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Abstract

In repeated games with public monitoring, the consideration of behavior strategies makes relevant the distinction between public and private strategies. Recently, Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7] have provided examples of games with equilibrium payoffs in private strategies which lie outside the set of Public Perfect Equilibrium payoffs. The present paper focuses on another distinction, that between mixed and behavior strategies. It is shown that, as far as with mixed strategies one is concerned, the restriction to public strategies is not a restriction at all. Our result provides a general explanation for the findings of Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7] as well as a general method for constructing examples of that sort.

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1 Introduction

Repeated Games with public monitoring are repeated games where, at the end of each stage, players do not observe each others' action, but only a public outcome, which is stochastically related to those. Abreu, Pearce and Stacchetti (henceforth, APS) [1] studied the set of pure-strategy (sequential) equilibrium payoff set of such games. Among other things, they showed that for any pure-strategy equilibrium payoff vector v, one can find a pure-strategy equilibrium profile σ which supports v and which is public in that strategies in σ depend on the publicly observed variable only. Hence, as far as with pure strategies one is concerned, the distinction between public and private strategies (strategies which, in addition, depend on the players' private information) does not matter.

The dynamic programming method of Abreu, Pearce and Stacchetti can be readily extended to study the equilibria associated to a special class of behavior strategies, namely those that depend on the publicly observed variable alone (for instance, Fudenberg, Levine and Maskin [4]). Such a set is typically referred to as the set of Public Perfect Equilibria (PPE). However, as soon as behavior strategies are introduced, the distinction between public and private strategies becomes relevant. In fact, Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7] have recently provided examples of games with equilibrium payoffs in private strategies which lie outside the set of PPE payoffs.

The present paper focuses on another distinction, that between mixed and behavior strategies, which appears to be relevant for the class of games under consideration. This point of view complements that of Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7] by showing that, as far as with mixed strategies one is concerned, the restriction to public strategies is not a restriction at all. Specifically, we show that if v is a Nash equilibrium payoff vector supported by private behavior strategies, then there exists a Nash equilibrium profile in public mixed strategies which supports v.

The main argument is based on Dalkey's theorem (Dalkey [3]), which states that in games with effectively perfect recall it is immaterial whether or not a player recalls his own past actions. Given a game with imperfect private monitoring satisfying the same assumptions as in Abreu, Pearce and Stacchetti [1], we consider a game that differs from the original one only in that players do not recall their own past actions. By construction, in such a game all the strategies are public strategies. After checking some measurability conditions, we mimic Dalkey's argument to show that this game and the original one have the same Nash equilibria in mixed strategies. From this, our result about the equivalence of public mixed strategies and private behavior strategies follows at once. By construction, the new game we introduced does not have perfect recall. As it is well-known, this implies that, generally speaking, its set of Nash equilibrium payoffs in mixed strategies and that in behavior strategies do not coincide. Hence, our result about the equivalence between this game and the original one implies at once that the set of PPE payoffs and the set of equilibrium payoffs in private behavior strategies do not coincide in the original game. In this respect, our result provides a general explanation for the findings of Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7] as well as a general method for constructing examples of that sort.

The paper proceeds as follows. Section 2 describes the model. Section 3 contains the proof of the equivalence stated above. Section 4 concludes by observing that such equivalence does not extend beyond the model of section 2.

2 The Model

The model described below is essentially the same as in APS [1] with the assumption about the existence of a pure-strategy Nash equilibrium for the stage game removed. When other assumptions are weakened, or additional interpretations proposed, a detailed discussion is given.

2.1 The Stage Game G

This is a finite-player game. Each player $i \in I$ has a finite set of actions, A_i; $a \in A = \times_i A_i$ denotes a profile of pure actions. The payoffs in the game depend on the action profile being played and on the realization $\omega \in \Omega$ of a state of the world. Such a state is randomly selected according to a measure $\mu(a)$ when profile a is played. We assume

- R1 Ω is a compact subset of a Polish space. $(\Omega, \mathcal{B}_\Omega)$ is a measurable space, with $\mathcal{B}_\Omega$ referring to the Borel σ-field.
- R2 The measures $\{\mu(a)\}_{a\in A}$ are mutually absolutely continuous. In particular, there exists a fixed measure μ such that $\mu(a) << \mu$, $\forall a \in A$.

