E}[Y_{i,j}]} &=: \gamma_n\left(1 + \frac{n^{-1/4}}{\sqrt{2}}\omega_{i,i}\right), i = j; \end{aligned}$$

where $\gamma_n := \frac{1}{2}\mathbb{E}[Y_{i,i}]/\mathbb{E}[Y_{i,j}]$.

It's easy to verify that for $i \geq j$, $\mathbb{E}[\omega_{i,j}] = 0$. And since $\mathbb{E}[\Gamma^{-1}(\alpha)] = \frac{1}{\alpha-1}, \mathrm{Var}[\Gamma^{-1}(\alpha)] = \frac{1}{(\alpha-1)^2(\alpha-2)}$, we have,

$$\mathrm{Var}[\omega_{i,j}] = 2\sqrt{n}\frac{\mathrm{Var}[Y_{i,j}]}{\mathbb{E}[Y_{i,j}]^2} = 1 + O\left(\frac{1}{\sqrt{n}}\right), i > j;$$
$$\mathrm{Var}[\omega_{i,i}] = 2\sqrt{n}\frac{\mathrm{Var}[Y_{i,i}]}{\mathbb{E}[Y_{i,i}]^2} = 2 + O\left(\frac{1}{\sqrt{n}}\right), i = j;$$
$$\gamma_n = 1 - \frac{\mu}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

In these notations, it follows that

$$\frac{\sqrt{n}}{2} \cdot 2^{-n}\mathbb{E}[Y_{i,j}]^{-n} \cdot Z_{\lfloor \frac{1}{2}(n+x\sqrt{n})\rfloor, \lfloor \frac{1}{2}(n-x\sqrt{n})\rfloor} = \frac{\sqrt{n}}{2}\mathfrak{Z}_n^{\omega,\gamma_n}(x; \frac{1}{\sqrt{2}} \cdot n^{-1/4}).$$

Note that now the weights $\omega_{i,j}$ on the off-diagonals are i.i.d with mean zero and variance asymptotically one, the weights $\omega_{i,i}$ on the diagonal are also i.i.d with mean zero but with variance asymptotically two. Re-examining the step 1 of the proof of Theorem 2.1.3, the difference we have here is that the weights $\omega_{i,i}$ and $\omega_{i,j}$ are not of the same distribution and $\omega_{i,i}$ are of variance asymptotically two instead of one, which could be overcome by Lemma 2.2.5.

Also for $\gamma_n = 1 - \frac{\mu}{\sqrt{n}} + O\left(\frac{1}{n}\right)$, we have the same local limit theorem as in Theorem 2.3.8. Thus the same argument for Theorem 2.1.3 still works and we get the desired convergence results for log-gamma polymer model. □

Chapter 3

Tightness for discrete Gibbsian line ensembles

3.1 Gibbsian line ensembles and the main result

We first introduce the basic notions of line ensembles in section 3.1.1 and then define the main objects of study in this thesis – Brownian and random walk Gibbsian line ensembles in section 3.1.2. Lastly in section 3.1.3, we list the assumptions A1-A4, under which we formulate the main result Theorem 3.1.13 of this thesis.

3.1.1 Basics about line ensembles

Definition 3.1.1. *Let Σ be an interval of $\mathbb{Z}$ and let Λ be a subset of $\mathbb{R}$. Consider the set $C(\Sigma \times \Lambda, \mathbb{R})$ of continuous functions $f : \Sigma \times \Lambda \to \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets of $\Sigma \times \Lambda$, and let $\mathcal{C}(\Sigma \times \Lambda, \mathbb{R})$ denote the sigma-field generated by Borel sets in $C(\Sigma \times \Lambda, \mathbb{R})$. A $\Sigma \times \Lambda$-indexed line ensemble $\mathcal{L}$ is a random variable on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in $C(\Sigma \times \Lambda, \mathbb{R})$ such that $\mathcal{L}$ is a measurable function from $\mathcal{B}$ to $\mathcal{C}(\Sigma \times \Lambda, \mathbb{R})$.*

For integers $k_1 < k_2$, let $[k_1, k_2]_{\mathbb{Z}} := \{k_1, k_1 + 1, \dots, k_2\}$. When Λ is a discrete subset of $\mathbb{R}$, it is possible to extend the line ensemble to one with Λ replaced by its convex hull (i.e. the minimal interval of $\mathbb{R}$ containing all points of Λ). Under this extension, the lines of $\mathcal{L}$ are extended by linear interpolation and the convergence of $\mathcal{L}$ implies the convergence of the extension. We will sometimes abuse notations and conflate a discrete line ensemble defined on a discrete set Λ with its linearly interpolated ensemble. Also we will generally write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$ even though it is not $\mathcal{L}$, but rather $\mathcal{L}(\omega)$ for each $\omega \in \Omega$ which is such a function. We will also sometimes specify a line ensemble by only giving its law without reference to the underlying probability space. We write $\mathcal{L}_i(\cdot) := \big(\mathcal{L}(\omega)\big)(i,\cdot)$ for the label $i \in \Sigma$ curve of the ensemble $\mathcal{L}$.

Definition 3.1.2. *Given a $\Sigma \times \Lambda$-indexed line ensemble $\mathcal{L}$ and a sequence of such ensembles $\{\mathcal{L}^N\}_{N\geq 1}$, we will say that $\mathcal{L}^N$ converges to $\mathcal{L}$ weakly as a line ensemble if for all bounded continuous functions $F : C(\Sigma \times \Lambda, \mathbb{R}) \to \mathbb{R}$, as $N \to \infty$,*

$$\int F\big(\mathcal{L}^N(\omega)\big) d\mathbb{P}^N(\omega) \to \int F\big(\mathcal{L}(\omega)\big) d\mathbb{P}(\omega).$$

This is equivalent to weak-$*$ convergence in $C(\Sigma \times \Lambda, \mathbb{R})$ endowed with the topology of uniform convergence on compact subsets of $\Sigma \times \Lambda$.

We will define two types of Gibbsian bridge line ensembles – those whose underlying path measures are given by Brownian motions, and those given by discrete time random walks. We start with the Brownian case and then define the random walk bridge case in a similar manner.

Definition 3.1.3. *Fix $k_1 \leq k_2$ with $k_1, k_2 \in \mathbb{Z}$, an interval $[a,b] \subset \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2-k_1+1}$. A $[k_1,k_2]_{\mathbb{Z}} \times [a,b]$-indexed line ensemble $\mathcal{L} = (\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2})$ is called a free Brownian bridge line ensemble with entrance data $\vec{x}$ and exit data $\vec{y}$ if its law $\mathbb{P}_{free}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$ is that of $k_2 - k_1 + 1$ independent standard Brownian bridges starting at time a at the points $\vec{x}$ and ending at time b at the points $\vec{y}$.*

A Hamiltonian $\mathbf{H}$ is defined to be a measurable function $\mathbf{H} : \mathbb{R} \to [0,\infty]$. Given a Hamiltonian $\mathbf{H}$ and two measurable function $f, g : (a,b) \to \mathbb{R} \cup \{\pm\infty\}$, we define the $\mathbf{H}$-Brownian bridge line ensemble with entrance data $\vec{x}$, exit data $\vec{y}$ and boundary data (f,g) to be a $[k_1,k_2]_{\mathbb{Z}} \times (a,b)$-indexed line ensemble $\mathcal{L} = (\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2})$ with law $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ given according to the following Radon-Nikodym derivative relation:

$$\frac{d\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}{d\mathbb{P}_{free}^{k_1,k_2,(a,b),\vec{x},\vec{y}}}(\mathcal{L}) := \frac{W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}.$$

Here we adopt convention that $\mathcal{L}_{k_1-1} = f$, $\mathcal{L}_{k_2+1} = g$ and define the Boltzmann weight

$$W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) := \exp\left\{ - \sum_{i=k_1-1}^{k_2} \int_a^b \mathbf{H}\Big(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u)\Big) du \right\}, \tag{3.1}$$

and the normalizing constant

$$Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g} := \mathbb{E}_{free}^{k_1,k_2,(a,b),\vec{x},\vec{y}}\Big[W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})\Big], \tag{3.2}$$

where $\mathcal{L}$ in the above expectation is distributed according to the measure $\mathbb{P}_{free}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$.

In Definition 3.1.3, we use Brownian bridges to build our line ensemble. We now describe how we may similarly construct discrete line ensembles in terms of random walk bridges. Random walks come in different flavors based on the choice of continuous versus discrete time, and continuous versus discrete jump distributions. In principle, for each such choice we can run the same type of construction as below. In this paper we focus on discrete time and continuous jump distributions, as it is suitable for our eventual application to study the line ensemble associated to the log-gamma directed polymers as introduced in [Sep] and further studied in [COSZ].

We start by defining H^{RW}-random walk bridges using the Hamiltonian function H^{RW}, as well as various line ensembles built off of them.

Definition 3.1.4. *Fix a discrete set of ordered times $\Lambda_d = \{u_m\}_{m\in I}$ where I is some non-empty interval of $\mathbb{Z}$. Here the subscript d denotes discrete. For $a < b$, let $\Lambda_d(a,b) = (a,b)\cap\Lambda_d$, $\Lambda_d[a,b] = [a,b]\cap\Lambda_d$ and likewise for half open / half close intervals.*

A random walk Hamiltonian is a measurable function $\mathrm{H}^{\mathrm{RW}} : \mathbb{R} \to (-\infty,\infty]$ such that

$$\int_{\mathbb{R}} \exp\big(-\mathrm{H}^{\mathrm{RW}}(x)\big)dx = 1.$$

Given $a < b$ in Λ_d and $x, y \in \mathbb{R}$ we define the H^{RW}-random walk bridge which starts at height x at time a, ends at height y at time b and jumps on the discrete set $\Lambda_d(a,b)$ to be a random function $S : \Lambda_d[a,b] \to \mathbb{R}$ with law given by

$$d\mathbb{P}^{\Lambda_d(a,b),x,y}_{\mathrm{H}^{\mathrm{RW}}}(S) := Z^{-1}\,\mathbb{1}\{S_a = x, S_b = y\}\cdot\exp\Big(-\sum_{m:u_m\in\Lambda_d[a,b)}\mathrm{H}^{\mathrm{RW}}(S_{u_{m+1}} - S_{u_m})\Big)\prod_{m:u_m\in\Lambda_d(a,b)} dS_{u_m}, \quad (3.3)$$

where Z is the normalization constant necessary to make this a probability distribution.

Fix $k_1 \le k_2$ with $k_1, k_2 \in \mathbb{Z}$, $a < b \in \Lambda_d$ and two vectors $\vec{x},\vec{y} \in \mathbb{R}^{k_2-k_1+1}$. A $[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)$-indexed line ensemble $\mathcal{L} = (\mathcal{L}_{k_1},\ldots,\mathcal{L}_{k_2})$ is called a free H^{RW}-random walk bridge line ensemble with entrance data $\vec{x}$ and exit data $\vec{y}$ if its law $\mathbb{P}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW}}}$ is such that each $\mathcal{L}_j$ is independent and distributed according to $\mathbb{P}^{\Lambda_d(a,b),x_j,y_j}_{\mathrm{H}^{\mathrm{RW}}}$.

A local interaction Hamiltonian is a function $\dot{\mathbf{H}} : (\mathbb{R}\cup\{\pm\infty\})^6 \to [0,\infty]$ (see Remark 3.1.5 for an explanation of the meaning of each slot of $\dot{\mathbf{H}}$). Given a local interaction Hamiltonian $\dot{\mathbf{H}}$ and two functions $f, g : \Lambda_d[a,b] \to \mathbb{R}\cup\{\pm\infty\}$, we define the $(\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}})$-random walk bridge line ensemble with entrance data $\vec{x}$, exit data $\vec{y}$ and boundary data (f,g) to be a $[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)$-indexed line ensemble $\mathcal{L} = (\mathcal{L}_{k_1},\ldots,\mathcal{L}_{k_2})$ with law $\mathbb{P}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}$ given according to the following Radon-Nikodym derivative relation:

$$\frac{d\mathbb{P}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}}{d\mathbb{P}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW}}}}(\mathcal{L}) = \frac{W^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}}(\mathcal{L})}{Z^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}}.$$

Here we adopt convention that $\mathcal{L}_{k_1-1} = f$, $\mathcal{L}_{k_2+1} = g$ and define the Boltzmann weight

$$W^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}}(\mathcal{L}) := \exp\Big\{-\sum_{k=k_1-1}^{k_2}\sum_{u_m\in\Lambda_d(a,b)}\dot{\mathbf{H}}\big(\dot{\square}(\mathcal{L},k,u_m)\big)\Big\}, \quad (3.4)$$

where

$$\dot{\square}(\mathcal{L},k,u_m) = \big(\mathcal{L}_k(u_{m-1}),\mathcal{L}_k(u_m),\mathcal{L}_k(u_{m+1}),\mathcal{L}_{k+1}(u_{m-1}),\mathcal{L}_{k+1}(u_m),\mathcal{L}_{k+1}(u_{m+1})\big), \quad (3.5)$$

see Figure 3.1 below for an illustration of the inputs for $\boxdot(\mathcal{L}, k, u_m)$*.*

The normalizing constant is equivalent to

$$Z^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}} = \mathbb{E}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW}}}\Big[W^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}}(\mathcal{L})\Big], \tag{3.6}$$

where $\mathcal{L}$ *in the above expectation is distributed according to the measure* $\mathbb{P}^{k_1,k_2,\Lambda_d[a,b],\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW}}}$*.*

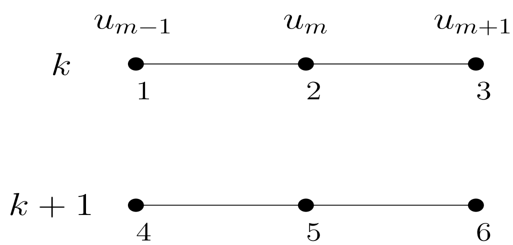


Figure 3.1: The points labeled with 1-6 correspond to the six inputs in (3.5).

Remark 3.1.5. *The function* $\boxdot(\mathcal{L}, k, u_m)$ *defined in* (3.5) *returns the point* $\mathcal{L}_k(u_m)$ *as well as five of its neighbors (in terms of the index space) corresponding to* k *and* $k+1$*, as well as* u_{m-1}, u_m, u_{m+1}*. The interaction Hamiltonians we consider are nearest neighbor in that a single curve at a single time will only depend on the curve above and below it at that same time or plus or minus one time increment. The function* $\boxdot(\mathcal{L}, k, u_m)$ *is used in specifying the interaction felt by curve* k *with respect to curve* $k+1$*. One could consider longer range interaction Hamiltonians, though it would require extending the boundary data necessary to specify the Gibbs property.*

3.1.2 H-Brownian Gibbs property and discrete $(\dot{\mathbf{H}}, \mathbf{H}^{\mathbf{RW}})$-Gibbs property

The Gibbs property for a line ensemble can be thought of as a spatial version of the Markov property whereby the distribution of a field in a given compact region depends entirely on the distribution of the field on the region's boundary. For a line ensemble which enjoys a Gibbs property, this distribution conditional on the boundary is specified exactly via the type of Radon-Nikodym derivative prescriptions in Definitions 3.1.3 and 3.1.4.

Definition 3.1.6 (**H**-Brownian Gibbs property)**.** *Let* Σ *be an interval in* $\mathbb{N}$*,* Λ *an interval in* $\mathbb{R}$ *and* $\mathbf{H}$ *a Hamiltonian function. A* $\Sigma \times \Lambda$*-indexed line ensemble* $\mathcal{L}$ *defined on a probability space* $(\Omega, \mathcal{B}, \mathbb{P})$ *has the* **H**-*Brownian Gibbs property if for all* $K = \{k_1, \cdots, k_2\} \subset \Sigma$ *and* $(a,b) \subset \Lambda$ *and any bounded continuous function* $F : C([k_1,k_2] \times (a,b), \mathbb{R}) \to \mathbb{R}$*,* $\mathbb{P}$*-almost surely*

$$\mathbb{E}\Big[F\big(\mathcal{L}|_{[k_1,k_2]_{\mathbb{Z}}\times(a,b)}\big)\big|\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times (a,b)\big)\Big] = \mathbb{E}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}\Big[F(\widetilde{\mathcal{L}})\Big]. \tag{3.7}$$

In the above,

$$\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times (a,b)\big) := \sigma\big(\mathcal{L}_k(u) : (k,u) \in \Sigma \times \Lambda \backslash [k_1,k_2]_{\mathbb{Z}} \times (a,b)\big)$$

is the sigma field generated by the line ensemble outside $[k_1,k_2]_{\mathbb{Z}} \times (a,b)$. The ensemble $\widetilde{\mathcal{L}}$ on the right-hand side of (3.7) *is independently drawn and the expectation is taken according to the $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ measure, where we have entrance data $\vec{x} = \big(\mathcal{L}_{k_1}(a),\ldots,\mathcal{L}_{k_2}(a)\big)$, exit data $\vec{y} = \big(\mathcal{L}_{k_1}(b),\ldots,\mathcal{L}_{k_2}(b)\big)$, upper boundary curve $f = \mathcal{L}_{k_1-1}|_{(a,b)}$ and lower boundary curve $g = \mathcal{L}_{k_2+1}|_{(a,b)}$. By convention if $k_1 - 1 \notin \Sigma$ then $\mathcal{L}_{k_1-1}$ is assumed to be everywhere $+\infty$ and likewise if $k_2+1 \notin \Sigma$ then $\mathcal{L}_{k_2+1}$ is assumed to be everywhere $-\infty$.*

The equality in (3.7) *is $\mathbb{P}$-almost surely as $\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times (a,b)\big)$-measurable random variables. The Gibbs property can alternatively be stated in terms of conditional distributions. In that case, conditional on the above defined entrance data $\vec{x}$, exit data $\vec{y}$ and boundary data (f,g), the law of $\mathcal{L}|_{[k_1,k_2]_{\mathbb{Z}}\times(a,b)}$ is given by $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$.*

Definition 3.1.7 (Discrete $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property)**.** *Fix an interval Σ of $\mathbb{N}$ and a discrete set of ordered times $\Lambda_d = \{u_m\}_{m\in I}$. Consider a $\Sigma \times \Lambda_d$-indexed line ensemble $\mathcal{L}$ defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$. For a random walk Hamiltonian H^{RW} and a local interaction Hamiltonian $\dot{\mathbf{H}}$ (see Definition 3.1.4), we say that that the line ensemble $\mathcal{L}$ enjoys the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-random walk Gibbs property if for any $k_1 < k_2$ with $k_1,k_2 \in \Sigma$, any $a < b$ with $a,b \in \Lambda_d$ and any continuous bounded function $F : C([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b),\mathbb{R}) \to \mathbb{R}$, $\mathbb{P}$-almost surely*

$$\mathbb{E}\Big[F\big(\mathcal{L}|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)}\big)\big|\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b)\big)\Big] = \mathbb{E}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}\Big[F(\widetilde{\mathcal{L}})\Big]. \tag{3.8}$$

In the above,

$$\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b)\big) := \sigma\big(\mathcal{L}_k(u) : (k,u) \in \Sigma \times \Lambda_d \backslash [k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b)\big) \tag{3.9}$$

is the sigma field generated by the (discrete) line ensemble outside $[k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b)$. The ensemble $\widetilde{\mathcal{L}}$ on the right-hand side of (3.8) *is independently drawn according to the $\mathbb{P}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}$ measure, where we have entrance data $\vec{x} = \big(\mathcal{L}_{k_1}(a),\ldots,\mathcal{L}_{k_2}(a)\big)$, exit data $\vec{y} = \big(\mathcal{L}_{k_1}(b),\ldots,\mathcal{L}_{k_2}(b)\big)$, upper boundary curve $f = \mathcal{L}_{k_1-1}|_{\Lambda_d[a,b]}$ and lower boundary curve $g = \mathcal{L}_{k_2+1}|_{\Lambda_d[a,b]}$. By convention if $k_1 - 1 \notin \Sigma$ then $\mathcal{L}_{k_1-1}$ is assumed to be everywhere $+\infty$ and likewise if $k_2+1 \notin \Sigma$ then $\mathcal{L}_{k_2+1}$ is assumed to be everywhere $-\infty$.*

The equality in (3.8) *is $\mathbb{P}$-almost surely as $\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b)\big)$-measurable random variables. The Gibbs property can alternatively be stated in terms of conditional distributions. In that case, conditional on the above defined entrance data $\vec{x}$, exit data $\vec{y}$ and boundary data (f,g), the law of $\mathcal{L}|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)}$ is given by $\mathbb{P}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}$.*

Remark 3.1.8. *An $\mathbf{H}$-Brownian bridge line ensemble naturally enjoys the $\mathbf{H}$-Brownian Gibbs property by definition. An $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-random walk bridge line ensemble enjoys the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-random walk Gibbs property, which follows from the locality of the interaction Hamiltonian and the random walk bridge Hamiltonian.*

Just as the strong Markov property extends the Markov property to stopping times, we may (following [CH14, CH16]) define stopping domains (Definition 3.1.9) and appeal to the strong Gibbs property

(Lemma 3.1.10). Note that in the case of discrete time Λ_d, the proof of this is considerably simpler than in continuous time (just as for the discrete versus continuous Markov processes). We do not provide the proof of this result as it is a simplified version of the proof of [CH14, Lemma 2.5].

Definition 3.1.9. *Continuing with the notation of Definition 3.1.7, a random vector $(l \le r) \in (\Lambda_d)^2$ is called a $[k_1,k_2]_{\mathbb{Z}}$-stopping domain if for all $a \le b \in \Lambda_d$,*

$$\{l \le a, r \ge b\} \in \mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(a,b)\big).$$

Also for fixed $k_1 \le k_2$, define the space

$$\mathbb{C}^{k_1,k_2} := \Big\{\big(l,r,f_{k_1},\cdots,f_{k_2}\big) : l \le r \in \Lambda_d \text{ and } (f_{k_1},\cdots,f_{k_2}) \in C([k_1,k_2]_{\mathbb{Z}} \times [l,r],\mathbb{R})\Big\}.$$

and let $\mathcal{B}\left(\mathbb{C}^{k_1,k_2}\right)$ denote the set of bounded continuous functions from $\mathbb{C}^{k_1,k_2} \to \mathbb{R}$.

Lemma 3.1.10. *Continuing with the notation of Definition 3.1.7 and 3.1.9, if (l,r) is a $[k_1,k_2]_{\mathbb{Z}}$-stopping domain for a line ensemble $\mathcal{L}$ which enjoys the $(\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}})$-random walk Gibbs property, then for any continuous bounded function $F \in \mathcal{B}(\mathbb{C}^{k_1,k_2})$, $\mathbb{P}$-almost surely*

$$\mathbb{E}\Big[F\big(l,r,\mathcal{L}|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(l,r)}\big)\big|\mathcal{F}_{ext}\big([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d(l,r)\big)\Big] = \mathbb{E}^{k_1,k_2,\Lambda_d(l,r),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}\Big[F(\widetilde{\mathcal{L}})\Big].$$

where we have entrance data $\vec{x} = \big(\mathcal{L}_{k_1}(l),\ldots,\mathcal{L}_{k_2}(l)\big)$, exit data $\vec{y} = \big(\mathcal{L}_{k_1}(r),\ldots,\mathcal{L}_{k_2}(r)\big)$, upper boundary curve $f = \mathcal{L}_{k_1-1}|_{\Lambda_d[l,r]}$ and lower boundary curve $g = \mathcal{L}_{k_2+1}|_{\Lambda_d[l,r]}$. The ensemble $\widetilde{\mathcal{L}}$ on the right-hand side above is independently draw according to the $\mathbb{P}^{k_1,k_2,\Lambda_d(l,r),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}$ measure.

3.1.3 Assumptions on $(\dot{\mathbf{H}}^N, \mathbf{H}^{\mathbf{RW},N})$ and the main result

Let $\{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}\}_{N\in\mathbb{N}}$ be a sequence of local interaction and random walk Hamiltonians and let $\Lambda_d^N = \frac{1}{N}\mathbb{Z}$. In this section we make four key assumptions on the interaction Hamiltonians $\dot{\mathbf{H}}^N$ and the underlying random walk bridge measure constructed with single jump distributions given by random walk Hamiltonians $\mathrm{H}^{\mathrm{RW},N}$. The two convexity assumptions A1 (convexity on $\dot{\mathbf{H}}^N$) and A2 (convexity of $\mathrm{H}^{\mathrm{RW},N}$) ensure the key monotone coupling for two line ensembles. A3 assumes that $\dot{\mathbf{H}}^N$ is approximating the exponential Hamiltonian function $\mathbf{H}(x) = e^x$. A4 assumes another key ingredient, the KMT type coupling between the underlying random walk bridge measures and Brownian bridges measure. The existence of such coupling for random walk bridges is the main topic studied as in [DW].

Assumption A1. $\dot{\mathbf{H}} : (\mathbb{R} \cup \{\pm\infty\})^6 \to \mathbb{R}$ satisfies the following properties:

(1) $\dot{\mathbf{H}}(\vec{a})$ is non-increasing in terms of a_1,a_2,a_3 and is non-decreasing in terms of a_4,a_5,a_6.

(2) Let $\vec{a}$, $\vec{b} \in (\mathbb{R} \cup \{\pm\infty\})^6$ and $\delta > 0$. Suppose $a_i \geq b_i$ for $i = 1, 2, \ldots, 6$ and $a_k = b_k$ for some $k \in \{1, 2, \ldots, 6\}$. Denote

$$a_i' = \begin{cases} a_k + \delta & i = k \\ a_i & i \neq k \end{cases}, \quad b_i' = \begin{cases} b_k + \delta & i = k \\ b_i & i \neq k \end{cases}.$$

Then for any $\vec{a}, \vec{b} \in (\mathbb{R} \cup \{\pm\infty\})^6$ and any $\delta > 0$, we require

$$-\dot{\mathbf{H}}(\vec{a}') + \dot{\mathbf{H}}(\vec{a}) \geq -\dot{\mathbf{H}}(\vec{b}') + \dot{\mathbf{H}}(\vec{b}).$$

Assumption A2. The random walk Hamiltonian function $\mathrm{H}^{\mathrm{RW}} : \mathbb{R} \to (-\infty, \infty]$ is convex.

For a convex Hamiltonian $\mathbf{H}$, it is known in the work of [CH16] that the $\mathbf{H}$-Brownian bridge line ensemble has certain monotonicity properties. For example, if f, g, $\vec{x}$ or $\vec{y}$ increase pointwise, then the resulting line ensemble can be coupled to the original one so as to dominate it pointwise. We will reprove these properties for our discrete line ensemble under Assumption A1 on $\dot{\mathbf{H}}$ and convexity of H^{RW}. Without such convexity, the constructive proof we give for monotonicity fails. We remark that [CD] involves a line ensemble which lacks this convexity. Therein they develop a new, weaker type of monotonicity (in terms of certain expectation values and up to certain constants) which turns out to be sufficient for proving tightness in the manner of [CH16].

We have the following lemma which allows us to couple different discrete line ensembles.

Lemma 3.1.11. *Fix $k_1 \leq k_2$, $a < b \in \Lambda_d$. For $i = 1, 2$ define pairs of vectors $(\vec{x}^i, \vec{y}^i) \in \mathbb{R}^{k_2 - k_1 + 1}$, pairs of measurable functions $(f^i, g^i) : \Lambda_d[a, b] \to \mathbb{R} \cup \{\pm\infty\}$. For $i \in \{1, 2\}$, let $\mathcal{Q}^i = \{\mathcal{Q}^i_j\}_{j=k_1}^{k_2}$ be a $[k_1, k_2]_{\mathbb{Z}} \times \Lambda_d(a, b)$-indexed line ensemble on a probability space $(\Omega^i, \mathcal{B}^i, \mathbb{P}^i)$ where $\mathbb{P}^i = \mathbb{P}^{k_1, k_2, \Lambda_d(a,b), \vec{x}^i, \vec{y}^i, f^i, g^i}_{\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}}}$.*

Assume H^{RW} satisfies Assumption A2, i.e. $\mathrm{H}^{\mathrm{RW}}(x)$ is convex in x and assume the local interaction Hamiltonian function $\dot{\mathbf{H}}$ satisfies Assumption A1. Assume that the $i = 1$ vectors and functions are pointwise greater than or equal to their $i = 2$ counterparts (e.g. $f^1(u) \geq f^2(u)$ for all $u \in \Lambda_d[a, b]$). Then there exists a coupling of the probability measure $\mathbb{P}^1$ and $\mathbb{P}^2$ such that almost surely $\mathcal{Q}^1_j(u) \leq \mathcal{Q}^2_j(u)$ for all $j \in [k_1, k_2]_{\mathbb{Z}}$ and $u \in \Lambda_d[a, b]$.

The proof of this lemma is given in Appendix following the Glauber dynamics approach of [CH14, CH16] which realizes the line ensemble as the invariant measure of a Markov chain on trajectories. Assumption A1 on $\dot{\mathbf{H}}$ and convexity of H^{RW} are sufficient conditions under which the Markov chains can be coupled, hence proving coupling of their invariant measures as well.

Note that this Lemma 3.1.11 still holds true when $\mathbf{H}(x)$ is $+\infty$ for $x > 0$ and 0 for $x < 0$. Moreover, generalizing Definition 3.1.7, we may define measures involving multiple boundary conditions for the lowest and highest indexed curves and note that the analogue of Lemma 3.1.11 could be generalized to this setting as well. Since the argument goes the same as that of Lemma 3.1.11, we don't pursue it here.

Definition 3.1.12. *Fix $k_1 \leq k_2 \in \Sigma$ and a local interaction Hamiltonian $\dot{\mathbf{H}}$. Fix two external Hamiltonians $\dot{\mathbf{H}}^{\hat{f}}, \dot{\mathbf{H}}^{\hat{g}} : \mathbb{R} \cup \{\pm\infty\} \to [0, \infty]$ and here the superscripts $\hat{f}$ and $\hat{g}$ distinguish between the Hamiltonian $\dot{\mathbf{H}}^{\hat{f}}$ felt by the lowest label curve, and $\dot{\mathbf{H}}^{\hat{g}}$ felt by the highest label curve, where $\hat{f}, \hat{g} : \Lambda_d(a, b) \to \mathbb{R} \cup \{\pm\infty\}$ are the extra upper and lower boundary functions respectively which play a similar role as f and g.*

Generalizing Definition 3.1.4, we define a measure $\mathbb{P}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g,\hat{f},\hat{g}}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}},\dot{\mathbf{H}}^{\hat{f}},\dot{\mathbf{H}}^{\hat{g}}}$ on curves $\mathcal{L}_{k_1}, \cdots, \mathcal{L}_{k_2} : (a, b) \to \mathbb{R}$ by specifying its Radon-Nikodym derivative with respect to $\mathbb{P}^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW}}}$ to be proportional (up to normalization to make the integral one) to the Boltzmann weight

$$
\begin{aligned}
W^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g,\dot{\mathbf{H}},\dot{\mathbf{H}}^{\hat{f}},\dot{\mathbf{H}}^{\hat{g}}}_{\dot{\mathbf{H}},\dot{\mathbf{H}}^{\hat{f}},\dot{\mathbf{H}}^{\hat{g}}}(\mathcal{L}) &:= W^{k_1,k_2,\Lambda_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}}(\mathcal{L}) \\
&\times \exp\Big\{ - \sum_{u_m \in \Lambda_d(a,b)} \Big(\dot{\mathbf{H}}^{\hat{f}}\big(\mathcal{L}_{k_1}(u_m) - \hat{f}(u_m)\big) + \dot{\mathbf{H}}^{\hat{g}}\big(\hat{g}(u_m) - \mathcal{L}_{k_2}(u_m)\big)\Big)\Big\}.
\end{aligned}
$$

The following assumption A3 compares the local interaction Hamiltonian $\dot{\mathbf{H}}^N$ and exponential Hamiltonian function $\mathbf{H}(x) = e^x$, which is a key feature of the KPZ line ensemble, as introduced in [CH16].

We first define modulus of continuity for a continuous function (or multiple functions). Let $K \geq 1$ and $r > 0$. For a K-continuous function $\vec{f} = (f_1, f_2, \cdots, f_K)$, the δ-modulus of continuity $\omega_{[a,b]}(\vec{f}, r)$ for K-continuous function $\vec{f}$ on $[a, b]$ is defined as

$$
\omega_{[a,b]}(\vec{f}, r) = \sup_{1 \leq i \leq K} \sup_{\substack{s,t \in [a,b] \\ |s-t| \leq r}} |f_i(u) - f_i(t)|. \tag{3.10}
$$

In this paper, since the functions f are normally piecewise linear (as an interpolation of the discrete line ensemble on the set Λ_d), we can restrict s, t to live in Λ_d.

Assumption A3. There exists a constant $C_1 > 0$ such that for any continuous function $\mathcal{L}_{k+1}$, it holds that

$$
\left| \frac{\displaystyle\sum_{u \in \Lambda_d^N(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L}, k, u))}{\displaystyle\int_a^b \exp(\mathcal{L}_{k+1} - \mathcal{L}_k(u))du} - 1 \right| \leq e^{C_1\left(\omega_{[a,b]}(\mathcal{L}_k, 1/N) + \omega_{[a,b]}(\mathcal{L}_{k+1}, 1/N) + 1/N\right)} - 1.
$$

The last Assumption A4 is a strong (KMT) coupling assumption, which serves as a key tool in the comparison between discrete line ensemble and $\mathbf{H}$-Brownian line ensemble with $\mathbf{H}(x) = e^x$. Below we start with notations for random walk bridges and then state Assumption A4.

Fix $N \in \mathbb{N}$, $L > 0$ with $NL \in \mathbb{N}$ and $z \in \mathbb{R}$. We define a random walk bridge process $\{\bar{S}^N_{L,z}(k/N)\}_{k=0}^{NL}$ with law equal to that of the random walk $\{S(k)\}_{k=0}^{NL}$ conditioned on $S(NL) = z$, denoted as follows,

$$\bar{S}^N_{L,z}(k/N) := X_1^N + X_2^N + \cdots + X_k^N \Big| X_1^N + X_2^N + \cdots + X_{NL}^N = z, \tag{3.11}$$

where X_i^N are i.i.d. random variables with probability density function given by $\exp\left(-\mathrm{H}^{\mathrm{RW},N}(x)\right)$. For general $u \in [0, L]$, $\bar{S}^N_{L,z}(u)$ is defined through linear interpolation.

Denote $B_L(\cdot)$ as a Brownian bridge with $B_L(0) = B_L(L) = 0$. Under suitable requirements on $\mathrm{H}^{\mathrm{RW},N}$, Donsker invariance principle says that $\bar{S}^N_{L,z}(u)$ converges weakly to $B_L(u) + \frac{u}{L} \cdot z$ as N goes to infinity, while KMT coupling provides a quantitative estimate for this convergence rate. For the original classical result on the case of random walks with exponential moment, see [KMT75, KMT76] and a recent treatment for the case of random walk bridges is considered in [DW].

Assumption A4. For any $b_1, b_2 > 0$ there exist constants $0 < a_1, a_2 < \infty$ (depending on b_1, b_2 but not on N) such that the following statement holds. For any $N \in \mathbb{N}$, any $L > 0$ with $NL \in \mathbb{N}$, there exists a probability space on which a Brownian bridge $B_L(u)$ and a family of random walk bridges $\{\bar{S}^N_{L,z}(u)\}_{z \in \mathbb{R}}$ are defined. Furthermore, for all $z \in \mathbb{R}$, one has the following estimate

$$\mathbb{P}\left[\sup_{0 \le u \le L} \left| B_L(u) + \frac{u}{L} \cdot z - \bar{S}^N_{L,z}(u) \right| \ge a_1 N^{-1/2} \log(NL)\right] \le a_2 (NL)^{-b_1} e^{b_2 z^2/L}.$$

Under assumptions A1-A4, we are ready to state the main result of this paper.

Theorem 3.1.13. *Denote* $[k_1, k_2]_{\mathbb{Z}} := \{k_1, \cdots, k_2\}$ *and denote* $\Lambda_d^N := \frac{1}{N}\mathbb{Z}$*, for* $N \in \mathbb{N}$*. Fix* $K \in \mathbb{N} \cup \{\infty\}$*, let* $\mathcal{L}^N$ *be a* $[1, K]_{\mathbb{Z}} \times \Lambda_d^N$*-indexed discrete line ensemble that enjoys the discrete* $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$*-Gibbs property with respect to some local interaction Hamiltonian* $\dot{\mathbf{H}}^N$ *and random walk Hamiltonian* $\mathrm{H}^{\mathrm{RW},N}$*. Here we adopt the convention that* $\mathcal{L}_0^N = +\infty$ *and* $\mathcal{L}_{K+1}^N = -\infty$*.*

Assume that $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ *satisfies assumptions A1-A4. Moreover, assume that* $\mathcal{L}_1^N(u) + u^2/2$ *(defined through linear interpolation) converges weakly to a stationary process, under the topology of uniform convergence on compact sets. Then we have*

1. *For any* $T > 0$ *and* $1 \le k \le K$*, the restriction of the line ensemble* $\mathcal{L}^N$ *to* $[1, k]_{\mathbb{Z}} \times [-T, T]$ *is sequentially compact as* N *varies.*

2. *Furthermore, any subsequential limit line ensemble* $\mathcal{L}^\infty$ *satisfies* $\mathbf{H}$*-Brownian Gibbs property with* $\mathbf{H}(x) = e^x$*.*

3.2 Proof of Theorem 3.1.13

In this chapter, we will first present three propositions, analogous to [CH16, Propositions 6.1, 6.2 and 6.3] in Section 4.8.1 and provide a few estimates on random walk bridges and $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-discrete Gibbs line ensembles in Section 4.8.2. Then we will deduce from them the proof of Theorem 3.1.13.

3.2.1 Three key propositions for random walk bridges

Fix $K \in \mathbb{N} \cup \{\infty\}$, for $N \in \mathbb{N}$, let $\mathcal{L}^N = \{\mathcal{L}_1^N, \cdots, \mathcal{L}_K^N\}$ be a discrete line ensemble which enjoys the discrete $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property with respect to some local interaction Hamiltonian $\dot{\mathbf{H}}^N$ and random walk Hamiltonian $\mathrm{H}^{\mathrm{RW},N}$ such that $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ satisfy assumptions A1-A4. We have the three following propositions.

Proposition 3.2.1. *Fix $k \in \{1, 2, \cdots K\}$. For each $\varepsilon > 0$, there exists $R_k = R_k(\varepsilon) > 0$ such that for any $x_0 > 0$ there exist $N_0(x_0, \varepsilon)$ such that for $N \geq N_0$ and $\overline{x} \in [-x_0, x_0]$,*

$$\mathbb{P}\Big(\inf_{u \in \Lambda_d^N[\overline{x}-1/2, \overline{x}+1/2]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) < -R_k\Big) < \varepsilon.$$

Proposition 3.2.2. *Fix $k \in \{1, 2, \cdots K\}$. For each $\varepsilon > 0$ and $\delta \in (0, 1/8)$, there exists $T_0 > 0$ such that for any $x_0 > T_0$ there exist $N_0(x_0, \varepsilon, \delta)$ such that $N \geq N_0$, $T \in [T_0, x_0]$ and $y_0 \in [-x_0, x_0 - T]$,*

$$\mathbb{P}\Big(\inf_{u \in \Lambda_d^N[y_0, y_0+T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) < -\delta T^2\Big) < \varepsilon.$$

Proposition 3.2.3. *Fix $k \in \{1, 2, \cdots K\}$. For each $\varepsilon > 0$, there exists $\hat{R}_k = \hat{R}_k(\varepsilon) > 0$ and $N_0(\varepsilon) > 0$ such that for any $x_0 > 0$ there exist $N_0(x_0, \varepsilon)$ such that $N \geq N_0$ and $\overline{x} \in [-x_0, x_0 - 1]$,*

$$\mathbb{P}\Big(\sup_{u \in \Lambda_d^N[\overline{x}, \overline{x}+1]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) > \hat{R}_k\Big) < \varepsilon.$$

The main ingredients in the proofs of these propositions are the discrete Gibbs property of the line ensembles, the monotone coupling Lemma 3.1.11, Assumption A3 and the strong approximation of the random walk bridges and Brownian bridges (Assumption A4). We also rely upon the arguments used in the proofs of [CH16, Propositions 6.1, 6.2 and 6.3]. The proofs of the above propositions are given in Chapter 4.9.

3.2.2 Estimates for random walk bridges and $(\dot{\mathbf{H}}^N, \mathbf{H}^{\mathbf{RW},N})$ line ensembles

In this section we prove a few lemmas which we will need in the proof of main Theorem 3.1.13.

Lemma 3.2.4. *The supremum of a Brownian bridge $B_L : [0, L] \to \mathbb{R}, B(0) = B(L) = 0$ satisfies that for all $s > 0$,*

$$\mathbb{P}\left(\sup_{u \in [0,L]} B_L(u) > s\right) \leq e^{-2s^2/L}.$$

Proof. This amounts to a use of reflection principle - see (3.40) in Chapter 4 of [KS]. □

The following lemma is an analogue for the random walk bridge case under Assumption A4.

Lemma 3.2.5. *Let $L > 0$, $z_0 \geq 0$ and $s_0 \geq 1$. Assume that $\mathrm{H}^{\mathrm{RW},N}$ satisfies Assumption A4 (KMT coupling). Then there exist $N_0 = N_0(L, s_0, z_0)$ such that the following holds. For any $|z| \leq z_0$, $1 \leq s \leq s_0$ and $N \geq N_0$, let $\bar{S}^N_{L,z}(u)$ be the random walk bridge defined as (3.11). Then*

$$\mathbb{P}\left(\sup_{u\in[0,L]}\left(\bar{S}^N_{L,z}(u) - \frac{u}{L}\cdot z\right) > s\right) \leq e^{-s^2/L}. \tag{3.12}$$

Proof. Let $b_1 = b_2 = 1$ and a_1, a_2 be the constants determined in Assumption A4. Then we take N_0 large enough such that the following two inequalities are true for all $N \geq N_0$:

$$\begin{aligned} a_1 N^{-1/2}\log(NL) \leq & 1 - \frac{\sqrt{3}}{2}, \\ a_2 (NL)^{-1} e^{z_0^2/L} \leq & \min_{1\leq s\leq s_0}\left(e^{-s^2/L} - e^{-3s^2/(2L)}\right). \end{aligned}$$

Then by Lemma 4.8.4, Assumption A4 and $s \geq 1$, we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{u\in[0,L]}\left(\bar{S}^N_{L,z}(u) - \frac{u}{L}\cdot z\right) > s\right) \\ \leq & \mathbb{P}\left(\sup_{u\in[0,L]} B_L(u) > \frac{\sqrt{3}}{2}s\right) + \mathbb{P}\left(\sup_{u\in[0,L]}\left|B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| > \left(1 - \frac{\sqrt{3}}{2}\right)s\right) \\ \leq & e^{-3s^2/(2L)} + \min_{1\leq s\leq s_0}\left(e^{-s^2/L} - e^{-3s^2/(2L)}\right) \\ \leq & e^{-s^2/L}. \end{aligned}$$

□

In the following we proceed to prove Proposition 4.8.8, which compares two normalizing constants from $\mathbf{H}$-Brownian line ensembles and $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-discrete line ensembles respectively under assumptions A3 and A4.

Lemma 3.2.6. *Fix $K \geq 1$ and $a < b \in \mathbb{R}$. Let $\mathcal{L} = \{\mathcal{L}_1, \dots \mathcal{L}_K\}$ be K real continuous functions defined on $[a, b]$. Let $\mathbf{H}(x) = e^x$ be a Hamiltonian function and $\Lambda^N_d = \frac{1}{N}\mathbb{Z}$. Assume $\dot{\mathbf{H}}^N$ is a sequence of Hamiltonian function satisfying Assumption A3. Thus there exists a constant $C_2 > 0$ such that for any $1 \leq k_1 \leq k_2 \leq K$, we have*

$$\left|W^{k_1,k_2,\Lambda^N_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N}(\mathcal{L}) - W^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}(\mathcal{L})\right| \leq C_2\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N\right), \tag{3.13}$$

where we adapt the convention $\mathcal{L}_{k_1-1} = f$ and $\mathcal{L}_{k_2+1} = g$ for $k_1 \geq 2, k_2 \leq K-1$ and the convention $\mathcal{L}_0 = \infty$ and $\mathcal{L}_{K+1} = -\infty$.

Proof. By Assumption A3, we have

$$\left|\frac{\log W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}-1\right| = \left|\frac{-\sum_{i=k_1-1}^{k_2}\sum_{u\in\Lambda_d^N(a,b)}\dot{\mathbf{H}}^N(\square(\mathcal{L},i,u))}{-\sum_{i=k_1-1}^{k_2}\int_a^b \exp\left(\mathcal{L}_{i+1}(u)-\mathcal{L}_i(u)\right)du}-1\right|$$
$$\leq e^{C_1\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right)}-1.$$

If we further assume $\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\leq 1$, by the mean value theorem

$$e^{C_1\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right)}-1\leq C_1e^{C_1}\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right).$$

Hence

$$\left|\frac{\log W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}-1\right|\leq C_1e^{C_1}\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right).$$

For the rest of the proof, we rely on the following fact (4.171) which we prove now. For any $\lambda\geq 0$ and any $|r|\leq 1/2$, from the mean value theorem,

$$|\lambda^{1+r}-\lambda| = |\lambda^{1+r'}\log\lambda|\cdot|r|$$

for some $|r'|\leq 1/2$. Note that

$$\sup_{\lambda\in[0,1]}|\lambda^{1+r'}\log a|\leq \sup_{\lambda\in[0,1]}|\lambda^{1/2}\log\lambda|\leq 1$$

Hence

$$|\lambda^{1+r}-\lambda|\leq|r|\quad\text{for all } a\in[0,1],|b|\leq 1/2. \tag{3.14}$$

Now by taking

$$\lambda = W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}),$$
$$1+r = \frac{\log W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})},$$

we have

$$\lambda^{1+r} = W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}).$$

Applying inequality (4.171) with above choice of a,b, we have

$$\left|W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})-W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})\right| = |a^{1+b}-a|\leq|b|$$
$$\leq C_1e^{C_1}\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right)$$

if the case $\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N \leq \min\{1, 1/(2C_1)e^{-C_1}\}$ holds.

On the other hand if the case $\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N\right) > \min\{1, 1/(2C_1)e^{-C_1}\}$ holds, since by definition it holds that $W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}), W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \in (0,1]$, we now have

$$\left|W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})\right| \leq \max\{2, 4C_1e^{C_1}\}\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N\right).$$

Therefore the desired result follows by taking $C_2 = \max\{4C_1e^{C_1}, 2\}$. □

By a similar argument, we now control the difference between $W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})$ and $W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')$ by the sup norm between $\mathcal{L}$ and $\mathcal{L}'$.

Lemma 3.2.7. *Fix $K \geq 1$ and $a < b \in \mathbb{R}$. Let $\mathcal{L} = \{\mathcal{L}_1, \dots \mathcal{L}_K\}$ and $\mathcal{L}' = \{\mathcal{L}'_1, \dots \mathcal{L}'_K\}$ be two collections of K real continuous functions defined on $[a,b]$ and let $\mathbf{H}(x) = e^x$ be a Hamiltonian function. There exists a constant $C_3 > 0$ such that for any $1 \leq k_1 \leq k_2 \leq K$,*

$$\left|W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')\right| \leq C_3 \sup_{\substack{a\leq u\leq b\\ k_1-1\leq k\leq k_2+1}} |\mathcal{L}_k(u) - \mathcal{L}'_k(u)|. \tag{3.15}$$

Proof. Note that we have for any $k_1 - 1 \leq i \leq k_2$

$$\begin{aligned}\left|\frac{\int_a^b \exp(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u))du}{\int_a^b \exp(\mathcal{L}'_{i+1}(u) - \mathcal{L}'_i(u))du} - 1\right| &\leq \left|\sup_{a\leq u\leq b} \frac{\exp(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u))}{\exp(\mathcal{L}'_{i+1}(u) - \mathcal{L}'_i(u))} - 1\right| \\ &\leq \exp\left(2 \sup_{a\leq u\leq b, i\leq k\leq i+1} |\mathcal{L}_k(u) - \mathcal{L}'_k(u)|\right) - 1.\end{aligned}$$

Therefore

$$\begin{aligned}\left|\frac{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')} - 1\right| &= \left|\frac{-\sum_{i=k_1-1}^{k_2}\int_a^b \exp\left(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u)\right)du}{-\sum_{i=k_1-1}^{k_2}\int_a^b \exp\left(\mathcal{L}'_{i+1}(u) - \mathcal{L}'_i(u)\right)du} - 1\right| \\ &\leq \exp\left(2 \sup_{a\leq u\leq b, k_1-1\leq i\leq k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|\right) - 1.\end{aligned}$$

If we further assume $\sup_{a\leq u\leq b, k_1-1\leq i\leq k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)| \leq 1$, by the mean value theorem we have

$$\begin{aligned}&\left|\frac{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')} - 1\right| \\ \leq& 2e^2 \cdot \sup_{\substack{a\leq u\leq b\\ k_1-1\leq i\leq k_2+1}} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|.\end{aligned}$$

Now by applying inequality (4.171) with $\lambda = W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')$ and

$$1 + r = \log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) / \log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}'),$$

we have

$$\left| W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}') - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \right| = |\lambda^{1+r} - \lambda| \leq |r|$$
$$\leq 2e^2 \cdot \sup_{\substack{a \leq u \leq b \\ k_1 - 1 \leq i \leq k_2 + 1}} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|$$

if the case $\sup_{a \leq u \leq b, k_1-1 \leq i \leq k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)| \leq \frac{1}{4} e^{-2}$ holds.

On the other hand if the case $\sup_{a \leq u \leq b, k_1-1 \leq i \leq k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)| > \frac{1}{4} e^{-2}$ holds, we simply use

$$\left| W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}') - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \right| \leq 2 \leq 8e^2 \sup_{a \leq u \leq b, k_1-1 \leq i \leq k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|.$$

Thus the desired result follows by taking $C_3 = 8e^2$. ☐

The following proposition shows that given the same boundary data and under assumptions A3 and A4, the normalizing constant $Z_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}$ of $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs line ensemble converges to $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ of $\mathbf{H}$-Brownian Gibbs line ensemble where $\mathbf{H}(x) = e^x$.

Proposition 3.2.8. *Suppose $\dot{\mathbf{H}}^N$ and $\mathrm{H}^{\mathrm{RW},N}$ satisfy Assumption A3 and Assumption A4 respectively. For any $k_0 \in \mathbb{N}$, $0 < L_1 < L_2$, $z_0 > 0$ and $\varepsilon > 0$, there exists N_0 and $\delta > 0$ such that the following statement holds. Fix $N \in \mathbb{N}$ with $N \geq N_0$, $k_1 \leq k_2 \in \mathbb{N}$ with $k_2 - k_1 \leq k_0$, $a < b \in \mathbb{R}$ with $L_1 \leq b - a \leq L_2$, $\vec{x} = \{x_{k_1}, \dots x_{k_2}\}, \vec{y} = \{y_{k_1}, \dots y_{k_2}\}$, with $\sup_{k_1 \leq i \leq k_2} |x_i - y_i| \leq z_0$ and two continuous functions $f, g : [a,b] \to \mathbb{R}$ with $\omega_{[a,b]}(f, 1/N) + \omega_{[a,b]}(g, 1/N) < \delta$. Denote by $\bar{S}^N : [k_1, k_2]_{\mathbb{Z}} \times \Lambda_d^N[a,b] \to \mathbb{R}$ and $B : [k_1, k_2]_{\mathbb{Z}} \times [a,b] \to \mathbb{R}$ the the random walk bridges constructed by $\mathrm{H}^{\mathrm{RW},N}$ and Brownian bridges in $[a,b]$ with entrance and exit data $\vec{x}$ and $\vec{y}$. In other words,*

$$\bar{S}^N(k,u) \overset{(d)}{=} x_k + \bar{S}^N_{L, y_k - x_k}(u-a),$$

with $L = b - a$ and

$$B(k,u) \overset{(d)}{=} \left(\frac{b-u}{b-a} \right) x_k + \left(\frac{u-a}{b-a} \right) y_k + B_L(u-a).$$

Here $\bar{S}^N_{L, y_k - x_k}$ is defined in (3.3) and different curves in $\bar{S}^N$ and B are independent. Then $\bar{S}^N$ and B can be coupled in a probability space and suppose J and J' are two events with $\mathbb{P}(J \Delta J') < \varepsilon'$, we have

$$\left| \mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) \cdot \mathbb{1}_J \right) - \mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \cdot \mathbb{1}_{J'} \right) \right| < \varepsilon + \varepsilon'. \tag{3.16}$$

In particular, by taking $J = J'$ to be the whole probability space, we have

$$\left| Z_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g} - Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g} \right| < \varepsilon. \tag{3.17}$$

Proof. Let δ_1, δ_2 be two small numbers to be determined. By taking $b_1 = 1$ and $b_2 = 1$ in Assumption A4, we can couple $\bar{S}^N$ and B in the same probability space such that for each $k \in [k_1, k_2]_{\mathbb{Z}}$,

$$\mathbb{P}\left(\sup_{a \le u \le b} \left|B_k(u) - \bar{S}_k^N(u)\right| > a_1 N^{-1/2}\log(NL_2)\right) \le a_2(NL_1)^{-1}e^{z_0^2/L_1}.$$

Define the events

$$\begin{aligned} A_{1,N} =& \{\omega_{(a,b),1/N}(B) < \delta_1\}, \\ A_{2,N} =& \left\{\sup_{a \le u \le b, k_1 \le k \le k_2} \left|B_k(u) - \bar{S}_k^N(u)\right| < \delta_2\right\}, \\ A_N =& A_{1,N} \cap A_{2,N}. \end{aligned}$$

Take N_0 large enough such that $a_1 N_0^{-1/2}\log(N_0 L_2) < \delta_2$, then

$$\mathbb{P}(A_{2,N}^-) \le k_0 \cdot \left(a_2(N_0L_1)^{-1}e^{z_0^2/L_1}\right).$$

Hence through taking N_0 large enough, we have

$$\mathbb{P}(A_{2,N}^-) \le \varepsilon.$$

Also, as N_0 is large enough depending on L_2, z_0, k_0 and δ_1, we have

$$\mathbb{P}(A_{1,N}^-) \le \varepsilon.$$

Note that as the event A_N occurs,

$$\omega_{[a,b]}(\bar{S}^N, 1/N) \le \delta_1 + 2\delta_2.$$

On the event A_N, applying Lemma 4.8.6 and 4.8.7, we see that

$$\begin{aligned} &\left|W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\right| \\ \le& \left|W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N)\right| \\ &+ \left|W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\right| \\ \le& C_2(\delta + \delta_1 + 2\delta_2 + 1/N) + C_3\delta_2. \end{aligned}$$

By choosing δ, δ_1, δ_2 and $1/N_0$ small enough, we have

$$\left|W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\right| < \varepsilon,$$

which implies that

$$\begin{aligned}
&\left|\mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N)\cdot\mathbb{1}_J\right)-\mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\cdot\mathbb{1}_{J'}\right)\right|\\
\le&\left|\mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N)\cdot\mathbb{1}_{J\cap J'\cap A_N}\right)-\mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\cdot\mathbb{1}_{J\cap J'\cap A_N}\right)\right|\\
+&\left|\mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N)\cdot\mathbb{1}_{J\cap J'\cap A_N^-}\right)\right|+\left|\mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\cdot\mathbb{1}_{J\cap J'\cap A_N^-}\right)\right|\\
+&\left|\mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N)\cdot\mathbb{1}_{J\setminus J'}\right)\right|+\left|\mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\cdot\mathbb{1}_{J'\setminus J}\right)\right|\\
\le&3\varepsilon+\varepsilon',
\end{aligned}$$

where we used inequality $\left|W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N)-W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\right|<\varepsilon$ in $J\cap J\cap A_N^-$ for the first term, $W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N), W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B)\in[0,1]$ and the bound of the event probability for the other four terms. ☐

3.2.3 Proof of Theorem 3.1.13 (1)

Let $\mathcal{L}^N$ be a sequence of $(\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N})$-discrete line ensembles which satisfies the assumptions in Theorem 3.1.13. In order to establish Theorem 3.1.13, we will first prove the following lower bound for the normalizing constant $Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}$ (Definition 3.6) of the discrete Gibbs line ensemble $\mathcal{L}^N$. The proof exploits the analogous result for the normalizing constant $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ (Definition 3.2) of a **H**-Brownian Gibbs line ensemble in [CH16, Proposition 6.4] and result that $Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}$ is approximating $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ implied by Proposition 4.8.8.

Lemma 3.2.9. *Fix $k_1\le k_2$ and an interval $[a,b]\in\mathbb{R}$. Then for all $\varepsilon>0$, there exists $\delta>0$, and $N_0(k_1,k_2,a,b,\varepsilon)$ such that for all $N>N_0(k_1,k_2,a,b,\varepsilon)$, we have*

$$\mathbb{P}\Big(Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}<\delta\Big)<\varepsilon,$$

where $\vec{x}=(\mathcal{L}_i^N(a))_{i=k_1}^{k_2}$, $\vec{y}=(\mathcal{L}_i^N(b))_{i=k_1}^{k_2}$, $f=\mathcal{L}_{k_1-1}^N$, $g=\mathcal{L}_{k_2+1}^N$.

Proof. Propositions 4.8.1, 4.8.2 and 4.8.3 imply that (given k_1,k_2,a,b,ε) there exists $M>0$ such that the event

$$\begin{aligned}
E=\Big\{\min_{u\in\Lambda_d^N[a,b]}\mathcal{L}_{k_1-1}^N(u)>-M\Big\}\cap\Big\{\max_{u\in\Lambda_d^N[a,b]}\mathcal{L}_{k_2+1}^N(u)<M\Big\}&\\
\cap\Big\{|\mathcal{L}_i^N(u)|\le M,\ \forall u\in\{a,b\},i\in\{k_1,\cdots,k_2\}\Big\}&,
\end{aligned}$$

has probability $\mathbb{P}(E)\ge 1-\varepsilon$.

For δ to be specified soon, define the event (with $\vec{x},\vec{y},f,g$ as in the statement of the lemma)

$$D=\Big\{Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}<\delta\Big\}.$$

Since $\mathbb{P}(E^{\neg}) \le \varepsilon$ and

$$\mathbb{P}(D) \le \mathbb{P}(D \cap E) + \mathbb{P}(E^{\neg}),$$

we only need to control $\mathbb{P}(D \cap E)$. Let us assume that E occurs. In that case, due to the monotonicity of Assumption A1 (1),

$$Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g} \ge Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},-M,M}.$$

Clearly given that E occurs, there exists some $\delta > 0$, depending on k_1, k_2, a, b and M, such that

$$Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},-M,M} > 2\delta.$$

By Proposition 4.8.8, we may show that

$$Z_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N[a,b],\vec{x},\vec{y},-M,M} > \delta$$

for $N \ge N_0(k_1, k_2, a, b, \varepsilon)$ large enough. Thus for such δ, $\mathbb{P}(D \cap E) = 0$ and we complete the proof. □

Now we proceed to the proof of Theorem 3.1.13(1). First let us recall the tightness criterion for k continuous functions. For a k-continuous function $\vec{f} = (f_1, f_2, \ldots, f_k)$ defined on an interval $[a, b]$, the $k-$modulus of continuity of $\vec{f}$ is defined in (3.10) as

$$\omega_{[a,b]}\big(\vec{f}, r\big) = \sup_{1\le i\le k} \sup_{\substack{s,t\in[a,b]\\ |s-t|\le r}} |f_i(s) - f_i(t)|.$$

Consider a sequence of probability measures $\mathbb{P}_N$ on k functions $\vec{f} = (f_1, \cdots, f_k)$ on the interval $[a, b]$ and define event

$$U_{[a,b]}\big(\vec{f}, \varrho, r\big) = \Big\{\omega_{[a,b]}\big(\vec{f}, r\big) \le \varrho\Big\}.$$

As an immediate generalization of [Bi, Theorem 8.2], a sequence $\mathbb{P}_N$ of probability measures on k functions $\vec{f} = (f_1, \cdots, f_k)$ is tight if, for each $1 \le i \le k$, the one-point distribution of $f_i(x)$ at a fixed $x \in [a, b]$ is tight and if, for each positive ϱ and η, there exists a $r > 0$ and integer N_0 such that

$$\mathbb{P}_N\big(U_{[a,b]}(\vec{f}, \varrho, r)\big) \ge 1 - \eta, \qquad \text{for } N \ge N_0.$$

We will apply this tightness criterion for the sequence of measures $\mathbb{P}_N$ of discrete line ensembles $\mathcal{L}^N$ restricted on $[-T, T]$. Denote $\vec{x} = (\mathcal{L}_1^N(-T), \cdots, \mathcal{L}_K^N(-T))$, $\vec{y} = (\mathcal{L}_1^N(T), \cdots, \mathcal{L}_K^N(T))$ and $g(u) = \mathcal{L}_{K+1}^N(u)$.

Recall that we denote $\mathcal{L}^N$ as a sequence of discrete Gibbs line ensembles which satisfy assumptions in the statement of main Theorem 3.1.13. Denote $\mathbb{P}_N$ and $\mathbb{E}_N$ as the corresponding probability measures and expectations. Proposition 4.8.1 and 4.8.3 show tightness for the one-point distribution, i.e. for each given $i \in \{1, \cdots, K\}$, the one-point distribution of $\mathcal{L}_i^N(u)$ is tight in $N \in \mathbb{N}$ for any u varies over $[-T, T]$.

In order to prove the tightness of the line ensemble $\{\mathcal{L}_i^N(u) : i \in \{1, \cdots, K\}, u \in [-T, T]\}$, it suffices to verify that, for all $\varrho, \eta > 0$, we can find a $r(\varrho, \eta)$ such that for $N \geq N_0(r, \varrho, \eta)$ large enough

$$\mathbb{P}_N\big(U_{[-T,T]}(\vec{f}, \varrho, r)\big) \geq 1 - \eta, \tag{3.18}$$

with U defined as above with $f_i = \mathcal{L}_i^N$ on the interval $[-T, T]$.

We introduce two notations. For $M > 0$, we define the event

$$S_{N,M} = \bigcap_{i=1}^{K} \big\{ -M \leq \mathcal{L}_i^N(-T), \mathcal{L}_i^N(T) \leq M \big\}.$$

Denote Z_N as a shorthand for

$$Z_N := Z^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}. \tag{3.19}$$

It is enough to prove that for any $\varrho, \eta > 0$, there exists $\delta, M, r > 0$ and $N_0(\varrho, \eta, \delta, M, r)$ large enough such that for all $N \geq N_0$

$$\mathbb{P}_N\Big(U_{[-T,T]}(\mathcal{L}^N, \varrho, r) \cap \{Z_N \geq \delta\} \cap S_{N,M}\Big) > 1 - \eta, \tag{3.20}$$

since (4.175) follows from (4.177).

Observe that the events $\{Z_N \geq \delta\} \cap S_{N,M}$ are $\mathcal{F}_{\mathrm{ext}}([1, K]_{\mathbb{Z}} \times \Lambda_d^N(-T, T))$-measurable, we can rewrite the left-hand side of (4.177) as

$$\mathbb{E}_N\Big[\mathbb{1}_{Z_N \geq \delta} \mathbb{1}_{S_{N,M}} \mathbb{E}_N\big[\mathbb{1}_{U_{[-T,T]}(\mathcal{L}^N,\varrho,r)} \big| \mathcal{F}_{\mathrm{ext}}([1, K]_{\mathbb{Z}} \times \Lambda_d^N(-T, T))\big]\Big]. \tag{3.21}$$

Denote

$$\mathbf{P}_N := \mathbf{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},+\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}, \tag{3.22}$$

thus by the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property enjoyed by the line ensemble $\mathcal{L}^N$, we have, $\mathbb{P}_N$-almost surely,

$$\mathbb{E}_N\big[\mathbb{1}_{U_{[-T,T]}(\mathcal{L}^N,\varrho,r)} \big| \mathcal{F}_{\mathrm{ext}}([1, K]_{\mathbb{Z}} \times \Lambda_d^N(-T, T))\big] = \mathbf{P}_N\big(U_{[-T,T]}(\mathcal{L}^N, \varrho, r)\big). \tag{3.23}$$

The proof of Theorem 3.1.13 will be completed by the following lemma.

Lemma 3.2.10. *Let $\varrho, \eta, \delta, M, T > 0$. There exists $r(\varrho, \eta, \delta, M, T)$ and $N_0(r, \varrho, \eta, \delta, M, T)$ large enough such that for all $N \geq N_0$,*

$$\mathbf{P}_N\big(U_{[-T,T]}(\mathcal{L}^N, \varrho, r)\big) \geq 1 - \eta/2.$$

provided that $\vec{x}, \vec{y} \in \mathbb{R}^K$ satisfy the condition $|x_i|, |y_i| \leq M$ for $1 \leq i \leq K$ and the condition $Z_N \geq \delta$ holds.

Let us assume the lemma for the moment and complete the proof of the claim in (4.177). By choosing r small enough (depending on $\varrho, \eta, \delta, M, T$) and N_0 large enough(depending on $r, \varrho, \eta, \delta, M, T$), using Lemma 4.8.10 and equality (4.180), we find that

$$(4.178) \geq (1 - \eta/2)\mathbb{E}_N[\mathbb{1}_{Z_N \geq \delta} \mathbb{1}_{S_{N,M}}].$$

By Lemma 4.8.9, there exists $\delta > 0$ and N_0, such that, for $N > N_0$, $\mathbb{P}_N(Z_N < \delta) \le \eta/4$. Propositions 4.8.1 and 4.8.3 imply that we may choose M, N_0 large enough and δ small enough so that $\mathbb{P}(S^{\neg}_{N,M}) \le \eta/4$. This implies that

$$\mathbb{P}_N(\{Z_N \ge \delta\} \cap S_{N,M}) \ge 1 - \eta/2, \tag{3.24}$$

and thus

$$(4.178) \ge (1-\eta/2)^2 > 1 - \eta$$

which completes the proof of claim (4.177) and hence Theorem 3.1.13(1).

Proof of Lemma 4.8.10. The proof is based on the estimates provided by KMT coupling (Assumption A4) and the estimates on modulus of continuity for free Brownian bridges.

Recall that the law of Bridge ensemble, $\mathbf{P}_N = \mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},+\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}$ is specified with Radon-Nikodym derivative with respect to free random walk bridges, (see Definition 3.1.4)

$$\frac{d\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},+\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}}{d\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}}(\bar{S}) = \frac{W(\bar{S})}{Z_N}.$$

We abbreviate $\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}$ as $\mathbf{P}_{free,N}$ and its expectation as $\mathbf{E}_{free,N}$. Let $U^{\neg}_{[-T,T]}(\mathcal{L}^N,\varrho,r)$ be the complement of $U_{[-T,T]}(\mathcal{L}^N,\varrho,r)$, i.e. $\Big\{\omega_{a,b}\big(\{\mathcal{L}_1,\cdots,\mathcal{L}_k\},r\big) > \varrho\Big\}$. Since W is always less than 1, on the event $\{Z_N \ge \delta\}$, we have

$$\begin{aligned}\mathbf{P}_N\big(U^{\neg}_{[-T,T]}(\mathcal{L}^N,\varrho,r)\big) &= \frac{\mathbf{E}_{free,N}[\mathbb{1}_{U^{\neg}}\cdot W(\bar{S})]}{Z_N} \\ &\le \frac{\mathbf{E}_{free,N}[\mathbb{1}_{U^{\neg}}]}{Z_N} \le \frac{1}{\delta}\mathbf{P}_{free,N}\big(U^{\neg}_{[-T,T]}(\bar{S},\varrho,r)\big).\end{aligned} \tag{3.25}$$

Therefore the proof of Lemma 4.8.10 is reduced to prove for fixed $\delta > 0, \varrho, \eta > 0$, there is a $r(\varrho,\eta)$ and $N_0(\delta,\varrho,\eta,r)$ large enough such that for all $N \ge N_0$, we have

$$\frac{1}{\delta}\mathbf{P}_{free,N}\big(U^{\neg}_{[-T,T]}(\bar{S},\varrho,r)\big) \le \frac{\eta}{2}$$

on the modulus of continuity for K-free random walk bridges sampled from measure $\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}$ where $\vec{x},\vec{y},g$ satisfy the assumptions in Lemma 4.8.10.

By applying KMT coupling (Assumption A4) to K independent random walk bridges, there exists a coupling between K-random walk bridges with measure $\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}$ and Brownian bridge on $[-T,T]$ with boundary value $\vec{x}$ and $\vec{y}$. We denote $\mathbb{P}_{\mathrm{cpl}}$ as the coupling measure and we use $\bar{S}$ and B to represent K-free random bridges and K-free Brownian bridges sampled from this coupled measure. Take $b_1 = 30$ and $b_2 = 1$ in Assumption A4. There exist $a, c > 0$, depending on T, K and M, and a coupling measure $\mathbb{P}_{\mathrm{cpl}}$ such that for all $N \ge 1$,

$$\mathbb{P}_{\mathrm{cpl}}\left(\sum_{1\le i\le K}\sup_{u\in[-T,T]}\big|\bar{S}(i,u) - B(i,u)\big| \ge \frac{a\log N}{\sqrt{N}}\right) \le \frac{c}{N^{30}}.$$

We also have the following estimates on the modulus of continuity for K-free Brownian bridges with boundary value the same as above. Let δ be the same as in the assumption of Lemma 4.8.10, for any $\varrho, \eta > 0$, there exist $r(\varrho, \eta, \delta, M, T) > 0$ such that

$$\mathbb{P}_{\text{cpl}}\left(U^{\neg}_{[-T,T]}\left(B, \frac{\varrho}{2}, r\right)\right) \leq \frac{\delta\eta}{4}. \tag{3.26}$$

Choose N_0 large enough, such that $\frac{a \log N}{\sqrt{N}} < \frac{\varrho}{4}$ and $\frac{c}{N^{30}} < \frac{\delta\eta}{4}$ for all $N > N_0$. Under the coupling measure $\mathbb{P}_{\text{cpl}}$, define event

$$D = \left\{ \sum_{1 \leq i \leq K} \sup_{u \in [-T,T]} \left|\bar{S}(i,u) - B(i,u)\right| < \frac{a \log N}{\sqrt{N}} \right\}.$$

Thus $\mathbb{P}_{\text{cpl}}(D^{\neg}) < \frac{c}{N^{30}} < \frac{\delta\eta}{4}$. On the event D, we have

$$\begin{aligned}
&\sup_{1 \leq i \leq K} \sup_{\substack{s,t \in [-T,T] \\ |s-t| \leq r}} |\bar{S}(i,s) - \bar{S}(i,t)| \\
\leq &\sup_{1 \leq i \leq K} \sup_{\substack{s,t \in [-T,T] \\ |s-t| \leq r}} |B(i,s) - B(i,t)| + \frac{2a \log N}{\sqrt{N}} \\
< &\sup_{1 \leq i \leq K} \sup_{\substack{s,t \in [-T,T] \\ |s-t| \leq r}} |B(i,s) - B(i,t)| + \frac{\varrho}{2}.
\end{aligned} \tag{3.27}$$

Therefore, on the event D, using (4.183) and (4.184), we have

$$\mathbb{P}_{\text{cpl}}\left(U^{\neg}_{[-T,T]}(\bar{S}, \varrho, r) \cap D\right) \leq \mathbb{P}_{\text{cpl}}\big(U^{\neg}_{[-T,T]}(B, \varrho/2, r)\big) \leq \frac{\delta\eta}{4}.$$

Since $\mathbb{P}_{\text{cpl}}(D^{\neg}) \leq \frac{\delta\eta}{4}$, we have

$$\mathbb{P}_{\text{cpl}}\big(U^{\neg}_{[-T,T]}(\bar{S}, \varrho, r)\big) \leq \frac{\delta\eta}{2},$$

which implies

$$\mathbf{P}_N\big(U^{\neg}_{[-T,T]}(\mathcal{L}^N, \varrho, r)\big) \leq \frac{\eta}{2},$$

hence completing the proof. □

3.2.4 Proof of Theorem 3.1.13 (2)

After establishing the tightness of $\mathcal{L}^N$ in the previous section, we now demonstrate that all sub-sequential limiting line ensembles enjoy the H-Brownian Gibbs property. The proof is an adaption of [CH16, Proposition 5.2(1)] and we deal with the fact that underlying path measure converge to Brownian bridge measures.

Proof of Theorem 3.1.13 (2). Without loss of generality, we assume that $\mathcal{L}^N$ converges weakly to a line ensemble $\mathcal{L}^\infty$. The topology is the sup norm on K bounded continuous functions with domain $[-T,T]$. Fix an index $i \in \{1, 2, \ldots, K-1\}$ and two times $a, b \in [-T,T]$ with $a < b$ and interaction Hamiltonian $\mathbf{H}(x) = e^x$.

We will show that the law of $\mathcal{L}^\infty$ is unchanged if $\mathcal{L}_i^\infty$ is resampled between a and b according to the law $\mathbb{P}_H^{i-1,i+2,(a,b),\mathcal{L}_i^\infty(a),\mathcal{L}_i^\infty(b),\mathcal{L}_{i-1}^\infty,\mathcal{L}_{i+1}^\infty}$. The argument can easily be generalized to multiple consecutive curves. Note that the H-Brownian Gibbs property is equivalent to this resampling invariance, hence finishing the proof.

Since the Banach space of K bounded continuous functions equipped with the sup norm (Definition 3.1.1), denoted by $(C[-T,T])^K$, is separable, the Skorohod representation theorem applies. Therefore there exists a probability space $(\Omega,\mathcal{B},\mathbb{P})$ on which all of $\mathcal{L}^N$ for $N\in\mathbb{N}\cup\{\infty\}$ are defined and almost surely $\mathcal{L}^N(\omega)\to\mathcal{L}^\infty(\omega)$ in the topology of $(C[-T,T])^K$.

Let $L=b-a$. Recall that $\bar{S}^N_{L,z}$ is the random walk bridge defined in (3.11). From Assumption A4, for each $N\geq L^{-1}$ there exists a probability space $(\Omega^N_{\text{cpl}},\mathcal{B}^N_{\text{cpl}},\mathbb{P}^N_{\text{cpl}})$ on which all of random walk bridges $\bar{S}^N_{L,z},\ z\in\mathbb{R}$ and a Brownian bridge B_L are defined. Moreover, by taking $b_1=30$ and $b_2=1$ in Assumption A4, there exits $0<a_1,a_2<\infty$ such that

$$\mathbb{P}^N_{\text{cpl}}\left(\sup_{0\leq u\leq L}\left|B_L(u)+\frac{u}{L}\cdot z-\bar{S}^N_{L,z}(u)\right|>a_1N^{-1}\log(NL)\right)\leq a_2(NL)^{-30}e^{z^2/L}.$$

We further put all such coupling together and construct a probability space $(\Omega_{\text{cpl}},\mathcal{B}_{\text{cpl}},\mathbb{P}_{\text{cpl}})$ on which all of $\bar{S}^N_{L,z},\ z\in\mathbb{R},\ N\geq L^{-1},\ z\in\mathbb{R}$ and a Brownian bridge B_L are defined and the above estimates hold with $\mathbb{P}^N_{\text{cpl}}$ replaced by $\mathbb{P}_{\text{cpl}}$. Suppose we have a bounded sequence $z_N\in\mathbb{R}$ converging to z_∞, then

$$\sum_{N\geq L^{-1}}\mathbb{P}_{\text{cpl}}\left(\sup_{0\leq u\leq L}\left|B_L(u)+\frac{u}{L}\cdot z_N-\bar{S}^N_{L,z_N}(u)\right|>a_1N^{-1}\log(NL)\right)<\infty.$$

Through the Borel-Cantelli lemma, one has, $\mathbb{P}_{\text{cpl}}$-almost surely

$$\begin{aligned}&\sup_{u\in[0,L]}\left|B_L(u)+\frac{u}{L}\cdot z_\infty-\bar{S}^N_{L,z_N}(u)\right|\\&\leq\sup_{u\in[0,L]}\left|B_L(u)+\frac{u}{L}\cdot z_N-\bar{S}^N_{L,z_N}(u)\right|+|z_N-z_\infty|\to 0.\end{aligned}\tag{3.28}$$

Let $\{(\bar{S}^{N,\ell}_{L,z},B^\ell_L)\}_{\ell\in\mathbb{N}}$ be a sequence of such coupling and independent between different ℓ. Let $\{U_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of independent random variables, each having the uniform distribution on $[0,1]$. We further augment the probability space $(\Omega,\mathcal{B},\mathbb{P})$ to include all such random variables in the independent manner.

In the first step, we define the ℓ-th candidate for the resampled bridge. As $u\in[a,b]$, define

$$\mathcal{L}_i^{N,\ell}(u)=\mathcal{L}_i^N(a)+\bar{S}^{N,\ell}_{L,\mathcal{L}_i^N(b)-\mathcal{L}_i^N(a)}(u-a),$$

and $\mathcal{L}_i^{N,\ell}(u)=\mathcal{L}_i^N(u)$ for $u\in[-T,a)\cup(b,T]$. Similarly, as $u\in[a,b]$, define

$$\mathcal{L}_i^{\infty,\ell}(u)=\mathcal{L}_i^\infty(a)+B^\ell_L(u-a)+\frac{u-a}{b-a}\cdot(\mathcal{L}_i^\infty(b)-\mathcal{L}_i^\infty(a)),$$

and $\mathcal{L}_i^{\infty,\ell}(u)=\mathcal{L}_i^\infty(u)$ for $u\in[-T,a)\cup(b,T]$.

In the second step, we check whether

$$U_\ell \leq W(N,\ell) := W_{\dot{\mathbf{H}}^N}^{i-1,i+1,\Lambda_d^N(a,b),\mathcal{L}_i^N(a),\mathcal{L}_i^N(b),\mathcal{L}_{i-1}^N,\mathcal{L}_{i+1}^N}(\mathcal{L}_i^{N,\ell}), \tag{3.29}$$

and **accept** the candidate resampling $\mathcal{L}_i^{N,\ell}$ if this event occurs. We define accordingly

$$W(\infty,\ell) := W_{\mathbf{H}}^{i-1,i+1,(a,b),\mathcal{L}_i^\infty(a),\mathcal{L}_i^\infty(b),\mathcal{L}_{i-1}^\infty,\mathcal{L}_{i+1}^\infty}(\mathcal{L}_i^{\infty,\ell}). \tag{3.30}$$

For $N \in \mathbb{N}\cup\{\infty\}$, define $\ell(N)$ to be the minimal value of ℓ for which we accept $\mathcal{L}_i^{N,\ell}$. Write $\mathcal{L}^{N,\mathrm{re}}$ for the line ensemble with the i-th line replaced by $\mathcal{L}_i^{N,\ell(N)}$. The random walk Gibbs property is equivalent to the fact that for $N \in \mathbb{N}$,

$$\mathcal{L}^{N,\mathrm{re}} \overset{(d)}{=} \mathcal{L}^N. \tag{3.31}$$

Our **goal** is to show the same equality holds for $N=\infty$, which verifies the $\mathbf{H}$-Brownian Gibbs property for the limiting line ensembles. For the moment we assume $\ell(N)$ converges to $\ell(\infty)$ with $\ell(\infty)$ bounded almost surely (which we will prove in the lemmas following later) and we complete the proof of Theorem 3.1.13(2) first.

