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Storable Votes

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STORABLE VOTES

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Abstract

Motivated by the need for more flexible decision-making mechanisms in the European Union, the paper proposes a simple but novel voting scheme for binary decisions taken by committees that meet regularly over time. At each meeting, committee members are allowed to store their vote for future use; the decision is then taken according to the majority of votes cast. The possibility of shifting votes intertemporally allows agents to concentrate their votes when preferences are more intense, and although the scheme will not in general achieve full efficiency, making votes storable typically leads to ex ante welfare gains. The analysis in the paper suggests that the result will hold if one of the following conditions is satisfied: (i) the number of voters is above a minimum threshold; (ii) preferences are not too polarized; (iii) the horizon is long enough.

I. INTRODUCTION

Consider a committee that meets regularly over time to vote up or down proposals that affect all of its members. The committee members are heterogenous and have different preferences over the policy to be enacted. Decisions are taken by majority vote, and as always a majority with weak preferences will win over a minority with more strongly held opinions. Think of this simple alternative: although each member continues to accrue one new vote at each meeting, he now has the option of storing his vote for future use. If a member abstains at the first meeting, he will be able to cast either *0*, *1* or *2* votes at the second; were he to abstain again, he would have up to *3* votes available for the next meeting. In other words, suppose votes are storable. Would this plausible procedural change improve the efficiency of the committee? If asked at some preliminary constitutional stage, would committee members prefer a system of storable votes? The purpose of this paper is to propose such a mechanism and begin addressing the questions it raises.

Its main results, in the simplified setting the paper studies, are promising. By allowing voters to shift their votes intertemporally, storable votes lead them to concentrate their votes on times when preferences are more intense, and therefore increase the probability of obtaining the desired decision when it is more important. Under plausible assumptions, ex ante welfare is higher than with non-storable votes. In addition, storable votes appear to behave well even if voters follow plausible rules-of-thumb, as opposed to fully rational strategies. Finally, at least in the examples analyzed in the paper, storable votes have better welfare properties than tradable votes, besides being more transparent and procedurally simpler and thus less objectionable both ethically and practically.

The research project was motivated by concerns with the mechanisms through which the European Union coordinates (or attempts to coordinate) the policies of its members. The problem of achieving a unified policy while respecting the sovereignty of heterogeneous countries is very difficult, and all reforms of European Union's institutions are caught between the need for faster decision-making that majority voting provides and the importance of

respecting each country's priorities lest the whole process of integration comes to an end.[1] Intuitively, a country should be able to weigh more heavily when a fundamental interest is threatened, but at a price: as in the case of money, the choice to obtain control over one item should come at the cost of smaller disposable resources available in the future. Storable votes fulfill this function. Other mechanisms may do so too, but storable votes have the advantage of being extremely simple: the mechanism is very natural, can be explained in few words and induces very intuitive behavior.

In line with this motivation, and notwithstanding the theoretical interest of the dynamic game induced by storable votes, the goal of this research remains very applied. The final ambition is to study the conditions under which a system of storable votes would be a desirable and feasible procedural reform over current practice, or equivalently to learn that such a reform would be unwise. The questions asked and the model designed reflect strongly these practical concerns. Thus, for example, the paper does not investigate how an optimal decentralized mechanism could be designed in the setting it studies. Storable votes share with other market-type systems of voting a fundamental weakness: a voter who chooses to exploit the mechanism alters the probability that all other votes be pivotal, and hence the value of those votes. The externality implies that the final outcome will fall short of full efficiency and that there is scope for a superior mechanism. But the externality does not imply that storable votes cannot dominate non-storable votes, and it is this more modest question that the current paper addresses.[2] In the same spirit, the paper does not allow for monetary side-payments. In practical applications, credit constraints and unequal wealth distribution would lead to uneven political influence, an inequitable - and in general inefficient - outcome that would be politically unacceptable.

Of course the importance of preserving strongly felt minority preferences extends much beyond the immediate challenges of the European Union, to the design of most democratic institutions. The paper refers to the specific example of the European Central Bank because it

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[1]See for example the surprising defense of veto power in *The Economist* (12, 8th-14th, 2001, p.34)

[2]Both observations - the inefficiency brought about by the externality and the welfare gain that may nonetheless remain relative to standard majority voting - have been made also in reference to tradable votes. See for example Philipson and Snyder (1996) and Piketty (1994).

provides a concrete example of a repeated binary voting game with fixed agenda, and it is this simple setting that this first model studies. But it should be clear that there is no reason why storable votes should not be studied eventually for potential applications to generic committees.

The idea of using more resources, here more votes, when a decision is valued more is very natural, but storable votes have no clear precedent in the literature. The two closest relations are vote trading (see for example Buchanan and Tullock, 1962, Coleman, 1966, Brams and Riker, 1973, Ferejohn 1974, Philipson and Snyder, 1996 and Piketty, 1994) and cumulative voting (Dodgson, 1884, Sawyer and MacRae, 1962, Brams, 1975, Cox, 1990, Guinier, 1994, Gerber, Morton and Rietz, 1998, among others). But storable votes are different from both.

Without recurring to monetary exchanges, trading votes by exchanging support for each others' favorite policies is common practice, and although a practice just as frequently decried and almost never granted institutional support[3], it has long been studied as a natural and potentially beneficent way of expressing voters' intensity of preferences. In a system of storable votes, votes are traded intertemporally, not interpersonally, but the two mechanisms seem similar, and it is interesting to compare them. They differ on two main grounds. First, because they rely on individuals acting alone, storable votes are a simpler and more transparent institution. When votes are exchanged interpersonally, traders must find each other, competitors must outbid one another and future promises must be enforced. How these steps take place depends on auxiliary, but essential, institutions;[4] different alternatives are possible and affect the outcome, creating potential sources of manipulation that contribute to the public's instinctive suspicion of tradable votes. These complications do not exist in the case of storable votes. Second, with both mechanisms individuals' decisions impose an externality on other voters, but the problem is different in the two cases. Vote trading results in coalitions, and individuals uninterested or unable to trade votes find their influence reduced both today and in the future. With storable votes, on the other hand, when an individual chooses to cast more than *1* vote in any period, the other committee members are automatically compensated by their increased

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[3]An interesting exception is "pairing" in the US Congress: two members who will be absent for a vote can choose to "pair" in a couple whose votes would cancel each other out. Their pairing is recorded and transmitted to the leadership. See Oleszek (2001).

[4]See for example the discussion in Philipson and Snyder, 1996.

influence in the other periods. At least in the simple example studied in the paper, storable votes are not only less tainted ethically, but welfare superior.[5]

Cumulative voting, on the other hand, is a system where voters elect a subset of candidates in a given election and are free to allocate a given stock of votes among them as they see fit. It is a static mechanism, where all voters choose simultaneously how to cast all their votes in a single election. Storable votes on the contrary are a dynamic mechanism, where votes are kept over time and over multiple binary decisions. As time passes, uncertainty is resolved both with respect to the voters' evolving preferences, and the stock of votes still available to one's opponents. The presence and the partial resolution of uncertainty lend importance to time and differentiate the two mechanisms.

Other voting schemes have some of the flavor of storable votes, but again all are different. Voting by successive veto (Mueller,1978, Moulin,1982) and voting by successive pair-wise elimination (Moulin,1979) are schemes where one of several possible alternatives is selected through a dynamic process of elimination. All information is known at the beginning, and the dynamic aspect allows to select subgame perfect equilibria (and hence restrict the set of possible outcomes). The main concern is the theoretical design of desirable schemes when voters must choose among more than two alternatives. Here instead, information is acquired over time, and the decision is binary each period. The observation that introducing a cost to voting selects voters with more intense preferences and thus may be efficiency-enhancing has been made before (for example, Börgers, 2001). But storable votes make that cost endogenous by allowing voters to choose how much future influence to renounce.

The dynamic nature of storable votes calls to mind the economic theory of optimal dynamic contracting (Townsend, 1982, Phelan and Townsend, 1991, Atkeson and Lucas, 1992). The problem addressed in these works is the design of the optimal incentive-compatible insurance mechanism when agents are subject to random shocks that are private information. The literature has found that such a mechanism must be dynamic: agents are effectively given a stock of claims to transfers; their use at any time reduces, according to an optimal formula, the

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[5]Notice that analyses of vote markets fall in the realm of second-best theory, and restricting some trades need not lead to worse outcomes.

amount of claims available for the future.[6] It seems plausible that the optimal mechanism for the problem studied in this paper will similarly be dynamic, and share some features of storable votes. Storable votes fulfill a function similar to that of simple borrowing and lending schemes in Townsend (1982), not optimal, but superior to one-period agreements.

The paper proceeds as follows. Section II describes the assumptions of the model. Section III discusses why storable votes lead to welfare gains in the simplest setting, when there are only *2* voters. Section IV extends the analysis to *N* voters and considers in detail the case of *3* voters. Section V compares storable and tradable votes, and section VI concludes. The Appendix presents the longer proofs.

II. A SIMPLE GAME

A committee of *n* individuals meets regularly to take a common decision *d* that can assume two values: $d \in \{0,1\}$. We can think of $d = 0$ as maintaining the status quo, and $d = 1$ as change. Each member's preferences are indexed by a parameter σ: in period *t*, individual *i*'s utility equals $\sigma_{i,t} d_t$. The parameter $\sigma_{i,t}$ is drawn from a continuous distribution $F(\sigma)$, defined over the support *[-1, 1]* and symmetric around zero. Thus whenever the realization of $\sigma_{i,t}$ is negative, *i* prefers $d_t = 0$; when $\sigma_{i,t}$ is positive, *i* prefers $d_t = 1,$ and the absolute value of $\sigma_{i,t}$ measures the intensity of *i*'s preferences. The distribution $F(\sigma)$ is common across all committee members and all periods, and $\sigma_{i,t}$ is independently distributed both across individuals and across time. The committee takes the decision every period for a total of *T* periods, where *T* is finite.

For concreteness, we can think of the committee as the Executive Council of the European Central Bank, meeting each month to decide whether to maintain current interest rates ($d = 0$), or to change them ($d = 1$), under the assumption that both the direction of the possible change and, more controversially, its size are known before the meeting. Each member of the Council has preferences over European monetary policy and these preferences need not be

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[6]Athey and Bagwell, 1999, study the optimal intertemporal contract supporting collusion in a cartel with private information about cost shocks. The question is different, but the logic, and the answer, are very similar. Dixit, Grossman and Gul, 2000, analyze the efficient equilibrium of a sharing game where two political parties have varying political strength over time. Again the result relies on dynamic ties: the efficient allocation is non-Markov, and parties' shares must reflect the history of the game.

homogenous, reflecting different needs of the national economies, possibly different objectives or different individual perceptions of Europe-wide conditions. Each member's preferences are summarized by $\sigma_{i,t}$. The assumption of i.i.d. shocks is not ideal in this context, but simplifies the analysis greatly without being fundamental to the logic of the scheme proposed.

Every period each committee member is given one vote. He can cast it in favor of the option he prefers, or store it for use at a later time. Thus in period *1*, a member can cast either *0* or *1* votes; if he decides to save his vote, in period *2* he will have a total of *2* votes at his disposal and will decide how many of these, if any, to use; and so on in all successive periods until time *T* when the game ends. We assume that votes can be stored but not borrowed to avoid the difficult problems that could arise in practice if one member were to run out of votes, but we will show later that the assumption is unimportant.[7] Subject to the budget constraint that the votes cast cannot exceed the number of votes available, each member is asked to indicate his preferred decision and the number of votes he is willing to spend to support his choice. When individuals vote, they know the realization of their current $\sigma_{i,t}$ and the probability distribution *F*, but cannot observe the preferences of the other members. On the other hand, because the initial allocation of votes and the history of the game are known, the number of votes that each player has at his disposal is common knowledge.

The committee selects *d* according to which of the two alternatives has received more votes. If the votes are equal, the preferences of members who have cast zero votes are considered; if the tie is still not broken, the decision is taken with a coin toss. The tie-breaking mechanism seems plausible and has some advantages in deriving analytical solutions, but does not affect the substance of the results.[8]

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[7]The constraint on borrowing is common to policy mechanisms that rely on market-type behavior (for example environmental regulation through tradable pollution licences), because it reduces the costs of mistakes and inexperience, and increases the credibility of the rules. In addition, when members are subject to appointments or elections, the inability to borrow from the future limits the extent to which current members can expropriate the power of their successors. In our model too, the constraint is imposed for purely pragmatic considerations.

