Essays on Approximation Algorithms for Robust Linear Optimization Problems

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ABSTRACT

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Solving optimization problems under uncertainty has been an important topic since the appearance of mathematical optimization in the mid 19th century. George Dantzig's 1955 paper, "Linear Programming under Uncertainty" is considered one of the ten most influential papers in Management Science [27]. The methodology introduced in Dantzig's paper is named stochastic programming, since it assumes an underlying probability distribution of the uncertain input parameters. However, stochastic programming suffers from the "curse of dimensionality", and knowing the exact distribution of the input parameter may not be realistic. On the other hand, robust optimization models the uncertainty using a deterministic uncertainty set. The goal is to optimize the worst-case scenario from the uncertainty set. In recent years, many studies in robust optimization have been conducted and we refer the reader to Ben-Tal and Nemirovski [4–6], El Ghaoui and Lebret [20], Bertsimas and Sim [15, 16], Goldfarb and Iyengar [24], Bertsimas et al. [8] for a review of robust optimization. Computing an optimal adjustable (or dynamic) solution to a robust optimization problem is generally hard. This motivates us to study the hardness of approximation of the problem and provide efficient approximation algorithms. In this dissertation, we consider adjustable robust linear optimization problems with packing and covering formulations and their approximation algorithms. In particular, we study the performances of static solution and affine solution as approximations for the adjustable robust problem.

Chapter 2 and 3 consider two-stage adjustable robust linear packing problem with uncertain

second-stage constraint coefficients. For general convex, compact and down-monotone uncertainty sets, the problem is often intractable since it requires to compute a solution for all possible realizations of uncertain parameters [23]. In particular, for a fairly general class of uncertainty sets, we show that the two-stage adjustable robust problem is NP-hard to approximate within a factor that is better than $\Omega(\log n)$, where n is the number of columns of the uncertain coefficient matrix. On the other hand, a static solution is a single (here and now) solution that is feasible for all possible realizations of the uncertain parameters and can be computed efficiently. We study the performance of static solution as an approximation for the adjustable robust problem and relate its optimality to a transformation of the uncertain set. With this transformation, we show that for a fairly general class of uncertainty sets, static solution is optimal for the adjustable robust problem. This is surprising since the static solution is widely perceived as highly conservative. Moreover, when the static solution is not optimal, we provide an instance-based tight approximation bound that is related to a measure of non-convexity of the transformation of the uncertain set. We also show that for two-stage problems, our bound is at least as good (and in many case significantly better) as the bound given by the symmetry of the uncertainty set [11, 12]. Moreover, our results can be generalized to the case where the objective coefficients and right-hand-side are also uncertainty.

In Chapter 3, we focus on the two-stage problems with a family of column-wise and constraint-wise uncertainty sets where any constraint describing the set involves entries of only a single column or a single row. This is a fairly general class of uncertainty sets to model constraint coefficient uncertainty. Moreover, it is the family of uncertainty sets that gives the previous hardness result. On the positive side, we show that a static solution is an $O\big(\log n \cdot \min(\log \Gamma, \log(m+n))\big)$-approximation for the two-stage adjustable robust problem where m and n denote the numbers of rows and columns of the constraint matrix and Γ is the maximum possible ratio of upper bounds of the uncertain constraint coefficients. Therefore, for constant Γ, surprisingly the performance bound for static solutions matches the hardness of approximation for the adjustable problem. Furthermore, in general the static solution provides nearly the best efficient approximation for the two-stage adjustable robust problem.

In Chapter 4, we extend our result in Chapter 2 to a multi-stage adjustable robust linear optimization problem. In particular, we consider the case where the choice of the uncertain constraint coefficient matrix for each stage is independent of the others. In real world applications, decision problems are often of multiple stages and a iterative implementation of two-stage solution may result in a suboptimal solution for multi-stage problem. We consider the static solution for the adjustable robust problem show that it is optimal for the adjustable robust problem when the uncertainty set for each stage is constraint-wise. We also give an approximation bound on the performance of static solution for multi-stage adjustable robust problem that is related to the measure of non-convexity introduced in Chapter 2.