From (4.185) and the independence among $\mathcal{L}^N$ and $\{\bar{S}_{L,z}^{N,\ell}, B_L^\ell\}_{\ell\in\mathbb{N}}$, one obtains almost surely

$$\sup_{u\in[a,b]}\left|B_L^{\ell(\infty)}(u-a)+\frac{u-a}{b-a}\cdot(\mathcal{L}_i^\infty(b)-\mathcal{L}_i^\infty(a))-\bar{S}^{N,\ell(N)}_{L,\mathcal{L}_i^N(b)-\mathcal{L}_i^N(a)}(u-a)\right|\to 0 \tag{3.32}$$

Here we used the independence among $\mathcal{L}^N$ and $\{S_{L,z}^{N,\ell}, B_L^\ell\}_{\ell\in\mathbb{N}}$ to ensure (4.185) can be applied to $z_N=\mathcal{L}_i^N(b)-\mathcal{L}_i^N(a)$, $z_\infty=\mathcal{L}_i^\infty(b)-\mathcal{L}_i^\infty(a)$ and the convergence still holds almost surely. Then $\mathcal{L}^{N,\mathrm{re}}$ converges to $\mathcal{L}^{\infty,\mathrm{re}}$ in $C[-T,T]$ almost surely and thus $\mathcal{L}^{\infty,\mathrm{re}} \overset{(d)}{=} \mathcal{L}^\infty$. We complete the proof of Theorem 3.1.13 (2). ☐

Lemma 3.2.11. *Almost surely $\ell(\infty)$ is finite.*

Proof. For fixed $\mathcal{L}_i^\infty(a),\mathcal{L}_i^\infty(b),\mathcal{L}_{i-1}^\infty,\mathcal{L}_{i+1}^\infty$, the law of $W(\infty,\ell)$ (randomness coming from B_L^ℓ) is supported in $(0,1)$. Hence, for some $\varepsilon>0$, $W(\infty,\ell)$ is at least ε with probability at least ε, which implies that $\ell(\infty)$ is finite almost surely. ☐

Lemma 3.2.12. *Almost surely for all ℓ, $\lim_{N\to\infty} W(N,\ell)=W(\infty,\ell)$.*

Proof. Let A be the intersection of the following events:

- $\displaystyle\sum_{i=1}^K \sup_{u\in[-T,T]}|\mathcal{L}_i^N(u)-\mathcal{L}_i^\infty(u)|\to 0.$
- For all $\ell\in\mathbb{N}$,
  $\displaystyle\sup_{u\in[a,b]}\left|B_L^\ell(u-a)+\frac{u-a}{b-a}\cdot(\mathcal{L}_i^\infty(b)-\mathcal{L}_i^\infty(a))-\bar{S}^{N,\ell}_{L,\mathcal{L}_i^N(b)-\mathcal{L}_i^N(a)}(u-a)\right|\to 0.$

One direct consequence is that as A occurs, $\mathcal{L}^{N,\ell}$ converges uniformly to $\mathcal{L}^{\infty,\ell}$ for all ℓ. In below we show that as A happens, $W(N,\ell) \to W(\infty,\ell)$. We estimate

$$
\begin{aligned}
&|W(N,\ell) - W(\infty,\ell)| \\
\leq &\left| W_{\dot{\mathbf{H}}^N}^{i-1,i+1,\Lambda_d^N(a,b),\mathcal{L}_i^N(a),\mathcal{L}_i^N(b),\mathcal{L}_{i-1}^N,\mathcal{L}_{i+1}^N}(\mathcal{L}^{N,\ell}) - W_{\mathbf{H}}^{i-1,i+1,(a,b),\mathcal{L}_i^N(a),\mathcal{L}_i^N(b),\mathcal{L}_{i-1}^N,\mathcal{L}_{i+1}^N}(\mathcal{L}^{N,\ell}) \right| \\
+ &\left| W_{\mathbf{H}}^{i-1,i+1,(a,b),\mathcal{L}_i^N(a),\mathcal{L}_i^N(b),\mathcal{L}_{i-1}^N,\mathcal{L}_{i+1}^N}(\mathcal{L}^{N,\ell}) - W_{\mathbf{H}}^{i-1,i+1,(a,b),\mathcal{L}_i^N(a),\mathcal{L}_i^N(b),\mathcal{L}_{i-1}^N,\mathcal{L}_{i+1}^N}(\mathcal{L}^{\infty,\ell}) \right| \\
+ &\left| W_{\mathbf{H}}^{i-1,i+1,(a,b),\mathcal{L}_i^N(a),\mathcal{L}_i^N(b),\mathcal{L}_{i-1}^N,\mathcal{L}_{i+1}^N}(\mathcal{L}^{\infty,\ell}) - W_{\mathbf{H}}^{i-1,i+1,(a,b),\mathcal{L}_i^\infty(a),\mathcal{L}_i^\infty(b),\mathcal{L}_{i-1}^\infty,\mathcal{L}_{i+1}^\infty}(\mathcal{L}^{\infty,\ell}) \right|.
\end{aligned}
$$

The first term is bounded by $C_2(\omega_{[a,b]}(\mathcal{L}_{i-1}^{N,\ell}, 1/N) + \omega_{[a,b]}(\mathcal{L}_i^{N,\ell}, 1/N) + \omega_{[a,b]}(\mathcal{L}_{i+1}^{N,\ell}, 1/N) + 1/N)$ by Lemma 4.8.6. Then by

$$
\omega_{[a,b]}(\mathcal{L}_i^{N,\ell}, 1/N) \leq \omega_{[a,b]}(\mathcal{L}_i^{\infty,\ell}, 1/N) + 2\|\mathcal{L}_i^{N,\ell} - \mathcal{L}_i^{\infty,\ell}\|_{C[a,b]},
$$

the first terms goes to zero. By Lemma 4.8.7, the second term is bounded by $\|\mathcal{L}_i^{N,\ell} - \mathcal{L}_i^{\infty,\ell}\|_{C[a,b]}$ which converges to zero. The last terms also converges to zero since $\mathcal{L}^{\infty,\ell}$ is a continuous line ensemble and $\|\mathcal{L}^N - \mathcal{L}^\infty\|_{C[-T,T]}$ goes to zero. □

Lemma 3.2.13. *Almost surely* $\lim_{N\to\infty} \ell(N) = \ell(\infty)$.

Proof. Let A' be the intersection of the event A above and

- $\ell(\infty) < \infty$
- $W(\infty, \ell(\infty)) > U_{\ell(\infty)}$

The last condition occurs with probability 1 since $W(\infty,\ell(\infty)) \in (0,1)$ and, conditioned on $\{W(\infty,\ell(\infty))\}_{j=1}^{\ell(\infty)}$, $U_{\ell(\infty)}$ is the uniform distribution in $[0, W(\infty,\ell(\infty))]$. Then from $W(N,\ell(\infty)) \to W(\infty,\ell(\infty))$, we have for N large enough $W(N,\ell(\infty)) > U_{\ell(\infty)}$ and then $\ell(N) \leq \ell(\infty)$. In particular,

$$
\limsup_{n\to\infty} \ell(N) \leq \ell(\infty).
$$

On the other hand, for all $1 \leq j \leq \ell(\infty) - 1$, one has $W(\infty, j) < U_j$. Therefore $W(N,j) < U_j$ for N large enough and hence

$$
\liminf_{n\to\infty} \ell(N) \geq \ell(\infty).
$$

□

3.3 Proof of Three key Propositions

In this chapter, we will prove Propositions 4.8.1, 4.8.2 and 4.8.3 by induction on the index $k \in \mathbb{N}$. The proof follows the same logic as used in [CH16, Proposition 6.1, 6.2 and 6.3] with certain modifications as needed for the discrete case. The induction proceeds in the following following order:

- We start by proving Proposition 4.8.1 for index k from the knowledge of all three propositions for index $k-1$.
- We deduce Proposition 4.8.2 for index k from Proposition 4.8.1 for index k and Proposition 4.8.2 for index $k-1$.
- We deduce Proposition 4.8.3 for index k from Proposition 4.8.1 for index $k, k-1, k-2$ and Proposition 4.8.1 for index $k-1$.

3.3.1 Proof of Proposition 4.8.1

This proof is similar to that of [CH16, Proposition 6.1]. The main technical difference is that we replace [CH16, Lemma 2.11] by Lemma 4.8.5 for the random walk bridge.

The $k=1$ case follows from the assumption of Theorem 3.1.13 since we assume that the $k=1$ indexed curve converges weakly as a process on $\mathbb{R}$ to a stationary process. Note the stationarity is need for the independence of $R_k(\varepsilon)$ with respect to x_0. We assume now that $k \geq 2$ and for $k-1$ all three propositions have been verified.

Let consider $\varepsilon > 0$ given in the hypothesis of Proposition 4.8.1. For the proof, we recall and define a few constant parameters. Let R_{k-1} and $\hat{R}_{k-1}$ be as in Proposition 4.8.1 and 4.8.3 for our given ε. Let $K > 0$ be such that $4(1-e^{-1/2})^{-1}e^{-K^2} = \varepsilon$. Let T_0 be the parameter provided by Proposition 4.8.2 and for any $\delta \in (1/128, 1/8)$. We also require that $T \in \Lambda_d^N$ large enough that

$$T > T_0, \quad Te^{-T^{1/2}} \leq \frac{1}{4}\log 2, \quad \hat{R}_{k-1} \leq \frac{1}{16}T^2 - KT^{1/2}, \quad R_{k-1} \leq \frac{1}{16}T^2. \tag{3.33}$$

Define

$$M = \frac{1}{8}T^2 - \hat{R}_{k-1} + (K+1)T^{1/2}, \qquad R_k = M + 2T^2 + KT^{1/2}. \tag{3.34}$$

We will prove that for $\varepsilon > 0$ given, if we choose R_k as above then, for any x_0, there exist $N_0(x_0, \varepsilon)$ such that for $N \geq N_0$ and $\overline{x} \in \Lambda_d^N[-x_0, x_0]$, it holds that

$$\mathbb{P}\Big(\inf_{u \in \Lambda_d^N[\overline{x}-1/2, \overline{x}+1/2]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2} \big) < -R_k \Big) < 10\varepsilon, \tag{3.35}$$

therefore it suffices to verify (4.192) to finish the proof of Proposition 4.8.1.

Consider arbitrary x_0 and $\overline{x} \in [-x_0, x_0]$. For T and M as above, define two events

$$E_k^{N,-} = \Big\{ \sup_{u \in \Lambda_d^N[\overline{x}-2T, \overline{x}-T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) > -M \Big\}$$

$$E_k^{N,+} = \Big\{ \sup_{u \in \Lambda_d^N[\overline{x}+T, \overline{x}+2T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) > -M \Big\}$$

and their intersection

$$E_k^N = E_k^{N,-} \cap E_k^{N,+}.$$

The following lemma 4.9.1 is one of the key step towards proving Proposition 4.8.1 and it shows that with high probability that the k indexed curve exceeds some level $-M$ at some point from the two outside region of $[\overline{x} - T, \overline{x} + T]$. Hence together with the re-sampling invariant nature provided by Gibbs property, likewise the value of the line ensemble restricted at interior of $[\overline{x} - T, \overline{x} + T]$ should not deviate from $-M$ about the same size as a random walk bridge does and this imply the desired Proposition 4.8.1. The above idea is summarized as the following two lemmas.

Lemma 3.3.1. *For any $\varepsilon > 0$, $x_0 > 0$, there exists $N_0(x_0, \varepsilon)$ such that for $N > N_0$, and $\overline{x} \in \Lambda_d^N[-x_0, x_0]$, we have*

$$\mathbb{P}\Big((E_k^N)^\neg\Big) \leq 8\varepsilon.$$

Lemma 3.3.2. *For any $\varepsilon > 0$, $x_0 > 0$, there exists $N_0(x_0, \varepsilon)$ such that for $N > N_0$ and $\overline{x} \in \Lambda_d^N[-x_0, x_0]$, we have*

$$\mathbb{P}\Big(\Big\{ \inf_{u \in \Lambda_d^N[\overline{x}-T, \overline{x}+T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) < -R_k \Big\} \cap E_k^N \Big) < 2\varepsilon.$$

Proof of Lemma 4.9.1. We will prove that there exists $N_0(x_0, \varepsilon)$ such that for $N > N_0$, we have

$$\mathbb{P}\Big((E_k^{N,-})^\neg\Big) \leq 4\varepsilon,$$

and the lemma immediately follows by the union bound since the analogous result holds for $E_k^{N,+}$.

Define the event

$$H_{k-1} := \Big\{ \mathcal{L}_{k-1}^N(u) + \frac{u^2}{2} < \hat{R}_{k-1} \text{ for } u = \overline{x} - 2T \text{ and } u = \overline{x} - T \Big\}.$$

By Proposition 4.8.3, there exists N_0 such that $\mathbb{P}\Big((H_{k-1})^\neg\Big) \leq 2\varepsilon$ for $N > N_0$. Hence it is enough to prove that there exists N_0 such that

$$\mathbb{P}\Big((E_k^{N,-})^\neg \cap H_{k-1}\Big) \leq 2\varepsilon.$$

We define the event

$$A := \Big\{ \mathcal{L}_{k-1}^N(\overline{x} - 3T/2) + \frac{(\overline{x} - 3T/2)^2}{2} < -T^2/16 \Big\}.$$

We would like to bound the probability of this event A by applying Proposition 4.8.2 with curve index $k-1$. In order to apply Proposition 4.8.2 for the index $k-1$, we need to make sure that $\overline{x}-3T/2$ lies in the interval $[y_0, y_0+T]$, where $y_0 \in [-x_0, x_0-T]$ with $x_0 = x_0(k-1)$ chosen in Proposition 4.8.2 for $(k-1)$-th labeled curve. And this holds true by choosing that $x_0(k) = \frac{1}{2}x_0(k-1)$ and $x_0(k-1) \geq 5T$ given that $\overline{x} \leq x_0(k)$.

Hence by applying Proposition 4.8.2 for any choice of $\delta \in (0, 1/16)$, there exists $N_0(\varepsilon, k, x_0(k))$ such that $\mathbb{P}(A) \leq \varepsilon$ for $N > N_0$. Now it is enough to prove

$$\mathbb{P}\Big((E_k^{N,-})^{\neg} \cap H_{k-1} \cap A^{\neg}\Big) \leq 2\varepsilon, \tag{3.36}$$

for which we will apply the Gibbs property of the line ensemble and the monotonicity Lemma 3.1.11 to control the probability of the event in (4.193) by the probability of an event on random walk bridges, which could be bounded in a similar way as in the case of Brownian bridges (which is dealt with in [CH16, Proof of Lemma 7.2]) due to the KMT coupling.

It is clear that the event $(E_k^{N,-})^{\neg} \cap H_{k-1}$ is $\mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)$-measurable, thus by the property of conditional expectation, we have

$$\mathbb{P}\Big((E_k^{N,-})^{\neg} \cap H_{k-1} \cap A^{\neg}\Big) = \mathbb{E}\Big[\mathbb{1}_{(E_k^{N,-})^{\neg} \cap H_{k-1}} \mathbb{E}\Big[\mathbb{1}_{A^{\neg}} \big| \mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)\Big]\Big]$$

Due to the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property enjoyed by discrete line ensemble $\mathcal{L}$, it holds that

$$\mathbb{E}\Big[\mathbb{1}_{A^{\neg}} \big| \mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)\Big] = \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),\mathcal{L}_k^N(\overline{x}-2T),\mathcal{L}_k^N(\overline{x}-T),\mathcal{L}_{k-2}^N,\mathcal{L}_k^N}\big(A^{\neg}\big).$$

In order to show (4.193), it suffices to show that under the condition that the event $(E_k^{N,-})^{\neg} \cap H_{k-1}$ occurs,

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),\mathcal{L}_k^N(\overline{x}-2T),\mathcal{L}_k^N(\overline{x}-T),\mathcal{L}_{k-2}^N,\mathcal{L}_k^N}\big(A^{\neg}\big) \leq \varepsilon,$$

which we prove in the following.

Assume $(E_k^{N,-})^{\neg} \cap H_{k-1}$ holds, thus monotone coupling Lemma 3.1.11 implies that we can construct a coupling of the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),\mathcal{L}_k^N(\overline{x}-2T),\mathcal{L}_k^N(\overline{x}-T),\mathcal{L}_{k-2}^N,\mathcal{L}_k^N}$$

on the curve $S = \mathcal{L}_{k-1}^N : \Lambda_d^N[\overline{x}-2T, \overline{x}-T] \to \mathbb{R}$ and the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}$$

on the curve $\widetilde{S} : \Lambda_d^N[\overline{x}-2T, \overline{x}-T] \to \mathbb{R}$ such that almost surely $S(x) \leq \widetilde{S}(x)$ in the interval $[\overline{x}-2T, \overline{x}-T]$.

Since the event $A^{\neg}$ becomes more probable under pointwise increase in $S(x)$, the existence of the coupling implies that

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{A^{\neg}} \big| \mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)\Big] \\ &\leq \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}(A^{\neg}),\end{aligned}$$

where in the RHS A is now defined with respect to $\widetilde{S}$.

Now we proceed to control

$$\mathbb{P}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(A^\neg)\leq\varepsilon \tag{3.37}$$

using estimates on Brownian bridges and KMT coupling.

Recall Definition 3.1.4 that the law of $\widetilde{S}$ is specified by its Radon-Nikodym derivative $Z_N^{-1}W_N(\widetilde{S})$ with respect to the free random walk bridges measure

$$\mathbb{P}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1}}_{free,\mathrm{H}^{\mathrm{RW},N}}.$$

For the rest of this proof, we will use $\mathbb{P}_N$ and $\mathbb{E}_N$ to denote the probability measure and expectation respectively for $\mathbb{P}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2+\hat{R}_{k-1},-(\overline{x}-T)^2+\hat{R}_{k-1}}_{free,\mathrm{H}^{\mathrm{RW},N}}$.

Since normalizing constant $Z_N=\mathbb{E}_N\left(W(\widetilde{S})\right)$ and $W(\widetilde{S})\leq 1$ always holds, we have

$$\begin{aligned}&\mathbb{P}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),(\overline{x}-2T)^2+\hat{R}_{k-1},-(\overline{x}-T)^2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(A^\neg)\\ &=\frac{\mathbb{E}_N\left(\mathbb{1}_{A^\neg}W(\widetilde{S})\right)}{\mathbb{E}_N\left(W(\widetilde{S})\right)}\leq\frac{\mathbb{E}_N\left(\mathbb{1}_{A^\neg}\right)}{Z_N}=\frac{\mathbb{P}_N(A^\neg)}{Z_N}.\end{aligned}$$

Now in order to prove (4.194), it suffices to verify the $Z_N\geq\frac{1}{4}\left(1-e^{-2}\right)$ and $\mathbb{P}_N\left(A^\neg\right)\leq e^{-K^2}$ for $N\geq N_0$ large enough through the choice of K such that $\frac{4e^{-K^2}}{(1-e^{-2})}\leq\varepsilon$.

Note that (see [CH16, (80)]), the lower bound $\frac{1}{2}(1-e^{-2})$ has been proved for the normalization constant Z of a line ensemble with H-Brownian Gibbs property conditioned on the same boundary conditions with Hamiltonian $H=e^x$. In light of Proposition 4.8.8, likewise we obtain for $N\geq N_0(k,x_0,T)$ large enough,

$$Z_N=Z^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2+\hat{R}_{k-1},-(\overline{x}-T)^2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}\geq\frac{1}{4}(1-e^{-2}).$$

It remains to show $\mathbb{P}_N\left(A^\neg\right)\leq e^{-K^2}$ for $N\geq N_0$ large enough. To this end, let $L:[\overline{x}-2T,\overline{x}-T]\to\mathbb{R}$ denote the linear interpolation between $L(\overline{x}-2T)=-(\overline{x}-2T)^2/2+\hat{R}_{k-1}$ and $L(\overline{x}-T)=-(\overline{x}-T)^2/2+\hat{R}_{k-1}$. By Lemma 4.8.5 we have

$$\mathbb{P}_N\left(\sup_{u\in[\overline{x}-2T,\overline{x}-T]}(\widetilde{S}(u)-L(u))\geq KT^{1/2}\right)\leq e^{-K^2}$$

Moreover we have

$$\inf_{u\in[\overline{x}-2T,\overline{x}-T]}(L(u)+u^2/2+M)\geq\hat{R}_{k-1}-\frac{1}{8}T^2+M=(K+1)T^{1/2},$$

the last equality follows from the definition of M in (4.191).

And if $A^{\neg}$ holds, we see that

$$\begin{aligned}
&\widetilde{S}(\overline{x}-3T/2) - L(\overline{x}-3T/2) \\
&= \left(\widetilde{S}(\overline{x}-3T/2) + \frac{(\overline{x}-3T/2)^2}{2}\right) - \left(L(\overline{x}-3T/2) + \frac{(\overline{x}-3T/2)^2}{2}\right) \\
&= \left(\widetilde{S}(\overline{x}-3T/2) + \frac{(\overline{x}-3T/2)^2}{2}\right) - \left(\hat{R}_{k-1} - \frac{T^2}{8}\right) \\
&\geq -\frac{T^2}{16} - \hat{R}_{k-1} + T^2/8 \geq KT^{1/2},
\end{aligned}$$

where the first equality is a direct computation of the value of the linear interpolation at $\overline{x}-3T/2$ and the last inequality follow from $A^{\neg}$. The above estimate implies that

$$A^{\neg} \subset \left(\sup_{u\in[\overline{x}-2T,\overline{x}-T]} (\widetilde{S}(u) - L(u)) \geq KT^{1/2}\right).$$

Hence we have

$$\mathbb{P}_N(A^{\neg}) \leq \mathbb{P}_N\left(\sup_{u\in[\overline{x}-2T,\overline{x}-T]} (\widetilde{S}(u) - L(u)) \geq KT^{1/2}\right) \leq e^{-K^2},$$

and this finishes the proof of Lemma 4.9.1. □

Proof of Lemma 4.9.2. When E_k^N holds, the curve $\mathcal{L}_k^N(u) + \frac{u^2}{2}$ rises above the level $-M$ on both of the intervals $[\overline{x}-2T,\overline{x}-T]$ and$[\overline{x}+T,\overline{x}+2T]$ at $\sigma_{\pm}$ respectively. By the strong Gibbs property, the restricted measure of $\mathcal{L}_k^N(\cdot)$ on $[\sigma_-,\sigma_+]$ is a re-weighted random walk bridge measure. Moreover, as $N\to\infty$, the underline path measure converge to Brownian bridge measure. Provided that the normalizing constant well behaves, the typical deviation for this weight random walk path should be controlled by the deviation for Brownian bridges up to some constants as $N \geq N_0$ for N_0 large enough. The estimates of this proof is carried out similarly to that of the previous lemma.

Define the event

$$F_{k-1}^N = \left\{\inf_{u\in\Lambda_d^N[\overline{x}-2T,\overline{x}+2T]} (\mathcal{L}_{k-1}^N(u) + u^2/2) \geq -M + 2T^{1/2}\right\}.$$

We can obtain an upper bound on the probability of the complement of F_{k-1}^N by applying Proposition 4.8.2 at index $k-1$. Recall that in the proof of Lemma 4.9.1, we use $x_0(k) = x_0(k-1)/2$. We may split the interval $[\overline{x}-2T,\overline{x}-T]$ into four consecutive intervals of length T and apply Proposition 4.8.2 to each of them. In order to do that task, one need to verify that each interval is contained in $[-x_0(k-1), x_0(k-1)+T]$. This condition is easily verified by our choice of parameter and thus there exists N_0 such that

$$\mathbb{P}\big((F_{k-1}^N)^{\neg}\big) \leq \varepsilon,$$

for $N \geq N_0$. We also define the event

$$G^N = \left\{\inf_{u\in\Lambda_d^N[\overline{x}-T,\overline{x}+T]} (\mathcal{L}_k^N(u) + u^2/2) \leq -R_k\right\}$$

We will prove that there exists $N_0(x_0, \varepsilon)$ such that

$$\mathbb{P}\big(E_k^N \cap F_{k-1}^N \cap G^N\big) \leq \varepsilon \tag{3.38}$$

for $N > N_0$. Then the right hand side of Lemma 4.9.2 is bounded above by

$$\mathbb{P}\big(E_k^N \cap F_{k-1}^N \cap G^N\big) + \mathbb{P}\big((F_{k-1}^N)^{\neg}\big) \leq 2\varepsilon,$$

as needed to complete to proof of the lemma.

We prove (4.195) by following a similar approach as in the previous proof but by using the strong Gibbs property. Define $\sigma_{-,k}$ to be the infimum over those $u \in \Lambda_d^N[\overline{x} - 2T, \overline{x} - T]$ such that $\mathcal{L}_k^N(u) + u^2/2 \geq -M$. Likewise define $\sigma_{+,k}$ to be the infimum over those $u \in \Lambda_d^N[\overline{x} + T, \overline{x} + 2T]$ such that $\mathcal{L}_k^N(u) + u^2/2 \geq -M$. It is easy to see that the interval $(\sigma_{-,k}, \sigma_{+,k})$ form a $\{k\}$-stopping domain (Definition 3.1.9) and the event $E_k^N \cap F_k^N$ is $\mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)$-measurable. These facts imply that

$$\mathbb{P}\big(E_k^N \cap F_{k-1}^N \cap G^N\big) = \mathbb{E}\bigg[\mathbb{1}_{E_k^N \cap F_k^N} \mathbb{E}\Big[\mathbb{1}_{G^N} \Big| \mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)\Big]\bigg]$$

and

$$\mathbb{E}\Big[\mathbb{1}_{G^N} \Big| \mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)\Big] = \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),\mathcal{L}_k^N(\sigma_{-,k}),\mathcal{L}_k^N(\sigma_{+,k}),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}(G^N).$$

To simplify the notation, we let $S : (\sigma_{-,k}, \sigma_{+,k}) \to \mathbb{R}$ be the curve distributed according to the given measure on the right hand side and G^N is defined now in terms of S.

Under the assumption that the event $E_k^N \cap F_{k-1}^N$ occurs, we know that $\mathcal{L}_k^N(\sigma_{\pm,k}) = (\sigma_{\pm,k})^2/2 - M$. By Lemma 3.1.11, there exists a coupling of the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),\mathcal{L}_k^N(\sigma_{-,k}),\mathcal{L}_k^N(\sigma_{+,k}),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}$$

on the curve S with the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),(\sigma_{-,k})^2/2-M,(\sigma_{+,k})^2/2-M,-\frac{u^2}{2}-M+2T^{1/2},-\infty}$$

on the curve $\widetilde{S}$ such that almost surely $S(u) \geq \widetilde{S}(u)$ for $u \in \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})$. Since the event G^N becomes more probable under the pointwise decrease in S, this implies that

$$\mathbb{E}\Big[\mathbb{1}_{G^N} \Big| \mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)\Big] \leq \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),(\sigma_{-,k})^2/2-M,(\sigma_{+,k})^2/2-M,\mathcal{L}_{k-1}^N,-\infty}(G^N)$$

As in the proof of Lemma 4.9.1, we have the law of $\widetilde{S}$ is defined by its Radon-Nikodym derivative $Z_N^{-1} W_N(\widetilde{S})$ with respect to the law of a rescaled random walk bridge with the same starting and ending points. Z_N is the expectation of $W_N(\widetilde{S})$ with respect to the random walk measure on $\widetilde{S}$. Now the proof proceeds the same way as previous lemma such that we need a lower bound for normalizing constant Z_N and the upper bound of the same event under the probability measure of free random walk bridges.

Analogous to Z_N, the control of Z such that $Z \geq \frac{1}{2}(1-2e^{-1/2})$ in the case of H- Gibbs line ensemble with $H = e^x$ is already proved in [CH16, Proposition 6.1]. Another application of Lemma 4.8.8 shows that $Z_N \geq \frac{1}{4}(1-2e^{-1/2})$ for $N \geq N_0$ large enough.

Let $L : (\sigma_{-,k}, \sigma_{+,k}) \to \mathbb{R}$ denote the linear interpolation between $L(\sigma_{-,k}) = -(\sigma_{-,k})^2/2 - M$ and $L(\sigma_{+,k}) = -(\sigma_{+,k})^2/2 - M$. We further find that

$$\mathbb{P}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),(\sigma_{-,k})^2/2-M,(\sigma_{+,k})^2/2-M,\mathcal{L}_{k-1}^N,-\infty}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}\left(\inf_{u\in\Lambda_d^N[\sigma_{-,k},\sigma_{+,k}]}(\widetilde{S}(u)-L(u)) \leq -KT^{1/2}\right)$$
$$\leq Z^{-1}e^{-K^2} \leq \varepsilon,$$

together with the convexity of $\frac{u^2}{2}$ we have

$$\inf_{u\in[\sigma_{-,n}^N,\sigma_{+,k}^N]}\left(\widetilde{S}(u)+\frac{u^2}{2}+M+2T^2\right) \geq \inf_{u\in[\sigma_{-,k}^N,\sigma_{+,k}^N]}\left(\widetilde{S}(u)-L(u)\right),$$

therefore by choosing $R_k = M + 2T^2 + KT^{1/2}$, we have

$$\mathbb{P}\left(\widetilde{S}(u)+\frac{u^2}{2} \leq -R_k\right) \leq \mathbb{P}\left(\inf_{u\in[\sigma_{-,k}^N,\sigma_{+,k}^N]}\left(\widetilde{S}(u)-L(u)\right) \leq -KT^{1/2}\right) \leq \varepsilon$$

and this completes our proof. □

3.3.2 Proof of Proposition 4.8.2

We prove this proposition by induction on the index k. In order to deduce the proposition for index k, we rely on Proposition 4.8.1 for index k, as well as Proposition 4.8.2 for index $k-1$. For the case $k=0$, it is easy to check that the result holds. We assume now that $k \geq 1$.

Consider $\varepsilon > 0$ and $\delta \in (0, 1/8)$ fixed from the statement of the proposition. We assume that $\varepsilon \in (0,1)$ since the case $\varepsilon \geq 1$ is trivial. By using Proposition 4.8.1 for index k, there exists a constant R_k such that, for all $x_0 > 0$ and $u \in \Lambda_d^N[-x_0, x_0]$,

$$\mathbb{P}\big(\mathcal{L}_k^N(u) + u^2/2 < -R_k\big) \leq \frac{\varepsilon\delta}{3}, \tag{3.39}$$

whenever $N \geq N_0(x_0, \varepsilon\delta/3)$. For $y_0, T > 0$, we define the event

$$C_{y_0,T}^N = \left\{\inf_{u\in\Lambda_d^N[y_0,y_0+T]} \mathcal{L}_{k-1}^N(u) + u^2/2 \geq -\frac{1}{2}\delta T^2\right\}.$$

We fix the constant $T_0 > 0$ large enough such that the following conditions hold:

1. $R_k \leq \frac{5}{8}\delta T_0^2$.

2. For all $T > T_0$, define

$$K(T) = \Big(\log\big(2(1-e^{-2})^{-1}3T\varepsilon^{-1}\big)\Big)^{1/2}, \tag{3.40}$$

and require that

$$\max\Big\{\big(\delta(T+1)\big)^{1/2}, K(T)\big(\delta(T+1)\big)^{1/2}, \delta^2(T+1)^2\Big\} \leq \frac{1}{8}\delta T^2,$$
$$\exp\big\{-(T+1)\delta e^{-1/8\delta T^2}\big\} \geq 1/2.$$

3. For all $x_0 \geq T_0$, $T \in [T_0, x_0]$ and $y_0 \in \Lambda_d^N[-x_0, x_0 - T]$
$$\mathbb{P}(C^N_{y_0,T}) \geq 1 - \frac{\varepsilon}{3}, \tag{3.41}$$
for $N \geq N_0(x_0, \varepsilon, \delta)$ large enough. The existence of such T_0 is a direct consequence of Proposition 4.8.2 for index $k-1$.

Define the event
$$E^N_{y_0,T} = \Big\{\inf_{u\in\Lambda_d^N[y_0,y_0+T]}(\mathcal{L}_k^N(u) + u^2/2) \leq -\delta T^2\Big\}.$$
We will show that
$$\mathbb{P}(E^N_{y_0,T}) < \varepsilon, \tag{3.42}$$
which proves the desired Proposition 4.8.2.

We will say that $u \in \mathbb{Z}\cap\Lambda_d^N[-x_0, x_0]$ is $(\varepsilon\delta/3)-$good if $\mathcal{L}_k^N(u) + u^2/2 \geq -R_k$ where R_k is define in (4.196). We say that $u \in \mathbb{Z}\cap\Lambda_d^N[-x_0, x_0]$ is $(\varepsilon\delta/3)-$bad if it is not $(\varepsilon\delta/3)-$good. Define $B^N_{y_0,T}$ the event that the number of $(\varepsilon\delta/3)-$bad u in $\mathbb{Z}\cap\Lambda_d^N[y_0, y_0+T]$ is at most $(T+1)\delta$. It is straight forward from (4.196) that the probability that any given $u \in \mathbb{Z}\cap\Lambda_d^N[-x_0, x_0]$ is $(\varepsilon\delta/3)-$good is at least $1-\varepsilon\delta/3$. The mean number of $(\varepsilon\delta/3)-$bad u is therefore at most $(T+1)\varepsilon\delta/3$. Thus by the Markov inequality,
$$\mathbb{P}(B^N_{y_0,T}) \geq 1 - \frac{\varepsilon}{3}. \tag{3.43}$$
On the other hand, we have
$$\mathbb{P}(E^N_{y_0,T}) \leq \mathbb{P}\big(E^N_{y_0,T}\cap B^N_{y_0,T}\cap C^N_{y_0,T}\big) + \mathbb{P}\big((B^N_{y_0,T}\cap C^N_{y_0,T})^\neg\big).$$
By the bounds (4.200) and (4.198), we find that
$$\mathbb{P}\big((B^N_{y_0,T}\cap C^N_{y_0,T})^\neg\big) \leq \frac{2}{3}\varepsilon.$$
Hence to prove (4.199), it remains to show that
$$\mathbb{P}\big(E^N_{y_0,T}\cap B^N_{y_0,T}\cap C^N_{y_0,T}\big) \leq \frac{\varepsilon}{3}. \tag{3.44}$$
The event $C^N_{y_0,T}$ depends on the curve of index $k-1$, hence it is $\mathcal{F}_{\text{ext}}(\{k\}\times\Lambda_d^N(y_0, y_0+T))-$measurable. Using the conditional expectation we have that
$$\mathbb{P}\big(E^N_{y_0,T}\cap B^N_{y_0,T}\cap C^N_{y_0,T}\big) = \mathbb{E}\Big[\mathbb{1}_{C^N_{y_0,T}}\mathbb{E}[1_{E^N_{y_0,T}\cap B^N_{y_0,T}}|\mathcal{F}_{\text{ext}}(\{k\}\times\Lambda_d^N(y_0, y_0+T))]\Big].$$

Then in order to prove (4.201), we only need to check that $\mathbb{P}$-almost surely

$$\mathbb{E}\Big[1_{E^N_{y_0,T}\cap B^N_{y_0,T}}\big|\mathcal{F}_{\mathrm{ext}}(\{k\}\times\Lambda^N_d(y_0,y_0+T))\Big]\le\frac{\varepsilon}{3}\mathbb{1}_{C^N_{y_0,T}}+\mathbb{1}_{(C^N_{y_0,T})^\neg}. \tag{3.45}$$

Since the bounds by $\mathbb{1}_{(C^N_{y_0,T})^\neg}$ is trivial, we need to prove that if the event $C^N_{y_0,T}$ holds then the left-hand side of (4.202) is bounded by $\varepsilon/3$. On this event, the $(\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N})-$ property for $\mathcal{L}^N$ implies that $\mathbb{P}-$almost surely

$$\mathbb{E}\Big[1_{E^N_{y_0,T}\cap B^N_{y_0,T}}\big|\mathcal{F}_{\mathrm{ext}}(\{k\}\times\Lambda^N_d(y_0,y_0+T))\Big]=\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E^N_{y_0,T}\cap B^N_{y_0,T}).$$

On the right hand side the event $E^N_{y_0,T}\cap B^N_{y_0,T}$ are now defined in terms of $S:[y_0,y_0+T]\to\mathbb{R}$ which is distributed according to $\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}$.

In order to establish this, we need to decompose the event $E^N_{y_0,T}\cap B^N_{y_0,T}$ further. For any subset $A\subset\mathbb{Z}\cap\Lambda^N_d[y_0,y_0+T]$, let G^N_A denote the event that the set of $(\varepsilon\delta/3)-$good $u\in\mathbb{Z}\cap\Lambda^N_d[y_0,y_0+T]$ is exactly the set A. Write l_A as maximal length of gaps between two consecutive points in A and denote by $S_{T,\delta}$ the set of all $A\subset\mathbb{Z}\cap\Lambda^N_d[y_0,y_0+T]$ such that $l_A\le(T+1)\delta$.

Observe that the event $B^N_{y_0,T}$ is a subset of the union of G^N_A over all $A\in S_{T,\delta}$. This is because having at most $(T+1)\delta$ integer $x\in\mathbb{Z}\cap\Lambda^N_d[y_0,y_0+T]$ which are $(\varepsilon\delta/3)-$bad implies that the maximal number of such consecutive integers is at most $(T+1)\delta$. This implies that

$$\begin{aligned}&\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E^N_{y_0,T}\cap B^N_{y_0,T})\\ \le&\sum_{A\in S_{T,\delta}}\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E^N_{y_0,T}\cap G^N_A)\\ =&\sum_{A\in S_{T,\delta}}p_A\cdot\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E^N_{y_0,T}|G^N_A),\end{aligned}$$

where

$$p_A=\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(G^N_A)$$

and

$$\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(\bullet|G^N_A)$$

is the measure conditioned on G^N_A occurring.

This conditioned measure is a special case of a general class of measure from Definition 3.1.12 such that

$$\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(\bullet|G^N_A)=\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^F,\widetilde{H}^G}(\bullet),$$

where $\widetilde{f}(u)=(-u^2/2-R_k)\cdot\mathbb{1}_{u\in\mathbb{Z}\cap A^\neg}+\infty\cdot\mathbb{1}_{u\notin\mathbb{Z}\cap A^\neg}$, $\widetilde{g}(u)=(-u^2/2-R_k)\cdot\mathbb{1}_{u\in\mathbb{Z}\cap A}-\infty\cdot\mathbb{1}_{u\notin\mathbb{Z}\cap A}$ and $\widetilde{H}^F(x)=\widetilde{H}^G(x)=\infty\cdot\mathbb{1}_{x\ge0}+0\cdot\mathbb{1}_{x<0}$ (which corresponds to conditioning on non-intersection).

We will prove that

$$\mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\vec{f},\vec{g}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^f,\widetilde{H}^g}(E^N_{y_0,T})\le\frac{\varepsilon}{3}. \tag{3.46}$$

Since $\sum_{A\in S_{T,\delta}} p_A \leq 1$, this will imply that

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E^N_{y_0,T}\cap B^N_{y_0,T}) \leq \frac{\varepsilon}{3}.$$

In order to prove (4.203), we will utilize the monotonicity from Lemma 3.1.11. Write $\widetilde{S}$ to denote the curve distributed according to this law and on the event $C^N_{y_0,T}$, we may couple the measure $\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^F,\widetilde{H}^G}$ on the curve $\widetilde{S}$ to the measure P (which we will introduce below) on the curve $\hat{S}$ so that $\widetilde{S}(u)\geq \hat{S}(u)$ for all $u\in\Lambda_d^N[y_0,y_0+T]$. Since the event $E^N_{y_0,T}$ is more probable as $\widetilde{S}$ decreases, this monotonicity implies that

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^F,\widetilde{H}^G}(E^N_{y_0,T}) \leq P(E^N_{y_0,T}). \tag{3.47}$$

The measure P on the curve $\hat{S}:[y_0,y_0+T]\to\mathbb{R}$ is defined as follows:

- For $u\in A$, fix $\hat{S}(u) = -\frac{3}{4}\delta T^2 - u^2/2$;
- For $a<a'$ that are consecutive elements in A, the law of $\hat{S}$ on the interval (a,a') is specified by requiring that it has Radon-Nikodym derivative $Z^{-1}W(S)$ with respect to the law of the random walk bridge $\mathbb{P}^{\Lambda_d^N[a,a'],-a^2/2-3/4\delta T^2,-a'^2/2-3/4\delta T^2}_{\mathrm{H}^{\mathrm{RW},N}}$. Here we set $S_0(u) = -\frac{u^2}{2}-\frac{1}{2}\delta T^2$, $S_1(u)=S(u)$ and the Boltzmann weight is given by

$$W^N(S) = \exp\Big\{-\sum_{u\in\Lambda_d[a,a']}\dot{\mathbf{H}}^N(\square(S,0,u))\Big\}\cdot \mathbb{1}_{S(b)+b^2/2<-5/8\delta T^2,\forall b\in\mathbb{Z}\cap\Lambda_d^N[a,a']}. \tag{3.48}$$

- For the minimal $a\in A$, the law of $\hat{S}$ on the interval $[y_0,a)$ is given by requiring that it has Radon-Nikodym derivative $Z^{-1}W(S)$ with respect to the law of the random walk bridge $\mathbb{P}^{\Lambda_d^N[y_0,a),\mathcal{L}_k^N(y_0),-a^2/2-3/4\delta T^2}_{\mathrm{H}^{\mathrm{RW},N}}$. Here we set $S_0(u) = -\frac{u^2}{2}-\frac{1}{2}\delta T^2$, $S_1(u)=S(u)$ and the Boltzmann weight is given by

$$W^N(S) = \exp\Big\{-\sum_{u\in\Lambda_d[y_0,a]}H(\square(S,0,u))\Big\}\cdot \mathbb{1}_{S(b)+b^2/2<-5/8\delta T^2,\forall b\in\mathbb{Z}\cap\Lambda_d^N[y_0,a]}. \tag{3.49}$$

- For the maximal $a\in A$, the law of $\hat{S}$ in $(a,y_0+T]$ is similarly defined as for the minimal $a\in A$.

The general monotone coupling in Lemma 3.1.11 implies the inequality (4.204). Hence it suffices to verify that

$$P(E^N_{y_0,T}) \leq \frac{\varepsilon}{3}. \tag{3.50}$$

The rest of the proof constitutes prove of the above claim.

Consider $a<a'$, two consecutive elements in A. Let $\hat{S}_{a,a'}$ the restriction of the curve $\hat{S}$ on the interval $[a,a']$. Note that $a'-a\leq (T+1)\delta$. Therefore by the KMT Assumption A4 and the estimate for Brownian motion, for $N\geq N_0$ with N_0 large enough, we know that

$$\mathbb{P}^{\Lambda_d^N[a,a'],-a^2/2-3/4\delta T^2,-a'^2/2-3/4\delta T^2}_{\mathrm{H}^{\mathrm{RW},N}}\left(\sup_{u\in\Lambda_d^N[a,a']}\big(\hat{S}_{a,a'}(u)-L(u)\big)\leq(\delta(T+1))^{1/2}\right)\geq 1-e^{-2}$$

where $L(u) : [a, a'] \to \mathbb{R}$ denotes the linear interpolation of $L(a) = -a^2 - 3/4\delta T^2$ and $L(a') = -(a')^2 - 3/4\delta T^2$.

By the concavity of $-u^2/2$ and the bound $(\delta(T+1))^{1/2} \le \delta T^2/8$ for T large enough, we see that for such a curve $\hat{S}_{a,a'}$ with

$$\hat{S}_{a,a'}(u) \le L(u) + \delta T^2/8 \le -u^2/2 - \frac{5}{8}\delta T^2,$$

By the same reasoning as in the proof of Proposition 4.8.1, we have for $N \ge N_0$ with N_0 large enough, $Z \ge \frac{1}{4}(1 - e^{-2})$. As a consequence,

$$P\Big(\inf_{u \in \Lambda_d^N[a,a']} \big(\hat{S}_{a,a'}(u) - L(u)\big) \le -K(T)(\delta(T+1))^{1/2}\Big) \le Z^{-1}\exp(-K(T)^2) \le \frac{\varepsilon}{3T},$$

where the final inequality is due to the definition of $K(T)$ given in (4.197). Since the curve $L(u)$ and $-u^2/2 - \frac{3}{4}\delta T^2$ differ by at most $(a'-a)^2$ on $[a, a']$, hence

$$P\Big(\inf_{u \in \Lambda_d^N[a,a']} \big(\hat{S}_{a,a'}(u) + u^2/2\big) \le -K(T)(\delta(T+1))^{1/2} - 3/4\delta T^2 - \delta^2(T+1)^2\Big) \le \frac{\varepsilon}{3T}.$$

Given that $\delta < 1/8$, by assumption on T_0, for $T \ge T_0$, we have $\delta^2(T+1)^2 \le 1/8\delta T^2$ and $K(T)(\delta(T+1))^{1/2} \le 1/8\delta T^2$. Thus

$$P\Big(\inf_{u \in \Lambda_d^N[a,a']} \big(\hat{S}_{a,a'}(u) + u^2/2\big) \le -\delta T^2\Big) \le \frac{\varepsilon}{3T}.$$

Since there are at most T pairs (a, a') consecutive elements in A, the above inequality implies that

$$P\Big(\inf_{u \in \Lambda_d^N[a_*,a^*]} \big(\hat{S}_{a,a'}(u) + u^2/2\big) \le -\delta T^2\Big) \le \frac{\varepsilon}{3}, \tag{3.51}$$

where a_*, a^* denote the minimal and maximal elements of A. Also $a_* \le y_0 + (T+1)\delta$ and $a^* \ge y_0 + T - (T+1)\delta$, thus

$$P\Big(\inf_{u \in \Lambda_d^N[y_0+(T+1)\delta,\, y_0+T-(T+1)\delta]} \big(\hat{S}_{a,a'}(u) + u^2/2\big) \le -\delta T^2\Big) \le \frac{\varepsilon}{3},$$

or equivalently

$$P\big(E^N_{y_0+(T+1)\delta,\, T-2(T+1)\delta}\big) \le \frac{\varepsilon}{3}.$$

So by slightly changing values of $T \in [T_0, x_0]$ and $y_0 \in [-x_0, x_0 - T]$, the argument above yields the conclusion $P(E^N_{y_0,T}) \le \varepsilon/3$ which completes the proof.

3.3.3 Proof of Proposition 4.8.3

This proof will generally follow the steps in the proof of [CH16, Proposition 6.3]. In fact, rather than trying to adapt everything to the discrete setting, we will use the strong coupling provided by Assumption A4 to deduce Lemma 4.9.3 from [CH16, Proposition 7.6]. The proof of that proposition is quite involved and lengthy and this saves us from being needed to redo or adapt it.

Our proof proceeds by induction on the curve index k. For the case $k=1$, Proposition 4.8.3 follows from assumption in Theorem 3.1.13. The general case is $k \geq 3$, and the case $k=2$ is a specialization of the $k \geq 3$ proof. So, from here on we will assume that $k \geq 3$.

In order to deduce the proposition for general index $k \geq 3$, we will apply Proposition 4.8.1 for indices $k-2, k-1$ and k and Proposition 4.8.3 for index $k-1$. The basic idea of the argument is to show that should the index k curve be too high at some time $u \in [\bar{x}, \bar{x}+\frac{1}{2}]$ then (up to the occurrence of certain events which we show are likely) so too must the index $k-1$ curve be high at some point between $[x, \bar{x}+2]$. This violates the index $k-1$ result of Proposition 4.8.3 assumed by the induction, and hence proves the index k case.

For arbitrary $x_0 > 0$ and $\overline{x} \in \Lambda_d^N[-x_0, x_0-1]$, and $k \in \mathbb{N}$ we will define the following events. These events will be determined by additional parameters K_k, R_k, $\hat{R}_{k-1}$ and $\hat{R}_k$ which we will specify later in the proof. Define the event $E_k^N(\hat{R}_k)$ (which we will later show is unlikely)

$$E_k^N(\hat{R}_k) = \left\{ \sup_{u \in \Lambda_d^N[\overline{x}, \overline{x}+\frac{1}{2}]} (\mathcal{L}_k^N(u) + u^2/2) \geq \hat{R}_k \right\}$$

and

$$\chi^N(\hat{R}_k) = \inf\left\{ u \in \Lambda_d^N[\overline{x}, \overline{x}+\frac{1}{2}] : (\mathcal{L}_k^N(u) + u^2/2) \geq \hat{R}_k \right\},$$

with the convention that if the infimum is not attained then $\chi(\hat{R}_k) = \overline{x}+1/2$. Note that almost surely $E_k^N(\hat{R}_k) = \{\chi^N(\hat{R}_k) < \overline{x}+1/2\}$. We will generally shorten $\chi^N(\hat{R}_k)$ by just writing χ. Likewise, for the above and below N-dependent events, we will typically drop the N superscript.

Let us further define events (which by our inductive hypotheses we will show to be typical)

$$Q_{k-2}^N(K_k) = \left\{ \inf_{u \in \Lambda_d^N[\overline{x}, \overline{x}+2]} (\mathcal{L}_{k-2}^N(u) + u^2/2) \geq -K_k \right\},$$

$$A_{k-1,k}^N(R_k) = \left\{ \mathcal{L}_{k-1}^N(\chi) + \chi^2/2 \geq -R_k \right\} \cap \left\{ \mathcal{L}_j^N(\overline{x}+2) + (\overline{x}+2)^2/2 \geq -R_k \text{ for } j = k, k-1 \right\}.$$

Lastly, define the event (which is atypical by the inductive hypotheses)

$$B_{k-1}^N(\hat{R}_{k-1}) = \left\{ \sup_{u \in \Lambda_d^N[\chi, \overline{x}+2]} (\mathcal{L}_{k-1}^N(u) + u^2/2) \geq \hat{R}_{k-1} \right\}.$$

Observe that the interval $[\chi, \overline{x}+2]$ forms a $\{k-1, k\}$-stopping domain for $\mathcal{L}$. Observe also that the events $E_k^N(\hat{R}_k), Q_{k-2}^N(K_k)$ and $A_{k-1,k}^N(R_k)$ are all $\mathcal{F}_{\text{ext}}(\{k-1,k\}, \Lambda_d^N(\chi, \overline{x}+2))$-measurable. The event $B_{k-1}(\hat{R}_{k-1})$ is, however, not measurable with respect to this external sigma-field. By using the strong Gibbs property, we have that $\mathbb{P}$-almost surely:

$$\mathbb{E}\left[B_{k-1}(\hat{R}_{k-1}) \middle| \mathcal{F}_{\text{ext}}(\{k-1,k\}, \Lambda_d^N(\chi, \overline{x}+2)) \right]$$
$$= \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(\mathcal{L}_{k-1}^N(\chi),\mathcal{L}_k^N(\chi)),(\mathcal{L}_{k-1}^N(\overline{x}+2),\mathcal{L}_k^N(\overline{x}+2)),\mathcal{L}_{k-2}^N,\mathcal{L}_{k+1}^N}(B_{k-1}(\hat{R}_{k-1})).$$

Given that the event $E_k^N(\hat{R}_k) \cap Q_{k-2}(K_k) \cap A_{k-1,k}(R_k)$ occurs, it follows that

$$\begin{aligned}
\mathcal{L}_{k-1}^N(\chi) &\geq -R_k - \chi^2/2 \\
\mathcal{L}_k^N(\chi) &\geq \hat{R}_k - \chi^2/2 \\
\mathcal{L}_{k-1}^N(\overline{x}+2) &\geq -R_k - (\overline{x}+2)^2/2 \\
\mathcal{L}_k^N(\overline{x}+2) &\geq -R_k - (\overline{x}+2)^2/2 \\
\mathcal{L}_{k-2}^N(x) &\geq -K_k - x^2/2 \quad \text{for all } x \in \Lambda_d^N[\chi, \overline{x}+2] \\
\mathcal{L}_k^N(x) &\geq -\infty \quad \text{for all } x \in \Lambda_d^N[\chi, \overline{x}+2]
\end{aligned}$$

Therefore, by Lemma 3.1.11, there exists a coupling of the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(\mathcal{L}_{k-1}^N(\chi),\mathcal{L}_k^N(\chi)),(\mathcal{L}_{k-1}^N(\overline{x}+2),\mathcal{L}_k^N(\overline{x}+2)),\mathcal{L}_{k-2}^N,\mathcal{L}_{k+1}^N}$$

on the curves $S_1, S_2 : [\chi, \overline{x}+2] \to \mathbb{R}$ with the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,(\chi,\overline{x}+2),(-R_k-\chi^2/2,\hat{R}_k-\chi^2/2),(-R_k-(\overline{x}+2)^2/2,-R_k-(\overline{x}+2)^2/2),-K_k-x^2/2,-\infty}$$

on the curves $\widetilde{S}_1, \widetilde{S}_2$ such that almost surely $S_i(x) \geq \widetilde{S}_i(x)$ for all $i \in \{1,2\}$ and $x \in \Lambda_d^N[\chi, \overline{x}+2]$. Since the event $B_{k-1}(\hat{R}_{k-1})$ becomes less probable as curves S_1, S_2 decrease, then

$$\begin{aligned}
&\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(\mathcal{L}_{k-1}^N(\chi),\mathcal{L}_k^N(\chi)),(\mathcal{L}_{k-1}^N(\overline{x}+2),\mathcal{L}_k^N(\overline{x}+2)),\mathcal{L}_{k-2}^N,\mathcal{L}_{k+1}^N}(B_{k-1}(\hat{R}_{n-1})) \\
&\geq p(R_k, K_k, \hat{R}_k) \mathbb{1}_{E_k^N(\hat{R}_k) \cap Q_{k-2}(K_n) \cap A_{k-1,k}(R_n)},
\end{aligned} \tag{3.52}$$

where $p(R_k, K_k, \hat{R}_k)$ is a shorthand for

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(-R_k-\chi^2/2,\hat{R}_k-\chi^2/2),(-R_k-(\overline{x}+2)^2/2,-R_k-(\overline{x}+2)^2/2),-K_k-x^2/2,-\infty}(B_{k-1}(\hat{R}_{k-1})).$$

We would like to choose other parameters $K_k, R_k, \hat{R}_k$ such that $p(R_k, K_k, \hat{R}_k) \geq 1/2$ for any $\chi \in \Lambda_d^N[\overline{x}, \overline{x}+1/2]$. This is a direct consequence of Lemma 4.9.3 below by applying Lemma 4.9.3 with the choice $\mu = 1/2$. Now by taking the expectation in (4.209), we obtain

$$\mathbb{P}(E_k^N(\hat{R}_k) \cap Q_{k-2}(K_k) \cap A_{k-1,k}(R_k)) \leq 2\mathbb{P}(B_{k-1}(\hat{R}_{k-1})).$$

We choose $\hat{R}_{k-1}$ so that $\mathbb{P}(B_{k-1}(\hat{R}_{k-1})) \leq 2\varepsilon$, which can be achieved for $N \geq N_0(x_0, \varepsilon)$ owing to Proposition 4.8.3 applied for index $k-1$. Thus, we deduce that

$$\mathbb{P}(E_k^N(\hat{R}_k) \cap Q_{k-2}(K_n) \cap A_{k-1,k}(R_n)) \leq 4\varepsilon.$$

Moreover from Proposition 4.8.1 for index $k-2$ there exist $N \geq N_0$ and $K_k > K^0(R_k)$ such that

$$\mathbb{P}(Q_{k-2}(K_k)) \geq 1 - 2\varepsilon.$$

Similar, by applying Proposition 4.8.1 for indices $k-1, k$, there exist $N > N_0$ and $\delta \hat{R}_k/2 > \hat{R}_{k-1}$ such that

$$\mathbb{P}(A_{k-1,k}) \geq 1 - 3\varepsilon.$$

Now we have for $N > N_0(x_0, \varepsilon, k)$,

$$\begin{aligned}\mathbb{P}(E_k^N(\hat{R}_k)) &\leq \mathbb{P}(E_k^N(\hat{R}_k) \cap Q_{k-2}(K_n) \cap A_{k-1,k}(R_n)) + \mathbb{P}((Q_{k-2}(K_n))^\neg \cup (A_{k-1,k}(R_n))^\neg) \\ &\leq 4\varepsilon + 2\varepsilon + 3\varepsilon.\end{aligned}$$

In the end, by a simple shifting $\overline{x}$ to $\overline{x} + \frac{1}{2}$, we obtain Proposition 4.8.3.

Lemma 3.3.3. *Suppose $\dot{\mathbf{H}}^N$ and $\mathrm{H}^{\mathrm{RW},N}$ satisfy assumptions A3 and A4 respectively. For any $\mu \in (0,1)$. There exists $\delta > 0$, $R^0 > 0$, and functions $K^0(R) > 0$ and $\hat{R}^0(R,K) > 0$ such that, for all $R > R^0$, $K > K^0(R)$, $\hat{R} \geq \hat{R}(R,K)$ and all $\overline{x} \in \mathbb{R}$ and $\chi \in \Lambda_d^N[\overline{x}, \overline{x}+1/2]$, we have the following estimate if provided that $N \geq N_0(R,K,\hat{R})$ with N_0 large enough.*

$$\mathbb{P}_N^{1,2}\left(\sup_{u \in \Lambda_d^N[\chi, \overline{x}+2]} (\bar{S}_1(u) + u^2/2) \geq \frac{1}{2}\delta\hat{R}\right) \geq \mu,$$

where $\mathbb{P}_N^{1,2}$ is a shorthand for the measure below on the curves $\bar{S}_1$ and $\bar{S}_2$,

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{1,2,\Lambda_d^N(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2),-K-x^2/2,-\infty}.$$

Now we only need to prove Lemma 4.9.3 to complete the full proof. The analogue of Lemma 4.9.3 when the underlying path measure is Brownian bridge measure has been proved in [CH16, Proposition 7.6] and we show that the same estimate also holds for random walk bridge case assuming the KMT coupling Assumption A4, hence finishing the proof.

Proof of Lemma 4.9.3. In order to simplify the notation we denote:

$$\mathbb{Q} := \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{1,2,\Lambda_d^N(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2),-K-x^2/2,-\infty},$$

and

$$\overline{\mathbb{Q}} := \mathbb{P}_{free, \mathrm{H}^{\mathrm{RW},N}}^{1,2,\Lambda_d^N(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2)}.$$

These measures are supported on two random walks bridges S_1, S_2. On the other hand we also consider the measures which are defined on two Brownian bridges B_1 and B_2 (Definition 3.1.3):

$$\mathbb{K} := \mathbb{P}_{\mathbf{H}}^{1,2,(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2),-K-x^2/2,-\infty},$$

and

$$\overline{\mathbb{K}} := \mathbb{P}_{free}^{1,2,(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2)}.$$

In this case, from [CH16, Proposition 7.6], we have for any $\mu \in (0,1)$, there exists $\delta > 0$, $R^0 > 0$, and functions $K^0(R) > 0$ and $\hat{R}^0(R,K) > 0$ such that, for all $R > R^0$,$K > K^0(R)$, $\hat{R} \geq \hat{R}(R,K)$ and all $\overline{x} \in \mathbb{R}$ and $\chi \in [\overline{x}, \overline{x}+1/2]$,

$$\mathbb{K}\left(\sup_{u\in[\chi,\overline{x}+2]}(B_1(u)+u^2/2) \geq \frac{1}{2}\delta\hat{R}\right) \geq \frac{\mu+1}{2}.$$

We actually consider the event :

$$J^N(\delta,\hat{R}) = \left\{\sup_{u\in[\chi,\overline{x}+2]}(B_1(u)+u^2/2) \geq \frac{1}{2}\delta\hat{R} + aN^{-1/3}\right\}$$

It is clear that as $N \to \infty$ and we have:

$$\mathbb{K}(J^N(\delta,\hat{R})) \to \mathbb{K}\big(\sup_{u\in[\chi,\overline{x}+2]}(B_1(u)+u^2/2) \geq \frac{1}{2}\delta\hat{R}\big).$$

Hence there exists N_0 such that for $N > N_0$, we have :

$$\mathbb{K}(J^N(\delta,\hat{R})) \geq \mu.$$

In the rest of the proof, we will compare $\mathbb{K}(J^N(\delta,\hat{R}))$ with $\mathbb{Q}(I^N(\delta,\hat{R}))$ where

$$I^N(\delta,\hat{R}) = \Big\{\sup_{u\in\Lambda_d^N[\chi,\overline{x}+2]}(\bar{S}_1(u)+u^2/2) \geq \frac{1}{2}\delta\hat{R}\Big\}.$$

Recall the definition of $\mathbb{Q}$, we have

$$\mathbb{Q}\left(I^N(\delta,\hat{R})\right) = \frac{\mathbb{E}_{\overline{\mathbb{Q}}}\big[\mathbb{1}_{I^N(\delta,\hat{R})}W\big]}{\mathbb{E}_{\overline{\mathbb{Q}}}\big[W\big]},$$

where

$$W = \exp\Big[-\sum_{k=0}^{2}\sum_{u\in\Lambda_d^N[\chi,\overline{x}+2]}\dot{\mathbf{H}}^N[\square(\bar{S},k,u)]\Big].$$

Here we denote $\bar{S}_0(u) = -K - u^2/2$ and $\bar{S}_3(u) = -\infty$. Recall that the measure $\overline{\mathbb{Q}}$ is the law of two independent random walk bridges S_1, S_2 starting at time χ at the point $(-R-\chi^2/2, \hat{R}-\chi^2/2)$ and ending at time $\overline{x}+2$ at the point $(-R-(\overline{x}+2)^2/2, -R-(\overline{x}+2)^2/2)$. Similarly, we have:

$$\mathbb{K}(J^N(\delta,\hat{R})) = \frac{\mathbb{E}_{\overline{\mathbb{K}}}\big[\mathbb{1}_{J^N(\delta,\hat{R})}\widehat{W}\big]}{\mathbb{E}_{\overline{\mathbb{K}}}\big[\widehat{W}\big]},$$

where

$$\widehat{W} = \exp\Big[-\sum_{i=0}^{2}\int_{\chi}^{\overline{x}+2} du\,\exp[B_i(u) - B_{i+1}(u)]\Big],$$

with the conventions $B_0(x) = -K - x^2/2$ and $B_3(x) = -\infty$. Recall that the measure $\overline{\mathbb{K}}$ is the law of two independent Brownian bridges starting at time χ at the point $(-R-\chi^2/2, \hat{R}-\chi^2/2)$ and ending at time $\overline{x}+2$ at the point $(-R-(\overline{x}+2)^2/2, -R-(\overline{x}+2)^2/2)$.

By applying the strong approximation in Assumption A4, we can enlarge the probability space $(\Omega, \mathbb{P})$ and couple the measure $\overline{\mathbb{K}}$ and $\overline{\mathbb{Q}}$ such that these exits $c(R, \hat{R}), a(R, \hat{R})$, for $i = 1, 2$,

$$\mathbb{P}\Big(\sup_{u\in[\overline{x},\overline{x}+2]} |\bar{S}_i(u) - B_i(u)| \geq \frac{a \log N}{\sqrt{N}}\Big) \leq cN^{-30}.$$

Such coupling will allow us to obtain the desired lower bound μ for $\mathbb{Q}(I^N(\delta, \hat{R}))$ with N large enough thus completing the proof. It amounts to find a lower bound for $Y'_N := \overline{\mathbb{E}}_{\mathbb{Q}}\big[\mathbb{1}_{I^N(\delta,\hat{R})} W\big]$, and an upper bound for $Y_N := \overline{\mathbb{E}}_{\mathbb{Q}}\big[W\big]$.

Let us first start with the lower bound for Y'_N. For $i = 1, 2$, let D_i be the event:

$$D_i := \left\{\sup_{u\in[\chi,\overline{x}+2]} |\bar{S}_i(u) - B_i(u)| \leq \frac{a \log N}{\sqrt{N}}\right\}.$$

Since $\chi \in [\overline{x}, \overline{x} + 1/2]$, it is clear that

$$\mathbb{P}(D_1, D_2) \geq (1 - cN^{-30})^2.$$

Consequently, we have

$$Y'_N \geq (1 - cN^{-30})^2 \overline{\mathbb{Q}}\big[\mathbb{1}_{I^N(\delta,\hat{R})} W \big| D_1, D_2\big]$$

Under the Assumption A3 about the Hamiltonian $\dot{\mathbf{H}}^N$ in Theorem 3.1.13, there exists a constant C such that

$$W \geq \exp\left[-\sum_{i=0}^{2} \int_{\chi}^{\overline{x}+2} du \exp\left(S_{i+1}(u) - S_i(u)\right)\right] \cdot \exp\left[-C\left(\omega_{\chi,\overline{x}+2}(\bar{S}_1, 1/N) + \omega_{\chi,\overline{x}+2}(\bar{S}_2, 1/N)\right)\right]$$

Assume that the events D_1 and D_2 occur, we obtain

$$\mathbb{1}_{I^N(\delta,\hat{R})} \geq \mathbb{1}_{J^N(\delta,\hat{R})}$$
$$\omega_{\chi,\overline{x}+2}(\bar{S}_i, 1/N) \leq \omega_{\chi,\overline{x}+2}(B_i, 1/N) + \frac{2a \log N}{\sqrt{N}}$$
$$\int_{\chi}^{\overline{x}+2} du \exp[\bar{S}_{i+1}(u) - \bar{S}_i(u)] \leq \int_{\chi}^{\overline{x}+2} du \exp[B_{i+1}(u) - B_i(u)] \exp\left(\frac{2a \log N}{\sqrt{N}}\right).$$

By the coupling between $\overline{\mathbb{Q}}$ and $\overline{\mathbb{K}}$, we have

$$Y'_N \geq (1 - cN^{-30})^2 \times \overline{\mathbb{K}}\left[\mathbb{1}_{J^N(\delta,\hat{R})} \exp\left[-\sum_{i=0}^{2} \exp\left(\frac{2a \log N}{\sqrt{N}}\right)\left(\int_{\chi}^{\overline{x}+2} du \exp\left(B_{i+1}(u) - B_i(u)\right)\right)\right]\right.$$
$$\left.\times \exp\left(-C\omega_{\chi,\overline{x}+2}(B_1, 1/N) - C\omega_{\chi,\overline{x}+2}(B_2, 1/N) - 4C\frac{a \log N}{\sqrt{N}}\right)\right].$$

Fixing an $\varepsilon > 0$ and conditioning on the event that $\{\omega_{\chi,\overline{x}+2}(B^i, 1/N) < \varepsilon\}$ for $i = 1, 2$ then the above inequality becomes:

$$Y'_N \geq A'(N, \varepsilon) \times \overline{\mathbb{K}}\left[\mathbb{1}_{J^N(\delta,\hat{R})} \exp\left[-\sum_{i=0}^{2} \exp\left(\frac{2a \log N}{\sqrt{N}}\right) \int_{\chi}^{\overline{x}+2} du \exp\left(B_{i+1}(u) - B_i(u)\right)\right]\right], \qquad (3.53)$$

where

$$A'(N,\varepsilon) = (1 - cN^{-30})^2 \times (\mathbb{P}(\{\omega_{\chi,\overline{x}+2}(B,1/N) < \varepsilon\}))^2 \times \exp\left(-2C\varepsilon - \frac{4aC\log N}{\sqrt{N}}\right).$$

Since $\exp\left(\frac{2a\log N}{\sqrt{N}}\right) > 1$, by the convexity of function x^α for $\alpha > 1$ we have,

$$Y'_N \geq A'(N,\varepsilon) \times \overline{\mathbb{K}}\left[\mathbb{1}_{J^N(\delta,\hat{R})}\widehat{W}\right]^{\left(\frac{2a\log N}{\sqrt{N}}\right)}. \tag{3.54}$$

Remark that for fixed $\varepsilon > 0$ and $N \to \infty$, $A'(N,\varepsilon) \to \exp(-2C\varepsilon)$.

Now we continue by establishing an upper bound for Y_N. Since almost surely W is bounded by 1, then

$$Y_N \leq \overline{\mathbb{Q}}\Big[W|D_1, D_2\Big]\mathbb{P}[D_1, D_2] + \mathbb{P}(D_1^-) + \mathbb{P}[D_2^-]. \tag{3.55}$$

As we have seen before, $\mathbb{P}[D_i^-] \leq cN^{-30}$. Moreover conditioning on the events D_1, D_2, it is straightforward that for $i = 1, 2$

$$\omega_{\chi,\overline{x}+2}(\bar{S}_i, 1/n) \geq \omega_{\chi,\overline{x}+2}(B_i, 1/N) - \frac{2a\log N}{\sqrt{N}}$$

$$\int_\chi^{\overline{x}+2} du \exp[\bar{S}_{i+1}(u) - \bar{S}_i(u)] \geq \int_\chi^{\overline{x}+2} du \exp[B_{i+1}(u) - B_i(u)] \exp\left(\frac{-2a\log N}{\sqrt{N}}\right).$$

By applying Assumption A3 and conditioning on the events $\{\omega_{\chi,\overline{x}+2}(B_i, 1/N) > \varepsilon\}$, we can easily get the following upper bound

$$\begin{aligned}\overline{\mathbb{Q}}\Big[W|D_1, D_2\Big] &\leq 2\mathbb{P}[\omega_B(\chi, \overline{x}+2, 1/N) > \varepsilon] \\ &\quad + \overline{\mathbb{K}}\Big[\exp\Big[-\sum_{i=0}^{2}\exp\left(\frac{-2a\log N}{\sqrt{N}}\right)\int_\chi^{\overline{x}+2} du \exp[B_{i+1}(u) - B_i(u)]\Big]\Big] \\ &\quad \times \exp\left(2C\varepsilon + 4C\frac{a\log N}{\sqrt{N}}\right),\end{aligned}$$

Since $0 < \exp(-2\alpha(2N)) < 1$, by the concavity of x^α for $0 < \alpha < 1$, we can deduce that

$$\overline{\mathbb{Q}}\Big[W|D_1, D_2\Big] \leq 2\mathbb{P}[\omega_{\chi,\overline{x}+2}(B, 1/N) > \varepsilon] + \overline{\mathbb{K}}\Big[\overline{W}\Big]^{\exp\left(\frac{-2a\log N}{\sqrt{N}}\right)} \times \exp\left(2C\varepsilon + 4C\frac{a\log N}{\sqrt{N}}\right). \tag{3.56}$$

Combining (4.212) and (4.213), we obtain:

$$Y_N \leq C(N,\varepsilon) + A(N,\varepsilon)\overline{\mathbb{K}}\Big[\overline{W}\Big]^{\exp\left(\frac{-2\log N}{\sqrt{N}}\right)}, \tag{3.57}$$

where

$$A(N,\varepsilon) = \exp\left(2C\varepsilon + \frac{4aC\log N}{\sqrt{N}}\right) \times \left(1 - cN^{-30}\right)^2,$$

and

$$C(N,\varepsilon) = 2cN^{-30} + \left(1 - cN^{-30}\right)^2 \times 2\mathbb{P}[\omega_{\chi,\overline{x}+2}(B, 1/N) > \varepsilon].$$

At the end, we obtain from (4.211) and (4.214) that

$$\mathbb{Q}[I^N(\delta,\hat{R})] = \frac{Y'_N}{Y_N} \geq \frac{A'(N,\varepsilon)\times \overline{\mathbb{K}}\Big[\mathbb{1}_{J^N(\delta,\hat{R})}\overline{W}\Big]^{\exp(\frac{2a\log N}{\sqrt{N}})}}{C(N,\varepsilon)+A(N,\varepsilon)\overline{\mathbb{K}}\Big[\widehat{W}\Big]^{\exp(\frac{-2a\log N}{\sqrt{N}})}}.$$

Since for fixed $\varepsilon > 0$, we have $A(N,\varepsilon) \to e^{2C\varepsilon}$, $A'(N,\varepsilon)\to e^{-2C\varepsilon}$ and $C(N,\varepsilon)\to 0$ as $N\to\infty$. Thus as $N \to \infty$,

$$\frac{Y'_N}{Y_N} \geq \frac{A'(N,\varepsilon)\times \overline{\mathbb{K}}\Big[\mathbb{1}_{J^N(\delta,\hat{R})}\widehat{W}\Big]^{\exp(\frac{2a\log N}{\sqrt{N}})}}{C(N,\varepsilon)+A(N,\varepsilon)\overline{\mathbb{K}}\Big[\widehat{W}\Big]^{\exp(\frac{-2a\log N}{\sqrt{N}})}} \to e^{-4C\varepsilon}\mathbb{K}\left(\sup_{x\in[\chi,\overline{x}+2]}\left(B_1(x)+x^2\right)\geq \frac{1}{2}\delta\hat{R}\right).$$

Recall that $\mathbb{K}\left(\sup_{x\in[\chi,\overline{x}+2]}\left(B_1(x)+\frac{1}{2}x^2\right)\geq \frac{1}{2}\delta\hat{R}\right) \geq \frac{\mu+1}{2}$, by choosing $\varepsilon(\mu)$ small enough we have for $N\geq N_0$ with $N_0\left(R,K,\hat{R}\right)$ large enough,

$$\mathbb{Q}[I^N(\delta,\hat{R})] \geq \mu,$$

which finishes the proof. □

3.4 Application for the log-gamma directed polymers

In this chapter, we first construct a discrete K-curve log-gamma line ensemble $\mathcal{L}_K = \{\mathcal{L}_{K,i}(n), 1\leq i\leq K, n\geq i\}$, which is assocaited with the log-gamma directed polymer, as introduced in [Sep]. Under weak noise scaling, the lowest indexed curve $\mathcal{L}_{K,1}$ of $\mathcal{L}_K$, converges weakly to solution of KPZ equation with narrow wedge initial data. This result follows from a weak KPZ universality result about directed polymers in [AKQ] (with slight modifications explained later in Proposition 4.10.9). When the same scaling is applied to the full line ensemble $\mathcal{L}_K$, we aim to apply our main Theorem 3.1.13 to show the tightness of the scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ and the H-Brownian Gibbs property with $\mathbf{H}(x) = e^x$ for all of its subsequential limit line ensembles.

There is an immediate problem about the attempt of directly applying Theorem 3.1.13. Note that the discrete $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property of $\mathcal{L}_K$ is a direct consequence of a Markov property in n for $\mathcal{L}_K(n)$, which has been proved only for $n\geq K$ in [COSZ, Theorem 3.7(i) and Theorem 3.9(i,iii)]. After applying the intermediate scaling to $\mathcal{L}_K(n)$, the restriction $n \geq K$ requires that the argument of scaled line ensemble $\overline{\mathcal{L}}^N(\cdot)$($N$ is the scaling parameter) is nonnegative, i.e. $\overline{\mathcal{L}}^N(\cdot)$ enjoys Gibbs property when restricted over nonnegative numbers in a lattice $\frac{2}{\sqrt{N}}\mathbb{Z}$. We resolved this difficulty by proving the Gibbs property for scaled line ensemble $\overline{\mathcal{L}}^N(\cdot)$ over an interval with left endpoint escaping to $-\infty$ (as N goes to infinity) when restricted for the first finite k curves with any $k\geq 1$. The proof relies on the observation (first proved in [CH14] for

Brownian Gibbs property) that $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property survives under weak convergence of discrete line ensembles and another observation in [COSZ] that $\mathcal{L}_K(n)$ could be well approximated by a Markov chain $\mathcal{J}_K^M(n)$ as M goes to infinity without restrictions on n.

We then proceed to the verification of the assumptions of Theorem 3.1.13 and as an application, we prove Theorem 4.10.11.

3.4.1 Definition of log-gamma line ensemble

Let us first introduce the discrete log-gamma line ensemble $\mathcal{L}_K = \{\mathcal{L}_{K,i}(n), 1 \le i \le K, n \ge i\}$. We explain the construction and illustrate the obstacle for the construction of line ensemble $\mathcal{L}_K(n)$ when $n < K$.

Definition 3.4.1. *Let γ be a positive real number. A random variable X has inverse gamma distribution with shape parameter $\gamma > 0$ if is supported on the positive reals and it has distribution*

$$\mathbb{P}(X \in dx) = \frac{1}{\Gamma(\gamma)} x^{-\gamma-1} \exp(-\frac{1}{x}) dx,$$

where $\Gamma(\gamma)$ is the Gamma function. We abbreviate with $X \overset{(d)}{=} \text{Inv-Gamma}(\gamma)$.

It could be directly computed that for any $k \in \mathbb{N}$ with $\gamma > k$,

$$\mathbb{E}[\text{Inv-Gamma}(\gamma)^k] = \frac{\Gamma(\gamma - k)}{\Gamma(\gamma)} = \frac{1}{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}. \tag{3.58}$$

Fix $K \in \mathbb{N}$ and consider a semi-infinite matrix $d = (d_{ij} : i \ge 1, 1 \le j \le K)$ of i.i.d random variables with distribution:

$$d_{ij} \overset{(d)}{=} \text{Inv-Gamma}(\gamma). \tag{3.59}$$

For each $n \ge 1$, form the $n \times K$ matrix $d^{n,K} = (d_{ij} : 1 \le i \le n, 1 \le j \le K)$. $d^{n,K}$ serves as a random environment for the log-gamma polymers such that d_{ij} is the weight a path collect at location (i,j) in $\mathbb{Z}^2$.

The following construction is also known as the geometric RSK correspondence [COSZ]. For $1 \le l \le k \le K$, let $\Pi^n_{k,l}$ denote the set of l-tuples $\pi = (\pi_1, \cdots, \pi_l)$ of non-intersecting lattice paths in $\mathbb{Z}^2$ such that for $1 \le r \le l$, π_r is a lattice path from $(1,r)$ to $(n, k+r-l)$. Remark that we need $n \ge l$ to avoid emptiness or trivialness of the paths collection $\Pi^n_{k,l}$. For $0 \le n < l < k \le K$, the set of paths $\Pi^n_{k,l}$ is empty. At $l = k$ there is a unique l-tuple such that all paths are horizontal. See the left figure in Table 4.1 below for an illustration.

Definition 3.4.2. *The weight of an l-tuples π of such paths is*

$$wt(\pi) := \prod_{r=1}^{l} \prod_{(i,j)\in\pi_r} d_{ij}.$$

For $1 \le l \le k \le K$, define

$$\tau_{k,l}(n) := \sum_{\pi \in \Pi^n_{k,l}} wt(\pi), \tag{3.60}$$

and we take the empty sum to equal zero for the empty sum case $0 \le n < l < k \le K$.