[8]Other tie-break rules - no weight on zero voters; status quo wins when votes are tied (with or without considering zero voters) - always yielded the same qualitative results. In the application of the game to the European Central Bank, an alternative model is also possible: we could assume that the decision is between a cut (*d=-1*) or an increase (*d=1*) in interest rates, with the status quo (*d=0*) prevailing in case of ties. However this set-up would minimize the role of the status quo, while in fact maintaining the status quo is often the preferred option for most central bankers. In any case, the results of the two models are identical up to a factor of proportionality in

The individuals' objective is to announce a policy preference and choose a number of votes each period so as to maximize the expected flow of utility over the whole time horizon. Given a common discount factor δ, the problem amounts to maximizing $EU^i = E(\sum_t \delta^t \sigma_{it} d_t)$, where E is the expectations operator, subject to the constraint that for each committee member the stock of available votes k_{it} equals the votes stored the previous period plus the allocation of *1* new vote ($k_{it} = k_{it-1} - v_{it-1} + 1$, where v_{it-1} are votes cast by i at t-1). The state of the game is given by the distribution of available votes among all members and calendar time; if we call K_t the vector of votes held by players at time t, we can write the expected value of the game to individual i when all players follow optimal strategies as $EV_t^i (K_t, t)$.

The goal of the paper is to compare the storable votes scheme to the more traditional case where votes are not transferable over time, and thus each individual always casts one vote in favor of his preferred alternative. The two games are obviously identical if the time horizon reduces to a single period, but differ otherwise. The storable votes game requires some thought: the choice of how many votes to cast reflects not only the current intensity of preferences, relative to expected future preferences, but also the probability that a vote be pivotal, today or in the future, and thus the expectation of the other players' voting behavior over time. It is a non-stationary dynamic game, where each individual's optimal strategy will be conditioned on the realization of his preference shock, on the distribution of available votes among all players and on calendar time. To build intuition for the results that will follow, it is good to begin with the simplest case.

III. TWO VOTERS.

III. A. Two Periods.

Two players i and j must take decision d. At period T-1, they are endowed with *1* vote each; they will both receive an additional vote at period T, but the game will then end. At T they will both spend all available votes on their preferred alternative; thus the only problem each player must solve is what preference to announce and whether to cast *1* or *0* votes in its support at T-1. Because preferences and votes are announced simultaneously and preferences shocks

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expected payoffs.

are i.i.d., a voter will always announce preferences truthfully: he cannot manipulate his opponent's strategy. The choice reduces to the number of votes to cast.

Consider individual i, and suppose for now that the realization of his preference shock σ_{iT-1} is positive. Thus i prefers $d = 1$. If i saves his vote, his expected utility is given by:

$$Eu^i_{T-1}(v_i = 0) = \sigma_{iT-1}(3/4 p_{j0} + p_{j1}/2) + \delta\left(p_{j0}EV^i_T(2,2) + p_{j1}EV^i_T(2,1)\right) \qquad (1)$$

where p_{jv} is the probability that j casts v votes at $T-1$, and $EV^i_T(s,k)$ is i's expected value of the game in the next and final period, given stocks of available votes s (for player i) and k (for player j). With σ_{iT-1} positive, i's expected current return is σ_{iT-1}, times the probability of obtaining the desired decision $d = 1$ when casting 0 votes. If j casts 0 votes too, with probability $\frac{1}{2}$ $d = 1$ is chosen because it is both individuals' desired option (given the symmetry of the distribution function $F(\sigma)$, and even if neither player is willing to spend any votes); with probability $\frac{1}{2}$ j prefers $d = 0$, but even in that case i has a fifty percent chance of winning the coin toss. Hence if j casts 0 votes, the probability that $d = 1$ equals $3/4$. If instead j casts 1 vote, then $d = 1$ is chosen only if it is j's preferred option, which occurs with probability $\frac{1}{2}$. Thus i's expected current return must equal σ_{iT-1} $(3/4 p_{jo} + 1/2 p_{j1})$, the first term in equation (1). As for next period, if i casts 0 votes today, he will then have 2 votes available, while j's votes will depend on j's current strategy.

Similarly, if individual i casts his vote at $T-1$, his expected utility is:

$$Eu^i_{T-1}(v_i = 1) = \sigma_{iT-1}(p_{j0} + 3/4 p_{j1}) + \delta\left(p_{j0}EV^i_T(1,2) + p_{j1}EV^i_T(1,1)\right) \qquad (2)$$

taking into account that i always obtains his preferred option at $T-1$ if j casts 0 votes, or with probability $3/4$ if j casts 1 vote. Comparing (1) and (2), we obtain that i will cast 1 vote at $T-1$ if and only if:

$$\sigma_{iT-1}(p_{j0} + p_{j1})/4 \geq \delta\left(p_{j0}\left(EV^i_T(2,2) - EV^i_T(1,2)\right) + p_{j1}\left(EV^i_T(2,1) - EV^i_T(1,1)\right)\right) \qquad (3)$$

Solving next period's expected values, we can obtain an explicit solution to the optimal strategy.

In period T, both players cast all votes they have, and the one with most votes wins with probability 1. Thus:

$$EV^i{}_T(2,1) = \int_{-1}^{0} \sigma dF(\sigma)(0) + \int_{0}^{1} \sigma dF(\sigma)(1) = \int_{0}^{1} \sigma dF(\sigma) \tag{4}$$

since whenever σ_{iT} is negative, i will be able to impose $d = 0$, and whenever σ_{iT} is positive, d will equal 1. The player with fewer votes will not be able to influence the choice of d, but half of the times his opponent's preferred choice matches his own. Hence:

$$EV^i{}_T(1,2) = 1/2 \int_{-1}^{0} \sigma dF(\sigma) + 1/2 \int_{0}^{1} \sigma dF(\sigma) = 0 \tag{5}$$

Finally, when the two players have the same number of votes, the value of the game at period T is identical to the value of the one-period non-storable votes game (with equal votes). Call the value of this latter game W, noticing that it is time independent and that any number of equal votes is equivalent. For any realization of σ_{iT}, player i expects to obtain his preferred value of d three quarters of the times, either because j has the same preference (fifty percent of the times), or because when they do disagree, the coin toss is in i's favor (twenty five percent of the times). That is:

$$EV^i{}_T(1,1) = EV^i{}_T(2,2) = \int_{-1}^{0} \sigma dF(\sigma)\big(3/4(0) + 1/4\,(1)\big) + \int_{0}^{1} \sigma dF(\sigma)\big(3/4\,(1) + 1/4\,(0)\big)$$

or:

$$EV^i{}_T(1,1) = EV^i{}_T(2,2) = 1/2 \int_{0}^{1} \sigma dF(\sigma) = W \tag{6}$$

Substituting (4), (5) and (6) in (3), we obtain:

$$\sigma_{iT-1}(p_{j0} + p_{j1})/4 \geq \delta(p_{j0} + p_{j1})W$$

or:

$$\sigma_{iT-1} \geq 4\delta W \qquad\qquad \text{if } \sigma_{iT-1} > 0 \tag{7}$$

It is easy to verify that if $\sigma_{i\,T\text{-}1}$ is negative the same logic leads voter i to cast his vote if

and only if $-\sigma_{i\,T-1} \geq 4\delta W$. [9]   Thus i's optimal strategy is to identify a threshold value $\alpha = 4\delta W$ > 0 and cast *1* vote whenever $|\sigma_{i\,T-1}|$ is larger or equal to α, and cast *0* votes whenever $|\sigma_{i\,T-1}|$ is smaller than α. Not surprisingly, the threshold α equals the average intensity of preferences (discounted), and is strictly smaller than *1* as long as there is any probability mass outside the extreme values -*1* and *1*. In the simple case where $F(\sigma)$ is Uniform and $\delta = 1$, $\alpha = 1/2$.

The conclusion was expected: if i's policy preference is particularly strong today, he will be willing to sacrifice some of his possible future power to increase his chances of obtaining the desired outcome; vice versa, if his policy preference is weak, he will prefer to abstain today and increase his influence tomorrow. It was this intuition that motivated the paper. We can now verify whether it leads indeed to desirable welfare properties (at least in this simple example).

Before the preference shock is realized, the expected value of the game for player i equals:

$$\begin{aligned}
EV_{T-1}(1,1) = &\int_0^\alpha \sigma dF(\sigma)(3/4\,p_{j0} + 1/2\,p_{j1}) + \int_{-\alpha}^0 \sigma dF(\sigma)(1 - 3/4\,p_{j0} - 1/2\,p_{j1}) + \\
&+ (2F(\alpha)-1)\delta W\big(p_{j0} + 2p_{j1}\big) + \int_\alpha^1 \sigma dF(\sigma)(p_{j0} + 3/4\,p_{j1}) + \\
&+ \int_{-1}^{-\alpha} \sigma dF(\sigma)(1 - p_{j0} - 3/4\,p_{j1}) + 2\big(1 - F(\alpha)\big)\delta W p_{j1}
\end{aligned} \tag{8}$$

The expression seems unwieldy, but is easily simplified. Voter j faces an identical problem and conditions his voting behavior on the same threshold α: he will vote *0* with probability *2[F(α)-1/2]*, and *1* with probability *2[1-F(α)]* (using, as in (8), the symmetry of the probability distribution).[10] Substituting these values for p_{j0} and p_{j1}, the expected value of the two-period game for either player becomes:

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[9] If $\sigma_{i\,T-1}$ is negative, i's expected utility from playing *1* or *0* is analogous to equations (1) and (2) above, but the negative preference shock now multiplies the corresponding probability of losing, as opposed to winning (since instantaneous utility is then different from zero only if i does not succeed in imposing his preference for $d = 0$). The probability of losing when casting *1* or *0* votes is the complement to *1* of the probability of winning we derived earlier, and the two expressions for expected utility are then immediately calculated.

[10] Notice that the equilibrium strategy is unique; in fact in this two-period example it is dominant.

$$EV_{T-1}(1,1) = \int_0^\alpha \sigma dF(\sigma)\big(F(\alpha) - 1/2\big) + \int_\alpha^1 \sigma dF(\sigma)\, F(\alpha) + \delta W \qquad (9)$$

We can now establish the first result of the paper. If W is the value of the one-shot non-storable votes game, call W_{T-1} the value of the two-period non-storable votes game (where the finite horizon yields the unique equilibrium value $W_{T-1} = W(1+\delta)$). Then we can state:

Proposition 1. *For any distribution $F(\sigma)$, $EV_{T-1}(1,1) > W_{T-1}$.*

Proof. Notice first that when $\alpha = 0$, $F(\alpha) = \frac{1}{2}$, and when $\alpha = 1$, $F(\alpha) = 1$. In both cases, $EV_{T-1}(1,1) = W_{T-1}$. To understand the behavior of $EV_{T-1}(1,1)$ in the interval $\alpha \in (0,1)$, differentiate $EV_{T-1}(1,1)$ with respect to α:

$$\frac{\partial EV_{T-1}}{\partial \alpha} = f(\alpha)\left(\int_0^1 \sigma dF(\sigma) - \alpha/2\right) \qquad (10)$$

where $f(\alpha)$ is the density $f(\sigma)$ evaluated at α, and thus is positive. (10) is positive at $\alpha = 0$ and has a single root; since we know $EV_{T-1}(1,1) = W_{T-1}$ at $\alpha = 0$ and at $\alpha = 1$, it follows that $EV_{T-1}(1,1) > W_{T-1}$ for all $\alpha \in (0,1)$. And since we know that α is strictly positive and smaller than 1, the conclusion follows. ∎

Proposition 1 states that the two-period game with two players has higher expected value when the votes can be stored. The result is not surprising and is clearly visible in expression (9): the possibility to store votes increases the likelihood that a player will win when his preference is stronger. The positive threshold α shifts probability mass from payoffs with relatively low value (when $|\sigma|$ is smaller than α) to payoffs with relatively higher value (when $|\sigma|$ is larger than α), and thus increases ex ante welfare. The analysis is simplified by assuming i.i.d. shocks and a symmetrical probability distribution, but intuitively it seems unlikely that the conclusion would be sensitive to these aspects of the problem.[11]

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[11]Correlation in players' preferences will reduce the incentive to vote, but seems unlikely to affect the nature of the result. Autocorrelation of each player's preferences over time creates a more difficult problem: it could induce non-truthful voting in an effort to manipulate future voting decisions by one's opponent.