Let $g_i(\omega, a)$ be the payoff that player i receives when a is played and ω realized. We assume also,

- P1 $g_i(\cdot, a)$ is continuous in ω for every $a \in A$.

Note that R1-P1 imply bounded payoffs, that is $\exists c \in R$ such that $g_i(\cdot, \cdot) \in [-c, c]$.

To complete the description of the stage game, denote by m_i a mixed action for player i. The set of mixed profiles is $\Delta(A) = \times_i \Delta(A_i)$, and a generic element in $\Delta(A)$ is denoted by m. The family $\{\mu(a)\}_{a\in A}$ is extended to a family $\{\mu(m)\}_{m\in\Delta(A)}$ in the obvious way. Finally, when profile m is played, player i's expected payoff is $\int g_i(\omega, m) d\mu(m) = E[g_i(\omega, m)]$.

By Nash, G has an equilibrium (in mixed strategies).

2.2 The Repeated Game $G^\infty(\delta)$ and the Signalling Structure

We denote by $G^\infty(\delta)$ the game consisting of the infinite repetition of G, where all players discount their payoffs at the common discount factor $\delta \in [0, 1)$. The signalling structure in $G^\infty(\delta)$ is described by the following assumptions

S1 At the end of each stage t, $t \in N$, ω is realized and observed by all players.

S2 In each stage t, the distribution on Ω depends only on the (mixed) action profile played in that stage. In other words, transition probabilities are state-independent.

S3 At the end of each stage t, player i observes the payoff he achieved in that stage. However, as in APS, we assume that such a payoff depends on the actions a_{-i} of player i's opponents only indirectly through the effect that these actions have on the distributions on Ω.

The last assumption was introduced by APS in [1], and is recurrent in the applied work on games with public information. It imposes that payoffs, though observed, do not carry player i additional information other than that derived from the realization of ω and the knowledge of (the realization of) his own action. It might be worth noticing that this is a "global" assumption. Payoffs do not carry additional information both on and off the equilibrium path. Because of this, one can replace S3 by the following alternative assumption, which still preserves the same information structure, but does not impose any restriction on the technology of the payoff functions.

S3' Stage payoffs are unobservable.

Summarizing, the information that player i has at the beginning of stage t consists of a t-vector $\omega^t = (\omega_0, \omega_1, ..., \omega_{t-1})$ of realizations of the publicly observed state along with a t-vector $a_i^t = (a_{i,1}, ..., a_{i,t-1})$ of realizations of his own (mixed) actions.

2.3 Strategies

A pure-strategy for player i in $G^\infty(\delta)$ is a sequence of measurable maps $\sigma_i = \{\sigma_{i,t}\}_0^\infty$, with $\sigma_{i,t} : \Omega^t \times A_i^{t-1} \longrightarrow A_i$. A mixed-strategy is a distribution over pure strategies. A behavior-strategy for player i is a sequence of measurable maps $\sigma_i = \{\sigma_{i,t}\}_0^\infty$, with $\sigma_{i,t} : \Omega^t \times A_i^{t-1} \longrightarrow \Delta(A_i)$. $\Omega^t \times A_i^{t-1}$ is a measurable space with the product σ-algebra $\left(\overset{t}{\otimes}\mathcal{B}_{\Omega^k}\right) \otimes \left(\overset{t-1}{\otimes}\mathcal{P}(A_i)\right)$, and $\Delta(A_i)$ has its usual Borel structure. The set of player i's pure strategies (mixed, behavior) in $G^\infty(\delta)$ is denoted by Σ_i ($\mathcal{M}_i, \mathcal{B}_i$, respectively). Here, we have adopted the convention of denoting by $\cdot_t$ the signal received after stage t is played. To ease the notation, we have defined $\sigma_{i,0}$ on ω_0 which is, clearly, arbitrary (similarly, APS [1]). Such a choice has, obviously, no consequences for the analysis.