Then at time n, *we can define an array* $z(n) = \{z_{k,l}(n) : 1 \le k \le K, 1 \le l \le k\}$ *by*

$$z_{k,l}(n) = \begin{cases} \dfrac{\tau_{k,l}(n)}{\tau_{k,l-1}(n)}, & l \le n \wedge k, \\ \\ \text{undefined}, & n < l \le k. \end{cases} \tag{3.61}$$

When $n < K$, *there are elements* $z_{k,l}(n) : n < l \le k \le K$ *that are undefined, even though strictly speaking one more element, namely* $z_{n+1,n+1}(n)$, *could be consistently defined as 1.*

Define the shape $y(n) := (y_1(n), \cdots, y_K(n))$ *of the array* $z(n)$ *as:*

$$y_i(n) := \begin{cases} z_{K,i}(n), & i \le n \wedge K, \\ \\ \text{undefined}, & n < i \le K. \end{cases} \tag{3.62}$$

When $n < K$, $y_i(n) := z_{K,i}(n)$ *can only be defined for* $1 \le i \le n$.

We denote $\mathbb{T}_K := (z_{kl} : 1 \le l \le k \le K \text{ and } z_{kl} \in (0, \infty))$ be the set of triangular arrays with positive real entries and $\mathbb{Y}_K := (y_l : 1 \le l \le K, y_l \in (0, \infty))$. We define $K(y, dz)$ as a kernel from $\mathbb{Y}_K$ to $\mathbb{T}_K$ as:

$$K(y, dz) = \prod_{1 \le l \le k < K} \exp\left(- \frac{z_{k,l}}{z_{k+1,l}} - \frac{z_{k+1,l+1}}{z_{k+1,l}} \right) \frac{dz_{k,l}}{z_{k,l}} \prod_{l=1}^{K} \delta_{y_l}(dz_{K,l}).$$

For $y \in \mathbb{Y}_K$, define

$$w(y) := \int_{\mathbb{T}_K} K(y, dz).$$

As proved in [COSZ, Theorem 3.7 and Theorem 3.9], the process $\{y(n), n \ge K\}$ is a Markov chain with the following transition kernel on $\mathbb{Y}_K$:

$$P(y, d\widetilde{y}) = \frac{w(\widetilde{y})}{w(y)} \prod_{i=1}^{K-1} \exp\left[- \frac{\widetilde{y}_{i+1}}{y_i} \right] \prod_{j=1}^{K} \left(\Gamma(\gamma)^{-1} \left(\frac{y_j}{\widetilde{y}_j}\right)^{\gamma} \exp\left(- \frac{y_j}{\widetilde{y}_j} \right) \right) \frac{d\widetilde{y}_j}{\widetilde{y}_j}. \tag{3.63}$$

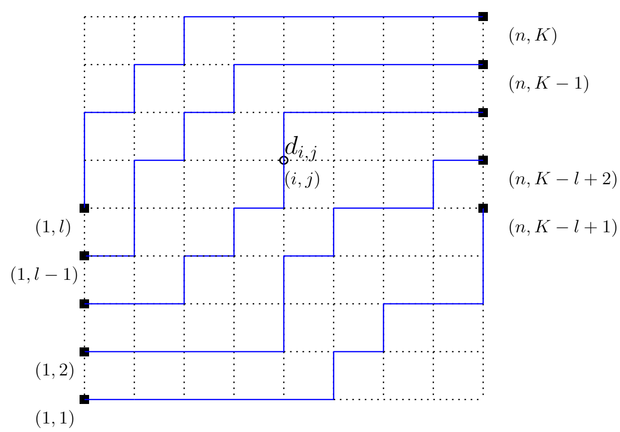
We summarize the the construction of log-gamma line ensemble in the following Table 4.1.

Definition 3.4.3. *We define the log-gamma line ensemble* $\mathcal{L}_{K,i}(n), 1 \le i \le K, n \ge 1$ *by*

$$\mathcal{L}_{K,i}(n) := \begin{cases} \log(y_i(n)) & 1 \le i \le n \wedge K \text{ and } n \ge 1, \\ \\ \text{undefined} & n < i \le K \text{ and } n \ge 1. \end{cases}$$

The following Lemma 4.10.4 shows that an $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property readily follows from the Markov property (4.220), i.e. any Markov chains with transition kernel (4.220) enjoy the same Gibbs property.

Paths in environment	Definition of log-gamma line ensemble $\mathcal{L}_K$



Define partition functions as

- $z_{K,l}(n) := \sum_{\pi \in \Pi^n_{K,l}} wt(\pi)$,
- $wt(\pi) := \prod_{r=1}^{l} \prod_{(i,j)\in\pi_r} d_{ij}$.

Define the *log-gamma line ensemble* $\mathcal{L}_K$ as

- $\mathcal{L}_{K,l}(n) := \log z_{K,l}(n) - \log z_{K,l-1}(n),\ l \le n$
- undefined for $l > n$.

Table 3.1: Summary of the process of constructing $\mathcal{L}(n), n \ge 1$. We regard n as the evolving coordinate. We take the empty sum to equal zero in the definition of $z_{K,l}(n)$ when $n < l$.

Lemma 3.4.4. *Fix $K \in \mathbb{N}$ and an initial state $\hat{y}^0$ in $\mathbb{Y}_K$. Let $\hat{y}(n), n \ge 1$ be the Markov chain on $\mathbb{T}_K$ with initial state $\hat{y}(0) = \hat{y}^0$ and transition kernel given by* (4.220)*. We define the line ensemble $\hat{\mathcal{L}}_{K,i}(n), 1 \le i \le K, n \ge 1$ by*

$$\hat{\mathcal{L}}_{K,i}(n) = \log(\hat{y}_i(n)).$$

Then the discrete line ensemble $\hat{\mathcal{L}}_K = \{\hat{\mathcal{L}}_{K,i}(n), 1 \le i \le K, n \ge 1\}$ enjoys a $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property with:

$$\mathrm{H}^{\mathrm{RW}}(x) = \log\Gamma(\gamma) + \gamma x + e^{-x}, \tag{3.64}$$

and

$$\dot{\mathbf{H}}(\square(\mathcal{L}, k, u)) = \exp\left(\mathcal{L}_{k+1}(u+1) - \mathcal{L}_k(u)\right). \tag{3.65}$$

Proof. Fix $1 \le k_1 < k_2 \le K$, $\Lambda_d = [1,\infty)_{\mathbb{Z}}$ and $a < b \in \Lambda_d$. We denote $\vec{x} = \left(\hat{\mathcal{L}}_{K,k_1}(a), \dots, \hat{\mathcal{L}}_{K,k_2}(a)\right)$ as the entrance data, $\vec{y} = \left(\hat{\mathcal{L}}_{K,k_1}(b), \dots, \hat{\mathcal{L}}_{K,k_2}(b)\right)$ as the exit data, $f = \hat{\mathcal{L}}_{K,k_1-1}|_{\Lambda_d[a,b]}$ be the upper boundary curve and $g = \hat{\mathcal{L}}_{K,k_2+1}|_{\Lambda_d[a,b]}$ be the lower boundary curve and we adopt the convention that $\hat{\mathcal{L}}_0 = +\infty$ and $\hat{\mathcal{L}}_{K+1} = -\infty$.

Given a continuous function $F : C([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d[a,b], \mathbb{R})$, by the Markov property of the process

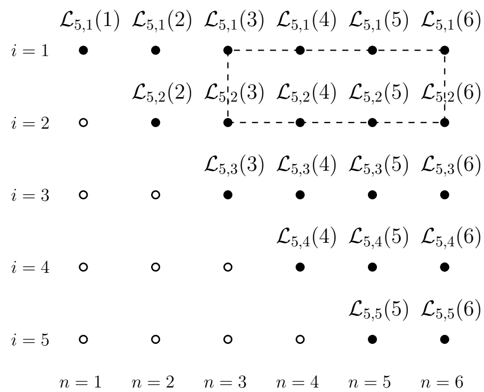


Figure 3.2: Illustration of the log-gamma line ensemble construction with $K = 5$. There are five curves $\mathcal{L}_{5,1}(n)$ through $\mathcal{L}_{5,5}(n)$. The values of $\mathcal{L}_{5,i}(n)$ are not defined on the hollow points. The Gibbs property holds for $n \geq 5$ as a consequence of Lemma 4.10.4 and the Markov property for $\mathcal{L}_5(n)$ when $n \geq 5$ with transition kernel from (4.220). We are in need of the same Gibbs property when $n < 5$, e.g. the dashed rectangle region.

$\{\hat{y}(n), n \geq 1\}$ with the transition kernel (4.220), we have:

$$\begin{aligned}
&\mathbb{E}\Big[F\big(\hat{\mathcal{L}}_K|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)}\big)\big|\mathcal{F}_{\text{ext}}\big([k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)\big)\Big] \\
&\propto \int F\big(\hat{\mathcal{L}}_K|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)}\big) \prod_{u\in\Lambda_d[a,b]}\prod_{i=k_1}^{k_2} \exp\left[-\dot{\mathbf{H}}(\boxdot(\hat{\mathcal{L}}_K,i,u))\right]\times \\
&\prod_{i=k_1}^{k_2}\exp\Big(-\mathrm{H}^{\mathrm{RW}}(\hat{\mathcal{L}}_{K,i}(u+1)-\hat{\mathcal{L}}_{K,i}(u))\Big)d\hat{\mathcal{L}}_{K,i}(u),
\end{aligned} \tag{3.66}$$

where $\dot{\mathbf{H}}$ and H^{RW} are given by (4.222) and (4.221) respectively. Note that the coefficient could be fixed as the normalizing constant for the density inside the integral to be a probability density function and it is obvious to see that the constant should only depend on $\vec{x}, \vec{y}, f, g$. Now we see that the density inside the integrand is exactly $\mathbb{P}^{k_1,k_2,\Lambda_d[a,b],\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}$, which proves the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property. ☐

Lemma 4.10.4 further implies that $\mathcal{L}_{K,i}(n)$ when restricted to $n \geq K$ enjoys $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property with $(H, \mathrm{H}^{\mathrm{RW}})$ as in (4.221) and (4.222). It seems that we are lack of Gibbs property when $n < K$ due to the fact that the Markov property for $y(n)$ only holds for $n \geq K$, see an illustration in Figure 4.1. But by taking a deeper exploration of the proof of [COSZ, Theorem 3.9(iii)], together with Proposition 4.10.7 that Gibbs property survives under weak convergence of line ensemble, we argue below that we still have the same $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property when $n < K$ for the well-defined region of $\mathcal{L}_K$.

Consider $\varrho_K = (\varrho_{K,i})_{1\leq i\leq K}$ with

$$\varrho_K = (\varrho_{K,i})_{1\leq i\leq K} = \left(\frac{K-1}{2}, \frac{K-1}{2} - 1, \cdots, -\frac{K-1}{2}\right).$$

Let $y^{0,M} := (e^{-M\varrho_{K,i}})_{1\le i\le K}$. Define $y^M(n) := (y_i^M(n))_{1\le i\le K, n\ge 1}$ as a Markov chain in $\mathbb{Y}_K$ with the initial state $y^M(0) = y^{0,M}$ and transition kernel given by (4.220). The following proposition is proved in [COSZ, Proposition 5.3], which says that $\{y^M(n), n\ge 1\}$ converges weakly to $y(n)$ on compact sets.

Proposition 3.4.5. *Let $\{y^M(n), n\ge 1\}$ be the Markov chain on $\mathbb{Y}_K$ defined above. And let $y_i(n)$, $1\le i\le n\wedge K, n\ge 1$ be defined as (4.219). Then for any $n_0\ge 1$, $\{y_i^M(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ converges weakly to $\{y_i(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ as M goes to infinity.*

Remark 3.4.6. *It is proved in [COSZ, Proposition 5.3] not only the weak convergence of the shape $y(n)$ but also the weak convergence of the array $z(n)$.*

Define $\mathcal{J}_{K,i}^M(n) := \log y_i^M(n)$. As a consequence, for all $n_0\ge 1$, $\{\mathcal{J}_{K,i}^M(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ converges weakly to $\{\mathcal{L}_{K,i}(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ as M goes to infinity. Now we are ready to prove the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property for $\mathcal{L}_K(n)$ over the well-defined region.

Proposition 3.4.7. *Fix $K\ge 1$. The line ensemble $\mathcal{L}_K(n)$ satisfies the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property in the region it is defined with $\dot{\mathbf{H}}$ and H^{RW} given by (4.222) and (4.221) respectively. Precisely, for any $1\le k_1 < k_2\le K$, $a < b\in\mathbb{N}$ with $(k_2+1)\wedge K\le a$, (4.223) holds with $\hat{\mathcal{L}}_K$ replaced by $\mathcal{L}_K$.*

Proof. From Proposition 4.10.5 and Lemma 4.10.4, $\mathcal{L}_K$ is the weak limit of the sequence of line ensembles $\mathcal{J}_K^M$ as M goes to infinity, which enjoys the same $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property. Therefore it suffices to show $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property is preserved under the weak limit. The proof proceeds similarly as the one of Theorem 3.1.13 (2), thus we omit repeated details and focus on the part that needs modification.

Let $C = [k_1, k_2]_{\mathbb{Z}}\times[a,b]_{\mathbb{Z}}$ be a compact region where $\mathcal{L}_K$ is defined. Through Skorohod representation theorem, in such region C one can couple $\mathcal{L}_K$ with the converging sequence $\mathcal{J}_K^M$ in the same probability space such that and the sequence $\mathcal{J}_K^M$ converges to $\mathcal{L}_K$ almost surely. Now as in Theorem 3.1.13 (2), one can reformulate the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property as a resampling invariance the same way as in (4.188) using the random walk bridges constructed from H^{RW}. Denote such random walk bridges by $S_{L,z}(\cdot)$, where $L\in\mathbb{N}$ is number of steps for the discrete bridges and $z\in\mathbb{R}$ is the location of the endpoint. In the proof of Theorem 3.1.13 (2), we rely on the KMT coupling Assumption A4 to ensure that the random walk bridges almost surely converge uniformly to a Brownian bridge as in (4.189). Here we need to show that such uniform converge to random walk bridge S_{L,z_∞} still holds for random walk bridges S_{L,z_M} as the location of the endpoint z_M converges to z_∞. This will be proved in Lemma 4.10.8 below, hence we can finish the proof by following the arguments for Theorem 3.1.13 (2). ☐

Lemma 3.4.8. *Fix $L\in\mathbb{N}$. Let $S_{L,z}(u), u\in[0,L]_{\mathbb{Z}}, z\in\mathbb{R}$ be the random walk bridges constructed from a smooth random walk Hamiltonian H^{RW}. Then we can couple $(S_{L,z})_{z\in\mathbb{R}}$ in the same probability space*

*$(\Omega_L, \mathcal{B}_L, \mathbb{P}_L)$ such that the following statement holds. For **all** $\omega \in \Omega_L$ and sequence z_j converging to z_∞, we have*

$$\limsup_{j\to\infty} \sup_{u\in[0,L]_{\mathbb{Z}}} |(S_{L,z_j}(\omega))(u) - (S_{L,z_\infty}(\omega))(u)| = 0. \tag{3.67}$$

Proof. We use induction argument in L. The lemma clearly holds for $L = 1$.

For $L \geq 2$, assume $L = 2m$ is even for simplicity. By the induction hypothesis, we can pick two such couplings $(S^1_{m,z}(\cdot))_{z\in\mathbb{R}}$ and $(S^2_{m,z}(\cdot))_{z\in\mathbb{R}}$ independently. For all $z \in \mathbb{R}$, we denote by $W_z = S_{L,z}(L/2)$ the middle point of the random walk bridge. We may couple all $(W_z)_{z\in\mathbb{R}}$ in the same probability space $(\Omega_{\text{mid}}, \mathcal{B}_{\text{mid}}, \mathbb{P}_{\text{mid}})$ through quantile coupling. As the law of W_z varies smoothly in z, for **all** $\omega \in \Omega_{\text{mid}}$ and $z_j \to z_\infty$, $W_{z_j}(\omega) \to W_{z_\infty}(\omega)$.

Now we couple $(W_z)_{z\in\mathbb{R}}$ with $(S^1_{m,z}(\cdot))_{z\in\mathbb{R}}$ and $(S^2_{m,z}(\cdot))_{z\in\mathbb{R}}$ independently. Define a L-step random walk bridges $S_{L,z}(u)$ for all $z \in \mathbb{R}$ by,

$$S_{L,z}(u) := \begin{cases} S^1_{m,W_z}(u) & 0 \leq u \leq m, \\ W_z + S^2_{m,z-W_z}(u-m) & m < u \leq 2m. \end{cases}$$

It can be checked directly that $S_{L,z}(\cdot)$ has the law of a random walk bridge with the location of the endpoint being z. Given a sequence $z_j \to z_\infty$ and an element ω in the probability space, we denote $W_j = W_{z_j}(\omega)$ and $W_\infty = W_{z_\infty}(\omega)$. Then

$$\begin{aligned}
&\limsup_{j\to\infty} \sup_{u\in[0,L]_{\mathbb{Z}}} |(S_{L,z_j}(\omega))(u) - (S_{L,z_\infty}(\omega))(u)| \\
\leq &\limsup_{j\to\infty} \sup_{u\in[0,m]_{\mathbb{Z}}} |(S^1_{m,W_j}(\omega))(u) - (S^1_{m,W_\infty}(\omega))(u)| + \limsup_{j\to\infty} |W_j - W_\infty| \\
&+ \limsup_{j\to\infty} \sup_{u\in[0,m]_{\mathbb{Z}}} |(S^2_{m,z_j-W_j}(\omega))(u) - (S^2_{m,z_\infty-W_\infty}(\omega))(u)| \\
=&0.
\end{aligned}$$

Here we used the induction hypothesis and $W_j \to W_\infty$. Therefore the proof is completed. ☐

3.4.2 Scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ and application of Theorem 3.1.13

Starting from now we set the parameter $\gamma = \gamma_N = \sqrt{N}$. We rewrite the log-gamma line ensemble as $\mathcal{L}^N_K = \{\mathcal{L}^N_{K,1}, \ \ldots \mathcal{L}^N_{K,K}\}$ to indicate the dependence on N.

The following result is a consequence of [AKQ, Theorem 2.7] with slight modifications of their arguments, which we explain in the proof below.

Proposition 3.4.9. *Consider the following weak noise scaling for lowest indexed curve of log-gamma line ensemble, $\mathcal{L}^N_{K,1}$, and denote*

$$\widetilde{\mathcal{L}}^N_1(t,u) := \mathcal{L}^N_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right) - (Nt+\sqrt{N}u)\left(\log 2 - \log(\sqrt{N}-1)\right) + \log\sqrt{N} - \log 2.$$

Then $\widetilde{\mathcal{L}}^N_1(t,u)$ converges weakly to $\log\mathcal{Z}_{\sqrt{2}}(t,u)$ in the topology of uniform convergence on compact sets of $\{t\in\mathbb{R}_+,\ u\in\mathbb{R}\}$.

Here $\mathcal{Z}_{\sqrt{2}}(t,u)$ is defined by the following chaos expansion with convention $t_0=0, u_0=0$,

$$\mathcal{Z}_{\sqrt{2}}(t,u) := \varrho(t,u) + \sum_{k=1}^{\infty}(\sqrt{2})^k \int_{\Delta_k(0,t]}\int_{\mathbb{R}^k}\prod_{i=1}^{k} W(t_i,u_i)\varrho(t_i-t_{i-1},u_i-u_{i-1})\varrho(t-t_k,u-u_k)du_i dt_i. \tag{3.68}$$

In the above expression, $W(t,u)$ is a white noise on $\mathbb{R}_+\times\mathbb{R}$ with covariance structure $\mathbb{E}[W(t,x)W(s,y)] = \delta(t-s)\delta(x-y)$. $\varrho(t,u)$ is the standard Gaussian heat kernel such that

$$\varrho(t,u) = \frac{e^{-u^2/2t}}{\sqrt{2\pi t}}.$$

And the integral is over a k-dimensional simplex $\Delta_k(0,t] = \{0=t_0<t_1<t_2<\cdots<t_k\le t\}$.

Proof. We prove the above result by explaining how the convergence follows from the arguments for [AKQ, Theorem 2.7]. Note that [AKQ, Theorem 2.7] shows that, under weak noise scaling (called intermediate disorder regime in [AKQ]), a modified point-to-point partition function of directed polymer model in dimension 1+1, $\frac{\sqrt{N}}{2}\mathfrak{Z}^{\omega_N}(Nt+\sqrt{N}u,\sqrt{N}u;N^{-1/4})$, converges to the chaos series $\mathcal{Z}_{\sqrt{2}}(t,x)$ in (4.225), which is the solution to stochastic heat equation with multiplicative noise. The random environment ω considered therein is of mean zero, variance one and has sixth moments. We record their convergence result below,

$$\frac{\sqrt{N}}{2}\mathfrak{Z}^{\omega}(Nt+\sqrt{N}u,\sqrt{N}u;N^{-1/4}) \Rightarrow \mathcal{Z}_{\sqrt{2}}(t,u).$$

If we choose ω_N to be an i.i.d. random environment with each weight $\omega_N(i,j)$ distributed as follows,

$$\omega_N(i,j) \overset{(d)}{=} N^{1/4}\left(\frac{\text{Inv-Gamma}(\sqrt{N})}{\mathbb{E}\left[\text{Inv-Gamma}(\sqrt{N})\right]} - 1\right). \tag{3.69}$$

By the definition of partition function $\mathfrak{Z}^{\omega_N}$ in [AKQ] and τ in (4.217), we have the following identification that

$$\begin{aligned}&2^{-(Nt+\sqrt{N}u)}\mathbb{E}[\text{Inv-Gamma}(\sqrt{N})]^{-(Nt+\sqrt{N}u)}\tau_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right)\\ =&\mathfrak{Z}^{\omega_N}(Nt+\sqrt{N}u,\sqrt{N}u;N^{-1/4}).\end{aligned}$$

From (4.226), it can be deduced using (4.215) that ω_N has mean zero and variance $1 + O(N^{-1/2})$. Moreover, the sixth moment of ω_N is $15+O(N^{-1/2})$. Under such conditions, one can run the same arguments in [AKQ] to obtain the convergence of $\frac{\sqrt{N}}{2}\mathfrak{z}^{\omega_N}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4})$, hence the convergence of $\tau_{Nt/2,1}$.

Moreover note that by the definition of $\mathcal{L}^N$, we have

$$\mathcal{L}^N_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right) = \log \tau_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right).$$

As $\mathbb{E}\left[\text{Inv-Gamma}(\sqrt{N})\right] = \frac{1}{\sqrt{N}-1}$,

$$\widetilde{\mathcal{L}}^N_1(t,u) = \log\left(\frac{\sqrt{N}}{2}\mathfrak{z}^{\omega_N}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4})\right),$$

which converges to $\log \mathcal{Z}_{\sqrt{2}}(t,u)$. □

In the following we perform one more rescaling in order to get rid of the $\sqrt{2}$.

Definition 3.4.10. *We define scaled log-gamma line ensemble as follows.*

$$\overline{\mathcal{L}}^N_i(t,u) := \mathcal{L}^N_{Nt/8,i}\left(\frac{1}{8}Nt+\frac{1}{2}\sqrt{N}u\right) - \left(\frac{1}{4}Nt+\frac{1}{2}\sqrt{N}u\right)\left(\log 2 - \log(\sqrt{N}-1)\right) + \log\sqrt{N} - \log 2.$$

We remark that $\overline{\mathcal{L}}^N_i(t,u)$ is defined for $t \in \frac{8}{N}\mathbb{N}$, $u \in \frac{2}{\sqrt{N}}\mathbb{Z}$ and $i \leq (Nt/8) \wedge (Nt/8+\sqrt{N}u/2)$ and that $\overline{\mathcal{L}}^N_1(t,u) = \widetilde{\mathcal{L}}^N_1(t/4,u/2)$.

Now we are ready to state the main application of Theorem 3.1.13 as follows. We fix $t=1$ for notation simplicity and the same result holds for any $t>0$ by the same argument modulo the modification that $\overline{\mathcal{L}}^N_1(t,u)+\frac{u^2}{2t}$ converges to a stationary process.

Theorem 3.4.11. *Fix $t=1$. Given $k\in\mathbb{N}$, $T>0$, the restriction of the line ensemble $\overline{\mathcal{L}}^N$ given by $\{\overline{\mathcal{L}}^N_j(1,u) : j\in\{1,\cdots,k\}, u\in[-T,T]\}$ is tight as N varies. Moreover, any subsequential limit line ensemble satisfies the H-Brownian Gibbs property with $\mathbf{H}(x)=e^x$.*

Proof. Recall the Gibbs property for $\mathcal{L}_K$ in Proposition 4.10.7. The random walk Gibbs property of $\overline{\mathcal{L}}^N_i(1,x)$ can be computed directly through change of variables. Fix $N\in\mathbb{N}$. To simplify notations, we denote $\overline{\mathcal{L}}_i(u) = \overline{\mathcal{L}}^N_i(1,u)$. Then $\overline{\mathcal{L}}_i(u)$ satisfies a $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-random walk Gibbs property over the region

$$\left\{u\in\frac{2}{\sqrt{N}}\mathbb{Z}, i\leq (N/8)\wedge(N/8+\sqrt{N}u/2)\right\}$$

with local interaction Hamiltonian

$$\dot{\mathbf{H}}^N(\square(\overline{\mathcal{L}},k,u)) = \frac{2}{\sqrt{N}-1}\exp\left(\overline{\mathcal{L}}_{k+1}(u+2/\sqrt{N}) - \overline{\mathcal{L}}_k(u)\right), \tag{3.70}$$

and random walk Hamiltonian

$$\mathrm{H}^{\mathrm{RW},N}(x) = \log\Gamma(\sqrt{N}) + \sqrt{N}\left(x - \log(\sqrt{N}-1) + \log 2\right) + \exp\left(-x + \log(\sqrt{N}-1) - \log 2\right). \quad (3.71)$$

Since for fixed i, $\overline{\mathcal{L}}_i^N(u)$ is defined on $\{u \geq -\frac{\sqrt{N}}{4} + 2i\}$, which diverges to $-\infty$. Hence for any choice of compact set $C = \{k_1, \cdots, k_2\} \times [a,b]$, the $\overline{\mathcal{L}}^N$ is well-defined for N large enough. This implies that for any compact set C, the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-random walk Gibbs property holds on C when N is larges enough.

We have shown that the scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ enjoys the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ Gibbs property with local interaction and random walk Hamiltonians given by (4.227) and (4.228) respectively. In light of Theorem 3.1.13, it suffices to verify assumptions A1-A4 and the convergence of the lowest indexed curve $\overline{\mathcal{L}}_1^N$.

Through the scaling invariance of the white noise, $W(at, bu) \overset{(d)}{=} (ab)^{-1/2}W(t,u)$, it follows that for any $\lambda, \beta > 0$, $\mathcal{Z}_\beta(\lambda^2 t, \lambda u) \overset{(d)}{=} \lambda^{-1}\mathcal{Z}_{\lambda^{1/2}\beta}(t,u)$. Together with Proposition 4.10.9, $\overline{\mathcal{L}}_1^N(t,u)$ converges weakly to $\log \mathcal{Z}_1(t,u) + \log 2$. In [AKQ, Proposition 2.3], it is also shown that for fixed $t > 0$, $\mathcal{Z}_\beta(t,u)/\varrho(t,u)$ is a stationary process in u. Therefore, $\overline{\mathcal{L}}_1^N(1,u) + \frac{u^2}{2}$ converges weakly to a stationary process. In the following we proceed to verify Assumptions A1-A4.

Assumptions A1(1) is explicit about $\dot{\mathbf{H}}^N$ in (4.227) and can be verified directly. Since $\dot{\mathbf{H}}^N$ depends only on the second and the sixth entry, to check Assumption A1(2), it's sufficient to consider $\vec{a} = (a_2, a_6)$, $\vec{b} = (b_2, b_6) \in (\mathbb{R} \cup \{\pm\infty\})^2$ with $a_2 \geq b_2$, $a_6 \geq b_6$ and $a_2 = b_2$ or $a_6 = b_6$. For the case $a_2 = b_2$, let $\delta >$ be a fixed number and

$$\vec{a}' = (a_2 + \delta, a_6),\ \vec{b}' = (b_2 + \delta, b_6),$$

as in Assumption A1 (2). Then

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) = \frac{2(e^{a_6 - a_2} - e^{a_6 - a_2 - \delta})}{\sqrt{N} - 1},$$
$$-\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}) = \frac{2(e^{b_6 - b_2} - e^{b_6 - b_2 - \delta})}{\sqrt{N} - 1}.$$

From the convexity of e^x, $a_6 \geq b_6$, $a_2 = b_2$ and $\delta > 0$, we have

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) \geq -\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}).$$

For the case $a_6 = b_6$, let $\delta > 0$ be a fixed number and

$$\vec{a}' = (a_2, a_6 + \delta),\ \vec{b}' = (b_2, b_6 + \delta),$$

as in Assumption A1 (2). Then

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) = \frac{2(e^{a_6 - a_2} - e^{a_6 - a_2 + \delta})}{\sqrt{N} - 1},$$
$$-\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}) = \frac{2(e^{b_6 - b_2} - e^{b_6 - b_2 + \delta})}{\sqrt{N} - 1}.$$

From the convexity of e^x, $a_6 = b_6$, $a_2 \geq b_2$ and $\delta > 0$, we have

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) \geq -\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}).$$

This finishes the verification of Assumption A1. Assumptions A2 can be checked directly from the form of $\mathrm{H}^{\mathrm{RW},N}$ (4.228).

Assumption A3 could can be verified through the definition of Riemann integral. We note that the mesh size for rescaled log-gamma line ensemble is $2/\sqrt{N}$ instead of $1/N$ as in Assumption A3. We could relabel N to solve is discrepancy. However, as long as the mesh size goes to zero and N goes to infinity it has no essential effect on the rest of the proof, we don't do such relabelling. Understanding the mesh size is $2/\sqrt{N}$, we calculate for any $0 \leq a < b \in \mathbb{R}_+$, and any continuous line ensemble $\mathcal{L}$

$$\begin{aligned}
\sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L}, k, u)) = & \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \frac{2}{\sqrt{N}-1} \exp\Big(\mathcal{L}_{k+1}(u + 2/\sqrt{N}) - \mathcal{L}_k(u)\Big) \\
\leq & \frac{\sqrt{N}}{\sqrt{N}-1} e^{\omega_{(a,b)}(\mathcal{L}_{k+1}, 2/\sqrt{N})} \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \frac{2}{\sqrt{N}} \exp\left(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)\right) \\
\leq & \frac{\sqrt{N}}{\sqrt{N}-1} e^{2\omega_{(a,b)}(\mathcal{L}_{k+1}, 2/\sqrt{N}) + \omega_{(a,b)}(\mathcal{L}_k, 2/\sqrt{N})} \int_a^b \exp(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)) du.
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L}, k, u)) \\
\leq & \exp\left(2\omega_{(a,b)}(\mathcal{L}_{k+1}, 2/\sqrt{N}) + \omega_{(a,b)}(\mathcal{L}_k, 2/\sqrt{N}) - \log(1 - 1/\sqrt{N})\right) \\
& \times \int_a^b \exp(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)) du.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L}, k, u)) \\
\geq & \exp\left(- 2\omega_{(a,b)}(\mathcal{L}_{k+1}, 2/\sqrt{N}) - \omega_{(a,b)}(\mathcal{L}_k, 2/\sqrt{N})\right) \int_a^b \exp(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)) du.
\end{aligned}$$

These two yield Assumption A3.

Now we turn to Assumption A4. For any $\gamma > 0$, we denote by

$$Y(\gamma) = -\log\big(\text{Inv-Gamma}(\gamma)\big)$$

the log-gamma random variable with parameter γ. It is easy to verify that the function $\exp(-\mathrm{H}^{\mathrm{RW},N}(x))$ is the density function of the random variable $-Y(\sqrt{N}) + \log(\sqrt{N}-1) - \log 2$. For any $N \geq 2$ let X_j^N be i.i.d. with $X_1^N \overset{(d)}{=} -Y(\sqrt{N}) + \log(\sqrt{N}-1) - \log 2$. For any $L > 0$ with $\frac{\sqrt{N}}{2}L \in \mathbb{N}$, we define random walk bridges $\bar{S}_{L,z}^N$ conditioned on arriving at z in $\sqrt{N}L/2$ steps as in (3.11):

$$\bar{S}_{L,z}^N\left(\frac{2}{\sqrt{N}}k\right) = X_1^N + X_2^N + \dots X_k^N \Big| X_1^N + X_2^N + \dots X_{\sqrt{N}L/2}^N = z, \tag{3.72}$$

And we still extend $\bar{S}_{L,z}^N(u)$ to $u \in [0, L]$ through linear interpolation. Now it suffices to verify that the random walk bridge defined in (4.229) satisfies the estimate in Assumption A4.

To this end, we rely on [DW, Corollary 8.1], which provides the desired estimates for normalized random variables. We start by recording the result of [DW, Corollary 8.1]. Let $m(\gamma)$ and $\sigma(\gamma)^2$ be the mean and variance of $Y(\gamma)$ respectively. For any $n \geq 1$, $0 \leq k \leq n$ and $z \in \mathbb{R}$, we define

$$S_{n,z}(k;\gamma) := X_1(\gamma) + X_2(\gamma) + \dots X_k(\gamma) \Big| X_1(\gamma) + X_2(\gamma) + \dots X_n(\gamma) = z, \tag{3.73}$$

where $X_j(\gamma)$ are i.i.d. with $X_1(\gamma) \overset{(d)}{=} (Y(\gamma) - m(\gamma))/\sigma(\gamma)$. For general $u \in [0, n]$, we define $S_{n,z}(u;\gamma)$ through linear interpolation. The following is a direct consequence of [DW, Corollary 8.1].

Corollary 3.4.12. *For any $b > 0$ and $\gamma_0 > 0$, there exists constants $0 < C, a, \alpha' < \infty$ such that for every positive integer n and $\gamma \geq \gamma_0$, there is a probability space on which are defined a Brownian bridge $B_1(\cdot)$ and a family of processes $\{S_{n,z}(\cdot;\gamma)\}_{z\in\mathbb{R}}$ such that for any $r \geq 0$,*

$$\mathbb{P}\left(\sup_{0\leq u\leq n}\left|\sqrt{n}B_1(u/n) + \frac{u}{n}\cdot z - S_{n,z}(u;\gamma)\right| \geq r\log n\right) \leq C n^{\alpha' - ar} e^{bz^2/n}. \tag{3.74}$$

Denote $q(N) := \log(\sqrt{N}-1) - \log 2 - m(\sqrt{N})$ and we will use m, σ, q as shorthand for $m(\sqrt{N}), \sigma(\sqrt{N}), q(N)$ to simply notations. Now we have $X_1^N \overset{(d)}{=} -\sigma X_1(\sqrt{N}) + q$, which implies

$$\bar{S}_{L,z}^N(u) \overset{(d)}{=} -\sigma S_{\frac{\sqrt{N}}{2}L, -\sigma^{-1}(z - \frac{\sqrt{N}}{2}Lq)}\left(\frac{\sqrt{N}}{2}u; \sqrt{N}\right) + \frac{\sqrt{N}}{2}uq. \tag{3.75}$$

However, Corollary 4.10.12 fails to directly apply here since the endpoint $-\sigma^{-1}(z - \frac{\sqrt{N}}{2}Lq)$ blows up as N goes to infinity. To overcome this issue, we apply a tilting trick and identify the random walk bridge $\bar{S}_{L,z}^N$ with another bridge for which Corollary 4.10.12 applies.

For two random variables X and X' with density functions f_X and $f_{X'}$ separately, we say X and X' are related through tilting if there exist $t \in \mathbb{R}$ and a positive constant C such that

$$f_X(x) = Ce^{tx} f_{X'}(x).$$

We need the following result.

Lemma 3.4.13. *Suppose X and X' are related through tilting. Then the random walk bridges, constructed by the distribution of X and X' separately, have the same distribution.*

Proof. This lemma could be proved by directly comparing the densities of the two bridges. □

Choose ξ and μ (depending on N) such that

$$\begin{aligned} m(\xi) =& m(\sqrt{N}) + q(N), \\ \mu =& \sigma(\xi)/\sigma(\sqrt{N}). \end{aligned}$$

Through direct verification, $X_1(\sqrt{N}) - q/\sigma$ and $\mu X_1(\xi)$ are related through tilting. Hence X_1^N and $-\sigma\mu X_1(\xi)$ are related through tilting. By Lemma 4.10.13,

$$\bar{S}^N_{L,z}(u) \overset{(d)}{=} -\sigma\mu S_{\sqrt{N}L/2,-\sigma^{-1}\mu^{-1}z}\left(\frac{\sqrt{N}u}{2};\xi\right).$$

Let $n = \frac{\sqrt{N}}{2}L$ and $u' = \frac{\sqrt{N}}{2}u$. By the scale invariance for Brownian bridges $\sqrt{L}B_1(u/L) \overset{(d)}{=} B_L(u)$,

$$\mu\sigma\left(\sqrt{n}B_1(u'/n) + \frac{u'}{n}\cdot\frac{z}{\mu\sigma} + S_{n,-\mu^{-1}\sigma^{-1}z}(u';\xi)\right) \overset{(d)}{=} 2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u).$$

In order to apply Lemma 4.10.12, we compute the asymptotics for ξ and μ. First by [DW, (8.10)], as N goes to infinity, for $m,\ \sigma$ and q we have

$$\begin{aligned} m(\sqrt{N}) =& \log\sqrt{N} + O(N^{-1/2}), \\ \sigma(\sqrt{N}) =& N^{-1/4} + O(N^{-3/4}), \\ q(N) =& -\log 2 + O(N^{-1/2}). \end{aligned}$$

Hence

$$\begin{aligned} \xi =& \frac{\sqrt{N}}{2} + O(1), \\ \mu =& 2^{1/2} + O(N^{-1/2}). \end{aligned}$$

Since ξ goes to infinity, (4.231) applies and we obtain the following estimate. For any $b > 0$, there exist $0 < C, a, \alpha' < \infty$ such that for every $N \geq 2$ and $L > 0$

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leq u\leq L}\left|2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq r\cdot\mu\sigma\log(\sqrt{N}L/2)\right) \\ \leq& C(\sqrt{N}L/2)^{\alpha'-ar}\exp\left(\frac{2bz^2}{\mu^2\sigma^2N^{1/2}L}\right). \end{aligned} \tag{3.76}$$

We are ready to verify Assumption A4. Fix $b_1, b_2 > 0$. From the asymptotic of σ and μ, we can choose $b > 0$ such that

$$\sup_{N\geq 2}\frac{2b}{\mu^2\sigma^2N^{1/2}} \leq b_2.$$

Let C, a and α' be determined through Corollary 4.10.12 with $\gamma_0 = \inf_N \xi$. Take $r > 0$ such that $\alpha' - ar = -b_1$ and then take a_1 such that

$$a_1 \geq 2^{3/2} r \sup_N \mu(N)\sigma(\sqrt{N})N^{-1/4}.$$

Take $a_2 = \{8, 2C\}$. Then by rewriting (4.233)

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&\frac{a_2}{2}(\sqrt{N}L/2)^{-b_1}e^{b_2z^2/L}.\end{aligned}$$

The last step is to change to coefficient of $B_L(t)$ to 1. From the asymptotic of μ and σ, there exists a constant C_0 such that for all N,

$$|2^{-1/2}\mu\sigma N^{1/4} - 1| \leq C_0 N^{-1/2}.$$

We compute that

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|C_0N^{-1/2}B_L(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ =&\mathbb{P}\left(\sup_{0\leq u\leq 1}|B_1(u)| \geq 2^{-1/2}a_1C_0^{-1}N^{1/4}L^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&4\exp\left(-\frac{1}{2}\cdot\frac{1}{2}a_1^2C_0^{-2}N^{1/2}L^{-1}(\log(\sqrt{N}L/2)^2)\right)\\ \leq&4(\sqrt{N}L/2)^{-\frac{1}{4}\cdot a_1^2C_0^{-2}N^{1/2}L^{-1}}.\end{aligned}$$

By taking N large enough depending on L and b_1, we can arrange

$$4(\sqrt{N}L/2)^{-\frac{1}{4}\cdot a_1^2C_0^{-2}N^{1/2}L^{-1}} \leq \frac{a_2}{2}(\sqrt{N}L/2)^{-b_1}.$$

Thus

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq a_1(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ &+\mathbb{P}\left(\sup_{0\leq u\leq L}\left|C_0N^{-1/2}B_L(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&a_2(\sqrt{N}L/2)^{-b_1}e^{b_2z^2/L}.\end{aligned}$$

This finishes the verification of Assumption A4 and the proof of Theorem 4.10.11 is completed. □

Chapter 4

KMT coupling for random walk bridges

4.1 General setup

In this section we describe the general setting of a random walk bridge that we consider and the specific assumptions we make about it. Our discussion naturally splits into two parts, depending on whether the jump of the random walk is continuous or discrete. In each case we formulate a precise list of assumptions and present the statements we can prove for the corresponding random walk bridges that satisfy them. In the last part of this section we give a brief explanation of the significance of our assumptions.

4.1.1 Continuous random walk bridges

We start by fixing some notation. Suppose that X is a continuous random variable with density $f_X(\cdot)$ and X_i are i.i.d. random variables with density f_X. For $n \in \mathbb{N}$ we define $S_n := X_1 + \cdots + X_n$ and also let $f_n(x)$ be the density of S_n.

For any random variable X and $t \in \mathbb{R}$ we define

$$M_X(t) := \mathbb{E}\left[e^{tX}\right], \quad \phi_X(t) := \mathbb{E}\left[e^{itX}\right], \quad \Lambda(t) := \log M_X(t), \quad \Lambda^*(t) := \sup_{x \in \mathbb{R}}\{tx - \Lambda(x)\}. \tag{4.1}$$

Let $\mathcal{D}_\Lambda := \{x : \Lambda(x) < \infty\}$ and $\mathcal{D}_{\Lambda^*} := \{x : \Lambda^*(x) < \infty\}$.

We make the following assumptions on the function $f_X(x)$.

Assumption C1. We assume that there are $\alpha \in [-\infty, \infty)$ and $\beta \in (\alpha, \infty]$ and that $f_X(x)$ is positive and continuous on (α, β) and zero outside this interval. In addition, we assume that $f_X(x)$ has a continuous extension to α if $\alpha > -\infty$ and to β if $\beta < \infty$.

Assumption C2. We assume that there is a $\lambda > 0$ such that $\mathbb{E}\left[e^{\lambda |X|}\right] < \infty$.

For each $n \geq 1$ we set $L_n = (n\alpha, n\beta)$, where α, β are as in Assumption C1. For $z \in L_n$ we let $S^{(n,z)} = \{S_m^{(n,z)}\}_{m=0}^n$ denote the process with the law of $\{S_m\}_{m=0}^n$ conditioned so that $S_n = z$. We call this process a *continuous random walk bridge* between the points $(0,0)$ and (n,z). Notice that this law is well-defined by Assumption C1. As a natural extension of this definition we define $S_t^{(n,z)}$ for non-integer t by linear interpolation. In addition, we will denote the density of $S_m^{(n,z)}$ by $f_{m,n-m}(\cdot|z)$.

If f_X satisfies Assumption C2 then $\mathcal{D}_\Lambda$ contains a neighborhood of 0. In addition, it is easy to see that $\mathcal{D}_\Lambda$ is a connected set and hence an interval. We denote (A_Λ, B_Λ) the interior of $\mathcal{D}_\Lambda$ where $A_\Lambda \in [-\infty, -\lambda]$ and $B_\Lambda \in [\lambda, \infty]$. We isolate some properties for the functions in (4.1) under the above assumptions in the following lemma.

Lemma 4.1.1. *Suppose that X is a random variable with density f_X, which satisfies Assumptions C1 and C2. Then $M_X(u)$ has an analytic continuation to the vertical strip $D := \{z : A_\Lambda < Re(z) < B_\Lambda\}$. Moreover, $\Lambda(\cdot)$ is a smooth function on (A_Λ, B_Λ) and $\Lambda''(x) > 0$ for all $x \in (A_\Lambda, B_\Lambda)$.*

Proof. Let $[a_n, b_n]$ be such that a_n strictly decreases to α and b_n strictly increases to β. For each $z \in D$ and $x \in (\alpha, \beta)$ we define $F(z,x) = e^{xz} f_X(x)$ and note that $F(z,x)$ is holomorphic in z for each x and continuous on $D \times [a_n, b_n]$. It follows from [SS03, Theorem 2.5.4] that the function

$$g_n(z) = \int_{a_n}^{b_n} F(z,x)dx$$

is holomorphic on D. If K is a compact subset of D, and $z \in K$ we note that

$$g(z) := \int_\alpha^\beta e^{xz} f_X(x)dx$$

is well defined because

$$\int_\alpha^\beta |e^{xz}| f_X(x)dx = \int_\alpha^\beta e^{xRe(z)} f_X(x)dx = M_X(Re(z)) < \infty.$$

which is true as $Re(z) \in (A_\Lambda, B_\Lambda)$.

Note that there is $[c,d] \subset (\alpha, \beta)$ such that if $z \in K$ then $Re(z) \in [c,d]$. In particular, we see that $e^{xRe(z)} \leq e^{cx} + e^{dx}$ and so by the dominated convergence theorem with dominating function $f_X(x) \cdot [e^{cx} + e^{dx}]$ we get that

$$\lim_{n \to \infty} g_n(z) = g(z),$$

where the convergence is uniform over K. It follows from [SS03, Theorem 2.5.2] that $g(z)$ is holomorphic in D. Clearly, $g(z) = M_X(z)$ when $z \in (A_\Lambda, B_\Lambda)$, which proves the first part of the lemma.

One can use further applications of the dominated convergence theorem to show that the derivatives of $g(z)$ are given by

$$g^{(n)}(z) = \int_{\alpha}^{\beta} \left[\frac{d^n}{dz^n} e^{xz} \right] f_X(x) dx = \int_{\alpha}^{\beta} x^n e^{xz} f_X(x) dx,$$

and the latter integral is absolutely convergent for $Re(z) \in (A_\Lambda, B_\Lambda)$. For example, see [Mat]. We next observe that for $x \in (A_\Lambda, B_\Lambda)$, using the continuity and positivity of f_X, we know that $g(x) > 0$ and so $\Lambda(x) = \log[g(x)]$ is a smooth function on (A_Λ, B_Λ). From the Chain rule, we see that

$$\Lambda''(y) = \frac{g''(y)g(y) - [g'(y)]^2}{g^2(y)} = \frac{1}{2g^2(y)} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} e^{(x_1+x_2)y} \left[x_1^2 + x_2^2 - 2x_1x_2 \right] f_X(x_1) f_X(x_2) dx_1 dx_2,$$

which is clearly positive. This suffices for the proof. □

If f_X satisfies Assumptions C1 and C2 then in view of Lemma 4.1.1 we know that $\Lambda'(x)$ is a strictly increasing function on (A_Λ, B_Λ). We let (A^*, B^*) denote the image of (A_Λ, B_Λ) under the map $\Lambda'(\cdot)$. In addition, we write $M_X(u)$ for all $u \in D = \{z \in \mathbb{C} : A_\Lambda < Re(z) < B_\Lambda\}$ to mean the (unique) analytic extension of $M_X(x)$ to D afforded by Lemma 4.1.1.

Assumption C3. We assume that the function $\Lambda(\cdot)$ is lower semi-continuous on $\mathbb{R}$.

Lemma 4.1.2. *Suppose that X is a random variable with density f_X, which satisfies Assumptions C1-C3. Then $(\alpha, \beta) \subset (A^*, B^*) \subset \mathcal{D}_{\Lambda^*}$ and for all $y \in (A^*, B^*)$ we have $\Lambda^*(y) = \eta y - \Lambda(\eta)$, where $\eta = (\Lambda')^{-1}(y)$.*

Proof. By Lemma 4.1.1 we know that $\Lambda'(\cdot)$ is a strictly increasing smooth function from (A_Λ, B_Λ) to (A^*, B^*), which implies that $(\Lambda')^{-1}(\cdot)$ is also a smooth increasing function from (A^*, B^*) to (A_Λ, B_Λ). The statements $(A^*, B^*) \subset \mathcal{D}_{\Lambda^*}$ and $\Lambda^*(y) = \eta y - \Lambda(\eta)$ for all $y \in (A^*, B^*)$ follow from [DZ, Lemma 2.2.5]. In the remainder we show that $(\alpha, \beta) \subset (A^*, B^*)$.

Let $z \in (\alpha, \beta)$ and suppose that $\varepsilon > 0$ is such that $(z - \varepsilon, z + \varepsilon) \subset (\alpha, \beta)$. Suppose first that $A_\Lambda > -\infty$. Then by Assumption C3, we know that $\liminf_{x_n \to A_\Lambda} \Lambda(x_n) = \infty$. This implies that

$$\lim_{x_n \to A_\Lambda} z x_n - \Lambda(x_n) = -\infty.$$

Conversely, if $A_\Lambda = -\infty$ and $x_n \to A_\Lambda$ then

$$\limsup_{x_n \to A_\Lambda} z x_n - \Lambda(x_n) = \limsup_{x_n \to A_\Lambda} z x_n - \log \left[\mathbb{E} \left[e^{x_n X} \right] \right] \leq$$

$$\limsup_{x_n \to A_\Lambda} z x_n - \log \left[e^{x_n (z - \varepsilon/2)} \cdot \mathbb{P}(X \in [z - \varepsilon, z - \varepsilon/2]) \right] \leq \frac{x_n \varepsilon}{2} - \log(\mathbb{P}(X \in [z - \varepsilon, z - \varepsilon/2])) = -\infty.$$

Similar considerations show that $\lim_{x_n \to B_\Lambda} z x_n - \Lambda(x_n) = -\infty$.

By Lemma 4.1.1 $zx - \Lambda(x)$ is smooth in (A_Λ, B_Λ) and from the above we conclude that its maximum is achieved at a point $x_z \in (A_\Lambda, B_\Lambda)$ with $0 = \frac{d}{dx}[zx - \Lambda(x)] = z - \Lambda'(x_z)$. This shows that $z \in (A^*, B^*)$.

□

Assumption C4. We assume that for every $B_\Lambda > t > s > A_\Lambda$ there exist positive constants $K_1(s,t)$ and $p_{s,t} > 0$ such that $|M_X(z)| \leq \frac{K_1(s,t)}{(1+|Im(z)|)^{p_{s,t}}}$, provided $s \leq Re(z) \leq t$.

Assumption C5. We suppose that there are constants $L, D, d > 0$ such that $f_X(x) \leq L$ for all $x \in \mathbb{R}$ and at least one of the following statements holds

$$1.\ f_X(x) \leq De^{-dx^2} \text{ for all } x \geq 0 \text{ or } 2.\ f_X(x) \leq De^{-dx^2} \text{ for all } x \leq 0. \tag{4.2}$$

Assumption C6. We assume that there are functions $\hat{C} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and $\hat{a} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that the following holds. For all $n \geq 1$, $z \in L_n$ and $\hat{b} > 0$ we have

$$\mathbb{E}\left[\exp\left(\hat{a}(\hat{b}) \max_{1\leq k\leq n} |S_k|\right) \Big| S_n = z\right] \leq \hat{C}(\hat{b}) \exp\left(\hat{b}(n + z^2/n)\right). \tag{4.3}$$

In the sequel we denote $u_z = (\Lambda')^{-1}(z)$, $\sigma_z^2 = \Lambda''(u_z)$ – these are well defined for densities f_X that satisfy Assumptions C1-C3 as follows from Lemmas 4.1.1 and 4.1.2. Using this notation we can formulate the main theorem we prove for continuous random walk bridges.

Theorem 4.1.3. *Suppose that X is a random variable whose density function f_X satisfies Assumptions C1-C6 and fix $p \in (\alpha, \beta)$. For every $b > 0$, there exist constants $0 < C, a, \alpha' < \infty$ (depending on b, p and the function $f_X(\cdot)$) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = \sigma_p^2$ and the family of processes $S^{(n,z)}$ for $z \in L_n$ such that*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha'(\log n)}e^{b|z-pn|^2/n}, \tag{4.4}$$

where $\Delta(n,z) = \Delta(n,z,B^\sigma,S^{(n,z)}) = \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \frac{t}{n}z - S_t^{(n,z)}\right|$.

In Section 4.1.3 we provide some explanation of the significance of Assumptions C1-C6.

4.1.2 Discrete random walk bridges

We start by fixing some notation. Suppose that X is a random variable such that $\mathbb{P}(X \in \mathbb{Z}) = 1$ and let $p_X(n) = \mathbb{P}(X = n)$ for $n \in \mathbb{Z}$ denote its probability mass function. We let X_i be an i.i.d. sequence of random variables with distribution function p_X. For $n \in \mathbb{N}$ we define $S_n = X_1 + \cdots + X_n$ and also let $p_n(\cdot)$ be the probability mass function of S_n.

Similarly to Section 4.1.1 we define

$$M_X(t) := \mathbb{E}\left[e^{tX}\right], \quad \phi_X(t) := \mathbb{E}\left[e^{itX}\right], \quad \Lambda(t) := \log M_X(t) \quad \Lambda^*(t) := \sup_{x\in\mathbb{R}}\{tx - \Lambda(x)\}. \tag{4.5}$$

Let $\mathcal{D}_\Lambda := \{x : \Lambda(x) < \infty\}$ and $\mathcal{D}_{\Lambda^*} := \{x : \Lambda^*(x) < \infty\}$.

We make the following assumptions on the function $p_X(x)$.

Assumption D1. We assume that $p_X(x)$ has a single interval of support, i.e. $I = \{x \in \mathbb{Z} : p_X(x) > 0\} = (\alpha - 1, \beta + 1) \cap \mathbb{Z}$ for some $\alpha \in \mathbb{Z} \cup \{-\infty\}$ and $\beta \in ((\alpha, \infty] \cap \mathbb{Z}) \cup \{\infty\}$.

Assumption D2. We assume that there is a $\lambda > 0$ such that $\mathbb{E}\left[e^{\lambda|X|}\right] < \infty$.

For each $n \geq 1$ we set $L_n = (n\alpha - 1, n\beta + 1) \cap \mathbb{Z}$, where α, β are as in Assumption D1. For $z \in L_n$ we let $S^{(n,z)} = \{S_m^{(n,z)}\}_{m=0}^n$ denote the process with the law of $\{S_m\}_{m=0}^n$ conditioned so that $S_n = z$. We call this process a *discrete random walk bridge* between the points $(0, 0)$ and (n, z). Notice that this law is well-defined by Assumption D1. As a natural extension of this definition we define $S_t^{(n,z)}$ for non-integer t by linear interpolation. In addition, we will denote the distribution function of $S_m^{(n,z)}$ by $p_{m,n-m}(\cdot|z)$.

If p_X satisfies Assumption D2 then $\mathcal{D}_\Lambda$ contains a neighborhood of 0. In addition, it is easy to see that $\mathcal{D}_\Lambda$ is a connected set and hence an interval. We denote (A_Λ, B_Λ) the interior of $\mathcal{D}_\Lambda$ where $A_\Lambda \in [-\infty, -\lambda]$ and $B_\Lambda \in [\lambda, \infty]$. We isolate some properties for the functions in (4.5) under the above assumptions in the following lemma.

Lemma 4.1.4. *Suppose that X is a random variable whose distribution function p_X satisfies Assumptions D1 and D2. Then $M_X(u)$ has an analytic continuation to the vertical strip $D := \{z : A_\Lambda < Re(z) < B_\Lambda\}$. Moreover, $\Lambda(\cdot)$ is a smooth function on (A_Λ, B_Λ) and $\Lambda''(x) > 0$ for all $x \in (A_\Lambda, B_\Lambda)$.*

Proof. The proof is analogous to that of Lemma 4.1.1. □

If p_X satisfies Assumptions D1 and D2 then in view of Lemma 4.1.4 we know that $\Lambda'(x)$ is a strictly increasing function on (A_Λ, B_Λ). We let (A^*, B^*) denote the image of (A_Λ, B_Λ) under the map $\Lambda'(\cdot)$. In addition, we write $M_X(u)$ for all $u \in D = \{z \in \mathbb{C} : A_\Lambda < Re(z) < B_\Lambda\}$ to mean the (unique) analytic extension of $M_X(x)$ to D afforded by Lemma 4.1.4.

Assumption D3. We assume that the function $\Lambda(\cdot)$ is lower semi-continuous on $\mathbb{R}$.

Lemma 4.1.5. *Suppose that X is a random variable whose distribution function p_X satisfies satisfies Assumptions D1-D3. Then $(\alpha, \beta) \subset (A^*, B^*) \subset \mathcal{D}_{\Lambda^*}$ and for all $y \in (A^*, B^*)$ we have $\Lambda^*(y) = \eta y - \Lambda(\eta)$, where $\eta = (\Lambda')^{-1}(y)$. Furthermore, $\Lambda^*(x)$ is lower semi-continuous. If $\alpha > -\infty$ then $\alpha \in \mathcal{D}_{\Lambda^*}$ and $\Lambda^*(\alpha) = -\log p_X(\alpha)$. Similarly, if $\beta < \infty$ then $\beta \in \mathcal{D}_{\Lambda^*}$ and $\Lambda^*(\beta) = -\log p_X(\beta)$.*

Proof. By Lemma 4.1.4 we know that $\Lambda'(\cdot)$ is a strictly increasing smooth function from (A_Λ, B_Λ) to (A^*, B^*), which implies that $(\Lambda')^{-1}(\cdot)$ is also a smooth increasing function from (A^*, B^*) to (A_Λ, B_Λ). The statements $(A^*, B^*) \subset \mathcal{D}_{\Lambda^*}$, $\Lambda^*(y) = \eta y - \Lambda(\eta)$ for all $y \in (A^*, B^*)$ and the lower semi-continuity of Λ^* follow from [DZ, Lemma 2.2.5]. We next show that $(\alpha, \beta) \subset (A^*, B^*)$.

Let $z \in (\alpha, \beta)$ and fix $k, m \in \mathbb{Z}$ such that $\alpha \leq k < z$ and $z > m \geq \beta$. Suppose first that $A_\Lambda > -\infty$. Then by Assumption D3, we know that $\liminf_{x_n \to A_\Lambda} \Lambda(x_n) = \infty$. This implies that

$$\lim_{x_n \to A_\Lambda} zx_n - \Lambda(x_n) = -\infty.$$

Conversely, if $A_\Lambda = -\infty$ and $x_n \to A_\Lambda$ then

$$\limsup_{x_n \to A_\Lambda} zx_n - \Lambda(x_n) = \limsup_{x_n \to A_\Lambda} zx_n - \log\left[\mathbb{E}\left[e^{x_n X}\right]\right] \leq$$

$$\limsup_{x_n \to A_\Lambda} zx_n - \log\left[e^{x_n k} \cdot \mathbb{P}(X = k)\right] \leq x_n(z - k) - \log(p_X(k)) = -\infty.$$

Similar considerations show that $\lim_{x_n \to B_\Lambda} zx_n - \Lambda(x_n) = -\infty$.

By Lemma 4.1.4 $zx - \Lambda(x)$ is smooth in (A_Λ, B_Λ) and from the above we conclude that its maximum is achieved at a point $x_z \in (A_\Lambda, B_\Lambda)$ with $0 = \frac{d}{dx}[zx - \Lambda(x)] = z - \Lambda'(x_z)$. This shows that $z \in (A^*, B^*)$.

Next suppose that $\alpha > -\infty$. Then we have $A_\Lambda = -\infty$. We have for any $x \in \mathbb{R}$ that

$$\alpha x - \Lambda(x) \leq \alpha x - \log\left[\mathbb{E}\left[e^{xX}\right]\right] \leq \alpha x - \log\left[e^{\alpha x} p_X(\alpha)\right] \leq -\log p_X(\alpha).$$

Furthermore, we have

$$\liminf_{x_n \to -\infty} \alpha x_n - \Lambda(x_n) \geq \liminf_{x_n \to -\infty} \alpha x_n - \log\left[e^{\alpha x_n} p_X(\alpha) + e^{(\alpha+1)x_n} \cdot (1 - \mathbb{P}(X = \alpha))\right] =$$

$$\liminf_{x_n \to -\infty} -\log\left[p_X(\alpha) + e^{x_n} \cdot (1 - p_X(\alpha))\right] = -\log p_X(\alpha).$$

Thus $\alpha \in \mathcal{D}_{\Lambda^*}$ and $\Lambda^*(\alpha) = -\log p_X(\alpha)$. Analogous arguments prove the statement for $\beta < \infty$. ☐

Assumption D4. We suppose that there are constants $D, d > 0$ such that at least one of the following statements holds

$$1.\ p_X(x) \leq De^{-dx^2} \text{ for all } x \geq 0 \text{ or } 2.\ p_X(x) \leq De^{-dx^2} \text{ for all } x \leq 0. \tag{4.6}$$

Assumption D5. We assume that there are functions $\hat{C} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and $\hat{a} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that the following holds. For all $n \geq 1$, $z \in L_n$ and $\hat{b} > 0$ we have

$$\mathbb{E}\left[\exp\left(\hat{a}(\hat{b}) \max_{1 \leq k \leq n} |S_k|\right) \Big| S_n = z\right] \leq \hat{C}(\hat{b}) \exp\left(\hat{b}(n + z^2/n)\right). \tag{4.7}$$

In the sequel we denote $u_z = (\Lambda')^{-1}(z)$, $\sigma_z^2 = \Lambda''(u_z)$ – these are well defined for distribution functions p_X that satisfy Assumptions D1-D3 as follows from Lemmas 4.1.4 and 4.1.5. Using this notation we can formulate the main theorem we prove for discrete random walk bridges.

Theorem 4.1.6. *Suppose that X is a random variable whose probability distribution function p_X satisfies Assumptions D1-D5 and fix $p \in (\alpha, \beta)$. For every $b > 0$, there exist constants $0 < C, a, \alpha' < \infty$ (depending on b, p and the function $p_X(\cdot)$) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = \sigma_p^2$ and the family of processes $S^{(n,z)}$ for $z \in L_n$ such that*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha'(\log n)}e^{b|z-pn|^2/n}, \tag{4.8}$$

where $\Delta(n,z) = \Delta(n,z,B^\sigma,S^{(n,z)}) = \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \frac{t}{n}z - S_t^{(n,z)}\right|$.

In Section 4.1.3 we provide some explanation of the significance of Assumptions D1-D5.

4.1.3 Significance of assumptions

Let us explain the role of the different Assumptions C1-C6 and D1-D5 that we made in the previous sections. Assumption C1 (resp. D1) ensures that the law of the random walk bridge $S^{(n,z)}$ is well defined. Without the assumption that the support of $f_X(\cdot)$ (resp. $p_X(\cdot)$) is a single interval one runs into the possibility of conditioning on events of zero probability (in the density sense for the continuous bridges). It is possible to relax this condition, by requiring that sufficiently many convolutions of $f_X(\cdot)$ (resp. p_X) with itself satisfy this assumption, but we will assume that $f_X(\cdot)$ (resp. p_X) satisfies it instead, as this somewhat simplifies our discussion.

Assumptions C2 and C4 (resp. D2) are essentially the same as those used in [KMT75, KMT76]. Since our result is related to [KMT75, KMT76, Theorem 1] it is natural to have these assumptions.

In the process of proving Theorem 4.1.3 (resp. Theorem 4.1.6) we will require detailed estimates on the conditional distributions $f_{m,n}(\cdot|z)$ (resp. $p_{m,n}(\cdot|z)$) for $m, n \geq 1$, which in turn would require estimates on $f_{n+m}(z)$ (resp. $p_{n+m}(z)$). Consequently, we will require large deviation estimates for the latter densities, which involve the rate function Λ. For this reason, it will be convenient for us to assume that Λ is lower semi-continuous, which is Assumption C3 (resp. D3).

Assumptions C5 and C6 (resp. D4 and D5) are more technical and more directly tied to the particular approach we take to proving Theorem 4.1.3 (resp. Theorem 4.1.6). It is possible that one can relax (or entirely remove) some of these assumptions, but one would need to implement different ideas than the ones we use. Our argument goes through a comparison of the distribution $f_{n,n}(\cdot|z)$ (resp. $p_{n,n}(\cdot|z)$) with a suitable Gaussian density, for which it is useful to know that $f_{n,n}(\cdot|z)$ (resp. $p_{n,n}(\cdot|z)$) has Gaussian tails – this is the essence of Assumption C5 (resp. D5). Our proof of Theorems 4.1.3 and 4.1.6 relies on an inductive argument on n. When we go from $n/2$ to n, Assumptions C1-C5 (resp. D1-D4) are enough to complete the induction step, provided z is close to the reference slope pn, but for points that are macroscopically away from this point, we require the estimates in Assumption C6 (resp. D5). Later in Section 4.6 we provide several easy to check conditions that imply Assumption C6 (resp. D5).

We want to emphasize that it is not enough to assume Assumptions C1-C5 (resp D1-D4), and obtain Theorem 4.1.3 (resp. Theorem 4.1.6) as we demonstrate in Section 4.6.2, by providing a counterexample. The counterexample is for the discrete setting of our problem but can be naturally adapted to the continuous one. This indicates that one should make additional assumptions on $f_X(\cdot)$ (resp. $p_X(\cdot)$) and our choice of Assumption C6 (resp. D5) is made because it is somewhat natural and satisfied by the distributions in the particular applications that we have in mind.

We end this section with the following remark.

Remark 4.1.7. *In the process of establishing the results necessary for the proofs of Theorems 4.1.3 and 4.1.6 we will obtain numerous constants that depend on the jump distribution f_X in the continuous and p_X in the discrete case. Some of the applications we have in mind are to situations when the jump distribution depends on a parameter that is allowed to vary in some (possibly infinite) interval. Consequently, we are interested in quantifying the dependence of our coupling constants on the functions f_X and p_X, through various observables of these distributions. In words, we are interested in showing that the coupling constants a, C and α' in Theorems 4.1.3 and 4.1.6 can be taken uniform even if f_X or p_X depend on some parameter so long as one has uniform control of several observables for f_X or p_X that will be made explicit in later sections. These more quantified versions of Theorems 4.1.3 and 4.1.6 can be found in Section 4.6 as Theorems 4.5.3 and 4.5.6 respectively. We provide an example of the situation described in this remark in Section 4.7.3.*

4.2 Midpoint distribution: Continuous case

We continue with the same notation as in Section 4.1.1. To ease the notation a bit we will write M, ϕ and Λ instead of M_X, ϕ_X and Λ_X. Let $f_{n,m}(x|y)$ be the density of S_m conditioned on $S_{n+m} = y$. Our goal in this section is to obtain several asymptotic statements about the distribution $f_{m,n}(\cdot|(m+n)z)$ and we start by analyzing $f_N(Nz)$.

4.2.1 Asymptotics of $f_N(Nz)$

In this section we assume that $f_X(\cdot)$ satisfies Assumptions C1-C4. For a fixed $z \in (A^*, B^*)$ we define

$$G_z(u) = \Lambda(u) - z \cdot u, \text{ for } u \in (A_\Lambda, B_\Lambda). \tag{4.9}$$

Definition 4.2.1. *Suppose that we are given $s, t \in \mathbb{R}$ such that $\alpha < s < t < \beta$, where α, β are as in Assumption C1. In addition, we denote $S = (\Lambda')^{-1}(s)$ and $T = (\Lambda')^{-1}(t)$ – these quantities are well-defined in view of Lemma 4.1.2. By Lemma 4.1.1 there exist $\infty > M_{s,t} \geq m_{s,t} > 0$ such that $M_{s,t} \geq \Lambda''(y) \geq m_{s,t}$ for all $y \in [S, T]$. We can pick $\delta_{s,t} > 0$ sufficiently small (depending on s, t and $f_X(\cdot)$) so that*

1. *If $D_{\delta_{s,t}}(S,T) := \{z \in \mathbb{C} : d(z, [S,T]) < \delta_{s,t}\}$ then $\overline{D}_{\delta_{s,t}}(S,T) \subset \{z \in \mathbb{C} : A_\Lambda < Re(z) < B_\Lambda\}$;*

2. *Re*$[M_X(u)] > 0$ *for all* $u \in \overline{D}_{\delta_{s,t}}(S,T)$;

3. $\delta_{s,t} < 1/2$;

4. $8\delta_{s,t} \cdot |\log(M_X(u))| < m_{s,t}$ *for all* $u \in \overline{D}_{\delta_{s,t}}(S,T)$.

Definition 4.2.2. *Suppose that we are given* $s,t \in \mathbb{R}$ *such that* $\alpha < s < t < \beta$*, where* α, β *are as in Assumption C1. In view of Assumption C4 there exists a constant* $K_{s,t} \geq 1$ *sufficiently large (depending on* s,t *and* $f_X(\cdot)$*) and* $p_{u_s,u_t} > 0$ *so that for every* $z \in [s,t]$ *we have*

$$\left| M(u_z + iy) \cdot e^{-z(u_z+iy)} e^{-G_z(u_z)} \right| \leq \frac{K_{s,t}}{(1+|y|)^{p_{u_s,u_t}}}.$$

Definition 4.2.3. *Suppose that we are given* $s,t \in \mathbb{R}$ *such that* $\alpha < s < t < \beta$*, where* α, β *are as in Assumption C1. Suppose that* $\delta_{s,t}$ *and* $K_{s,t}$ *satisfy the conditions in Definitions 4.2.1 and 4.2.2. Denote* $\varepsilon_{s,t} = \delta_{s,t}^4$ *and* $R_{s,t} = [4K_{s,t}]^{2/p_{s,t}}$*. Then we can find* $q_{s,t} \in (0,1)$ *(depending on* $s, t, \delta_{s,t}, K_{s,t}$ *and* $f_X(\cdot)$*)such that for every* $z \in [s,t]$ *and* $y \in [\varepsilon_{s,t}, R_{s,t}]$ *we have*

$$\left| \mathbb{E}\left[e^{(u_z+iy)X} \right] \right| e^{-zu_z} e^{-G_z(u_z)} \leq q_{s,t}.$$

To see why the above is true, notice that

$$\left| \mathbb{E}\left[e^{(u_z+iy)X} \right] \right| e^{-zu_z} e^{-G_z(u_z)} < \mathbb{E}\left[\left| e^{(u_z+iy)X} \right| \right] e^{-zu_z} e^{-G_z(u_z)} = 1,$$

where the above inequality is strict for any $y \neq 0$ *as the contrary would imply* $X \in 2\pi y^{-1} \cdot \mathbb{Z}$ *almost surely, which is not true. This combined with the continuity of* $\mathbb{E}\left[e^{(u_z+iy)X}\right]$ *in* y *and* z *ensures the existence of* $q_{s,t}$ *with the desired properties.*

We are interested in proving the following statement.

Proposition 4.2.4. *Suppose that* f_X *satisfies Assumptions C1-C4. Fix* $\beta > t > s > \alpha$ *and* $z \in [s,t]$*. Then there exists* $N_0 \in \mathbb{N}$ *such that if* $N \geq N_0$ *one has*

$$f_N(Nz) = \frac{1}{\sqrt{2\pi N}\sigma_z} \cdot \exp\left(NG_z(u_z) + \delta_1(z,N)\right), \text{ where } \delta_1(z,N) = O(N^{-1/2}). \tag{4.10}$$

The number N_0 *and the constant in the big O notation depend on* f_X, s *and* t *only through the constants* $\delta_{s,t}, m_{s,t}, K_{s,t}$ *and* $q_{s,t}$ *as in Definitions 4.2.1, 4.2.2 and 4.2.3 in addition to* $p_{\min(u_s,0),\max(u_t,0)}$ *from Assumption C4.*

Proof. From Assumption C4 and [Dur, Theorem 3.3.5] we know that for N sufficiently large (specifically it suffices to take $N > p^{-1}_{\min(u_s,0),\max(u_t,0)}$) then

$$f_N(Nz) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iyNz} \left(\phi(y)\right)^N dy.$$

Performing the change of variables $u = iy$ we see that

$$f_N(Nz) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} M^N(u)e^{-uNz}du. \tag{4.11}$$

Let us shift the u contour in (4.11) to the vertical contour passing through u_z. In view of Lemma 4.1.1, we do not pass any poles in the process of deformation and so by Cauchy's theorem the value of the integral remains unchanged. The decay necessary to deform the contours near $\pm i\infty$ comes from Assumption C4 and our assumption that N is sufficiently large. The result is

$$f_N(Nz) = \frac{e^{NG_z(u_z)}}{2\pi i}\int_{u_z-i\infty}^{u_z+i\infty} M(u)^N e^{-uNz}e^{-NG_z(u_z)}du. \tag{4.12}$$

For the given s,t as in the statement of the proposition we define $\delta_{s,t}, m_{s,t}, K_{s,t}, \varepsilon_{s,t}, R_{s,t}$ and $q_{s,t}$ as in Definitions 4.2.1, 4.2.2 and 4.2.3. To ease notation we will drop s,t from the notation of these quantities. We will also denote by $C_{s,t}$ the supremum of $|\log(M(u))|$ as u varies over $\overline{D_\delta}$. Notice that by construction we have

$$\varepsilon < \delta/2 \text{ and } \varepsilon \cdot 8C_{s,t}\cdot\delta^{-3} < m.$$

From (4.12) we have $f_N(Nz) = (I) + (II),$ where

$$\begin{aligned}(I) = \frac{e^{NG_z(u_z)}}{2\pi i}\int_{u_z-i\varepsilon}^{u_z+i\varepsilon} e^{N[G_z(u)-G_z(u_z)]}du, (II) = \frac{e^{NG_z(u_z)}}{2\pi i}\int_{u_z-i\infty}^{u_z-i\varepsilon}\left[M(u)e^{-uz}e^{-G_z(u_z)}\right]^N du\\ + \frac{e^{NG_z(u_z)}}{2\pi i}\int_{u_z+i\varepsilon}^{u_z+i\infty}\left[M(u)e^{-uz}e^{-G_z(u_z)}\right]^N du.\end{aligned} \tag{4.13}$$

We will first obtain estimates on (I), which will require analyzing the power series expansion of $G_z(u_z + ir) - G_z(u_z)$ around the point u_z. Note that by definition

$$G_z(u_z + ir) - G_z(u_z) = -\frac{r^2\sigma_z^2}{2} + \sum_{n=3}^{\infty}\frac{\Lambda^{(n)}(u_z)}{n!}(ir)^n.$$

From the Cauchy inequalities [SS03, Corollary 2.4.3] and our choice of ε we conclude that for $|r| \leq \varepsilon$

$$\left|G_z(u_z + ir) - G_z(u_z) + \frac{r^2\sigma_z^2}{2}\right| \leq C_{s,t}|r|^3\sum_{n=3}^{\infty}\frac{|\varepsilon|^{n-3}}{\delta^n} \leq 2\delta^{-3}C_{s,t}|r|^3 =: C(s,t,\delta)|r|^3. \tag{4.14}$$

Changing variables in (4.13) and using (4.14) we obtain

$$\frac{e^{NG_z(u_z)}}{2\pi\sqrt{N}}\int_{-\varepsilon N^{1/2}}^{\varepsilon N^{1/2}}\exp\left[-\frac{x^2\sigma_z^2}{2} - \frac{C(s,t,\delta)}{\sqrt{N}}|x|^3\right]dx \leq (I) \leq \frac{e^{NG_z(u_z)}}{2\pi\sqrt{N}}\int_{-\varepsilon N^{1/2}}^{\varepsilon N^{1/2}}\exp\left[-\frac{x^2\sigma_z^2}{2} + \frac{C(s,t,\delta)}{\sqrt{N}}|x|^3\right]dx.$$

Using the inequality $|e^A - 1| \leq |A|e^{|A|}$ for all $A \in \mathbb{R}$, we obtain

$$\left|(I) - \frac{e^{NG_z(u_z)}}{2\pi\sqrt{N}}\int_{-\varepsilon N^{1/2}}^{\varepsilon N^{1/2}}\exp\left[-\frac{\sigma_z^2x^2}{2}\right]dx\right| \leq \frac{e^{NG_z(u_z)}}{2\pi\sqrt{N}}\int_{-\varepsilon N^{1/2}}^{\varepsilon N^{1/2}}\frac{C(s,t,\delta)|x|^3}{\sqrt{N}}\exp\left[-\frac{\sigma_z^2x^2}{2} + \frac{C(s,t,\delta)}{\sqrt{N}}|x|^3\right]dx.$$

Notice that by our choice of ε we have for $|x| \leq \varepsilon N^{1/2}$ that

$$-\frac{\sigma_z^2 x^2}{2} + C(s,t,\delta)|x|^3 N^{-1/2} \leq -\frac{\sigma_z^2 x^2}{4},$$

which implies from above that

$$\left|(I) - \frac{e^{NG_z(u_z)}}{2\pi\sigma_z\sqrt{N}} \cdot \left(1 - 2\bar{\Phi}\left(\varepsilon\sqrt{N}\right)\right)\right| \leq \frac{e^{NG_z(u_z)}}{2\pi\sqrt{N}} \int_{\mathbb{R}} \frac{C(s,t,\delta)}{\sqrt{N}} |x|^3 \exp\left[-\frac{\sigma_z^2 x^2}{4}\right] dx, \tag{4.15}$$

where $\bar{\Phi}(x) = \mathbb{P}(Z > x)$ with Z being a Gaussian variable with mean zero and variance 1.

Using a simple change of variables we have

$$\int_{\mathbb{R}} |x|^3 \exp\left[-\frac{\sigma_z^2}{4} x^2\right] dx = \frac{4}{\sigma_z} \int_0^\infty y^3 e^{-y^2} dy = \frac{2}{\sigma_z}.$$

Combining the latter with the inequality $\bar{\Phi}(x) \leq 2e^{-x^2/2}$ for all $x \geq 0$ and (4.15) we get

$$\left|(I) - \frac{e^{NG_z(u_z)}}{2\pi\sigma_z\sqrt{N}}\right| \leq \frac{e^{NG_z(u_z)}}{2\pi\sigma_z\sqrt{N}} \cdot \left(\frac{2C(s,t,\delta)}{\sqrt{N}} + 4\exp\left(-\varepsilon N/2\right)\right) \tag{4.16}$$

We can now make N_0 sufficiently large so that for all $z \in [s,t]$ and $N \geq N_0$

$$(I) = \frac{e^{NG_z(u_z)}}{2\pi\sigma_z\sqrt{N}} \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]. \tag{4.17}$$

We next forcus on estimating (II). We first note by construction we have

$$\left|M(u_z + iy) \cdot e^{-z(u_z+iy)} e^{-G_z(u_z)}\right| \leq \frac{K}{(1+|y|)^p}, \text{ where}$$

$p = p_{\min(u_s,0),\max(u_t,0)}$ is as in Assumption C4. The latter implies that if $N \geq N_0 > 2/p$ we have

$$\int_{|y|>R} \left|M(u_z + iy) \cdot e^{-z(u_z+iy)} e^{-G_z(u_z)}\right|^N dy \leq 2K^N \frac{R^{1-pN}}{pN-1} \leq 2K^N R^{-pN/2} = 2 \cdot 4^{-N}. \tag{4.18}$$

Suppose next that $y \in [\varepsilon, R]$. Then by definition we have

$$\int_{\varepsilon \leq |y| \leq R} \left|M(u_z + iy) \cdot e^{-z(u_z+iy)} e^{-G_z(u_z)}\right|^N dy \leq 2Rq^N. \tag{4.19}$$

Combining (4.18) and (4.19) we get

$$|(II)| \leq \frac{e^{NG_z(u_z)}}{2\pi} \cdot [2Rq^N + 2 \cdot 4^{-N}] \leq \frac{e^{NG_z(u_z)}}{2\pi\sigma_z N}, \tag{4.20}$$

where the last inequality holds provided N_0 is sufficiently large and $N \geq N_0$. Combining (4.17) and (4.20) yields (4.10). □

4.2.2 Asymptotics of $f_{n,m}(\cdot|(m+n)z)$

We start with a useful definition.