Chapters 5 considers a two-stage adjustable robust linear covering problem with uncertain right-hand-side parameter. As mentioned earlier, such problems are often intractable due to astronomically many extreme points of the uncertainty set. We introduce a new approximation framework where we consider a “simple” set that is “close” to the original uncertainty set. Moreover, the adjustable robust problem can be solved efficiently over the extended set. We show that the approximation bound is related to a geometric factor that represents the Banach-Mazur distance between the two sets. Using this framework, we provide approximation bounds that are better than the bounds given by an affine policy in [7] for a large class of interesting uncertainty sets. For instance, we provide an approximation solution that gives a $m^{1/4}$-approximation for the two-stage adjustable robust problem with hypersphere uncertainty set, while the affine policy has an approximation ratio of $O(\sqrt{m})$. Moreover, our bound for general p-norm ball is $m^{\frac{p-1}{p^2}}$ as opposed to $m^{\frac{1}{p}}$ given by an affine policy.

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Brian Yin Lu

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To my parents and my wife

Chapter 1

Introduction

1.1 Background and Motivation

This thesis is devoted to approximation algorithms for adjustable robust linear problems. Research in this area was ignited by the emergence of optimization problems under uncertainty in the input parameters. Such uncertainty arises naturally in many real world optimization problems. For example, in a Knapsack problem, the item sizes and the size of the knapsack may be uncertain; In a machine scheduling problem, the processing time for the arriving jobs may be uncertain. George Dantzig's 1955 paper, "Linear Programming under Uncertainty" is considered one of the ten most influential papers in Management Science [27]. The methodology introduced in Dantzig's paper is named stochastic programming, since it assumes an underlying probability distribution of the uncertain input parameters. The objective of stochastic programming is to optimize the expected value subject to chance constraints. We refer the readers to Kall and Wallace [29], Prekopa [31], Shapiro [32], Shapiro et al. [33] for a thorough introduction of stochastic programming. Several empirical algorithms such as sample average approximation and stochastic gradient descent have shown theoretical and numerical success. However, stochastic programming suffers from the "curse of dimensionality" and is intractable in general. Moreover, knowing the exact distribution of the input parameter may not be realistic, and one may only have partial information such as the moments of the uncertain parameters or bounds on such quantities.

On the other hand, robust optimization models the uncertain parameters using a deterministic uncertainty set. The goal is to optimize the objective value corresponding to the worst-case scenario from the uncertainty set. Soyster [34] first considers robust linear optimization problem in the early 1970s. The author shows that there is a compact linear programming formulation for robust problem with certain uncertainty sets. In fact, robust optimization is computationally tractable for a large class of problems if we want to compute a static solution which is feasible for all scenarios. However, interestingly enough, the methodology went unnoticed for more than 20 years after its debut. It was until late 1990s that researches in this field have become active again. The series of Ben-Tal and Nemirovski [4–6], El Ghaoui and Lebret [20], Bertsimas and Sim [15, 16], Goldfarb and Iyengar [24], Bertsimas et al. [8] give a solid review of robust optimization, and most of these studies focus on the robustification of optimization problems and tractable approaches in formulation.

In general, computing an adjustable (or dynamic) optimal solution for the robust optimization problem is intractable. In fact, Feige et al. [23] show that it is hard to even approximate a two-stage robust fractional set covering problem with uncertain right-hand-side within a factor better than $\Omega(\log m/\log\log m)$, where m is the number of elements. This motivates us to consider approximation algorithms for the problem. Static robust solutions and affine adjustable robust solutions are two approaches that have been studied in literatures. In a static robust solution, we compute a single optimal solution that is independent of the choice of the uncertain parameters. Therefore, it is feasible for all possible scenarios in the uncertainty set. Bertsimas and Goyal [9], Bertsimas et al. [12] consider a two-stage adjustable robust covering problem with uncertain right-hand-side and relate the performance of static solution to the symmetry of the uncertainty set. They show that the static robust solution provides a 2-approximation for the two-stage adjustable problem if the uncertainty set is symmetric. However, the gap can be arbitrarily large for a general convex uncertainty set. Ben-Tal and Nemirovski [5] consider an adjustable robust packing problem with constraint uncertainty set. They show that the static robust solution is optimal for the two-stage adjustable robust problem if the uncertainty set is constraint-wise, i.e., the choice of each row in the uncertain coefficient matrix is independent of the other rows (a Cartesian product of row uncertainty sets). This motivates us to study the optimality conditions of static robust solution for general convex, compact uncertainty sets. As mentioned earlier, Soyster [34] considers column-wise uncertainty sets and shows that the static robust solution corresponds to a hypercube uncertainty set and can be solved by a single LP. This is a fairly general class of uncertainty sets. However, to the best of our knowledge, no result for the performance of static solution as an approximation to the adjustable robust problem with such uncertainty sets is known yet.