Notice that the proof of Proposition 1 does not rely on the equilibrium value of α - any threshold strictly between 0 and 1 would lead to welfare gains. This is potentially important - if practical applications of the mechanism are considered, its robustness to mistakes should be investigated further. Threshold strategies are very intuitive rules of thumb, and if they are followed, we can say something more. When the two thresholds are equal (as they must be in equilibrium), the result always holds. When they differ, call α voter i's threshold, and β voter j's. Then:

$$EV^i_{T-1}(1,1) = \int_0^{\alpha} \sigma dF(\sigma)\left(F(\beta) - 1/2\right) + \int_{\alpha}^1 \sigma dF(\sigma) \ F(\beta) + \delta W\left(1 + 2\left(F(\alpha) - F(\beta)\right)\right) \qquad (11)$$

It is not difficult to verify that if $\delta=1$, $EV^i_{T-1}(1,1) > W_{T-1}$ for all $\alpha \in (0,1)$, independently of β (and $EV^i_{T-1}(1,1) = W_{T-1}$ at $\alpha = 0$ or 1).[12] If the second period is discounted, limits begin to appear as to how different the thresholds can be, but even for very small δ there is a sizable range of acceptable thresholds values, i.e. values consistent with welfare gains. In Figure 1, the area between two curves labeled with the same δ value corresponds to the acceptable area at that δ when $F(\sigma)$ is Uniform.

III.B. T Periods.

Although the logic of the problem is unchanged, the analysis becomes quickly more complicated as the number of periods increases because the number of states to be considered increases very rapidly with the length of the horizon. Starting from state (k^i, s^j) at t, we need to evaluate $(k^i+1)(s^j+1)$ possible states at time $t+1$, and of course the expected value of each of these states must be solved backwards from all the possible options it itself can give rise to, and so on at all times, using as anchor the expected values of all possible different states in the terminal periods. It is clear that the only possible solution method must be recursive. But here

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[12]When $\delta=1$, $EV^i_{T-1}(1,1) \geq W_{T-1}$ if $1/2 \int_{\alpha}^1 \sigma dF(\sigma) \geq \int_0^1 \sigma dF(\sigma)[F(\alpha) - 1]$, a condition that does not depend on β and is readily satisfied.

we encounter another problem: the game is non-stationary, and the equilibrium strategies depend both on the current state and on calendar time. To calculate the expected values of future states, we need to weigh them by their probability of realization, and hence by the probabilities of the players' alternative strategies in equilibrium. And these change over time, even for given states. It is then a welcome surprise to be able to obtain analytical results.

Call v_t the vector of strategies $(v_t{}^i, v_t{}^j)$, i.e. the number of votes cast by the *2* voters at time t. After the realization of the preference shock σ_{it}, player *i*'s expected value of the game at state K_t is given by:

$$EU_t{}^i(v_t; \sigma_{it}, K_t) = \max_{v_t{}^i} \left\{ E_{v_t{}^j} u^i(v_t; \sigma_{it}, K_t) + \delta E_{v_t{}^j}[E_{\sigma_{t+1}} V_{t+1}{}^i(K_{t+1}, \sigma_{t+1})] \right\} \qquad (12)$$

The first expectation inside the curly brackets is taken over $v_t{}^j$, *j*'s strategy; The second expectation is taken both over $v_t{}^j$, because *j*'s current strategy affects the state of the game next period. and σ_{t+1} (the vector $(\sigma_{i\,t+1}, \sigma_{j\,t+1})$), because next period preferences affect both strategies and payoffs next period. Given the possible existence of multiple equilibria, we restrict our analysis to Markov strategies - i.e. strategies that depend on past history only to the extent that it is reflected in the current state. It is possible to show that the following results must hold:

Proposition 2: *(i) At any t, there exist pure Markov strategies $v_t{}^*$ such that*:

$$EU_t{}^i(v_t{}^{i*}, v_t{}^{j*}; \sigma_{it}, K_t) \geq EU_t{}^i(v_t{}^i, v_t{}^{j*}; \sigma_{it}, K_t) \qquad \forall v_t{}^i \neq v_t{}^{i*}$$

$$EU_t{}^i(v_t{}^{i*}, v_t{}^{j*}; \sigma_{jt}, K_t) \geq EU_t{}^i(v_t{}^{i*}, v_t{}^j; \sigma_{jt}, K_t) \qquad \forall v_t{}^j \neq v_t{}^{j*} .$$

(ii) $v_t{}^{i*}$ is monotonically increasing in $|\sigma_{i\,t}|$, the player's intensity of preferences.

(The proofs are in the Appendix).

In addition to establishing that an equilibrium exists, and that it exists in pure strategies, Proposition 2 states that in equilibrium a voter will never casts fewer votes when the intensity of his preferences is higher than when it is lower. This simple conclusion allows us to characterize equilibrium strategies a bit more precisely. Notice that the number of votes that a player has at his disposal is always finite, while the support of $|\sigma_{i\,t}|$, the segment *[0,1]*, is continuous. Thus at any state of the game and time *t*, each voter must identify a series of thresholds that divide the

segment *[0,1]* into a finite number of intervals. For all realizations of $/\sigma_{i\,t}/$ in a given interval, i casts the same number of votes, but higher intervals must correspond to a larger number of votes. The thresholds are functions of the state of the game, calendar time and the opponent's equilibrium strategies, and although their number cannot be larger than the number of votes the player has available, it can well be smaller - some feasible number of votes may never be cast in equilibrium.

Proposition 2 does not state that the equilibrium is unique, and uniqueness is not required for what follows.[13] The important point is that Proposition 2 allows us to characterize each player's expected instantaneous utility in equilibrium. Consider for example the symmetrical state (k_t, k_t), where both voters enter the period with identical stocks of votes. Call $Eg_t^i(k_t, k_t)$ i's expected one period equilibrium utility (or payoff) before the realization of the preference shock when both players play optimal strategies, and $\alpha_{v-1}(K,t) \le \alpha_v(K,t)$ the equilibrium thresholds such that i will cast *v-1* votes for all $/\sigma_{i\,t}/ \in [\alpha_{v-1}(K,t), \alpha_v(K,t))$. Then in a symmetrical equilibrium:

$$Eg_t^i(k_t,k_t) = \int_0^{\alpha_1(t,k)} \sigma dF(\sigma)\big(F(\alpha_1(t,k)) - 1/2\big) + \int_{\alpha_1(t,k)}^{\alpha_2(t,k)} \sigma dF(\sigma)\big(F(\alpha_1(t,k)) + F(\alpha_2(t,k)) - 1\big) + \ldots$$

$$\ldots\ldots + \int_{\alpha_{k-1}(t,k)}^{\alpha_k(t,k)} \sigma dF(\sigma)\big(F(\alpha_k(t,k)) + F(\alpha_{k-1}(t,k)) - 1\big) + \int_{\alpha_k(t,k)}^1 \sigma dF(\sigma)F(\alpha_k(t,k))$$

(13)

where $0 \le \alpha_v(k,t) \le \alpha_{v+1}(k,t) \le 1, \ \forall t, \ \forall v \in \{1,..,k-1\}$.[14] A more cumbersome but analogous expression describes expected one-period equilibrium payoffs in asymmetrical states.

Player i's expected value of the game at state (k_t^i, k_t^j) before the realization of the preference shock, is given by:

$$EV_t^i(s_t^i, k_t^j) =$$
$$Eg_t^i(s_t^i, k_t^j) + \delta EV_{t+1}^i(s_t^i - v_t^i * + 1, k_t^j - v_t^j * + 1)$$

(14)

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[13]All numerical exercises with *2* voters we studied did however have a unique equilibrium.

[14]Equation (13) is analogous to (8) above. When i casts v_i votes, he obtains the decision he prefers with probability $1 * prob(v_j < v_i) + 3/4 * prob(v_j = v_i) + 1/2 * prob(v_j > v_i)$, or, exploiting Proposition 2, $\frac{1}{2} [F(\alpha_i) + F(\alpha_{i+1})]$. For each interval of σ_i values corresponding to a given strategy, i's expected return is weighted by the probability of the decision he prefers minus the probability of the decision he opposes (to account for negative realizations of σ_i - see (8)), or $F(\alpha_i) + F(\alpha_{i+1}) - 1$.

where, with abuse of notation, we use a single expectations operator although EV_{t+1} must be calculated by taking expectations over both σ_t (which determines j's current strategy) and σ_{t+1}. (See the Appendix for a more detailed description of how to proceed).

Expressions (13), and its analogue in asymmetrical states, and (14) allow us to establish the main result of this section. Define as symmetrical any equilibrium where i and j play the same strategies if there are at the same state at the same calendar time. Then:

Proposition 3. *For any distribution $F(\sigma)$ and any $t < T$, in any symmetrical equilibrium $EV_t(1,1) > W_t$.*

Proof. Intuitively, the objective is to reduce the expected value of the game at the initial period t to the sum of the expected one-period equilibrium payoffs corresponding to each possible state in all future periods. Exactly as in the 2-period case, when the state is symmetric, the possibility of storing votes when preferences are weak makes expected one-period payoffs higher than in the game with non-storable votes. The problem comes in non-symmetrical states: it is the prospect of being the weaker player in these states, possibly protracted over time and absent by assumption from the game with non storable votes, that creates concerns. But notice that in any symmetrical equilibrium and from any symmetrical state (k_t, k_t), the probability of reaching state $(s_{t+\tau}{}^i, k_{t+\tau}{}^j)$ is identical to the probability of reaching state $(k_{t+\tau}{}^i, s_{t+\tau}{}^j)$. Thus when evaluating possible future states, a player will give the same weight to the two opposite asymmetrical states and in effect consider their mean expected payoff. All we require then is that this mean payoff be higher, or at least not smaller, than the expected payoff with non-storable votes. It is this observation that allows us to establish the Proposition.

The intuition is formalized in the following two results:

Lemma 1.
    i) $Eg_t{}^i(k_t, k_t) \geq W \quad \forall t,$ with strict inequality at T-1.

    ii) $Eg_t{}^i(s_t{}^i, k_t{}^j) + Eg_t{}^i(k_t{}^i, s_t{}^j) \geq 2W \quad \forall t.$

Lemma 2. *Suppose the following inequalities hold at $t+1$:*

$$i) \quad EV_{t+1}^{i}(k_{t+1}, k_{t+1}) > W_{t+1}$$

$$ii) \quad EV_{t+1}^{i}(s_{t+1}^{i}, k_{t+1}^{j}) + EV_{t+1}^{i}(k_{t+1}^{i}, s_{t+1}^{j}) \geq 2W_{t+1}$$

Then they must hold at t.

The proofs of the two lemmas amount to manipulating expected equilibrium payoffs (expression (13) and its counterpart in asymmetrical states) and the dynamic programming equation (14). They can be found in the Appendix.

Once the two lemmas are established, Proposition 2 follows immediately. Because all votes are cast at T, $EV_T(s_T^i, k_T^j) + EV_T(k_T^i, s_T^j) = 2W$; in addition in all symmetrical equilibria at T-1, $EV_{T-1}(k_T, k_T) = Eg_{T-1}(k_T, k_T) + \delta W > W_{T-1}$ by Lemma 1. By induction, the inequalities hold at all previous times t, and in particular $EV_t(1,1) > W_t$ at any $t < T$. ∎

The result confirms that the intuition highlighted so clearly in the 2-period example extends to a longer horizon. Indeed we would expect the ratio $EV_t(1,1)/W_t$ to be higher the smaller is t, or the larger is $(T-t)$, the horizon remaining before the end of the game, a property that is not implied directly by the two lemmas but that was satisfied by all numerical simulations we investigated.

Notice that once again the proof of Proposition 3 makes no use of the exact values of the equilibrium thresholds, but holds for all monotone symmetrical thresholds. For example, some positive welfare gains would still be realized if a voter holding k votes followed this simple rule of thumb: at any t, divide the interval *[0,1]* in *k+1* subintervals of equal size, and once σ_i is realized cast v_i votes, where v_i satisfies: $v_i/(k+1) \leq |\sigma_i| < (v_i+1)/(k+1)$, $v_i \in \{0,1,..,k\}$. If instead off equilibrium voters' strategies are not symmetric, we can use the proof of Lemma 2 (ii) to show that the average expected one-period payoff (averaged over the two voters) cannot be inferior to the expected payoff under nonstorable votes. By induction this will hold for the average ex ante value of the full game (although not necessarily for each individual player).