Definition 4.2.5. *Suppose that $f_X(\cdot)$ satisfies Assumptions C1-C4 and that $\beta > t > s > \alpha$ are given. Then in view of Lemmas 4.1.1 and 4.1.2 we know that $F(z) := G_z(u_z)$ is smooth on (α, β) and so for each $k \geq 0$ exists $M_{s,t}^{(k)} > 0$ such that $|F^{(k)}(z)| \leq M_{s,t}^{(k)}$ for all $z \in [s,t]$.*

We have the following asymptotic estimate for $f_{n,m}(\cdot|(m+n)z)$.

Proposition 4.2.6. *Suppose that f_X satisfies Assumptions C1-C4. Fix s,t such that $\beta > t > s > \alpha$ and let N_0 be as in the statement of Proposition 4.2.4. Then there exists $M > 0$ such that the following holds. Suppose that $m,n \geq N_0$ are such that $|m-n| \leq 1$ and denote $N = n+m$. In addition, let z,x be such that $xN/n, (z-x)N/m, z \in [s,t]$. Then we have*

$$f_{n,m}(Nx|Nz) = \frac{2}{\sqrt{2\pi N}\sigma_z} \cdot \exp\left(-N \cdot \frac{4}{2\sigma_z^2}\left[x - \frac{z}{2}\right]^2 + \delta_2(N,x,z)\right), \tag{4.21}$$

where

$$|\delta_2(N,x,z)| \leq M \cdot \left(\frac{1}{\sqrt{N}} + N\left|x - \frac{z}{2}\right|^3\right). \tag{4.22}$$

The constant M depends on s,t and also on $f_X(\cdot)$, where the dependence on the latter is only through the constants in the statement of Proposition 4.2.4 and $M_{s,t}^{(3)}, M_{s,t}^{(4)}$ in Definition 4.2.5.

Proof. Set $\phi = \frac{m}{n}$ and $\psi = \frac{n}{m}$. From Proposition 4.2.4 we know that for $m,n \geq N_0$ we have

$$\begin{aligned} f_{n,m}(Nx|Nz) = \frac{f_n(Nx)f_m(N(z-x))}{f_N(Nz)} = \frac{f_n(n[xN/n])f_m(m[(z-x)N/m])}{f_N(Nz)} = \\ e^{N\left(\frac{F[x(1+\phi)]}{1+\phi} + \frac{F[(z-x)(1+\psi)]}{1+\psi} - F(z)\right)} \cdot \frac{2\sigma_z}{\sqrt{2\pi N}\sigma_{x(1+\phi)} \cdot \sigma_{(z-x)(1+\psi)}} \cdot \exp\left[O\left(\frac{1}{\sqrt{N}}\right)\right], \end{aligned} \tag{4.23}$$

where the constant in the big O notation depends on s,t and the constants in the statement of Proposition 4.2.4.

Notice that $F'(z) = \partial_z[\Lambda(z) - zu_z] = -u_z$, where the last equality used that $\Lambda'(u_z) = z$. In addition, differentiating the last expression shows that $\partial_z u_z = \frac{1}{\Lambda''(u_z)} = \frac{1}{\sigma_z^2}$. This means that $F''(z) = -\frac{1}{\sigma_z^2}$ and $F'(z) = -u_z$. This shows that F is a strictly concave function in z and its second derivative is bounded from above by $-1/M_{s,t}$ as in Definition 4.2.1.

Let us write $x = \frac{z}{1+\phi} + r$ and denote

$$h(r) := \frac{F(z + (1+\phi)r)}{1+\phi} + \frac{F(z - r(1+\psi))}{1+\psi} - F(z).$$

Then $h(0) = h'(0) = 0$ and

$$h''(r) = (1+\phi)F''(z+(1+\phi)r) + (1+\psi)F''(z+(1+\psi)r), \text{ hence } h''(0) = -\frac{2+\phi+\psi}{\sigma_z^2}.$$

Next we have

$$h'''(r) = (1+\phi)^2 F'''(z+(1+\phi)r) + (1+\psi)^2 F'''(z+(1+\psi)r),$$

In view of Definition 4.2.5 there exists a constant K depending only on $M_{s,t}^{(3)}$ such that $\left|h^{(3)}(r)\right| \leq K$, provided $z+(1+\phi)r, z+(1+\psi)r \in [s,t]$. Then we see that

$$e^{N\left(\frac{F[x(1+\phi)]}{1+\phi}+\frac{F[(z-x)(1+\psi)]}{1+\psi}-F(z)\right)} = \exp\left(Nh\left(x - \frac{z}{1+\phi}\right)\right) = \exp\left(-N\frac{2+\phi+\psi}{2\sigma_z^2}\left[x-\frac{z}{1+\phi}\right]^2 + O\left(N\left|x-\frac{z}{1+\phi}\right|^3\right)\right), \tag{4.24}$$

where the constant in the big O notation is just K.

We claim that

$$\frac{\sigma_z^2}{\sigma_{x(1+\phi)}\cdot\sigma_{(z-x)(1+\psi)}} = \exp\left[O\left(\frac{1}{\sqrt{N}} + N\left|x-\frac{z}{1+\phi}\right|^3\right)\right]. \tag{4.25}$$

Combining (4.23), (4.24) and (4.25) gives (4.21). In the remainder we establish (4.25).

Squaring the left side of (4.25) and taking logarithm gives

$$\log[-F''(x(1+\phi))] + \log[-F''((z-x)(1+\psi))] - 2\log[-F''(z)].$$

Let us set $x = \frac{z}{1+\phi} + r$ and denote

$$g(r) = \log[-F''(z+r(1+\phi))] + \log[-F''(z-r(1+\psi))] - 2\log[-F''(z)].$$

Then $g(0)=0$ and

$$g'(r) = -\frac{(1+\phi)F'''(z+r(1+\phi))}{F''(z+r(1+\phi))} + \frac{(1+\psi)F'''(z-r(1+\psi))}{F''(z+r(1+\psi))}.$$

This implies that

$$g'(0) = (\psi-\phi)\frac{F^{'''}(z)}{F''(z)}.$$

As discussed before $|F''(z)| \geq 1/M_{s,t}$ for all $z \in [s,t]$ and so we conclude that $|g'(0)| \leq \frac{K_2}{N}$ for some constant K_2 that depends on $s, t, M_{s,t}$ and $M_{s,t}^{(3)}$. On the other hand, it is easy to see that $|g''(r)| \leq K_3$ for some constant that depends on $s, t, M_{s,t}, M_{s,t}^{(3)}$ and $M_{s,t}^{(4)}$. This implies

$$|g(r)| \leq r \cdot \frac{K_2}{N} + r^2 K_3,$$

which implies that

$$\frac{\sigma_z^2}{\sigma_{x(1+\phi)}\cdot\sigma_{(z-x)(1+\psi)}} = \exp\left[O\left(\frac{1}{N}\left|x-\frac{z}{1+\phi}\right| + \left|x-\frac{z}{1+\phi}\right|^2\right)\right].$$

The latter inequality implies (4.25) and concludes the proof of the proposition. □

4.2.3 Tails of $f_{n,m}(\cdot|(m+n)z)$

In this section we will further assume that $f_X(\cdot)$ satisfies Assumption C5 and use that to deduce tail estimates for $f_{n,m}(\cdot|(m+n)z)$. We start with a couple of lemmas.

Lemma 4.2.7. *Suppose that f_X satisfies Assumption C5. Then for all $N \geq 1$*

$$f_N(x) \leq \begin{cases} W^N e^{-dN^{-1}x^2} \text{ for all } x \geq 0 \text{ if C5.1 holds and} \\ W^N e^{-dN^{-1}x^2} \text{ fro all } x \leq 0 \text{ if C5.2 holds,} \end{cases} \tag{4.26}$$

where $W = D\frac{\sqrt{\pi}}{\sqrt{d}} + 1 + D$.

Proof. By symmetry it is clearly enough to consider the case when C5.A.1 holds. Suppose that $C_1, C_2, c_1, c_2 > 0$ and h_1, h_2 are probability density functions such that

$$h_i(x) \leq C_i e^{-c_i x^2} \text{ for all } x \geq 0 \text{ and } i = 1, 2.$$

In addition, set $g(y) = \int_{\mathbb{R}} h_1(y-x)h_2(x)dx$ and $h_i^1(x) = h_i(x) \cdot \mathbf{1}_{x\geq 0}$ and $h_i^2 = h_i(x) \cdot \mathbf{1}_{x<0}$ for $i = 1, 2$. We thus obtain for $y \geq 0$

$$g(y) = \int_0^\infty h_1^1(y-x)h_2^1(x)dx + \int_0^\infty h_1^2(y-x)h_2^1(x)dx + \int_0^\infty h_1^1(x)h_2^2(y-x)dx \leq$$

$$C_1C_2\int_0^y e^{-c_1(y-x)^2}e^{-c_2x^2}dx + C_2\int_y^\infty h_1^2(y-x)e^{-c_2x^2}dx + C_1\int_y^\infty e^{-c_1x^2}h_2^2(y-x)dx.$$

Using that h_i are probability density functions we get

$$\int_y^\infty e^{-c_ix^2}h_j^2(y-x)dx \leq e^{-c_iy^2}.$$

Using that the convolution of two Gaussian densities is again a Gaussian density we get

$$\int_0^y e^{-c_1(y-x)^2}e^{-c_2x^2}dx \leq \int_{\mathbb{R}} e^{-c_1(y-x)^2}e^{-c_2x^2}dx = \frac{\sqrt{\pi}}{\sqrt{c_1+c_2}}\exp\left(-\frac{y^2c_1c_2}{c_1+c_2}\right). \tag{4.27}$$

Combining all of the above we get

$$g(y) \leq C_1C_2\frac{\sqrt{\pi}}{\sqrt{c_1+c_2}}\exp\left(-\frac{y^2c_1c_2}{c_1+c_2}\right) + C_2e^{-c_2y^2} + C_1e^{-c_1y^2}. \tag{4.28}$$

We now proceed to prove (4.26) by induction on N with base case $N = 1$, being true by assumption. Suppose the result holds true for N. Setting $h_1(x) = f_X(x)$ and $h_2(x) = f_N(x)$ and applying (4.28) we obtain for any $y \geq 0$ that

$$f_{N+1}(y) \leq \frac{DW^N\sqrt{\pi}}{\sqrt{d+d/N}}\exp\left(-\frac{y^2d(d/N)}{d+d/N}\right) + W^Ne^{-(d/N)y^2} + De^{-dy^2} \leq$$

$$\leq W^N\left[D\frac{\sqrt{\pi}}{\sqrt{d}} + 1 + D\right]e^{-d(N+1)^{-1}y^2} = W^{N+1}e^{-d(N+1)^{-1}y^2}.$$

This proves (4.26) for the case $N + 1$ and the general result proceeds by induction on N. ☐

Lemma 4.2.8. *Suppose that f_X satisfies Assumption C5 and $\alpha > -\infty$ or $\beta < \infty$ or both. Then for all $N \geq 1$*

$$f_N(x) \leq \begin{cases} \frac{L^N}{(N-1)!}(x - N\alpha)^{N-1} \text{ for all } x > N\alpha \text{ if } \alpha > -\infty \\ \frac{L^N}{(N-1)!}(N\beta - x)^{N-1} \text{ for all } x < N\beta \text{ if } \beta < \infty \ . \end{cases}$$

Proof. By symmetry it is clearly enough to consider the case $\alpha > -\infty$ and prove the first statement of the lemma. By shifting X by $-\alpha$ we may assume that $\alpha = 0$. We proceed by induction on N with base case $N = 1$ being true by assumption. We now suppose that the result holds true for N and let $y > 0$. Then

$$f_{N+1}(y) = \int_0^y f_N(x) f_1(y - x) \leq \int_0^y \frac{L^N}{(N-1)!} x^{N-1} \cdot L dx = \frac{L^{N+1}}{N!} y^N.$$

This proves the induction step and the general result follows by induction. ☐

We next summarize a couple of parameter choices for future use.

Definition 4.2.9. *Suppose that $f_X(\cdot)$ satisfies Assumptions C1-C5. Fix t, s such that $\beta > t > s > \alpha$. Then in view of Proposition 4.2.4 we can find $C_1 > 1$ sufficiently large depending on the constants in that proposition and $M^{(0)}_{s,t}$ in Definition 4.2.5 so that*

$$C_1^{-N} \leq f_N(Nz)$$

for all $z \in [s, t]$ and $N \geq N_0$ (where N_0 is as in the statement of Proposition 4.2.4).

We can also find $\varepsilon_1 > 0$ sufficiently small so that $48C_1^2 L \cdot \varepsilon_1 \leq 1$, $s \geq \alpha + 3\varepsilon_1$ and $t \leq \beta - 3\varepsilon_1$, where L is as in Assumption C5.

We can also find $R_1 > 1$ sufficiently large so that

$$[s, t] \subset [-R_1, R_1] \text{ and } WC_1 e^{-dR_1^2/2} \leq 1,$$

where $W = D\frac{\sqrt{\pi}}{\sqrt{d}} + 1 + D$ with D, d as in Assumption C5.

Finally, given the above choice of ε_1 and R_1 we can define the variables $\hat{s}, \hat{t}$ as follows:

- *$\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$ if $\alpha > -\infty$ and $\beta < \infty$;*
- *$\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = 3\max(t, 0) - \alpha - \varepsilon_1$ if $\alpha > -\infty$ and $\beta = \infty$;*
- *$\hat{s} = 3\min(0, s) - \beta + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$ if $\alpha = -\infty$ and $\beta < \infty$;*
- *$\hat{s} = -6R_1$ and $\hat{t} = 6R_1$ if $\alpha = -\infty$ and $\beta = \infty$.*

Definition 4.2.10. *Suppose that $f_X(\cdot)$ satisfies Assumptions C1-C5. Fix t, s such that $\beta > t > s > \alpha$ and let $C_1, \varepsilon_1, R_1, \hat{s}$ and $\hat{t}$ be as in Definition 4.2.9. For future reference we summarize the following list of constants:*

1. *the constants in Assumptions C1 and C5;*

2. $C_1, \varepsilon_1, R_1, \hat{t}, \hat{s}$ *as in Definition 4.2.9;*

3. $p_{\min(u_{\hat{s}},0),\max(u_{\hat{t}},0)}$ *from Assumption C4;*

4. $M_{\hat{s},\hat{t}}, m_{\hat{s},\hat{t}}, \delta_{\hat{s},\hat{t}}$ *as in Definition 4.2.1;*

5. $K_{\hat{s},\hat{t}}, p_{\hat{s},\hat{t}}$ *as in Definition 4.2.2;*

6. $q_{\hat{s},\hat{t}}$ *as in Definition 4.2.3;*

7. $M^{(0)}_{\hat{s},\hat{t}}, M^{(1)}_{\hat{s},\hat{t}}, M^{(2)}_{\hat{s},\hat{t}}, M^{(3)}_{\hat{s},\hat{t}}, M^{(4)}_{\hat{s},\hat{t}}$ *from Definition 4.2.5.*

We can now prove the following complement to Proposition 4.2.6, which establishes tail estimates for the midpoint density of a continuous random walk bridge.

Proposition 4.2.11. *Suppose that $f_X(\cdot)$ satisfies Assumptions C1-C5. Fix s, t such that $\beta > t > s > \alpha$. There exist constants $A, a > 0$ and $N_1 \in \mathbb{N}$, such that the following holds. Suppose that $m, n \geq N_1$ are such that $|m - n| \leq 1$ and denote $N = n + m$. In addition, let $z \in [s, t]$. Then we have for any $x \in \mathbb{R}$*

$$f_{n,m}(Nx|Nz) \leq A \cdot \exp\left(-aN\left[x - \frac{z}{2}\right]^2\right). \tag{4.29}$$

The constants a, A and N_1 depend on the values s, t and the function $f_X(\cdot)$, where the dependence on the latter is through the constants in Definition 4.2.10.

Proof. Denote $\phi = \frac{m}{n}$ and $\psi = \frac{n}{m}$. For clarity we will split the proof into several cases.
Case 1. Suppose first that $\alpha > -\infty$. From the first line of (4.23) we know that

$$f_{n,m}(Nx|Nz) = \frac{f_n(Nx) \cdot f_m(N(z-x))}{f_N(Nz)}, \tag{4.30}$$

and the latter expression is zero unless $Nx \geq n\alpha$ and $N(z - x) \geq m\alpha$. We will assume that x satisfies these inequalities as otherwise (4.29) trivially holds for any $A, a > 0$. From Definition 4.2.9 we know that for all $N \geq N_0$ we have

$$f_{n,m}(Nx|Nz) \leq C_1^N f_n(Nx) \cdot f_m(N(z-x)). \tag{4.31}$$

In particular, since f_n and f_m are uniformly bounded by a constant (namely L), we see that we can make (4.29) true for all small $N \geq N_0$ by choosing A sufficiently large and $a \leq 1$. We will thus focus on showing (4.29) for sufficiently large $N \geq N_0$.

Suppose that $Nx \leq n\alpha + n\varepsilon_1$, where ε_1 is as in Definition 4.2.9. From Lemma 4.2.8 and the inequality

$$\frac{1}{(N-1)!} = \frac{1}{\Gamma(N)} \leq \frac{e^{N-1}}{N^{N-1}}, \tag{4.32}$$

which can be found in [LC] we conclude that

$$f_n(Nx) \leq L\left(\frac{Len\varepsilon_1}{n}\right)^{n-1} \leq L\,(L\varepsilon_1 e)^{n-1}.$$

The above, combined with the definition of ε_1 and (4.31) imply

$$f_{n,m}(Nx|Nz) \leq C_1^N \cdot L\,(L\varepsilon_1 e)^{n-1} \cdot L \leq 16C_1^4 L^2 2^{-N},$$

while for $N \geq N_1$ with N_1 sufficiently large depending on α we have

$$A \cdot \exp\left(-aN\left[x - \frac{z}{2}\right]^2\right) \geq A \cdot \exp\left(-aN\frac{\varepsilon_1^2}{4}\right).$$

It follows from the above inequalities that (4.29) holds provided we take $A \geq 16C_1^4L^2$, a sufficiently small and $Nx \in [n\alpha, n\alpha + n\varepsilon_1]$. Analogous arguments applied to $z - x$ in place of x show that for the same A and a we have (4.29) provided that $N(z-x) \in [m\alpha, m\alpha + m\varepsilon_1]$. We may thus assume that $Nx \geq n\alpha + n\varepsilon_1$ and $N(z-x) \geq m\alpha + m\varepsilon_1$.

We next consider the cases $\beta = \infty$ and $\beta < \infty$ separately starting with the former.

Case 1.A. If $\beta = \infty$ then we let N_1 be sufficiently large so that $N_1 \geq N_0$, where N_0 is as in the statement of Proposition 4.2.4 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = 3\max(t, 0) - \alpha - \varepsilon_1$.

Then from Proposition 4.2.4 (see also equation (4.23)) we know that we have for $m, n \geq N_1$ and $Nx \geq n\alpha + n\varepsilon_1$ and $N(z-x) \geq m\alpha + m\varepsilon_1$ that

$$f_{n,m}(Nx|Nz) \leq C_2 \exp\left[N\left(\frac{F(x(1+\phi))}{1+\phi} + \frac{F((z-x)(1+\psi))}{1+\psi} - F(z)\right)\right], \tag{4.33}$$

where the constant C_2 depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.2.1 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = 3\max(t,0) - \alpha - \varepsilon_1$. As in the proof of Proposition 4.2.6 we write $x = \frac{z}{1+\phi} + r$ and denote

$$h(r) = \frac{F[z + (1+\phi)r]}{1+\phi} + \frac{F[z - r(1+\psi)]}{1+\psi} - F(z).$$

Then $h(0) = h'(0) = 0$ and

$$h''(r) = (1+\phi)F''(z + (1+\phi)r) + (1+\psi)F''(z + (1+\psi)r) = -\left[\frac{1+\phi}{\sigma^2_{z+(1+\phi)r}} + \frac{1+\psi}{\sigma^2_{z+(1+\psi)r}}\right].$$

The above shows that $h(r)$ is strictly concave and its second derivative is less than $-d_2$ for some $d_2 > 0$ (depending on $M_{\hat{s},\hat{t}}$ alone) on the interval $z + (1+\phi)r, z + (1+\psi)r \in [\hat{s}, \hat{t}]$. Putting this in (4.33) we conclude

$$f_{n,m}(Nx|Nz) \leq C_2 \exp\left(-\frac{d_2}{2} \cdot N \cdot \left[x - \frac{z}{1+\phi}\right]^2\right),$$

which implies (4.29) in this case.

Case 1.B. We suppose that $\beta < \infty$. As before we know that (4.29) holds for any $A, a > 0$ if $Nx > n\beta$ or $N(z-x) > m\beta$ and so we may assume that $Nx \leq n\beta$ and $N(z-x) \leq m\beta$.

Suppose that $Nx \geq n\beta - n\varepsilon_1$. Then from Lemma 4.2.8, (4.31) and (4.32) we know that

$$f_{n,m}(Nx|Nz) \leq C_1^N \cdot L\,(Le\varepsilon_2)^{n-1} \cdot L \leq 16C^4L^2 2^{-N},$$

while for $N \geq N_1$ with N_1 sufficiently large depending on β we have

$$A \cdot \exp\left(-aN\left[x - \frac{z}{2}\right]^2\right) \geq A\exp\left(-aN\frac{\varepsilon_1^2}{4}\right).$$

It follows from the above inequalities that (4.29) holds provided we take $A \geq 16C^4L^2$, a sufficiently small and $Nx \in [n\beta - n\varepsilon_1, n\beta]$. Analogous arguments applied to $z - x$ in place of x show that for the same A and a we have (4.29) provided that $N(z-x) \in [m\beta - m\varepsilon_1, m\beta]$. We may thus assume that $Nx \in [n\alpha + n\varepsilon_1, n\beta - n\varepsilon_1]$ and $N(z-x) \in [m\alpha + m\varepsilon_1, m\beta - m\varepsilon_1]$.

We let N_1 be sufficiently large so that $N_1 \geq N_0$, where N_0 is as in the statement of Proposition 4.2.4 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$.

Then from Proposition 4.2.4 (see also equation (4.23)) we know that for $m, n \geq N_1$ and $Nx \in [n\alpha + n\varepsilon_1, n\beta - n\varepsilon_1]$ and $N(z-x) \in [m\alpha + m\varepsilon_1, m\beta - m\varepsilon_1]$ that

$$f_{n,m}(Nx|Nz) \leq C_2 \exp\left[N\left(\frac{F(x(1+\phi))}{1+\phi} + \frac{F((z-x)(1+\psi))}{1+\psi} - F(z)\right)\right], \tag{4.34}$$

where the constant C_2 depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.2.1 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$.

Repeating the same arguments that follow (4.33) and using the strict negativity of $F''(z)$ for $z \in [\hat{s}, \hat{t}]$ we conclude that

$$f_{n,m}(Nx|Nz) \leq C_2 \exp\left(-\frac{d_2}{2} \cdot N \cdot \left[x - \frac{z}{1+\phi}\right]^2\right),$$

which implies (4.29) in this case. Overall, we conclude (4.29) under the condition that $\alpha > -\infty$.

Case 2. Suppose now that $\alpha = -\infty$.

Case 2.A. If $\beta < \infty$ then we can conclude (4.29) by the same arguments as those in Case 1.A.

Case 2.B. Suppose that $\beta = \infty$. By symmetry it suffices to consider the case when Assumption C5.1 holds. Let R_1 be as in Definition 4.2.9. Then from Lemma 4.2.7 and (4.31) we know that for $x \geq R_1$ and $N \geq N_0$

$$f_{n,m}(Nx|Nz) \leq C_1^N \cdot W^n e^{-dNx^2} \cdot L \leq L \cdot e^{-dNx^2/2},$$

while

$$A \cdot \exp\left(-aN\left[x - \frac{z}{2}\right]^2\right) \geq A\exp\left(-aN[x + R_1/2]^2\right).$$

It follows from the above inequalities that (4.29) holds provided we take $A \geq L$, a sufficiently small (say $a \leq d/8$) and $x \geq R_1$. Analogous arguments applied to $z - x$ in place of x show that for the same A and a we have (4.29) provided that $z - x \geq R_1$. We may thus assume that $x, z - x \in [-2R_1, 2R_1]$.

We let N_1 be sufficiently large so that $N_1 \geq N_0$, where N_0 is as in the statement of Proposition 4.2.4 for the values $\hat{s} = -6R_1$ and $\hat{t} = 6R_1$. Then from Proposition 4.2.4 (see also equation (4.23)) we know that for $m, n \geq N_1$ and $x \in [-2R_1, 2R_1]$

$$f_{n,m}(Nx|Nz) \leq C_2 \exp\left[N\left(\frac{F(x(1+\phi))}{1+\phi} + \frac{F((z-x)(1+\psi))}{1+\psi} - F(z)\right)\right], \tag{4.35}$$

where the constant C_2 depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.2.1 for the values $\hat{s} = -6R_1$ and $\hat{t} = 6R_1$. Repeating the same arguments that follow (4.33) and using the strict negativity of $F''(z)$ for $z \in [\hat{s}, \hat{t}]$ we conclude that

$$f_{n,m}(Nx|Nz) \leq C \exp\left(-\frac{d_2}{2} \cdot N \cdot \left[x - \frac{z}{1+\phi}\right]^2\right),$$

which implies (4.29) in this case. Overall, we conclude (4.29) when $\alpha = -\infty$ and $\beta = \infty$. □

4.3 Midpoint distribution: Discrete case

We continue with the same notation as in Section 4.1.2. To ease the notation a bit we will write M, ϕ and Λ instead of M_X, ϕ_X and Λ_X. Let $p_{n,m}(x|y)$ be the distribution of S_m conditioned on $S_{n+m} = y$. Our goal in this section is to obtain several asymptotic statements about the distribution $p_{m,n}(\cdot|l)$ and we start by analyzing $p_N(l)$.

4.3.1 Asymptotics of $p_N(l)$

In this section we assume that $p_X(\cdot)$ satisfies Assumptions D1-D3. For a fixed $z \in (A^*, B^*)$ we define

$$G_z(u) = \Lambda(u) - z \cdot u, \text{ for } u \in (A_\Lambda, B_\Lambda). \tag{4.36}$$

Definition 4.3.1. *Suppose that we are given $s, t \in \mathbb{R}$ such that $\alpha < s < t < \beta$, where α, β are as in Assumption D1. In addition, we denote $S = (\Lambda')^{-1}(s)$ and $T = (\Lambda')^{-1}(t)$ – these quantities are well-defined in view of Lemma 4.1.5. By Lemma 4.1.4 there exist $\infty > M_{s,t} \geq m_{s,t} > 0$ such that $M_{s,t} \geq \Lambda''(y) \geq m_{s,t}$ for all $y \in [S, T]$. We can pick $\delta_{s,t} > 0$ sufficiently small (depending on s, t and $p_X(\cdot)$) so that*

1. *If $D_{\delta_{s,t}}(S,T) := \{z \in \mathbb{C} : d(z, [S,T]) < \delta_{s,t}\}$ then $\overline{D}_{\delta_{s,t}}(S,T) \subset \{z \in \mathbb{C} : A_\Lambda < Re(z) < B_\Lambda\}$;*
2. *$Re[M_X(u)] > 0$ for all $u \in \overline{D}_{\delta_{s,t}}(S,T)$;*
3. *$\delta_{s,t} < 1/2$;*
4. *$8\delta_{s,t} \cdot |\log(M_X(u))| \leq m_{s,t}$ for all $u \in \overline{D}_{\delta_{s,t}}(S,T)$.*

Definition 4.3.2. *Suppose that we are given $s, t \in \mathbb{R}$ such that $\alpha < s < t < \beta$, where α, β are as in Assumption D1. Suppose that $\delta_{s,t}$ satisfies the conditions in Definitions 4.3.1 and let $\varepsilon_{s,t} = \delta_{s,t}^4$. Then we can find $q_{s,t} \in (0,1)$ (depending on $s, t, \delta_{s,t}$ and $f_X(\cdot)$)such that for every $z \in [s,t]$ and $y \in [\varepsilon_{s,t}, \pi]$ we have*

$$\left| \mathbb{E}\left[e^{(u_z+iy)X} \right] \right| e^{-zu_z} e^{G_z(u_z)} \leq q_{s,t}.$$

To see why the above is true, notice that

$$\left| \mathbb{E}\left[e^{(u_z+iy)X} \right] \right| e^{-zu_z} e^{G_z(u_z)} < \mathbb{E}\left[\left| e^{(u_z+iy)X} \right| \right] e^{-zu_z} e^{G_z(u_z)} = 1,$$

where the above inequality is strict for any $y \neq 0$ as the contrary would imply $X \in 2\pi y^{-1} \cdot \mathbb{Z}$ almost surely, which is not true. This combined with the continuity of $\mathbb{E}\left[e^{(u_z+iy)X}\right]$ in y and z ensures the existence of $q_{s,t}$ with the desired properties.

We are interested in proving the following statement.

Proposition 4.3.3. *Suppose that p_X satisfies Assumptions D1-D3. Fix $\beta > t > s > \alpha$. Then there exists N_0 such that if $N \geq N_0$, $l \in \mathbb{Z}$ and $z = l/N \in [s,t]$ one has*

$$p_N(l) = \frac{1}{\sqrt{2\pi N}\sigma_z} \cdot \exp\left(NG_z(u_z) + \delta_1(z,N)\right), \text{ where } \delta_1(z,N) = O(N^{-1/2}). \tag{4.37}$$

The number N_0 and the constant in the big O notation depend on f_X, s and t only through the constants $\delta_{s,t}, m_{s,t}$ and $q_{s,t}$ as in Definitions 4.3.1 and 4.3.2.

Proof. To simplify the notation, we drop the dependence on X. For any $m \in \mathbb{Z}$ and $N \geq 1$ we have

$$p_N(l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itl} \cdot (\phi(t))^N \, dt.$$

Performing the change of variables $u = it$ we see that

$$p_N(l) = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} M^N(u) e^{-ul} du. \tag{4.38}$$

Consider the rectangular contour R consisting of straight segments connecting $-i\pi$ to $u_z - i\pi$, to $u_z + i\pi$, to $i\pi$ back to $-i\pi$ with a positive orientation. It follows by Lemma 4.1.4 that $M^N(u)e^{-ul}$ is analytic in a neighborhood enclosing that rectangle and so by Cauchy's theorem the integral over R vanishes. In addition, the integral over the top segment and the bottom segment are equal and hence their sum vanishes (as they have opposite orientation). The conclusion is

$$p_N(l) = \frac{e^{NG_z(u_z)}}{2\pi i} \int_{u_z - i\pi}^{u_z + i\pi} M(u)^N e^{-uNz} e^{-NG_z(u_z)} du. \tag{4.39}$$

For the given s, t as in the statement of the proposition we define $\delta_{s,t}, m_{s,t}, \varepsilon_{s,t}$ and $q_{s,t}$ as in Definitions 4.3.1 and 4.3.2. To ease notation we will drop s, t from the notation for these quantities. We will also denote by $C_{s,t}$ the supremum of $|\log(M(u))|$ as u varies over $\overline{D_\delta}$. Notice that by construction we have

$$\varepsilon < \delta/2 \text{ and } \varepsilon \cdot 8C_{s,t} \cdot \delta^{-3} < m.$$

From (4.39) we have $p_N(m) = (I) + (II),$ where

$$(I) = \frac{e^{NG_z(u_z)}}{2\pi i} \int_{u_z - i\varepsilon}^{u_z + i\varepsilon} e^{N[G_z(u) - G_z(u_z)]} du, (II) = \frac{e^{NG_z(u_z)}}{2\pi i} \int_{u_z - i\pi}^{u_z - i\varepsilon} \left[M(u) e^{-uz} e^{-G_z(u_z)} \right]^N du$$
$$+ \frac{e^{NG_z(u_z)}}{2\pi i} \int_{u_z + i\varepsilon}^{u_z + i\pi} \left[M(u) e^{-uz} e^{-G_z(u_z)} \right]^N du. \tag{4.40}$$

Arguing as in the proof of Proposition 4.2.4, we have for N_0 sufficiently large and $N \geq N_0$

$$(I) = \frac{e^{NG_z(u_z)}}{2\pi\sigma_z\sqrt{N}} \left[1 + O\left(\frac{1}{\sqrt{N}} \right) \right], \tag{4.41}$$

where the constant in the big O notation depends on the constants in this proposition.

We next forcus on estimating (II). Suppose that $\pm y \in [\varepsilon, \pi]$. Then by definition we have

$$\left| M(u_z + iy) e^{-z(u_z + iy)} e^{G_z(u_z)} \right| \leq q.$$

The above implies that

$$|(II)| \leq \frac{e^{NG_z(u_z)}}{2\pi} \cdot 2\pi q^N \leq \frac{e^{NG_z(u_z)}}{2\pi\sigma_z N}, \tag{4.42}$$

where the last inequality holds provided N_0 is sufficiently large and $N \geq N_0$. Combining (4.41) and (4.42) yields (4.37). □

4.3.2 Asymptotics of $p_{n,m}(\cdot|l)$

We start with a useful definition.

Definition 4.3.4. *Suppose that $p_X(\cdot)$ satisfies Assumptions D1-D3 and that $\beta > t > s > \alpha$ are given. Then in view of Lemmas 4.1.4 and 4.1.5 we know that $F(z) := G_z(u_z)$ is smooth on (α, β) and so for each $k \geq 0$ exists $M_{s,t}^{(k)} > 0$ such that $|F^{(k)}(z)| \leq M_{s,t}^{(k)}$ for all $z \in [s,t]$.*

We have the following asymptotic estimate for $p_{n,m}(\cdot|l)$.

Proposition 4.3.5. *Suppose that p_X satisfies Assumptions D1-D3. Fix s,t such that $\beta > t > s > \alpha$ and let N_0 be as in the statement of Proposition 4.3.3. Then there exists $M > 0$ such that the following holds. Suppose that $m, n \geq N_0$ are such that $|m - n| \leq 1$ and denote $N = n + m$. In addition, let $k, l \in \mathbb{Z}$ be such that if $z := l/N$ and $x := k/N$, then $z, xN/n$ and $(z - x)N/m \in [s,t]$. Then*

$$p_{n,m}(k|l) = \frac{2}{\sqrt{2\pi N}\sigma_z} \cdot \exp\left(-N \cdot \frac{4}{2\sigma_z^2} \left[x - \frac{z}{2} \right]^2 + \delta_2(N, x, z) \right), \tag{4.43}$$

where

$$|\delta_2(N, x, z)| \leq M \cdot \left(\frac{1}{\sqrt{N}} + N \left| x - \frac{z}{2} \right|^3 \right). \tag{4.44}$$

The constant M depends on s,t and also on $p_X(\cdot)$, where the dependence on the latter is only through the constants in the statement of Proposition 4.3.3 and $M_{s,t}^{(3)}, M_{s,t}^{(4)}$ in Definition 4.3.4.

Proof. Set $\phi = \frac{m}{n}$ and $\psi = \frac{n}{m}$. From Proposition 4.3.3 we know that for $m, n \geq N_0$ we have

$$
\begin{aligned}
p_{n,m}(k|l) &= \frac{p_n(k)p_m(l-k)}{p_N(l)} = \\
&e^{N\left(\frac{F[x(1+\phi)]}{1+\phi} + \frac{F[(z-x)(1+\psi)]}{1+\psi} - F(z)\right)} \cdot \frac{2\sigma_z}{\sqrt{2\pi N}\sigma_{x(1+\phi)} \cdot \sigma_{(z-x)(1+\psi)}} \cdot \exp\left[O\left(\frac{1}{\sqrt{N}}\right)\right],
\end{aligned} \tag{4.45}
$$

where the constant in the big O notation depends on s, t and the constants in the statement of Proposition 4.3.3. From here the proof of the proposition follows the same arguments as in the proof of Proposition 4.2.6. □

4.3.3 Tails of $p_{n,m}(\cdot|l)$

In this section we will further assume that $p_X(\cdot)$ satisfies Assumption D4 and use that to deduce tail estimates for $p_{n,m}(\cdot|l)$. We start with a couple of lemmas.

Lemma 4.3.6. *Suppose that p_X satisfies Assumption D4. Then for all $N \geq 1$ and $x \in \mathbb{Z}$*

$$
p_N(x) \leq \begin{cases} W^N e^{-dN^{-1}x^2} & \text{for all } x \geq 0 \text{ if D4.1 holds and} \\ W^N e^{-dN^{-1}x^2} & \text{fro all } x \leq 0 \text{ if D4.2 holds,} \end{cases}
$$

where $W = D\frac{\sqrt{\pi}}{\sqrt{d}} + 1 + 2D$.

Proof. By symmetry it is clearly enough to consider the case when Assumption D4.1 holds. We proceed by induction on N with base case $N = 1$ being true by assumption. Suppose the result holds true for N and let $y \geq 0$. Then we have

$$
p_{N+1}(y) = \sum_{x=0}^{y} p_N(x)p_1(y-x) + \sum_{x=y}^{\infty} p_N(x)p_1(y-x) + \sum_{x=y}^{\infty} p_N(y-x)p_1(x).
$$

By induction hypothesis and Assumption D4.1 we have

$$
\sum_{x=y}^{\infty} p_N(x)p_1(y-x) \leq W^N e^{-dN^{-1}y^2} \text{ and } \sum_{x=y}^{\infty} p_N(y-x)p_1(x) \leq De^{-dy^2}.
$$

Denote $f(x) = e^{-dN^{-1}x^2}e^{-d(x-y)^2}$ and note that the function has a unique maximum on $[0, y]$, given by $x_{\max} = \frac{Ny}{N+1}$, and $f(x_{\max}) = e^{-d(N+1)^{-1}y^2}$. We thus have

$$
\sum_{x=0}^{y} f(x) \leq \int_0^y f(u)du + e^{-d(N+1)^{-1}y^2} \leq \left(\frac{\sqrt{\pi}}{\sqrt{d+d/N}} + 1\right) \cdot e^{-d(N+1)^{-1}y^2},
$$

where in the last inequality we used (4.27). The latter implies that

$$
\sum_{x=0}^{y} p_N(x)p_1(y-x) \leq W^N D\left(\frac{\sqrt{\pi}}{\sqrt{d+d/N}} + 1\right) \cdot e^{-d(N+1)^{-1}y^2}.
$$

Combining all of the above we see that

$$p_{N+1}(y) \leq W^N e^{-d(N+1)^{-1}y^2}\left[1 + DW^{-N} + D\left(\frac{\sqrt{\pi}}{\sqrt{d + d/N}} + 1\right)\right] \leq W^{N+1} e^{-d(N+1)^{-1}y^2}.$$

This proves the case $N+1$ and the general result follows by induction. □

We next summarize a couple of parameter choices for future use.

Definition 4.3.7. *Suppose that $p_X(\cdot)$ satisfies Assumptions D1-D4. Fix t, s such that $\beta > t > s > \alpha$. Then in view of Proposition 4.3.3 we can find $C_1 > 1$ sufficiently large depending on the constants in that proposition and $M^{(0)}_{s,t}$ in Definition 4.3.4 so that*

$$C_1^{-N} \leq p_N(z)$$

for all $z \in [s, t] \cap \mathbb{Z}$ and $N \geq N_0$ (where N_0 is as in the statement of Proposition 4.3.3).

We can also find $\varepsilon_1 > 0$ sufficiently small so that $s \geq \alpha + 3\varepsilon_1$ and $t \leq \beta - 3\varepsilon_1$.

We can also find $R_1 > 1$ sufficiently large so that

$$[s, t] \subset [-R_1, R_1] \text{ and } WC_1 e^{-dR_1^2/2} \leq 1,$$

where $W = D\frac{\sqrt{\pi}}{\sqrt{d}} + 1 + 2D$ with D, d as in Assumption D4.

Finally, given the above choice of ε_1 and R_1 we can define the variables $\hat{s}, \hat{t}$ as follows:

- *$\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$ if $\alpha > -\infty$ and $\beta < \infty$;*
- *$\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = 3\max(t, 0) - \alpha - \varepsilon_1$ if $\alpha > -\infty$ and $\beta = \infty$;*
- *$\hat{s} = 3\min(0, s) - \beta + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$ if $\alpha = -\infty$ and $\beta < \infty$;*
- *$\hat{s} = -6R_1$ and $\hat{t} = 6R_1$ if $\alpha = -\infty$ and $\beta = \infty$.*

Definition 4.3.8. *Suppose that $p_X(\cdot)$ satisfies Assumptions D1-D4. Fix t, s such that $\beta > t > s > \alpha$ and let $C_1, \varepsilon_1, R_1, \hat{s}$ and $\hat{t}$ be as in Definition 4.3.7. For future reference we summarize the following list of constants:*

1. *the constants in Assumptions D1 and D4;*
2. *$C_1, \varepsilon_1, R_1, \hat{t}, \hat{s}$ as in Definition 4.3.7;*
3. *$M_{\hat{s},\hat{t}}, m_{\hat{s},\hat{t}}, \delta_{\hat{s},\hat{t}}$ as in Definition 4.3.1;*
4. *$q_{\hat{s},\hat{t}}$ as in Definition 4.3.2;*
5. *$M^{(0)}_{\hat{s},\hat{t}}, M^{(1)}_{\hat{s},\hat{t}}, M^{(2)}_{\hat{s},\hat{t}}, M^{(3)}_{\hat{s},\hat{t}}, M^{(4)}_{\hat{s},\hat{t}}$ from Definition 4.3.4.*

We can now prove the following complement to Proposition 4.3.5, which establishes tail estimates for the midpoint density of a discrete random walk bridge.

Proposition 4.3.9. *Suppose that p_X satisfies Assumptions D1-D4. Fix s, t such that $\beta > t > s > \alpha$. There exist constants A, a and $N_1 \in \mathbb{N}$ such that the following holds. Suppose that $m, n \geq 1$ are such that $|m - n| \leq 1$ and denote $N = n + m,$. In addition, let $l \in \mathbb{Z}$ be such that $z := l/N \in [s, t]$. Then for any $k \in \mathbb{Z}$ and $x = k/N$ we have*

$$p_{n,m}(k|l) \leq A \cdot \exp\left(-aN\left[x - \frac{z}{2}\right]^2\right). \tag{4.46}$$

The constants a, A and N_1 depend on the values s, t and the function $p_X(\cdot)$, where the dependence on the latter is through the constants in Definition 4.3.8.

Proof. Denote $\phi = \frac{m}{n}$ and $\psi = \frac{n}{m}$. For clarity we split the proof into several cases.

Case 1. Suppose first that $\alpha > -\infty$. From the first line of (4.45) we know that

$$p_{n,m}(k|l) = \frac{p_n(k) \cdot p_m(l-k)}{p_N(l)}, \tag{4.47}$$

and the latter expression is zero unless $k \geq n\alpha$ and $l - k \geq m\alpha$. We will assume that k satisfies these inequalities as otherwise (4.46) trivially holds for any $A, a > 0$. From Definition 4.3.7 we know that for all $N \geq N_0$ we have

$$p_{n,m}(k|l) \leq C_1^N p_n(k) \cdot p_m(l) \leq C_1^N. \tag{4.48}$$

The latter implies that (4.46) is true for all small $N \geq N_0$ by choosing A sufficiently large and $a \leq 1$. We will thus focus on showing (4.46) for sufficiently large $N \geq N_0$.

Recall that $F(z) = G_z(u_z) = -\Lambda_X^*(z)$ is defined for $z \in (\alpha, \beta)$ but by Lemma 4.1.5 we can continuously extend it to α (and to β provided $\beta < \infty$) by setting $F(\alpha) = \log p_X(\alpha)$ (and $F(\beta) = \log p_X(\beta)$ if $\beta < \infty$). We next observe that for any $m, n \geq 1$, $n\beta \geq k \geq n\alpha$ and $m\beta \geq l - k \geq m\alpha$

$$p_n(k) \leq e^{nF(k/n)} \text{ and } p_m(l-k) \leq e^{mF((k-l)/m)}. \tag{4.49}$$

Indeed, focusing on the first inequality, the statement is true for $k \neq \alpha n$ and $k \neq \beta n$ from (4.38) and the fact that the integrand in that equation is bounded in absolute value by 1 as shown in Definition 4.3.2. The statement is also true for $k = \alpha n$ and $k = \beta n$ by our extension of F above.

Suppose that $Nx \leq n\alpha + n\varepsilon_1$, where ε_1 is as in Definition 4.3.7. From (4.49) and Proposition 4.3.3 we know that there is a $C > 0$, depending on $m_{\hat{s},\hat{t}}$, such that for $m, n \geq N_0$

$$p_{n,m}(k|l) \leq C\sqrt{N} \cdot \exp\left[N\left(\frac{F[x(1+\phi)]}{1+\phi} + \frac{F[(z-x)(1+\psi)]}{1+\psi} - F(z)\right)\right]. \tag{4.50}$$

Similarly to the proof of Proposition 4.2.6 we write $x = \frac{z}{1+\phi} + r_x$ and denote

$$h(r) = \frac{F[z + (1+\phi)r]}{1+\phi} + \frac{F[z - r(1+\psi)]}{1+\psi} - F(z).$$

Notice that since $k \leq n\alpha + n\varepsilon_1$ we have that $r_x \geq \frac{2\varepsilon_1}{1+\phi} \geq \frac{2\varepsilon_1}{3}$. In addition, we have

$$h''(r) = (1+\phi)F''(z + (1+\phi)r) + (1+\psi)F''(z + (1+\psi)r) \leq 0$$

for all $r \in [0, r_x]$ and so by the continuity of F and its smoothness on (α, β) we conclude

$$\frac{F[x(1+\phi)]}{1+\phi} + \frac{F[(z-x)(1+\psi)]}{1+\psi} - F(z) = \int_0^{r_x} \int_0^y h''(r) dr dy \leq \int_0^{\varepsilon_1/3} \int_0^y h''(r) dr dy$$
$$\leq \int_0^{\varepsilon_1/3} \int_0^y \left[-\frac{2}{M_{\hat{s},\hat{t}}} \right] dr dy = -\frac{\varepsilon_1^2}{9M_{\hat{s},\hat{t}}}.$$

Applying the above in (4.50) we conclude

$$p_{n,m}(k|l) \leq C\sqrt{N} \cdot \exp\left(-\frac{\varepsilon_1^2 N}{9M_{\hat{s},\hat{t}}} \right). \tag{4.51}$$

On the other hand, for N_1 sufficiently large depending on α and $N \geq N_1$ we have

$$A \cdot \exp\left(-aN \left[x - \frac{z}{2} \right]^2 \right) \geq A \cdot \exp\left(-aN \frac{\varepsilon_1^2}{4} \right).$$

It follows from the above inequalities that (4.46) holds provided we take $A = 1$, a sufficiently small, N_1 sufficiently large and $Nx \in [n\alpha, n\alpha + n\varepsilon_1]$ for $m, n \geq N_1$. Analogous arguments applied to $z - x$ in place of x show that for the same A and a we have (4.46) provided that $N(z-x) \in [m\alpha, m\alpha + m\varepsilon_1]$. We may thus assume that $Nx \geq n\alpha + n\varepsilon_1$ and $N(z-x) \geq m\alpha + m\varepsilon_1$.

We next consider the cases $\beta = \infty$ and $\beta < \infty$ separately starting with the former.

Case 1.A. If $\beta = \infty$ then we let N_1 be sufficiently large so that $N_1 \geq N_0$, where N_0 is as in the statement of Proposition 4.3.3 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = 3\max(t, 0) - \alpha - \varepsilon_1$.

Then from Proposition 4.3.3 (see also equation (4.45)) we know that we have for $m, n \geq N_1$ and $Nx \geq n\alpha + n\varepsilon_1$ and $N(z-x) \geq m\alpha + m\varepsilon_1$ that

$$p_{n,m}(k|l) \leq C_2 \exp\left[N \left(\frac{F(x(1+\phi))}{1+\phi} + \frac{F((z-x)(1+\psi))}{1+\psi} - F(z) \right) \right], \tag{4.52}$$

where the constant C_2 depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.3.1 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = 3\max(t, 0) - \alpha - \varepsilon_1$. From here the proof continues as that of Case 1.A. in Proposition 4.2.11.

Case 1.B. We suppose that $\beta < \infty$. As before we know that (4.46) holds for any $A, a > 0$ if $Nx > n\beta$ or $N(z-x) > m\beta$ and so we may assume that $Nx \leq n\beta$ and $N(z-x) \leq m\beta$.

Suppose $Nx \geq n\beta - n\varepsilon_1$. We can repeat our arguments from before and see that (4.51) holds in this case as well. On the other hand, for $N \geq N_1$ with N_1 sufficiently large depending on β we have

$$A \cdot \exp\left(-aN \left[x - \frac{z}{2} \right]^2 \right) \geq A \exp\left(-aN \frac{\varepsilon_1^2}{4} \right).$$

It follows from the above inequalities that (4.46) holds provided we take $A = 1$, a sufficiently small, N_1 sufficiently large and $Nx \in [m\beta, m\beta - m\varepsilon_1]$ for $m, n \geq N_1$. Analogous arguments applied to $z - x$ in place of x show that for the same A and a we have (4.46) provided that $N(z-x) \in [m\beta - m\varepsilon_1, m\beta]$. We may thus assume that $Nx \in [n\alpha + n\varepsilon_1, n\beta - n\varepsilon_1]$ and $N(z-x) \in [m\alpha + m\varepsilon_1, m\beta - m\varepsilon_1]$.

We let N_1 be sufficiently large so that $N_1 \geq N_0$, where N_0 is as in the statement of Proposition 4.3.3 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$.

Then from Proposition 4.3.3 (see also equation (4.45)) we know that for $m, n \geq N_1$ and $Nx \in [n\alpha + n\varepsilon_1, n\beta - n\varepsilon_1]$ and $N(z-x) \in [m\alpha + m\varepsilon_1, m\beta - m\varepsilon_1]$ that

$$p_{n,m}(k|l) \leq C_2 \exp\left[N\left(\frac{F(x(1+\phi))}{1+\phi} + \frac{F((z-x)(1+\psi))}{1+\psi} - F(z)\right)\right], \tag{4.53}$$

where the constant C_2 depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.3.1 for the values $\hat{s} = \alpha + \varepsilon_1$ and $\hat{t} = \beta - \varepsilon_1$. From here the proof continues as that of Case 1.B. in Proposition 4.2.11. Overall, we conclude (4.46) under the condition that $\alpha > -\infty$.

Case 2. Suppose now that $\alpha = -\infty$.

Case 2.A. If $\beta < \infty$ then we can conclude (4.46) by the same arguments as those in Case 1.A.

Case 2.B. Suppose that $\beta = \infty$. By symmetry it suffices to consider the case when Assumption D4.1 holds. Let R_1 be as in Definition 4.3.7. Then from Lemma 4.3.6 and (4.48) we know that for $x \geq R_1$ and $N \geq N_0$

$$p_{n,m}(k|l) \leq C_1^N \cdot W^n e^{-dNx^2} \leq e^{-dNx^2/2},$$

while

$$A \cdot \exp\left(-aN\left[x - \frac{z}{2}\right]^2\right) \geq A \exp\left(-aN[x + R_1/2]^2\right).$$

It follows from the above inequalities that (4.46) holds provided we take $A = 1$, a sufficiently small (say $a \leq d/8$) and $x \geq R_1$. Analogous arguments applied to $z - x$ in place of x show that for the same A and a we have (4.46) provided that $z - x \geq R_1$. We may thus assume that $x, z - x \in [-2R_1, 2R_1]$.

We let N_1 be sufficiently large so that $N_1 \geq N_0$, where N_0 is as in the statement of Proposition 4.3.3 for the values $\hat{s} = -6R_1$ and $\hat{t} = 6R_1$. Then from Proposition 4.3.3 (see also equation (4.45)) we know that for $m, n \geq N_1$ and $x \in [-2R_1, 2R_1]$

$$p_{n,m}(k|l) \leq C_2 \exp\left[N\left(\frac{F(x(1+\phi))}{1+\phi} + \frac{F((z-x)(1+\psi))}{1+\psi} - F(z)\right)\right],$$

where the constant C_2 depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.3.1 for the values $\hat{s} = -6R_1$ and $\hat{t} = 6R_1$. From here the proof proceeds as that of Case 2.B. in Proposition 4.2.11.

□

4.4 Gaussian coupling

In this section we isolate some results about the quantile coupling of random variables with certain estimates on their probabilities to Gaussian random variables. We start by isolating some results about Gaussian random variables. We denote by $\Phi(x)$ and $\phi(x)$ the cumulative distribution function and density of a standard normal random variable. The following two lemmas can be found in [MZ, Section 4.2].

Lemma 4.4.1. *There is a constant $c > 1$ such that for all $x \geq 0$ we have*

$$\frac{1}{c(1+x)} \leq \frac{1-\Phi(x)}{\phi(x)} \leq \frac{c}{1+x}, \tag{4.54}$$

Lemma 4.4.2. *For all $A > 0$, $n \geq 64A^2$ and $0 \leq x \leq \frac{1}{8A}$ we have*

$$\log\left(\frac{\Phi(-\sqrt{n}x+u)}{\Phi(-\sqrt{n}x)}\right) = \log\left(\frac{1-\Phi(\sqrt{n}x-u)}{1-\Phi(\sqrt{n}x)}\right) \geq A(nx^3+n^{-1/2}) \tag{4.55}$$

and

$$\log\left(\frac{\Phi(-\sqrt{n}x-u)}{\Phi(-\sqrt{n}x)}\right) = \log\left(\frac{1-\Phi(\sqrt{n}x+u)}{1-\Phi(\sqrt{n}x)}\right) \leq -A(nx^3+n^{-1/2}), \tag{4.56}$$

where $u = 2A(\sqrt{n}x^2 + n^{-1/2})$.

From Rolle's theorem one deduces the following simple result.

Lemma 4.4.3. *Let $R > 0$ be given. There exists a positive constant c_1 such that for $x, y \in [-R, R]$*

$$|\Phi(x) - \Phi(y)| \leq c_1|x-y|. \tag{4.57}$$

The following is an analogue of [LF, Lemma 6.9]. We include it here for the sake of completeness.

Lemma 4.4.4. *Let $M_0 > 0$, $\varepsilon_0 > 0$, $\widetilde{c} \in (0,1)$, $b' > 0$ and $c' > 0$ be given. Then we can find constants $c_2, \varepsilon_2 > 0$, $N_2 \in \mathbb{N}$ such that the following holds for every positive integer $n \geq N_2$ and every $\sigma^2 \in [\widetilde{c}, \widetilde{c}^{-1}]$. Suppose that X is an integer random variable and for all $x \in \{y : y \in \mathbb{Z}, |y| \leq n\varepsilon_0\}$*

$$\mathbb{P}(X = x) = \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{x^2}{2n\sigma^2} + \delta(x)\right), \tag{4.58}$$

where

$$|\delta(x)| \leq M_0\left[\frac{1}{\sqrt{n}} + \frac{|x|^3}{n^2}\right]. \tag{4.59}$$

Assume additionally that for any $m \in \mathbb{Z}$

$$\mathbb{P}(X = m) \leq c' e^{-b'm^2/n}. \tag{4.60}$$

Then for any $|x| \leq \varepsilon_2 n$ we have

$$F\left(x - c_2\left(1+\frac{x^2}{n}\right)\right) \leq \mathbb{P}(X \leq x-1) \leq \mathbb{P}(X \leq x+1) \leq F\left(x + c_2\left(1+\frac{x^2}{n}\right)\right), \tag{4.61}$$

where $F(x)$ is the cumulative distribution function of a $N(0, \sigma^2 n)$ random variable.

Proof. For convenience we denote $G(x) = \mathbb{P}(X \leq x)$, $\bar{F}(x) = 1 - F(x)$, $f(x) = F'(x) = \frac{e^{-x^2/(2\sigma^2 n)}}{\sqrt{2\pi n\sigma^2}}$ and $\bar{G}(x) = 1 - G(x)$. Throughout C, c will stand for generic constants that depend on $M_0, \widetilde{c}, \varepsilon_0, b', c'$ unless otherwise specified.

By symmetry we can assume $x \geq 0$. It suffices to prove (4.61) only for integer values of x and for n sufficiently large. In particular, we assume that N_2 is sufficiently large so that $\varepsilon_0 n \geq n^{5/8} \geq \sqrt{3\widetilde{c}} \cdot n^{1/2} \geq 1$ for all $n \geq N_2$. We prove (4.61) in three cases depending on the size of $|x|$.

We first consider the case $x \leq \sqrt{3\widetilde{c}} \cdot n^{1/2}$. We then have

$$\bar{F}(x) = \sum_{j>x} f(j) + \sum_{j>x} [\mathbb{P}(X = j) - f(j)] = \int_x^\infty f(x)dx + \sum_{j>x} [\mathbb{P}(X = j) - f(j)] + O\left(\frac{1}{\sqrt{n}}\right), \tag{4.62}$$

where in the last equality we used that $f(x)$ is decreasing for $x \geq 0$ and its integral over any unit interval is at most $\frac{1}{\sqrt{2\pi n\sigma^2}}$. Using that $f(x)$ is decreasing for all $x \geq 0$ we get

$$\sum_{j>x} |\mathbb{P}(X = j) - f(j)| \leq \sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} |e^{\delta(j)} - 1| + \mathbb{P}(X \geq n^{2/3}) + \int_{n^{2/3}-1}^\infty f(x)dx \tag{4.63}$$

We next increase N_2 so that $N_2^{-1/3} M_0 \leq \frac{\widetilde{c}}{4} \leq \frac{1}{4\sigma^2}$ and use the inequality $|e^x - 1| \leq |x|e^{|x|}$ to estimate

$$\sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} |e^{\delta(j)} - 1| \leq \sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} |\delta(j)| e^{|\delta(j)|} \leq \sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{4n\sigma^2} + \frac{M_0}{\sqrt{n}}}}{\sqrt{2\pi\sigma^2 n}} \left[\frac{M_0}{\sqrt{n}} + \frac{M_0|j|^3}{n^2}\right]. \tag{4.64}$$

Since $f(x)$ is decreasing for all $x \geq 0$

$$\sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{4n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \left[\frac{M_0}{\sqrt{n}}\right] \leq \frac{\sqrt{2}M_0}{\sqrt{n}} \cdot \int_0^\infty \frac{e^{-\frac{u^2}{4n\sigma^2}}}{\sqrt{4\pi\sigma^2 n}} du = \frac{M_0}{\sqrt{2n}}. \tag{4.65}$$

Analogously, by using that $x^3 e^{-x^2/2}$ is decreasing for all $x \geq \sqrt{3}$ we have

$$\begin{aligned} \sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{4n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \left[\frac{M_0|j|^3}{n^2}\right] &= \sum_{j=x+1}^{\lfloor \sqrt{3\widetilde{c}n}\varepsilon + 1 \rfloor} \frac{e^{-\frac{j^2}{4n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \left[\frac{M_0|j|^3}{n^2}\right] + \sum_{j=\lfloor \sqrt{3\widetilde{c}n} \rfloor + 2}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{4n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \left[\frac{M_0|j|^3}{n^2}\right] \\ &\leq \frac{M_0[2\sqrt{3\widetilde{c}}\varepsilon]^3}{\sqrt{2n}} + \frac{M_0}{n^2} \int_0^\infty \frac{u^3 e^{-\frac{u^2}{4n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} du = \frac{M_0[2\sqrt{3\widetilde{c}}\varepsilon]^3}{\sqrt{2n}} + \frac{2(\sqrt{2\sigma^2 n})^3 M_0}{\sqrt{\pi} n^2}. \end{aligned} \tag{4.66}$$

Finally, we have that by taking N_2 larger we can ensure using (4.54) and (4.60) that

$$\mathbb{P}(X \geq n^{2/3}) \leq \frac{c' e^{-b' \lfloor n^{2/3} \rfloor^2 / n}}{1 - e^{-b' \lfloor n^{2/3} \rfloor / n}} \leq Ce^{-cn^{1/3}}, \int_{n^{2/3}-1}^\infty f(x)dx = 1 - \Phi\left(\frac{n^{2/3} - 1}{\sigma^2 \sqrt{n}}\right) \leq Ce^{-cn^{1/3}}. \tag{4.67}$$

Combining (4.62), (4.63), (4.64), (4.65), (4.66) and (4.67) we conclude for $|x| \leq \sqrt{3\widetilde{c}} n^{1/2}$ and n large

$$|G(x) - F(x)| = \left|\bar{G}(x) - \bar{F}(x)\right| \leq \frac{C}{\sqrt{n}},$$

which implies (4.61) in view of Lemma 4.4.3.

Next we consider the case $n^{5/8} \geq x \geq \sqrt{3\widetilde{c}} \cdot n^{1/2}$. In this case we have

$$\bar{G}(x) = \sum_{j>x} f(j) + \sum_{j>x} [\mathbb{P}(X=j) - f(j)] = \bar{F}(x) + \sum_{j>x} [\mathbb{P}(X=j) - f(j)] + O\left(\frac{e^{-\frac{x^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \right), \tag{4.68}$$

where in the last equality we used that $f(y)$ is decreasing on $[x, \infty)$ and its integral over any unit interval is at most $\frac{e^{-\frac{x^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}}$. Notice that for $x+1 \leq j \leq n^{2/3}$ we have $|\delta(j)| \leq C|j|^3/n^2 \leq C$, where $C = M_0 \cdot [1 + (3\widetilde{c})^{-3/2}]$. This means that $\left|e^{\delta(j)} - 1\right| \leq e^C |\delta(j)| \leq C|j|^3/n^2$ and so

$$\sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} |e^{\delta(j)} - 1| \leq C \sum_{j=x+1}^{\lfloor n^{2/3} \rfloor} \frac{e^{-\frac{j^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \left[\frac{j^3}{n^2} \right] \leq \frac{C}{n^2} \int_x^\infty \frac{u^3 e^{-\frac{u^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} du \leq \frac{Cx^2}{n^{3/2}} e^{-\frac{x^2}{2n\sigma^2}}. \tag{4.69}$$

From (4.67) we know that by possibly making N_2 larger we can ensure

$$\mathbb{P}(X \geq n^{2/3}) \leq Ce^{-cn^{1/3}} \leq \frac{1}{\sqrt{n}} \cdot e^{-\frac{x^2}{2n\sigma^2}} \text{ and } \int_{n^{2/3}-1}^\infty f(x)dx \leq Ce^{-cn^{1/3}} \leq \frac{1}{\sqrt{n}} \cdot e^{-\frac{x^2}{2n\sigma^2}}. \tag{4.70}$$

Combining (4.68), (4.63), (4.69) and (4.70) we conclude for $n^{5/8} \geq x \geq \sqrt{3\widetilde{c}} \cdot n^{1/2}$ and n sufficiently large

$$|G(x) - F(x)| = \left|\bar{G}(x) - \bar{F}(x)\right| \leq C \left[1 + \frac{x^2}{n^{3/2}} \right] \cdot \frac{e^{-\frac{x^2}{2n\sigma^2}}}{\sqrt{2\pi\sigma^2 n}} \leq C \cdot \frac{x^3}{n^2} \cdot \bar{F}(x),$$

where in the last inequality we used (4.54). The above inequality implies that for all n large

$$\begin{aligned} \bar{G}(x) &\leq \left[1 + C\frac{x^3}{n^2} \right] \bar{F}(x) \leq e^{Cx^3/n^2} \bar{F}(x) \leq \bar{F}\left(x - C\left[1 + \frac{x^2}{n} \right] \right) \\ \bar{G}(x) &\geq \left[1 - C\frac{x^3}{n^2} \right] \bar{F}(x) \geq e^{-Cx^3/n^2} \bar{F}(x) \geq \bar{F}\left(x + C\left[1 + \frac{x^2}{n} \right] \right), \end{aligned} \tag{4.71}$$

where the right most inequalities used Lemma 4.4.2. From (4.71) we conclude (4.61) for some large c_0 and all $n^{5/8} \geq x \geq \sqrt{3\widetilde{c}} \cdot n^{1/2}$ provided n is large enough.

We finally consider the case $n\varepsilon_2 \geq x \geq n^{5/8}$, where ε_2 is to be chosen sufficiently small as follows. Consider the functions $h_\pm(z) = -\frac{z^2}{2\sigma^2} \pm 2M_0 \frac{z^3}{\sqrt{n}}$. Then

$$h'_\pm(z) = -\frac{z}{\sigma^2} \pm 6M_0 \frac{z^2}{\sqrt{n}} \leq -\widetilde{c}z \pm 6M_0 \frac{z^2}{\sqrt{n}} \leq z\left[\pm 6M_0 \frac{z}{\sqrt{n}} - \widetilde{c} \right]$$

and we can choose $\varepsilon_1 \leq \min(\varepsilon_0, 1)$ sufficiently small (depending on M_0 and $\widetilde{c}$) such that the functions $h_\pm(z)$ are decreasing and moreover $-\frac{3z^2}{2} \leq h_-(z) \leq h_+(z) \leq -\frac{z^2}{4\sigma^2}$ for $0 < z \leq \varepsilon_1\sqrt{n}$. We next pick $\varepsilon_2 > 0$ (depending on $\widetilde{c}$, M_0, b' and c') so that $\varepsilon_2 \leq \varepsilon_1/2$ and for all $n \geq \varepsilon_2^{-6}$

$$\mathbb{P}(X \geq n\varepsilon_1) \leq \frac{c' e^{-b'\lfloor n\varepsilon_1 \rfloor^2/n}}{1 - e^{-b'\lfloor n\varepsilon_1 \rfloor/n}} \leq \frac{e^{h_+(\sqrt{n}\varepsilon_2)}}{\sqrt{2\pi\sigma^2 n}} \leq \frac{e^{h_+(x/\sqrt{n})}}{\sqrt{2\pi\sigma^2 n}}. \tag{4.72}$$

Using the inequality $e^{\delta(j)} \leq \exp\left(2M_0 \frac{j^3}{n^2}\right)$ for $x+1 \leq j \leq \varepsilon_1 n$ and the fact that $h_+(z)$ is decreasing on $0 < z \leq \varepsilon_1\sqrt{n}$ by our choice of ε_1 we see that

$$\bar{G}(x) = \sum_{j=x+1}^{\lfloor n\varepsilon_1 \rfloor} f(j)e^{\delta(j)} + \mathbb{P}(X \geq n\varepsilon_1) \leq \int_{x/\sqrt{n}}^{\sqrt{n}\varepsilon_1} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} + \mathbb{P}(X \geq n\varepsilon_1) \leq \int_{x-1/\sqrt{n}}^{\sqrt{n}\varepsilon_1} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}}, \tag{4.73}$$

where in the last inequality we used (4.72). Using that $2x \leq 2\varepsilon_2 n \leq \varepsilon_1 n$ we have

$$\int_{x-1/\sqrt{n}}^{\sqrt{n}\varepsilon_1} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} = \int_{(x-2)/\sqrt{n}}^{2x/\sqrt{n}} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} + \int_{2x/\sqrt{n}}^{\sqrt{n}\varepsilon_1} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} - \int_{(x-2)/\sqrt{n}}^{x-1/\sqrt{n}} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} \leq$$

$$\int_{(x-2)/\sqrt{n}}^{2x/\sqrt{n}} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} + \int_{2x/\sqrt{n}}^{\sqrt{n}\varepsilon_1} \frac{e^{-u^2/4\sigma^2}du}{\sqrt{2\pi\sigma^2}} - \int_{(x-2)/\sqrt{n}}^{x-1/\sqrt{n}} \frac{e^{-u^2/2\sigma^2}du}{\sqrt{2\pi\sigma^2}} \leq \int_{(x-2)/\sqrt{n}}^{2x/\sqrt{n}} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}},$$

where the last inequality holds provided n is sufficiently large in view of (4.54). Combining the above with Lemma 4.4.2 we see that by possibly making ε_2 smaller and N_2 larger we can ensure that

$$\bar{G}(x) \leq \int_{(x-2)/\sqrt{n}}^{2x/\sqrt{n}} \frac{e^{h_+(u)}du}{\sqrt{2\pi\sigma^2}} \leq e^{16M_0x^3/n^2} \cdot \bar{F}(x-2) \leq \bar{F}\left(x - C\left[1 + \frac{x^2}{n}\right]\right). \tag{4.74}$$

To get the lower bound notice that $2x \leq 2\varepsilon_2 n \leq \varepsilon_1 n$ and so

$$\bar{G}(x) \geq \sum_{j=x+1}^{\lfloor n\varepsilon_1 \rfloor} f(j)e^{\delta(j)} \geq \int_{\frac{x+1}{\sqrt{n}}}^{\frac{2x}{\sqrt{n}}} \frac{e^{h_-(u)}du}{\sqrt{2\pi\sigma^2}} \geq e^{-16M_0x^3/n^2} \cdot \bar{F}(x+1) \geq \bar{F}\left(x + C\left[1 + \frac{x^2}{n}\right]\right). \tag{4.75}$$

From (4.74) and (4.75) we conclude (4.61) for some large c_2 and all $\varepsilon_2 n \geq x \geq n^{5/8}$ provided $n \geq N_2$ with N_2 large enough and ε_2 small enough. This suffices for the proof. □

As an immediate corollary to the above lemma we have the following statement.

Corollary 4.4.5. *Let $M_0 > 0$, $\varepsilon_0 > 0$, $\tilde{c} \in (0,1)$, $b' > 0$ and $c' > 0$ be given. Then we can find constants $c_2, \varepsilon_2 > 0$, $N_2 \in \mathbb{N}$ such that the following holds for every positive integer $n \geq N_2$ and every $\sigma^2 \in [\tilde{c}, \tilde{c}^{-1}]$. Suppose that X is a continuous random variable with density g and for all $x \in \{y : y \in \mathbb{R}, |y| \leq n\varepsilon_0\}$*

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{x^2}{2n\sigma^2} + \delta(x)\right), \tag{4.76}$$

where

$$|\delta(x)| \leq M_0\left[\frac{1}{\sqrt{n}} + \frac{|x|^3}{n^2}\right]. \tag{4.77}$$

Assume additionally that for any $x \in \mathbb{R}$

$$g(x) \leq c'e^{-b'x^2/n}. \tag{4.78}$$

Then for any $|x| \leq \varepsilon_2 n$ we have

$$F\left(x - c_2\left(1 + \frac{x^2}{n}\right)\right) \leq \mathbb{P}(X \leq x) \leq F\left(x + c_2\left(1 + \frac{x^2}{n}\right)\right), \tag{4.79}$$

where $F(x)$ is the cumulative distribution function of a $N(0, \sigma^2 n)$ random variable.

Proof. By our assumptions we know that $W = \lfloor X \rfloor$ is an integer valued random variable that satisfies the conditions of Lemma 4.4.4. The result now follows from (4.61) and the fact that $\mathbb{P}(W \leq x-1) \leq \mathbb{P}(X \leq x) \leq \mathbb{P}(W \leq x+1)$. □

4.5 Strong coupling

We formulate quantified refinements of Theorems 4.1.3 and 4.1.6 as Theorems 4.5.3 and 4.5.6, respectively, below and present their proof. As usual we split our discussion depending on whether our random walk bridge has continuous or discrete jumps.

4.5.1 Continuous case

We use the same notation as in Sections 4.1.1 and 4.2.

Lemma 4.5.1. *Suppose that f_X satisfies Assumptions C1-C5 and fix $p \in (\alpha, \beta)$. Let $s = p - \varepsilon'$ and $t = p + \varepsilon'$, where $\varepsilon' > 0$ is sufficiently small so that $\alpha < s < t < \beta$. Then there exists $\varepsilon_3 \in (0, \varepsilon')$ and $N_3 \in \mathbb{N}$ such that for every $b_1 > 0$ there exist constants $0 < c_1, a_1 < \infty$ such that the following holds. Suppose that m, n are integers such that $m, n \geq N_3$ with $|m - n| \leq 1$, set $N = m + n$. We can define a probability space on which are defined a standard normal random variable ξ and a collection of random variables $W = W^{(m,n,z)}$ for all $z \in \{x \in L_N : |x - pN| \leq \varepsilon_3 N\}$ such that the law of $W^{(m,n,z)}$ is the same as that of $S_n^{(N,z)}$ and such that we have almost surely*

$$\mathbb{E}\left[e^{a_1|Z-W|}\Big|W\right] \leq c_1 \cdot \exp\left(b_1 \frac{(W-pn)^2 + (z-pN)^2}{N}\right), \tag{4.80}$$

where

$$Z = Z^{(m,n,z)} = \frac{z}{2} + \frac{\sqrt{N}\sigma_p}{2} \cdot \xi, \text{ so that } Z \sim N\left(\frac{z}{2}, \frac{\sigma_p^2 N}{4}\right).$$

The constants ε_3 and N_3 depend on the values p, s, t and the function $f_X(\cdot)$, where the dependence on the latter is through the constants in Definition 4.2.10.

Proof. Notice that we only need to prove the lemma for N sufficiently large. In order to simplify the notation we will assume that $n = m = N/2$ (the other cases can be handled similarly).

We apply Propositions 4.2.6 and 4.2.11 for the variables s and t. This implies that provided $N_3 \geq \max(N_0, N_1)$ as in the statements of those propositions and $n \geq N_3$ we have that the random variable $S_n^{(N,z)} - z/2$ satisfies the conditions of Corollary 4.4.5 for $M_0 = M$ as in Proposition 4.2.6, $\varepsilon_0 = \varepsilon'$ as in the statement of this proposition, $\widetilde{c} = (1/2) \cdot \min(m_{\hat{s},\hat{t}}, M_{\hat{s},\hat{t}}^{-1})$ as in Definition 4.2.1 for the variables $\hat{s}, \hat{t}$ as in Definition 4.2.9, $b' = a$ and $c' = A$ as in the statement of Proposition 4.2.11. We consequently, let c_2, N_2, ε_2 be as in the statement of that corollary for the above constants.

In what follows we fix $\varepsilon_3 \leq 4^{-1}\min(\varepsilon_2, \varepsilon')$ sufficiently small so that $\varepsilon_3 M \leq 1/M_{\hat{s},\hat{t}}$ where M is as in the statement of Proposition 4.2.6 and $M_{\hat{s},\hat{t}}$ is as in Definition 4.2.1 for the variables $\hat{s}, \hat{t}$ as in Definition 4.2.9. Observe that the choice of ε_3 implies that $\varepsilon_3 M \leq 1/\sigma^2_{z/N}$ for all $|z - pN| \leq N\varepsilon_3$. We also set $N_3 = \max(N_0, N_1, N_2)$.

We denote by Φ the cumulative distribution function of a normal random variable with mean 0 and variance 1. Let $G_{n,m,z}$ denote the cumulative distribution function of $S_n^{(N,z)}$. In addition, let $G^{\varepsilon_3,+}_{n,m,z}$ and $G^{\varepsilon_3,-}_{n,m,z}$ denote the cumulative distribution function of $S_n^{(N,z)}$ conditioned on $\{S_n^{(N,z)} > z/2 + 2\varepsilon_3 n\}$ and $\{S_n^{(N,z)} < z/2 - 2\varepsilon_3 n\}$ respectively. For convenience we let $A < B$ be the unique real numbers such that

$$1 - \Phi(B) = \mathbb{P}(S_n^{(N,z)} > z/2 + 2\varepsilon_3 n), \quad \Phi(A) = \mathbb{P}(S_n^{(N,z)} < z/2 + 2\varepsilon_3 n).$$

We now turn to defining our probability space. We let U_1, U_2, U_3 be three independent uniform $(0,1)$ random variables and set $\xi = \Phi^{-1}(U_1)$. In addition, we set $W_+ = \left(G^{\varepsilon_3,+}_{n,m,z}\right)^{-1}(U_2)$ and $W_- = \left(G^{\varepsilon_3,-}_{n,m,z}\right)^{-1}(U_3)$. Given a realization of ξ, W_- and W_+ we define a random variable W as follows

- if $A \leq \xi \leq B$ we set $W = (G_{n,m,z})^{-1}(U_1)$;
- if $\xi > B$ we set $W = W_+$;
- if $\xi < A$ we set $W = W_-$.

It is easy to see that as defined W indeed has the same distribution as $S_n^{(N,z)}$. In words, W is *quantile coupled* to ξ near 0 and independent from it for large values.

We denote

$$Z = Z_{n,z} = z/2 + \frac{\sigma_p\sqrt{N}}{2}\cdot\xi, \qquad \hat{Z} = \hat{Z}_{n,z} = z/2 + \frac{\sigma_{z/N}\sqrt{N}}{2}\cdot\xi.$$

and write $F = F_{n,z}$ for the cumulative distribution function of $\hat{Z}$. It is easy to check that our construction satisfies the following property. If $y \in [z/2 - 2n\varepsilon_3, z/2 + 2n\varepsilon_3]$ and $x > 0$ is fixed and

$$F(y - x) \leq G_{n,m,z}(y) \leq F(y + x),$$

then

$$|\hat{Z} - W| \leq x \qquad \text{on the event } A \leq \xi \leq B\ . \tag{4.81}$$

By our choice of ε_3, N_3 and c_2 and Corollary 4.4.5 applied to $S_n^{(N,z)} - z/2$ we have that for all $y \in [z/2 - 2n\varepsilon_3, z/2 + 2n\varepsilon_3]$

$$F\left(y - c_2\left[1 + \frac{(y - z/2)^2}{n}\right]\right) \leq G_{n,m,z}(y) \leq F\left(y + c_2\left[1 + \frac{(y - z/2)^2}{n}\right]\right). \tag{4.82}$$

Combining (4.81) and (4.82) we get

$$|\hat{Z} - W| \leq c_2\left[1 + \frac{(W - z/2)^2}{n}\right] \text{ almost surely on the event } A \leq \xi \leq B, \tag{4.83}$$

for all $n \geq N_3$, provided that $|z - pN| \leq \varepsilon_3 N$, $|W - z/2| \leq 2\varepsilon_3 n$.

We next claim that $|A| = O(\sqrt{N})$ and $|B| = O(\sqrt{N})$. To see the latter notice that

$$\mathbb{P}(\xi \geq B) = \mathbb{P}(W \geq z/2 + 2n\varepsilon_3) = 1 - \mathbb{P}(W - z/2 \leq 2n\varepsilon_3) \geq 1 - \mathbb{P}\left(\hat{Z} - \frac{z}{2} \leq 2n\varepsilon_3 + c_2\left(1 + \frac{4n^2(\varepsilon_3)^2}{n}\right)\right)$$

$$= \mathbb{P}\left(\frac{\sigma_{z/N}\sqrt{N}}{2} \cdot \xi \geq 2n\varepsilon_3 + c_2[1 + 4n(\varepsilon_3)^2]\right) \geq \mathbb{P}(\xi \geq \widetilde{C}\sqrt{N}),$$

for some positive constant $\widetilde{C}$. The inequality in the first line follows from Corollary 4.4.5 applied to $W - z/2$. The above implies that $B \leq \widetilde{C}\sqrt{N}$ and an anologous argument shows that $A \geq -\widetilde{C}\sqrt{N}$ for some possibly larger $\widetilde{C}$. We conclude that there is a constant $\widetilde{C} > 0$ such that $|\xi| \leq \widetilde{C}\sqrt{N}$ on the event $A \leq \xi \leq B$.