Ben-Tal et al. [3] introduce an affine adjustable solution (also known as affine policy) to approximate two-stage adjustable robust covering problem with uncertain right-hand-side. This approach assumes an affine relationship between the second-stage variable and the uncertain right-hand-side. Such solution is preferred in its computational tractability and strong empirical performance. Bertsimas et al. [13], Iancu et al. [28] consider single dimension multi-stage problem and give optimality conditions for affine policy. When the geometric properties of the uncertainty set are known, Bertsimas and Bidkhori [7] consider a two-stage adjustable robust covering problem with uncertain right-hand-side and provide an approximation bound on the power of affine policy that depends on the simplex dilation factor, the translation factor and symmetry of the uncertainty set. They also compute the above geometric properties for several specific uncertainty sets. For general uncertainty sets, Bertsimas and Goyal [10] give a generic bound of $O(\sqrt{m})$ on the performance of affine policy in regardless of the structure of the uncertainty set, where m is the dimension of the uncertain right-hand-side. Moreover, they show that the bound is tight when the uncertainty set is the intersection of the unit ℓ_2-norm ball and the positive orthant, i.e.,

$$\mathcal{U} = \{\mathbf{h} \in \mathbb{R}^m_+ \mid ||\mathbf{h}||_2 \leq 1, \mathbf{h} \geq \mathbf{0}\}. \tag{1.1.1}$$

Note that the above set has infinitely many extreme points. The authors also show that affine policy is optimal if the uncertainty set is a simplex. However, for uncertainty sets with even $(m+3)$ extreme points, affine policy can still be sub-optimal. The worst case of affine policy holds for the uncertainty sets with huge number of extreme points. That motivates us to find new policies where we can have a good approximation for the adjustable problem even that the number of extreme points can be very large.

1.2 Preliminaries

1.2.1 Basic Notation

We denote the set of real numbers by $\mathbb{R}$, the n-dimensional Euclidean space by $\mathbb{R}^n$, and the Euclidean space of the set of matrices of dimension m by n by $\mathbb{R}^{m\times n}$. We also denote the entry-wise non-negative counterpart of these sets with subscript "$+$", e.g., $\mathbb{R}^{m\times n}_+$ means set of m by n matrices with non-negative entries. Vectors and matrices are in bold fonts, e.g., $\mathbf{x} \in \mathbb{R}^m$ implies that $\mathbf{x}$ is a m-dimensional vector. As a conventional routine, $\mathbf{e}$ denotes vector of all ones (of appropriate dimension), while $\mathbf{e}_i$ denotes the standard unit vector in the i^{th} coordinate, i.e., one at the i^{th} entry and zeros elsewhere. We denote $[n]$ as the set of numbers $\{1,2,\ldots,n\}$. The superscript "T" denotes the transpose operation. The inner product of vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ is denoted by $\mathbf{x}^T\mathbf{y} = \sum_{j=1}^n x_j y_j$. The Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ is denoted by $||\mathbf{x}||_2 = (\mathbf{x}^T\mathbf{x})^{1/2}$. We use $||\mathbf{x}||_1$ to denote the ℓ_1-norm of $\mathbf{x}$, i.e., $||\mathbf{x}||_1 = \sum_{j=1}^n |x_j|$. $||\mathbf{x}||_\infty$ denotes the infinity norm, i.e., the largest component of $\mathbf{x}$ in magnitude, i.e., $||\mathbf{x}||_\infty = \max_j |x_j|$. For $\mathbf{x} \in \mathbb{R}^m$, $\mathsf{diag}(\mathbf{x})$ denotes a $m \times m$ matrix with diagonal whose diagonal entries are the elements of $\mathbf{x}$ and off-diagonal entries are zeros.