Finally, we have assumed so far that votes accrue to voters over time and future allocations cannot be borrowed. Relaxing this constraint would increase the set of possible states, but at any state equilibrium strategies would still take the form of monotone thresholds (see the

proof of Proposition 2). Because this is all is needed in Proposition 3, both the proof and the Proposition would remain identical.[15]

IV. *N* PLAYERS.

Because the voting mechanism proposed here is new, it is important to study its properties in the reasonably transparent case of *2* voters. Still, most applications of the scheme should be to situations involving larger groups. There are two reasons to anticipate complications. First, as mentioned earlier, there is an externality built into the scheme that must prevent the achievement of the first-best,[16] and welfare properties will be hard to characterize. Second, the logic of the mechanism exploits the value of casting more votes than one's opponents, i.e. the resulting increase in the probability of being pivotal. But such an increase depends on the number of votes, on the number of voters and, not surprisingly, on whether the committee is of even or odd size.

As a simple illustration, Figure 2 reports the percentage increase in the ex ante probability of obtaining one's desired outcome from holding *2* votes when everybody else has a single one, in the one period game where all votes are cast.[17] As expected, the marginal impact of an extra vote (typically) diminishes with the size of the committee. However, when the size is small enough the extra vote is much more valuable if the number of voters is even rather than odd, and, unexpectedly, the impact increases, instead of falling, when the size of the committee goes from *3* to *5* voters.

These effects depend on the discreteness of the number of votes and voters and are independent of any potential inefficiency. In other words, even if voters chose their voting

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[15]Of course this does not mean that the welfare gain would remain identical. But notice that the second best nature of the problem implies that borrowing need not increase the expected value of the game.

[16]It is easy to see for example how diffuse but weak preferences might lose against a determined minority even in those cases where a static utilitarian concept (the sum of utilities) would dictate otherwise. The correct efficient yardstick is more complex because it involves the full intertemporal problem, but the principle is unchanged.

[17]The figure holds for any $F(\sigma)$ symmetrical around zero. The relevant formulas are in the Appendix. .

strategies cooperatively - ie even if the externality were taken into account and the thresholds set optimally - the expected welfare gains would still depend on *n*. Since in addition the severity of the externality itself changes with *n,* the number of voters affects the final welfare properties of the scheme through two separate channels which are not easy to disentangle and characterize systematically. Figure 3 provides a clear summary of these observations. It depicts the ratio $EV_{T-1}(1,1,1,...,1)/ W_{T-1}(n)$ in the next to last period of the game when $F(\sigma)$ is Uniform and $\delta=1$, as function of the number of voters. Three features are particularly noticeable: First, the ratio is larger than *1*, i.e. the welfare gains are positive, for all *n* different from *3* or *5*. Second, the ratio behaves differently for *n* odd and *n* even: especially when the number of players is small, the welfare gains from the scheme are much higher for *n* even than for *n-1* or *n+1*. However, and this is the third feature, the ratio increases with the number of players if *n* is odd and decreases if *n* is even, finally converging to a value larger than *1* for all *n* large enough. If $F(\sigma)$ is unimodal with a peak at *0*, the graph shifts upward (the more so the larger the probability mass around *0*), and eventually welfare gains become positive for all *n*, but the qualitative results are otherwise unchanged.[18]

All considered, we are lead to conjecture that the argument for storable votes may be weakest when the number of voters is small and odd. To understand why, in the remainder of the paper we concentrate on the case of *n=3*, and begin by deriving one-period equilibrium payoffs. We must proceed in three steps. First, we verify that a voter's equilibrium strategy continues to be a non-decreasing function of his intensity of preferences.

Lemma 3. *At any t and for any number of voters n, voter i's equilibrium strategy $v_t^{i*} (K_t, t, n, \sigma_{it})$ is monotonically increasing in $|\sigma_{it}|$.*

The reasoning is identical to that exploited in the case of *2* voters (see the proof of Proposition 2 in the Appendix). The generalization to *n* voters results simply from noticing that the probability of obtaining one' preferred outcome must be monotonically (if possibly weakly) increasing in the number of votes cast, for any *n*, and the proof is not repeated here. Notice that, as in the case of *2*

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[18]The relevant formulas are in the Appendix. It can be shown analytically that the ratio $EV_{T-1}(1,1,1,...,1)/ W_{T-1}(n)$ converges to a number higher than *1* as the number of players becomes arbitrarily large, for any distribution $F(\sigma)$ (Casella and Gelman, in progress).

voters, monotonicity implies that at any t each voter divides the segment *[0,1]* into a finite number of intervals, identified by optimal thresholds $\alpha_v{}^i$. The thresholds remain a function of the state K and of calendar time t.

Second, we derive the probability of obtaining the desired outcome $d_i{}^*$ for player i casting v votes in an arbitrary state K at time t when facing two potential opponents j and z. For any K and for any v, there are only three alternatives that are relevant to i: either his vote is decisive on its own, regardless of the others' preferences; or i obtains the outcome he prefers if and only if he has at least *1* ally; or his vote is irrelevant to the final decision. When i needs at least *1* ally, with i.i.d shocks and symmetrical $F(\sigma)$, the probability of obtaining $d_i{}^*$ equals *3/4* (*1* minus the probability that both opponents agree against i, or *1/4*); as in the case of two voters this is the relevant reference point because it corresponds to the case of non-storable votes. When i's vote is decisive on its own, he obtains his favorite outcome with probability *1*; this occurs if the sum of i's opponents' votes is smaller than v, or if it is equal to v and i wins the coin toss. Finally, i's vote is irrelevant when one of the other players is sure to determine alone the final decision, i.e. when the difference between i's opponents votes is larger than v, or when it is equal to v but the player casting the largest number of votes wins the coin toss. In the special case where $v = 0$, player i is never decisive on its own, but the tie-break rule states that he will always be consulted in case of ties among his opponents. We can then conclude:

Lemma 4. *Consider an arbitrary state K. If player i, facing two potential opponents j and z, casts v votes, the probability that he obtains his preferred decision $d_i{}^*$ equals:*

$$prob(d=d_i^*)=\begin{cases} 3/4 - 1/4\ prob(|v_j - v_z| > v) - 1/8\ prob(|v_j - v_z| = v) + \\ + 1/4\ prob(v_j + v_z < v) + 1/8\ prob(v_j + v_z = v) \qquad \text{if } v \neq 0 \\ \\ 3/4 - 1/4\ prob(|v_j - v_z| > v) \qquad\qquad\qquad\qquad \text{if } v = 0 \end{cases}$$

where v_j and v_z are the number of votes cast be j and z respectively.

Given Lemmas 3 and 4, constructing expected one-period equilibrium payoffs is not difficult. Consider for example symmetrical state *(k,k,k)* at time t. Then:

$$Eg(k_t, k_t, k_t) = \sum_{v=0}^{k} \int_{\alpha_v}^{\alpha_{v+1}} \alpha dF(\sigma) \left[2 prob(d_i^* | v_i = v) - 1 \right] \tag{15}$$

An equivalent equation holds for asymmetrical states, and we can write generally:

$$Eg(k_t, y_t, z_t) =$$

$$W + \int_0^{\alpha_1} \alpha dF(\sigma) \left[-\frac{1}{2} prob(|v_j - v_z| > 0) \right] + \sum_{v=1}^{k} \int_{\alpha_v}^{\alpha_{v+1}} \alpha dF(\sigma) [-\frac{1}{2} prob(|v_j - v_z| > v) - \tag{16}$$

$$-\frac{1}{4} prob(|v_j - v_z| = v) + \frac{1}{2} prob(v_j + v_z < v) + \frac{1}{4} prob(v_j + v_z) = v]$$

where $W = 1/2 \int_0^1 \alpha dF(\sigma)$, the expected payoff with non-storable votes (and $\alpha_{k+1} \equiv 1$).[19]

As in the case of *2* voters, relative to non-storable votes, storable votes shift the probability of obtaining one's desired outcome away from low and towards high realizations of the preference parameter - away from instances where preferences are weaker and towards instances where they are more intense. This is clearly visible in (16): the negative terms are larger the closer to zero are the intervals over which the preference parameter is integrated, while the opposite is true for the positive terms. The fundamental mechanism that delivered welfare gains in the case of *2* voters (and that supports intuitively the idea of storable votes) remains unchanged.

What does change however is the extent of this shift - i.e. the change in the probability of being pivotal. With *3* voters, gains and losses - relative to non-storable votes - require more extreme asymmetry than with *2* voters: while before a voter controlled the final outcome whenever he cast more votes than his opponent, now he must cast more votes than both his opponents combined; while before he had no say if he cast less votes than his opponent, now this only occurs if his votes and those of one of his opponent, combined, are less than the votes cast

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[19]Notice that *W* is unchanged for *n=3* and *n=2*. For arbitrary *n:*

$$W(n) = \int_0^1 \alpha dF(\sigma)(1/2)^{n-1} \left(\frac{n-1}{n-2+I_n}{2} \right) \quad .$$

by the third voter. Two remarks follow immediately. First, the probability of these more extreme events will in general be lower than the corresponding critical probabilities in the case of *2* voters. And since it is the shift in the probability of being pivotal that underlies potential welfare gains, we should expect these gains to be lower than with *2* voters. Second, there is an asymmetry built into the critical probabilities - relative to non-storable votes, a voter increases his chances of being pivotal only when he casts at least as many votes as his two opponents combined, but has no say if *either one* of his two opponents is decisive on his own. In symmetrical states, the ex ante probability of the latter event must then be twice as large than the probability of the former, implying that the negative terms in (16) are larger than the positive terms. But because the number of votes a player chooses to cast depends on the strength of his preferences, in equilibrium the negative terms multiply smaller integrals than the positive ones, and it does not follow that $Eg(k_p,k_p,k_l)$ is smaller than W - indeed typically it is larger.[20] More problematic are asymmetrical states, where the dominant voting behavior of a single voter is not necessarily compensated by correspondingly stronger preferences.

Consider a specific example - the two-period horizon game where all voters start out with a single vote. In the first period, each player votes *0* if his draw of $/\sigma/$ is less than a threshold α_1, and *1* otherwise (by Lemma 3). Following (16), the first period expected payoff is::

$$Eg_{T-1}(1,1,1) = W - \int_0^{\alpha_1} \alpha dF(\sigma)4[1 - F(\alpha_1)][F(\alpha_1) - 1/2] + \int_{\alpha_1}^1 \alpha dF(\sigma)2[F(\alpha_1) - 1/2]^2 \qquad (17)$$

which can be shown to be larger than W for any $\alpha_1 \in (0,1)$ if for instance $F(\sigma)$ is Uniform. But now evaluate expected payoffs in the last period.   The possible states at *T* are: *(1,1,1), (2,2,2), (1,2,2)* and its permutations, and *(1,1,2)* and its permutations. In the last period, players always cast all their available votes. Thus the expected payoffs associated with symmetrical states at *T* must equal W : $Eg_T(1,1,1) = Eg_T(2,2,2) = W$. If the state is *(1,2,2)*, each voter obtains his preferred outcome if at least one of the other voters agrees, an event that occurs with probability *3/4*, exactly as in the case of symmetrical states. Thus $Eg_T(1,2,2) = Eg_T(2,1,2) = Eg_T(2,2,1) = W$.

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[20]In all numerical exercises, modeling $F(\sigma)$ as a Beta distribution with curvature parameter *b* we found $Eg(k_p,k_p,k_l) \geq W$ as long as $b \geq 1$ i.e. unless $F(\sigma)$ is convex and bi-modal, with maxima at the two extremes.