The latter implies that almost surely on the event $A \leq \xi \leq B$ we have

$$\mathbb{E}\left[e^{|Z-\hat{Z}|}\Big|W\right] \leq \mathbb{E}\left[e^{|\xi||\sigma_p - \sigma_{z/N}|}\Big|W\right] \leq \mathbb{E}\left[e^{\widetilde{C}\sqrt{N}|\sigma_p - \sigma_{z/N}|}\Big|W\right].$$

From Lemma 4.1.1 we know that we can find a constant $c_p > 0$, that depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.2.1 as well as $M^{(3)}_{\hat{s},\hat{t}}$ as in Definition 4.2.5 for the variables $\hat{s}, \hat{t}$ as in Definition 4.2.9, such that $|\sigma_p - \sigma_{z/N}|^2 \leq c_p|p - z/N|^2$ for all $|z - pN| \leq \varepsilon_3 N$. Combining the latter with the Cauchy-Schwarz inequality, (4.83) and the triangle inequality we conclude that there are constants $C, c > 0$ such that if $|W - z/2| \leq 2\varepsilon_3 n$ and $|z - pN| \leq \varepsilon_3 N$ then

$$\mathbb{E}\left[e^{|W-Z|}\Big|W\right] \leq \mathbb{E}\left[e^{|W-\hat{Z}|+|Z-Z|}\Big|W\right] \leq C\exp\left(\frac{c_p(z-pN)^2}{N} + \frac{c(W-z/2)^2}{n}\right).$$

Applying Jensen's inequality to the above we have for any $v \in \mathbb{N}$ that

$$\mathbb{E}\left[e^{(1/v)|W-Z|}\Big|W\right] \leq \mathbb{E}\left[e^{|W-Z|}\Big|W\right]^{1/v} \leq C^{1/v}\exp\left(\frac{c_p(z-pN)^2}{Nv} + \frac{c(W-z/2)^2}{nv}\right),$$

and if we further use that $(x+y)^2 \leq 2x^2 + 2y^2$ above we see that

$$\mathbb{E}\left[e^{(1/v)|W-Z|}\Big|W\right] \leq C^{1/v} \cdot \exp\left(\frac{[c_p + c](z-pN)^2}{Nv} + \frac{4c(W-pn)^2}{Nv}\right), \tag{4.84}$$

provided $n \geq N_3$, $|w - z/2| \leq 2\varepsilon_3 n$ and $|z - pN| \leq \varepsilon_3 N$.

Suppose now that b_1 is given, and let v be sufficiently large so that

$$\frac{c_p + c}{v} \leq b_1 \text{ and } \frac{4c}{v} \leq b_1.$$

If $a_1 \leq 1/v$ we see from (4.84) that

$$\mathbb{E}\left[e^{a_1|W-Z|}\Big|W\right] \leq C \cdot \exp\left(\frac{b_1(z-pN)^2}{N} + \frac{b_1(w-pn)^2}{N}\right), \tag{4.85}$$

provided $n \geq N_3$, $|w - z/2| \leq 2\varepsilon_3 n$ and $|z - pN| \leq \varepsilon_3 N$.

Suppose now that $|W - z/2| > 2\varepsilon_3 n$ and suppose for concreteness that $W - z/2 \geq 2\varepsilon_3 n$. On the event $\{W > z/2 + 2\varepsilon_3 n\}$ we have that W and Z are independent with Z having the distribution of a normal random variable with mean $z/2$ and variance $\frac{\sigma_p^2 N}{4}$ conditioned on being larger than $s := z/2 + \frac{\sigma_p\sqrt{N}}{2} \cdot B$. It follows that almost surely on $\{W > z/2 + 2\varepsilon_3 n\}$

$$\mathbb{E}\left[e^{|W-Z|}\middle|W\right] \leq e^{|W-z/2|} \cdot \int_B^\infty \frac{e^{\frac{\sigma_p\sqrt{N}}{2}|y|}e^{-y^2/2}}{\sqrt{2\pi}} \cdot (1 - \Phi(B))^{-1}.$$

From our earlier work we know that $B \leq \widetilde{C}\sqrt{N}$ for some $\widetilde{C} > 0$. This implies that

$$1 - \Phi(B) \geq e^{-cN\varepsilon_3^2},$$

for some sufficiently large $c > 0$. Combining the last two inequalities gives for some new $c > 0$

$$\mathbb{E}\left[e^{|W-Z|}\middle|W\right] \leq \exp\left(cN + |W - z/2|\right) \leq \exp\left((c + 5/4)N + \frac{(z-pN)^2}{N} + \frac{(W-pn)^2}{N}\right),$$

where the last inequality uses the triangle inequality and the fact that $\sqrt{ab} \leq a + b$ for $a, b \geq 0$. Applying Jensen's inequality to the above we have for any $v \in \mathbb{N}$ that

$$\mathbb{E}\left[e^{(1/v)|W-Z|}\middle|W\right] \leq \mathbb{E}\left[e^{|W-Z|}\middle|W\right]^{1/v} \leq \exp\left(\frac{(c+5/4)N}{v} + \frac{(z-pN)^2}{vN} + \frac{(W-pn)^2}{vN}\right).$$

In particular, suppose that v is sufficiently large so that

$$\frac{1}{v} \leq \frac{b_1}{2} \text{ and } \frac{c+5/4}{v} \leq \frac{b_1\varepsilon_3^2}{8}$$

and $a_1 \leq 1/v$. We then have from the above inequality that

$$\mathbb{E}\left[e^{a_1|W-Z|}\middle|W\right] \leq \exp\left(\frac{b_1\varepsilon_3^2}{16}N + \frac{b_1(z-pN)^2}{2N} + \frac{b_1(W-pn)^2}{2N}\right) \leq \exp\left(\frac{b_1(z-pN)^2}{N} + \frac{b_1(W-pn)^2}{N}\right),$$

where in the last inequality we used that $|W - z/2| \geq 2\varepsilon_3 n$ and $|z/2 - pn| \leq \varepsilon_3 n$. We conclude that (4.85) holds even when $W - z/2 > 2\varepsilon_3 n$. An analogous argument shows that (4.85) also holds when $W - z/2 < -2\varepsilon_3 n$, and so almost surely for all W. This suffices for the proof.

□

We also isolate for future use the following statement.

Lemma 4.5.2. *Assume the same notation as in Lemma 4.5.1. There exist positive constants b_2, c_2, N_4 such that for every integers $m, n \geq N_4$, $N = m + n$ such that $|m - n| \leq 1$, every z such that $|z - pN| \leq \varepsilon' N$ and $w \in \mathbb{R}$,*

$$f_{m,n}(w|z) \leq c_2 N^{-1/2} \exp\left(-b_2\frac{(w-(z/2))^2}{N}\right).$$

The constants b_2, c_2, N_4 depend on s, t, p and the constants in Definition 4.2.10.

Proof. This is an immediate corollary of Propositions 4.2.6 and 4.2.11. □

We now turn to the main theorem of this section.

Theorem 4.5.3. *Suppose that f_X satisfies Assumptions C1-C6 and fix $p \in (\alpha, \beta)$. Let $s = p - \varepsilon'$ and $t = p + \varepsilon'$, where $\varepsilon' > 0$ is sufficiently small so that $\alpha < s < t < \beta$. For every $b > 0$, there exist constants $0 < C, a, \alpha' < \infty$ such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = \sigma_p^2$ and the family of processes $S^{(n,z)}$ for $z \in L_n$ such that*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha'(\log n)}e^{b|z-pn|^2/n}, \tag{4.86}$$

where $\Delta(n,z) = \Delta(n, z, B^\sigma, S^{(n,z)}) = \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \frac{t}{n}z - S_t^{(n,z)}\right|$. The constants C, a, α' depend on b as well as s, t, p and f_X through the constants in Definition 4.2.10 and the functions in Assumption C6.

Proof. It suffices to prove the theorem when b is sufficiently small. For the remainder we fix $b > 0$ such that $b < b_2/37$, where b_2 is the constant from Lemma 4.5.2. Let ε_3 and N_3 be as in Lemma 4.5.1 and N_4 as in Lemma 4.5.2 for our choice of s, t and put $N_5 = \max(N_3, N_4)$.

In this proof, by an *n-coupling* we will mean a probability space on which are defined a Brownian bridge B^σ and the family of processes $\{S^{(n,z)} : z \in L_n\}$. Notice that for any n-coupling if $z \in L_n$, $S_t = S_t^{(n,z)}$ then

$$\Delta(n,z) = \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \frac{t}{n}z - S_t^{(n,z)}\right| \leq |z| + \max_{0\leq k\leq n}|S_k^{(n,z)}| + \sup_{0\leq t\leq n}|\sqrt{n}B^\sigma_{t/n}|$$

which implies

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq \mathbb{E}\left[\exp\left(3a\sup_{0\leq t\leq 1}\sqrt{n}|B^\sigma_t|\right)\right] + \exp(3a|z|) + \mathbb{E}\left[\exp\left(3a\max_{1\leq k\leq n}|S_k|\right)\Big|S_n = z\right].$$

Note that if $|z - pn| \geq \varepsilon_3 n$ we have

$$b|z-pn|^2/n \geq \frac{b\varepsilon_3^2 n}{2} + \frac{b|z-pn|^2}{2n} \geq \frac{b\varepsilon_3^2 n}{2} + b\kappa\frac{z^2}{n},$$

where κ is sufficiently small so that

$$\kappa < 1/2, \qquad \frac{p}{1-2\kappa} \in [p - \varepsilon_3, p + \varepsilon_3], \text{ and } \varepsilon_3/2 - \kappa(\pm\varepsilon_3 + p)^2 > 0.$$

In view of the above and Assumption C6 there exists $\hat{a}$ small enough and $\hat{C}$ large enough depending on b such that if $a < \hat{a}$ we can ensure that

$$\exp(3a|z|) + \mathbb{E}\left[\exp\left(3a\max_{1\leq k\leq n}|S_k|\right)\Big|S_n = z\right] \leq \hat{C}e^{b|z-pn|^2/n},$$

provided that $|z - pn| \geq \varepsilon_3 n$.

Further we know that there exist positive constants $\widetilde{c}$ and u such that $\mathbb{E}\left[\exp\left(\sup_{0\le t\le 1} y|B^\sigma_t|\right)\right] \le \widetilde{c}e^{uy^2}$ for any $y > 0$ (see e.g. (6.5) in [LF]). Clearly, there exists $\hat{a}_2$ (depending on b) such that if $0 < a < \hat{a}_2$ then $18ua^2 \le b\varepsilon_3^2$. This implies that if $a < a_0 := \min(\hat{a}, \hat{a}_2)$ then

$$\mathbb{E}\left[\exp\left(3a \sup_{0\le t\le 1} \sqrt{n}|B^\sigma_t|\right)\right] + \exp(3a|z|) + \mathbb{E}\left[\exp\left(3a \max_{1\le k\le n} |S_k|\right) \Big| S_n = z\right] \le [\hat{C} + \widetilde{c}]e^{b|z-pn|^2/n},$$

provided that $|z - pn| \ge \varepsilon_3 n$.

The latter has the following implication. Firstly, (4.86) will hold for any n-coupling with $C = \hat{C}_1 := \widetilde{c} + \hat{C}$, $\alpha' = 0$ and $a \in (0, a_0)$ if $z \in L_n$ satisfies $|z - pn| \ge \varepsilon_3 n$. Moreover, we can find a constant $\hat{C}_2 > 1$ such that if $a < a_0$, $|z - pn| \le \varepsilon_3 n$ and $n \le 4N_5$ then

$$\mathbb{E}\left[\exp\left(3a \sup_{0\le t\le 1} \sqrt{n}|B^\sigma_t|\right)\right] + \exp(3a|z|) + \mathbb{E}\left[\exp\left(3a \max_{1\le k\le n} |S_k|\right) \Big| S_n = z\right] \le \hat{C}_2.$$

For the remainder of the proof we take $b_1 = b/20$ and let a_1, c_1 be as in Lemma 4.5.1 for this value of b_1. We will take $a = (1/2) \cdot \min(a_0, a_1)$ and $C = \max(\hat{C}_1, \hat{C}_2)$ as above and show how to construct the n-coupling so that (4.86) holds for some α'.

We will show that for every positive integer s, there exist n-couplings for all $n \le 2^s$ such that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] e^{-b|z-pn|^2/n} \le A^{s-1} \cdot C, \qquad \forall z \in L_n, \tag{4.87}$$

where $A = 1 + 2c_1(1 + 8c_2 b^{-1/2})$. The theorem clearly follows from this claim.

We proceed by induction on s with base case $s = 1$ being true by our choice of C above. We suppose our claim is true for s and let $2^s < n \le 2^{s+1}$. We will show how to construct a probability space on which we have a Brownian bridge and a family of processes $\{S^{(n,z)} : |z - pn| \le \varepsilon_3 n\}$, which satisfy (4.87). Afterwards we can adjoin (after possibly enlarging the probability space) the processes for $|z| > n\varepsilon_3$. Since $C \ge \hat{C}_1$ and $a < a_0$ we know that (4.87) will continue to hold for these processes as well. Hence, we assume that $|z - pn| \le \varepsilon_3 n$.

If $2^{s+1} \le 4N_5$ then by our choice of $C \ge \hat{C}_2$ and the fact that $A > 1$ we will have that (4.87) holds for any coupling provided $|z - pn| \le \varepsilon_3 n$. We may thus assume that $2^s > 2N_5$. For simplicity we assume that $n = 2k$, where $k \ge N_5$ is an integer such that $2^{s-1} < k \le 2^s$ (if n is odd we write $n = k + (k + 1)$ and do a similar argument).

We define the n-coupling as follows:

- Choose two independent k-couplings

$$\left(\{S^{1(k,z)}\}_{z\in L_k}, B^1\right), \qquad \left(\{S^{2(k,z)}\}_{z\in L_k}, B^2\right), \text{ satisfying (4.87).}$$

  Such a choice is possible by the induction hypothesis.

- We let W^z and ξ be as in the statement of Lemma 4.5.1, and set $Z^z = \frac{z}{2} + \frac{\sqrt{n}\sigma_p}{2} \cdot \xi$. Assume, as we may, that all of these random variables are independent of the two k-couplings chosen above. Observe that by our choice of a and $k \geq N_5$ we have that

$$\mathbb{E}\left[e^{a|Z^z - W^z|}\Big|W^z\right] \leq c_1 \cdot \exp\left(\frac{b}{20} \cdot \frac{(W^z - kp)^2 + (z-np)^2}{n}\right). \tag{4.88}$$

- Let

$$B_t = \begin{cases} 2^{-1/2}B^1_{2t} + t\sqrt{p(1-p)}\xi & 0 \leq t \leq 1/2, \\ 2^{-1/2}B^2_{2(t-1/2)} + (1-t)\sqrt{p(1-p)}\xi & 1/2 \leq t \leq 1. \end{cases} \tag{4.89}$$

  By Lemma 6.5 in [LF], B_t is a Brownian bridge with variance σ^2.

- Let $S_k^{(n,z)} = W^z$, and

$$S_m^{(n,z)} = \begin{cases} S_m^{1(k,W^z)} & 0 \leq m \leq k, \\ W^z + S_{m-k}^{2(k,z-W^z)}, & k \leq m \leq n. \end{cases}$$

  What we have done is that we first chose the value of $S_k^{(n,z)}$ from the conditional distribution of S_k, given $S_n = z$. Conditioned on the midpoint $S_k^{(n,z)} = W^z$ the two halves of the random walk bridge are independent and upto a trivial shift we can use $S^{1(k,W^z)}$ and $S^{2(k,z-W^z)}$ to build them.

The above defines our coupling and what remains to be seen is that it satisfies (4.87) with $s+1$.

Note that

$$\Delta(n,z,S^{(n,z)},B) \leq |Z^z - W^z| + \max\left(\Delta(k,W^z,S^{1(k,W^z)},B^1), \Delta(k,z-W^z,S^{2(k,z-W^z)},B^2)\right)$$

and therefore almost surely

$$\mathbb{E}\left[e^{a\Delta(n,z)}\Big|W^z\right] \leq \mathbb{E}\left[e^{a|Z^z-W^z|}\Big|W^z\right] \times CA^{s-1}\left(e^{b|W^z-kp|^2/k} + e^{b|z-W^z-kp|^2/k}\right).$$

In deriving the last expression we used that our two k-couplings satisfy (4.87) and the simple inequality $\mathbb{E}[e^{\max(Z_1,Z_2)}] \leq \mathbb{E}[e^{Z_1}] + \mathbb{E}[e^{Z_2}]$. Taking expectation on both sides above we see that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq C \cdot (2c_1) \cdot A^{s-1}\mathbb{E}\left[\exp\left(\frac{9}{4} \cdot \frac{b\max(|W^z-kp|^2, |z-W^z-kp|^2)}{n}\right)\right]. \tag{4.90}$$

In deriving the last expression we used (4.88) and the simple inequality $x^2 + y^2 \leq 5\max(x^2,(x-y)^2)$ as well as that $k = n/2$.

We finally estimate the expectation in (4.90) by splitting it over W^z such that $|W^z - z/2| > |z-pn|/6$ and $|W^z - z/2| \leq |z-pn|/6$; we call the latter events E_1 and E_2 respectively. Notice that if $|W^z - z/2| \leq |z-pn|/6$ we have $\max(|W^z-pk|^2, |z-W^z-pk|^2) \leq (2|z-pn|/3)^2$; hence

$$\mathbb{E}\left[\exp\left(\frac{9}{4} \cdot \frac{\max(|W^z-kp|^2, |z-W^z-kp|^2)}{n}\right) \cdot \mathbf{1}\{E_2\}\right] \leq \exp\left(\frac{|z-pn|^2}{n}\right). \tag{4.91}$$

To handle the case $|W^z - z/2| > |z - pn|/6$ we use Lemma 4.5.2, from which we know that

$$f_{m,n}(W_z|z) \leq c_2 n^{-1/2} \exp\left(-b_2 \frac{(W^z - (z/2))^2}{n}\right).$$

Using the latter together with the fact that for $|W^z - z/2| > |z - pn|/6$ we have that $(W^z - z/2)^2 > \frac{1}{16}\max\left((W^z - kp)^2, |z - W^z - kp|^2\right)$ we see that

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{9}{4} \cdot \frac{b \max(|W^z - kp|^2, |z - W^z - kp|^2)}{n}\right) \cdot \mathbf{1}\{E_1\}\right] \leq \\ &c_2 n^{-1/2} \int_{\mathbb{R}} \exp\left(-\frac{b}{16} \cdot \frac{(y - kp)^2}{n}\right) dy = c_2 n^{-1/2} 4 \frac{\pi^{1/2} n^{1/2}}{b^{1/2}} \leq 8 c_2 b^{-1/2}. \end{aligned} \tag{4.92}$$

Combining the above estimates we see that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq C \cdot (2c_1) \cdot A^{s-1}\left[\exp\left(\frac{|z - pn|^2}{n}\right) + 8c_2 b^{-1/2}\right] \leq C \cdot A^s \exp\left(\frac{|z - pn|^2}{n}\right).$$

The above concludes the proof.

□

4.5.2 Discrete case

We use the same notation as in Sections 4.1.2 and 4.3.

Lemma 4.5.4. *Suppose that p_X satisfies Assumptions D1-D4 and fix $p \in (\alpha, \beta)$. Let $s = p - \varepsilon'$ and $t = p + \varepsilon'$, where $\varepsilon' > 0$ is sufficiently small so that $\alpha < s < t < \beta$. Then there exists $\varepsilon_3 \in (0, \varepsilon')$ and $N_3 \in \mathbb{N}$ such that for every $b_1 > 0$ there exist constants $0 < c_1, a_1 < \infty$ such that the following holds. Suppose that m, n are integers such that $m, n \geq N_3$ with $|m - n| \leq 1$, set $N = m + n$. We can define a probability space on which are defined a standard normal random variable ξ and a collection of random variables $W = W^{(m,n,z)}$ for all $z \in \{x \in L_N : |x - pN| \leq \varepsilon_3 N\}$ such that the law of $W^{(m,n,z)}$ is given by $p_{n,m}(\cdot|z)$ and such that we have almost surely*

$$\mathbb{E}\left[e^{a_1|Z-W|}\middle|W\right] \leq c_1 \cdot \exp\left(b_1 \frac{(W - pn)^2 + (z - pN)^2}{N}\right), \tag{4.93}$$

where

$$Z = Z^{(m,n,z)} = \frac{z}{2} + \frac{\sqrt{N}\sigma_p}{2} \cdot \xi, \text{ so that } Z \sim N\left(\frac{z}{2}, \frac{\sigma_p^2 N}{4}\right).$$

The constants ε_3 and N_3 depend on the values p, s, t and the function $p_X(\cdot)$, where the dependence on the latter is through the constants in Definition 4.3.8.

Proof. Notice that we only need to prove the lemma for N sufficiently large. In order to simplify the notation we will assume that $n = m = N/2$ (the other cases can be handled similarly).

We apply Propositions 4.3.5 and 4.3.9 for the variables s and t. This implies that provided $N_3 \geq \max(N_0, N_1)$ as in the statements of those propositions and $n \geq N_3$ we have that the random variable $S_n^{(N,z)} - z/2$ satisfies the conditions of Lemma 4.4.4 for $M_0 = M$ as in Proposition 4.3.5, $\varepsilon_0 = \varepsilon'$ as in the statement of this proposition, $\widetilde{c} = (1/2) \cdot \min(m_{\hat{s},\hat{t}}, M_{\hat{s},\hat{t}}^{-1})$ as in Definition 4.3.1 for the variables $\hat{s}, \hat{t}$ as in Definition 4.3.7, $b' = a$ and $c' = A$ as in the statement of Proposition 4.3.9. We consequently, let c_2, N_2, ε_2 be as in the statement of that corollary for the above constants.

In what follows we fix $\varepsilon_3 \leq 4^{-1} \min(\varepsilon_2, \varepsilon')$ sufficiently small so that $\varepsilon_3 M \leq 1/M_{\hat{s},\hat{t}}$ where M is as in the statement of Proposition 4.3.5 and $M_{\hat{s},\hat{t}}$ is as in Definition 4.3.1 for the variables $\hat{s}, \hat{t}$ as in Definition 4.3.7. Observe that the choice of ε_3 implies that $\varepsilon_3 M \leq 1/\sigma_{z/N}^2$ for all $|z - pN| \leq N\varepsilon_3$. We also set $N_3 = \max(N_0, N_1, N_2)$.

Let $\hat{A} = \{x \in \mathbb{Z} : x \in [z/2 - 2\varepsilon_3 n, z/2 + 2\varepsilon_3 n]\}$ and let $\hat{a}_1, \dots, \hat{a}_k$ be an enumeration of the elements in $\hat{A}$ in increasing order. Let $G = G_{n,z}$ denote the cumulative distribution function of $S_n^{(N,z)}$. In addition, we let Φ denote the cumulative distribution function of a standard normal random variable. Since Φ is strictly increasing and $p_{n,m}(\hat{a}|z) > 0$ for all $\hat{a} \in \hat{A}$ we can define the unique real numbers r_{j-} and r_j for $j = 1, \dots, k$ that satisfy

$$\Phi(r_{j-}) = G(\hat{a}_j-), \qquad \Phi(r_j) = G(\hat{a}_j).$$

Suppose that we have a probability space that supports three independent variables W_-, W_+ and ξ, where ξ is a standard normal random variable, W_- has the distribution of $S_n^{(N,z)}$ conditioned on being less than $\hat{a}_1$ and W_+ has the distribution of $S_n^{(N,z)}$ conditioned on being larger than $\hat{a}_k$. Set

$$Z = Z_{n,z} = z/2 + \frac{\sigma_p \sqrt{N}}{2} \cdot \xi, \qquad \hat{Z} = \hat{Z}_{n,z} = z/2 + \frac{\sigma_{z/N} \sqrt{N}}{2} \cdot \xi.$$

Given a realization of ξ, W_- and W_+ we define a random variable W as follows.

- if $r_{j-} < \xi \leq r_j$ we set $W = \hat{a}_j$;
- if $\xi \leq r_{1-}$ we set $W = W_-$;
- if $\xi \geq r_k$ we set $W = W_+$.

It is easy to see that as defined W indeed has the same distribution as $S_n^{(N,z)}$. In words, W is *quantile coupled* to ξ near 0 and independent from it for large values.

We denote

$$Z = Z_{n,z} = z/2 + \frac{\sigma_p \sqrt{N}}{2} \cdot \xi, \qquad \hat{Z} = \hat{Z}_{n,z} = z/2 + \frac{\sigma_{z/N} \sqrt{N}}{2} \cdot \xi.$$

and write $F = F_{n,z}$ for the distribution function of $\hat{Z}$. It is easy to check that our construction satisfies the following property. If $j = 1, \dots, k$ and

$$F(\hat{a}_j - x) \leq G(\hat{a}_j-) < G(\hat{a}_j) \leq F(\hat{a}_j + x),$$

then

$$|\hat{Z} - W| = |\hat{Z} - \hat{a}_j| \leq x \qquad \text{on the event } \{W = \hat{a}_j\} \text{ for } j = 1, \dots, k. \tag{4.94}$$

By our choice of ε_3, N_3 and c_2 and Lemma 4.4.4 we have that for all $j = 1, \dots, k$ and $n \geq N_3$

$$F\left(\hat{a}_j - c_2\left[1 + \frac{(\hat{a}_j - z/2)^2}{n}\right]\right) \leq G(\hat{a}_j) \leq F\left(\hat{a}_j + c_2\left[1 + \frac{(\hat{a}_j - z/2)^2}{n}\right]\right). \tag{4.95}$$

Combining (4.94) and (4.95) we get

$$|\hat{Z} - W| \leq c_2\left[1 + \frac{(W - z/2)^2}{n}\right] \text{ on the event } W \in \hat{A}, \tag{4.96}$$

for all $n \geq N_3$, provided that $|z - pN| \leq \varepsilon_3 N$, $|W - z/2| \leq 2\varepsilon_3 n$.

We next claim that $|r_{1-}| = O(\sqrt{N})$ and $|r_k| = O(\sqrt{N})$. To see the latter notice that

$$\mathbb{P}(\xi \geq r_k) = \mathbb{P}(W \geq z/2 + 2n\varepsilon_3) = 1 - \mathbb{P}(W - z/2 \leq 2n\varepsilon_3) \geq 1 - \mathbb{P}\left(\hat{Z} - \frac{z}{2} \leq 2n\varepsilon_3 + c_2\left(1 + \frac{4n^2(\varepsilon_3)^2}{n}\right)\right)$$

$$= \mathbb{P}\left(\frac{\sigma_{z/N}\sqrt{N}}{2} \cdot \xi \geq 2n\varepsilon_3 + c_2[1 + 4n(\varepsilon_3)^2]\right) \geq \mathbb{P}(\xi \geq \widetilde{C}\sqrt{N}),$$

for some positive constant $\widetilde{C}$. The inequality in the first line follows from Lemma 4.4.4 applied to $W - z/2$. The above implies that $r_k \leq \widetilde{C}\sqrt{N}$ and an anologous argument shows that $r_{1-} \geq -\widetilde{C}\sqrt{N}$ for some possibly larger $\widetilde{C}$. We conclude that there is a constant $\widetilde{C} > 0$ such that $|\xi| \leq \widetilde{C}\sqrt{N}$ on the event $W \in \hat{A}$.

The latter implies that almost surely on the event $W \in \hat{A}$ we have

$$\mathbb{E}\left[e^{|Z - \hat{Z}|}\Big|W\right] \leq \mathbb{E}\left[e^{|\xi||\sigma_p - \sigma_{z/N}|}\Big|W\right] \leq \mathbb{E}\left[e^{\widetilde{C}\sqrt{N}|\sigma_p - \sigma_{z/N}|}\Big|W\right].$$

From Lemma 4.1.4 we know that we can find a constant $c_p > 0$, that depends on $m_{\hat{s},\hat{t}}$ and $M_{\hat{s},\hat{t}}$ as in Definition 4.3.1 as well as $M^{(3)}_{\hat{s},\hat{t}}$ as in Definition 4.3.4 for the variables $\hat{s}, \hat{t}$ as in Definition 4.3.7, such that $|\sigma_p - \sigma_{z/N}|^2 \leq c_p|p - z/N|^2$ for all $|z - pN| \leq \varepsilon_3 N$. Combining the latter with the Cauchy-Schwarz inequality, (4.96) and the triangle inequality we conclude that there are constants $C, c > 0$ such that if $|W - z/2| \leq 2\varepsilon_3 n$ and $|z - pN| \leq \varepsilon_3 N$ then

$$\mathbb{E}\left[e^{|W - Z|}\Big|W\right] \leq \mathbb{E}\left[e^{|W - \hat{Z}| + |Z - Z|}\Big|W\right] \leq C\exp\left(\frac{c_p(z - pN)^2}{N} + \frac{c(W - z/2)^2}{n}\right).$$

Applying Jensen's inequality to the above we have for any $v \in \mathbb{N}$ that

$$\mathbb{E}\left[e^{(1/v)|W - Z|}\Big|W\right] \leq \mathbb{E}\left[e^{|W - Z|}\Big|W\right]^{1/v} \leq C^{1/v}\exp\left(\frac{c_p(z - pN)^2}{Nv} + \frac{c(W - z/2)^2}{nv}\right),$$

and if we further use that $(x + y)^2 \leq 2x^2 + 2y^2$ above we see that

$$\mathbb{E}\left[e^{(1/v)|W - Z|}\Big|W\right] \leq C^{1/v} \cdot \exp\left(\frac{[c_p + c](z - pN)^2}{Nv} + \frac{4c(W - pn)^2}{Nv}\right), \tag{4.97}$$

provided $n \geq N_3$, $|w - z/2| \leq 2\varepsilon_3 n$ and $|z - pN| \leq \varepsilon_3 N$.

Suppose now that b_1 is given, and let v be sufficiently large so that

$$\frac{c_p + c}{v} \leq b_1 \text{ and } \frac{4c}{v} \leq b_1.$$

If $a_1 \leq 1/v$ we see from (4.97) that

$$\mathbb{E}\left[e^{a_1|W-Z|}\middle|W\right] \leq C \cdot \exp\left(\frac{b_1(z-pN)^2}{N} + \frac{b_1(w-pn)^2}{N}\right), \tag{4.98}$$

provided $n \geq N_3$, $|w - z/2| \leq 2\varepsilon_3 n$ and $|z - pN| \leq \varepsilon_3 N$.

Suppose now that $|W - z/2| > 2\varepsilon_3 n$ and suppose for concreteness that $W - z/2 \geq 2\varepsilon_3 n$. On the event $\{W > z/2 + 2\varepsilon_3 n\}$ we have that W and Z are independent with Z having the distribution of a normal random variable with mean $z/2$ and variance $\frac{\sigma_p^2 N}{4}$ conditioned on being larger than $s := z/2 + \frac{\sigma_p\sqrt{N}}{2} \cdot r_k$. It follows that almost surely on $\{W > z/2 + 2\varepsilon_3 n\}$

$$\mathbb{E}\left[e^{|W-Z|}\middle|W\right] \leq e^{|W-z/2|} \cdot \int_{r_k}^{\infty} \frac{e^{\frac{\sigma_p\sqrt{N}}{2}|y|}e^{-y^2/2}}{\sqrt{2\pi}} \cdot (1 - \Phi(r_k))^{-1}.$$

From our earlier work we know that $r_k \leq \widetilde{C}\sqrt{N}$ for some $\widetilde{C} > 0$. This implies that

$$1 - \Phi(r_k) \geq e^{-cN\varepsilon_3^2},$$

for some sufficiently large $c > 0$. Combining the last two inequalities gives for some new $c > 0$

$$\mathbb{E}\left[e^{|W-Z|}\middle|W\right] \leq \exp\left(cN + |W - z/2|\right) \leq \exp\left((c + 5/4)N + \frac{(z-pN)^2}{N} + \frac{(W-pn)^2}{N}\right),$$

where the last inequality uses the triangle inequality and the fact that $\sqrt{ab} \leq a + b$ for $a, b \geq 0$. Applying Jensen's inequality to the above we have for any $v \in \mathbb{N}$ that

$$\mathbb{E}\left[e^{(1/v)|W-Z|}\middle|W\right] \leq \mathbb{E}\left[e^{|W-Z|}\middle|W\right]^{1/v} \leq \exp\left(\frac{(c+5/4)N}{v} + \frac{(z-pN)^2}{vN} + \frac{(W-pn)^2}{vN}\right).$$

In particular, suppose that v is sufficiently large so that

$$\frac{1}{v} \leq \frac{b_1}{2} \text{ and } \frac{c + 5/4}{v} \leq \frac{b_1\varepsilon_3^2}{8}$$

and $a_1 \leq 1/v$. We then have from the above inequality that

$$\mathbb{E}\left[e^{a_1|W-Z|}\middle|W\right] \leq \exp\left(\frac{b_1\varepsilon_3^2}{16}N + \frac{b_1(z-pN)^2}{2N} + \frac{b_1(W-pn)^2}{2N}\right) \leq \exp\left(\frac{b_1(z-pN)^2}{N} + \frac{b_1(W-pn)^2}{N}\right),$$

where in the last inequality we used that $|W - z/2| \geq 2\varepsilon_3 n$ and $|z/2 - pn| \leq \varepsilon_3 n$. We conclude that (4.98) holds even when $W - z/2 > 2\varepsilon_3 n$. An analogous argument shows that (4.98) also holds when $W - z/2 < -2\varepsilon_3 n$, and so almost surely for all W. This suffices for the proof.

□

We also isolate for future use the following statement.

Lemma 4.5.5. *Assume the same notation as in Lemma 4.5.4. There exist positive constants b_2, c_2, N_4 such that for every integers $m, n \geq N_4$, $N = m + n$ such that $|m - n| \leq 1$, every $z \in \{x \in L_N : |x - pN| \leq \varepsilon_3 N\}$ and $w \in \mathbb{Z}$,*

$$p_{m,n}(w|z) \leq c_2 N^{-1/2} \exp\left(-b_2 \frac{(w - (z/2))^2}{N}\right).$$

The constants b_2, c_2, N_4 depend on s, t, p and the constants in Definition 4.3.8.

Proof. This is an immediate corollary of Propositions 4.3.5 and 4.3.9. □

We now turn to the main theorem of this section.

Theorem 4.5.6. *Suppose that p_X satisfies Assumptions D1-D5 and fix $p \in (\alpha, \beta)$. Let $s = p - \varepsilon'$ and $t = p + \varepsilon'$, where $\varepsilon' > 0$ is sufficiently small so that $\alpha < s < t < \beta$. For every $b > 0$, there exist constants $0 < C, a, \alpha' < \infty$ such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = \sigma_p^2$ and the family of processes $S^{(n,z)}$ for $z \in L_n$ such that*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha'(\log n)} e^{b|z-pn|^2/n}, \tag{4.99}$$

where $\Delta(n, z) = \Delta(n, z, B^\sigma, S^{(n,z)}) = \sup_{0 \leq t \leq n} \left|\sqrt{n} B^\sigma_{t/n} + \frac{t}{n} z - S_t^{(n,z)}\right|$. The constants C, a, α' depend on b as well as s, t, p and p_X through the constants in Definition 4.3.8 and the functions in Assumption D5.

Proof. It suffices to prove the theorem when b is sufficiently small. For the remainder we fix $b > 0$ such that $b < b_2/37$, where b_2 is the constant from Lemma 4.5.5. Let ε_3 and N_3 be as in Lemma 4.5.4 and N_4 as in Lemma 4.5.5 for our choice of s, t and put $N_5 = \max(N_3, N_4)$.

In this proof, by an *n-coupling* we will mean a probability space on which are defined a Brownian bridge B^σ and the family of processes $\{S^{(n,z)} : z \in L_n\}$. Notice that for any n-coupling if $z \in L_n$, $S_t = S_t^{(n,z)}$ then

$$\Delta(n, z) = \sup_{0 \leq t \leq n} \left|\sqrt{n} B^\sigma_{t/n} + \frac{t}{n} z - S_t^{(n,z)}\right| \leq |z| + \max_{0 \leq k \leq n} |S_k^{(n,z)}| + \sup_{0 \leq t \leq n} |\sqrt{n} B^\sigma_{t/n}|$$

which implies

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq \mathbb{E}\left[\exp\left(3a \sup_{0 \leq t \leq 1} \sqrt{n} |B^\sigma_t|\right)\right] + \exp(3a|z|) + \mathbb{E}\left[\exp\left(3a \max_{1 \leq k \leq n} |S_k|\right) \Big| S_n = z\right].$$

Note that if $|z - pn| \geq \varepsilon_3 n$ we have

$$b|z - pn|^2/n \geq \frac{b\varepsilon_3^2 n}{2} + \frac{b|z - pn|^2}{2n} \geq \frac{b\varepsilon_3^2 n}{2} + b\kappa \frac{z^2}{n},$$

where κ is sufficiently small so that

$$\kappa < 1/2, \qquad \frac{p}{1 - 2\kappa} \in [p - \varepsilon_3, p + \varepsilon_3], \text{ and } \varepsilon_3/2 - \kappa(\pm\varepsilon_3 + p)^2 > 0.$$

In view of the above and Assumption D5 there exists $\hat{a}$ small enough and $\hat{C}$ large enough depending on b such that if $a < \hat{a}$ we can ensure that

$$\exp(3a|z|) + \mathbb{E}\left[\exp\left(3a \max_{1\le k\le n}|S_k|\right)\Big|S_n = z\right] \le \hat{C}e^{b|z-pn|^2/n},$$

provided that $|z - pn| \ge \varepsilon_3 n$.

Further we know that there exist positive constants $\widetilde{c}$ and u such that $\mathbb{E}\left[\exp\left(\sup_{0\le t\le 1} y|B_t^\sigma|\right)\right] \le \widetilde{c}e^{uy^2}$ for any $y > 0$ (see e.g. (6.5) in [LF]). Clearly, there exists $\hat{a}_2$ (depending on b) such that if $0 < a < \hat{a}_2$ then $18ua^2 \le b\varepsilon_3^2$. This implies that if $a < a_0 := \min(\hat{a}, \hat{a}_2)$ then

$$\mathbb{E}\left[\exp\left(3a \sup_{0\le t\le 1}\sqrt{n}|B_t^\sigma|\right)\right] + \exp(3a|z|) + \mathbb{E}\left[\exp\left(3a \max_{1\le k\le n}|S_k|\right)\Big|S_n = z\right] \le [\hat{C} + \widetilde{c}]e^{b|z-pn|^2/n},$$

provided that $|z - pn| \ge \varepsilon_3 n$.

The latter has the following implication. Firstly, (4.99) will hold for any n-coupling with $C = \hat{C}_1 := \widetilde{c} + \hat{C}$, $\alpha' = 0$ and $a \in (0, a_0)$ if $z \in L_n$ satisfies $|z - pn| \ge \varepsilon_3 n$. Moreover, we can find a constant $\hat{C}_2 > 1$ such that if $a < a_0$, $|z - pn| \le \varepsilon_3 n$ and $n \le 4N_5$ then

$$\mathbb{E}\left[\exp\left(3a \sup_{0\le t\le 1}\sqrt{n}|B_t^\sigma|\right)\right] + \exp(3a|z|) + \mathbb{E}\left[\exp\left(3a \max_{1\le k\le n}|S_k|\right)\Big|S_n = z\right] \le \hat{C}_2.$$

For the remainder of the proof we take $b_1 = b/20$ and let a_1, c_1 be as in Lemma 4.5.4 for this value of b_1. We will take $a = (1/2)\cdot\min(a_0, a_1)$ and $C = \max(\hat{C}_1, \hat{C}_2)$ as above and show how to construct the n-coupling so that (4.99) holds for some α'.

We will show that for every positive integer s, there exist n-couplings for all $n \le 2^s$ such that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right]e^{-b|z-pn|^2/n} \le A^{s-1}\cdot C, \qquad \forall z \in L_n, \tag{4.100}$$

where $A = 1 + 2c_1(1 + c_2(8b^{-1/2} + 2))$. The theorem clearly follows from this claim.

We proceed by induction on s with base case $s = 1$ being true by our choice of C above. We suppose our claim is true for s and let $2^s < n \le 2^{s+1}$. We will show how to construct a probability space on which we have a Brownian bridge and a family of processes $\{S^{(n,z)} : |z - pn| \le \varepsilon_3 n\}$, which satisfy (4.100). Afterwards we can adjoin (after possibly enlarging the probability space) the processes for $|z| > n\varepsilon_3$. Since $C \ge \hat{C}_1$ and $a < a_0$ we know that (4.100) will continue to hold for these processes as well. Hence, we assume that $|z - pn| \le \varepsilon_3 n$.

If $2^{s+1} \le 4N_5$ then by our choice of $C \ge \hat{C}_2$ and the fact that $A > 1$ we will have that (4.100) holds for any coupling provided $|z - pn| \le \varepsilon_3 n$. We may thus assume that $2^s > 2N_5$. For simplicity we assume that $n = 2k$, where $k \ge N_5$ is an integer such that $2^{s-1} < k \le 2^s$ (if n is odd we write $n = k + (k + 1)$ and do a similar argument).

We define the n-coupling as follows:

- Choose two independent k-couplings

$$\left(\{S^{1(k,z))}\}_{z\in L_k}, B^1\right), \quad \left(\{S^{2(k,z))}\}_{z\in L_k}, B^2\right), \text{ satisfying (4.87).}$$

  Such a choice is possible by the induction hypothesis.

- We let W^z and ξ be as in the statement of Lemma 4.5.4, and set $Z^z = \frac{z}{2} + \frac{\sqrt{n}\sigma_p}{2}\cdot\xi$. Assume, as we may, that all of these random variables are independent of the two k-couplings chosen above. Observe that by our choice of a and $k \geq N_5$ we have that

$$\mathbb{E}\left[e^{a|Z^z - W^z|}\middle|W^z\right] \leq c_1 \cdot \exp\left(\frac{b}{20}\cdot\frac{(W^z-kp)^2+(z-np)^2}{n}\right). \tag{4.101}$$

- Let

$$B_t = \begin{cases} 2^{-1/2}B^1_{2t} + t\sqrt{p(1-p)}\xi & 0 \leq t \leq 1/2, \\ 2^{-1/2}B^2_{2(t-1/2)} + (1-t)\sqrt{p(1-p)}\xi & 1/2 \leq t \leq 1. \end{cases} \tag{4.102}$$

  By Lemma 6.5 in [LF], B_t is a Brownian bridge with variance σ^2.

- Let $S_k^{(n,z)} = W^z$, and

$$S_m^{(n,z)} = \begin{cases} S_m^{1(k,W^z)} & 0 \leq m \leq k, \\ W^z + S_{m-k}^{2(k,z-W^z)}, & k \leq m \leq n. \end{cases}$$

  What we have done is that we first chose the value of $S_k^{(n,z)}$ from the conditional distribution of S_k, given $S_n = z$. Conditioned on the midpoint $S_k^{(n,z)} = W^z$ the two halves of the random walk bridge are independent and upto a trivial shift we can use $S^{1(k,W^z)}$ and $S^{2(k,z-W^z)}$ to build them.

The above defines our coupling and what remains to be seen is that it satisfies (4.100) with $s+1$.

Note that

$$\Delta(n,z,S^{(n,z)},B) \leq |Z^z - W^z| + \max\left(\Delta(k,W^z,S^{1(k,W^z)},B^1), \Delta(k,z-W^z,S^{2(k,z-W^z)},B^2)\right)$$

and therefore almost surely

$$\mathbb{E}\left[e^{a\Delta(n,z)}\middle|W^z\right] \leq \mathbb{E}\left[e^{a|Z^z-W^z|}\middle|W^z\right] \times CA^{s-1}\left(e^{b|W^z-kp|^2/k} + e^{b|z-W^z-kp|^2/k}\right).$$

In deriving the last expression we used that our two k-couplings satisfy (4.100) and the simple inequality $\mathbb{E}[e^{\max(Z_1,Z_2)}] \leq \mathbb{E}[e^{Z_1}] + \mathbb{E}[e^{Z_2}]$. Taking expectation on both sides above we see that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq C\cdot(2c_1)\cdot A^{s-1}\mathbb{E}\left[\exp\left(\frac{9}{4}\cdot\frac{b\max(|W^z-kp|^2,|z-W^z-kp|^2)}{n}\right)\right]. \tag{4.103}$$

In deriving the last expression we used (4.101) and the simple inequality $x^2 + y^2 \leq 5 \max(x^2, (x-y)^2)$ as well as that $k = n/2$.

We finally estimate the expectation in (4.103) by splitting it over W^z such that $|W^z - z/2| > |z-pn|/6$ and $|W^z - z/2| \leq |z-pn|/6$; we call the latter events E_1 and E_2 respectively. Notice that if $|W^z - z/2| \leq |z-pn|/6$ we have $\max(|W^z - pk|^2, |z - W^z - pk|^2) \leq (2|z-pn|/3)^2$; hence

$$\mathbb{E}\left[\exp\left(\frac{9}{4} \cdot \frac{\max(|W^z - kp|^2, |z - W^z - kp|^2)}{n}\right) \cdot \mathbf{1}\{E_2\}\right] \leq \exp\left(\frac{|z-pn|^2}{n}\right). \tag{4.104}$$

To handle the case $|W^z - z/2| > |z - pn|/6$ we use Lemma 4.5.5, from which we know that

$$p_{m,n}(W^z|z) \leq c_2 n^{-1/2} \exp\left(-b_2 \frac{(W^z - (z/2))^2}{n}\right).$$

Using the latter together with the fact that for $|W^z - z/2| > |z - pn|/6$ we have that $(W^z - z/2)^2 > \frac{1}{16} \max\left((W^z - kp)^2, |z - W^z - kp|^2\right)$ we see that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{9}{4} \cdot \frac{b\max(|W^z - kp|^2, |z - W^z - kp|^2)}{n}\right) \cdot \mathbf{1}\{E_1\}\right] \leq \\ &c_2 n^{-1/2} \sum_{y \in \mathbb{Z}} \exp\left(-\frac{b}{16} \cdot \frac{(y-kp)^2}{n}\right) \leq c_2 n^{-1/2}\left[2 + 4\frac{\pi^{1/2} n^{1/2}}{b^{1/2}}\right] \leq c_2(8b^{-1/2} + 2).\end{aligned} \tag{4.105}$$

Combining the above estimates we see that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq C \cdot (2c_1) \cdot A^{s-1}\left[\exp\left(\frac{|z-pn|^2}{n}\right) + c_2(8b^{-1/2}+2)\right] \leq C \cdot A^s \exp\left(\frac{|z-pn|^2}{n}\right).$$

The above concludes the proof.

□

4.6 Assumptions D5 and C6

4.6.1 Strongly unimodal distributions

In this section we give sufficient conditions for the technical Assumptions D5 and C6 to hold.

4.6.1.1 Continuous case

The goal of this section is to give general conditions under which a distribution satisfying Assumptions C1-C5 will also satisfy Assumption C6. We use the same notation as in Sections 4.1.1 and 4.2.

Let us introduce some useful notation. Let f be a continuous probability density function on $\mathbb{R}$. We say that f is *unimodal* if there exists at least one real number M such that

$$f(x) \leq f(y) \text{ for all } x \leq y \leq M, \text{ and } f(x) \leq f(y) \text{ for all } x \geq y \geq M.$$

We further say that $f(\cdot)$ is *strongly unimodal* if the convolution of $f(\cdot)$ with any unimodal distribution function $h(\cdot)$ on $\mathbb{R}$ is again unimodal. In [Ibr56], the authors proved that $f(\cdot)$ is strongly unimodal if and only if it is log-concave, i.e. $\log f$ is concave.

Definition 4.6.1. *Suppose that f_X satisfies Assumptions C1-C5 and $\alpha = -\infty$, $\beta = \infty$. It follows from Assumption C2 that X has all finite moments and we let $\mu = \mathbb{E}[X]$. In addition, we have $\Lambda'(0) = \frac{M_X'(0)}{M_X(0)} = \mu$ and so $u_\mu = (\Lambda')^{-1}(\mu) = 0$ and $G_\mu(u_\mu) = \Lambda(u_\mu) - u_\mu \cdot \mu = 0$. The latter and Proposition 4.2.4 imply that there is a constant $\Delta > 0$ such that for all $n \geq 1$ we have*

$$\inf_{x \in [-1,1]} f_n(n\mu + x) \geq n^{-1/2}\Delta.$$

Indeed, the latter is obvious from (4.10) for all large n and for small n we can deduce it from the continuity and positivity of $f_n(n\mu + x)$ on the interval $[-1, 1]$ from Assumption C1. The above implies that we can find a constant $R > 0$ such that $R > |\mu| + 1 + \Delta^{-1}$.

In view of Proposition 4.2.4 applied to $s = -2R$ and $t = 2R$ we also deduce that there are positive constants C_R and c_R such that for all $n \geq 1$ and $z \in [-2R, 2R]$

$$f_n(nz) \geq C_R n^{-1/2} e^{-c_R n}.$$

As before the above follows from Proposition 4.2.4 provided n is sufficiently large, while for small n it follows from the continuity and positivity of $f_n(nz)$ on $[-2R, 2R]$.

Finally, given the above constants, λ as in Assumption C2 and L as in Assumption C5, we can find constants $\hat{C}_R$ and $\hat{c}_R$ such that for all $n \geq 1$ we have

$$\mathbb{E}[e^{\lambda|X|}]^n \left[\frac{4n^{3/2}}{\Delta} + LC_R^{-1}\sqrt{n}e^{c_R n}\right] \leq \hat{C}_R \cdot e^{\hat{c}_R n}.$$

The main result of the section is as follows.

Lemma 4.6.2. *Suppose that f_X satisfies Assumptions C1-C5. Then it will also satisfy Assumption C6 if any of the following hold*

- *$\alpha > -\infty$;*
- *$\beta < \infty$;*
- *$\alpha = -\infty$, $\beta = \infty$ and the density function $f(x)$ of X is a strongly unimodal function.*

Moreover, if $\alpha > -\infty$ then we can take $\hat{a}(\hat{b}) = \frac{\hat{b}}{1+\hat{b}+|\alpha|}$ and $\hat{C}(\hat{b}) = 1$; if $\beta < \infty$ then we can take $\hat{a}(\hat{b}) = \frac{\hat{b}}{1+\hat{b}+|\beta|}$ and $\hat{C}(\hat{b}) = 1$. If $\alpha = -\infty$ and $\beta = \infty$ then we can choose $\hat{a}(\hat{b}) = \lambda v^{-1}$ and $\hat{C}(\hat{b}) = \hat{C}_R^{1/v}$, where v is a large enough integer such that $c_R v^{-1} \leq \hat{b}/2$ and $\lambda v^{-1} \leq \hat{b}/2$ with $c_R, \hat{C}_R$ as in Definition 4.6.1 and λ as in Assumption C2.

Proof. Assume first that $\alpha > -\infty$. Then we have for any $k \in \{1, \dots, n\}$ and $z \in L_n$ that

$$S_k \geq -k\alpha \text{ and } S_n - S_k \geq -(n-k)\alpha \text{ almost surely.}$$

The latter implies that

$$|z| + n|\alpha| \geq |S_k|,$$

which means using the ineqiality $|xy| \leq x^2 + y^2$ that

$$\mathbb{E}\left[\exp\left(\hat{a} \max_{1 \leq k \leq n} |S_k|\right) \Big| S_n = z\right] \leq \exp\left(\hat{a}|z| + \hat{a}|\alpha|n\right) \leq \exp\left(\hat{a}|z|^2/n + \hat{a}n + \hat{a}|\alpha|n\right).$$

Thus if we choose $\hat{C} = 1$ and $\hat{a} = \frac{\hat{b}}{1+\hat{b}+|\alpha|}$ we would obtain (4.3). An analogous argument establishes (4.3) when $\beta < \infty$.

In the remainder we focus on the last case. Notice that by assumption $f_m(x)$ are unimodal functions for any $m \geq 1$. For future use we call $\mu = \mathbb{E}[X]$ and for $|t| \leq \lambda$ as in Assumption C2 we set $M_{|X|}(t) = \mathbb{E}\left[e^{t|X|}\right]$. We also let Δ, R, C_R, c_R, $\hat{C}_R$ and $\hat{c}_R$ be as in Definition 4.6.1.

By definition we have for $m \geq 1$ that

$$\inf_{x \in [-1,1]} f_m\left(m\mu + x\right) \geq m^{-1/2} \cdot \Delta,$$

The latter implies that if M_m is any real number such that $f_m(x) \leq f_m(y)$ for all $x \leq y \leq M_m$ and $f_m(x) \leq f_m(y)$ for all $x \geq y \geq M_m$, we then have $|M_m| \leq Rm$. Indeed, if we suppose for example that $M_m > Rm$ then this would mean that $f_m(t + m\mu) \geq f_m(m\mu)$ for all $t \in [0, (1 + \Delta^{-1})m]$, so

$$\int_0^{(1+\Delta^{-1})m} f_m(t + m\mu) \geq \left((1 + \Delta^{-1})m + 1\right) \cdot f_m(m\mu) \geq (1 + \Delta^{-1})m^{1/2} \cdot \Delta > 1,$$

which is impossible. One rules out the case $M_m < -Rm$ in a similar fashion.

Let us now fix $n \geq 1$, $1 \leq m < n$, $|z| > 2Rn$ and $\lambda > 0$ as in Assumption C2. We then have that

$$\begin{aligned} &\mathbb{E}\left[e^{\lambda|S_m|}\Big|S_n = z\right] = (I) + (II) + (III), \text{ where } (I) = \frac{\int_{|t| \leq |z|+Rn} f_m(t) f_{n-m}(z-t) e^{\lambda|t|} dt}{\int_{\mathbb{R}} f_m(t) f_{n-m}(z-t) dt}, \\ &(II) = \frac{\int_{t > |z|+Rn} f_m(t) p_{n-m}(z-t) e^{\lambda|t|} dt}{\int_{\mathbb{R}} f_m(t) f_{n-m}(z-t) dt}, (III) = \frac{\int_{t < -|z|-Rn} f_m(t) f_{n-m}(z-t) e^{\lambda|t|} dt}{\int_{\mathbb{R}} f_m(t) f_{n-m}(z-t) dt}. \end{aligned} \tag{4.106}$$

Firstly, we have the trivial bound

$$(I) \leq e^{\lambda Rn + \lambda|z|}. \tag{4.107}$$

In addition, if $z < -2Rn$ then by the unimodality of the density function $f_{n-m}(\cdot)$ we get

$$(II) \leq \frac{\int_{t>|z|+Rn} f_m(t) f_{n-m}(z-t) e^{\lambda|t|} dt}{\int_{m\mu}^{m\mu+1} f_m(t) f_{n-m}(z-t) dt} \leq \frac{\sqrt{n}}{c} \cdot \int_{t>|z|+Rn} f_m(t) e^{\lambda|t|} dt \leq \frac{\sqrt{n}}{\Delta} \mathbb{E}\left[e^{\lambda|S_m|}\right] \leq \frac{\sqrt{n}}{\Delta} M_{|X|}(\lambda)^n.$$

On the other hand, if $z > 2Rn$ we have by the unimodality of $f_m(\cdot)$ that

$$(II) \leq \frac{\int_{t > Rn + |z|} f_m(t) f_{n-m}(z-t) e^{\lambda |t|} dt}{\int_{\mu(n-m)}^{\mu(n-m)+1} f_m(z-t) f_{n-m}(t) dt} \leq \frac{\sqrt{n}}{\Delta} \cdot \int_{t > Rn + |z|} f_{n-m}(z-t) e^{\lambda |t| dt} \leq \frac{\sqrt{n}}{\Delta} e^{\lambda z} M_{|X|}(\lambda)^n.$$

Applying the same arguments to (III) and combining the cases $z > 2Rn$ and $z < -2Rn$ we conclude that if $|z| > 2Rn$ we have

$$(II) + (III) \leq \frac{4\sqrt{n}}{\Delta} e^{\lambda |z|} M_{|X|}(\lambda)^n \tag{4.108}$$

Combining (4.107) and (4.108) and the inequality

$$\mathbb{E}\left[\exp\left(\lambda \max_{1 \leq k \leq n} |S_k| \right) \Big| S_n = z \right] \leq \sum_{m=1}^{n} \mathbb{E}\left[e^{\lambda |S_m|} \Big| S_n = z \right],$$

we conclude that if $|z| > 2Rn$ then

$$\mathbb{E}\left[\exp\left(\lambda \max_{1 \leq k \leq n} |S_k| \right) \Big| S_n = z \right] \leq \frac{4n^{3/2}}{\Delta} e^{\lambda |z|} M_{|X|}(\lambda)^n. \tag{4.109}$$

Suppose now that $|z| \leq 2Rn$. Then by definition we have

$$\mathbb{E}\left[e^{\lambda |S_m|} \Big| S_n = z \right] = \frac{\int_{\mathbb{R}} f_m(t) f_{n-m}(z-t) e^{\lambda |t|} dt}{f_n(z)} \leq C_R^{-1} \sqrt{n} e^{c_R n} \int_{\mathbb{R}} f_m(t) f_{n-m}(z-t) e^{\lambda |t|} dt$$
$$\leq L C_R^{-1} \sqrt{n} e^{c_R n} \int_{\mathbb{R}} f_m(t) e^{\lambda |t|} dt \leq L C_R^{-1} \sqrt{n} e^{c_R n} M_{|X|}(\lambda)^n,$$

where L is as in Assumption C5.

Combining the latter with (4.109) we conclude that for any $z \in \mathbb{R}$ we have

$$\mathbb{E}\left[\exp\left(\lambda \max_{1 \leq k \leq n} |S_k| \right) \Big| S_n = z \right] \leq \left[\frac{4n^{3/2}}{\Delta} + L C_R^{-1} \sqrt{n} e^{c_R n} \right] \cdot M_{|X|}(\lambda)^n \cdot e^{\lambda |z|} \leq \hat{C}_R \cdot e^{\hat{c}_R n + \lambda |z|}. \tag{4.110}$$

From Jensen's inequality and (4.110) we know that for any $v \in \mathbb{N}$

$$\mathbb{E}\left[\exp\left(\lambda v^{-1} \max_{1 \leq k \leq n} |S_k| \right) \Big| S_n = z \right] \leq \hat{C}_R^{1/v} \cdot e^{v^{-1} \hat{c}_R n + v^{-1} \lambda |z|}. \tag{4.111}$$

Suppose now that $\hat{b} > 0$ is given. Then we can choose v sufficiently large so that $\lambda / v \leq \hat{b}/2$ and $c_R / v \leq \hat{b}/2$. Consequently, if we set $\hat{a} = \lambda v^{-1}$ and $\hat{C} = \hat{C}_R^{1/v}$ we would have in view of (4.111)

$$\mathbb{E}\left[\exp\left(\hat{a} \max_{1 \leq k \leq n} |S_k| \right) \Big| S_n = z \right] \leq \hat{C} \cdot e^{(\hat{b}/2)(|z| + n)} \leq \hat{C} \cdot e^{\hat{b}(n + z^2/n)},$$

where we used that $|z|/2 \leq z^2/n + n/2$ as follows by the Cauchy-Schwarz inequality. □

4.6.1.2 Discrete case

In this section we give general conditions under which a distribution satisfying Assumptions D1-D4 will also satisfy Assumption D5. We use the same notation as in Sections 4.1.2 and 4.3.

We first introduce some useful notation. Let $p(n)$ be a probability mass function on $\mathbb{Z}$. We say that p is *unimodal* if there exists at least one integer M such that

$$p(n) \geq p(n-1) \text{ for all } n \leq M, \text{ and } p(n+1) \leq p(n) \text{ for all } n \geq M.$$

We further say that $p(\cdot)$ is *strongly unimodal* if the convolution of $p(\cdot)$ with any unimodal distribution function $h(\cdot)$ on $\mathbb{Z}$ is again unimodal. In [KG71, Theorem 3], inspired by the classical work of [Ibr56], the authors proved that $p(\cdot)$ is strongly unimodal if and only if

$$p(n)^2 \geq p(n-1)p(n+1) \text{ for all } n \in \mathbb{Z}. \tag{4.112}$$

Definition 4.6.3. *Suppose that p_X satisfies Assumptions D1-D4 and $\alpha = -\infty$, $\beta = \infty$. It follows from Assumption D2 that X has all finite moments and we let $\mu = \mathbb{E}[X]$. In addition, we have $\Lambda'(0) = \frac{M_X'(0)}{M_X(0)} = \mu$ and so $u_\mu = (\Lambda')^{-1}(\mu) = 0$ and $G_\mu(u_\mu) = \Lambda(u_\mu) - u_\mu \cdot \mu = 0$. The latter and Proposition 4.3.3 imply that there is a constant $\Delta > 0$ such that for all $n \geq 1$ we have*

$$p_n(\lfloor \mu m \rfloor) \geq n^{-1/2}\Delta.$$

Indeed, the latter is obvious from (4.37) for all large n and for small n we can deduce it from the positivity of $p_n(\lfloor \mu n \rfloor)$ from Assumption C1. The above implies that we can find a constant $R > 0$ such that $R > |\mu| + 1 + \Delta^{-1}$.

In view of Proposition 4.3.3 applied to $s = -2R$ and $t = 2R$ we also deduce that there are positive constants C_R and c_R such that for all $n \geq 1$ and $z \in [-2R, 2R] \cap L_n$

$$p_n(z) \geq C_R n^{-1/2} e^{-c_R n}.$$

As before the above follows from Proposition 4.3.3 provided n is sufficiently large, while for small n it follows from the positivity of $p_n(z)$ on $[-2R, 2R] \cap L_n$.

Finally, given the above constants, we can find constants $\hat{C}_R$ and $\hat{c}_R$ such that for all $n \geq 1$ we have

$$\mathbb{E}[e^{\lambda |X|}]^n \left[\frac{4n^{3/2}}{\Delta} + LC_R^{-1}\sqrt{n}e^{c_R n} \right] \leq \hat{C}_R \cdot e^{\hat{c}_R n}.$$

The main result of the section is as follows.

Lemma 4.6.4. *Suppose that p_X satisfies Assumptions D1-D4. Then it will also satisfy Assumption D5 if any of the following hold*

- $\alpha > -\infty$;
- $\beta < \infty$;
- $\alpha = -\infty$, $\beta = \infty$ *and $p_X(n)$ is a strongly unimodal function.*

Moreover, if $\alpha > -\infty$ then we can take $\hat{a}(\hat{b}) = \frac{\hat{b}}{1+\hat{b}+|\alpha|}$ and $\hat{C}(\hat{b}) = 1$; if $\beta < \infty$ then we can take $\hat{a}(\hat{b}) = \frac{\hat{b}}{1+\hat{b}+|\beta|}$ and $\hat{C}(\hat{b}) = 1$. If $\alpha = -\infty$ and $\beta = \infty$ then we can choose $\hat{a}(\hat{b}) = \lambda v^{-1}$ and $\hat{C}(\hat{b}) = \hat{C}_R^{1/v}$, where v is a large enough integer such that $c_R v^{-1} \leq \hat{b}/2$ and $\lambda v^{-1} \leq \hat{b}/2$ with $c_R, \hat{C}_R$ as in Definition 4.6.3 and λ as in Assumption D2.

Proof. The cases $\alpha > -\infty$ and $\beta < \infty$ can be handled exactly the same as in the proof of Lemma 4.6.2. We focus on the case $\alpha = -\infty$ and $\beta = \infty$ in the remainder.

Notice that by assumption $p_m(n)$ are unimodal functions for any $m \geq 1$. For future use we call $\mu = \mathbb{E}[X]$ and for $|t| \leq \lambda$ as in Assumption D2 we set $M_{|X|}(t) = \mathbb{E}\left[e^{t|X|}\right]$.

By definition we have for $m \geq 1$ that

$$p_m\left(\lfloor m\mu \rfloor\right) \geq m^{-1/2} \cdot \Delta,$$

The latter implies that if M_m is any integer such that $p_m(x) \leq p_m(y)$ for all $x \leq y \leq M_m$ and $p_m(x) \leq p_m(y)$ for all $x \geq y \geq M_m$, we then have $|M_m| \leq Rm$. Indeed, if we suppose for example that $M_m > Rm$ then $p_m(n + \lfloor m\mu \rfloor) \geq p_m(\lfloor m\mu \rfloor)$ for all $n = 0, \dots, \lfloor (1+\Delta^{-1})m \rfloor$ and so

$$\sum_{n=0}^{\lfloor (1+\Delta^{-1})m \rfloor} p_m(n + \lfloor m\mu \rfloor) \geq (\lfloor (1+\Delta^{-1})m \rfloor + 1) \cdot p_m(\lfloor m\mu \rfloor) \geq (1+\Delta^{-1})m \frac{\Delta}{\sqrt{m}} > 1,$$

which is impossible. One rules out the case $M_m < -Rm$ in a similar fashion.

Let us now fix $n \geq 1$, $1 \leq m < n$, $|z| > 2Rn$ and $\lambda > 0$ as in Assumption D2. We then have that

$$\mathbb{E}\left[e^{\lambda|S_m|}\Big|S_n = z\right] = (I) + (II) + (III), \text{ where } (I) = \frac{\sum_{|k| \leq |z|+Rn} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}}{\sum_{k\in\mathbb{Z}} p_m(k)p_{n-m}(z-k)},$$
$$(II) = \frac{\sum_{k > |z|+Rn} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}}{\sum_{k\in\mathbb{Z}} p_m(k)p_{n-m}(z-k)}, (III) = \frac{\sum_{k < -|z|-Rn} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}}{\sum_{k\in\mathbb{Z}} p_m(k)p_{n-m}(z-k)}. \quad (4.113)$$

Firstly, we have the trivial bound

$$(I) \leq e^{\lambda Rn + \lambda|z|}. \quad (4.114)$$

In addition, we have that if $z < -2Rn$ then by the unimodality of the sequence $p_{n-m}(\cdot)$ we get

$$(II) \leq \frac{\sum_{k > Rn+|z|} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}}{p_m(M_m)p_{n-m}(z-M_m)} \leq \frac{\sqrt{n}}{c} \cdot \sum_{k > Rn+|z|} p_m(k)e^{\lambda|k|} \leq \frac{\sqrt{n}}{c}\mathbb{E}\left[e^{\lambda|S_m|}\right] \leq \frac{\sqrt{n}}{c} M_{|X|}(\lambda)^n.$$

On the other hand, if $z > 2Rn$ we have by the unimodality of $p_m(\cdot)$ that

$$(II) \leq \frac{\sum_{k > Rn+|z|} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}}{p_m(z-M_{n-m})p_{n-m}(M_{n-m})} \leq \frac{\sqrt{n}}{c} \cdot \sum_{k > Rn+|z|} p_{n-m}(z-k)e^{\lambda|k|} \leq \frac{\sqrt{n}}{c} e^{\lambda z} M_{|X|}(\lambda)^n.$$

Applying the same arguments to (III) and combining the cases $z > 2Rn$ and $z < -2Rn$ we conclude that if $|z| > 2Rn$ we have

$$(II) + (III) \leq \frac{4\sqrt{n}}{c} e^{\lambda|z|} M_{|X|}(\lambda)^n \tag{4.115}$$

Combining (4.114) and (4.115) and the inequality

$$\mathbb{E}\left[\exp\left(\lambda \max_{1\leq k\leq n} |S_k|\right) \Big| S_n = z\right] \leq \sum_{m=1}^{n} \mathbb{E}\left[e^{\lambda|S_m|}\Big|S_n = z\right], \tag{4.116}$$

we conclude that if $|z| > 2Rn$ then

$$\mathbb{E}\left[\exp\left(\lambda \max_{1\leq k\leq n} |S_k|\right) \Big| S_n = z\right] \leq \frac{4n^{3/2}}{c} e^{\lambda|z|} M_{|X|}(\lambda)^n \cdot e^{\lambda Rn}, \tag{4.117}$$

where we apply inequality $e^x + e^y \leq e^{x+y}$ for $x, y \geq 1$.

Suppose now that $|z| \leq 2Rn$. Then by definition we have

$$\mathbb{E}\left[e^{\lambda|S_m|}\Big|S_n = z\right] = \frac{\sum_{k\in\mathbb{Z}} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}}{p_n(z)} \leq C_R^{-1}\sqrt{n}e^{c_R n} \sum_{k\in\mathbb{Z}} p_m(k)p_{n-m}(z-k)e^{\lambda|k|}$$

$$\leq C_R^{-1}\sqrt{n}e^{c_R n} \sum_{k\in\mathbb{Z}} p_m(k)e^{\lambda|k|} = C_R^{-1}\sqrt{n}e^{c_R n} M_{|X|}(\lambda)^n.$$

Combining the latter with (4.117) we conclude that for any $z \in \mathbb{R}$ we have

$$\mathbb{E}\left[\exp\left(\lambda \max_{1\leq k\leq n} |S_k|\right) \Big| S_n = z\right] \leq \left[\frac{4n^{3/2}}{\Delta} + C_R^{-1}\sqrt{n}e^{c_R n}\right] \cdot M_{|X|}(\lambda)^n \cdot e^{\lambda|z|} \leq \hat{C}_R \cdot e^{\hat{c}_R n + \lambda|z|}. \tag{4.118}$$

From here the proof proceeds as that of Lemma 4.6.2. □

4.6.2 Insufficiency of Assumptions D1-D4

In this section we construct a probability distribution p_X, which satisfies Assumptions D1-D4, but for which the statement of Theorem 4.1.6 does not hold. The example illustrates that in general one needs further assumptions on p_X in order to ensure the strong coupling of random walk bridges with step distribution p_X and Brownian bridges of fixed variance.

We will use the same notation as in Section 4.1.1. Suppose that $A = \{x \in \mathbb{Z} : x = 3^n + n \text{ for some } n \in \mathbb{N}\}$ and $B = \{x \in \mathbb{Z} : x = -3^n \text{ for some } n \in \mathbb{N}\}$. For convenience we denote $a_n = 3^n + n$ and $b_n = -3^n$ for $n \geq 1$ and note that these are distinct integers. We define a weight function w as follows

$$w(x) = \begin{cases} \exp(-x^2) \text{ if } x \in A \cup B, \\ \exp(-g(x)) \text{ if } \notin A \cup B, \text{ where } g(x) = 10^{10^{|x|}} \end{cases} \tag{4.119}$$

Observe that $w(x) > 0$ for all $x \in \mathbb{Z}$ and $w(x) \leq e^{-x^2}$ for all $x \in \mathbb{Z}$. This means that $Z := \sum_{x\in\mathbb{Z}} w(x) < \infty$ and the function

$$p_X(x) := w(x) \cdot Z^{-1} \tag{4.120}$$

defines a probability mass function on $\mathbb{Z}$. We note that p_X satisfies Assumption D1, with $\alpha = -\infty$ and $\beta = \infty$; Assumption D2 with any $\lambda > 0$, in particular we have $\mathcal{D}_\Lambda = \mathbb{R}$ and so by Lemma 4.1.4 we know that Λ_X is continuous on $\mathbb{R}$ so that Assumption D3 is also satisfied. Finally, by definition $p_X(x) \leq Z^{-1}e^{-x^2}$ and so Assumption D4 is satisfied with $D = Z^{-1}$ and $d = 1$. Overall, we see that p_X satisfies Assumptions D1-D4.

Suppose now that $S^{(n,z)}$ is a random walk bridge whose steps size is p_X. We want to show that for any $a, c, C > 0$ and $\sigma > 0$ and any coupling of $S^{(2,z)}$ with a Brownian bridge B^σ of variance σ^2 there exists a $z \in \mathbb{Z}$ such that

$$\mathbb{E}\left[e^{a\Delta(2,z)}\right] \geq Ce^{c|z|^2}, \tag{4.121}$$

where $\Delta(n,z) = \Delta(n,z,B^\sigma,S^{(n,z)}) = \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \frac{t}{n}z - S_t^{(n,z)}\right|$. The latter statement implies that we cannot couple the bridge of size two to any fixed variance Brownian bridge uniformly in the endpoint z, which means that Theorem 4.1.6 fails to hold for this bridge.

Remark 4.6.5. *Let us heuristically explain why the above example breaks the coupling. The distribution in (4.120) satisfies the condition that it has spikes at the points in A and B and is extremely small away from those sets. The latter means that for certain large enough z, we will have that conditional on $X_1 + X_2 = z$, with overwhelming probability $X_1 = 3^z + z$ and $X_2 = -3^z$ or $X_1 = -3^z$ and $X_2 = 3^z + z$. The latter implies that the midpoint of the bridge is essentially a Bernoulli variable that takes the values $3^z + z$ and -3^z with equal probability. This makes its variance increase as we increase z, which makes a close coupling to a Brownian bridge of fixed variance impossible.*

The main take-away point is that while p_X may be an extremely well-behaved distribution, the conditional distribution of the midpoint of a Bridge with step size p_X can become quite singular in the presence of spikes in p_X. This means that one needs better control of the conditional distribution, and one way to achieve this is to assume p_X has no spikes. This is one reason behind our introduction of the strongly log-concave distributions in Section 4.6.1 above.

In the remainder we prove (4.121). We will prove that there are large enough z such that

$$\mathbb{E}\left[e^{a|S_1^{(2,z)} - \sqrt{2}B^\sigma_{1/2} - z/2|}\right] \geq Ce^{c|z|^2},$$

which certainly implies (4.121). Using that $e^{a|x-y|} \geq e^{a|x| - a|y|} \geq e^{(a/2)|x|} - e^{a|y|}$ we see that

$$\mathbb{E}\left[e^{a|S_1^{(2,z)} - \sqrt{2}B^\sigma_{1/2} - z/2|}\right] \geq \mathbb{E}\left[e^{(a/2)|S_1^{(2,z)}|}\right] + \mathbb{E}\left[e^{a|\sqrt{2}B^\sigma_{1/2}| + |az/2|}\right] = \mathbb{E}\left[e^{(a/2)|S_1^{(2,z)}|}\right] - \mathbb{E}\left[e^{a|\sqrt{2}B^\sigma_{1/2}| + |az/2|}\right].$$

Furthermore we have

$$\mathbb{E}\left[e^{a|\sqrt{2}B^\sigma_{1/2}| + |az/2|}\right] \leq e^{|az|/2} \cdot \left[\mathbb{E}\left[\exp\left(a\sqrt{2}B^\sigma_{1/2}\right)\right] + \mathbb{E}\left[\exp\left(-a\sqrt{2}B^\sigma_{1/2}\right)\right]\right] = 2e^{|az|/2 + a^2\sigma^2/4}.$$

Combining the above statements we see that to prove (4.121) it is enough to show that for any fixed $a, c, C > 0$ we can find large enough z so that

$$\mathbb{E}\left[e^{a|S_1^{(2,z)}|}\right] \geq Ce^{c|z|^2}. \tag{4.122}$$

This is the statement we will establish.

We claim that if $z = 2 \cdot 3^m$ with m sufficiently large we have

$$3p_1(a_z)p_1(b_z) \geq p_2(z). \tag{4.123}$$

If true the above would imply

$$\mathbb{E}\left[e^{a|S_1^{(2,z)}|}\right] = \sum_{k\in\mathbb{Z}} \frac{p_1(k)p_1(z-k)e^{ak}}{p_2(z)} \geq \frac{p_1(a_z)p_1(b_z)e^{aa_z}}{p_2(z)} \geq \frac{1}{3} \cdot e^{a(3^z+z)},$$

which certainly implies (4.122). We thus focus on (4.123).

We have for all $m \geq 2$ that

$$\begin{aligned} &p_2(z) \leq (I) + (II), \text{ where } (I) = 2\sum_{r=1}^{\infty} \mathbf{1}\{k \notin A, z-k \notin A\}p_1(k)p_1(z-k), \\ &(II) = 2\sum_{r=1}^{\infty} p_1(a_r)p_1(z-a_r). \end{aligned} \tag{4.124}$$

If $r \leq m$ then we have $4 \leq a_r \leq 3^m + m$ and so $3^{m+1}+m \geq 2\cdot 3^m - 4 \geq z - a_r \geq 2\cdot 3^m - 3^m - m \geq 3^{m-1}+m$. This means that $z - a_r \notin A \cup B$ and so

$$\sum_{r=1}^{m} p_1(a_r)p_1(z-a_r) \leq \sum_{r=1}^{m} p_1(z-a_r) \leq Z^{-1}\cdot m \cdot \exp\left(-g(3^{m-1}+m)\right). \tag{4.125}$$

If $m < r < 2\cdot 3^m$ then we have $z - a_r = 2\cdot 3^m - 3^r - r$ and so

$$-3^{r-1} > z - a_r > -3^r.$$

This means that $z - a_r \notin A \cup B$ and so

$$\sum_{r=m+1}^{z-1} p_1(a_r)p_1(z-a_r) \leq \sum_{r=m+1}^{z-1} p_1(z-a_r) \leq Z^{-1}\cdot(z-m)\cdot \exp\left(-g(3^m)\right). \tag{4.126}$$

If $2\cdot 3^m < r$ then

$$-3^r > z - a_r = 2\cdot 3^m - 3^r - r > -3^{r+1}.$$

This means that $z - a_r \notin A \cup B$ and so

$$\sum_{r=z+1}^{\infty} p_1(a_r)p_1(z-a_r) \leq \sum_{r=z+1}^{\infty} p_1(z-a_r) \leq Z^{-1}\cdot \sum_{r=z+1}^{\infty} \exp\left(-g(3^r)\right). \tag{4.127}$$

Combining (4.125), (4.126) and (4.127) we have

$$(II) - 2p_1(a_z)\cdot p_1(z-a_z) \leq Z^{-1}\cdot \sum_{r=z+1}^{\infty} \exp\left(-g(3^r)\right) + Z^{-1}\cdot z\cdot \exp(-g(z/6)) \leq e^{-g(z/10)}, \tag{4.128}$$

where the last inequality holds provided m (and hence z) is sufficiently large. On the other hand,

$$p_1(a_z) \cdot p_1(z - a_z) = \exp(-a_z^2) \cdot \exp(-b_z^2) = \exp\left(-(3^z + z)^2 - 3^{2z}\right) \geq 10 \cdot e^{-g(z/10)},$$

for all large enough m and so we conclude that for all large m and $z = 2 \cdot 3^m$ we have

$$(II) \leq (2.2) \cdot p_1(a_z) \cdot p_1(z - a_z). \tag{4.129}$$

We next focus on (I). Notice that if $k \leq 3^m$ then $z - k \geq 3^m$ and so

$$\sum_{r=1}^{z/2} \mathbf{1}\{k \notin A, z - k \notin A\} p_1(k) p_1(z - k) \leq \sum_{r=1}^{z/2} \mathbf{1}\{k \notin A, z - k \notin A\} p_1(z - k) \leq (z/2) \cdot \exp(-g(z/2)).$$

In addition, we have

$$\sum_{r=z/2+1}^{\infty} \mathbf{1}\{k \notin A, z - k \notin A\} p_1(k) p_1(z - k) \leq$$

$$\sum_{r=z/2+1}^{\infty} \mathbf{1}\{k \notin A, z - k \notin A\} p_1(k) \leq \sum_{r=z/2+1}^{\infty} \exp(-g(r)) \leq \exp(-g(z/3)),$$

where the last inequality holds for all large enough m. Combining the latter we get for all large m

$$(I) \leq z \cdot \exp(-g(z/2)) + 2 \cdot \exp(-g(z/3)) \leq \exp(-g(z/10)) \leq (0.1) \cdot p_1(a_z) \cdot p_1(z - a_z). \tag{4.130}$$

Combining (4.129) and (4.130) we conclude (4.123), which concludes our proof.