We conclude the section with the following

Definition 1 *A strategy (pure or behavior) σ_i is said to be a public strategy if for any $\omega^t \in \Omega^t$ and any $t \in N$, $\sigma_{i,t}(\omega^t, a_i^t) = \sigma_{i,t}(\omega^t, \bar{a}_i^t)$ for any pair $(a_i^t, \bar{a}_i^t) \in A_i^t \times A_i^t$.*

3 Equivalence of Public Mixed-Strategies and Private Behavior-Strategies

The following definitions and theorem are well-known (see, for instance, Mertens, Sorin and Zamir [8]). They are included here for the ease of the reader.

Definition 2 *A game is said to have* ***(effectively) perfect recall*** *for player i if player i* ***(knowing the pure-strategy he is using)*** *can deduce from any signal he may get along some feasible play, the sequence of previous signals he got along that play. A game is said to have (effectively) perfect recall if it has (effectively) perfect recall for each player $i \in I$.*

For games with effectively perfect recall for player i, we have the following important theorem due to Dalkey [3].

Theorem 3 ***(Dalkey [3])*** *In a game with effectively perfect recall for player i, player i's pure-strategy set is the same (up to duplications) whether or not he recalls his own past moves.*

To prove the theorem ([8], p. 64), start by noticing that – as in Section 2.3 – a pure-strategy for player i has the form $a_n = \sigma(\omega_0, \omega_1, ..., \omega_{n-1}; a_1, ..., a_{n-1})$, $a_k \in A_i$ for each k. We want to define a new strategy, ζ, which does not depend on $(a_1, ..., a_{n-1})$. For each initial signal ω_0, define $a_1 = \sigma(\omega_0) = \zeta(\omega_0)$. Proceed by defining $a_2 = \sigma(\omega_0, \omega_1; a_1) = \zeta(\omega_0, \omega_1; \zeta(\omega_0))$. Then, inductively, ζ is defined. Finally, note that, whatever the other players' strategy, ζ generates the same probability distribution on plays as σ.

Now, let us return to our problem. Alongside with $G^\infty(\delta)$, let us introduce a new game, $G_P^\infty(\delta)$, which is such that players do not recall the realizations of their own mixed actions, and is otherwise identical to $G^\infty(\delta)$. Observe that, by construction, there are no private strategies in $G_P^\infty(\delta)$. Though patently artificial, this line of reasoning is useful to establish the desired equivalence by means of Dalkey's theorem.

Let V_P be the Nash equilibrium payoff set of $G_P^\infty(\delta)$, and let V be that of $G^\infty(\delta)$. Both sets are nonempty as the infinite repetition of the stage game Nash equilibrium is a Nash equilibrium (the same) in both cases. We have,

Theorem 4 *For the model described in Section 2,*

$$V = V_P$$

In particular, every $v \in V$ can be obtained by public strategies.

Proof. Let σ_i be a pure-strategy in $G^\infty(\delta)$, and let ζ_i be defined as in the proof of Dalkey's theorem. We begin by establishing that ζ_i is indeed a sequence of measurable maps.

ζ_i is defined by means of the following composition

$$\begin{tikzcd}
\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1} \arrow[d, "i^t \otimes \left(\overset{t-1}{\otimes}[\zeta_k \circ \pi^k]\right)"'] \arrow[dr, "\zeta_{i,t}"] & \\
\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1} \times A_1 \times \cdots \times A_{t-1} \arrow[r, "\sigma_{i,t}"] & A
\end{tikzcd}$$

where

i^t is the identity on $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1}$

$\pi^k : \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1} \longrightarrow \Omega_0 \times \Omega_1 \times \cdots \times \Omega_k$ is the projection on the first $k+1$ factors