If the state is *(1,1,2)*, however, players with a single vote only obtain their preferred choice with probability *5/8* (if the player with *2* votes agrees - with probability *½* - or if he disagrees, but the two smaller players agree with each other and win the coin toss - with probability *1/4 (1/2) = 1/8*), and the player with *2* votes with probability *7/8* (he wins unless the *2* smaller voters agree with each other and not with him and win the coin toss, with probability *1/4 (1/2)=1/8*). It follows that $Eg_T(1,1,2) = ½ W$ and $Eg_T(2,1,1) = 3/2 W$. This is important. In the case of *2* voters it was possible to show that average expected payoffs in asymmetrical states are never lower than *W* (where, starting at time *0,* each player rationally assigns the same probability to being the weaker or the stronger player). This very strong result does not hold with *3* voters: $1/3(2 \ Eg_T(1,1,2)+ Eg_T(2,1,1)) = 5/6 \ W < W.$

The strong welfare result, and the simple proof, obtained with *2* voters will not carry over now. Whether or not it remains true that storable votes deliver ex ante welfare gains must depend on the exact value of the equilibrium thresholds, a much more difficult result to establish with any generality. In the simple two-period case we are considering here, the question is whether the higher expected payoff at *T-1* is sufficient to compensate for the possibility of a welfare loss at *T*. If $F(\sigma)$ is Uniform, for example, and $\delta=1$, the ratio $EV_{T-1}(1,1,1)/W_{T-1}$ is reproduced in Figure 4, as function of the threshold α_1. As the figure shows, the ratio is larger than *1* if α_1 is larger than *.5*. But, following the same procedure described in the case of *2* players, it is easy to calculate that the equilibrium threshold in this case equals *.364*, yielding a $EV_{T-1}(1,1,1)/W_{T-1}$ ratio of *.99* (the same value shown in Figure 3 for *n=3*).[21]

However it should be clear from our discussion that the negative conclusion does not hold generally - because it depends on the exact value of the threshold, it must depend on the parameters of the model. Possible welfare losses, relative to non-storable votes, arise from those states where asymmetrical strategies that result in one player's dominant power occur with large probabilities, and are not compensated by sufficient differences in strength of preferences. But

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[21] $EV_{T-1}(1,1,1) = Eg_{T-1}(1,1,1) + \delta W\left(1- 4[F(\alpha_1)- 1/2][1- F(\alpha_1)]^2\right)$ where Eg_{T-1} *(1,1,1)* is given by (17). If the *3* players chose the threshold cooperatively, they would set $\alpha_1=.695$ and realize positive expected welfare gains: by reducing the ex ante probability of casting a vote at *T-1*, they would reduce the likelihood of state *(1,1,2)* at *T*. Notice that $\alpha_1=.695$ is not an equilibrium.

the probability of reaching those states, as well as the probability of very asymmetrical strategies given the state, are crucially influenced by the variance of the intensity of preferences and the length of the horizon. The lower the variance, the less likely it is that very asymmetrical states will be reached; and the longer the horizon the higher the option value of a vote, and the less likely that asymmetrical states will translate into very asymmetrical strategies.

To evaluate this intuition, numerical exercises were run with different time horizons and different $F(\sigma)$. $F(\sigma)$ was modeled as a Beta distribution, with the parameter b summarizing its curvature: $b=1$ corresponds to the Uniform; as b increases, the relative probability mass around zero increases and the variance falls. The results are in Figure 5, where the expected value of the storable votes game at time 0, relative to the value with non-storable votes, is plotted for different horizons and different distributions. Two regularities emerge. First, at any horizon, an increase in b is associated with better ex ante welfare properties for storable relative to non-storable votes. As expected, the larger frequence of weak preferences leads players to expect higher vote savings and a lower likelihood of states dominated by a single voter. The final consequence is the improved expected performance of storable votes. Indeed, although the figure does not show it, for b high enough, the ratio EV_0/W_0 is larger than 1 for all T. Second, for $b>1$ storable votes are associated with ex ante welfare gains if the horizon T is longer than a critical value $\overline{T}(b)$, where

$\overline{T}$ is lower the larger is b. For $b=1$, the conclusion might still hold - for $T > 3$ the ratio EV_0/W_0 is monotonically increasing in T, as in the case of higher b's - but the simulated horizon is not long enough to reflect it.[22]

Better welfare properties for longer time horizons pose a commitment problem - as time passes and the end of the game approaches voters may wish to renegotiate. We ignore this aspect

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[22]Three further comments. First, reducing δ (increasing future discounting) raises EV_0/W_0, presumably because the more asymmetrical states are expected to arise at the end of the horizon and are then discounted more. But the effect is not large because it is countered in part by the reduction in vote saving, and hence the higher likelihood of these same states. Second, as the figure shows, EV_0/W_0 reaches a minimum at $T = 3$ for all b's. Although a higher T implies more vote saving, it also multiplies possible asymmetrical states - according to the numerical exercises, the second effect dominates for $T=3$, but becomes relatively less important as T increases further. Finally, multiple equilibria are possible, but only in one case did we in fact find two equilibria: $b=1$ and state *(5,4,3)* at *T-1* (which requires *T=6*). In one equilibrium, voter *i* never plays *1* and player *j* never plays *1* or *2*. In the other equilibrium, player *j* never plays *3* and player *z* never plays *1* or *2*. Although the first equilibrium leads to slightly better welfare properties, the effect washes out almost completely in the calculation of EV_0/W_0. The figure uses the second equilibrium.

here - storable votes are evaluated ex ante at some constitutional stage, and we take the possibility of commitment as granted. This said, the conclusion is pleasing - the very idea of storable votes relies on the possibility of intertemporal trades and we should expect it to yield its potential benefits only if there is enough time for these trades to be possible. The lack of welfare gains with a *2*-period horizon is much less of a concern if longer horizons improve the performance of the mechanism.

V. STORABLE V/S TRADABLE VOTES

As discussed in the Introduction, storable votes inevitably bring to mind vote trading. How related are the two mechanisms? Having ruled out monetary side-payments, the natural comparison is to log-rolling: exchanges of current for future votes. To study the two schemes side by side we make them comparable through two assumptions: first, we ignore the problem of enforcement posed by vote trading in our finite horizon setting - debtors would be sure to renege in the last period, with the usual cascading effect. We posit instead the existence of credible outside enforcement. Second, with vote trading players who buy votes are effectively borrowing against their future voting allocation. It seems appropriate then to allow borrowing also in the case of storable votes. We simply assume that when votes are storable the entire stock of votes is allocated to each player at time *0*, for him to distribute over future decisions as he sees fit.

Begin with the simplest case: *2* voters and *2* periods. With storable votes, each player enters the game with *2* votes. Following the usual steps, it is easy to establish that voter *i* will choose to cast *2* votes if $|\sigma_i| \geq 4\delta W$ and *0* votes otherwise. Although the initial number of votes is different, the outcome is identical to the case studied earlier where each voter was endowed with *1* new vote each period. The expected value of the game is again given by equation (9).

Suppose now that votes are tradable, but not storable. At time *1*, each voter has *1* vote and three options: he can offer to sell his vote now (*S*) (and have *2* votes next period, when his opponent will have *0*, if the trade takes place); do nothing (*N*) (and have *1* vote each period, just like his opponent), or offer to buy a vote (*B*) (and have *0* votes next period, when his opponent will have *2*). Call p_B the probability that a player offers to buy, p_S the probability that he offers to sell, and $p_N = 1 - p_B - p_S$ the probability that he does nothing. Given σ_i, positive for simplicity, the three alternatives lead to expected utilities:

$$Eu_i|S = p_B(\sigma_i 1/2 + \delta 2W) + (1 - p_B)(\sigma_i 3/4 + \delta W)$$
$$Eu_i|N = \sigma_i 3/4 + \delta W \tag{18}$$
$$Eu_i|B = p_S\sigma_i + (1 - p_S)(\sigma_i 3/4 + \delta W)$$

It is possible of course that both voters would want to buy, or both would want to sell. With a longer number of periods, prices would emerge, but in this simple example no voter can offer more (or less) than a *1*-to-*1* exchange between a vote today and a vote tomorrow. Thus if both voters find themselves on the same side of the trade no exchange can be concluded, and this is reflected in (18). Given these equations, it is easy to see that the optimal strategy is to offer to buy a vote whenever $|\sigma_i| \geq 4\delta W$, and offer to sell otherwise. It follows that there is a deviation from the reference case of no-trade only when at time *1* one voter's preference intensities are above the threshold, and the other's below. But this is exactly what happens with storable votes (in all other cases, the players' strategies cancel each other). And because the threshold is the same, it is not surprising then that the expected value of the game is given once again by equation (9). With *2* players and *2* periods, the two voting mechanisms are identical.[23] Indeed, one can conjecture that with *2* voters the result should continue to hold for any arbitrary time horizon *T*.

Will this be true with *3* voters? The question is interesting because it is with more than *2* voters that the inefficiency built into vote trading comes into being. If the severity of the externality differs between tradable and storable votes, it is then that such a difference will appear. Consider a *3*-voters, *2*-period game. With storable votes each player again enters the game with *2* votes. The optimal strategy in the first period is to abstain if $|\sigma_i| < \alpha^{SV}$ and cast *2* votes otherwise, where $\alpha^{SV} \equiv 4\delta W(1-p_0^2)/(1-p_2^2)$, and $p_0 = 1 - p_2 = 2[F(\alpha^{SV}) - 1/2]$ (the superscript *SV* stands for "storable votes"). The expected payoff at *T-1* and the expected value of the game are given by:

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[23]One important caveat. For consistency with the rest of the paper, we are maintaining the information assumption made all along: a voter makes decisions knowing his preferences but not his opponent's. In the case of tradable votes, this can result in trades between voters on the same side of an issue, albeit with different intensities. Whether a voter would want to reveal his true position is an interesting question not pursued here (see for example the discussion in Mueller, 1989).

$$Eg_{T-1}^{SV} = W + \int_{\alpha^{SV}}^{1} \sigma dF(\sigma)\frac{p_0^2}{2} - \int_{0}^{\alpha^{SV}} \sigma dF(\sigma)p_0 p_2$$

$$EV_{T-1}^{SV} = Eg_{T-1}^{SV} + \delta W(1 - p_0 p_2^2)$$

(19)

with the probabilities defined above.

Consider now the case of tradable votes. The horizon continues to be *2* periods, implying that again each player can buy or sell at most *1* vote at *T-1*. As in the case of *2* voters, there are three possible alternatives: each player can offer to sell, offer to buy, or do nothing. When a transaction is proposed, it is concluded only if there is at least another voter who has made the complementary proposal. But now a new difficulty emerges: a voter willing to transact may be shut out of the market because his *2* opponents trade among themselves. Ruling out side-payments, no price and no bargaining can emerge in the *2*-period game, and we assume that if *2* willing buyers, for example, face a single seller, the successful one will be chosen with a coin toss. Taking this into account, expected utilities from any of the three actions can be calculated as usual. Consider for example voter *i,* offering to sell his vote. He succeeds in selling if: (i) both other voters offer to buy (with probability $p_B^{\,2}$); (ii) one offers to buy and one does nothing (with probability $2p_B p_N$); or (iii) one offers to buy, one offers to sell and the coin toss is favorable to *i* (with probability $p_B p_S$). If *i* succeeds, then he is left with *0* votes today in exchange for *2* tomorrow, while his opponents have *2* and *1* votes today and *0* and *1* tomorrow. In both period, the voter who controls *2* votes alone determines the outcome; thus *i*'s expected utility, conditional on succeeding in selling, equals $\sigma_i/2 + 2\delta W$ (with σ_i positive for simplicity). If *i*'s offer is not accepted, that may be either because no-one is interested in buying (with probability $(1-p_B)^2$), in which case no transaction takes place and *i*'s expected utility equals $\sigma_i 3/4 + \delta W$, or because the other *2* voters trade among themselves (with probability $p_B p_S$), in which case *i* carries no weight either this period or the next and his expected utility equals $\sigma_i/2$. Thus we can write:

$$Eu_i|S = (p_B^2 + 2p_B p_N + p_B p_S)(\sigma_i/2 + 2\delta W) + (1 - p_B)^2(\sigma_i 3/4 + \delta W) +$$
$$+ p_B p_S \sigma_i/2$$

(20)

The expected utilities associates with the two remaining alternatives are calculated analogously:

$$Eu_i|B = (p_S^2 + 2p_S p_N + p_B p_S)\sigma_i + (1-p_s)^2(\sigma_i 3/4 + \delta W) + p_B p_S \sigma_i / 2$$
$$Eu_i|N = p_B p_S \sigma_i + (1 - 2 p_B p_S)(\sigma_i 3/4 + \delta W)$$

(20)'

Given these equations it is easy to establish that it is never optimal to do nothing. There is a single relevant threshold α^M such that voter i offers to sell his vote if $|\sigma_i| < \alpha^M$, and offers to buy otherwise (the superscript M stands for "market"), where once again $\alpha^M = 4\delta W$. Expected payoff at T-1 and the expected value of the game are given by:

$$Eg_{T-1}^M = W + \int_{\alpha^M}^1 \sigma dF(\sigma) \frac{p_S^2}{2} - \int_0^{\alpha^M} \sigma dF(\sigma) \frac{(1+p_S)}{2} p_B$$
$$EV_{T-1}^M = Eg_{T-1}^M + \delta W(1 - p_S p_B)$$

(21)

where $p_S = 1 - p_B = 2[F(\alpha^M) - 1/2]$.