4.7 Examples

In this section we present several examples of distributions that satisfy Assumptions C1-C6 in Section 4.7.1 and Assumptions D1-D5 in Section 4.7.2. The goal is to illustrate how to verify that a given distribution satisfies the assumptions and in particular prove Theorems 4.1.3 and 4.1.6. In Section 4.7.3 we discuss an example with the log-gamma distribution with parameter $\gamma > 0$. The log-gamma distribution is of interest to us due to connections to integrable probability and the example we consider is the principal one that motivated our quantified Theorem 4.5.3.

4.7.1 Examples: continuous jumps

We continue with the notation from Section 4.1.1.

Example 1. We consider the distributions in Theorem 4.1.3. By assumption we know that X is a continuous random variable with density $f_X(\cdot)$, which has a compact interval of support $[\alpha, \beta]$ and which is continuously differentiable and positive on (α, β) with a bounded derivative. Since the derivative of f_X is bounded and continuous on (α, β) we conclude that f_X can be continuously extended to $[\alpha, \beta]$ and so Assumption C1 is satisfied. In addition, since X is uniformly bounded, we see that Assumption C2 is satisfied for any $\lambda > 0$ and so $\mathcal{D}_\Lambda = \mathbb{R}$. The latter and Lemma 4.1.1 imply that $\Lambda(\cdot)$ is continuous on $\mathbb{R}$ and so Assumption C3 holds.

We next observe using integration by parts that if $z \in \mathbb{C}$ and $z \neq 0$ we have

$$\int_\alpha^\beta f_X(x)e^{xz}dx = f_X(\beta) \cdot \frac{e^{\beta z}}{z} - f_X(\alpha) \cdot \frac{e^{\alpha z}}{z} - \int_\alpha^\beta f_X'(x) \cdot \frac{e^{xz}}{z} dx.$$

Let us fix $s, t \in \mathbb{R}$ with $\alpha < s < t < \beta$ and suppose that $z = u + iv$ with $u \in [s, t]$. Then the boundedness of $f_X(\cdot)$ and $f_X'(\cdot)$ and the above equation imply that

$$\left| \int_\alpha^\beta f_X(x)e^{xz}dx \right| \leq \frac{K_1(s,t)}{1 + |v|},$$

for some sufficiently large constant $K_1(s,t)$ and so Assumption C4 holds with $p_{s,t} = 1$.

As $f_X(\cdot)$ has compact support and is bounded, Assumption C5 holds as well. In view of Lemma 4.6.2 Assumption C6 also holds. Overall, we conclude that f_X satisfies Assumptions C1-C6 and so by Theorem 4.1.3 we conclude Theorem 4.1.3.

The above example illustrates that our strong coupling result holds for essentially any compactly supported density with a bounded continuous derivative. We next illustrate a case when the support is not compact using the usual exponential distribution.

Example 2. Suppose that X has exponential distribution with parameter $\mu > 0$, i.e. $f_X(x) = \mathbf{1}\{x > 0\} \cdot \mu e^{-\mu x}$. In this case Assumption C1 holds trivially with $\alpha = 0$ and $\beta = \infty$. In addition, we have $M_X(t) = \frac{\mu}{\mu - t}$ and so Assumption C2 holds with any $0 < \lambda < \mu$. Next, we have that $\Lambda(x) = \log(\mu) - \log(\mu - t)$ is lower semi-continuous on $\mathcal{D}_\lambda = (-\infty, \mu)$ and Assumption C3 holds.

Let us fix $s, t \in \mathbb{R}$ with $0 < s < t < \infty$ and suppose that $z = u + iv$ with $u \in [s, t]$. Then we have

$$|M_X(z)| = \left| \frac{\mu}{\mu - z} \right| \leq \frac{K_1(s,t)}{1 + |v|}$$

for some sufficiently large constant $K_1(s,t)$ and so Assumption C4 holds with $p_{s,t} = 1$. Assumption C5 holds trivially as $f_X(x) = 0$ for $x \leq 0$ and Assumption C6 is satisfied in view of Lemma 4.6.2. Overall, we conclude that f_X satisfies Assumptions C1-C6 and so Theorem 4.1.3 holds for random walk bridges with exponential jumps.

4.7.2 Discrete case, geometric distribution

We continue with the notation from Section 4.1.2.

Example 1. We consider the distributions in Theorem 4.1.6. By assumption we know that X is an integer valued random variable with probability mass function $p_X(\cdot)$ such that $p_X(x) > 0$ for all $x \in \mathbb{Z} \cap [\alpha, \beta]$ and $\mathbb{P}(X \in [\alpha, \beta]) = 1$. The latter iplies that p_X satisfies Assumption D1. In addition, since X is uniformly bounded, we see that Assumption D2 is satisfied for any $\lambda > 0$ and so $\mathcal{D}_\Lambda = \mathbb{R}$. The latter and Lemma 4.1.4 imply that $\Lambda(\cdot)$ is continuous on $\mathbb{R}$ and so Assumption D3 holds. As $p_X(\cdot)$ is compactly supported and bounded, Assumption D4 holds as well. In view of Lemma 4.6.4 Assumption D5 also holds. Overall, we conclude that p_X satisfies Assumptions D1-D5 and so by Theorem 4.1.6 we conclude Theorem 4.1.6.

The above example illustrates that our strong coupling result holds for essentially any integer valued variable with a single compact (integer) interval of support. We next illustrate a case when the support is not compact using the usual geometric distribution.

Example 2. Suppose that X has geometric distribution with parameter $q \in (0, 1)$, i.e. $p_X(n) = q \cdot (1-q)^n$ for $n \geq 0$. In this case Assumption D1 holds trivially with $\alpha = 0$ and $\beta = \infty$. In addition, we have $M_X(t) = \frac{q}{1-(1-q)e^t}$ and so Assumption D2 holds with any $0 < \lambda < -\log(1-q)$. Next, we have that $\Lambda(x) = \log(q) - \log(1-(1-q)e^t)$ is lower semi-continuous on $\mathcal{D}_\lambda = (-\infty, -\log(1-q))$ and Assumption D3 holds.

Assumption D4 holds trivially as $p_X(x) = 0$ for $x < 0$ and Assumption D5 is satisfied in view of Lemma 4.6.4. Overall, we conclude that p_X satisfies Assumptions D1-D5 and so Theorem 4.1.6 holds for random walk bridges with geometric jumps.

4.7.3 Example: log-gamma distribution

The log-gamma density function with parameter $\gamma > 0$ is given by

$$f_\gamma(x) = \frac{1}{\Gamma(\gamma)} \exp\left(\gamma x - e^x\right) \text{ for } x \in \mathbb{R}. \tag{4.131}$$

If ξ is a random variable with density f_γ one readily observes that

$$M_\xi(t) = \frac{\Gamma(\gamma + t)}{\Gamma(\gamma)}, \text{ and so } M_\xi(t) < \infty \text{ for } t > -\gamma. \tag{4.132}$$

The above formula also implies that

$$\mathbb{E}[\xi] = m_\gamma = \psi^{(0)}(\gamma) \text{ and } Var(\xi) = \sigma_\gamma^2 = \psi^{(1)}(\gamma), \tag{4.133}$$

where $\psi^{(k)}$ denote the polygamma functions given by

$$\psi^{(-1)}(z) = \log \Gamma(z) \text{ and } \psi^{(k)}(z) = \frac{d^{k+1}}{dz^{k+1}} \psi^{(-1)}(z), \text{ for } k \geq 0. \tag{4.134}$$

We consider in this section random walk bridges as in the setup of Section 4.1.1, whose jump has distribution $X = \frac{\xi - m_\gamma}{\sigma_\gamma}$. To indicate the dependence of the bridges on γ we write $S_\gamma^{(n,z)}$ to denote a process whose law is given by that of a random walk bridge with step distribution X and which is condititioned to end at z after n steps. The main result we wish to establish is the following.

Corollary 4.7.1. *For any $b > 0$ and $\gamma_0 > 0$ there exist constants $0 < C, a, \alpha' < \infty$ such that for every positive integer n and $\gamma \geq \gamma_0$, there is a probability space on which are defined a Brownian bridge B^σ with $\sigma = 1$ and a family of processes $S_\gamma^{(n,z)}$ for $z \in \mathbb{R}$ such that*

$$\mathbb{E}[e^{a\Delta(n,z)}] \leq Ce^{\alpha'(\log n)}e^{bz^2/n}, \tag{4.135}$$

where $\Delta(n,z) = \sup_{0 \leq t \leq n} |\sqrt{n}B_{t/n} + \frac{t}{n}z - S_{\gamma,t}^{(n,z)}|$.

In the remainder of this section we provide the proof of Corollary 4.7.1. The goal is to show that the density

$$f_X(x) = \frac{\sigma_\gamma}{\Gamma(\gamma)} \exp\left(\gamma(\sigma_\gamma x + m_\gamma) - e^{\sigma_\gamma x + m_\gamma}\right) \tag{4.136}$$

satisfies Assumptions C1-C6 and that the constants in Definition 4.2.10 and the functions in Assumption 6 can be chosen uniformly in $\gamma \geq \gamma_0$. If true then Corollary 4.7.1 will follow from Theorem 4.5.3 applied to $p = 0$ and $\varepsilon' = 1$. For clarity we split the proof into several steps and use the same notation as in Section 4.1.1. .

Step 1. In this step we summarize several statements that we will need throughout the proof.

From (4.132) we have

$$M_X(t) = e^{-m_\gamma t/\sigma_\gamma}\frac{\Gamma(\gamma + t/\sigma_\gamma)}{\Gamma(\gamma)} \text{ and } \Lambda(t) = \log[M_X(t)] = \psi^{(-1)}\left(\gamma + \frac{t}{\sigma_\gamma}\right) - \psi^{(-1)}(\gamma) - \frac{m_\gamma t}{\sigma_\gamma}, \tag{4.137}$$

Using (4.136) we have

$$\frac{d}{dx}\log f_X(x) = \sigma_\gamma\left(\gamma - e^{\sigma_\gamma x + m_\gamma}\right) \text{ and } \frac{d^2}{dx^2}\log f_X(x) = -\sigma_\gamma^2 e^{m_\gamma} \cdot e^{\sigma_\gamma x}. \tag{4.138}$$

From [**?**, Lemma 3] we have for $x > 0$

$$\begin{aligned} \log(x) - \frac{1}{x} \leq &\psi^{(0)}(x) \leq \log(x) - \frac{1}{2x} \\ \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \leq &\psi^{(k)}(x) \leq \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \text{ for } k \in \mathbb{N}. \end{aligned} \tag{4.139}$$

Using (4.139) and [AS, (6.3.18)] we know that

$$\sigma_\gamma = \gamma^{-1/2} + O(\gamma^{-1}) \text{ and } m_\gamma = \log\gamma - \frac{1}{2\gamma} + O(\gamma^{-2}) \text{ as } \gamma \to \infty. \tag{4.140}$$

We have the following series representation for $\psi^{(0)}(z)$ for $z \neq 0, -1, -2, \ldots$, see e.g. [AS, 6.3.16],

$$\psi^{(0)}(z) = -\gamma_E + \sum_{n=0}^{\infty}\left[\frac{1}{n+1} - \frac{1}{n+z}\right], \tag{4.141}$$

where γ_E is the Euler constant.

Step 2. In this step we demonstrate that $f_X(\cdot)$ satisfies Assumptions C1-C5.

From (4.136) we know that Assumption C1 holds with $\alpha = -\infty$ and $\beta = \infty$. In addition, from (4.137) we know that Assumption C2 holds for any $0 < \lambda < \sigma_\gamma \cdot \gamma$, in particular it holds when $\lambda = 2^{-1} \cdot \sigma_\gamma \cdot \gamma$. We have $\mathcal{D}_\Lambda = (-\gamma\sigma_\gamma, \infty)$ and $\Lambda(\cdot)$ is lower semi-continuous on $\mathbb{R}$. This verifies Assumption C3.

We isolate the verification of Assumption C4 in the following lemma.

Lemma 4.7.2. *For any $\gamma > 0$ and $-\sigma_\gamma \cdot \gamma < S < T < \infty$ there is a $K_1(S,T,\gamma) > 0$ such that*

$$|M_X(z)| \leq \frac{K_1}{1+|v|}, \text{ where } z = u + iv \text{ with } s \leq u \leq t. \tag{4.142}$$

Proof. From (4.141) we have

$$|M_X(z)| = |M_X(u)| \cdot \left|\frac{M_X(z)}{M_X(u)}\right| = |M_X(u)| \exp\left(\int_0^v \sum_{n=0}^{\infty} Re\left[\frac{i}{n+1} - \frac{i}{n+\gamma+(u+iy)/\sigma_\gamma}\right] dy\right).$$

We observe that

$$Re\left[\frac{i}{n+1} - \frac{i}{n+\gamma+(u+iy)/\sigma_\gamma}\right] = \frac{-y}{(n+\gamma+u/\sigma_\gamma)^2 + y^2}.$$

Combining the last two statements we see

$$|M_X(z)| \leq |M_X(u)| \cdot \exp\left(\int_0^v \frac{-y \cdot dy}{[a^2+y^2]}\right) = \frac{|M_X(u)|}{\sqrt{v^2+a^2}}, \tag{4.143}$$

where $a = \gamma + u/\sigma_\gamma$. The last line proves (4.142). □

In view of (4.142) we conclude that f_X satisfies Assumption C4. We next verify Assumption C5.

Lemma 4.7.3. *For any $\gamma_0 > 0$ there exist constants $L, D, d > 0$ such that*

$$f_X(x) \leq L \text{ for all } x \in \mathbb{R} \text{ and } f_X(x) \leq De^{-dx^2} \text{ for all } x \geq 0. \tag{4.144}$$

Proof. From (4.138) we know that f_X is log-concave and has a unique maximum when $x = x_c = \sigma_\gamma^{-1} \cdot [\log(\gamma) - m_\gamma]$. In particular, this implies that

$$f_X(x) \leq f_X(x_c) = \frac{\sigma_\gamma}{\Gamma(\gamma)} \exp\left(\gamma \log(\gamma) - \gamma\right).$$

The right side above is uniformly bounded on $[\gamma_0, M]$ for any finite M, and as $\gamma \to \infty$ we have by Stirling's approximation formula (see e.g. [AS, 6.1.37]) and (4.140) that

$$\frac{\sigma_\gamma}{\Gamma(\gamma)} \exp\left(\gamma \log(\gamma) - \gamma\right) \sim \frac{1}{\sqrt{2\pi}} \text{ as } \gamma \to \infty.$$

Overall we conclude that we can find L sufficiently large depending on γ_0 alone so that the left inequality in (4.144) holds.

We next fix $x \geq 0$. We have

$$\frac{f_X(x)}{f_X(0)} = \exp\left(\gamma\sigma_\gamma x - e^{\sigma_\gamma x + m_\gamma} + e^{m_\gamma}\right) \leq \exp\left(-\frac{e^{m_\gamma}\sigma_\gamma^2}{2}x^2\right),$$

where in the last inequality we used that $e^a \geq 1 + a + \frac{a^2}{2}$ for $a \geq 0$. We observe by (4.139) that

$$\frac{e^{m_\gamma}\sigma_\gamma^2}{2} \geq \frac{1}{2}e^{-1/\gamma},$$

and so we conclude that

$$f_X(x) \leq f_X(0) \cdot \exp\left(-e^{-1/\gamma_0} \cdot x^2/2\right).$$

This proves the right inequality in (4.144) with $D = L$ and $d = e^{-1/\gamma_0}/2$. □

Step 3. In what follows we fix $-\infty < s < t < \infty$ and set $S_\gamma = u_s = (\Lambda')^{-1}(s)$ and $T_\gamma = u_t = (\Lambda')^{-1}(t)$. We write below $C(\gamma_0, s, t)$ to mean a generic positive constant that depend on s, t and γ_0, whose value may change from line to line. The goal of this step is to show

$$\gamma + S_\gamma\sigma_\gamma^{-1} \geq C(s,t,\gamma_0) \cdot \gamma \text{ and } \gamma + T_\gamma\sigma_\gamma^{-1} \leq C(s,t,\gamma_0) \cdot \gamma. \tag{4.145}$$

From (4.137) we know that

$$\Lambda'(S_\gamma) = \frac{\psi^{(0)}(\gamma + S_\gamma\sigma_\gamma^{-1}) - \psi^{(0)}(\gamma)}{\sigma_\gamma} = s \text{ and } \Lambda'(T_\gamma) = \frac{\psi^{(0)}(\gamma + T_\gamma\sigma_\gamma^{-1}) - \psi^{(0)}(\gamma)}{\sigma_\gamma} = t. \tag{4.146}$$

Combining (4.146) and (4.139) we conclude that

$$\begin{aligned} &\log\left[\gamma + S_\gamma\sigma_\gamma^{-1}\right] - \log[\gamma] - \frac{1}{2(\gamma + S_\gamma\sigma_\gamma^{-1})} + \frac{1}{\gamma} \geq \sigma_\gamma \cdot s \\ &\log\left[\gamma + T_\gamma\sigma_\gamma^{-1}\right] - \log[\gamma] - \frac{1}{\gamma + T_\gamma\sigma_\gamma^{-1}} + \frac{1}{2\gamma} \leq \sigma_\gamma \cdot t. \end{aligned} \tag{4.147}$$

From the first line in (4.147) we see that

$$\log\left[\gamma + S_\gamma\sigma_\gamma^{-1}\right] \geq \log[\gamma] + \sigma_\gamma \cdot s - \frac{1}{\gamma_0}.$$

Exponentiating both sides above and using (4.140) we get the first left part of (4.145).

On the other hand, from the second line in (4.147) we have

$$\log\left[\gamma + T_\gamma\sigma_\gamma^{-1}\right] \leq \log[\gamma] + \sigma_\gamma \cdot t + \frac{1}{\gamma + T_\gamma\sigma_\gamma^{-1}}.$$

Using the left part of (4.145) we have $\gamma + T_\gamma\sigma_\gamma^{-1} \geq \gamma + S_\gamma\sigma_\gamma^{-1} \geq C(s,t,\gamma_0) \cdot \gamma$ and so if we exponentiate both sides of the above equation we conclude the right side of (4.145).

Step 4. In this step we show that we can find $\infty > M_{s,t} > m_{s,t} > 0$ that depend on s,t and γ_0 alone such that if $\gamma \geq \gamma_0$ and $x \in [S_\gamma, T_\gamma]$ we have

$$M_{s,t} \geq \Lambda''(x) \geq m_{s,t}. \tag{4.148}$$

From (4.139) we have that for $x \in [S_\gamma, T_\gamma]$

$$\frac{1}{\sigma_\gamma^2} \cdot \left[\frac{1}{\gamma + S_\gamma \sigma_\gamma^{-1}} + \frac{1}{(\gamma + S_\gamma \sigma_\gamma^{-1})^2} \right] \geq \Lambda''(S_\gamma) \geq \Lambda''(x) = \frac{1}{\sigma_\gamma^2} \cdot \psi^{(1)}\left(\gamma + x\sigma_\gamma^{-1}\right) \geq$$
$$\Lambda''(T_\gamma) \geq \frac{1}{\sigma_\gamma^2} \cdot \left[\frac{1}{\gamma + T_\gamma \sigma_\gamma^{-1}} + \frac{1}{2(\gamma + T_\gamma \sigma_\gamma^{-1})^2} \right].$$

The above inequalities together with (4.145) and (4.140) imply (4.148).

Step 5. We have from (4.140) and (4.145) that there is $\delta^1_{s,t} \in (0,1)$ sufficiently small depending on s,t and γ_0 such that

$$\gamma + \min(S_\gamma, 0) \cdot \sigma_\gamma^{-1} \geq 2\delta^1_{s,t} \cdot \sigma_\gamma^{-1}. \tag{4.149}$$

We fix such a $\delta^1_{s,t}$ and denote $S'_\gamma = S_\gamma - \delta^1_{s,t}$, and $T'_\gamma = T_\gamma + \delta^1_{s,t}$. Notice that if $D_{\delta^1_{s,t}}(S_\gamma, T_\gamma)$ is as in Definition 4.2.1 then $\overline{D}_{\delta^1_{s,t}} \subset \{z \in \mathbb{C} : -\gamma \cdot \sigma_\gamma < Re(z) < \infty\}$. In this step we show that we can find $\hat{M}(s,t,\gamma_0) > 0$, depending on s,t and γ_0, such that

$$|\Lambda(z)| \leq \hat{M}_0(s,t,\gamma_0) \text{ for all } z \in \overline{D}_{\delta^1_{s,t}}(\min(0,S_\gamma), \max(T_\gamma, 0)). \tag{4.150}$$

From (4.137) and (4.141) we have for $x \in (-\gamma \cdot \sigma_\gamma, \infty)$ that

$$\Lambda'(x) = \frac{1}{\sigma_\gamma} \cdot \left[\psi^{(0)}(\gamma + x\sigma_\gamma^{-1}) - \psi^{(0)}(\gamma)\right] \text{ and } \Lambda''(x) = \frac{1}{\sigma_\gamma^2} \cdot \sum_{n=0}^{\infty} \frac{1}{(n + \gamma + x\sigma_\gamma^{-1})^2} > 0,$$

which implies that $x = 0$ is the unique minimizer of $\Lambda(x)$ and the maximum of this function on $[S'_\gamma, T'_\gamma]$ is obtained either when $x = S'_\gamma$ or $x = T'_\gamma$. Furthermore, it follows from (4.140), (4.145) and (4.147) that there is a sufficiently large positive constant $\hat{C}(s,t,\gamma_0) > 0$ such that

$$\hat{C}(s,t,\gamma_0) \geq T'_\gamma > S'_\gamma \geq -\hat{C}(s,t,\gamma_0). \tag{4.151}$$

Combining (4.151) with (4.149) and (4.139) we conclude that there is a sufficiently large positive constant $\hat{M}_1(s,t,\gamma_0) > 0$ such that for $x \in [\min(0, S'_\gamma), \max(T'_\gamma, 0)]$ we have

$$|\Lambda'(x)| \leq \hat{M}_1(s,t,\gamma_0). \tag{4.152}$$

Combining (4.151) and (4.152) with the fact that $\Lambda(0) = 0$ we conclude that there is a sufficiently large constant $\hat{M}_0(s,t,\gamma_0) > 0$ such that for $x \in [\min(0, S'_\gamma), \max(T'_\gamma, 0)]$ we have

$$|\Lambda(x)| \leq \hat{M}_0(s,t,\gamma_0). \tag{4.153}$$

Now we suppose that $x \in [\min(0, S'_\gamma), \max(T'_\gamma, 0)]$ and note that

$$\Lambda'(x+iy) = \frac{1}{\sigma_\gamma^2} \cdot \sum_{n=0}^{\infty} \frac{\sigma_\gamma^{-1} y^2 + x(n+\gamma+x\sigma_\gamma^{-1})}{(n+\gamma)[(n+\gamma+x\sigma_\gamma^{-1})^2 + \sigma_\gamma^{-2} y^2]} + \frac{i}{\sigma_\gamma^2} \sum_{n=0}^{\infty} \frac{y}{(n+\gamma+x\sigma_\gamma^{-1})^2 + \sigma_\gamma^{-2} y^2}. \tag{4.154}$$

where we used (4.141). In particular, we see that

$$\frac{1}{\sigma_\gamma^2} \cdot \sum_{n=0}^{\infty} \frac{\sigma_\gamma^{-1} y^2 + |x|(n+\gamma+x\sigma_\gamma^{-1})}{(n+\gamma)[(n+\gamma+x\sigma_\gamma^{-1})^2 + \sigma_\gamma^{-2} y^2]} \leq \frac{1}{\sigma_\gamma^3 \cdot \gamma} \cdot \sum_{n=0}^{\infty} \frac{y^2}{(n+\gamma+\sigma_\gamma^{-1} x)^2} + \\ + \frac{1}{\sigma_\gamma^2} \cdot \sum_{n=0}^{\infty} \frac{|x|}{(n+\gamma+x\sigma_\gamma^{-1})^2} + \frac{x^2}{\sigma_\gamma^3 \cdot \gamma} \cdot \sum_{n=0}^{\infty} \frac{1}{(n+\gamma+\sigma_\gamma^{-1} x)^2}$$

and also

$$\frac{1}{\sigma_\gamma^2} \sum_{n=0}^{\infty} \frac{|y|}{(n+\gamma+x\sigma_\gamma^{-1})^2 + y^2} \leq \frac{1}{\sigma_\gamma^2} \sum_{n=0}^{\infty} \frac{|y|}{(n+\gamma+x\sigma_\gamma^{-1})^2}.$$

We use that

$$\sum_{n=0}^{\infty} \frac{1}{(n+\gamma+x\sigma_\gamma^{-1})^2} \leq \frac{1}{(\gamma+x\sigma_\gamma^{-1})^2} + \int_0^\infty \frac{1}{(\gamma+x\sigma_\gamma^{-1}+u)^2} du = \frac{1}{(\gamma+x\sigma_\gamma^{-1})^2} + \frac{1}{\gamma+x\sigma_\gamma^{-1}}.$$

Substituting the above inequalities into (4.154) we get for $x \in [\min(0, S'_\gamma), \max(T'_\gamma, 0)]$

$$|\Lambda'(x+iy)| \leq \left[\frac{1}{(\gamma+x\sigma_\gamma^{-1})^2} + \frac{1}{\gamma+x\sigma_\gamma^{-1}} \right] \cdot \left[\frac{y^2}{\sigma_\gamma^3 \cdot \gamma} + \frac{|x|}{\sigma_\gamma^2} + \frac{x^2}{\sigma_\gamma^3 \cdot \gamma} + \frac{|y|}{\sigma_\gamma^2} \right].$$

From (4.145) we have $\gamma + S'_\gamma \sigma_\gamma^{-1} \geq C(s,t,\gamma_0) \cdot \gamma$ and so the above inequality implies

$$|\Lambda'(x+iy)| \leq \frac{C(s,t,\gamma_0)}{\gamma} \cdot \left[\frac{y^2}{\sigma_\gamma^3 \cdot \gamma} + \frac{|x|}{\sigma_\gamma^2} + \frac{x^2}{\sigma_\gamma^3 \cdot \gamma} + \frac{|y|}{\sigma_\gamma^2} \right].$$

If we finally combine the latter with (4.151) and (4.139) we see that

$$|\Lambda'(x+iy)| \leq C(s,t,\gamma_0) \cdot [1+y^2]. \tag{4.155}$$

In view of (4.153) and (4.155) we know that by possibly making $\hat{M}_0(s,t,\gamma_0)$ larger we can ensure that (4.150) holds.

Step 6. In this step we show that we can choose the constants in Definitions 4.2.1 and 4.2.2 uniformly in $\gamma \geq \gamma_0$. We fix $m_{s,t}$ and $M_{s,t}$ as in (4.148) above. From (4.150) and the fact that $x = 0$ is the unique minimizer of $\Lambda(x)$ on $[\min(0, S'_\gamma), \max(T'_\gamma, 0)]$ we get

$$e^{\hat{M}_0(s,t,\gamma_0)} \geq M_X(x) \geq 1. \tag{4.156}$$

Also we have

$$|M_X(x) - M_X(x+iy)| = M_X(x) \cdot \left| 1 - \exp\left(\int_0^y i\Lambda'(x+iu) du \right) \right| \leq C(s,t,\gamma_0) \cdot |y|.$$

The latter implies that we can pick $0 < \delta_{s,t} \leq \delta^1_{s,t}$ sufficiently small depending on s, t and γ_0 so that

$$8\delta_{s,t} \cdot \hat{M}_0(s,t,\gamma_0) < m_{s,t} \text{ and } |M_X(x) - M_X(x+iy)| < 1/2. \tag{4.157}$$

In particular, the latter together with (4.150) and (4.156) imply that for $z \in \overline{D}_{\delta_{s,t}}(S_\gamma, T_\gamma)$ we have $Re[M_X(z)] \geq 1/2$ and $8\delta_{s,t} \cdot |\Lambda(z)| < m_{s,t}$. Thus $\delta_{s,t}$ satisfies the conditions in Definition 4.2.1.

Note that by (4.143) we have

$$|M_X(x+iy)| \leq |M_X(x)| \cdot \exp\left(\int_0^v \frac{-u \cdot du}{[a^2+u^2]}\right) = \frac{|M_X(x)|}{\sqrt{y^2+a^2}}, \tag{4.158}$$

where $a = \gamma + x \cdot \sigma_\gamma^{-1}$. Combining the latter with (4.145) we conclude that there is $K_{s,t}$ depending on s, t and γ_0 such that for all $x \in [S_\gamma, T_\gamma]$ we have

$$\left| M(x+iy) \cdot e^{-\Lambda'(x)\cdot(x+iy)} e^{-\Lambda(x)+x\Lambda'(x)} \right| = \frac{1}{\sqrt{y^2 + (\gamma + S_\gamma \cdot \sigma_\gamma^{-1})^2}} \leq \frac{K_{s,t}}{1+|y|}.$$

This fixes $K_{s,t}$ in Definition 4.2.2 and $p_{u_s,u_t} = 1$.

Step 7. In this step we show that we can choose $q_{s,t}$ in Definition 4.2.3 uniformly in $\gamma \geq \gamma_0$.

Let $\varepsilon_{s,t}$ and $R_{s,t}$ be as in the statement of Definition 4.2.3 for the constants $\delta_{s,t}$ and $K_{s,t}$ in Step 6. In view of (4.154) we have for any $x \in [S_\gamma, T_\gamma]$ that

$$\frac{d}{dy} Re[\Lambda(x+iy)] = \frac{1}{\sigma_\gamma^2} \sum_{n=0}^\infty \frac{-y}{(n+\gamma+x\sigma_\gamma^{-1})^2 + \sigma_\gamma^{-2} y^2}, \tag{4.159}$$

which implies that $Re[\Lambda(x+iy)]$ is decreasing in y on $[0,\infty)$ and increasing in y on $(-\infty, 0)$. Let us first consider the case $y \geq \varepsilon_{s,t}$. The above inequality implies that

$$Re[\Lambda(x+iy)] - \Lambda(x) \leq Re[\Lambda(x+i\varepsilon_{s,t})] - \Lambda(x) \leq \int_0^{\varepsilon_{s,t}} \sum_{n=0}^\infty \frac{-u\sigma_\gamma^{-2} du}{(\gamma + S_\gamma \sigma_\gamma^{-1} + n)^2} =$$
$$= -\frac{\varepsilon_{s,t}^2}{2\sigma_\gamma^2} \sum_{n=0}^\infty \frac{1}{(\gamma + S_\gamma\sigma_\gamma^{-1} + n)^2} \leq -\frac{\varepsilon_{s,t}^2}{2\sigma_\gamma^2} \cdot \int_1^\infty \frac{dv}{(\gamma + S_\gamma \sigma_\gamma^{-1} + v)^2} = \frac{-\varepsilon_{s,t}^2}{2\sigma_\gamma^2(\gamma + S_\gamma\sigma_\gamma^{-1} + 1)}.$$

Combining the latter with (4.140) and (4.145) we conclude that there is $q_{s,t} \in (0,1)$ that depends on s, t and γ_0 such that

$$Re[\Lambda(x+iy)] - \Lambda(x) \leq \log[q_{s,t}],$$

In particular, exponentiating both sides we see that for $x \in [S_\gamma, T_\gamma]$ and $y \geq \varepsilon_{s,t}$ we have

$$\left| \frac{M_X(x+iy)}{M_X(x)} \right| \leq q_{s,t}. \tag{4.160}$$

Since $|M_X(x+iy)| = |M_X(x-iy)|$ we conclude that (4.160) holds for $|y| \geq \varepsilon_{s,t}$, which verifies that $q_{s,t}$ satisfies the conditions in Definition 4.2.3.

Step 8. In this step we show that we can choose the constants in Definition 4.2.5 uniformly in $\gamma \geq \gamma_0$. We first show that we can find constants $\hat{M}_k(s,t,\gamma_0) > 0$ for $k = 0,1,2,3,4$ such that

$$|\Lambda^{(k)}(x)| \leq \hat{M}_k(s,t,\gamma_0) \text{ for all } x \in [S_\gamma, T_\gamma]. \tag{4.161}$$

Indeed for $k=0$, $k=1$ and $k=2$ this follows from (4.153), (4.152) and (4.148) respectively. Next we have for $k = 3,4$ that

$$|\Lambda^{(k)}(x)| = \frac{1}{\sigma_\gamma^k}\left|\psi^{(k-1)}(\gamma + S_\gamma\sigma_\gamma^{-1})\right| \leq \frac{1}{\sigma_\gamma^k}\cdot\left[\frac{(k-2)!}{[\gamma + x\sigma_\gamma^{-1}]^{k-1}} + \frac{(k-1)!}{[\gamma + S_\gamma\sigma_\gamma^{-1}]^k}\right],$$

where in the last inequalitly we used (4.139). Using (4.145) and (4.140) we conclude (4.161) for $k=3$ and $k=4$ as well.

Next we recall that $F(z) = G_z(u_z) = \Lambda(u_z) - u_z \cdot z$. We claim that for $k = 0,1,2,3,4$ we can find constants $M_{s,t}^{(k)}$ that depend on s,t and γ_0 such that if $z \in [s,t]$ we have

$$|F^{(k)}(x)| \leq M_{s,t}^{(k)} \text{ for all } x \in [S_\gamma, T_\gamma]. \tag{4.162}$$

If $z \in [s,t]$ then $u_z \in [S_\gamma, T_\gamma]$ and then in view of (4.150) and (4.151) we can find $M_{s,t}^{(0)}$ satisfying (4.162). We next use that $u_z = (\Lambda')^{-1}(z)$ to get

$$F'(z) = -u_z,\;\; F''(z) = -\frac{1}{\Lambda''(u_z)}\;\; F^{(3)}(z) = \frac{\Lambda^{(3)}(u_z)}{[\Lambda''(u_z)]^3}\;\; F^{(4)}(z) = \frac{\Lambda^{(4)}(u_z)\cdot\Lambda''(u_z) - 3\Lambda^{(3)}(u_z)}{[\Lambda''(u_z)]^5}.$$

The latter equalities together with (4.161) and (4.148) prove (4.162). The constants in (4.162) satisfy the conditions in Definition 4.2.5.

Step 9. In this step we show that we can choose the constants in Definitions 4.2.9 and 4.2.10 uniformly in $\gamma \geq \gamma_0$. Observe that by Steps 6. and 7. we can choose the constant N_0 in Proposition 4.2.4 depending on s,t and γ_0 alone and the same is true for the constant C_1. Since D, d and L in Assumption C5 were chosen uniformly in Lemma 4.7.3 in Step 2. we conclude that we can pick R_1 in Definition 4.2.9 depending on s,t and γ_0 alone. We now let $\hat{s} = -6R_1$ and $\hat{t} = 6R_1$. Then from Steps 6. and 7. we can pick all the remaining constants in Definition 4.2.10 uniformly in $\gamma \geq \gamma_0$.

Step 10. In this step we show that for any $r > 0$ there is a constant $\Delta_0 > 0$ that depends on r and γ_0 alone such that

$$\inf_{x\in[-r,r]} f_X(x) \geq \Delta_0. \tag{4.163}$$

We begin by proving a useful lemma.

Lemma 4.7.4. *The function $f_\gamma(x)$ converges uniformly over compact sets to $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ as $\gamma \to \infty$.*

Proof. Let us fix $R > 0$ and assume $x \in [-R, R]$. The functional equation $\Gamma(z+1) = z\Gamma(z)$ and [Bat, Theorem 1.6] give

$$\frac{F_\gamma(x) \cdot \gamma^{-\gamma+1/2} e^\gamma}{\sqrt{2\pi}} \cdot \frac{\sqrt{\gamma}}{\sqrt{\gamma+1}} \leq f_\gamma(x) \leq \frac{F_\gamma(x) \cdot \gamma^{-\gamma+1/2} e^\gamma}{\sqrt{2\pi}},$$

where $F_\gamma(x) = \sigma_\gamma \cdot \exp(\gamma(\sigma_\gamma x + m_\gamma) - e^{\sigma_\gamma x + m_\gamma})$. In addition, we have from (4.139) that

$$\gamma(\sigma_\gamma x + m_\gamma) - e^{\sigma_\gamma x + m_\gamma} = -\frac{x^2}{2} - e^{m_\gamma} + \gamma m_\gamma + O(\gamma^{-1/2}),$$

where the constant in the big O notation depends on R. Combining the latter with (4.140) we see that we can find a constant $C > 0$ depending on R such that

$$\frac{e^{-x^2/2 - C\gamma^{-1/2}} \cdot \sigma_\gamma \gamma^{1/2}}{\sqrt{2\pi}} \cdot \frac{\sqrt{\gamma}}{\sqrt{\gamma+1}} \leq f_\gamma(x) \leq \frac{e^{-x^2/2 + C\gamma^{-1/2}} \cdot \sigma_\gamma \gamma^{1/2}}{\sqrt{2\pi}},$$

from which we conclude the statement of the lemma after applying (4.140). □

Let us fix $r > 0$. By Lemma 4.7.4 we know that there is $\gamma_1 \geq \gamma_0$, depending on r, such that if $\gamma \geq \gamma_1$ then

$$\inf_{x \in [-r,r]} f_X(x) \geq \frac{1}{2\sqrt{2\pi}} \cdot e^{-r^2/2}.$$

Then since $f_X(x)$ is jointly continuous in x and γ and positive on $[-r, r] \times [\gamma_0, \gamma_1]$ there exists a positive constant Δ_1 depending on γ_0 and r such that

$$\inf_{x \in [-r,r]} f_1^\gamma(x) \geq \Delta_1, \tag{4.164}$$

for all $\gamma \in [\gamma_0, \gamma_1]$. In particular, we deduce that (4.163) holds with $\Delta_0 = \min\left(\Delta_1, \frac{1}{2\sqrt{2\pi}} \cdot e^{-r^2/2}\right)$.

Step 11. Let us denote $f_n^\gamma(\cdot)$ the density of $S_n = X_1 + \cdots + X_n$ where X_i are i.i.d. with distribution f_X. In this step we show that there is a positive constant Δ that depends on γ_0 such that

$$\inf_{x \in [-1,1]} f_n^\gamma(x) \geq \Delta \cdot n^{-1/2}. \tag{4.165}$$

We apply Proposition 4.2.4 to the distribution f_X and for the values $s = -1$ and $t = 1$. From our work in Steps 6. and 7. we know that we can find N_0 and $C_0 > 0$ depending on γ_0 such that for $N \geq N_0$ we have

$$f_N^\gamma(Nz) \geq \frac{C_0}{\sqrt{2\pi N \Lambda''(u_z)}} \cdot \exp\left(N G_z(u_z)\right).$$

In particular, using (4.148), the fact that $G_z(u_0) = 0$ and (4.162) we conclude that there is a constant $\Delta' > 0$ depending on γ_0 such that

$$\inf_{x \in [-1,1]} f_N^\gamma(x) \geq \Delta' \cdot N^{-1/2} \text{ for } \gamma \geq \gamma_0 \text{ and } N \geq N_0. \tag{4.166}$$

Next, we let $\Delta_0 \in (0,1)$ be sufficiently small so that (4.163) holds with $r = N_0$. Then we have for $1 \leq n \leq N_0$ and $x \in [-1,1]$ that

$$f_n^\gamma(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_X(x_1) \cdots f_X(x_{n-1}) \cdot f_X(x - x_1 - \cdots - x_{n-1}) dx_1 \cdots dx_n \geq$$
$$\geq \int_0^1 \cdots \int_0^1 f_X(x_1) \cdots f_X(x_{n-1}) \cdot f_X(x - x_1 - \cdots - x_{n-1}) dx_1 \cdots dx_n \geq (\Delta_0)^n,$$

In particular, we conclude from the latter and (4.166) that (4.165) holds for all $n \geq 1$ with $\Delta = \min(\Delta_0^{N_0}, \Delta')$.

Step 12. In this and the next step we show that we can choose the constants in Definition 4.6.1 uniformly in $\gamma \geq \gamma_0$. From (4.165) we can choose $\Delta > 0$ depending on γ_0 alone so that it satisfies the conditions of that definition. We also set $R = 2 + \Delta^{-1}$ in that definition. We may now apply Proposition 4.2.4 to the distribution f_X for the values $s = -2R$ and $t = 2R$. From our work in Steps 6. and 7. we know that we can find $N_0(R)$ and $C_0(R) > 0$ depending on γ_0 such that for $N \geq N_0(R)$ we have

$$f_N(Nz) \geq \frac{C_0(R)}{\sqrt{2\pi N \Lambda''(u_z)}} \cdot \exp\left(N G_z(u_z)\right).$$

In particular, using (4.148) and (4.162) we conclude there are positive constants C_R and c_R such that

$$f_N^\gamma(Nz) \geq C_R \cdot N^{-1/2} e^{-c_R N} \text{ for } \gamma \geq \gamma_0,\ z \in [-2R, 2R] \text{ and } N \geq N_0(R). \tag{4.167}$$

Furthermore, we can apply (4.163) to $r = 2R + N_0(R)$ to obtain the existence of a positive constant $\Delta_0(R) \in (0,1)$ such that

$$\inf_{x \in [-r,r]} f_1^\gamma(x) \geq \Delta_0(R).$$

Consequently, we have for $z \in [-2R, 2R]$ and $1 \leq n \leq N_0(R)$ that

$$f_n^\gamma(nz) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_X(x_1) \cdots f_X(x_{n-1}) \cdot f_X(nx - x_1 - \cdots - x_{n-1}) dx_1 \cdots dx_n \geq$$
$$\geq \int_0^1 \cdots \int_0^1 f_X(x_1) \cdots f_X(x_{n-1}) \cdot f_X(nx - x_1 - \cdots - x_{n-1}) dx_1 \cdots dx_n \geq (\Delta_0(R))^n.$$

The latter implies that (4.167) continues to hold for $1 \leq N \leq N_0(R)$ as well provided we make C_R small enough (and positive) depending on γ_0. This fixes the chooice of Δ, C_R and c_R.

Step 13. As we mentioned in Step 2. Assumption C2 holds for any $\lambda \in (0, \gamma\sigma_\gamma^{-1})$. Consequently, by (4.140) we can find $\lambda_0 > 0$ depending on γ_0 such that f_X satisfies Assumption C2 for $\lambda = \lambda_0$ and $\gamma \cdot \sigma_\gamma > 2\lambda_0$. We fix this choice for λ. Notice that by (4.137) and (4.139) we have for $x \in [-\lambda, \lambda]$ that $|\Lambda'(x)| \leq C(\gamma_0)$ for some $C(\gamma_0) > 0$. The latter and $\Lambda(0) = 0$ imply that for $x \in [-\lambda, \lambda]$ we have

$$|\Lambda(x)| \leq C(\gamma_0) \tag{4.168}$$

for some possibly different $C(\gamma_0) > 0$.

Finally, given λ and Δ, c_R, C_R as in Step 12. and L as in Lemma 4.7.3 we can find positive constants $\hat{C}_R$ and $\hat{c}_R$ that depend on γ_0 alone such that for all $n \geq 1$

$$\mathbb{E}[e^{\lambda|X|}]^n \left[\frac{4n^{3/2}}{\Delta} + LC_R^{-1}\sqrt{n}e^{c_R n}\right] \leq \hat{C}_R \cdot e^{\hat{c}_R n}.$$

In deriving the last expression we used (4.168) and the simple inequality $\mathbb{E}[e^{\lambda|X|}] \leq e^{\Lambda(\lambda)} + e^{\Lambda(-\lambda)}$.

From the proof of Lemma 4.7.3 we know that $f_X(x)$ is log-concave and so Lemma 4.6.2 is applicable. From that lemma we conclude that we can find functions $\hat{a}$ and $\hat{C}$ that satisfy the conditions of Assumption C6. Moreover, from the fact that $\lambda, \hat{C}_R$ and $\hat{c}_R$ are all independent of γ provided $\gamma \geq \gamma_0$, the lemma implies that the same is true for $\hat{a}$ and $\hat{C}$.

Summarizing all of our work in this section, we see that f_X satisfies Assumptions C1-C6 and so we can apply Theorem 4.5.3 to it. Since the constants C, a, α' in that theorem depend only on the parameters in Definition 4.2.10 and the functions in Assumption 6, and the latter can be chosen uniformly in $\gamma \geq \gamma_0$ this implies that the same is true for C, a, α'. We conclude that Theorem 4.5.3 implies Corollary 4.7.1. This suffices for the proof.

4.8 Proof of Theorem 3.1.13

In this chapter, we will first present three propositions, analogous to [CH16, Propositions 6.1, 6.2 and 6.3] in Section 4.8.1 and provide a few estimates on random walk bridges and $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-discrete Gibbs line ensembles in Section 4.8.2. Then we will deduce from them the proof of Theorem 3.1.13.

4.8.1 Three key propositions for random walk bridges

Fix $K \in \mathbb{N} \cup \{\infty\}$, for $N \in \mathbb{N}$, let $\mathcal{L}^N = \{\mathcal{L}_1^N, \cdots, \mathcal{L}_K^N\}$ be a discrete line ensemble which enjoys the discrete $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property with respect to some local interaction Hamiltonian $\dot{\mathbf{H}}^N$ and random walk Hamiltonian $\mathrm{H}^{\mathrm{RW},N}$ such that $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ satisfy assumptions A1-A4. We have the three following propositions.

Proposition 4.8.1. *Fix $k \in \{1, 2, \cdots K\}$. For each $\varepsilon > 0$, there exists $R_k = R_k(\varepsilon) > 0$ such that for any $x_0 > 0$ there exist $N_0(x_0, \varepsilon)$ such that for $N \geq N_0$ and $\overline{x} \in [-x_0, x_0]$,*

$$\mathbb{P}\Big(\inf_{u \in \Lambda_d^N[\overline{x}-1/2, \overline{x}+1/2]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) < -R_k\Big) < \varepsilon.$$

Proposition 4.8.2. *Fix $k \in \{1, 2, \cdots K\}$. For each $\varepsilon > 0$ and $\delta \in (0, 1/8)$, there exists $T_0 > 0$ such that for any $x_0 > T_0$ there exist $N_0(x_0, \varepsilon, \delta)$ such that $N \geq N_0$, $T \in [T_0, x_0]$ and $y_0 \in [-x_0, x_0 - T]$,*

$$\mathbb{P}\Big(\inf_{u \in \Lambda_d^N[y_0, y_0+T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) < -\delta T^2\Big) < \varepsilon.$$

Proposition 4.8.3. *Fix $k \in \{1, 2, \cdots K\}$. For each $\varepsilon > 0$, there exists $\hat{R}_k = \hat{R}_k(\varepsilon) > 0$ and $N_0(\varepsilon) > 0$ such that for any $x_0 > 0$ there exist $N_0(x_0, \varepsilon)$ such that $N \geq N_0$ and $\overline{x} \in [-x_0, x_0 - 1]$,*

$$\mathbb{P}\Big(\sup_{u \in \Lambda_d^N[\overline{x}, \overline{x}+1]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) > \hat{R}_k\Big) < \varepsilon.$$

The main ingredients in the proofs of these propositions are the discrete Gibbs property of the line ensembles, the monotone coupling Lemma 3.1.11, Assumption A3 and the strong approximation of the random walk bridges and Brownian bridges (Assumption A4). We also rely upon the arguments used in the proofs of [CH16, Propositions 6.1, 6.2 and 6.3]. The proofs of the above propositions are given in Chapter 4.9.

4.8.2 Estimates for random walk bridges and $(\dot{\mathbf{H}}^N, \mathbf{H}^{\mathbf{RW},N})$ line ensembles

In this section we prove a few lemmas which we will need in the proof of main Theorem 3.1.13.

Lemma 4.8.4. *The supremum of a Brownian bridge $B_L : [0, L] \to \mathbb{R}, B(0) = B(L) = 0$ satisfies that for all $s > 0$,*

$$\mathbb{P}\left(\sup_{u \in [0,L]} B_L(u) > s\right) \leq e^{-2s^2/L}.$$

Proof. This amounts to a use of reflection principle - see (3.40) in Chapter 4 of [KS]. ☐

The following lemma is an analogue for the random walk bridge case under Assumption A4.

Lemma 4.8.5. *Let $L > 0$, $z_0 \geq 0$ and $s_0 \geq 1$. Assume that $\mathrm{H}^{\mathrm{RW},N}$ satisfies Assumption A4 (KMT coupling). Then there exist $N_0 = N_0(L, s_0, z_0)$ such that the following holds. For any $|z| \leq z_0$, $1 \leq s \leq s_0$ and $N \geq N_0$, let $\bar{S}_{L,z}^N(u)$ be the random walk bridge defined as (3.11). Then*

$$\mathbb{P}\left(\sup_{u \in [0,L]} \left(\bar{S}_{L,z}^N(u) - \frac{u}{L} \cdot z\right) > s\right) \leq e^{-s^2/L}. \tag{4.169}$$

Proof. Let $b_1 = b_2 = 1$ and $a_1,\ a_2$ be the constants determined in Assumption A4. Then we take N_0 large enough such that the following two inequalities are true for all $N \geq N_0$:

$$\begin{aligned} a_1 N^{-1/2} \log(NL) \leq & 1 - \frac{\sqrt{3}}{2}, \\ a_2 (NL)^{-1} e^{z_0^2/L} \leq & \min_{1 \leq s \leq s_0} \left(e^{-s^2/L} - e^{-3s^2/(2L)}\right). \end{aligned}$$

Then by Lemma 4.8.4, Assumption A4 and $s \geq 1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{u\in[0,L]}\left(\bar{S}^N_{L,z}(u)-\frac{u}{L}\cdot z\right)>s\right)\\
\leq&\mathbb{P}\left(\sup_{u\in[0,L]}B_L(u)>\frac{\sqrt{3}}{2}s\right)+\mathbb{P}\left(\sup_{u\in[0,L]}\left|B_L(u)+\frac{u}{L}\cdot z-\bar{S}^N_{L,z}(u)\right|>\left(1-\frac{\sqrt{3}}{2}\right)s\right)\\
\leq&e^{-3s^2/(2L)}+\min_{1\leq s\leq s_0}\left(e^{-s^2/L}-e^{-3s^2/(2L)}\right)\\
\leq&e^{-s^2/L}.
\end{aligned}$$

□

In the following we proceed to prove Proposition 4.8.8, which compares two normalizing constants from **H**-Brownian line ensembles and $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-discrete line ensembles respectively under assumptions A3 and A4.

Lemma 4.8.6. *Fix $K \geq 1$ and $a < b \in \mathbb{R}$. Let $\mathcal{L} = \{\mathcal{L}_1, \dots \mathcal{L}_K\}$ be K real continuous functions defined on $[a, b]$. Let $\mathbf{H}(x) = e^x$ be a Hamiltonian function and $\Lambda^N_d = \frac{1}{N}\mathbb{Z}$. Assume $\dot{\mathbf{H}}^N$ is a sequence of Hamiltonian function satisfying Assumption A3. Thus there exists a constant $C_2 > 0$ such that for any $1 \leq k_1 \leq k_2 \leq K$, we have*

$$\left|W^{k_1,k_2,\Lambda^N_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N}(\mathcal{L})-W^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}(\mathcal{L})\right|\leq C_2\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right), \tag{4.170}$$

where we adapt the convention $\mathcal{L}_{k_1-1} = f$ and $\mathcal{L}_{k_2+1} = g$ for $k_1 \geq 2, k_2 \leq K-1$ and the convention $\mathcal{L}_0 = \infty$ and $\mathcal{L}_{K+1} = -\infty$.

Proof. By Assumption A3, we have

$$\begin{aligned}
\left|\frac{\log W^{k_1,k_2,\Lambda^N_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N}(\mathcal{L})}{\log W^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}(\mathcal{L})}-1\right| &= \left|\frac{-\sum_{i=k_1-1}^{k_2}\sum_{u\in\Lambda^N_d(a,b)}\dot{\mathbf{H}}^N(\square(\mathcal{L},i,u))}{-\sum_{i=k_1-1}^{k_2}\int_a^b\exp\left(\mathcal{L}_{i+1}(u)-\mathcal{L}_i(u)\right)du}-1\right|\\
&\leq e^{C_1\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right)}-1.
\end{aligned}$$

If we further assume $\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N \leq 1$, by the mean value theorem

$$e^{C_1\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right)}-1\leq C_1e^{C_1}\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right).$$

Hence

$$\left|\frac{\log W^{k_1,k_2,\Lambda^N_d(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N}(\mathcal{L})}{\log W^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}(\mathcal{L})}-1\right|\leq C_1e^{C_1}\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1},1/N)+1/N\right).$$

For the rest of the proof, we rely on the following fact (4.171) which we prove now. For any $\lambda \geq 0$ and any $|r| \leq 1/2$, from the mean value theorem,

$$|\lambda^{1+r} - \lambda| = |\lambda^{1+r'} \log \lambda| \cdot |r|$$

for some $|r'| \leq 1/2$. Note that

$$\sup_{\lambda \in [0,1]} |\lambda^{1+r'} \log a| \leq \sup_{\lambda \in [0,1]} |\lambda^{1/2} \log \lambda| \leq 1$$

Hence

$$|\lambda^{1+r} - \lambda| \leq |r| \quad \text{for all } a \in [0,1], |b| \leq 1/2. \tag{4.171}$$

Now by taking

$$\begin{aligned} \lambda =& W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}), \\ 1 + r =& \frac{\log W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}, \end{aligned}$$

we have

$$\lambda^{1+r} = W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}).$$

Applying inequality (4.171) with above choice of a, b, we have

$$\begin{aligned} &\left| W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \right| = |a^{1+b} - a| \leq |b| \\ \leq& C_1 e^{C_1} \left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N \right) \end{aligned}$$

if the case $\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N \leq \min\{1, 1/(2C_1)e^{-C_1}\}$ holds.

On the other hand if the case $\left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N\right) > \min\{1, 1/(2C_1)e^{-C_1}\}$ holds, since by definition it holds that $W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}), W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \in (0,1]$, we now have

$$\left| W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \right| \leq \max\{2, 4C_1 e^{C_1}\} \left(\omega_{[a,b]}(\mathcal{L}^{k_1-1,k_2+1}, 1/N) + 1/N \right).$$

Therefore the desired result follows by taking $C_2 = \max\{4C_1 e^{C_1}, 2\}$. ☐

By a similar argument, we now control the difference between $W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})$ and $W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')$ by the sup norm between $\mathcal{L}$ and $\mathcal{L}'$.

Lemma 4.8.7. *Fix $K \geq 1$ and $a < b \in \mathbb{R}$. Let $\mathcal{L} = \{\mathcal{L}_1, \dots \mathcal{L}_K\}$ and $\mathcal{L}' = \{\mathcal{L}'_1, \dots \mathcal{L}'_K\}$ be two collections of K real continuous functions defined on $[a,b]$ and let $\mathbf{H}(x) = e^x$ be a Hamiltonian function. There exists a constant $C_3 > 0$ such that for any $1 \leq k_1 \leq k_2 \leq K$,*

$$\left| W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}') \right| \leq C_3 \sup_{\substack{a \leq u \leq b \\ k_1-1 \leq k \leq k_2+1}} |\mathcal{L}_k(u) - \mathcal{L}'_k(u)|. \tag{4.172}$$

Proof. Note that we have for any $k_1 - 1 \le i \le k_2$

$$\left| \frac{\int_a^b \exp(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u))du}{\int_a^b \exp(\mathcal{L}'_{i+1}(u) - \mathcal{L}'_i(u))du} - 1 \right| \le \left| \sup_{a \le u \le b} \frac{\exp(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u))}{\exp(\mathcal{L}'_{i+1}(u) - \mathcal{L}'_i(u))} - 1 \right|$$

$$\le \exp\left(2 \sup_{a \le u \le b, i \le k \le i+1} |\mathcal{L}_k(u) - \mathcal{L}'_k(u)|\right) - 1.$$

Therefore

$$\left| \frac{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L})}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')} - 1 \right| = \left| \frac{-\sum_{i=k_1-1}^{k_2} \int_a^b \exp\left(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u)\right) du}{-\sum_{i=k_1-1}^{k_2} \int_a^b \exp\left(\mathcal{L}'_{i+1}(u) - \mathcal{L}'_i(u)\right) du} - 1 \right|$$

$$\le \exp\left(2 \sup_{a \le u \le b, k_1 - 1 \le i \le k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|\right) - 1.$$

If we further assume $\sup_{a \le u \le b, k_1-1 \le i \le k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)| \le 1$, by the mean value theorem we have

$$\left| \frac{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')}{\log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')} - 1 \right|$$

$$\le 2e^2 \cdot \sup_{\substack{a \le u \le b \\ k_1-1 \le i \le k_2+1}} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|.$$

Now by applying inequality (4.171) with $\lambda = W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}')$ and

$$1 + r = \log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) / \log W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}'),$$

we have

$$\left| W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}') - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \right| = |\lambda^{1+r} - \lambda| \le |r|$$

$$\le 2e^2 \cdot \sup_{\substack{a \le u \le b \\ k_1-1 \le i \le k_2+1}} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|$$

if the case $\sup_{a \le u \le b, k_1-1 \le i \le k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)| \le \frac{1}{4} e^{-2}$ holds.

On the other hand if the case $\sup_{a \le u \le b, k_1-1 \le i \le k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)| > \frac{1}{4} e^{-2}$ holds, we simply use

$$\left| W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}') - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\mathcal{L}) \right| \le 2 \le 8e^2 \sup_{a \le u \le b, k_1-1 \le i \le k_2+1} |\mathcal{L}_i(u) - \mathcal{L}'_i(u)|.$$

Thus the desired result follows by taking $C_3 = 8e^2$. ☐

The following proposition shows that given the same boundary data and under assumptions A3 and A4, the normalizing constant $Z_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}$ of $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs line ensemble converges to $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ of $\mathbf{H}$-Brownian Gibbs line ensemble where $\mathbf{H}(x) = e^x$.

Proposition 4.8.8. *Suppose* $\dot{\mathbf{H}}^N$ *and* $\mathrm{H}^{\mathrm{RW},N}$ *satisfy Assumption A3 and Assumption A4 respectively. For any* $k_0 \in \mathbb{N}$, $0 < L_1 < L_2$, $z_0 > 0$ *and* $\varepsilon > 0$, *there exists* N_0 *and* $\delta > 0$ *such that the following statement holds. Fix* $N \in \mathbb{N}$ *with* $N \geq N_0$, $k_1 \leq k_2 \in \mathbb{N}$ *with* $k_2 - k_1 \leq k_0$, $a < b \in \mathbb{R}$ *with* $L_1 \leq b - a \leq L_2$, $\vec{x} = \{x_{k_1}, \ldots x_{k_2}\}, \vec{y} = \{y_{k_1}, \ldots y_{k_2}\}$, *with* $\sup_{k_1 \leq i \leq k_2} |x_i - y_i| \leq z_0$ *and two continuous functions* $f, g : [a, b] \to \mathbb{R}$ *with* $\omega_{[a,b]}(f, 1/N) + \omega_{[a,b]}(g, 1/N) < \delta$. *Denote by* $\bar{S}^N : [k_1, k_2]_{\mathbb{Z}} \times \Lambda_d^N[a, b] \to \mathbb{R}$ *and* $B : [k_1, k_2]_{\mathbb{Z}} \times [a, b] \to \mathbb{R}$ *the the random walk bridges constructed by* $\mathrm{H}^{\mathrm{RW},N}$ *and Brownian bridges in* $[a, b]$ *with entrance and exit data* $\vec{x}$ *and* $\vec{y}$. *In other words,*

$$\bar{S}^N(k, u) \overset{(d)}{=} x_k + \bar{S}^N_{L, y_k - x_k}(u - a),$$

with $L = b - a$ *and*

$$B(k, u) \overset{(d)}{=} \left(\frac{b - u}{b - a}\right) x_k + \left(\frac{u - a}{b - a}\right) y_k + B_L(u - a).$$

Here $\bar{S}^N_{L, y_k - x_k}$ *is defined in* (3.3) *and different curves in* $\bar{S}^N$ *and* B *are independent. Then* $\bar{S}^N$ *and* B *can be coupled in a probability space and suppose* J *and* J' *are two events with* $\mathbb{P}(J \Delta J') < \varepsilon'$, *we have*

$$\left| \mathbb{P}\left(W^{k_1, k_2, \Lambda_d^N(a,b), \vec{x}, \vec{y}, f, g}_{\dot{\mathbf{H}}^N}(\bar{S}^N) \cdot \mathbb{1}_J \right) - \mathbb{P}\left(W^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}_{\mathbf{H}}(B) \cdot \mathbb{1}_{J'} \right) \right| < \varepsilon + \varepsilon'. \tag{4.173}$$

In particular, by taking $J = J'$ *to be the whole probability space, we have*

$$\left| Z^{k_1, k_2, \Lambda_d^N(a,b), \vec{x}, \vec{y}, f, g}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}} - Z^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}_{\mathbf{H}} \right| < \varepsilon. \tag{4.174}$$

Proof. Let δ_1, δ_2 be two small numbers to be determined. By taking $b_1 = 1$ and $b_2 = 1$ in Assumption A4, we can couple $\bar{S}^N$ and B in the same probability space such that for each $k \in [k_1, k_2]_{\mathbb{Z}}$,

$$\mathbb{P}\left(\sup_{a \leq u \leq b} \left| B_k(u) - \bar{S}^N_k(u) \right| > a_1 N^{-1/2} \log(N L_2) \right) \leq a_2 (N L_1)^{-1} e^{z_0^2 / L_1}.$$

Define the events

$$\begin{aligned}
A_{1,N} =& \{\omega_{(a,b), 1/N}(B) < \delta_1\}, \\
A_{2,N} =& \left\{ \sup_{a \leq u \leq b, k_1 \leq k \leq k_2} \left| B_k(u) - \bar{S}^N_k(u) \right| < \delta_2 \right\}, \\
A_N =& A_{1,N} \cap A_{2,N}.
\end{aligned}$$

Take N_0 large enough such that $a_1 N_0^{-1/2} \log(N_0 L_2) < \delta_2$, then

$$\mathbb{P}(A^{\neg}_{2,N}) \leq k_0 \cdot \left(a_2 (N_0 L_1)^{-1} e^{z_0^2 / L_1} \right).$$

Hence through taking N_0 large enough, we have

$$\mathbb{P}(A_{2,N}^{-}) \leq \varepsilon.$$

Also, as N_0 is large enough depending on L_2, z_0, k_0 and δ_1, we have

$$\mathbb{P}(A_{1,N}^{-}) \leq \varepsilon.$$

Note that as the event A_N occurs,

$$\omega_{[a,b]}(\bar{S}^N, 1/N) \leq \delta_1 + 2\delta_2.$$

On the event A_N, applying Lemma 4.8.6 and 4.8.7, we see that

$$\begin{aligned}
&\left| W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \right| \\
\leq &\left| W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) \right| \\
&+ \left| W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \right| \\
\leq &C_2(\delta + \delta_1 + 2\delta_2 + 1/N) + C_3\delta_2.
\end{aligned}$$

By choosing δ, δ_1, δ_2 and $1/N_0$ small enough, we have

$$\left| W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \right| < \varepsilon,$$

which implies that

$$\begin{aligned}
&\left| \mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) \cdot \mathbb{1}_J \right) - \mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \cdot \mathbb{1}_{J'} \right) \right| \\
\leq &\left| \mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) \cdot \mathbb{1}_{J\cap J'\cap A_N} \right) - \mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \cdot \mathbb{1}_{J\cap J'\cap A_N} \right) \right| \\
+ &\left| \mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) \cdot \mathbb{1}_{J\cap J'\cap A_N^{-}} \right) \right| + \left| \mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \cdot \mathbb{1}_{J\cap J'\cap A_N^{-}} \right) \right| \\
+ &\left| \mathbb{P}\left(W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) \cdot \mathbb{1}_{J\setminus J'} \right) \right| + \left| \mathbb{P}\left(W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \cdot \mathbb{1}_{J'\setminus J} \right) \right| \\
\leq &3\varepsilon + \varepsilon',
\end{aligned}$$

where we used inequality $\left| W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N) - W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \right| < \varepsilon$ in $J \cap J \cap A_N^{-}$ for the first term, $W_{\dot{\mathbf{H}}^N}^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}(\bar{S}^N), W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}(B) \in [0,1]$ and the bound of the event probability for the other four terms. □

4.8.3 Proof of Theorem 3.1.13 (1)

Let $\mathcal{L}^N$ be a sequence of $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-discrete line ensembles which satisfies the assumptions in Theorem 3.1.13. In order to establish Theorem 3.1.13, we will first prove the following lower bound for the normalizing constant $Z^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}$ (Definition 3.6) of the discrete Gibbs line ensemble $\mathcal{L}^N$. The proof exploits the analogous result for the normalizing constant $Z^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}$ (Definition 3.2) of a **H**-Brownian Gibbs line ensemble in [CH16, Proposition 6.4] and result that $Z^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}$ is approximating $Z^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\mathbf{H}}$ implied by Proposition 4.8.8.

Lemma 4.8.9. *Fix $k_1 \le k_2$ and an interval $[a,b] \in \mathbb{R}$. Then for all $\varepsilon > 0$, there exists $\delta > 0$, and $N_0(k_1,k_2,a,b,\varepsilon)$ such that for all $N > N_0(k_1,k_2,a,b,\varepsilon)$, we have*

$$\mathbb{P}\Big(Z^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}} < \delta\Big) < \varepsilon,$$

where $\vec{x} = (\mathcal{L}_i^N(a))_{i=k_1}^{k_2}$, $\vec{y} = (\mathcal{L}_i^N(b))_{i=k_1}^{k_2}$, $f = \mathcal{L}_{k_1-1}^N$, $g = \mathcal{L}_{k_2+1}^N$.

Proof. Propositions 4.8.1, 4.8.2 and 4.8.3 imply that (given k_1,k_2,a,b,ε) there exists $M > 0$ such that the event

$$E = \Big\{\min_{u\in\Lambda_d^N[a,b]} \mathcal{L}_{k_1-1}^N(u) > -M\Big\} \cap \Big\{\max_{u\in\Lambda_d^N[a,b]} \mathcal{L}_{k_2+1}^N(u) < M\Big\} \cap \Big\{|\mathcal{L}_i^N(u)| \le M,\ \forall u \in \{a,b\}, i \in \{k_1,\cdots,k_2\}\Big\},$$

has probability $\mathbb{P}(E) \ge 1-\varepsilon$.

For δ to be specified soon, define the event (with $\vec{x},\vec{y},f,g$ as in the statement of the lemma)

$$D = \Big\{Z^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}} < \delta\Big\}.$$

Since $\mathbb{P}(E^\neg) \le \varepsilon$ and

$$\mathbb{P}(D) \le \mathbb{P}(D\cap E) + \mathbb{P}(E^\neg),$$

we only need to control $\mathbb{P}(D\cap E)$. Let us assume that E occurs. In that case, due to the monotonicity of Assumption A1 (1),

$$Z^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}} \ge Z^{k_1,k_2,\Lambda_d^N(a,b),\vec{x},\vec{y},-M,M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}.$$

Clearly given that E occurs, there exists some $\delta > 0$, depending on k_1,k_2,a,b and M, such that

$$Z^{k_1,k_2,(a,b),\vec{x},\vec{y},-M,M}_{\mathbf{H}} > 2\delta.$$

By Proposition 4.8.8, we may show that

$$Z^{k_1,k_2,\Lambda_d^N[a,b],\vec{x},\vec{y},-M,M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}} > \delta$$

for $N \ge N_0(k_1,k_2,a,b,\varepsilon)$ large enough. Thus for such δ, $\mathbb{P}(D\cap E) = 0$ and we complete the proof. □

Now we proceed to the proof of Theorem 3.1.13(1). First let us recall the tightness criterion for k continuous functions. For a k-continuous function $\vec{f} = (f_1,f_2,\ldots,f_k)$ defined on an interval $[a,b]$, the

$k-$modulus of continuity of $\vec{f}$ is defined in (3.10) as

$$\omega_{[a,b]}(\vec{f},r) = \sup_{1\le i\le k} \sup_{\substack{s,t\in[a,b]\\ |s-t|\le r}} |f_i(s)-f_i(t)|.$$

Consider a sequence of probability measures $\mathbb{P}_N$ on k functions $\vec{f}=(f_1,\cdots,f_k)$ on the interval $[a,b]$ and define event

$$U_{[a,b]}(\vec{f},\varrho,r) = \Big\{\omega_{[a,b]}(\vec{f},r)\le \varrho\Big\}.$$

As an immediate generalization of [Bi, Theorem 8.2], a sequence $\mathbb{P}_N$ of probability measures on k functions $\vec{f}=(f_1,\cdots,f_k)$ is tight if, for each $1\le i\le k$, the one-point distribution of $f_i(x)$ at a fixed $x\in[a,b]$ is tight and if, for each positive ϱ and η, there exists a $r>0$ and integer N_0 such that

$$\mathbb{P}_N\big(U_{[a,b]}(\vec{f},\varrho,r)\big)\ge 1-\eta, \qquad \text{for } N\ge N_0.$$

We will apply this tightness criterion for the sequence of measures $\mathbb{P}_N$ of discrete line ensembles $\mathcal{L}^N$ restricted on $[-T,T]$. Denote $\vec{x}=(\mathcal{L}_1^N(-T),\cdots,\mathcal{L}_K^N(-T))$, $\vec{y}=(\mathcal{L}_1^N(T),\cdots,\mathcal{L}_K^N(T))$ and $g(u)=\mathcal{L}_{K+1}^N(u)$.

Recall that we denote $\mathcal{L}^N$ as a sequence of discrete Gibbs line ensembles which satisfy assumptions in the statement of main Theorem 3.1.13. Denote $\mathbb{P}_N$ and $\mathbb{E}_N$ as the corresponding probability measures and expectations. Proposition 4.8.1 and 4.8.3 show tightness for the one-point distribution, i.e. for each given $i\in\{1,\cdots,K\}$, the one-point distribution of $\mathcal{L}_i^N(u)$ is tight in $N\in\mathbb{N}$ for any u varies over $[-T,T]$.

In order to prove the tightness of the line ensemble $\{\mathcal{L}_i^N(u): i\in\{1,\cdots,K\}, u\in[-T,T]\}$, it suffices to verify that, for all $\varrho,\eta>0$, we can find a $r(\varrho,\eta)$ such that for $N\ge N_0(r,\varrho,\eta)$ large enough

$$\mathbb{P}_N\big(U_{[-T,T]}(\vec{f},\varrho,r)\big)\ge 1-\eta, \tag{4.175}$$

with U defined as above with $f_i=\mathcal{L}_i^N$ on the interval $[-T,T]$.

We introduce two notations. For $M>0$, we define the event

$$S_{N,M} = \bigcap_{i=1}^{K}\big\{-M\le \mathcal{L}_i^N(-T),\mathcal{L}_i^N(T)\le M\big\}.$$

Denote Z_N as a shorthand for

$$Z_N := Z^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},\infty,g}_{\dot{\mathbf{H}}^N,\mathbf{H}^{\mathrm{RW},N}}. \tag{4.176}$$

It is enough to prove that for any $\varrho,\eta>0$, there exists $\delta,M,r>0$ and $N_0(\varrho,\eta,\delta,M,r)$ large enough such that for all $N\ge N_0$

$$\mathbb{P}_N\Big(U_{[-T,T]}(\mathcal{L}^N,\varrho,r)\cap\{Z_N\ge\delta\}\cap S_{N,M}\Big)>1-\eta, \tag{4.177}$$

since (4.175) follows from (4.177).

Observe that the events $\{Z_N \geq \delta\} \cap S_{N,M}$ are $\mathcal{F}_{\text{ext}}([1,K]_{\mathbb{Z}} \times \Lambda_d^N(-T,T))$-measurable, we can rewrite the left-hand side of (4.177) as

$$\mathbb{E}_N\Big[\mathbb{1}_{Z_N \geq \delta}\mathbb{1}_{S_{N,M}}\mathbb{E}_N\big[\mathbb{1}_{U_{[-T,T]}(\mathcal{L}^N,\varrho,r)}\big|\mathcal{F}_{\text{ext}}([1,K]_{\mathbb{Z}} \times \Lambda_d^N(-T,T))\big]\Big]. \tag{4.178}$$

Denote

$$\mathbf{P}_N := \mathbf{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},+\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}, \tag{4.179}$$

thus by the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property enjoyed by the line ensemble $\mathcal{L}^N$, we have, $\mathbb{P}_N$-almost surely,

$$\mathbb{E}_N\big[\mathbb{1}_{U_{[-T,T]}(\mathcal{L}^N,\varrho,r)}\big|\mathcal{F}_{\text{ext}}([1,K]_{\mathbb{Z}} \times \Lambda_d^N(-T,T))\big] = \mathbf{P}_N\big(U_{[-T,T]}(\mathcal{L}^N,\varrho,r)\big). \tag{4.180}$$

The proof of Theorem 3.1.13 will be completed by the following lemma.

Lemma 4.8.10. *Let $\varrho,\eta,\delta,M,T>0$. There exists $r(\varrho,\eta,\delta,M,T)$ and $N_0(r,\varrho,\eta,\delta,M,T)$ large enough such that for all $N \geq N_0$,*

$$\mathbf{P}_N\big(U_{[-T,T]}(\mathcal{L}^N,\varrho,r)\big) \geq 1-\eta/2.$$

provided that $\vec{x},\vec{y} \in \mathbb{R}^K$ satisfy the condition $|x_i|,|y_i| \leq M$ for $1 \leq i \leq K$ and the condition $Z_N \geq \delta$ holds.

Let us assume the lemma for the moment and complete the proof of the claim in (4.177). By choosing r small enough (depending on ϱ,η,δ,M,T) and N_0 large enough(depending on $r,\varrho,\eta,\delta,M,T$), using Lemma 4.8.10 and equality (4.180), we find that

$$(4.178) \geq (1-\eta/2)\mathbb{E}_N[\mathbb{1}_{Z_N \geq \delta}\mathbb{1}_{S_{N,M}}].$$

By Lemma 4.8.9, there exists $\delta>0$ and N_0, such that, for $N>N_0$, $\mathbb{P}_N(Z_N<\delta) \leq \eta/4$. Propositions 4.8.1 and 4.8.3 imply that we may choose M, N_0 large enough and δ small enough so that $\mathbb{P}(S_{N,M}^{\neg}) \leq \eta/4$. This implies that

$$\mathbb{P}_N(\{Z_N \geq \delta\} \cap S_{N,M}) \geq 1-\eta/2, \tag{4.181}$$

and thus

$$(4.178) \geq (1-\eta/2)^2 > 1-\eta$$

which completes the proof of claim (4.177) and hence Theorem 3.1.13(1).

Proof of Lemma 4.8.10. The proof is based on the estimates provided by KMT coupling (Assumption A4) and the estimates on modulus of continuity for free Brownian bridges.

Recall that the law of Bridge ensemble, $\mathbf{P}_N = \mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},+\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}$ is specified with Radon-Nikodym derivative with respect to free random walk bridges, (see Definition 3.1.4)

$$\frac{d\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y},+\infty,g}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}}{d\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}}(\bar{S}) = \frac{W(\bar{S})}{Z_N}.$$

We abbreviate $\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}$ as $\mathbf{P}_{free,N}$ and its expectation as $\mathbf{E}_{free,N}$. Let $U^{\neg}_{[-T,T]}(\mathcal{L}^N,\varrho,r)$ be the complement of $U_{[-T,T]}(\mathcal{L}^N,\varrho,r)$, i.e. $\Big\{\omega_{a,b}\big(\{\mathcal{L}_1,\cdots,\mathcal{L}_k\},r\big)>\varrho\Big\}$. Since W is always less than 1, on the event $\{Z_N\geq\delta\}$, we have

$$\begin{aligned}\mathbf{P}_N\big(U^{\neg}_{[-T,T]}(\mathcal{L}^N,\varrho,r)\big)&=\frac{\mathbf{E}_{free,N}[\mathbb{1}_{U^{\neg}}\cdot W(\bar{S})]}{Z_N}\\&\leq\frac{\mathbf{E}_{free,N}[\mathbb{1}_{U^{\neg}}]}{Z_N}\leq\frac{1}{\delta}\mathbf{P}_{free,N}\big(U^{\neg}_{[-T,T]}(\bar{S},\varrho,r)\big).\end{aligned}\tag{4.182}$$

Therefore the proof of Lemma 4.8.10 is reduced to prove for fixed $\delta>0,\varrho,\eta>0$, there is a $r(\varrho,\eta)$ and $N_0(\delta,\varrho,\eta,r)$ large enough such that for all $N\geq N_0$, we have

$$\frac{1}{\delta}\mathbf{P}_{free,N}\big(U^{\neg}_{[-T,T]}(\bar{S},\varrho,r)\big)\leq\frac{\eta}{2}$$

on the modulus of continuity for K-free random walk bridges sampled from measure $\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}$ where $\vec{x},\vec{y},g$ satisfy the assumptions in Lemma 4.8.10.