$\otimes$ denotes tensor products

Start by noticing that $(\zeta_{i,1}, ..., \zeta_{i,t-1})$ Borel $\Longrightarrow \zeta_{i,t}$ Borel [$\zeta_{i,k}$ is a map from $\left(\overset{k}{\times}\Omega_j, \overset{k}{\otimes}\mathcal{B}_{\Omega^j}\right) \longrightarrow (A, \mathcal{P}(A))$]. In fact, i^t and π^k are measurable (product σ-algebra). Hence, each $\zeta_{i,k} \circ \pi^k$ is Borel being the composition of Borel maps, and so is $\xi = i^t \otimes \left(\overset{t-1}{\otimes}[\zeta_{i,k} \circ \pi^k]\right)$ being a tensor product of measurable maps. Hence, $\zeta_{i,t} = \sigma_{i,t} \circ \xi$ is measurable. In particular, $\zeta_{i,t}$ is $\overset{t}{\times}\mu$–measurable. In fact, since $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1}$ is Polish, every Borel subset of $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_{t-1}$ is universally measurable.

Now, observe that $\zeta_{i,1}$ is trivially measurable since $\zeta_{i,1} = \sigma_{i,1}$. Hence, by induction, the proof is complete.

This shows that ζ_i is a feasible strategy in $G^\infty(\delta)$, and that any pure-strategy profile in $G^\infty(\delta)$ can be replaced by a profile of pure strategies which (like our ζ_i above) depend on the publicly observed variable only.

Next, observe that ζ_i is a strategy in $G_P^\infty(\delta)$, and that any strategy in $G_P^\infty(\delta)$ is clearly feasible in $G^\infty(\delta)$. Hence, by combining these two observation, we can conclude that $G^\infty(\delta)$ and $G_P^\infty(\delta)$ have the same reduced normal form, and, hence, the same Nash equilibrium payoff set in mixed strategies. That is, $V = V_P$.

Finally, let $v \in V$ be a Nash equilibrium payoff vector in $G^\infty(\delta)$, possibly in private behavior strategies. By the preceding, there exists a profile of mixed strategies in $G_P^\infty(\delta)$ which supports v as a Nash equilibrium payoff vector. Clearly, such a profile is a Nash equilibrium profile in $G^\infty(\delta)$ as well [because $G^\infty(\delta)$ and $G_P^\infty(\delta)$ have the same reduced normal form]. This completes the proof. $\blacksquare$

The theorem says that, for the class of games described in Section 2, the restriction to public strategies is not a restriction at all as long as one is willing to trade private behavior strategies with public mixed strategies.

Now, we want to clarify the relation between our result and those of Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7]. To this end, we are going to deal first with the case in which the following full-support assumption, originally introduced in APS [1] (used also in some of the examples of [5] and [7]), is satisfied.

R3 *Supp* $\mu(a) = \Omega, \forall a \in A$.

In such a case, the set of Nash equilibrium payoffs of $G^{\infty}(\delta)$ is the same as its set of Perfect Bayesian equilibrium payoffs. Clearly, our conclusion that $G^{\infty}(\delta)$ and $G_P^{\infty}(\delta)$ have the same reduced normal form, and, hence, the same Nash equilibrium payoff set in mixed strategies is not affected by the introduction of **R3**.