Comparing equations (19) and (21) is very instructive. It is particularly easy when $F(\sigma)$ is Uniform, because in that case $\alpha^{SV} = \alpha^M = .5$, and $p_0 = p_S$, $p_2 = p_B$. The expected value of the game is unequivocally lower with tradable than with storable votes, a result that arises because expected payoffs are lower in both periods. And the reason is simple: tradable votes require two sides for a trade: with 3 voters, the buyer guarantees himself control over the public decision in the first period, and the seller in the second. The third voter, excluded from the transaction, has no voice in either periods. With storable votes, on the other hand, each voter decides his allocation of votes on his own. If a voter decides to abstain in the first period, neither one of his opponents experiences a decline in the probability of obtaining the desired outcome in that period; nor can the abstaining voter be sure of controlling the public decision in period 2. In other words, the externality is less severe with storable than with tradable votes. The intuition seems robust: the welfare results are unchanged for all $F(\sigma)$ we have tried.[24] The probability of being rationed when votes are tradable remains positive for all finite number of players, and although the game will be more complicated, the logic seems unchanged. Similarly, the emergence of prices when trading occurs over a longer horizon should be matched by an equivalent flexibility

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[24] As b increases, α^M becomes larger than α^{SV}, implying that the ex ante probability of putting one's vote up for sale when votes are tradable increases more than the probability of abstaining.

in the intertemporal program with storable votes. We cannot draw general conclusions at this point, but there is no obvious reason why the welfare results should be reversed.

VI. CONCLUSIONS

We have discussed a very simple - indeed natural - voting mechanism for committees that meet repeatedly over time: voters are allowed to store their votes and shift them intertemporally. As a result, voters cast more votes when their preferences are more intense, and the probability of obtaining their preferred decision shifts from times when preferences are weaker to times when they are stronger. Relative to non-storable votes, ex ante welfare rises.

This transparent intuition appears clearly, and can be proven rigorously, in the case of *2* voters. When the number of voters is larger, some complications arise but the analysis of the paper suggests that the conclusion should continue to hold if one of the following conditions is satisfied: (i) the number of voters is above a minimum threshold; (ii) preferences are not too polarized; (iii) the horizon is long enough.

The model studied here is very simple, and some of its restrictive assumptions will have to be relaxed before the promise of the voting scheme can be confirmed. Some needed extensions are immediate generalizations of this initial model. The importance of the horizon length suggests allowing for infinite horizon, keeping the analysis tractable, for example, by having votes expire after a fixed number of periods, or by studying overlapping generations of committee members with fixed terms. Different information assumptions should be studied - what if opponents' preferences are known? What if at least their signs, if not their intensities, are known? In most applications, correlation among preferences should be allowed, either over time or across voters.

Other extensions require introducing new issues. How robust are the results to endogenous agenda? Because storable votes derive their value from intertemporal planning, influencing the order in which votes will be called ought to be very important. An individual or a group controlling the agenda might be able, for example, to exhaust opponents' votes before an issue he considers crucial is decided. But the opposite could be argued too - the ability to shift votes intertemporally provides everybody with more flexibility and might in fact neutralize the

advantage enjoyed by those who set the agenda.

A related if different question is the impact of storable votes on minorities. Advocates of cumulative voting, the static counterpart of storable votes, have stressed their potential for increasing the power of minorities (Guinier, 1994), an observation confirmed at least partially by formal and experimental analyses (Cox, 1990, Gerber, Morton and Rietz, 1998). Others have expressed concern that when voting is costly the voters most likely to express their votes might be those with most extreme preferences (Campbell, 1999, Osborne, Rosenthal and Turner, 2000). Would decisions be dominated by extremists? What would the welfare implications be then? Notice that the outcome is not obvious: when votes are storable, the cost of voting is endogenous and the majority can control a small minority at relatively low cost, if the coordination problem is not too severe.

Finally, we have maintained the assumption that aggregating voters' preferences is made difficult by their divergence. Alternatively, we could model the voting problem as a common value problem: voters have the same preferences but receive different signals about the optimal choice (for example Feddersen and Pesendorfer, 1997, Piketty, 1999). Piketty (1994) has argued that in this case market-type mechanisms applied to voting, in particular spot markets for votes, are less efficient than simple majority voting, because they induce abstentions and thus reduce the amount of information transmitted through voting. To what extent would this argument apply to storable votes? What if both private and common values are present?

These questions are important and difficult, and will need to be addressed. For now, we can only conclude by stating once more that storable votes are not the most efficient mechanism theoretically possible. But they are very simple, could realistically be implemented and appear to take us part of the way towards efficiency without violating our ethical priors. They deserve more study.

APPENDIX

Proof of Proposition 2. *i) Existence.* Formally, we are looking for a perfect Bayesian equilibrium of a multi-stage game. An important simplification is that players' types are i.i.d. across different periods: the game has no updating of information on players' types, and if we restrict our focus to Markov strategies the only intertemporal link across periods is the evolution of the state variable - the accumulation or depletion of the votes' stock (which is common knowledge). It follows that we can find a perfect equilibrium by backward induction. In period *T*, the dominant strategy is to cast all remaining votes. In period *T-1*, given the state K_{T-1} the continuation value of the game depends only on the strategies at *T-1* (and on the expected value of $|\sigma_T| = 2W$, an exogenous parameter); the one-period payoff depends on the realization of one's own type $\sigma_{i,T-1}$, and on the strategies at *T-1*; thus, given the state, the total payoff of the game at *T-1*, $EU_{T-1}(v_{T-1}; \sigma_{i,T-1}, K_{T-1})$, depends only on current strategies. We can study the game at *T-1* as a one-stage simultaneous move game. Notice that the game satisfies a number of conditions: (i) *j*'s type does not enter *i*'s payoff directly (*j*'s strategy does); (ii) players' types are independently distributed; (iii) $F(\sigma)$ is assumed to be continuous and atomless; (iv) for each player, the strategies' space is finite. By using the notion of distributional strategies - joint distributions on actions and types - Milgrom and Weber (1985) have shown that these conditions guarantee that an equilibrium exists, and that all equilibrium strategies are empirically indistinguishable from pure strategies. (See also Fudenberg and Tirole, 1992, section 6.8). But if the game at *T-1* has an equilibrium, then equilibrium strategies at *T-1* can be anticipated, as function of the types' realizations and the state at *T-1*. Again, expected types' realizations are exogenous and K_{T-1} is determined completely by state and strategies at *T-2*. Hence, given $\sigma_{i,T-2}$ and K_{T-2}, we can study the game at *T-2* as a one-stage game and rely once more on Milgrom and Weber's result. With a finite horizon *T*, the complete game has a finite number of stages and states, and using backward induction we can replicate the procedure for each of them. ∎

ii) Monotonicity. Suppose $\sigma_{i\,t} > 0$. When casting *v* votes, player *i*'s expected utility at *t* is given by: $\sigma_{i\,t}\,prob(d_t=1\,/v_t^i=v)+\delta EV_{t+1}(k_t^i-v+1,k_t^j-v_t^j+1)$, where $prob(d_t=1\,/v_t^i=v) = [prob(v_t^j<v) + 3/4\,prob(v_t^j=v) + \frac{1}{2}\,prob(v_t^j>v)]$. Notice that $prob(d_t=1\,/v_t^i=v+1) - prob(d_t=1\,/v_t^i=v) = 1/4$

$[prob(v_t^j=v)+prob(v_t^j=v+1)] \geq 0$: as expected $prob(d_t=1/v_t^i=v)$ is monotonically increasing in v. Call v' (v'') the equilibrium number of votes cast by player i when $\sigma_{it} = \sigma'$ (σ'') (with $\sigma'>\sigma''>0$). By definition of equilibrium, the following two inequalities must hold:

$$\sigma' \, prob(d_t = 1|v') + \delta EV_{t+1}(k_t^i - v'+1, k_t^j - v^j + 1) \geq$$
$$\sigma' \, prob(d_t = 1|v'') + \delta EV_{t+1}(k_t^i - v''+1, k_t^j - v^j + 1)$$

$$\sigma'' \, prob(d_t = 1|v'') + \delta EV_{t+1}(k_t^i - v''+1, k_t^j - v^j + 1) \geq$$
$$\sigma'' \, prob(d_t = 1|v') + \delta EV_{t+1}(k_t^i - v'+1, k_t^j - v^j + 1)$$

Adding the two inequalities, we obtain:

$$(\sigma'-\sigma'')[\, prob(d_t = 1|v') - prob(d_t = 1|v'')] \geq 0$$

But with $\sigma' > \sigma''$ and $prob(d_t=1/v)$ monotonically increasing in v, this implies $v' \geq v''$, establishing the result. The logic is identical, with the appropriate sign changes, for $\sigma_{it} < 0$ Notice that the proof holds for any strategy chosen by j, implying that all best response functions must be monotonically increasing, a stronger result than stated in the Proposition. ■

Proof of Lemma 1. Begin by proving part *i)* for symmetrical states. Recall that expected one-period equilibrium payoff is given by (13), reflecting the optimal thresholds chosen by the voters. Define a function $\Psi(\alpha_v, ..., \alpha_k)$, representing (fictional) expected payoff when thresholds α_1 ,..., α_{v-1} are set to zero, and all other thresholds are kept at their equilibrium values (and where to simplify notation we ignore the time subscript). By construction $\Psi(\alpha_1, ..., \alpha_k) = Eg(k,k)$. We can show that the following two conditions hold:

   a) $\Psi(\alpha_k) \geq W$
   b) $\Psi(\alpha_v,....,\alpha_k) \geq \Psi(\alpha_{v+1},....,\alpha_k)$

To establish *a)*, notice that given (12) and the definition of $\Psi(\alpha_k)$, we can write:

$$\Psi(\alpha_k) \equiv \int_0^{\alpha_k} \sigma dF(\sigma)\big(F(\alpha_k)-1/2\big) + \int_{a_k}^1 \sigma dF(\sigma)F(\alpha_k)$$

where $\alpha_k \in [0,1]$. At $\alpha_k = 0$ or $\alpha_k = 1$, $\Psi(\alpha_k) = W$; in addition it is easy to verify that $\partial \Psi(\alpha_k)/\partial \alpha_k$ is positive at $\alpha_k = 0$ and has a single root in the interval $\alpha_k \in (0,1]$. Hence $\Psi(\alpha_k) > W \; \forall \, \alpha_k \in (0,1)$

and $\Psi(\alpha_k) = W$ *if* $\alpha_k = 0$ *or 1*. But from (12) we also know:

$$\Psi(\alpha_v,...,\alpha_k) \geq \Psi(\alpha_{v+1},...,\alpha_k) \Leftrightarrow$$

$$\int_0^{\alpha_v} \sigma dF(\sigma)\big(F(\alpha_v) - 1/2\big) + \int_{\alpha_v}^{\alpha_{v+1}} \sigma dF(\sigma)\big(F(\alpha_v) + F(\alpha_{v+1}) - 1\big) \geq \int_0^{\alpha_{v+1}} \sigma dF(\sigma)\big(F(\alpha_{v+1}) - 1/2\big) \quad \text{(A1)}$$

The left-hand side of (A1) is identical to the right-hand side if $\alpha_v = 0$ or $\alpha_v = \alpha_{v+1}$. At $\alpha_v = 0$, the left-hand side is increasing in α_v and again it can easily be shown that the derivative has a single root. Hence $\Psi(\alpha_v, ..., \alpha_k) > \Psi(\alpha_{v+1}, ..., \alpha_k) \; \forall \alpha_v \in (0, \alpha_{v+1})$ and $\Psi(\alpha_v, ..., \alpha_k) = \Psi(\alpha_{v+1}, ..., \alpha_k)$ for $\alpha_v = 0$ or $\alpha_v = \alpha_{v+1}$, and *b)* is established.