By applying KMT coupling (Assumption A4) to K independent random walk bridges, there exists a coupling between K-random walk bridges with measure $\mathbb{P}^{1,K,\Lambda_d^N(-T,T),\vec{x},\vec{y}}_{free,\mathrm{H}^{\mathrm{RW},N}}$ and Brownian bridge on $[-T,T]$ with boundary value $\vec{x}$ and $\vec{y}$. We denote $\mathbb{P}_{\mathrm{cpl}}$ as the coupling measure and we use $\bar{S}$ and B to represent K-free random bridges and K-free Brownian bridges sampled from this coupled measure. Take $b_1=30$ and $b_2=1$ in Assumption A4. There exist $a,c>0$, depending on T,K and M, and a coupling measure $\mathbb{P}_{\mathrm{cpl}}$ such that for all $N\geq1$,

$$\mathbb{P}_{\mathrm{cpl}}\left(\sum_{1\leq i\leq K}\sup_{u\in[-T,T]}\big|\bar{S}(i,u)-B(i,u)\big|\geq\frac{a\log N}{\sqrt{N}}\right)\leq\frac{c}{N^{30}}.$$

We also have the following estimates on the modulus of continuity for K-free Brownian bridges with boundary value the same as above. Let δ be the same as in the assumption of Lemma 4.8.10, for any $\varrho,\eta>0$, there exist $r(\varrho,\eta,\delta,M,T)>0$ such that

$$\mathbb{P}_{\mathrm{cpl}}\left(U^{\neg}_{[-T,T]}\left(B,\frac{\varrho}{2},r\right)\right)\leq\frac{\delta\eta}{4}.\tag{4.183}$$

Choose N_0 large enough, such that $\frac{a\log N}{\sqrt{N}}<\frac{\varrho}{4}$ and $\frac{c}{N^{30}}<\frac{\delta\eta}{4}$ for all $N>N_0$. Under the coupling measure $\mathbb{P}_{\mathrm{cpl}}$, define event

$$D=\left\{\sum_{1\leq i\leq K}\sup_{u\in[-T,T]}\big|\bar{S}(i,u)-B(i,u)\big|<\frac{a\log N}{\sqrt{N}}\right\}.$$

Thus $\mathbb{P}_{\text{cpl}}(D^{-}) < \frac{c}{N^{30}} < \frac{\delta\eta}{4}$. On the event D, we have

$$\begin{aligned}
&\sup_{1\le i\le K}\sup_{\substack{s,t\in[-T,T]\\|s-t|\le r}}|\bar{S}(i,s)-\bar{S}(i,t)|\\
&\le \sup_{1\le i\le K}\sup_{\substack{s,t\in[-T,T]\\|s-t|\le r}}|B(i,s)-B(i,t)|+\frac{2a\log N}{\sqrt{N}}\\
&< \sup_{1\le i\le K}\sup_{\substack{s,t\in[-T,T]\\|s-t|\le r}}|B(i,s)-B(i,t)|+\frac{\varrho}{2}. \qquad (4.184)
\end{aligned}$$

Therefore, on the event D, using (4.183) and (4.184), we have

$$\mathbb{P}_{\text{cpl}}\Big(U^{-}_{[-T,T]}(\bar{S},\varrho,r)\cap D\Big)\le \mathbb{P}_{\text{cpl}}\big(U^{-}_{[-T,T]}(B,\varrho/2,r)\big)\le\frac{\delta\eta}{4}.$$

Since $\mathbb{P}_{\text{cpl}}(D^{-})\le\frac{\delta\eta}{4}$, we have

$$\mathbb{P}_{\text{cpl}}\big(U^{-}_{[-T,T]}(\bar{S},\varrho,r)\big)\le\frac{\delta\eta}{2},$$

which implies

$$\mathbf{P}_N\big(U^{-}_{[-T,T]}(\mathcal{L}^N,\varrho,r)\big)\le\frac{\eta}{2},$$

hence completing the proof. □

4.8.4 Proof of Theorem 3.1.13 (2)

After establishing the tightness of $\mathcal{L}^N$ in the previous section, we now demonstrate that all sub-sequential limiting line ensembles enjoy the H-Brownian Gibbs property. The proof is an adaption of [CH16, Proposition 5.2(1)] and we deal with the fact that underlying path measure converge to Brownian bridge measures.

Proof of Theorem 3.1.13 (2). Without loss of generality, we assume that $\mathcal{L}^N$ converges weakly to a line ensemble $\mathcal{L}^\infty$. The topology is the sup norm on K bounded continuous functions with domain $[-T,T]$. Fix an index $i\in\{1,2,\dots,K-1\}$ and two times $a,b\in[-T,T]$ with $a<b$ and interaction Hamiltonian $\mathbf{H}(x)=e^x$. We will show that the law of $\mathcal{L}^\infty$ is unchanged if $\mathcal{L}^\infty_i$ is resampled between a and b according to the law $\mathbb{P}_H^{i-1,i+2,(a,b),\mathcal{L}^\infty_i(a),\mathcal{L}^\infty_i(b),\mathcal{L}^\infty_{i-1},\mathcal{L}^\infty_{i+1}}$. The argument can easily be generalized to multiple consecutive curves. Note that the H-Brownian Gibbs property is equivalent to this resampling invariance, hence finishing the proof.

Since the Banach space of K bounded continuous functions equipped with the sup norm (Definition 3.1.1), denoted by $(C[-T,T])^K$, is separable, the Skorohod representation theorem applies. Therefore there exists a probability space $(\Omega,\mathcal{B},\mathbb{P})$ on which all of $\mathcal{L}^N$ for $N\in\mathbb{N}\cup\{\infty\}$ are defined and almost surely $\mathcal{L}^N(\omega)\to\mathcal{L}^\infty(\omega)$ in the topology of $(C[-T,T])^K$.

Let $L=b-a$. Recall that $\bar{S}^N_{L,z}$ is the random walk bridge defined in (3.11). From Assumption A4, for each $N\ge L^{-1}$ there exists a probability space $(\Omega^N_{\text{cpl}},\mathcal{B}^N_{\text{cpl}},\mathbb{P}^N_{\text{cpl}})$ on which all of random walk bridges

$\bar{S}^N_{L,z},\ z\in\mathbb{R}$ and a Brownian bridge B_L are defined. Moreover, by taking $b_1=30$ and $b_2=1$ in Assumption A4, there exits $0<a_1,a_2<\infty$ such that

$$\mathbb{P}^N_{\text{cpl}}\left(\sup_{0\le u\le L}\left|B_L(u)+\frac{u}{L}\cdot z-\bar{S}^N_{L,z}(u)\right|>a_1N^{-1}\log(NL)\right)\le a_2(NL)^{-30}e^{z^2/L}.$$

We further put all such coupling together and construct a probability space $(\Omega_{\text{cpl}},\mathcal{B}_{\text{cpl}},\mathbb{P}_{\text{cpl}})$ on which all of $\bar{S}^N_{L,z},\ z\in\mathbb{R},\ N\ge L^{-1},\ z\in\mathbb{R}$ and a Brownian bridge B_L are defined and the above estimates hold with $\mathbb{P}^N_{\text{cpl}}$ replaced by $\mathbb{P}_{\text{cpl}}$. Suppose we have a bounded sequence $z_N\in\mathbb{R}$ converging to z_∞, then

$$\sum_{N\ge L^{-1}}\mathbb{P}_{\text{cpl}}\left(\sup_{0\le u\le L}\left|B_L(u)+\frac{u}{L}\cdot z_N-\bar{S}^N_{L,z_N}(u)\right|>a_1N^{-1}\log(NL)\right)<\infty.$$

Through the Borel-Cantelli lemma, one has, $\mathbb{P}_{\text{cpl}}$-almost surely

$$\begin{aligned}&\sup_{u\in[0,L]}\left|B_L(u)+\frac{u}{L}\cdot z_\infty-\bar{S}^N_{L,z_N}(u)\right|\\&\le\sup_{u\in[0,L]}\left|B_L(u)+\frac{u}{L}\cdot z_N-\bar{S}^N_{L,z_N}(u)\right|+|z_N-z_\infty|\to 0.\end{aligned}\tag{4.185}$$

Let $\{(\bar{S}^{N,\ell}_{L,z},B^\ell_L)\}_{\ell\in\mathbb{N}}$ be a sequence of such coupling and independent between different ℓ. Let $\{U_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of independent random variables, each having the uniform distribution on $[0,1]$. We further augment the probability space $(\Omega,\mathcal{B},\mathbb{P})$ to include all such random variables in the independent manner.

In the first step, we define the ℓ-th candidate for the resampled bridge. As $u\in[a,b]$, define

$$\mathcal{L}^{N,\ell}_i(u)=\mathcal{L}^N_i(a)+\bar{S}^{N,\ell}_{L,\mathcal{L}^N_i(b)-\mathcal{L}^N_i(a)}(u-a),$$

and $\mathcal{L}^{N,\ell}_i(u)=\mathcal{L}^N_i(u)$ for $u\in[-T,a)\cup(b,T]$. Similarly, as $u\in[a,b]$, define

$$\mathcal{L}^{\infty,\ell}_i(u)=\mathcal{L}^\infty_i(a)+B^\ell_L(u-a)+\frac{u-a}{b-a}\cdot(\mathcal{L}^\infty_i(b)-\mathcal{L}^\infty_i(a)),$$

and $\mathcal{L}^{\infty,\ell}_i(u)=\mathcal{L}^\infty_i(u)$ for $u\in[-T,a)\cup(b,T]$.

In the second step, we check whether

$$U_\ell\le W(N,\ell):=W^{i-1,i+1,\Lambda^N_d(a,b),\mathcal{L}^N_i(a),\mathcal{L}^N_i(b),\mathcal{L}^N_{i-1},\mathcal{L}^N_{i+1}}_{\dot{\mathbf{H}}^N}(\mathcal{L}^{N,\ell}_i),\tag{4.186}$$

and **accept** the candidate resampling $\mathcal{L}^{N,\ell}_i$ if this event occurs. We define accordingly

$$W(\infty,\ell):=W^{i-1,i+1,(a,b),\mathcal{L}^\infty_i(a),\mathcal{L}^\infty_i(b),\mathcal{L}^\infty_{i-1},\mathcal{L}^\infty_{i+1}}_{\mathbf{H}}(\mathcal{L}^{\infty,\ell}_i).\tag{4.187}$$

For $N\in\mathbb{N}\cup\{\infty\}$, define $\ell(N)$ to be the minimal value of ℓ for which we accept $\mathcal{L}^{N,\ell}_i$. Write $\mathcal{L}^{N,\text{re}}$ for the line ensemble with the i-th line replaced by $\mathcal{L}^{N,\ell(N)}_i$. The random walk Gibbs property is equivalent to the fact that for $N\in\mathbb{N}$,

$$\mathcal{L}^{N,\text{re}}\overset{(d)}{=}\mathcal{L}^N.\tag{4.188}$$

Our **goal** is to show the same equality holds for $N = \infty$, which verifies the **H**-Brownian Gibbs property for the limiting line ensembles. For the moment we assume $\ell(N)$ converges to $\ell(\infty)$ with $\ell(\infty)$ bounded almost surely (which we will prove in the lemmas following later) and we complete the proof of Theorem 3.1.13(2) first.

From (4.185) and the independence among $\mathcal{L}^N$ and $\{\bar{S}^{N,\ell}_{L,z}, B^\ell_L\}_{\ell\in\mathbb{N}}$, one obtains almost surely

$$\sup_{u\in[a,b]}\left|B^{\ell(\infty)}_L(u-a)+\frac{u-a}{b-a}\cdot(\mathcal{L}^\infty_i(b)-\mathcal{L}^\infty_i(a))-\bar{S}^{N,\ell(N)}_{L,\mathcal{L}^N_i(b)-\mathcal{L}^N_i(a)}(u-a)\right|\to 0 \tag{4.189}$$

Here we used the independence among $\mathcal{L}^N$ and $\{S^{N,\ell}_{L,z}, B^\ell_L\}_{\ell\in\mathbb{N}}$ to ensure (4.185) can be applied to $z_N = \mathcal{L}^N_i(b) - \mathcal{L}^N_i(a)$, $z_\infty = \mathcal{L}^\infty_i(b) - \mathcal{L}^\infty_i(a)$ and the convergence still holds almost surely. Then $\mathcal{L}^{N,\text{re}}$ converges to $\mathcal{L}^{\infty,\text{re}}$ in $C[-T,T]$ almost surely and thus $\mathcal{L}^{\infty,\text{re}} \overset{(d)}{=} \mathcal{L}^\infty$. We complete the proof of Theorem 3.1.13 (2). □

Lemma 4.8.11. *Almost surely $\ell(\infty)$ is finite.*

Proof. For fixed $\mathcal{L}^\infty_i(a), \mathcal{L}^\infty_i(b), \mathcal{L}^\infty_{i-1}, \mathcal{L}^\infty_{i+1}$, the law of $W(\infty,\ell)$ (randomness coming from B^ℓ_L) is supported in $(0,1)$. Hence, for some $\varepsilon > 0$, $W(\infty,\ell)$ is at least ε with probability at least ε, which implies that $\ell(\infty)$ is finite almost surely. □

Lemma 4.8.12. *Almost surely for all ℓ, $\lim_{N\to\infty} W(N,\ell) = W(\infty,\ell)$.*

Proof. Let A be the intersection of the following events:

- $\displaystyle\sum_{i=1}^K \sup_{u\in[-T,T]} |\mathcal{L}^N_i(u) - \mathcal{L}^\infty_i(u)| \to 0.$
- For all $\ell \in \mathbb{N}$,
  $\displaystyle\sup_{u\in[a,b]}\left|B^\ell_L(u-a)+\frac{u-a}{b-a}\cdot(\mathcal{L}^\infty_i(b)-\mathcal{L}^\infty_i(a))-\bar{S}^{N,\ell}_{L,\mathcal{L}^N_i(b)-\mathcal{L}^N_i(a)}(u-a)\right|\to 0.$

One direct consequence is that as A occurs, $\mathcal{L}^{N,\ell}$ converges uniformly to $\mathcal{L}^{\infty,\ell}$ for all ℓ. In below we show that as A happens, $W(N,\ell) \to W(\infty,\ell)$. We estimate

$$\begin{aligned}
&|W(N,\ell) - W(\infty,\ell)|\\
\leq&\left|W^{i-1,i+1,\Lambda^N_d(a,b),\mathcal{L}^N_i(a),\mathcal{L}^N_i(b),\mathcal{L}^N_{i-1},\mathcal{L}^N_{i+1}}_{\dot{\mathbf{H}}^N}(\mathcal{L}^{N,\ell}) - W^{i-1,i+1,(a,b),\mathcal{L}^N_i(a),\mathcal{L}^N_i(b),\mathcal{L}^N_{i-1},\mathcal{L}^N_{i+1}}_{\mathbf{H}}(\mathcal{L}^{N,\ell})\right|\\
+&\left|W^{i-1,i+1,(a,b),\mathcal{L}^N_i(a),\mathcal{L}^N_i(b),\mathcal{L}^N_{i-1},\mathcal{L}^N_{i+1}}_{\mathbf{H}}(\mathcal{L}^{N,\ell}) - W^{i-1,i+1,(a,b),\mathcal{L}^N_i(a),\mathcal{L}^N_i(b),\mathcal{L}^N_{i-1},\mathcal{L}^N_{i+1}}_{\mathbf{H}}(\mathcal{L}^{\infty,\ell})\right|\\
+&\left|W^{i-1,i+1,(a,b),\mathcal{L}^N_i(a),\mathcal{L}^N_i(b),\mathcal{L}^N_{i-1},\mathcal{L}^N_{i+1}}_{\mathbf{H}}(\mathcal{L}^{\infty,\ell}) - W^{i-1,i+1,(a,b),\mathcal{L}^\infty_i(a),\mathcal{L}^\infty_i(b),\mathcal{L}^\infty_{i-1},\mathcal{L}^\infty_{i+1}}_{\mathbf{H}}(\mathcal{L}^{\infty,\ell})\right|.
\end{aligned}$$

The first term is bounded by $C_2(\omega_{[a,b]}(\mathcal{L}_{i-1}^{N,\ell}, 1/N)+\omega_{[a,b]}(\mathcal{L}_{i}^{N,\ell}, 1/N)+\omega_{[a,b]}(\mathcal{L}_{i+1}^{N,\ell}, 1/N)+1/N)$ by Lemma 4.8.6. Then by

$$\omega_{[a,b]}(\mathcal{L}_i^{N,\ell}, 1/N) \le \omega_{[a,b]}(\mathcal{L}_i^{\infty,\ell}, 1/N) + 2\|\mathcal{L}_i^{N,\ell} - \mathcal{L}_i^{\infty,\ell}\|_{C[a,b]},$$

the first terms goes to zero. By Lemma 4.8.7, the second term is bounded by $\|\mathcal{L}_i^{N,\ell} - \mathcal{L}_i^{\infty,\ell}\|_{C[a,b]}$ which converges to zero. The last terms also converges to zero since $\mathcal{L}^{\infty,\ell}$ is a continuous line ensemble and $\|\mathcal{L}^N - \mathcal{L}^\infty\|_{C[-T,T]}$ goes to zero. □

Lemma 4.8.13. *Almost surely* $\lim_{N\to\infty} \ell(N) = \ell(\infty)$.

Proof. Let A' be the intersection of the event A above and

- $\ell(\infty) < \infty$
- $W(\infty, \ell(\infty)) > U_{\ell(\infty)}$

The last condition occurs with probability 1 since $W(\infty, \ell(\infty)) \in (0,1)$ and, conditioned on $\{W(\infty, \ell(\infty))\}_{j=1}^{\ell(\infty)}$, $U_{\ell(\infty)}$ is the uniform distribution in $[0, W(\infty, \ell(\infty))]$. Then from $W(N, \ell(\infty)) \to W(\infty, \ell(\infty))$, we have for N large enough $W(N, \ell(\infty)) > U_{\ell(\infty)}$ and then $\ell(N) \le \ell(\infty)$. In particular,

$$\limsup_{n\to\infty} \ell(N) \le \ell(\infty).$$

On the other hand, for all $1 \le j \le \ell(\infty) - 1$, one has $W(\infty, j) < U_j$. Therefore $W(N, j) < U_j$ for N large enough and hence

$$\liminf_{n\to\infty} \ell(N) \ge \ell(\infty).$$

□

4.9 Proof of Three key Propositions

In this chapter, we will prove Propositions 4.8.1, 4.8.2 and 4.8.3 by induction on the index $k \in \mathbb{N}$. The proof follows the same logic as used in [CH16, Proposition 6.1, 6.2 and 6.3] with certain modifications as needed for the discrete case. The induction proceeds in the following following order:

- We start by proving Proposition 4.8.1 for index k from the knowledge of all three propositions for index $k-1$.
- We deduce Proposition 4.8.2 for index k from Proposition 4.8.1 for index k and Proposition 4.8.2 for index $k-1$.

- We deduce Proposition 4.8.3 for index k from Proposition 4.8.1 for index $k, k-1, k-2$ and Proposition 4.8.1 for index $k-1$.

4.9.1 Proof of Proposition 4.8.1

This proof is similar to that of [CH16, Proposition 6.1]. The main technical difference is that we replace [CH16, Lemma 2.11] by Lemma 4.8.5 for the random walk bridge.

The $k = 1$ case follows from the assumption of Theorem 3.1.13 since we assume that the $k = 1$ indexed curve converges weakly as a process on $\mathbb{R}$ to a stationary process. Note the stationarity is need for the independence of $R_k(\varepsilon)$ with respect to x_0. We assume now that $k \geq 2$ and for $k-1$ all three propositions have been verified.

Let consider $\varepsilon > 0$ given in the hypothesis of Proposition 4.8.1. For the proof, we recall and define a few constant parameters. Let R_{k-1} and $\hat{R}_{k-1}$ be as in Proposition 4.8.1 and 4.8.3 for our given ε. Let $K > 0$ be such that $4(1-e^{-1/2})^{-1}e^{-K^2} = \varepsilon$. Let T_0 be the parameter provided by Proposition 4.8.2 and for any $\delta \in (1/128, 1/8)$. We also require that $T \in \Lambda_d^N$ large enough that

$$T > T_0, \quad Te^{-T^{1/2}} \leq \frac{1}{4}\log 2, \quad \hat{R}_{k-1} \leq \frac{1}{16}T^2 - KT^{1/2}, \quad R_{k-1} \leq \frac{1}{16}T^2. \tag{4.190}$$

Define

$$M = \frac{1}{8}T^2 - \hat{R}_{k-1} + (K+1)T^{1/2}, \qquad R_k = M + 2T^2 + KT^{1/2}. \tag{4.191}$$

We will prove that for $\varepsilon > 0$ given, if we choose R_k as above then, for any x_0, there exist $N_0(x_0, \varepsilon)$ such that for $N \geq N_0$ and $\overline{x} \in \Lambda_d^N[-x_0, x_0]$, it holds that

$$\mathbb{P}\Big(\inf_{u \in \Lambda_d^N[\overline{x}-1/2, \overline{x}+1/2]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) < -R_k\Big) < 10\varepsilon, \tag{4.192}$$

therefore it suffices to verify (4.192) to finish the proof of Proposition 4.8.1.

Consider arbitrary x_0 and $\overline{x} \in [-x_0, x_0]$. For T and M as above, define two events

$$E_k^{N,-} = \Big\{\sup_{u \in \Lambda_d^N[\overline{x}-2T, \overline{x}-T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) > -M\Big\}$$

$$E_k^{N,+} = \Big\{\sup_{u \in \Lambda_d^N[\overline{x}+T, \overline{x}+2T]} \big(\mathcal{L}_k^N(u) + \frac{u^2}{2}\big) > -M\Big\}$$

and their intersection

$$E_k^N = E_k^{N,-} \cap E_k^{N,+}.$$

The following lemma 4.9.1 is one of the key step towards proving Proposition 4.8.1 and it shows that with high probability that the k indexed curve exceeds some level $-M$ at some point from the two outside region of $[\overline{x}-T,\overline{x}+T]$. Hence together with the re-sampling invariant nature provided by Gibbs property, likewise the value of the line ensemble restricted at interior of $[\overline{x}-T,\overline{x}+T]$ should not deviate from $-M$ about the same size as a random walk bridge does and this imply the desired Proposition 4.8.1. The above idea is summarized as the following two lemmas.

Lemma 4.9.1. *For any $\varepsilon>0$, $x_0>0$, there exists $N_0(x_0,\varepsilon)$ such that for $N>N_0$, and $\overline{x}\in\Lambda_d^N[-x_0,x_0]$, we have*

$$\mathbb{P}\Big((E_k^N)^\neg\Big)\leq 8\varepsilon.$$

Lemma 4.9.2. *For any $\varepsilon>0$, $x_0>0$, there exists $N_0(x_0,\varepsilon)$ such that for $N>N_0$ and $\overline{x}\in\Lambda_d^N[-x_0,x_0]$, we have*

$$\mathbb{P}\Big(\Big\{\inf_{u\in\Lambda_d^N[\overline{x}-T,\overline{x}+T]}(\mathcal{L}_k^N(u)+\frac{u^2}{2})<-R_k\Big\}\cap E_k^N\Big)<2\varepsilon.$$

Proof of Lemma 4.9.1. We will prove that there exists $N_0(x_0,\varepsilon)$ such that for $N>N_0$, we have

$$\mathbb{P}\Big((E_k^{N,-})^\neg\Big)\leq 4\varepsilon,$$

and the lemma immediately follows by the union bound since the analogous result holds for $E_k^{N,+}$.

Define the event

$$H_{k-1}:=\Big\{\mathcal{L}_{k-1}^N(u)+\frac{u^2}{2}<\hat{R}_{k-1}\text{ for }u=\overline{x}-2T\text{ and }u=\overline{x}-T\Big\}.$$

By Proposition 4.8.3, there exists N_0 such that $\mathbb{P}\Big((H_{k-1})^\neg\Big)\leq 2\varepsilon$ for $N>N_0$. Hence it is enough to prove that there exists N_0 such that

$$\mathbb{P}\Big((E_k^{N,-})^\neg\cap H_{k-1}\Big)\leq 2\varepsilon.$$

We define the event

$$A:=\Big\{\mathcal{L}_{k-1}^N(\overline{x}-3T/2)+\frac{(\overline{x}-3T/2)^2}{2}<-T^2/16\Big\}.$$

We would like to bound the probability of this event A by applying Proposition 4.8.2 with curve index $k-1$. In order to apply Proposition 4.8.2 for the index $k-1$, we need to make sure that $\overline{x}-3T/2$ lies in the interval $[y_0,y_0+T]$, where $y_0\in[-x_0,x_0-T]$ with $x_0=x_0(k-1)$ chosen in Proposition 4.8.2 for $(k-1)$-th labeled curve. And this holds true by choosing that $x_0(k)=\frac{1}{2}x_0(k-1)$ and $x_0(k-1)\geq 5T$ given that $\overline{x}\leq x_0(k)$.

Hence by applying Proposition 4.8.2 for any choice of $\delta\in(0,1/16)$, there exists $N_0(\varepsilon,k,x_0(k))$ such that $\mathbb{P}(A)\leq\varepsilon$ for $N>N_0$. Now it is enough to prove

$$\mathbb{P}\Big((E_k^{N,-})^\neg\cap H_{k-1}\cap A^\neg\Big)\leq 2\varepsilon,\tag{4.193}$$

for which we will apply the Gibbs property of the line ensemble and the monotonicity Lemma 3.1.11 to control the probability of the event in (4.193) by the probability of an event on random walk bridges, which could be bounded in a similar way as in the case of Brownian bridges (which is dealt with in [CH16, Proof of Lemma 7.2]) due to the KMT coupling.

It is clear that the event $(E_k^{N,-})^\neg \cap H_{k-1}$ is $\mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)$-measurable, thus by the property of conditional expectation, we have

$$\mathbb{P}\Big((E_k^{N,-})^\neg \cap H_{k-1} \cap A^\neg\Big) = \mathbb{E}\Big[\mathbb{1}_{(E_k^{N,-})^\neg \cap H_{k-1}} \mathbb{E}\Big[\mathbb{1}_{A^\neg}\big|\mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)\Big]\Big]$$

Due to the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property enjoyed by discrete line ensemble $\mathcal{L}$, it holds that

$$\mathbb{E}\Big[\mathbb{1}_{A^\neg}\big|\mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)\Big] = \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),\mathcal{L}_k^N(\overline{x}-2T),\mathcal{L}_k^N(\overline{x}-T),\mathcal{L}_{k-2}^N,\mathcal{L}_k^N}\big(A^\neg\big).$$

In order to show (4.193), it suffices to show that under the condition that the event $(E_k^{N,-})^\neg \cap H_{k-1}$ occurs,

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),\mathcal{L}_k^N(\overline{x}-2T),\mathcal{L}_k^N(\overline{x}-T),\mathcal{L}_{k-2}^N,\mathcal{L}_k^N}\big(A^\neg\big) \leq \varepsilon,$$

which we prove in the following.

Assume $(E_k^{N,-})^\neg \cap H_{k-1}$ holds, thus monotone coupling Lemma 3.1.11 implies that we can construct a coupling of the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),\mathcal{L}_k^N(\overline{x}-2T),\mathcal{L}_k^N(\overline{x}-T),\mathcal{L}_{k-2}^N,\mathcal{L}_k^N}$$

on the curve $S = \mathcal{L}_{k-1}^N : \Lambda_d^N[\overline{x}-2T, \overline{x}-T] \to \mathbb{R}$ and the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}$$

on the curve $\widetilde{S} : \Lambda_d^N[\overline{x}-2T, \overline{x}-T] \to \mathbb{R}$ such that almost surely $S(x) \leq \widetilde{S}(x)$ in the interval $[\overline{x}-2T, \overline{x}-T]$.

Since the event $A^\neg$ becomes more probable under pointwise increase in $S(x)$, the existence of the coupling implies that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_{A^\neg}\big|\mathcal{F}_{\text{ext}}\Big(\{k-1\}, \Lambda_d^N(\overline{x}-2T, \overline{x}-T)\Big)\Big] \\
&\leq \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}(A^\neg),
\end{aligned}$$

where in the RHS A is now defined with respect to $\widetilde{S}$.

Now we proceed to control

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}(A^\neg) \leq \varepsilon \tag{4.194}$$

using estimates on Brownian bridges and KMT coupling.

Recall Definition 3.1.4 that the law of $\widetilde{S}$ is specified by its Radon-Nikodym derivative $Z_N^{-1} W_N(\widetilde{S})$ with respect to the free random walk bridges measure

$$\mathbb{P}_{free, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2/2+\hat{R}_{k-1},-(\overline{x}-T)^2/2+\hat{R}_{k-1}}.$$

For the rest of this proof, we will use $\mathbb{P}_N$ and $\mathbb{E}_N$ to denote the probability measure and expectation respectively for $\mathbb{P}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2+\hat{R}_{k-1},-(\overline{x}-T)^2+\hat{R}_{k-1}}_{free,\mathrm{H}^{\mathrm{RW},N}}$.

Since normalizing constant $Z_N = \mathbb{E}_N\left(W(\widetilde{S})\right)$ and $W(\widetilde{S}) \leq 1$ always holds, we have

$$\begin{aligned}
&\mathbb{P}^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),(\overline{x}-2T)^2+\hat{R}_{k-1},-(\overline{x}-T)^2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(A^\neg) \\
&= \frac{\mathbb{E}_N\left(\mathbb{1}_{A^\neg}W(\widetilde{S})\right)}{\mathbb{E}_N\left(W(\widetilde{S})\right)} \leq \frac{\mathbb{E}_N\left(\mathbb{1}_{A^\neg}\right)}{Z_N} = \frac{\mathbb{P}_N(A^\neg)}{Z_N}.
\end{aligned}$$

Now in order to prove (4.194), it suffices to verify the $Z_N \geq \frac{1}{4}\left(1-e^{-2}\right)$ and $\mathbb{P}_N\left(A^\neg\right) \leq e^{-K^2}$ for $N \geq N_0$ large enough through the choice of K such that $\frac{4e^{-K^2}}{(1-e^{-2})} \leq \varepsilon$.

Note that (see [CH16, (80)]), the lower bound $\frac{1}{2}(1-e^{-2})$ has been proved for the normalization constant Z of a line ensemble with H-Brownian Gibbs property conditioned on the same boundary conditions with Hamiltonian $H = e^x$. In light of Proposition 4.8.8, likewise we obtain for $N \geq N_0(k, x_0, T)$ large enough,

$$Z_N = Z^{k-1,k-1,\Lambda_d^N(\overline{x}-2T,\overline{x}-T),-(\overline{x}-2T)^2+\hat{R}_{k-1},-(\overline{x}-T)^2+\hat{R}_{k-1},+\infty,-\frac{x^2}{2}-M}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}} \geq \frac{1}{4}(1-e^{-2}).$$

It remains to show $\mathbb{P}_N\left(A^\neg\right) \leq e^{-K^2}$ for $N \geq N_0$ large enough. To this end, let $L : [\overline{x}-2T, \overline{x}-T] \to \mathbb{R}$ denote the linear interpolation between $L(\overline{x}-2T) = -(\overline{x}-2T)^2/2+\hat{R}_{k-1}$ and $L(\overline{x}-T) = -(\overline{x}-T)^2/2+\hat{R}_{k-1}$. By Lemma 4.8.5 we have

$$\mathbb{P}_N\left(\sup_{u\in[\overline{x}-2T,\overline{x}-T]}(\widetilde{S}(u)-L(u)) \geq KT^{1/2}\right) \leq e^{-K^2}$$

Moreover we have

$$\inf_{u\in[\overline{x}-2T,\overline{x}-T]}(L(u)+u^2/2+M) \geq \hat{R}_{k-1} - \frac{1}{8}T^2 + M = (K+1)T^{1/2},$$

the last equality follows from the definition of M in (4.191).

And if $A^\neg$ holds, we see that

$$\begin{aligned}
&\widetilde{S}(\overline{x}-3T/2) - L\left(\overline{x}-3T/2\right) \\
&= \left(\widetilde{S}(\overline{x}-3T/2) + \frac{(\overline{x}-3T/2)^2}{2}\right) - \left(L\left(\overline{x}-3T/2\right) + \frac{(\overline{x}-3T/2)^2}{2}\right) \\
&= \left(\widetilde{S}(\overline{x}-3T/2) + \frac{(\overline{x}-3T/2)^2}{2}\right) - \left(\hat{R}_{k-1} - \frac{T^2}{8}\right) \\
&\geq -\frac{T^2}{16} - \hat{R}_{k-1} + T^2/8 \geq KT^{1/2},
\end{aligned}$$

where the first equality is a direct computation of the value of the linear interpolation at $\overline{x}-3T/2$ and the last inequality follow from $A^\neg$. The above estimate implies that

$$A^\neg \subset \left(\sup_{u\in[\overline{x}-2T,\overline{x}-T]}(\widetilde{S}(u)-L(u)) \geq KT^{1/2}\right).$$

Hence we have

$$\mathbb{P}_N(A^\neg) \le \mathbb{P}_N\left(\sup_{u\in[\overline{x}-2T,\overline{x}-T]}(\widetilde{S}(u)-L(u)) \ge KT^{1/2}\right) \le e^{-K^2},$$

and this finishes the proof of Lemma 4.9.1. □

Proof of Lemma 4.9.2. When E_k^N holds, the curve $\mathcal{L}_k^N(u)+\frac{u^2}{2}$ rises above the level $-M$ on both of the intervals $[\overline{x}-2T,\overline{x}-T]$ and$[\overline{x}+T,\overline{x}+2T]$ at $\sigma_\pm$ respectively. By the strong Gibbs property, the restricted measure of $\mathcal{L}_k^N(\cdot)$ on $[\sigma_-,\sigma_+]$ is a re-weighted random walk bridge measure. Moreover, as $N\to\infty$, the underline path measure converge to Brownian bridge measure. Provided that the normalizing constant well behaves, the typical deviation for this weight random walk path should be controlled by the deviation for Brownian bridges up to some constants as $N\ge N_0$ for N_0 large enough. The estimates of this proof is carried out similarly to that of the previous lemma.

Define the event

$$F_{k-1}^N = \left\{\inf_{u\in\Lambda_d^N[\overline{x}-2T,\overline{x}+2T]}(\mathcal{L}_{k-1}^N(u)+u^2/2) \ge -M+2T^{1/2}\right\}.$$

We can obtain an upper bound on the probability of the complement of F_{k-1}^N by applying Proposition 4.8.2 at index $k-1$. Recall that in the proof of Lemma 4.9.1, we use $x_0(k)=x_0(k-1)/2$. We may split the interval $[\overline{x}-2T,\overline{x}-T]$ into four consecutive intervals of length T and apply Proposition 4.8.2 to each of them. In order to do that task, one need to verify that each interval is contained in $[-x_0(k-1),x_0(k-1)+T]$. This condition is easily verified by our choice of parameter and thus there exists N_0 such that

$$\mathbb{P}\big((F_{k-1}^N)^\neg\big) \le \varepsilon,$$

for $N\ge N_0$. We also define the event

$$G^N = \left\{\inf_{u\in\Lambda_d^N[\overline{x}-T,\overline{x}+T]}(\mathcal{L}_k^N(u)+u^2/2) \le -R_k\right\}$$

We will prove that there exists $N_0(x_0,\varepsilon)$ such that

$$\mathbb{P}\big(E_k^N\cap F_{k-1}^N\cap G^N\big) \le \varepsilon \tag{4.195}$$

for $N>N_0$. Then the right hand side of Lemma 4.9.2 is bounded above by

$$\mathbb{P}\big(E_k^N\cap F_{k-1}^N\cap G^N\big)+\mathbb{P}\big((F_{k-1}^N)^\neg\big) \le 2\varepsilon,$$

as needed to complete to proof of the lemma.

We prove (4.195) by following a similar approach as in the previous proof but by using the strong Gibbs property. Define $\sigma_{-,k}$ to be the infimum over those $u\in\Lambda_d^N[\overline{x}-2T,\overline{x}-T]$ such that $\mathcal{L}_k^N(u)+u^2/2\ge -M$.

Likewise define $\sigma_{+,k}$ to be the infimum over those $u \in \Lambda_d^N[\overline{x}+T, \overline{x}+2T]$ such that $\mathcal{L}_k^N(u) + u^2/2 \geq -M$. It is easy to see that the interval $(\sigma_{-,k}, \sigma_{+,k})$ form a $\{k\}$-stopping domain (Definition 3.1.9) and the event $E_k^N \cap F_k^N$ is $\mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)$-measurable. These facts imply that

$$\mathbb{P}\big(E_k^N \cap F_{k-1}^N \cap G^N\big) = \mathbb{E}\left[\mathbb{1}_{E_k^N \cap F_k^N} \mathbb{E}\Big[\mathbb{1}_{G^N} \Big| \mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)\Big]\right]$$

and

$$\mathbb{E}\Big[\mathbb{1}_{G^N} \Big| \mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)\Big] = \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),\mathcal{L}_k^N(\sigma_{-,k}),\mathcal{L}_k^N(\sigma_{+,k}),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}(G^N).$$

To simplify the notation, we let $S : (\sigma_{-,k}, \sigma_{+,k}) \to \mathbb{R}$ be the curve distributed according to the given measure on the right hand side and G^N is defined now in terms of S.

Under the assumption that the event $E_k^N \cap F_{k-1}^N$ occurs, we know that $\mathcal{L}_k^N(\sigma_{\pm,k}) = (\sigma_{\pm,k})^2/2 - M$. By Lemma 3.1.11, there exists a coupling of the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),\mathcal{L}_k^N(\sigma_{-,k}),\mathcal{L}_k^N(\sigma_{+,k}),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}$$

on the curve S with the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),(\sigma_{-,k})^2/2-M,(\sigma_{+,k})^2/2-M,-\frac{u^2}{2}-M+2T^{1/2},-\infty}$$

on the curve $\widetilde{S}$ such that almost surely $S(u) \geq \widetilde{S}(u)$ for $u \in \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})$. Since the event G^N becomes more probable under the pointwise decrease in S, this implies that

$$\mathbb{E}\Big[\mathbb{1}_{G^N} \Big| \mathcal{F}_{\text{ext}}\Big(\{k\}, \Lambda_d^N(\sigma_{-,k}, \sigma_{+,k})\Big)\Big] \leq \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),(\sigma_{-,k})^2/2-M,(\sigma_{+,k})^2/2-M,\mathcal{L}_{k-1}^N,-\infty}(G^N)$$

As in the proof of Lemma 4.9.1, we have the law of $\widetilde{S}$ is defined by its Radon-Nikodym derivative $Z_N^{-1} W_N(\widetilde{S})$ with respect to the law of a rescaled random walk bridge with the same starting and ending points. Z_N is the expectation of $W_N(\widetilde{S})$ with respect to the random walk measure on $\widetilde{S}$. Now the proof proceeds the same way as previous lemma such that we need a lower bound for normalizing constant Z_N and the upper bound of the same event under the probability measure of free random walk bridges.

Analogous to Z_N, the control of Z such that $Z \geq \frac{1}{2}(1 - 2e^{-1/2})$ in the case of H- Gibbs line ensemble with $H = e^x$ is already proved in [CH16, Proposition 6.1]. Another application of Lemma 4.8.8 shows that $Z_N \geq \frac{1}{4}(1 - 2e^{-1/2})$ for $N \geq N_0$ large enough.

Let $L : (\sigma_{-,k}, \sigma_{+,k}) \to \mathbb{R}$ denote the linear interpolation between $L(\sigma_{-,k}) = -(\sigma_{-,k})^2/2 - M$ and $L(\sigma_{+,k}) = -(\sigma_{+,k})^2/2 - M$. We further find that

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k,k,\Lambda_d^N(\sigma_{-,k},\sigma_{+,k}),(\sigma_{-,k})^2/2-M,(\sigma_{+,k})^2/2-M,\mathcal{L}_{k-1}^N,-\infty}\left(\inf_{u \in \Lambda_d^N[\sigma_{-,k},\sigma_{+,k}]} (\widetilde{S}(u) - L(u)) \leq -KT^{1/2}\right)$$
$$\leq Z^{-1} e^{-K^2} \leq \varepsilon,$$

together with the convexity of $\frac{u^2}{2}$ we have

$$\inf_{u\in[\sigma_{-,n}^N,\sigma_{+,k}^N]}\left(\widetilde{S}(u)+\frac{u^2}{2}+M+2T^2\right)\geq\inf_{u\in[\sigma_{-,k}^N,\sigma_{+,k}^N]}\left(\widetilde{S}(u)-L(u)\right),$$

therefore by choosing $R_k=M+2T^2+KT^{1/2}$, we have

$$\mathbb{P}\left(\widetilde{S}(u)+\frac{u^2}{2}\leq -R_k\right)\leq\mathbb{P}\left(\inf_{u\in[\sigma_{-,k}^N,\sigma_{+,k}^N]}\left(\widetilde{S}(u)-L(u)\right)\leq -KT^{1/2}\right)\leq\varepsilon$$

and this completes our proof. □

4.9.2 Proof of Proposition 4.8.2

We prove this proposition by induction on the index k. In order to deduce the proposition for index k, we rely on Proposition 4.8.1 for index k, as well as Proposition 4.8.2 for index $k-1$. For the case $k=0$, it is easy to check that the result holds. We assume now that $k\geq 1$.

Consider $\varepsilon>0$ and $\delta\in(0,1/8)$ fixed from the statement of the proposition. We assume that $\varepsilon\in(0,1)$ since the case $\varepsilon\geq 1$ is trivial. By using Proposition 4.8.1 for index k, there exists a constant R_k such that, for all $x_0>0$ and $u\in\Lambda_d^N[-x_0,x_0]$,

$$\mathbb{P}\big(\mathcal{L}_k^N(u)+u^2/2<-R_k\big)\leq\frac{\varepsilon\delta}{3},\tag{4.196}$$

whenever $N\geq N_0(x_0,\varepsilon\delta/3)$. For $y_0,T>0$, we define the event

$$C_{y_0,T}^N=\left\{\inf_{u\in\Lambda_d^N[y_0,y_0+T]}\mathcal{L}_{k-1}^N(u)+u^2/2\geq-\frac{1}{2}\delta T^2\right\}.$$

We fix the constant $T_0>0$ large enough such that the following conditions hold:

1. $R_k\leq\frac{5}{8}\delta T_0^2$.
2. For all $T>T_0$, define
$$K(T)=\Big(\log\big(2(1-e^{-2})^{-1}3T\varepsilon^{-1}\big)\Big)^{1/2},\tag{4.197}$$
and require that
$$\max\Big\{\big(\delta(T+1)\big)^{1/2},K(T)\big(\delta(T+1)\big)^{1/2},\delta^2(T+1)^2\Big\}\leq\frac{1}{8}\delta T^2,$$
$$\exp\big\{-(T+1)\delta e^{-1/8\delta T^2}\big\}\geq 1/2.$$
3. For all $x_0\geq T_0$, $T\in[T_0,x_0]$ and $y_0\in\Lambda_d^N[-x_0,x_0-T]$
$$\mathbb{P}(C_{y_0,T}^N)\geq 1-\frac{\varepsilon}{3},\tag{4.198}$$
for $N\geq N_0(x_0,\varepsilon,\delta)$ large enough. The existence of such T_0 is a direct consequence of Proposition 4.8.2 for index $k-1$.

Define the event

$$E^N_{y_0,T} = \left\{ \inf_{u\in\Lambda^N_d[y_0,y_0+T]} (\mathcal{L}^N_k(u) + u^2/2) \leq -\delta T^2 \right\}.$$

We will show that

$$\mathbb{P}(E^N_{y_0,T}) < \varepsilon, \tag{4.199}$$

which proves the desired Proposition 4.8.2.

We will say that $u \in \mathbb{Z}\cap\Lambda^N_d[-x_0,x_0]$ is $(\varepsilon\delta/3)-$good if $\mathcal{L}^N_k(u)+u^2/2 \geq -R_k$ where R_k is define in (4.196). We say that $u \in \mathbb{Z}\cap\Lambda^N_d[-x_0,x_0]$ is $(\varepsilon\delta/3)-$bad if it is not $(\varepsilon\delta/3)-$good. Define $B^N_{y_0,T}$ the event that the number of $(\varepsilon\delta/3)-$bad u in $\mathbb{Z}\cap\Lambda^N_d[y_0,y_0+T]$ is at most $(T+1)\delta$. It is straight forward from (4.196) that the probability that any given $u \in \mathbb{Z}\cap\Lambda^N_d[-x_0,x_0]$ is $(\varepsilon\delta/3)-$good is at least $1-\varepsilon\delta/3$. The mean number of $(\varepsilon\delta/3)-$bad u is therefore at most $(T+1)\varepsilon\delta/3$. Thus by the Markov inequality,

$$\mathbb{P}(B^N_{y_0,T}) \geq 1 - \frac{\varepsilon}{3}. \tag{4.200}$$

On the other hand, we have

$$\mathbb{P}(E^N_{y_0,T}) \leq \mathbb{P}\big(E^N_{y_0,T}\cap B^N_{y_0,T}\cap C^N_{y_0,T}\big) + \mathbb{P}\big((B^N_{y_0,T}\cap C^N_{y_0,T})^\neg\big).$$

By the bounds (4.200) and (4.198), we find that

$$\mathbb{P}\big((B^N_{y_0,T}\cap C^N_{y_0,T})^\neg\big) \leq \frac{2}{3}\varepsilon.$$

Hence to prove (4.199), it remains to show that

$$\mathbb{P}\big(E^N_{y_0,T}\cap B^N_{y_0,T}\cap C^N_{y_0,T}\big) \leq \frac{\varepsilon}{3}. \tag{4.201}$$

The event $C^N_{y_0,T}$ depends on the curve of index $k-1$, hence it is $\mathcal{F}_{\text{ext}}(\{k\}\times\Lambda^N_d(y_0,y_0+T)-$measurable. Using the conditional expectation we have that

$$\mathbb{P}\big(E^N_{y_0,T}\cap B^N_{y_0,T}\cap C^N_{y_0,T}\big) = \mathbb{E}\Big[\mathbb{1}_{C^N_{y_0,T}}\mathbb{E}[1_{E^N_{y_0,T}\cap B^N_{y_0,T}}|\mathcal{F}_{\text{ext}}(\{k\}\times\Lambda^N_d(y_0,y_0+T))]\Big].$$

Then in order to prove (4.201), we only need to check that $\mathbb{P}$-almost surely

$$\mathbb{E}\Big[1_{E^N_{y_0,T}\cap B^N_{y_0,T}}\big|\mathcal{F}_{\text{ext}}(\{k\}\times\Lambda^N_d(y_0,y_0+T))\Big] \leq \frac{\varepsilon}{3}\mathbb{1}_{C^N_{y_0,T}} + \mathbb{1}_{(C^N_{y_0,T})^\neg}. \tag{4.202}$$

Since the bounds by $\mathbb{1}_{(C^N_{y_0,T})^\neg}$ is trivial, we need to prove that if the event $C^N_{y_0,T}$ holds then the left-hand side of (4.202) is bounded by $\varepsilon/3$. On this event, the $(\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N})-$ property for $\mathcal{L}^N$ implies that $\mathbb{P}-$almost surely

$$\mathbb{E}\Big[1_{E^N_{y_0,T}\cap B^N_{y_0,T}}\big|\mathcal{F}_{\text{ext}}(\{k\}\times\Lambda^N_d(y_0,y_0+T))\Big] = \mathbb{P}^{k,k,\Lambda^N_d[y_0,y_0+T],\mathcal{L}^N_k(y_0),\mathcal{L}^N_k(y_0+T),\mathcal{L}^N_{k-1},\mathcal{L}^N_{k+1}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E^N_{y_0,T}\cap B^N_{y_0,T}).$$

On the right hand side the event $E_{y_0,T}^N \cap B_{y_0,T}^N$ are now defined in terms of $S : [y_0, y_0 + T] \to \mathbb{R}$ which is distributed according to $\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}$.

In order to establish this, we need to decompose the event $E_{y_0,T}^N \cap B_{y_0,T}^N$ further. For any subset $A \subset \mathbb{Z} \cap \Lambda_d^N[y_0, y_0+T]$, let G_A^N denote the event that the set of $(\varepsilon\delta/3)-$good $u \in \mathbb{Z} \cap \Lambda_d^N[y_0, y_0+T]$ is exactly the set A. Write l_A as maximal length of gaps between two consecutive points in A and denote by $S_{T,\delta}$ the set of all $A \subset \mathbb{Z} \cap \Lambda_d^N[y_0, y_0+T]$ such that $l_A \leq (T+1)\delta$.

Observe that the event $B_{y_0,T}^N$ is a subset of the union of G_A^N over all $A \in S_{T,\delta}$. This is because having at most $(T+1)\delta$ integer $x \in \mathbb{Z} \cap \Lambda_d^N[y_0, y_0+T]$ which are $(\varepsilon\delta/3)-$bad implies that the maximal number of such consecutive integers is at most $(T+1)\delta$. This implies that

$$\begin{aligned}
&\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E_{y_0,T}^N \cap B_{y_0,T}^N) \\
\leq &\sum_{A\in S_{T,\delta}} \mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E_{y_0,T}^N \cap G_A^N) \\
= &\sum_{A\in S_{T,\delta}} p_A \cdot \mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E_{y_0,T}^N | G_A^N),
\end{aligned}$$

where

$$p_A = \mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(G_A^N)$$

and

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(\bullet | G_A^N)$$

is the measure conditioned on G_A^N occurring.

This conditioned measure is a special case of a general class of measure from Definition 3.1.12 such that

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(\bullet | G_A^N) = \mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^F,\widetilde{H}^G}(\bullet),$$

where $\widetilde{f}(u) = (-u^2/2 - R_k) \cdot \mathbb{1}_{u\in\mathbb{Z}\cap A^\neg} + \infty \cdot \mathbb{1}_{u\notin\mathbb{Z}\cap A^\neg}$, $\widetilde{g}(u) = (-u^2/2 - R_k) \cdot \mathbb{1}_{u\in\mathbb{Z}\cap A} - \infty \cdot \mathbb{1}_{u\notin\mathbb{Z}\cap A}$ and $\widetilde{H}^F(x) = \widetilde{H}^G(x) = \infty \cdot \mathbb{1}_{x\geq 0} + 0 \cdot \mathbb{1}_{x<0}$ (which corresponds to conditioning on non-intersection).

We will prove that

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\vec{f},\vec{g}}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^f,\widetilde{H}^g}(E_{y_0,T}^N) \leq \frac{\varepsilon}{3}. \tag{4.203}$$

Since $\sum_{A\in S_{T,\delta}} p_A \leq 1$, this will imply that

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N}}(E_{y_0,T}^N \cap B_{y_0,T}^N) \leq \frac{\varepsilon}{3}.$$

In order to prove (4.203), we will utilize the monotonicity from Lemma 3.1.11. Write $\widetilde{S}$ to denote the curve distributed according to this law and on the event $C_{y_0,T}^N$, we may couple the measure $\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^F,\widetilde{H}^G}$ on the curve $\widetilde{S}$ to the measure P (which we will introduce below) on the curve $\hat{S}$ so that $\widetilde{S}(u) \geq \hat{S}(u)$ for all $u \in \Lambda_d^N[y_0, y_0+T]$. Since the event $E_{y_0,T}^N$ is more probable as $\widetilde{S}$ decreases, this monotonicity implies that

$$\mathbb{P}^{k,k,\Lambda_d^N[y_0,y_0+T],\mathcal{L}_k^N(y_0),\mathcal{L}_k^N(y_0+T),\mathcal{L}_{k-1}^N,\mathcal{L}_{k+1}^N}_{\dot{\mathbf{H}}^N,\mathrm{H}^{\mathrm{RW},N},\widetilde{H}^F,\widetilde{H}^G}(E_{y_0,T}^N) \leq P(E_{y_0,T}^N). \tag{4.204}$$

The measure P on the curve $\hat{S} : [y_0, y_0+T] \to \mathbb{R}$ is defined as follows:

- For $u \in A$, fix $\hat{S}(u) = -\frac{3}{4}\delta T^2 - u^2/2$;
- For $a < a'$ that are consecutive elements in A, the law of $\hat{S}$ on the interval (a, a') is specified by requiring that it has Radon-Nikodym derivative $Z^{-1}W(S)$ with respect to the law of the random walk bridge $\mathbb{P}^{\Lambda_d^N[a,a'],-a^2/2-3/4\delta T^2,-a'^2/2-3/4\delta T^2}_{\mathrm{H}^{\mathrm{RW},N}}$. Here we set $S_0(u) = -\frac{u^2}{2} - \frac{1}{2}\delta T^2$, $S_1(u) = S(u)$ and the Boltzmann weight is given by

$$W^N(S) = \exp\Big\{ - \sum_{u\in\Lambda_d[a,a']} \dot{\mathbf{H}}^N(\square(S,0,u))\Big\} \cdot \mathbb{1}_{S(b)+b^2/2<-5/8\delta T^2, \forall b\in\mathbb{Z}\cap\Lambda_d^N[a,a']}. \tag{4.205}$$

- For the minimal $a \in A$, the law of $\hat{S}$ on the interval $[y_0, a)$ is given by requiring that it has Radon-Nikodym derivative $Z^{-1}W(S)$ with respect to the law of the random walk bridge $\mathbb{P}^{\Lambda_d^N[y_0,a),\mathcal{L}_k^N(y_0),-a^2/2-3/4\delta T^2}_{\mathrm{H}^{\mathrm{RW},N}}$. Here we set $S_0(u) = -\frac{u^2}{2} - \frac{1}{2}\delta T^2$, $S_1(u) = S(u)$ and the Boltzmann weight is given by

$$W^N(S) = \exp\Big\{ - \sum_{u\in\Lambda_d[y_0,a]} H(\square(S,0,u))\Big\} \cdot \mathbb{1}_{S(b)+b^2/2<-5/8\delta T^2, \forall b\in\mathbb{Z}\cap\Lambda_d^N[y_0,a]}. \tag{4.206}$$

- For the maximal $a \in A$, the law of $\hat{S}$ in $(a, y_0+T]$ is similarly defined as for the minimal $a \in A$.

The general monotone coupling in Lemma 3.1.11 implies the inequality (4.204). Hence it suffices to verify that

$$P(E_{y_0,T}^N) \leq \frac{\varepsilon}{3}. \tag{4.207}$$

The rest of the proof constitutes prove of the above claim.

Consider $a < a'$, two consecutive elements in A. Let $\hat{S}_{a,a'}$ the restriction of the curve $\hat{S}$ on the interval $[a, a']$. Note that $a' - a \leq (T+1)\delta$. Therefore by the KMT Assumption A4 and the estimate for Brownian motion, for $N \geq N_0$ with N_0 large enough, we know that

$$\mathbb{P}^{\Lambda_d^N[a,a'],-a^2/2-3/4\delta T^2,-a'^2/2-3/4\delta T^2}_{\mathrm{H}^{\mathrm{RW},N}}\left(\sup_{u\in\Lambda_d^N[a,a']}\big(\hat{S}_{a,a'}(u) - L(u)\big) \leq (\delta(T+1))^{1/2}\right) \geq 1 - e^{-2}$$

where $L(u) : [a, a'] \to \mathbb{R}$ denotes the linear interpolation of $L(a) = -a^2 - 3/4\delta T^2$ and $L(a') = -(a')^2 - 3/4\delta T^2$.

By the concavity of $-u^2/2$ and the bound $(\delta(T+1))^{1/2} \leq \delta T^2/8$ for T large enough, we see that for such a curve $\hat{S}_{a,a'}$ with

$$\hat{S}_{a,a'}(u) \leq L(u) + \delta T^2/8 \leq -u^2/2 - \frac{5}{8}\delta T^2,$$

By the same reasoning as in the proof of Proposition 4.8.1, we have for $N \geq N_0$ with N_0 large enough, $Z \geq \frac{1}{4}(1-e^{-2})$. As a consequence,

$$P\Big(\inf_{u\in\Lambda_d^N[a,a']}\big(\hat{S}_{a,a'}(u)-L(u)\big)\leq -K(T)(\delta(T+1))^{1/2}\Big)\leq Z^{-1}\exp(-K(T)^2)\leq\frac{\varepsilon}{3T},$$

where the final inequality is due to the definition of $K(T)$ given in (4.197). Since the curve $L(u)$ and $-u^2/2-\frac{3}{4}\delta T^2$ differ by at most $(a'-a)^2$ on $[a,a']$, hence

$$P\Big(\inf_{u\in\Lambda_d^N[a,a']}\big(\hat{S}_{a,a'}(u)+u^2/2\big)\leq -K(T)(\delta(T+1))^{1/2}-3/4\delta T^2-\delta^2(T+1)^2\Big)\leq\frac{\varepsilon}{3T}.$$

Given that $\delta<1/8$, by assumption on T_0, for $T\geq T_0$, we have $\delta^2(T+1)^2\leq 1/8\delta T^2$ and $K(T)(\delta(T+1))^{1/2}\leq 1/8\delta T^2$. Thus

$$P\Big(\inf_{u\in\Lambda_d^N[a,a']}\big(\hat{S}_{a,a'}(u)+u^2/2\big)\leq -\delta T^2\Big)\leq\frac{\varepsilon}{3T}.$$

Since there are at most T pairs (a,a') consecutive elements in A, the above inequality implies that

$$P\Big(\inf_{u\in\Lambda_d^N[a_*,a^*]}\big(\hat{S}_{a,a'}(u)+u^2/2\big)\leq -\delta T^2\Big)\leq\frac{\varepsilon}{3},\tag{4.208}$$

where a_*, a^* denote the minimal and maximal elements of A. Also $a_*\leq y_0+(T+1)\delta$ and $a^*\geq y_0+T-(T+1)\delta$, thus

$$P\Big(\inf_{u\in\Lambda_d^N[y_0+(T+1)\delta,y_0+T-(T+1)\delta]}\big(\hat{S}_{a,a'}(u)+u^2/2\big)\leq -\delta T^2\Big)\leq\frac{\varepsilon}{3},$$

or equivalently

$$P\big(E^N_{y_0+(T+1)\delta,T-2(T+1)\delta}\big)\leq\frac{\varepsilon}{3}.$$

So by slightly changing values of $T\in[T_0,x_0]$ and $y_0\in[-x_0,x_0-T]$, the argument above yields the conclusion $P(E^N_{y_0,T})\leq\varepsilon/3$ which completes the proof.

4.9.3 Proof of Proposition 4.8.3

This proof will generally follow the steps in the proof of [CH16, Proposition 6.3]. In fact, rather than trying to adapt everything to the discrete setting, we will use the strong coupling provided by Assumption A4 to deduce Lemma 4.9.3 from [CH16, Proposition 7.6]. The proof of that proposition is quite involved and lengthy and this saves us from being needed to redo or adapt it.

Our proof proceeds by induction on the curve index k. For the case $k=1$, Proposition 4.8.3 follows from assumption in Theorem 3.1.13. The general case is $k\geq 3$, and the case $k=2$ is a specialization of the $k\geq 3$ proof. So, from here on we will assume that $k\geq 3$.

In order to deduce the proposition for general index $k\geq 3$, we will apply Proposition 4.8.1 for indices $k-2, k-1$ and k and Proposition 4.8.3 for index $k-1$. The basic idea of the argument is to show that should the index k curve be too high at some time $u \in [\bar{x}, \bar{x}+\frac{1}{2}]$ then (up to the occurrence of certain events which we show are likely) so too must the index $k-1$ curve be high at some point between $[x, \bar{x}+2]$. This violates the index $k-1$ result of Proposition 4.8.3 assumed by the induction, and hence proves the index k case.

For arbitrary $x_0 > 0$ and $\overline{x} \in \Lambda_d^N[-x_0, x_0-1]$, and $k \in \mathbb{N}$ we will define the following events. These events will be determined by additional parameters K_k, R_k, $\hat{R}_{k-1}$ and $\hat{R}_k$ which we will specify later in the proof. Define the event $E_k^N(\hat{R}_k)$ (which we will later show is unlikely)

$$E_k^N(\hat{R}_k) = \left\{ \sup_{u \in \Lambda_d^N[\overline{x}, \overline{x}+\frac{1}{2}]} (\mathcal{L}_k^N(u) + u^2/2) \geq \hat{R}_k \right\}$$

and

$$\chi^N(\hat{R}_k) = \inf\left\{ u \in \Lambda_d^N[\overline{x}, \overline{x}+\frac{1}{2}] : (\mathcal{L}_k^N(u) + u^2/2) \geq \hat{R}_k \right\},$$

with the convention that if the infimum is not attained then $\chi(\hat{R}_k) = \overline{x}+1/2$. Note that almost surely $E_k^N(\hat{R}_k) = \left\{\chi^N(\hat{R}_k) < \overline{x}+1/2\right\}$. We will generally shorten $\chi^N(\hat{R}_k)$ by just writing χ. Likewise, for the above and below N-dependent events, we will typically drop the N superscript.

Let us further define events (which by our inductive hypotheses we will show to be typical)

$$\begin{aligned} Q_{k-2}^N(K_k) &= \left\{ \inf_{u \in \Lambda_d^N[\overline{x}, \overline{x}+2]} (\mathcal{L}_{k-2}^N(u) + u^2/2) \geq -K_k \right\}, \\ A_{k-1,k}^N(R_k) &= \left\{ \mathcal{L}_{k-1}^N(\chi) + \chi^2/2 \geq -R_k \right\} \cap \left\{ \mathcal{L}_j^N(\overline{x}+2) + (\overline{x}+2)^2/2 \geq -R_k \text{ for } j = k, k-1 \right\}. \end{aligned}$$

Lastly, define the event (which is atypical by the inductive hypotheses)

$$B_{k-1}^N(\hat{R}_{k-1}) = \left\{ \sup_{u \in \Lambda_d^N[\chi, \overline{x}+2]} (\mathcal{L}_{k-1}^N(u) + u^2/2) \geq \hat{R}_{k-1} \right\}.$$

Observe that the interval $[\chi, \overline{x}+2]$ forms a $\{k-1, k\}$-stopping domain for $\mathcal{L}$. Observe also that the events $E_k^N(\hat{R}_k), Q_{k-2}^N(K_k)$ and $A_{k-1,k}^N(R_k)$ are all $\mathcal{F}_{\mathrm{ext}}\big(\{k-1,k\}, \Lambda_d^N(\chi, \overline{x}+2)\big)$-measurable. The event $B_{k-1}(\hat{R}_{k-1})$ is, however, not measurable with respect to this external sigma-field. By using the strong Gibbs property, we have that $\mathbb{P}$-almost surely:

$$\begin{aligned} &\mathbb{E}\left[B_{k-1}(\hat{R}_{k-1}) \middle| \mathcal{F}_{\mathrm{ext}}\big(\{k-1,k\}, \Lambda_d^N(\chi, \overline{x}+2)\big) \right] \\ &= \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(\mathcal{L}_{k-1}^N(\chi),\mathcal{L}_k^N(\chi)),(\mathcal{L}_{k-1}^N(\overline{x}+2),\mathcal{L}_k^N(\overline{x}+2)),\mathcal{L}_{k-2}^N,\mathcal{L}_{k+1}^N}(B_{k-1}(\hat{R}_{k-1})). \end{aligned}$$

Given that the event $E_k^N(\hat{R}_k) \cap Q_{k-2}(K_k) \cap A_{k-1,k}(R_k)$ occurs, it follows that

$$\begin{aligned}
\mathcal{L}_{k-1}^N(\chi) &\geq -R_k - \chi^2/2 \\
\mathcal{L}_k^N(\chi) &\geq \hat{R}_k - \chi^2/2 \\
\mathcal{L}_{k-1}^N(\overline{x}+2) &\geq -R_k - (\overline{x}+2)^2/2 \\
\mathcal{L}_k^N(\overline{x}+2) &\geq -R_k - (\overline{x}+2)^2/2 \\
\mathcal{L}_{k-2}^N(x) &\geq -K_k - x^2/2 \quad \text{for all } x \in \Lambda_d^N[\chi, \overline{x}+2] \\
\mathcal{L}_k^N(x) &\geq -\infty \quad \text{for all } x \in \Lambda_d^N[\chi, \overline{x}+2]
\end{aligned}$$

Therefore, by Lemma 3.1.11, there exists a coupling of the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(\mathcal{L}_{k-1}^N(\chi),\mathcal{L}_k^N(\chi)),(\mathcal{L}_{k-1}^N(\overline{x}+2),\mathcal{L}_k^N(\overline{x}+2)),\mathcal{L}_{k-2}^N,\mathcal{L}_{k+1}^N}$$

on the curves $S_1, S_2 : [\chi, \overline{x}+2] \to \mathbb{R}$ with the measure

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,(\chi,\overline{x}+2),(-R_k-\chi^2/2,\hat{R}_k-\chi^2/2),(-R_k-(\overline{x}+2)^2/2,-R_k-(\overline{x}+2)^2/2),-K_k-x^2/2,-\infty}$$

on the curves $\widetilde{S}_1, \widetilde{S}_2$ such that almost surely $S_i(x) \geq \widetilde{S}_i(x)$ for all $i \in \{1,2\}$ and $x \in \Lambda_d^N[\chi, \overline{x}+2]$. Since the event $B_{k-1}(\hat{R}_{k-1})$ becomes less probable as curves S_1, S_2 decrease, then

$$\begin{aligned}
&\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(\mathcal{L}_{k-1}^N(\chi),\mathcal{L}_k^N(\chi)),(\mathcal{L}_{k-1}^N(\overline{x}+2),\mathcal{L}_k^N(\overline{x}+2)),\mathcal{L}_{k-2}^N,\mathcal{L}_{k+1}^N}(B_{k-1}(\hat{R}_{n-1})) \\
&\geq p(R_k, K_k, \hat{R}_k) \mathbb{1}_{E_k^N(\hat{R}_k) \cap Q_{k-2}(K_n) \cap A_{k-1,k}(R_n)},
\end{aligned} \tag{4.209}$$

where $p(R_k, K_k, \hat{R}_k)$ is a shorthand for

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{k-1,k,\Lambda_d^N(\chi,\overline{x}+2),(-R_k-\chi^2/2,\hat{R}_k-\chi^2/2),(-R_k-(\overline{x}+2)^2/2,-R_k-(\overline{x}+2)^2/2),-K_k-x^2/2,-\infty}(B_{k-1}(\hat{R}_{k-1})).$$

We would like to choose other parameters $K_k, R_k, \hat{R}_k$ such that $p(R_k, K_k, \hat{R}_k) \geq 1/2$ for any $\chi \in \Lambda_d^N[\overline{x}, \overline{x}+1/2]$. This is a direct consequence of Lemma 4.9.3 below by applying Lemma 4.9.3 with the choice $\mu = 1/2$. Now by taking the expectation in (4.209), we obtain

$$\mathbb{P}(E_k^N(\hat{R}_k) \cap Q_{k-2}(K_k) \cap A_{k-1,k}(R_k)) \leq 2\mathbb{P}(B_{k-1}(\hat{R}_{k-1})).$$

We choose $\hat{R}_{k-1}$ so that $\mathbb{P}(B_{k-1}(\hat{R}_{k-1})) \leq 2\varepsilon$, which can be achieved for $N \geq N_0(x_0, \varepsilon)$ owing to Proposition 4.8.3 applied for index $k-1$. Thus, we deduce that

$$\mathbb{P}(E_k^N(\hat{R}_k) \cap Q_{k-2}(K_n) \cap A_{k-1,k}(R_n)) \leq 4\varepsilon.$$

Moreover from Proposition 4.8.1 for index $k-2$ there exist $N \geq N_0$ and $K_k > K^0(R_k)$ such that

$$\mathbb{P}(Q_{k-2}(K_k)) \geq 1 - 2\varepsilon.$$

Similar, by applying Proposition 4.8.1 for indices $k-1, k$, there exist $N > N_0$ and $\delta \hat{R}_k/2 > \hat{R}_{k-1}$ such that

$$\mathbb{P}(A_{k-1,k}) \geq 1 - 3\varepsilon.$$

Now we have for $N > N_0(x_0, \varepsilon, k)$,

$$\begin{aligned}\mathbb{P}(E_k^N(\hat{R}_k)) &\leq \mathbb{P}(E_k^N(\hat{R}_k) \cap Q_{k-2}(K_n) \cap A_{k-1,k}(R_n)) + \mathbb{P}((Q_{k-2}(K_n))^\neg \cup (A_{k-1,k}(R_n))^\neg) \\ &\leq 4\varepsilon + 2\varepsilon + 3\varepsilon.\end{aligned}$$

In the end, by a simple shifting $\overline{x}$ to $\overline{x} + \frac{1}{2}$, we obtain Proposition 4.8.3.

Lemma 4.9.3. *Suppose $\dot{\mathbf{H}}^N$ and $\mathrm{H}^{\mathrm{RW},N}$ satisfy assumptions A3 and A4 respectively. For any $\mu \in (0,1)$. There exists $\delta > 0$, $R^0 > 0$, and functions $K^0(R) > 0$ and $\hat{R}^0(R, K) > 0$ such that, for all $R > R^0$, $K > K^0(R)$, $\hat{R} \geq \hat{R}(R, K)$ and all $\overline{x} \in \mathbb{R}$ and $\chi \in \Lambda_d^N[\overline{x}, \overline{x} + 1/2]$, we have the following estimate if provided that $N \geq N_0(R, K, \hat{R})$ with N_0 large enough.*

$$\mathbb{P}_N^{1,2}\left(\sup_{u \in \Lambda_d^N[\chi, \overline{x}+2]} (\bar{S}_1(u) + u^2/2) \geq \frac{1}{2}\delta\hat{R}\right) \geq \mu,$$

where $\mathbb{P}_N^{1,2}$ is a shorthand for the measure below on the curves $\bar{S}_1$ and $\bar{S}_2$,

$$\mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{1,2,\Lambda_d^N(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2),-K-x^2/2,-\infty}.$$

Now we only need to prove Lemma 4.9.3 to complete the full proof. The analogue of Lemma 4.9.3 when the underlying path measure is Brownian bridge measure has been proved in [CH16, Proposition 7.6] and we show that the same estimate also holds for random walk bridge case assuming the KMT coupling Assumption A4, hence finishing the proof.

Proof of Lemma 4.9.3. In order to simplify the notation we denote:

$$\mathbb{Q} := \mathbb{P}_{\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N}}^{1,2,\Lambda_d^N(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2),-K-x^2/2,-\infty},$$

and

$$\overline{\mathbb{Q}} := \mathbb{P}_{free, \mathrm{H}^{\mathrm{RW},N}}^{1,2,\Lambda_d^N(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2)}.$$

These measures are supported on two random walks bridges S_1, S_2. On the other hand we also consider the measures which are defined on two Brownian bridges B_1 and B_2 (Definition 3.1.3):

$$\mathbb{K} := \mathbb{P}_{\mathbf{H}}^{1,2,(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2),-K-x^2/2,-\infty},$$

and

$$\overline{\mathbb{K}} := \mathbb{P}_{free}^{1,2,(\chi,\overline{x}+2),(-R-\chi^2/2,\hat{R}-\chi^2/2),(-R-(\overline{x}+2)^2/2,-R-(\overline{x}+2)^2/2)}.$$

In this case, from [CH16, Proposition 7.6], we have for any $\mu \in (0,1)$, there exists $\delta > 0$, $R^0 > 0$, and functions $K^0(R) > 0$ and $\hat{R}^0(R,K) > 0$ such that, for all $R > R^0$,$K > K^0(R)$, $\hat{R} \geq \hat{R}(R,K)$ and all $\overline{x} \in \mathbb{R}$ and $\chi \in [\overline{x}, \overline{x}+1/2]$,

$$\mathbb{K}\left(\sup_{u \in [\chi, \overline{x}+2]} (B_1(u) + u^2/2) \geq \frac{1}{2}\delta\hat{R}\right) \geq \frac{\mu+1}{2}.$$

We actually consider the event :

$$J^N(\delta,\hat{R}) = \left\{\sup_{u \in [\chi, \overline{x}+2]} (B_1(u) + u^2/2) \geq \frac{1}{2}\delta\hat{R} + aN^{-1/3}\right\}$$

It is clear that as $N \to \infty$ and we have:

$$\mathbb{K}(J^N(\delta,\hat{R})) \to \mathbb{K}\big(\sup_{u \in [\chi, \overline{x}+2]} (B_1(u) + u^2/2) \geq \frac{1}{2}\delta\hat{R}\big).$$

Hence there exists N_0 such that for $N > N_0$, we have :

$$\mathbb{K}(J^N(\delta,\hat{R})) \geq \mu.$$

In the rest of the proof, we will compare $\mathbb{K}(J^N(\delta,\hat{R}))$ with $\mathbb{Q}(I^N(\delta,\hat{R}))$ where

$$I^N(\delta,\hat{R}) = \Big\{\sup_{u \in \Lambda_d^N[\chi, \overline{x}+2]} (\bar{S}_1(u) + u^2/2) \geq \frac{1}{2}\delta\hat{R}\Big\}.$$

Recall the definition of $\mathbb{Q}$, we have

$$\mathbb{Q}\left(I^N(\delta,\hat{R})\right) = \frac{\mathbb{E}_{\overline{\mathbb{Q}}}\big[\mathbb{1}_{I^N(\delta,\hat{R})} W\big]}{\mathbb{E}_{\overline{\mathbb{Q}}}\big[W\big]},$$

where

$$W = \exp\Big[-\sum_{k=0}^{2} \sum_{u \in \Lambda_d^N[\chi, \overline{x}+2]} \dot{\mathbf{H}}^N[\square(\bar{S},k,u)]\Big].$$

Here we denote $\bar{S}_0(u) = -K - u^2/2$ and $\bar{S}_3(u) = -\infty$. Recall that the measure $\overline{\mathbb{Q}}$ is the law of two independent random walk bridges S_1, S_2 starting at time χ at the point $(-R-\chi^2/2, \hat{R}-\chi^2/2)$ and ending at time $\overline{x}+2$ at the point $(-R-(\overline{x}+2)^2/2, -R-(\overline{x}+2)^2/2)$. Similarly, we have:

$$\mathbb{K}(J^N(\delta,\hat{R})) = \frac{\mathbb{E}_{\overline{\mathbb{K}}}\big[\mathbb{1}_{J^N(\delta,\hat{R})} \widehat{W}\big]}{\mathbb{E}_{\overline{\mathbb{K}}}\big[\widehat{W}\big]},$$

where

$$\widehat{W} = \exp\Big[-\sum_{i=0}^{2} \int_{\chi}^{\overline{x}+2} du \exp[B_i(u) - B_{i+1}(u)]\Big],$$

with the conventions $B_0(x) = -K - x^2/2$ and $B_3(x) = -\infty$. Recall that the measure $\overline{\mathbb{K}}$ is the law of two independent Brownian bridges starting at time χ at the point $(-R-\chi^2/2, \hat{R}-\chi^2/2)$ and ending at time $\overline{x}+2$ at the point $(-R-(\overline{x}+2)^2/2, -R-(\overline{x}+2)^2/2)$.

By applying the strong approximation in Assumption A4, we can enlarge the probability space $(\Omega, \mathbb{P})$ and couple the measure $\overline{\mathbb{K}}$ and $\overline{\mathbb{Q}}$ such that these exits $c(R, \hat{R}), a(R, \hat{R})$, for $i = 1, 2$,

$$\mathbb{P}\Big(\sup_{u\in[\overline{x},\overline{x}+2]} |\bar{S}_i(u) - B_i(u)| \geq \frac{a\log N}{\sqrt{N}}\Big) \leq cN^{-30}.$$

Such coupling will allow us to obtain the desired lower bound μ for $\mathbb{Q}(I^N(\delta, \hat{R}))$ with N large enough thus completing the proof. It amounts to find a lower bound for $Y'_N := \overline{\mathbb{E}}_{\mathbb{Q}}\big[\mathbb{1}_{I^N(\delta,\hat{R})}W\big]$, and an upper bound for $Y_N := \overline{\mathbb{E}}_{\mathbb{Q}}\big[W\big]$.

Let us first start with the lower bound for Y'_N. For $i = 1, 2$, let D_i be the event:

$$D_i := \left\{\sup_{u\in[\chi,\overline{x}+2]} |\bar{S}_i(u) - B_i(u)| \leq \frac{a\log N}{\sqrt{N}}\right\}.$$

Since $\chi \in [\overline{x}, \overline{x} + 1/2]$, it is clear that

$$\mathbb{P}(D_1, D_2) \geq (1 - cN^{-30})^2.$$

Consequently, we have

$$Y'_N \geq (1 - cN^{-30})^2\overline{\mathbb{Q}}\big[\mathbb{1}_{I^N(\delta,\hat{R})}W\big|D_1, D_2\big]$$

Under the Assumption A3 about the Hamiltonian $\dot{\mathbf{H}}^N$ in Theorem 3.1.13, there exists a constant C such that

$$W \geq \exp\left[-\sum_{i=0}^{2}\int_{\chi}^{\overline{x}+2} du\exp\left(S_{i+1}(u) - S_i(u)\right)\right] \cdot \exp\left[-C\left(\omega_{\chi,\overline{x}+2}(\bar{S}_1, 1/N) + \omega_{\chi,\overline{x}+2}(\bar{S}_2, 1/N)\right)\right]$$

Assume that the events D_1 and D_2 occur, we obtain

$$\begin{aligned}\mathbb{1}_{I^N(\delta,\hat{R})} &\geq \mathbb{1}_{J^N(\delta,\hat{R})}\\ \omega_{\chi,\overline{x}+2}(\bar{S}_i, 1/N) &\leq \omega_{\chi,\overline{x}+2}(B_i, 1/N) + \frac{2a\log N}{\sqrt{N}}\\ \int_{\chi}^{\overline{x}+2} du\exp[\bar{S}_{i+1}(u) - \bar{S}_i(u)] &\leq \int_{\chi}^{\overline{x}+2} du\exp[B_{i+1}(u) - B_i(u)]\exp\left(\frac{2a\log N}{\sqrt{N}}\right).\end{aligned}$$

By the coupling between $\overline{\mathbb{Q}}$ and $\overline{\mathbb{K}}$, we have

$$\begin{aligned}Y'_N \geq (1 - cN^{-30})^2 \times \overline{\mathbb{K}}\Bigg[\mathbb{1}_{J^N(\delta,\hat{R})}\exp\Bigg[-\sum_{i=0}^{2}\exp\left(\frac{2a\log N}{\sqrt{N}}\right)\left(\int_{\chi}^{\overline{x}+2} du\exp\left(B_{i+1}(u) - B_i(u)\right)\right)\Bigg]\\ \times\exp\left(-C\omega_{\chi,\overline{x}+2}(B_1, 1/N) - C\omega_{\chi,\overline{x}+2}(B_2, 1/N) - 4C\frac{a\log N}{\sqrt{N}}\right)\Bigg].\end{aligned}$$

Fixing an $\varepsilon > 0$ and conditioning on the event that $\{\omega_{\chi,\overline{x}+2}(B^i, 1/N) < \varepsilon\}$ for $i = 1, 2$ then the above inequality becomes:

$$Y'_N \geq A'(N, \varepsilon) \times \overline{\mathbb{K}}\Bigg[\mathbb{1}_{J^N(\delta,\hat{R})}\exp\Bigg[-\sum_{i=0}^{2}\exp\left(\frac{2a\log N}{\sqrt{N}}\right)\int_{\chi}^{\overline{x}+2} du\exp\left(B_{i+1}(u) - B_i(u)\right)\Bigg]\Bigg], \tag{4.210}$$

where

$$A'(N,\varepsilon) = (1-cN^{-30})^2 \times (\mathbb{P}(\{\omega_{\chi,\overline{x}+2}(B,1/N) < \varepsilon\}))^2 \times \exp\left(-2C\varepsilon - \frac{4aC\log N}{\sqrt{N}}\right).$$

Since $\exp\left(\frac{2a\log N}{\sqrt{N}}\right) > 1$, by the convexity of function x^α for $\alpha > 1$ we have,

$$Y'_N \geq A'(N,\varepsilon) \times \overline{\mathbb{K}}\left[\mathbb{1}_{J^N(\delta,\hat{R})}\widehat{W}\right]^{\left(\frac{2a\log N}{\sqrt{N}}\right)}. \tag{4.211}$$

Remark that for fixed $\varepsilon > 0$ and $N \to \infty$, $A'(N,\varepsilon) \to \exp(-2C\varepsilon)$.