Under **R3**, the set of PPE payoffs in $G^{\infty}(\delta)$ is the set of sequential equilibria which depend on the public variable only. By our reasoning, it is immediate to observe that such a set coincides with the set of behavior-strategy Nash equilibrium payoffs in $G_P^{\infty}(\delta)$. Now, let us focus on $G_P^{\infty}(\delta)$. By construction $G_P^{\infty}(\delta)$ does not have perfect recall (because players do not recall their own actions). Therefore, generally speaking (i.e., for arbitrary payoff functions and discount factors), the set of behavior-strategy Nash equilibrium payoffs of $G_P^{\infty}(\delta)$ is a proper subset of its set of Nash equilibrium payoffs. The difference between the two is given by those Nash equilibrium payoffs supported by mixed strategies that cannot be supported by behavior strategies in $G_P^{\infty}(\delta)$. By Theorem 4, such equilibria still correspond to some equilibria in $G^{\infty}(\delta)$. The latter are exactly those sequential equilibria in $G^{\infty}(\delta)$ which cannot be supported by public behavior strategies. In our terminology, this can be rephrased by saying that, under **R3**, the examples of Kandori and Obara [5] and Mailath, Matthews and Sekiguchi [7] exploit exactly the fact that $G_P^{\infty}(\delta)$ does not have perfect recall. In other words, in order to construct such examples, it suffices to pick payoffs in $G^{\infty}(\delta)$ in such a way that the set of behavior-strategy Nash equilibrium payoffs of $G_P^{\infty}(\delta)$ is a proper subset of its set of Nash equilibrium payoffs. The equilibrium payoffs in $G_P^{\infty}(\delta)$ which are not behavior-strategy equilibria are exactly those which lie outside the set of PPE payoffs in the original game $G^{\infty}(\delta)$.

The situation is slightly more complicated without the full support assumption. To see this, suppose that v is a Nash equilibrium payoff in $G_P^{\infty}(\delta)$, and assume that v is not a sequential equilibrium payoff in $G_P^{\infty}(\delta)$. By Theorem 4, v is a Nash equilibrium in $G^{\infty}(\delta)$ as well. However, one cannot conclude that v is not a sequential equilibrium payoff in $G^{\infty}(\delta)$. In fact, while $G^{\infty}(\delta)$ and $G_P^{\infty}(\delta)$ have the same reduced normal form, the set of strategies in $G^{\infty}(\delta)$ is bigger than the set of strategies in $G_P^{\infty}(\delta)$. Hence, the set of players' equilibrium beliefs in $G^{\infty}(\delta)$ is potentially bigger that the set of players equilibrium beliefs in $G_P^{\infty}(\delta)$. This opens the possibility that there is an equilibrium belief in $G^{\infty}(\delta)$ (of course, associated to private behavior strategies) which makes v a sequential equilibrium in $G^{\infty}(\delta)$. In fact, this is exactly what happens in the third example of Mailath, Matthews and Sekiguchi [7], where they exhibit a pure strategy sequential equilibrium which cannot be replicated by public strategies. To be sure, the (payoff-) equivalent public-strategy profile still exists, but it cannot generate the same belief as the private-strategy profile at some history off the equilibrium path.

4 Conclusion

In this paper, we have shown the equivalence, from the viewpoint of the Nash equilibrium payoffs, between public mixed strategies and private behavior strategies. In the proof, it was crucial for the use of Dalkey's theorem that players' information patterns in $G^{\infty}(\delta)$ and $G_P^{\infty}(\delta)$ differ only because of a player's knowledge of his own actions, a property that was delivered by either assumption **S3** or **S3'** of Section 3.

To see that the equivalence result does not extend if either assumption is removed, consider the following example[1]. Suppose that players are engaged in a repeated Bertrand competition, that $\mu(a) = \mu$, $\forall a \in A$ and that payoffs are independent of ω, but assume that payoffs are observable. Suppose also that the discount factor is sufficiently high so that the monopolistic outcome is an equilibrium outcome. Clearly, such an outcome emerges as an equilibrium only because, by observing his per-period payoff, a player can infer with certainty whether or not the other players conformed to the monopolistic outcome. In fact, it is supported by a profile where player i cooperates at $t+1$ if the other did so at t, and reverts to the static Nash equilibrium if cooperation did not take place at t. It is clear that such a profile cannot be replicated by any profile in public strategies, since that would violate the measurability condition of a player's strategy with respect to his information. In other words, for the model just described – where both **S3** and **S3'** are violated – the equivalence between $G^{\infty}(\delta)$ and $G_P^{\infty}(\delta)$ fails since we have at least one element in $G^{\infty}(\delta)$ that cannot be obtained in $G_P^{\infty}(\delta)$.

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