Finally, it follows that *a)* and *b)* can both hold with equality only if all thresholds are either 0 or *1* or if there exist an α_v and α_{v+1} such that $\alpha_1 = ... = \alpha_v = 0$ and $\alpha_{v+1} = ... = \alpha_k = 1$, with $v \in \{1,...,k-1\}$, i.e. only if the same strategy is followed for all realizations of σ_i. If at least one threshold is strictly between *0* and *1*, then the inequality in part *i)* of Lemma 1 is strict. We expect that to be the case in all symmetrical states, but it is particularly easy, and sufficient for our purposes, to show that this must be true at *T-1*. Suppose both players are endowed with k votes at *T-1*. We show that casting v votes for all realizations of σ_i cannot be an equilibrium. Given σ_{iT-1}, which we suppose positive for simplicity, the expected utility of voter i casting v votes is given by: $EU^i_{T-1}(v) = \sigma_{iT-1} [prob(v_t^j < v) + 3/4 \, prob(v_t^j = v) + \frac{1}{2} \, prob(v_t^j > v)] + \delta W [prob(v_t^j = v) + 2prob(v_t^j > v)]$. It is easy to establish then that:

$$EU^i_{T-1}(v'+1) - EU^i_{T-1}(v') = (\sigma_{iT-1}/4 - \delta W) [prob(v_t^j = v'+1) + prob(v_t^j = v')]$$
$$EU^i_{T-1}(v') - EU^i_{T-1}(v'-1) = (\sigma_{iT-1}/4 - \delta W) [prob(v_t^j = v') + prob(v_t^j = v'-1)]$$

Recall that $4\delta W \in (0,1)$. Take any v' such that $prob(v_t^j = v') > 0$. It is immediate to show that if $\sigma_i < 4\delta W$, then player i must prefer $v'-1$ to v', and if $\sigma_i > 4\delta W$ he must prefer $v'+1$ to v. If v' equals *0* or k only one direction of deviation is feasible, but in all cases no v can be the equilibrium strategy for all σ_i. Part *i)* of lemma 1 is established. ∎

ii) To establish part *ii)* of the lemma for asymmetrical states, we follow the same logic. Suppose $s > k$, and denote $\{\gamma_1, \gamma_2,..., \gamma_{k+1}\}$ the equilibrium thresholds for the player holding s votes at t, and $\{\beta_1, \beta_2, ...,\beta_k\}$ the equilibrium thresholds for the player holding k votes (where,

again, to simplify notation time subscripts are omitted). Notice that the player holding s votes will never cast more than $k+1$. We can write:

$$Eg^i(k^i,s^j) + Eg^i(s^i,k^j) = \int_0^{\beta_1} \alpha dF(\sigma)\left(F(\gamma_1)-1/2\right) + \int_{\beta_1}^{\beta_2} \alpha dF(\sigma)\left(F(\gamma_1)+F(\gamma_2)-1\right)+..$$

$$...+\int_{\beta_k}^1 \alpha dF(\sigma)\left(F(\gamma_k)+F(\gamma_{k+1})-1\right) + \int_0^{\gamma_1} \alpha dF(\sigma)\left(F(\beta_1)-1/2\right) + \int_{\gamma_1}^{\gamma_2} \alpha dF(\sigma)\left(F(\beta_1)+F(\beta_2)-1\right)+....$$

$$...+\int_{\gamma_v}^{\gamma_{v+1}} \alpha dF(\sigma)\left(F(\beta_v)+F(\beta_{v+1})-1\right)+.....+\int_{\gamma_k}^{\gamma_{k+1}} \alpha dF(\sigma)\left(F(\beta_k)\right) + \int_{\gamma_{k+1}}^1 \alpha dF(\sigma) \qquad (A2)$$

Define a function $\Psi(\beta_1,...,\beta_v,\gamma_1,..,\gamma_{v+1})$, representing the (fictional) sum of expected payoffs in (A2) when thresholds $\beta_{v+1},...,\beta_{k-1}, \gamma_{v+2},.., \gamma_{k+1}$ are set to 1 (which is now more convenient than setting the omitted thresholds to 0), and all other thresholds are kept at their equilibrium values. By construction, $\Psi(\beta_1,..,\beta_k,\gamma_1,..,\gamma_{k+1}) \equiv Eg^i(s^i,k^j)+Eg^i(k^i,s^j)$. As in the symmetrical case, we proceed by first evaluating the function Ψ at the smallest threshold and then adding successively higher ones. We can show that the following three conditions hold:

   a) $\Psi(\gamma_1) \geq 2W;\ \ \Psi(\beta_1) \geq 2W$
   b) $\Psi(\gamma_1,\beta_1) \geq \Psi(\gamma_1);\ \ \Psi(\gamma_1,\beta_1) \geq \Psi(\beta_1)$
   c)   (i) $\Psi(\beta_1,...,\beta_{v+1},\gamma_1,...,\gamma_m) \geq \Psi(\beta_1,...,\beta_v,\gamma_1,...,\gamma_m)$ where either $\beta_{v+1} > \gamma_m$ or $\gamma_m = 1$
        (ii) $\Psi(\beta_1,...,\beta_v,\gamma_1,...,\gamma_{m+1}) \geq \Psi(\beta_1,...,\beta_v,\gamma_1,...,\gamma_m)$ where either $\gamma_{m+1} > \beta_v$ or $\beta_v = 1$

.

To verify a), notice that given (A2) we can write:

$$\Psi(\gamma_1) \equiv \int_0^1 \alpha dF(\sigma)\left(F(\gamma_1)-1/2\right) + \int_0^{\gamma_1} \alpha dF(\sigma)1/2 + \int_{\gamma_1}^1 \alpha dF(\sigma)$$

Differentiating $\Psi(\gamma_1)$ with respect to γ_1, it is easy to see that $\Psi(\gamma_1) > 2W\ \forall \gamma_1 \in (0,1)$ and $\Psi(\gamma_1) = 2W$ if $\gamma_1 = 0$ or 1.   The same reasoning, and a corresponding equation, establish $\Psi(\beta_1) \geq 2W$. To verify b), notice that from (A2) we also know:

$$\Psi(\beta_1,\gamma_1) \geq \Psi(\gamma_1) \Leftrightarrow \int_{0_1}^1 \alpha dF(\sigma)\left(F(\beta_1)-1/2\right) \geq \int_0^{\beta_1} \alpha dF(\sigma)1/2$$

an inequality that holds strictly for all $\beta_1 \in (0,1)$, and weakly at $\beta_1 = 0$ or1. Again an equivalent

condition establishes $\Psi(\beta_l, \gamma_l) \geq \Psi(\beta_l)$.

Finally, to verify *c)*, consider case (i) first. We can derive from (A2):

$$\Psi(\beta_1,...,\beta_{\nu+1},\gamma_1,...,\gamma_m) \geq \Psi(\beta_1,...,\beta_\nu,\gamma_1,...,\gamma_m) \Leftrightarrow$$

$$\int\limits_{\beta_{\nu+1}}^{1} \sigma dF(\sigma)(F(\gamma_{\nu+2})-F(\gamma_\nu)) + \int\limits_{\gamma_\nu}^{\gamma_{\nu+2}} \sigma dF(\sigma)(F(\beta_{\nu+1})-1) \geq 0 \qquad (A3)$$

where $\gamma_{\nu+2} \leq \gamma_m$ and hence either $\gamma_{\nu+2} \leq \beta_{\nu+1}$ or $\gamma_{\nu+2}=1$ (and similarly either $\gamma_\nu \leq \beta_{\nu+1}$ or $\gamma_\nu =1$,

where $\gamma_\nu \leq \gamma_{\nu+2}$). At $\gamma_{\nu+2}=1$, the inequality in (A3) becomes:

$$\int\limits_{\beta_{\nu+1}}^{1} \sigma dF(\sigma)(1- F(\gamma_\nu)) - \int\limits_{\gamma_\nu}^{1} \sigma dF(\sigma)(1- F(\beta_{\nu+1})) \geq 0$$

a condition that is satisfied for all $\gamma_\nu \leq \beta_{\nu+1}$ or at $\gamma_\nu =1$. If $\gamma_{\nu+2} \neq 1$, notice that (A3) holds with

equality at $\gamma_{\nu+2} = \gamma_\nu$ and becomes:

$$\int\limits_{\beta_{\nu+1}}^{1} \sigma dF(\sigma)(F(\beta_{\nu+1})- F(\gamma_\nu)) - \int\limits_{\gamma_\nu}^{\beta_{\nu+1}} \sigma dF(\sigma)(1- F(\beta_{\nu+1})) \geq 0$$

at $\gamma_{\nu+2} = \beta_{\nu+1}$, an inequality that is satisfied for all $\beta_{\nu+1} \geq \gamma_\nu$. But differentiating (A3) with respect

to $\gamma_{\nu+2}$, we can easily establish that for all $\beta_{\nu+1} < 1$ the derivative has a unique root which must be

a maximum. Hence if the inequality is satisfied at $\gamma_{\nu+2} = \gamma_\nu$ and at $\gamma_{\nu+2} = \beta_{\nu+1}$, it must be satisfied

everywhere. For $\beta_{\nu+1} = 1$, (A3) equals *0* for any $\gamma_{\nu+2}$, γ_ν. We conclude that condition *b)* (i) is

established. The proof of condition *b)* (ii) proceeds identically, and we leave it to the reader. ∎

Proof of Lemma 2. Once again, we proceed in two steps. First, we consider symmetrical states

and show that if the conditions in Lemma 2 hold, then $EV_t^i(k_t,k_t) > W_t^i(k,k)$. Then we prove the

corresponding result for asymmetrical states. What follows could be written in matrix form, but

the expanded notation, though cumbersome, is more transparent and is maintained here.

Consider state (k_t, k_t). We can write:

$$EV_t^i(k_t,k_t) = Eg_t^i(k_t,k_t) + \delta$$

$$[p_{i0}(p_{j0}EV_{t+1}^i(k_t+1,k_t+1) + p_{j1}EV_{t+1}^i(k_t+1,k_t)+....+p_{jk}EV_{t+1}^i(k_t+1,1)) +$$

$$+ p_{i1}(p_{j0}EV_{t+1}^i(k_t,k_t+1) + p_{j1}EV_{t+1}^i(k_t,k_t)+....+p_{jk}EV_{t+1}^i(k_t,1)) +.......$$

$$.................+p_{ik}(p_{j0}EV_{t+1}^i(1,k_t+1) + p_{j1}EV_{t+1}^i(1,k_t)+....+p_{jk}EV_{t+1}^i(1,1))]$$

or, more compactly:

$$EV_t^i(k_t, k_t) = Eg_t^i(k_t, k_t) + \delta\left[\sum_{v_i=0}^{k} p_{iv_i}\left(\sum_{v_j=0}^{k} p_{jv_j} EV_{t+1}^i(k_t - v^i + 1, k_t - v^j + 1)\right)\right] \quad (A4)$$

where p_{jv} is the probability that $|\sigma_{j,t}|$ falls into the interval that corresponds to j's optimal strategy v. But the game is symmetric, and starting from the symmetrical state (k_t, k_t), $p_{iv} = p_{jv}$ for all v. We can thus collect terms and rewrite (A4) as:

$$EV_t^i(k_t^i, k_t^j) = Eg_t^i(k_t, k_t) + \delta$$
$$[\sum_{v=0}^{k-1}\left[p_{iv}\sum_{r=v+1}^{k} p_{jr}\left(EV_{t+1}^i(k - v + 1, k - r + 1) + EV_{t+1}^i(k - r + 1, k - v + 1)\right)\right] +$$
$$+ \sum_{v=0}^{k} p_{iv}p_{jv} EV_{t+1}^i(k - v + 1, k - v + 1)] \quad (A5)$$

Substituting the conditions stated in Lemma 2, we then obtain:

$$EV_t^i(k_t, k_t) > Eg_t^i(k_t, k_t) + \delta W_{t+1}\left[2\sum_{v=0}^{k-1}\sum_{r=v+1}^{k} p_{iv}p_{jr} + \sum_{v=0}^{k} p_{iv}p_{jv}\right]$$

Once again using $p_{iv}p_{jr} = p_{ir}p_{jv}$, it is not difficult to verify that the probabilities, which span all possible equilibrium strategies, sum up to 1. Hence :

$$EV_t^i(k_t, k_t) > Eg_t^i(k_t, k_t) + \delta W_{t+1}$$

But we know by Lemma 1 that $Eg_t^i(k_t, k_t) \geq W$. Hence $EV_t^i(k_t, k_t) > W_t(k, k)$, and the first part of Lemma 2 is established.