Now we continue by establishing an upper bound for Y_N. Since almost surely W is bounded by 1, then

$$Y_N \leq \overline{\mathbb{Q}}\Big[W|D_1,D_2\Big]\mathbb{P}[D_1,D_2] + \mathbb{P}(D_1^-) + \mathbb{P}[D_2^-]. \tag{4.212}$$

As we have seen before, $\mathbb{P}[D_i^-] \leq cN^{-30}$. Moreover conditioning on the events D_1, D_2, it is straightforward that for $i = 1,2$

$$\omega_{\chi,\overline{x}+2}(\bar{S}_i,1/n) \geq \omega_{\chi,\overline{x}+2}(B_i,1/N) - \frac{2a\log N}{\sqrt{N}}$$
$$\int_\chi^{\overline{x}+2} du \exp[\bar{S}_{i+1}(u) - \bar{S}_i(u)] \geq \int_\chi^{\overline{x}+2} du \exp[B_{i+1}(u) - B_i(u)] \exp\left(\frac{-2a\log N}{\sqrt{N}}\right).$$

By applying Assumption A3 and conditioning on the events $\{\omega_{\chi,\overline{x}+2}(B_i,1/N) > \varepsilon\}$, we can easily get the following upper bound

$$\begin{aligned}\overline{\mathbb{Q}}\Big[W|D_1,D_2\Big] &\leq 2\mathbb{P}[\omega_B(\chi,\overline{x}+2,1/N) > \varepsilon] \\ &\quad + \overline{\mathbb{K}}\Bigg[\exp\left[-\sum_{i=0}^{2}\exp\left(\frac{-2a\log N}{\sqrt{N}}\right)\int_\chi^{\overline{x}+2} du \exp[B_{i+1}(u) - B_i(u)]\right]\Bigg] \\ &\quad \times \exp\left(2C\varepsilon + 4C\frac{a\log N}{\sqrt{N}}\right),\end{aligned}$$

Since $0 < \exp(-2\alpha(2N)) < 1$, by the concavity of x^α for $0 < \alpha < 1$, we can deduce that

$$\overline{\mathbb{Q}}\Big[W|D_1,D_2\Big] \leq 2\mathbb{P}[\omega_{\chi,\overline{x}+2}(B,1/N) > \varepsilon] + \overline{\mathbb{K}}\Big[\overline{W}\Big]^{\exp\left(\frac{-2a\log N}{\sqrt{N}}\right)} \times \exp\left(2C\varepsilon + 4C\frac{a\log N}{\sqrt{N}}\right). \tag{4.213}$$

Combining (4.212) and (4.213), we obtain:

$$Y_N \leq C(N,\varepsilon) + A(N,\varepsilon)\overline{\mathbb{K}}\Big[\overline{W}\Big]^{\exp\left(\frac{-2\log N}{\sqrt{N}}\right)}, \tag{4.214}$$

where

$$A(N,\varepsilon) = \exp\left(2C\varepsilon + \frac{4aC\log N}{\sqrt{N}}\right) \times \left(1 - cN^{-30}\right)^2,$$

and

$$C(N,\varepsilon) = 2cN^{-30} + \left(1 - cN^{-30}\right)^2 \times 2\mathbb{P}[\omega_{\chi,\overline{x}+2}(B,1/N) > \varepsilon].$$

At the end, we obtain from (4.211) and (4.214) that

$$\mathbb{Q}[I^N(\delta,\hat{R})] = \frac{Y_N'}{Y_N} \geq \frac{A'(N,\varepsilon)\times \overline{\mathbb{K}}\Big[\mathbb{1}_{J^N(\delta,\hat{R})}\overline{W}\Big]^{\exp(\frac{2a\log N}{\sqrt{N}})}}{C(N,\varepsilon)+A(N,\varepsilon)\overline{\mathbb{K}}\Big[\widehat{W}\Big]^{\exp(\frac{-2a\log N}{\sqrt{N}})}}.$$

Since for fixed $\varepsilon > 0$, we have $A(N,\varepsilon)\to e^{2C\varepsilon}$, $A'(N,\varepsilon)\to e^{-2C\varepsilon}$ and $C(N,\varepsilon)\to 0$ as $N\to\infty$. Thus as $N\to\infty$,

$$\frac{Y_N'}{Y_N} \geq \frac{A'(N,\varepsilon)\times \overline{\mathbb{K}}\Big[\mathbb{1}_{J^N(\delta,\hat{R})}\widehat{W}\Big]^{\exp(\frac{2a\log N}{\sqrt{N}})}}{C(N,\varepsilon)+A(N,\varepsilon)\overline{\mathbb{K}}\Big[\widehat{W}\Big]^{\exp(\frac{-2a\log N}{\sqrt{N}})}} \to e^{-4C\varepsilon}\mathbb{K}\left(\sup_{x\in[\chi,\overline{x}+2]}\left(B_1(x)+x^2\right)\geq \frac{1}{2}\delta\hat{R}\right).$$

Recall that $\mathbb{K}\left(\sup_{x\in[\chi,\overline{x}+2]}\left(B_1(x)+\frac{1}{2}x^2\right)\geq \frac{1}{2}\delta\hat{R}\right)\geq \frac{\mu+1}{2}$, by choosing $\varepsilon(\mu)$ small enough we have for $N\geq N_0$ with $N_0\left(R,K,\hat{R}\right)$ large enough,

$$\mathbb{Q}[I^N(\delta,\hat{R})]\geq\mu,$$

which finishes the proof. □

4.10 Application for the log-gamma directed polymers

In this chapter, we first construct a discrete K-curve log-gamma line ensemble $\mathcal{L}_K = \{\mathcal{L}_{K,i}(n), 1\leq i\leq K, n\geq i\}$, which is assocaited with the log-gamma directed polymer, as introduced in [Sep]. Under weak noise scaling, the lowest indexed curve $\mathcal{L}_{K,1}$ of $\mathcal{L}_K$, converges weakly to solution of KPZ equation with narrow wedge initial data. This result follows from a weak KPZ universality result about directed polymers in [AKQ] (with slight modifications explained later in Proposition 4.10.9). When the same scaling is applied to the full line ensemble $\mathcal{L}_K$, we aim to apply our main Theorem 3.1.13 to show the tightness of the scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ and the H-Brownian Gibbs property with $\mathbf{H}(x)=e^x$ for all of its subsequential limit line ensembles.

There is an immediate problem about the attempt of directly applying Theorem 3.1.13. Note that the discrete $(\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}})$-Gibbs property of $\mathcal{L}_K$ is a direct consequence of a Markov property in n for $\mathcal{L}_K(n)$, which has been proved only for $n\geq K$ in [COSZ, Theorem 3.7(i) and Theorem 3.9(i,iii)]. After applying the intermediate scaling to $\mathcal{L}_K(n)$, the restriction $n\geq K$ requires that the argument of scaled line ensemble $\overline{\mathcal{L}}^N(\cdot)$($N$ is the scaling parameter) is nonnegative, i.e. $\overline{\mathcal{L}}^N(\cdot)$ enjoys Gibbs property when restricted over nonnegative numbers in a lattice $\frac{2}{\sqrt{N}}\mathbb{Z}$. We resolved this difficulty by proving the Gibbs property for scaled line ensemble $\overline{\mathcal{L}}^N(\cdot)$ over an interval with left endpoint escaping to $-\infty$ (as N goes to infinity) when restricted for the first finite k curves with any $k\geq 1$. The proof relies on the observation (first proved in [CH14] for

Brownian Gibbs property) that $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property survives under weak convergence of discrete line ensembles and another observation in [COSZ] that $\mathcal{L}_K(n)$ could be well approximated by a Markov chain $\mathcal{J}_K^M(n)$ as M goes to infinity without restrictions on n.

We then proceed to the verification of the assumptions of Theorem 3.1.13 and as an application, we prove Theorem 4.10.11.

4.10.1 Definition of log-gamma line ensemble

Let us first introduce the discrete log-gamma line ensemble $\mathcal{L}_K = \{\mathcal{L}_{K,i}(n), 1 \le i \le K, n \ge i\}$. We explain the construction and illustrate the obstacle for the construction of line ensemble $\mathcal{L}_K(n)$ when $n < K$.

Definition 4.10.1. *Let γ be a positive real number. A random variable X has inverse gamma distribution with shape parameter $\gamma > 0$ if is supported on the positive reals and it has distribution*

$$\mathbb{P}(X \in dx) = \frac{1}{\Gamma(\gamma)} x^{-\gamma-1} \exp(-\frac{1}{x}) dx,$$

where $\Gamma(\gamma)$ is the Gamma function. We abbreviate with $X \overset{(d)}{=} \text{Inv-Gamma}(\gamma)$.

It could be directly computed that for any $k \in \mathbb{N}$ with $\gamma > k$,

$$\mathbb{E}[\text{Inv-Gamma}(\gamma)^k] = \frac{\Gamma(\gamma - k)}{\Gamma(\gamma)} = \frac{1}{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}. \tag{4.215}$$

Fix $K \in \mathbb{N}$ and consider a semi-infinite matrix $d = (d_{ij} : i \ge 1, 1 \le j \le K)$ of i.i.d random variables with distribution:

$$d_{ij} \overset{(d)}{=} \text{Inv-Gamma}(\gamma). \tag{4.216}$$

For each $n \ge 1$, form the $n \times K$ matrix $d^{n,K} = (d_{ij} : 1 \le i \le n, 1 \le j \le K)$. $d^{n,K}$ serves as a random environment for the log-gamma polymers such that d_{ij} is the weight a path collect at location (i, j) in $\mathbb{Z}^2$.

The following construction is also known as the geometric RSK correspondence [COSZ]. For $1 \le l \le k \le K$, let $\Pi^n_{k,l}$ denote the set of l-tuples $\pi = (\pi_1, \cdots, \pi_l)$ of non-intersecting lattice paths in $\mathbb{Z}^2$ such that for $1 \le r \le l$, π_r is a lattice path from $(1, r)$ to $(n, k + r - l)$. Remark that we need $n \ge l$ to avoid emptiness or trivialness of the paths collection $\Pi^n_{k,l}$. For $0 \le n < l < k \le K$, the set of paths $\Pi^n_{k,l}$ is empty. At $l = k$ there is a unique l-tuple such that all paths are horizontal. See the left figure in Table 4.1 below for an illustration.

Definition 4.10.2. *The weight of an l-tuples π of such paths is*

$$wt(\pi) := \prod_{r=1}^{l} \prod_{(i,j)\in\pi_r} d_{ij}.$$

For $1 \le l \le k \le K$, define

$$\tau_{k,l}(n) := \sum_{\pi \in \Pi^n_{k,l}} wt(\pi), \tag{4.217}$$

and we take the empty sum to equal zero for the empty sum case $0 \leq n < l < k \leq K$.

Then at time n, *we can define an array* $z(n) = \{z_{k,l}(n) : 1 \leq k \leq K, 1 \leq l \leq k\}$ *by*

$$z_{k,l}(n) = \begin{cases} \dfrac{\tau_{k,l}(n)}{\tau_{k,l-1}(n)}, & l \leq n \wedge k, \\ \\ \text{undefined}, & n < l \leq k. \end{cases} \tag{4.218}$$

When $n < K$, *there are elements* $z_{k,l}(n) : n < l \leq k \leq K$ *that are undefined, even though strictly speaking one more element, namely* $z_{n+1,n+1}(n)$, *could be consistently defined as 1.*

Define the shape $y(n) := (y_1(n), \cdots, y_K(n))$ *of the array* $z(n)$ *as:*

$$y_i(n) := \begin{cases} z_{K,i}(n), & i \leq n \wedge K, \\ \\ \text{undefined}, & n < i \leq K. \end{cases} \tag{4.219}$$

When $n < K$, $y_i(n) := z_{K,i}(n)$ *can only be defined for* $1 \leq i \leq n$.

We denote $\mathbb{T}_K := (z_{kl} : 1 \leq l \leq k \leq K \text{ and } z_{kl} \in (0, \infty))$ be the set of triangular arrays with positive real entries and $\mathbb{Y}_K := (y_l : 1 \leq l \leq K, y_l \in (0, \infty))$. We define $K(y, dz)$ as a kernel from $\mathbb{Y}_K$ to $\mathbb{T}_K$ as:

$$K(y, dz) = \prod_{1 \leq l \leq k < K} \exp\left(- \frac{z_{k,l}}{z_{k+1,l}} - \frac{z_{k+1,l+1}}{z_{k+1,l}} \right) \frac{dz_{k,l}}{z_{k,l}} \prod_{l=1}^{K} \delta_{y_l}(dz_{K,l}).$$

For $y \in \mathbb{Y}_K$, define

$$w(y) := \int_{\mathbb{T}_K} K(y, dz).$$

As proved in [COSZ, Theorem 3.7 and Theorem 3.9], the process $\{y(n), n \geq K\}$ is a Markov chain with the following transition kernel on $\mathbb{Y}_K$:

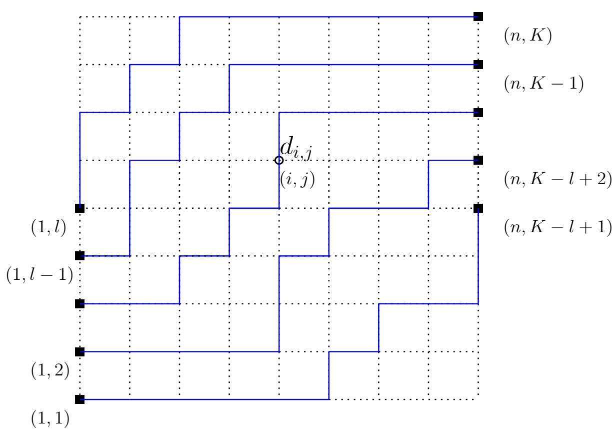
$$P(y, d\widetilde{y}) = \frac{w(\widetilde{y})}{w(y)} \prod_{i=1}^{K-1} \exp\left[- \frac{\widetilde{y}_{i+1}}{y_i} \right] \prod_{j=1}^{K} \left(\Gamma(\gamma)^{-1} \big(\frac{y_j}{\widetilde{y}_j}\big)^{\gamma} \exp\Big(-\frac{y_j}{\widetilde{y}_j}\Big) \right) \frac{d\widetilde{y}_j}{\widetilde{y}_j}. \tag{4.220}$$

We summarize the the construction of log-gamma line ensemble in the following Table 4.1.

Definition 4.10.3. *We define the log-gamma line ensemble* $\mathcal{L}_{K,i}(n), 1 \leq i \leq K, n \geq 1$ *by*

$$\mathcal{L}_{K,i}(n) := \begin{cases} \log(y_i(n)) & 1 \leq i \leq n \wedge K \text{ and } n \geq 1, \\ \\ \text{undefined} & n < i \leq K \text{ and } n \geq 1. \end{cases}$$

The following Lemma 4.10.4 shows that an $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property readily follows from the Markov property (4.220), i.e. any Markov chains with transition kernel (4.220) enjoy the same Gibbs property.

<table>
<tr><th>Paths in environment</th><th>Definition of log-gamma line ensemble $\mathcal{L}_K$</th></tr>
<tr><td>
</td><td>Define partition functions as

- $z_{K,l}(n) := \sum_{\pi \in \Pi^n_{K,l}} wt(\pi)$,
- $wt(\pi) := \prod_{r=1}^{l} \prod_{(i,j)\in\pi_r} d_{ij}$.

Define the *log-gamma line ensemble* $\mathcal{L}_K$ as

- $\mathcal{L}_{K,l}(n) := \log z_{K,l}(n) - \log z_{K,l-1}(n),\ l \le n$
- undefined for $l > n$.</td></tr>
</table>

Table 4.1: Summary of the process of constructing $\mathcal{L}(n), n \ge 1$. We regard n as the evolving coordinate. We take the empty sum to equal zero in the definition of $z_{K,l}(n)$ when $n < l$.

Lemma 4.10.4. *Fix $K \in \mathbb{N}$ and an initial state $\hat{y}^0$ in $\mathbb{Y}_K$. Let $\hat{y}(n), n \ge 1$ be the Markov chain on $\mathbb{T}_K$ with initial state $\hat{y}(0) = \hat{y}^0$ and transition kernel given by* (4.220). *We define the line ensemble $\hat{\mathcal{L}}_{K,i}(n), 1 \le i \le K, n \ge 1$ by*

$$\hat{\mathcal{L}}_{K,i}(n) = \log(\hat{y}_i(n)).$$

Then the discrete line ensemble $\hat{\mathcal{L}}_K = \{\hat{\mathcal{L}}_{K,i}(n), 1 \le i \le K, n \ge 1\}$ enjoys a $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property with:

$$\mathrm{H}^{\mathrm{RW}}(x) = \log\Gamma(\gamma) + \gamma x + e^{-x}, \tag{4.221}$$

and

$$\dot{\mathbf{H}}(\square(\mathcal{L}, k, u)) = \exp\big(\mathcal{L}_{k+1}(u+1) - \mathcal{L}_k(u)\big). \tag{4.222}$$

Proof. Fix $1 \le k_1 < k_2 \le K$, $\Lambda_d = [1,\infty)_{\mathbb{Z}}$ and $a < b \in \Lambda_d$. We denote $\vec{x} = \big(\hat{\mathcal{L}}_{K,k_1}(a), \dots, \hat{\mathcal{L}}_{K,k_2}(a)\big)$ as the entrance data, $\vec{y} = \big(\hat{\mathcal{L}}_{K,k_1}(b), \dots, \hat{\mathcal{L}}_{K,k_2}(b)\big)$ as the exit data, $f = \hat{\mathcal{L}}_{K,k_1-1}|_{\Lambda_d[a,b]}$ be the upper boundary curve and $g = \hat{\mathcal{L}}_{K,k_2+1}|_{\Lambda_d[a,b]}$ be the lower boundary curve and we adopt the convention that $\hat{\mathcal{L}}_0 = +\infty$ and $\hat{\mathcal{L}}_{K+1} = -\infty$.

Given a continuous function $F : C([k_1,k_2]_{\mathbb{Z}} \times \Lambda_d[a,b], \mathbb{R})$, by the Markov property of the process

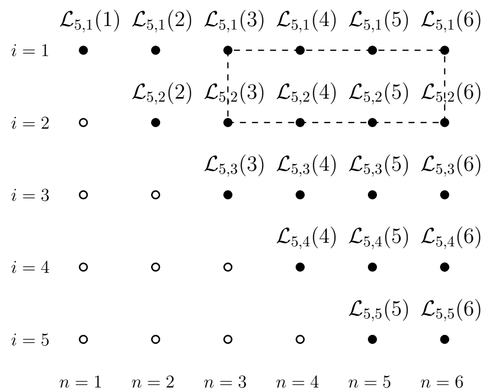


Figure 4.1: Illustration of the log-gamma line ensemble construction with $K = 5$. There are five curves $\mathcal{L}_{5,1}(n)$ through $\mathcal{L}_{5,5}(n)$. The values of $\mathcal{L}_{5,i}(n)$ are not defined on the hollow points. The Gibbs property holds for $n \geq 5$ as a consequence of Lemma 4.10.4 and the Markov property for $\mathcal{L}_5(n)$ when $n \geq 5$ with transition kernel from (4.220). We are in need of the same Gibbs property when $n < 5$, e.g. the dashed rectangle region.

$\{\hat{y}(n), n \geq 1\}$ with the transition kernel (4.220), we have:

$$\begin{aligned}
&\mathbb{E}\Big[F\big(\hat{\mathcal{L}}_K|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)}\big)\big|\mathcal{F}_{\text{ext}}\big([k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)\big)\Big] \\
&\propto \int F\big(\hat{\mathcal{L}}_K|_{[k_1,k_2]_{\mathbb{Z}}\times\Lambda_d(a,b)}\big) \prod_{u\in\Lambda_d[a,b]}\prod_{i=k_1}^{k_2} \exp\Big[-\dot{\mathbf{H}}(\boxdot(\hat{\mathcal{L}}_K,i,u))\Big]\times \\
&\prod_{i=k_1}^{k_2}\exp\Big(-\mathrm{H}^{\mathrm{RW}}(\hat{\mathcal{L}}_{K,i}(u+1)-\hat{\mathcal{L}}_{K,i}(u))\Big)d\hat{\mathcal{L}}_{K,i}(u),
\end{aligned} \tag{4.223}$$

where $\dot{\mathbf{H}}$ and H^{RW} are given by (4.222) and (4.221) respectively. Note that the coefficient could be fixed as the normalizing constant for the density inside the integral to be a probability density function and it is obvious to see that the constant should only depend on $\vec{x},\vec{y},f,g$. Now we see that the density inside the integrand is exactly $\mathbb{P}^{k_1,k_2,\Lambda_d[a,b],\vec{x},\vec{y},f,g}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}$, which proves the $(\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}})$-Gibbs property. □

Lemma 4.10.4 further implies that $\mathcal{L}_{K,i}(n)$ when restricted to $n \geq K$ enjoys $(\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}})$-Gibbs property with $(H,\mathrm{H}^{\mathrm{RW}})$ as in (4.221) and (4.222). It seems that we are lack of Gibbs property when $n < K$ due to the fact that the Markov property for $y(n)$ only holds for $n \geq K$, see an illustration in Figure 4.1. But by taking a deeper exploration of the proof of [COSZ, Theorem 3.9(iii)], together with Proposition 4.10.7 that Gibbs property survives under weak convergence of line ensemble, we argue below that we still have the same $(\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}})$-Gibbs property when $n < K$ for the well-defined region of $\mathcal{L}_K$.

Consider $\varrho_K = (\varrho_{K,i})_{1\leq i\leq K}$ with

$$\varrho_K = (\varrho_{K,i})_{1\leq i\leq K} = \left(\frac{K-1}{2}, \frac{K-1}{2}-1, \cdots, -\frac{K-1}{2}\right).$$

Let $y^{0,M} := (e^{-M\varrho_{K,i}})_{1\le i\le K}$. Define $y^M(n) := (y_i^M(n))_{1\le i\le K, n\ge 1}$ as a Markov chain in $\mathbb{Y}_K$ with the initial state $y^M(0) = y^{0,M}$ and transition kernel given by (4.220). The following proposition is proved in [COSZ, Proposition 5.3], which says that $\{y^M(n), n\ge 1\}$ converges weakly to $y(n)$ on compact sets.

Proposition 4.10.5. *Let $\{y^M(n), n\ge 1\}$ be the Markov chain on $\mathbb{Y}_K$ defined above. And let $y_i(n)$, $1\le i\le n\wedge K, n\ge 1$ be defined as (4.219). Then for any $n_0\ge 1$, $\{y_i^M(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ converges weakly to $\{y_i(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ as M goes to infinity.*

Remark 4.10.6. *It is proved in [COSZ, Proposition 5.3] not only the weak convergence of the shape $y(n)$ but also the weak convergence of the array $z(n)$.*

Define $\mathcal{J}_{K,i}^M(n) := \log y_i^M(n)$. As a consequence, for all $n_0\ge 1$, $\{\mathcal{J}_{K,i}^M(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ converges weakly to $\{\mathcal{L}_{K,i}(n) : 1\le i\le n\wedge K,\ 1\le n\le n_0\}$ as M goes to infinity. Now we are ready to prove the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property for $\mathcal{L}_K(n)$ over the well-defined region.

Proposition 4.10.7. *Fix $K\ge 1$. The line ensemble $\mathcal{L}_K(n)$ satisfies the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property in the region it is defined with $\dot{\mathbf{H}}$ and H^{RW} given by (4.222) and (4.221) respectively. Precisely, for any $1\le k_1 < k_2\le K$, $a < b\in\mathbb{N}$ with $(k_2+1)\wedge K\le a$, (4.223) holds with $\hat{\mathcal{L}}_K$ replaced by $\mathcal{L}_K$.*

Proof. From Proposition 4.10.5 and Lemma 4.10.4, $\mathcal{L}_K$ is the weak limit of the sequence of line ensembles $\mathcal{J}_K^M$ as M goes to infinity, which enjoys the same $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property. Therefore it suffices to show $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property is preserved under the weak limit. The proof proceeds similarly as the one of Theorem 3.1.13 (2), thus we omit repeated details and focus on the part that needs modification.

Let $C = [k_1, k_2]_{\mathbb{Z}}\times[a,b]_{\mathbb{Z}}$ be a compact region where $\mathcal{L}_K$ is defined. Through Skorohod representation theorem, in such region C one can couple $\mathcal{L}_K$ with the converging sequence $\mathcal{J}_K^M$ in the same probability space such that and the sequence $\mathcal{J}_K^M$ converges to $\mathcal{L}_K$ almost surely. Now as in Theorem 3.1.13 (2), one can reformulate the $(\dot{\mathbf{H}}, \mathrm{H}^{\mathrm{RW}})$-Gibbs property as a resampling invariance the same way as in (4.188) using the random walk bridges constructed from H^{RW}. Denote such random walk bridges by $S_{L,z}(\cdot)$, where $L\in\mathbb{N}$ is number of steps for the discrete bridges and $z\in\mathbb{R}$ is the location of the endpoint. In the proof of Theorem 3.1.13 (2), we rely on the KMT coupling Assumption A4 to ensure that the random walk bridges almost surely converge uniformly to a Brownian bridge as in (4.189). Here we need to show that such uniform converge to random walk bridge S_{L,z_∞} still holds for random walk bridges S_{L,z_M} as the location of the endpoint z_M converges to z_∞. This will be proved in Lemma 4.10.8 below, hence we can finish the proof by following the arguments for Theorem 3.1.13 (2). ☐

Lemma 4.10.8. *Fix $L\in\mathbb{N}$. Let $S_{L,z}(u), u\in[0,L]_{\mathbb{Z}}, z\in\mathbb{R}$ be the random walk bridges constructed from a smooth random walk Hamiltonian H^{RW}. Then we can couple $(S_{L,z})_{z\in\mathbb{R}}$ in the same probability space*

$(\Omega_L, \mathcal{B}_L, \mathbb{P}_L)$ *such that the following statement holds. For* ***all*** $\omega \in \Omega_L$ *and sequence* z_j *converging to* z_∞*, we have*

$$\limsup_{j\to\infty} \sup_{u\in[0,L]_{\mathbb{Z}}} |(S_{L,z_j}(\omega))(u) - (S_{L,z_\infty}(\omega))(u)| = 0. \tag{4.224}$$

Proof. We use induction argument in L. The lemma clearly holds for $L = 1$.

For $L \geq 2$, assume $L = 2m$ is even for simplicity. By the induction hypothesis, we can pick two such couplings $(S^1_{m,z}(\cdot))_{z\in\mathbb{R}}$ and $(S^2_{m,z}(\cdot))_{z\in\mathbb{R}}$ independently. For all $z \in \mathbb{R}$, we denote by $W_z = S_{L,z}(L/2)$ the middle point of the random walk bridge. We may couple all $(W_z)_{z\in\mathbb{R}}$ in the same probability space $(\Omega_{\text{mid}}, \mathcal{B}_{\text{mid}}, \mathbb{P}_{\text{mid}})$ through quantile coupling. As the law of W_z varies smoothly in z, for **all** $\omega \in \Omega_{\text{mid}}$ and $z_j \to z_\infty$, $W_{z_j}(\omega) \to W_{z_\infty}(\omega)$.

Now we couple $(W_z)_{z\in\mathbb{R}}$ with $(S^1_{m,z}(\cdot))_{z\in\mathbb{R}}$ and $(S^2_{m,z}(\cdot))_{z\in\mathbb{R}}$ independently. Define a L-step random walk bridges $S_{L,z}(u)$ for all $z \in \mathbb{R}$ by,

$$S_{L,z}(u) := \begin{cases} S^1_{m,W_z}(u) & 0 \leq u \leq m, \\ W_z + S^2_{m,z-W_z}(u-m) & m < u \leq 2m. \end{cases}$$

It can be checked directly that $S_{L,z}(\cdot)$ has the law of a random walk bridge with the location of the endpoint being z. Given a sequence $z_j \to z_\infty$ and an element ω in the probability space, we denote $W_j = W_{z_j}(\omega)$ and $W_\infty = W_{z_\infty}(\omega)$. Then

$$\begin{aligned}
&\limsup_{j\to\infty} \sup_{u\in[0,L]_{\mathbb{Z}}} |(S_{L,z_j}(\omega))(u) - (S_{L,z_\infty}(\omega))(u)| \\
\leq &\limsup_{j\to\infty} \sup_{u\in[0,m]_{\mathbb{Z}}} |(S^1_{m,W_j}(\omega))(u) - (S^1_{m,W_\infty}(\omega))(u)| + \limsup_{j\to\infty} |W_j - W_\infty| \\
&+ \limsup_{j\to\infty} \sup_{u\in[0,m]_{\mathbb{Z}}} |(S^2_{m,z_j-W_j}(\omega))(u) - (S^2_{m,z_\infty-W_\infty}(\omega))(u)| \\
= &0.
\end{aligned}$$

Here we used the induction hypothesis and $W_j \to W_\infty$. Therefore the proof is completed. □

4.10.2 Scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ and application of Theorem 3.1.13

Starting from now we set the parameter $\gamma = \gamma_N = \sqrt{N}$. We rewrite the log-gamma line ensemble as $\mathcal{L}^N_K = \{\mathcal{L}^N_{K,1}, \ \ldots \mathcal{L}^N_{K,K}\}$ to indicate the dependence on N.

The following result is a consequence of [AKQ, Theorem 2.7] with slight modifications of their arguments, which we explain in the proof below.

Proposition 4.10.9. *Consider the following weak noise scaling for lowest indexed curve of log-gamma line ensemble, $\mathcal{L}^N_{K,1}$, and denote*

$$\widetilde{\mathcal{L}}^N_1(t,u) := \mathcal{L}^N_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right) - (Nt+\sqrt{N}u)\left(\log 2 - \log(\sqrt{N}-1)\right) + \log\sqrt{N} - \log 2.$$

Then $\widetilde{\mathcal{L}}^N_1(t,u)$ converges weakly to $\log \mathcal{Z}_{\sqrt{2}}(t,u)$ in the topology of uniform convergence on compact sets of $\{t\in\mathbb{R}_+,\ u\in\mathbb{R}\}$.

Here $\mathcal{Z}_{\sqrt{2}}(t,u)$ is defined by the following chaos expansion with convention $t_0=0, u_0=0$,

$$\mathcal{Z}_{\sqrt{2}}(t,u) := \varrho(t,u) + \sum_{k=1}^{\infty}(\sqrt{2})^k \int_{\Delta_k(0,t]}\int_{\mathbb{R}^k}\prod_{i=1}^{k} W(t_i,u_i)\varrho(t_i-t_{i-1},u_i-u_{i-1})\varrho(t-t_k,u-u_k)du_i dt_i. \tag{4.225}$$

In the above expression, $W(t,u)$ is a white noise on $\mathbb{R}_+\times\mathbb{R}$ with covariance structure $\mathbb{E}[W(t,x)W(s,y)] = \delta(t-s)\delta(x-y)$. $\varrho(t,u)$ is the standard Gaussian heat kernel such that

$$\varrho(t,u) = \frac{e^{-u^2/2t}}{\sqrt{2\pi t}}.$$

And the integral is over a k-dimensional simplex $\Delta_k(0,t] = \{0=t_0<t_1<t_2<\cdots<t_k\le t\}$.

Proof. We prove the above result by explaining how the convergence follows from the arguments for [AKQ, Theorem 2.7]. Note that [AKQ, Theorem 2.7] shows that, under weak noise scaling (called intermediate disorder regime in [AKQ]), a modified point-to-point partition function of directed polymer model in dimension 1+1, $\frac{\sqrt{N}}{2}\mathfrak{Z}^{\omega_N}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4})$, converges to the chaos series $\mathcal{Z}_{\sqrt{2}}(t,x)$ in (4.225), which is the solution to stochastic heat equation with multiplicative noise. The random environment ω considered therein is of mean zero, variance one and has sixth moments. We record their convergence result below,

$$\frac{\sqrt{N}}{2}\mathfrak{Z}^{\omega}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4}) \Rightarrow \mathcal{Z}_{\sqrt{2}}(t,u).$$

If we choose ω_N to be an i.i.d. random environment with each weight $\omega_N(i,j)$ distributed as follows,

$$\omega_N(i,j) \overset{(d)}{=} N^{1/4}\left(\frac{\text{Inv-Gamma}(\sqrt{N})}{\mathbb{E}\left[\text{Inv-Gamma}(\sqrt{N})\right]} - 1\right). \tag{4.226}$$

By the definition of partition function $\mathfrak{Z}^{\omega_N}$ in [AKQ] and τ in (4.217), we have the following identification that

$$\begin{aligned}&2^{-(Nt+\sqrt{N}u)}\mathbb{E}[\text{Inv-Gamma}(\sqrt{N})]^{-(Nt+\sqrt{N}u)}\tau_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right)\\ =&\mathfrak{Z}^{\omega_N}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4}).\end{aligned}$$

From (4.226), it can be deduced using (4.215) that ω_N has mean zero and variance $1 + O(N^{-1/2})$. Moreover, the sixth moment of ω_N is $15 + O(N^{-1/2})$. Under such conditions, one can run the same arguments in [AKQ] to obtain the convergence of $\frac{\sqrt{N}}{2}\mathfrak{z}^{\omega_N}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4})$, hence the convergence of $\tau_{Nt/2,1}$.

Moreover note that by the definition of $\mathcal{L}^N$, we have

$$\mathcal{L}^N_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right) = \log \tau_{Nt/2,1}\left(\frac{1}{2}Nt+\sqrt{N}u\right).$$

As $\mathbb{E}\left[\text{Inv-Gamma}(\sqrt{N})\right] = \frac{1}{\sqrt{N}-1}$,

$$\widetilde{\mathcal{L}}^N_1(t,u) = \log\left(\frac{\sqrt{N}}{2}\mathfrak{z}^{\omega_N}(Nt+\sqrt{N}u, \sqrt{N}u; N^{-1/4})\right),$$

which converges to $\log \mathcal{Z}_{\sqrt{2}}(t,u)$. □

In the following we perform one more rescaling in order to get rid of the $\sqrt{2}$.

Definition 4.10.10. *We define scaled log-gamma line ensemble as follows.*

$$\overline{\mathcal{L}}^N_i(t,u) := \mathcal{L}^N_{Nt/8,i}\left(\frac{1}{8}Nt+\frac{1}{2}\sqrt{N}u\right) - \left(\frac{1}{4}Nt+\frac{1}{2}\sqrt{N}u\right)\left(\log 2 - \log(\sqrt{N}-1)\right) + \log\sqrt{N} - \log 2.$$

We remark that $\overline{\mathcal{L}}^N_i(t,u)$ is defined for $t \in \frac{8}{N}\mathbb{N}$, $u \in \frac{2}{\sqrt{N}}\mathbb{Z}$ and $i \le (Nt/8) \wedge (Nt/8+\sqrt{N}u/2)$ and that $\overline{\mathcal{L}}^N_1(t,u) = \widetilde{\mathcal{L}}^N_1(t/4,u/2)$.

Now we are ready to state the main application of Theorem 3.1.13 as follows. We fix $t = 1$ for notation simplicity and the same result holds for any $t > 0$ by the same argument modulo the modification that $\overline{\mathcal{L}}^N_1(t,u) + \frac{u^2}{2t}$ converges to a stationary process.

Theorem 4.10.11. *Fix $t = 1$. Given $k \in \mathbb{N}$, $T > 0$, the restriction of the line ensemble $\overline{\mathcal{L}}^N$ given by $\{\overline{\mathcal{L}}^N_j(1,u) : j \in \{1,\cdots,k\}, u \in [-T,T]\}$ is tight as N varies. Moreover, any subsequential limit line ensemble satisfies the H-Brownian Gibbs property with $\mathbf{H}(x) = e^x$.*

Proof. Recall the Gibbs property for $\mathcal{L}_K$ in Proposition 4.10.7. The random walk Gibbs property of $\overline{\mathcal{L}}^N_i(1,x)$ can be computed directly through change of variables. Fix $N \in \mathbb{N}$. To simplify notations, we denote $\overline{\mathcal{L}}_i(u) = \overline{\mathcal{L}}^N_i(1,u)$. Then $\overline{\mathcal{L}}_i(u)$ satisfies a $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-random walk Gibbs property over the region

$$\left\{u \in \frac{2}{\sqrt{N}}\mathbb{Z}, i \le (N/8) \wedge (N/8+\sqrt{N}u/2)\right\}$$

with local interaction Hamiltonian

$$\dot{\mathbf{H}}^N(\square(\overline{\mathcal{L}},k,u)) = \frac{2}{\sqrt{N}-1}\exp\left(\overline{\mathcal{L}}_{k+1}(u+2/\sqrt{N}) - \overline{\mathcal{L}}_k(u)\right), \tag{4.227}$$

and random walk Hamiltonian

$$\mathrm{H}^{\mathrm{RW},N}(x) = \log\Gamma(\sqrt{N}) + \sqrt{N}\Big(x - \log(\sqrt{N}-1) + \log 2\Big) + \exp\Big(-x + \log(\sqrt{N}-1) - \log 2\Big). \qquad (4.228)$$

Since for fixed i, $\overline{\mathcal{L}}_i^N(u)$ is defined on $\{u \geq -\frac{\sqrt{N}}{4} + 2i\}$, which diverges to $-\infty$. Hence for any choice of compact set $C = \{k_1, \cdots, k_2\} \times [a,b]$, the $\overline{\mathcal{L}}^N$ is well-defined for N large enough. This implies that for any compact set C, the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$-random walk Gibbs property holds on C when N is larges enough.

We have shown that the scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ enjoys the $(\dot{\mathbf{H}}^N, \mathrm{H}^{\mathrm{RW},N})$ Gibbs property with local interaction and random walk Hamiltonians given by (4.227) and (4.228) respectively. In light of Theorem 3.1.13, it suffices to verify assumptions A1-A4 and the convergence of the lowest indexed curve $\overline{\mathcal{L}}_1^N$.

Through the scaling invariance of the white noise, $W(at, bu) \overset{(d)}{=} (ab)^{-1/2}W(t,u)$, it follows that for any $\lambda, \beta > 0$, $\mathcal{Z}_\beta(\lambda^2 t, \lambda u) \overset{(d)}{=} \lambda^{-1}\mathcal{Z}_{\lambda^{1/2}\beta}(t,u)$. Together with Proposition 4.10.9, $\overline{\mathcal{L}}_1^N(t,u)$ converges weakly to $\log\mathcal{Z}_1(t,u) + \log 2$. In [AKQ, Proposition 2.3], it is also shown that for fixed $t > 0$, $\mathcal{Z}_\beta(t,u)/\varrho(t,u)$ is a stationary process in u. Therefore, $\overline{\mathcal{L}}_1^N(1,u) + \frac{u^2}{2}$ converges weakly to a stationary process. In the following we proceed to verify Assumptions A1-A4.

Assumptions A1(1) is explicit about $\dot{\mathbf{H}}^N$ in (4.227) and can be verified directly. Since $\dot{\mathbf{H}}^N$ depends only on the second and the sixth entry, to check Assumption A1(2), it's sufficient to consider $\vec{a} = (a_2, a_6)$, $\vec{b} = (b_2, b_6) \in (\mathbb{R} \cup \{\pm\infty\})^2$ with $a_2 \geq b_2$, $a_6 \geq b_6$ and $a_2 = b_2$ or $a_6 = b_6$. For the case $a_2 = b_2$, let $\delta >$ be a fixed number and

$$\vec{a}' = (a_2 + \delta, a_6),\ \vec{b}' = (b_2 + \delta, b_6),$$

as in Assumption A1 (2). Then

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) = \frac{2(e^{a_6 - a_2} - e^{a_6 - a_2 - \delta})}{\sqrt{N} - 1},$$
$$-\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}) = \frac{2(e^{b_6 - b_2} - e^{b_6 - b_2 - \delta})}{\sqrt{N} - 1}.$$

From the convexity of e^x, $a_6 \geq b_6$, $a_2 = b_2$ and $\delta > 0$, we have

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) \geq -\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}).$$

For the case $a_6 = b_6$, let $\delta > 0$ be a fixed number and

$$\vec{a}' = (a_2, a_6 + \delta),\ \vec{b}' = (b_2, b_6 + \delta),$$

as in Assumption A1 (2). Then

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) = \frac{2(e^{a_6 - a_2} - e^{a_6 - a_2 + \delta})}{\sqrt{N} - 1},$$
$$-\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}) = \frac{2(e^{b_6 - b_2} - e^{b_6 - b_2 + \delta})}{\sqrt{N} - 1}.$$

From the convexity of e^x, $a_6 = b_6$, $a_2 \geq b_2$ and $\delta > 0$, we have

$$-\dot{\mathbf{H}}^N(\vec{a}') + \dot{\mathbf{H}}^N(\vec{a}) \geq -\dot{\mathbf{H}}^N(\vec{b}') + \dot{\mathbf{H}}^N(\vec{b}).$$

This finishes the verification of Assumption A1. Assumptions A2 can be checked directly from the form of $\mathrm{H}^{\mathrm{RW},N}$ (4.228).

Assumption A3 could can be verified through the definition of Riemann integral. We note that the mesh size for rescaled log-gamma line ensemble is $2/\sqrt{N}$ instead of $1/N$ as in Assumption A3. We could relabel N to solve is discrepancy. However, as long as the mesh size goes to zero and N goes to infinity it has no essential effect on the rest of the proof, we don't do such relabelling. Understanding the mesh size is $2/\sqrt{N}$, we calculate for any $0 \leq a < b \in \mathbb{R}_+$, and any continuous line ensemble $\mathcal{L}$

$$\begin{aligned}
\sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L},k,u)) = & \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \frac{2}{\sqrt{N}-1} \exp\Big(\mathcal{L}_{k+1}(u+2/\sqrt{N}) - \mathcal{L}_k(u)\Big) \\
\leq & \frac{\sqrt{N}}{\sqrt{N}-1} e^{\omega_{(a,b)}(\mathcal{L}_{k+1},2/\sqrt{N})} \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \frac{2}{\sqrt{N}} \exp\left(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)\right) \\
\leq & \frac{\sqrt{N}}{\sqrt{N}-1} e^{2\omega_{(a,b)}(\mathcal{L}_{k+1},2/\sqrt{N}) + \omega_{(a,b)}(\mathcal{L}_k,2/\sqrt{N})} \int_a^b \exp(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)) du.
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L},k,u)) \\
\leq & \exp\left(2\omega_{(a,b)}(\mathcal{L}_{k+1},2/\sqrt{N}) + \omega_{(a,b)}(\mathcal{L}_k,2/\sqrt{N}) - \log(1-1/\sqrt{N})\right) \\
& \times \int_a^b \exp(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)) du.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \sum_{u \in \Lambda_d^{\sqrt{N}/2}(a,b)} \dot{\mathbf{H}}^N(\boxdot(\mathcal{L},k,u)) \\
\geq & \exp\left(-2\omega_{(a,b)}(\mathcal{L}_{k+1},2/\sqrt{N}) - \omega_{(a,b)}(\mathcal{L}_k,2/\sqrt{N})\right) \int_a^b \exp(\mathcal{L}_{k+1}(u) - \mathcal{L}_k(u)) du.
\end{aligned}$$

These two yield Assumption A3.

Now we turn to Assumption A4. For any $\gamma > 0$, we denote by

$$Y(\gamma) = -\log\big(\text{Inv-Gamma}(\gamma)\big)$$

the log-gamma random variable with parameter γ. It is easy to verify that the function $\exp(-\mathrm{H}^{\mathrm{RW},N}(x))$ is the density function of the random variable $-Y(\sqrt{N}) + \log(\sqrt{N}-1) - \log 2$. For any $N \geq 2$ let X_j^N be i.i.d. with $X_1^N \overset{(d)}{=} -Y(\sqrt{N}) + \log(\sqrt{N}-1) - \log 2$. For any $L > 0$ with $\frac{\sqrt{N}}{2}L \in \mathbb{N}$, we define random walk bridges $\bar{S}^N_{L,z}$ conditioned on arriving at z in $\sqrt{N}L/2$ steps as in (3.11):

$$\bar{S}^N_{L,z}\left(\frac{2}{\sqrt{N}}k\right) = X_1^N + X_2^N + \dots X_k^N \Big| X_1^N + X_2^N + \dots X^N_{\sqrt{N}L/2} = z, \tag{4.229}$$

And we still extend $\bar{S}^N_{L,z}(u)$ to $u \in [0, L]$ through linear interpolation. Now it suffices to verify that the random walk bridge defined in (4.229) satisfies the estimate in Assumption A4.

To this end, we rely on [DW, Corollary 8.1], which provides the desired estimates for normalized random variables. We start by recording the result of [DW, Corollary 8.1]. Let $m(\gamma)$ and $\sigma(\gamma)^2$ be the mean and variance of $Y(\gamma)$ respectively. For any $n \geq 1$, $0 \leq k \leq n$ and $z \in \mathbb{R}$, we define

$$S_{n,z}(k;\gamma) := X_1(\gamma) + X_2(\gamma) + \dots X_k(\gamma) \Big| X_1(\gamma) + X_2(\gamma) + \dots X_n(\gamma) = z, \tag{4.230}$$

where $X_j(\gamma)$ are i.i.d. with $X_1(\gamma) \overset{(d)}{=} (Y(\gamma) - m(\gamma))/\sigma(\gamma)$. For general $u \in [0, n]$, we define $S_{n,z}(u;\gamma)$ through linear interpolation. The following is a direct consequence of [DW, Corollary 8.1].

Corollary 4.10.12. *For any $b > 0$ and $\gamma_0 > 0$, there exists constants $0 < C, a, \alpha' < \infty$ such that for every positive integer n and $\gamma \geq \gamma_0$, there is a probability space on which are defined a Brownian bridge $B_1(\cdot)$ and a family of processes $\{S_{n,z}(\cdot;\gamma)\}_{z\in\mathbb{R}}$ such that for any $r \geq 0$,*

$$\mathbb{P}\left(\sup_{0\leq u\leq n} \left|\sqrt{n}B_1(u/n) + \frac{u}{n}\cdot z - S_{n,z}(u;\gamma)\right| \geq r\log n\right) \leq Cn^{\alpha' - ar}e^{bz^2/n}. \tag{4.231}$$

Denote $q(N) := \log(\sqrt{N}-1) - \log 2 - m(\sqrt{N})$ and we will use m, σ, q as shorthand for $m(\sqrt{N}), \sigma(\sqrt{N}), q(N)$ to simply notations. Now we have $X_1^N \overset{(d)}{=} -\sigma X_1(\sqrt{N}) + q$, which implies

$$\bar{S}^N_{L,z}(u) \overset{(d)}{=} -\sigma S_{\frac{\sqrt{N}}{2}L, -\sigma^{-1}(z - \frac{\sqrt{N}}{2}Lq)}\left(\frac{\sqrt{N}}{2}u; \sqrt{N}\right) + \frac{\sqrt{N}}{2}uq. \tag{4.232}$$

However, Corollary 4.10.12 fails to directly apply here since the endpoint $-\sigma^{-1}(z - \frac{\sqrt{N}}{2}Lq)$ blows up as N goes to infinity. To overcome this issue, we apply a tilting trick and identify the random walk bridge $\bar{S}^N_{L,z}$ with another bridge for which Corollary 4.10.12 applies.

For two random variables X and X' with density functions f_X and $f_{X'}$ separately, we say X and X' are related through tilting if there exist $t \in \mathbb{R}$ and a positive constant C such that

$$f_X(x) = Ce^{tx}f_{X'}(x).$$

We need the following result.

Lemma 4.10.13. *Suppose X and X' are related through tilting. Then the random walk bridges, constructed by the distribution of X and X' separately, have the same distribution.*

Proof. This lemma could be proved by directly comparing the densities of the two bridges. ☐

Choose ξ and μ (depending on N) such that

$$\begin{aligned} m(\xi) =& m(\sqrt{N}) + q(N), \\ \mu =& \sigma(\xi)/\sigma(\sqrt{N}). \end{aligned}$$

Through direct verification, $X_1(\sqrt{N}) - q/\sigma$ and $\mu X_1(\xi)$ are related through tilting. Hence X_1^N and $-\sigma\mu X_1(\xi)$ are related through tilting. By Lemma 4.10.13,

$$\bar{S}_{L,z}^N(u) \overset{(d)}{=} -\sigma\mu S_{\sqrt{N}L/2,-\sigma^{-1}\mu^{-1}z}\left(\frac{\sqrt{N}u}{2};\xi\right).$$

Let $n = \frac{\sqrt{N}}{2}L$ and $u' = \frac{\sqrt{N}}{2}u$. By the scale invariance for Brownian bridges $\sqrt{L}B_1(u/L) \overset{(d)}{=} B_L(u)$,

$$\mu\sigma\left(\sqrt{n}B_1(u'/n) + \frac{u'}{n}\cdot\frac{z}{\mu\sigma} + S_{n,-\mu^{-1}\sigma^{-1}z}(u';\xi)\right) \overset{(d)}{=} 2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}_{L,z}^N(u).$$

In order to apply Lemma 4.10.12, we compute the asymptotics for ξ and μ. First by [DW, (8.10)], as N goes to infinity, for m, σ and q we have

$$\begin{aligned} m(\sqrt{N}) =& \log\sqrt{N} + O(N^{-1/2}), \\ \sigma(\sqrt{N}) =& N^{-1/4} + O(N^{-3/4}), \\ q(N) =& -\log 2 + O(N^{-1/2}). \end{aligned}$$

Hence

$$\begin{aligned} \xi =& \frac{\sqrt{N}}{2} + O(1), \\ \mu =& 2^{1/2} + O(N^{-1/2}). \end{aligned}$$

Since ξ goes to infinity, (4.231) applies and we obtain the following estimate. For any $b > 0$, there exist $0 < C, a, \alpha' < \infty$ such that for every $N \geq 2$ and $L > 0$

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leq u\leq L}\left|2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}_{L,z}^N(u)\right| \geq r\cdot\mu\sigma\log(\sqrt{N}L/2)\right) \\ \leq& C(\sqrt{N}L/2)^{\alpha'-ar}\exp\left(\frac{2bz^2}{\mu^2\sigma^2N^{1/2}L}\right). \end{aligned} \tag{4.233}$$

We are ready to verify Assumption A4. Fix $b_1, b_2 > 0$. From the asymptotic of σ and μ, we can choose $b > 0$ such that

$$\sup_{N\geq 2}\frac{2b}{\mu^2\sigma^2N^{1/2}} \leq b_2.$$

Let C, a and α' be determined through Corollary 4.10.12 with $\gamma_0 = \inf_N \xi$. Take $r > 0$ such that $\alpha' - ar = -b_1$ and then take a_1 such that

$$a_1 \geq 2^{3/2} r \sup_N \mu(N)\sigma(\sqrt{N})N^{-1/4}.$$

Take $a_2 = \{8, 2C\}$. Then by rewriting (4.233)

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&\frac{a_2}{2}(\sqrt{N}L/2)^{-b_1}e^{b_2 z^2/L}.\end{aligned}$$

The last step is to change to coefficient of $B_L(t)$ to 1. From the asymptotic of μ and σ, there exists a constant C_0 such that for all N,

$$|2^{-1/2}\mu\sigma N^{1/4} - 1| \leq C_0 N^{-1/2}.$$

We compute that

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|C_0 N^{-1/2}B_L(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ =&\mathbb{P}\left(\sup_{0\leq u\leq 1}|B_1(u)| \geq 2^{-1/2}a_1 C_0^{-1}N^{1/4}L^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&4\exp\left(-\frac{1}{2}\cdot\frac{1}{2}a_1^2 C_0^{-2}N^{1/2}L^{-1}(\log(\sqrt{N}L/2)^2)\right)\\ \leq&4(\sqrt{N}L/2)^{-\frac{1}{4}\cdot a_1^2 C_0^{-2}N^{1/2}L^{-1}}.\end{aligned}$$

By taking N large enough depending on L and b_1, we can arrange

$$4(\sqrt{N}L/2)^{-\frac{1}{4}\cdot a_1^2 C_0^{-2}N^{1/2}L^{-1}} \leq \frac{a_2}{2}(\sqrt{N}L/2)^{-b_1}.$$

Thus

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq a_1(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&\mathbb{P}\left(\sup_{0\leq u\leq L}\left|2^{-1/2}\mu\sigma N^{1/4}B_L(u) + \frac{u}{L}\cdot z - \bar{S}^N_{L,z}(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ &+\mathbb{P}\left(\sup_{0\leq u\leq L}\left|C_0 N^{-1/2}B_L(u)\right| \geq \frac{a_1}{2}(\sqrt{N}/2)^{-1/2}\log(\sqrt{N}L/2)\right)\\ \leq&a_2(\sqrt{N}L/2)^{-b_1}e^{b_2 z^2/L}.\end{aligned}$$

This finishes the verification of Assumption A4 and the proof of Theorem 4.10.11 is completed. □

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Appendix

In this chapter we prove the monotone coupling lemma, Lemma 3.1.11 by adapting the approach of [CH16] which generalized the argument in [CH14] from non-intersecting Brownian bridges to H-Brownian line ensembles where H is a convex Hamiltonian function. In [CH16, CH14], they first construct monotone couplings along a sequence of Markov chains with only finite state spaces and the invariant measures converge to the bridge line ensembles (either non-intersecting or H-Brownian) and hence demonstrate the monotone coupling in the limit. We take the same approach and recall that in our case H^{RW} (see (3.3)) is the Hamiltonian corresponding to an underlying random walk which is continuous in space. We will approximate this continuous-in-space underlying random walk by a sequence of finite-state discrete random walks, from which it suffices to construct the monotone coupling along this sequence and the two convexity assumptions A1 and A2 guarantee that path measure stay coupled in order.

We start by proving monotone coupling for finite-state discrete random walks. Let X be a random varibable, by abuse of notation, we say a line ensemble enjoys (H, X)-Gibbs property if the free random walk bridge measures (Definition 3.1.4) are constructed using the law of X. We also replace H^{RW} by X when the the law of X plays the role of the law defined through H^{RW}.

Lemma .0.1. *Let $\delta, M > 0$ be any positive numbers and suppose that the underlying random walk X takes value in $\delta\mathbb{Z}\cap[-M, M]$. To modify convexity Assumption A2 for such random walk, we assume $-\log \mathbb{P}(X = k\delta)$ is a convex function in $k \in \mathbb{Z}$ with the convention $-\log 0 = \infty$. Further assume that the Hamiltonian $\dot{\mathbf{H}}$ satisfies Assumption A1. Then the result in Lemma 3.1.11 holds for $(\dot{\mathbf{H}}, X)$-Gibb line ensembles.*

Precisely, fix $k_1 \le k_2$ and $a < b \in \Lambda_d$. For $i = 1, 2$ define pairs of vectors $(\vec{x}^i, \vec{y}^i) \in (\delta\mathbb{Z})^{k_2-k_1+1}$, pairs of measurable functions $(f^i, g^i) : \Lambda_d[a, b] \to \mathbb{R} \cup \{\pm\infty\}$. We define $\mathbb{P}^i = \mathbb{P}_{H,X}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^i,\vec{y}^i,f^i,g^i}$ for $i = 1, 2$, which denotes the probability measure for the (H, X)-random walk bridge ensemble with X playing the role of defining the underlying random walk as the random walk Hamiltonian as H^{RW}.

For $i \in \{1, 2\}$, let $\mathcal{Q}^i = \{\mathcal{Q}^i_j\}_{j=k_1}^{k_2}$ be a $[k_1, k_2]_{\mathbb{Z}} \times \Lambda_d[a, b]$-indexed line ensemble on a probability space $(\Omega^i, \mathcal{B}^i, \mathbb{P}^i)$. Assume that the $i = 1$ vectors and functions are pointwise greater than or equal to their $i = 2$ counterparts (e.g. $f^1(s) \ge f^2(s)$ for all $s \in \Lambda_d[a, b]$). Then there exists a coupling of the probability measure $\mathbb{P}^1$ and $\mathbb{P}^2$ such that almost surely $\mathcal{Q}^1_j(s) \ge \mathcal{Q}^2_j(s)$ for all $j \in [k_1, k_2]_{\mathbb{Z}}$ and $s \in \Lambda_d[a, b]$.

Proof. By abuse of notation, we denote

$$\mathrm{H}^{\mathrm{RW}}(k\delta) = -\log \mathbb{P}(X = k\delta), \tag{1}$$

with the convention $-\log 0 = \infty$. Equivalently,

$$\mathbb{P}(X = k\delta) = \exp(-\mathrm{H}^{\mathrm{RW}}(k\delta)).$$

We will construct the monotone coupling associated to the random walk with same distribution as X. It is enough to demonstrate that if $\vec{x}^1 \ge \vec{x}^2, \vec{y}^1 \ge \vec{y}^2$ and $(f^1, g^1) \ge (f^2, g^2)$, then there is a probability space $(\Omega, \mathbb{P})$ on which both $\mathcal{Q}^1 = (\mathcal{Q}^1_j)_{j=k_1}^{k_2}$ and $\mathcal{Q}^2 = (\mathcal{Q}^2_j)_{j=k_1}^{k_2}$ are defined and enjoy the respective marginals $\mathbb{P}^1$ and $\mathbb{P}^2$ as defined above.

In order to construct such coupling, we will use a Markov chain argument. Let us first introduce the dynamics of the Markov chain on the random walk bridges $\mathcal{Q}^1$ and $\mathcal{Q}^2$. At time $t = 0$, the initial configuration of $(\mathcal{Q}^1)_0, (\mathcal{Q}^2)_0$ are chosen to be the lowest as possible trajectory of random walk bridges having the given endpoints and satisfying the boundary conditions. It is obvious that $(\mathcal{Q}^1_j(u))_0 \geq (\mathcal{Q}^2_j(u))_0$ for all $s \in \Lambda_d[a,b]$. The dynamics of the Markov chain are follows: for each $u_m \in \Lambda_d[a,b]$ and each $j \in [k_1,k_2]_{\mathbb{Z}}$ and each $l \in \{+1,-1\}$, there are an independent exponential clock with rate one and an independent uniform random variables on $[0,1]$, $U^{j,u_m,l}$.

When the clock labeled (u_m, j, l) rings, one tries to update $\mathcal{Q}^i_j(u_m)$ to $\widetilde{\mathcal{Q}}^i_j(u_m) = \mathcal{Q}^i_j(u_m) + l\delta$. This proposed update is accepted according to the rule $R_i \geq U^{j,u_m,l}$, where

$$R_i = \frac{W_{\dot{\mathbf{H}}}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^i,\vec{y}^i}(\widetilde{\mathcal{Q}}^i)\mathbb{P}_{free,X}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^i,\vec{y}^i}(\widetilde{\mathcal{Q}}^i)}{W_{\dot{\mathbf{H}}}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^i,\vec{y}^i}(\mathcal{Q}^i)\mathbb{P}_{free,X}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^i,\vec{y}^i}(\mathcal{Q}^i)}.$$

The two discrete line ensembles $\mathcal{Q}^i$, with $i = 1, 2$, are now coupled through the same collection of clocks and uniform random variables $U^{j,u_m,l}$. We will prove that this dynamics preserves the ordering between two line ensembles, i.e. for all time t, $(\mathcal{Q}^1_j(s))_t \geq (\mathcal{Q}^2_j(s))_t$. Since at each update step, we could only change $\mathcal{Q}^i_j(u_m)$ to $\widetilde{\mathcal{Q}}^i_j(u_m) = \mathcal{Q}^i_j(u_m) + \delta$ or $\widetilde{\mathcal{Q}}^i_j(u_m) = \mathcal{Q}^i_j(u_m) - \delta$, hence there are only two cases that the ordering could probably be violated. The first case is when the clock $(u_m, j, +)$ rings and $\mathcal{Q}^1_j(u_m) = \mathcal{Q}^2_j(u_m) = z$, then the ordering will not be hold if $R_1 \leq U^{j,u_m,l} \leq R_2$ and $R_1 < R_2$. We will prove now that when $(u_m, j, +)$ rings we always have $R_1 \geq R_2$, hence the monotonicity is preserved. From the definition of R_i, one obtains (we omit the index k_1, k_2, a, b for notation simplicity)

$$R_1 = R_{1,\mathrm{RW}} \cdot R_{1,\dot{\mathbf{H}}},$$

where

$$R_{1,\mathrm{RW}} = \frac{\exp\Big(-\mathrm{H}^{\mathrm{RW}}\big(z+\delta-\mathcal{Q}^1_j(u_{m-1})\big) - \mathrm{H}^{\mathrm{RW}}\big(\mathcal{Q}^1_j(u_{m+1}) - z - \delta\big)\Big)}{\exp\Big(-\mathrm{H}^{\mathrm{RW}}\big(z-\mathcal{Q}^1_j(u_{m-1})\big) - \mathrm{H}^{\mathrm{RW}}\big(\mathcal{Q}^1_j(u_{m+1}) - z\big)\Big)},$$

and

$$\begin{aligned} R_{1,\dot{\mathbf{H}}} =& \exp\Big(-\dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j, u_m)\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j, u_m)\big)\Big) \\ &\times \exp\Big(-\dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j, u_{m-1})\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j, u_{m-1})\big)\Big) \\ &\times \exp\Big(-\dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j, u_{m+1})\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j, u_{m+1})\big)\Big) \\ &\times \exp\Big(-\dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j-1, u_m)\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j-1, u_m)\big)\Big) \\ &\times \exp\Big(-\dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j-1, u_{m-1})\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j-1, u_{m-1})\big)\Big) \\ &\times \exp\Big(-\dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j-1, u_{m+1})\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j-1, u_{m+1})\big)\Big). \end{aligned}$$

$R_{2,\mathrm{RW}}$ and $R_{2,\dot{\mathbf{H}}}$ can be defined by replacing $\mathcal{Q}^1$ by $\mathcal{Q}^2$. Since H^{RW} is convex and $\mathcal{Q}^1 \geq \mathcal{Q}^2$, we have

$$\begin{aligned} &-\mathrm{H}^{\mathrm{RW}}(z+\delta-\mathcal{Q}^1_j(u_{m-1})) + \mathrm{H}^{\mathrm{RW}}(z-\mathcal{Q}^1_j(u_{m-1})) \\ \geq& -\mathrm{H}^{\mathrm{RW}}(z+\delta-\mathcal{Q}^2_j(u_{m-1})) + \mathrm{H}^{\mathrm{RW}}(z-\mathcal{Q}^2_j(u_{m-1})) \end{aligned}$$

and

$$\begin{aligned} &-\mathrm{H}^{\mathrm{RW}}(\mathcal{Q}^1_j(u_{m+1})-z-\delta) + \mathrm{H}^{\mathrm{RW}}(\mathcal{Q}^1_j(u_{m+1})-z) \\ \geq& -\mathrm{H}^{\mathrm{RW}}(\mathcal{Q}^2_j(u_{m+1})-z-\delta) + \mathrm{H}^{\mathrm{RW}}(\mathcal{Q}^2_j(u_{m+1})-z). \end{aligned}$$

This yields $R_{1,\mathrm{RW}} \geq R_{2,\mathrm{RW}}$. By Assumption A2 and $\mathcal{Q}^1 \geq \mathcal{Q}^2$, we have each individual term in $R_{1,\dot{\mathbf{H}}}$ is greater than or equal to the one in $R_{2,\dot{\mathbf{H}}}$. For instance, take

$$\vec{a} = \boxdot(\mathcal{Q}^1, j, u_m), \quad \vec{a'} = \boxdot(\widetilde{\mathcal{Q}}^1, j, u_m),$$

and

$$\vec{b} = \boxdot(\mathcal{Q}^2, j, u_m), \quad \vec{b'} = \boxdot(\widetilde{\mathcal{Q}}^2, j, u_m).$$

Then Assumption A1 (2) implies

$$\begin{aligned} &\exp\Big(- \dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^1, j, u_m)\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^1, j, u_m)\big)\Big) \\ \geq &\exp\Big(- \dot{\mathbf{H}}\big(\boxdot(\widetilde{\mathcal{Q}}^2, j, u_m)\big) + \dot{\mathbf{H}}\big(\boxdot(\mathcal{Q}^2, j, u_m)\big)\Big). \end{aligned}$$

The other five terms are dealt with similarly. This implies $R_1 \geq R_2$ and therefore the monotonicity is well preserved after this update.

For the other case when $(u_m, j, -)$ rings, the proof is similar. We have proved that for all $t > 0$, under our dynamics, we always have $(\mathcal{Q}^1_j(t_m))_t \geq (\mathcal{Q}^2_j(t_m))_t$.

Since the jumps of the random walk H^{RW} can only take finite size (cutoff by M), the Markov chain has finite states, moreover it is irreducible and aperiodic with an invariant measure given by the measure $\mathbb{P}^1$ and $\mathbb{P}^2$. Hence by taking t to infinity, we can obtain the desired coupling and hence complete the proof. □

Let's recall the notations in Lemma 3.1.11. Fix $k_1 \leq k_2$, $a < b \in \Lambda_d$. For $i = 1, 2$ define pairs of vectors $(\vec{x}^i, \vec{y}^i) \in \mathbb{R}^{k_2-k_1+1}$, pairs of measurable functions $(f^i, g^i) : \Lambda_d[a,b] \to \mathbb{R} \cup \{\pm\infty\}$. For $i \in \{1, 2\}$, let $\mathcal{Q}^i = \{\mathcal{Q}^i_j\}_{j=k_1}^{k_2}$ be a $[k_1,k_2]_{\mathbb{Z}} \times \Lambda_d[a,b]$-indexed line ensemble on a probability space $(\Omega^i, \mathcal{B}^i, \mathbb{P}^i)$ where $\mathbb{P}^i = \mathbb{P}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^i,\vec{y}^i,f^i,g^i}_{\dot{\mathbf{H}},\mathrm{H}^{\mathrm{RW}}}$. Fix $k_2 \geq k_1$ and vectors $\vec{x} \geq \vec{y}$ in $\mathbb{R}^{k_2-k_1+1}$.

To prove Lemma 3.1.11, we approximate the underlying random walk X by random variables X^ℓ as ℓ goes to infinity and argue X^ℓ satisfies the assumption in Lemma .0.1 as follows. For any $\ell \geq 1$, define discrete random variables X^ℓ by

$$\mathbb{P}(X^\ell = k\ell^{-1}) = \begin{cases} C_\ell^{-1} \exp(-\mathrm{H}^{\mathrm{RW}}(k\ell^{-1})) & k \in [-\ell^2, \ell^2]_{\mathbb{Z}}, \\ 0 & \text{others}, \end{cases}$$

where C_ℓ is the normalizing constant $C_\ell = \sum_{k=-\ell^2}^{\ell^2} \exp(-\mathrm{H}^{\mathrm{RW}}(k\ell^{-1}))$. Choose boundary vectors $\vec{x}^{i,\ell}, \vec{y}^{i,\ell} \in (\ell^{-1}\mathbb{Z})^{k_2-k_1+1}$ such that $\vec{x}^{i,\ell} \geq \vec{y}^{i,\ell}$ and $\vec{x}^{i,\ell} \to \vec{x}^i$, $\vec{y}^{i,\ell} \to \vec{y}^i$. Let

$$\mathbb{P}^{i,\ell} = \mathbb{P}^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i,\ell},\vec{y}^{i,\ell},f^i,g^i}_{\dot{\mathbf{H}},X^\ell}, \ i = 1, 2$$

and let $\mathcal{Q}^{i,\ell}_j(u)$ be the corresponding line ensembles.

Lemma .0.2. *Under the assumptions in Lemma 3.1.11, $\mathcal{Q}^{i,\ell}$ converges weakly to $\mathcal{Q}^i$ as ℓ goes to infinity.*

Proof. To simplify the notation, we assume $\Lambda_d[a,b] = [0,m]_{\mathbb{Z}}$ for some $m \in \mathbb{N}$ and denote $f(x) = \exp(-\mathrm{H}^{\mathrm{RW}}(x))$. The proof for the general case is the same except notation changes. We start by showing that $\ell^{-1}C_\ell$ converges to 1 as ℓ goes to infinity. Because H^{RW} is convex, there exists $M_0 > 0$ such that $f(x)$ is non-increasing for $x \geq M_0$ and is non-decreasing for $x \leq -M_0$.

$$\begin{aligned} \ell^{-1}C_\ell =& \frac{1}{\ell} \sum_{x \in [-\ell,\ell] \cap \ell^{-1}\mathbb{Z}} f(x) \\ =& \underbrace{\frac{1}{\ell} \sum_{x \in [-M,M] \cap \ell^{-1}\mathbb{Z}} f(x)}_{\mathrm{I}} + \underbrace{\frac{1}{\ell} \sum_{x \in ([-\ell,-M) \cup (M,\ell]) \cap \ell^{-1}\mathbb{Z}} f(x)}_{\mathrm{II}}. \end{aligned}$$

Since $f(x)$ is monotone for $|x| \geq M_0$, the second term II converges uniform in ℓ to zero as M goes to infinity. For any fixed $M > 0$, I converges to the integral of $f(x)$ in $[-M, M]$. As a result, $\ell^{-1}C_\ell$ converges to the total integral of $f(x)$ which is 1.

Next, let $S^\ell_m := X^\ell_1 + X^\ell_2 + \cdots + X^\ell_m$. Here $X^\ell_1, X^\ell_2, \ldots, X^\ell_m$ are i.i.d. random variables with the same law as X^ℓ. We claim that for any $z \in \mathbb{R}$ and any sequence $z^\ell \in \ell^{-1}\mathbb{Z}$ with $z^\ell \to z$,

$$\lim_{\ell\to\infty} \ell\mathbb{P}(S^\ell_m = z^\ell) = \prod_{j=1}^{m-1} \int_{x_j\in\mathbb{R}} dx_j \prod_{j=1}^{m} f(x_j - x_{j-1}), \tag{2}$$

where we used the convention $x_0 = 0$ and $x_m = z$. From the definition,

$$\left(\ell^{-1}C_\ell\right)^m \cdot \ell\mathbb{P}(S^\ell_m = z^\ell) = \frac{1}{\ell^{m-1}} \sum_{\substack{y_k\in[-\ell,\ell]\cap\ell^{-1}\mathbb{Z}\\ k=1,2,\ldots,m-1}} f(y_1)f(y_2)\ldots f(y_{m-1})f(z^\ell - \sum_{k=1}^{m-1} y_k).$$

For any $m' \in [1, m]_\mathbb{Z}$ and $M > 0$,

$$\begin{aligned}
&\frac{1}{\ell^{m-1}} \sum_{\substack{y_{m'}\in[-\ell,\ell]\cap\ell^{-1}\mathbb{Z}\\ |y_{m'}|>M}} \sum_{\substack{y_k\in[-\ell,\ell]\cap\ell^{-1}\mathbb{Z}\\ k=1,2,\ldots,m-1,k\neq m'}} f(y_1)f(y_2)\ldots f(y_{m-1})f(z^\ell - \sum_{k=1}^{m-1} y_k)\\
\leq &\max_{x\in\mathbb{R}} f(x) \cdot \frac{1}{\ell^{m-1}} \sum_{\substack{y_{m'}\in[-\ell,\ell]\cap\ell^{-1}\mathbb{Z}\\ |y_{m'}|>M}} \sum_{\substack{y_k\in[-\ell,\ell]\cap\ell^{-1}\mathbb{Z}\\ k=1,2,\ldots,m-1,k\neq m'}} f(y_1)f(y_2)\ldots f(y_{m-1})\\
= &\max_{x\in\mathbb{R}} f(x) \cdot (\ell^{-1}C_\ell)^{m-2} \left(\frac{1}{\ell} \sum_{y\in([-\ell,-M)\cup(M,\ell])\cap\ell^{-1}\mathbb{Z}} f(y) \right),
\end{aligned}$$

which converges uniformly in ℓ to zero as M goes to infinity. Therefore,

$$\ell\mathbb{P}(S^\ell_m = z^\ell) = \left(\ell^{-1}C_\ell\right)^{-m} \cdot \frac{1}{\ell^{m-1}} \sum_{\substack{y_k\in[-M,M]\cap\ell^{-1}\mathbb{Z}\\ k=1,2,\ldots,m-1}} f(y_1)f(y_2)\ldots f(y_{m-1})f(z^\ell - \sum_{k=1}^{m-1} y_k) + o(1),$$

and (2) follows. By a similar argument, we have for any $a_1, a_2, \ldots, a_{m-1} \in \mathbb{R}$,

$$\ell\mathbb{P}(S^\ell_1 \leq a_1, S^\ell_2 \leq a_2, \ldots, S^\ell_{m-1} \leq a_{m-1}, S^\ell_m = z^\ell)$$

converges to

$$\prod_{j=1}^{m-1} \int_{x_j\leq a_j} dx_j \prod_{j=1}^{m} f(x_j - x_{j-1}),$$

where we use the convention $x_0 = 0$ and $x_m = z$. As a result, the free bridges of X^ℓ converge to the free bridges of X.

Lastly, let $\mathbb{E}^{i,\ell}$ be the expectation with respect to $\mathbb{P}^{i,\ell}$ and $\mathbb{E}^i$ be the expectation with respect to $\mathbb{P}^i$. Also, let $\mathbb{E}^{i,\ell}_{free}$ be the expectation with respect to free random walk bridges of X^ℓ and $\mathbb{E}^i_{free}$ be the expectation with respect the free random walk bridges of X. Denote the normalizing constants

$$\begin{aligned}
Z^{i,\ell} =&\mathbb{E}^{i,\ell}_{free}[W^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i,\ell},\vec{y}^{i,\ell},f^i,g^i}_{\dot{\mathbf{H}}}],\\
Z^{i} =&\mathbb{E}^{i}_{free}[W^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i},\vec{y}^{i},f^i,g^i}_{\dot{\mathbf{H}}}].
\end{aligned}$$

Because the Boltzmann weight is a bounded continuous functional,

$$\lim_{\ell\to\infty} Z^{i,\ell} = Z^{i,\ell}.$$

By the same reason, for any other bounded continuous functional F,

$$\lim_{\ell\to\infty} \mathbb{E}^{i,\ell}_{free}[F \times W^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i,\ell},\vec{y}^{i,\ell},f^i,g^i}_{\dot{\mathbf{H}}}] = \mathbb{E}^{i}_{free}[F \times W^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i},\vec{y}^{i},f^i,g^i}_{\dot{\mathbf{H}}}].$$

This implies that

$$\mathbb{E}^{i,\ell}[F(\mathcal{Q}^{i,\ell})] = \frac{1}{Z^{i,\ell}} \mathbb{E}^{i,\ell}_{free}[F \times W^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i,\ell},\vec{y}^{i,\ell},f^i,g^i}_{\dot{\mathbf{H}}}]$$

converges to

$$\mathbb{E}^{i}[F(\mathcal{Q}^{i})] = \frac{1}{Z^{i}} \mathbb{E}^{i}_{free}[F \times W^{k_1,k_2,\Lambda_d[a,b],\vec{x}^{i},\vec{y}^{i},f^i,g^i}_{\dot{\mathbf{H}}}]$$

Thus $\mathcal{Q}^{i,\ell}$ converges weakly to $\mathcal{Q}^i$. □

Proof of Lemma 3.1.11. Since $\mathcal{Q}^{i,\ell}$ converges weakly to $\mathcal{Q}^i$, $\mathcal{Q}^{1,\ell}$ and $\mathcal{Q}^{2,\ell}$ are tight in ℓ separately. This implies that for all $\eta > 0$, there exists a compact set K_η such that

$$\mathbb{P}^{i,\ell}\left(\mathcal{Q}^{1,\ell} \in K_\eta\right) \geq 1 - \eta \text{ for } i = 1, 2.$$

Because of Assumption A2, X^ℓ satisfies the assumption in Lemma .0.1. Denote by $\mathbb{P}^\ell$ the coupling constructed in Lemma .0.1. Then

$$\begin{aligned}\mathbb{P}^\ell\left((\mathcal{Q}^1, \mathcal{Q}^2) \notin K_\eta \times K_\eta\right) \leq & \mathbb{P}^\ell\left(\mathcal{Q}^1 \notin K_\eta\right) + \mathbb{P}^\ell\left(\mathcal{Q}^2 \notin K_\eta\right) \\ & \mathbb{P}^{1,\ell}\left(\mathcal{Q}^1 \notin K_\eta\right) + \mathbb{P}^{2,\ell}\left(\mathcal{Q}^2 \notin K_\eta\right) \leq 2\eta.\end{aligned}$$

Hence $(\mathcal{Q}^{1,\ell}, \mathcal{Q}^{2,\ell})$ is also tight in ℓ. Denote by $(\mathcal{Q}^1, \mathcal{Q}^2)$ a subsequential limit of $(\mathcal{Q}^{1,\ell}, \mathcal{Q}^{2,\ell})$ as ℓ goes to infinity and denote by $\mathbb{P}$ their joint law. It remains to show that almost surely for all $j \in [k_1, k_2]_{\mathbb{Z}}$ and all $u \in [a, b]$, $\mathcal{Q}^1_j(u) \geq \mathcal{Q}^2_j(u)$.

For any $m \in \mathbb{N}$, define the function $\phi_m(s)$ as

$$\phi_m(s) = \begin{cases} 1 & s \geq \frac{1}{m}, \\ 2m\left(s - \frac{1}{2m}\right) & \frac{1}{2m} < s < \frac{1}{m}, \\ 0 & s < \frac{1}{m}. \end{cases}$$

Define $h_m : C[a, b] \times C[a, b] \to \mathbb{R}$ as

$$h_m(f^1, f^2) = \phi_m\left(\sup_{u\in[a,b]} (f^2(u) - f^1(u))\right).$$

h_m is a bounded functional defined on $C[a, b] \times C[a, b]$ which is continuous with respect to the sup norm. From the definition, $h_m(f^1, f^2) \neq 0$ only if $\sup_{u\in[a,b]}(f^2(u) - f_1(u)) > 0$. Fix $j \in [k_1, k_2]_{\mathbb{Z}}$. Because $\mathcal{Q}^{1,\ell}_j \geq \mathcal{Q}^{2,\ell}_j$ almost surely, we have

$$\mathbb{E}^\ell\left[h_m(\mathcal{Q}^{1,\ell}_j, \mathcal{Q}^{2,\ell}_j)\right] = 0.$$

Here $\mathbb{E}^\ell$ is the expectation with respect to $\mathbb{P}^\ell$. Let ℓ go to infinity, we deduce

$$\mathbb{E}\left[h_m(\mathcal{Q}^1_j, \mathcal{Q}^2_j)\right] = 0.$$

Here $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. As m goes to infinity, $h_m(f^1, f^2)$ converges pointwisely to the indicator function of the set

$$\{\sup_{u\in[a,b]}(f^2(u) - f^1(u)) > 0\}.$$

By the Dominant Convergence Theorem, we obtain

$$\mathbb{P}\left[\sup_{u\in[a,b]}(\mathcal{Q}_j^2(u) - \mathcal{Q}_j^1(u)) > 0\right] = 0.$$

Hence almost surely $\mathcal{Q}_j^1(u) \geq \mathcal{Q}_j^2(u)$ for all $u \in [a, b]$ and all $j \in [k_1, k_2]_{\mathbb{Z}}$. This finishes the proof of Lemma 3.1.11. □