The logic of the proof is identical in the asymmetrical state (s_t^i, k_t^j). We can write:

$$EV_t^i(s_t^i, k_t^j) + EV_t^i(k_t^i, s_t^j) = Eg_t^i(s_t^i, k_t^j) + Eg_t^i(k_t^i, s_t^j) + \delta$$
$$[p_{i0}(s_t^i, k_t^j)\left(p_{j0}(s_t^i, k_t^j)EV_{t+1}^i(s_t^i + 1, k_t^j + 1) + \ldots + p_{jk}(s_t^i, k_t^j)EV_{t+1}^i(s_t^i + 1, 1)\right) + \ldots$$
$$\ldots + p_{is}(s_t^i, k_t^j)\left(p_{j0}(s_t^i, k_t^j)EV_{t+1}^i(1, k_t^j + 1) + \ldots + p_{jk}(s_t^i, k_t^j)EV_{t+1}^i(1, 1)\right)] + \delta \quad (A6)$$
$$[p_{i0}(k_t^i, s_t^j)\left(p_{j0}(k_t^i, s_t^j)EV_{t+1}^i(k_t^i + 1, s_t^j + 1) + \ldots + p_{js}(k_t^i, s_t^j)EV_{t+1}^i(k_t^i + 1, 1)\right) + \ldots$$
$$\ldots + p_{ik}(k_t^i, s_t^j)\left(p_{j0}(k_t^i, s_t^j)EV_{t+1}^i(1, s_t^j + 1) + \ldots + p_{js}(k_t^i, s_t^j)EV_{t+1}^i(1, 1)\right)]$$

As always, the probability that a given strategy is chosen by either player is a function of the state; and since we are considering two different states this dependence is recognized explicitly. Using $p_{iv}(s_t{}^i, k_t{}^j) = p_{jv}(k_t{}^i, s_t{}^j)$ $\forall v, s, k, t$, we can simplify (A6):

$$EV_t{}^i(s_t{}^i, k_t{}^j) + EV_t{}^i(k_t{}^i, s_t{}^j) = Eg_t{}^i(s_t{}^i, k_t{}^j) + Eg_t{}^i(k_t{}^i, s_t{}^j) + \delta$$

$$[p_{i0}(s_t{}^i, k_t{}^j)\sum_{v=0}^{k} p_{jv}(s_t{}^i, k_t{}^j)\left(EV_{t+1}{}^i(s_t{}^i + 1, k_t{}^j - v + 1) + EV_{t+1}{}^i(k_t{}^i - v + 1, s_t{}^j + 1)\right) + ..$$

$$...+ p_{is}(s_t{}^i, k_t{}^j)\sum_{v=0}^{k} p_{jv}(s_t{}^i, k_t{}^j)\left(EV_{t+1}{}^i(1, k_t{}^j - v + 1) + EV_{t+1}{}^i(k_t{}^i - v + 1, 1)\right)]$$

(A7)

More compactly, we can write:

$$EV_t{}^i(s_t{}^i, k_t{}^j) + EV_t{}^i(k_t{}^i, s_t{}^j) = Eg_t{}^i(s_t{}^i, k_t{}^j) + Eg_t{}^i(k_t{}^i, s_t{}^j) + \delta$$

$$\sum_{r=0}^{s} p_{ir}\sum_{v=0}^{k} p_{jv}\left(EV_{t+1}{}^i(s_t{}^i - r + 1, k_t{}^j - v + 1) + EV_{t+1}{}^i(k_t{}^i - v + 1, s_t{}^j - r + 1)\right)$$

(A8)

(All probabilities are now conditional on the same state, and in absence of ambiguity the notation is simplified). Substituting the conditions in Lemma 2, we then derive:

$$EV_t{}^i(s_t{}^i, k_t{}^j) + EV_t{}^i(k_t{}^i, s_t{}^j) \geq Eg_t{}^i(s_t{}^i, k_t{}^j) + Eg_t{}^i(k_t{}^i, s_t{}^j) + 2\delta W_{t+1}\sum_{r=0}^{s}\sum_{v=0}^{k} p_{ir} p_{jv}$$

It is easy to verify that the probabilities sum up to *1*. Hence:

$$EV_t{}^i(s_t{}^i, k_t{}^j) + EV_t{}^i(k_t{}^i, s_t{}^j) \geq \left(Eg_t{}^i(s_t{}^i, k_t{}^j) + \delta W_{t+1}\right) +$$
$$+ \left(Eg_t{}^i(k_t{}^i, s_t{}^j) + \delta W_{t+1}\right)$$

But by Lemma 1, the second part of Lemma 2 is then established. ∎

Derivation of Figure 2*:* When every voter casts *1* vote, the probability of obtaining one's preferred decision equals the probability that at least *(n-1)/2* other members agree, if *n* is odd; and the probability that either *n/2* other members agree, or that exactly *(n/2)-1* do and the tie-break is favorable, if *n* is even. Or:

$$1/2\left[1+(1/2)^{n-1}\begin{pmatrix}n-1\\ \dfrac{n-2+I_n}{2}\end{pmatrix}\right] \qquad I_n\equiv1 \text{ for } n \text{ odd, and } 0 \text{ for } n \text{ even.}$$

When voter i alone controls 2 votes, he needs the agreement of at least $(n/2)+1$ others if n is even, and of at least $(n-1)/2$ or exactly $(n-3)/2$ plus a favorable tie-break, if n is odd. The probability of i's preferred outcome is then:

$$\left[1/2+(1/2)^{n-1}\begin{pmatrix}n-1\\ \dfrac{n-2}{2}\end{pmatrix}\right] \text{ for } n \text{ even;} \qquad 1/2\left[1+(1/2)^{n-1}\begin{pmatrix}n-1\\ \dfrac{n-3}{2}\end{pmatrix}\dfrac{2n}{n-1}\right] \text{ for } n \text{ odd.}$$

It is not difficult to see that the probability of winning goes from $3/4$ to $7/8$ when n is 3, a percentage increase of $1/6$; and from $11/16$ to $13/16$ when n is 5, a percentage increase of $2/11$.

Derivation of Figure 3: When votes are storable and all players can either cast 1 vote or abstain, the probability of obtaining one's preferred outcome when voting equals:

$$\sum_{x=0}^{n-1} p_1^x p_0^{n-x-1}\begin{pmatrix}n-1\\ x\end{pmatrix}\left[1/2\left(1+(1/2)^x\begin{pmatrix}x\\ \dfrac{x-I_x}{2}\end{pmatrix}\right)\right] \qquad I_x\equiv1 \text{ for } x \text{ odd, and } 0 \text{ for } x \text{ even.}$$

The corresponding probability when abstaining equals:

$$\sum_{x=0}^{n-1} p_1^x p_0^{n-x-1}\begin{pmatrix}n-1\\ x\end{pmatrix}\left[1/2\left(1+(1-I_x)(1/2)^{n-1}\begin{pmatrix}x\\ \dfrac{x}{2}\end{pmatrix}\begin{pmatrix}n-x-1\\ \dfrac{n-x-2+I_n}{2}\end{pmatrix}\right)\right] \qquad I_n\equiv1 \text{ for } n \text{ odd, and } 0 \text{ for } n \text{ even.}$$

As stated in Lemma 3, the probabilities $p_1(n)$ and $p_0(n)$ continue to depend on a threshold $\alpha(n)$ such that $p_1(n) = 2[1-F(\alpha(n))]$ and $p_0(n) = 2[F(\alpha(n))-1/2]$. On the basis of these equations, it is possible to derive expected payoffs and the expected value of the game.

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Figure 1

2 players, 2 periods
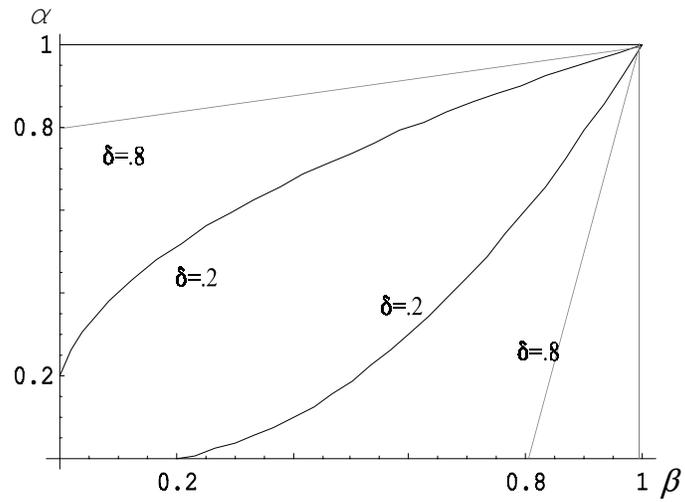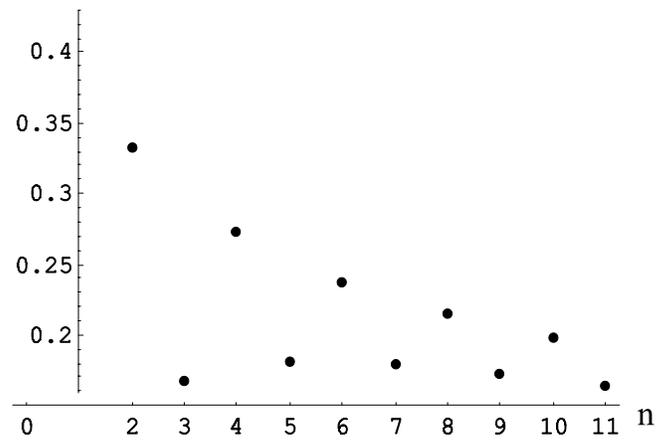*Acceptable thresholds range*
F(σ) Uniform

Figure 2

One period game.
The marginal value of an extra vote

Percentage increase in the probability of obtaining the desired outcome from holding 2 votes when everybody else has a single one, as function of the total number of voters.
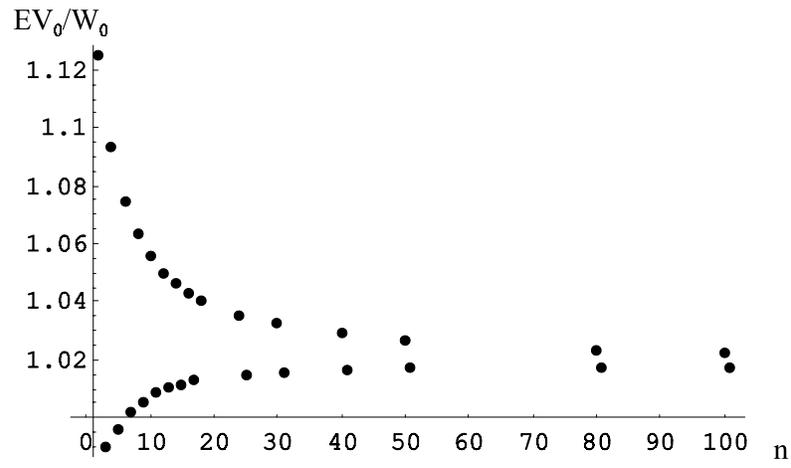
Figure 3

2 periods, n players
$F(\sigma)$*Uniform;* $\delta=1$

Figure 4

3 players, 2 periods.
F(σ)Uniform; δ=1
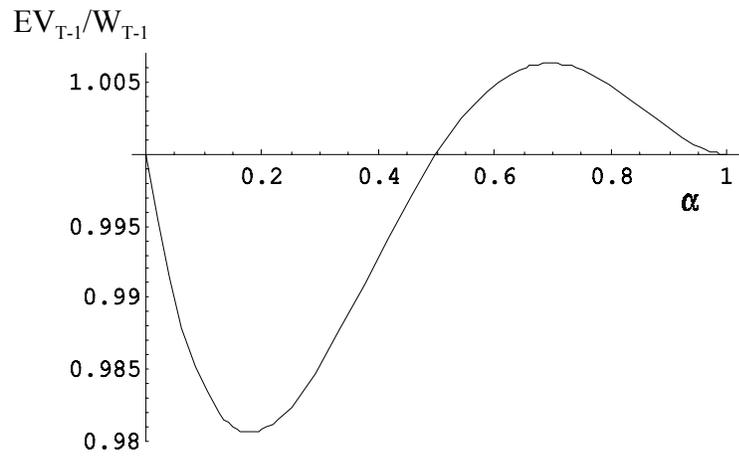
Figure 5

3 players, T periods
F(σ) Beta; δ=1

EV$_0$/W$_0$

1.01

b=3

b=2

3

4

5

6

T

0.99

0.98

b=1
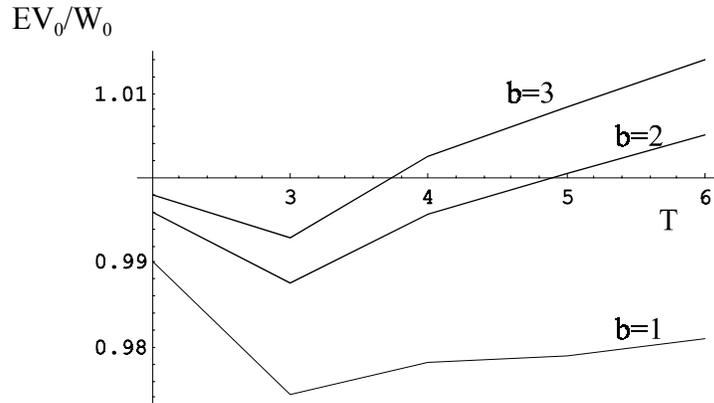
Beta distribution: $f(\sigma) = \dfrac{1}{\int\limits_{-1}^{1}(1-\sigma^2)^{b-1}d\sigma}(1-\sigma^2)^{b-1}$

b=3

0.8

b=1

0.6

b=2

0.4

0.2

−1

−0.5

0.5

1