1.2.2 Robust Packing Problems

In Chapters 2 and 3, we consider the following two-stage adjustable robust linear packing problems $\Pi_{\mathsf{AR-pack}}$ under uncertain constraint coefficients.

$$
\begin{aligned}
z_{\mathsf{AR-pack}} = \max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} + \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}(\mathbf{B})} \mathbf{d}^T\mathbf{y}(\mathbf{B}) & \\
\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}) \ \leq \ \mathbf{h} & \\
\mathbf{x} \ \in \ \mathbb{R}^{n_1} & \\
\mathbf{y}(\mathbf{B}) \ \in \ \mathbb{R}_+^{n_2}, &
\end{aligned} \tag{1.2.1}
$$

where $\mathbf{A} \in \mathbb{R}^{m\times n_1}, \mathbf{c} \in \mathbb{R}^{n_1}, \mathbf{d} \in \mathbb{R}_+^{n_2}$ and $\mathbf{h} \in \mathbb{R}^m$. The second-stage constraint matrix $\mathbf{B} \in \mathbb{R}_+^{m\times n_2}$ is uncertain and belongs to a full dimensional compact convex uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n_2}$ in the non-negative orthant. The decision variables $\mathbf{x}$ represent the first-stage decisions before the constraint matrix $\mathbf{B}$ is revealed, and $\mathbf{y}(\mathbf{B})$ represent the second-stage or recourse decision variables after observing the uncertain constraint matrix $\mathbf{B} \in \mathcal{U}$. Therefore, the (adjustable) second-stage decisions depend on the uncertainty realization. We can assume without loss of generality that $\mathcal{U}$ is *down-monotone* (see Appendix A.1).

We would like to emphasize that the second-stage objective coefficients $\mathbf{d}$, constraint coefficients $\mathbf{B}$, and the second-stage decision variables $\mathbf{y}(\mathbf{B})$ are all non-negative. Also, the uncertainty set $\mathcal{U}$ of second-stage constraint matrices is contained in the non-negative orthant. Therefore, the model is slightly restrictive and does not allow us to handle arbitrary two-stage linear problems. For instance, we can not handle covering constraints involving second-stage variables, or lower bounds on second-stage decision variables. Note that there is no restrictions on the first-stage constraint coefficients $\mathbf{A}$ or objective coefficients $\mathbf{c}$ until later in this thesis. Also, the first-stage decision variables $\mathbf{x}$ and right-hand-side $\mathbf{h}$ are not necessarily non-negative.

Our model is still fairly general and captures important applications including resource allocation and revenue management problems. For instance, in the resource allocation problem considered in [37], m corresponds to the number of resources with capacities $\mathbf{h}$. The linear constraints correspond to capacity constraints on the resources, the first-stage matrix $\mathbf{A}$ denotes the resource requirements of known first-stage demands and $\mathbf{B}$ denotes the uncertain resource requirements for future demands. In the framework of (1.2.1), we want to compute first-stage (fractional) allocation decisions $\mathbf{x}$ such that the worst case total revenue over all possible future demand arrivals from $\mathcal{U}$ is maximized.

As another example, consider a multi-server scheduling problem as in [14] where jobs arrive with uncertain processing times and we need to make the scheduling decisions to maximize the utility. The first-stage matrix $\mathbf{A}$ denotes the known processing time of first-stage jobs, $\mathbf{h}$ denotes the available timespan and $\mathbf{B}$ represents the time requirements of unknown arriving jobs. If we employ a pathwise enumeration for the uncertain time requirement, such stochastic project scheduling problem can be modeled as two-stage packing linear programming problems with uncertain constraint coefficients as in (1.2.1).

Computing an optimal adjustable robust solution is intractable in general. In Chapter 2, we show that $\Pi_{\mathsf{AR-pack}}$ (1.2.1) is hard to approximate within any factor that is better than $\Omega(\log n)$. Therefore, we consider a static robust optimization approach to approximate $\Pi_{\mathsf{AR-pack}}$. The corresponding static robust optimization problem Π_{Rob} can be formulated as follows.

$$
\begin{aligned}
z_{\mathsf{Rob}} = \max_{\mathbf{x}}\ & \mathbf{c}^T\mathbf{x} + \mathbf{d}^T\mathbf{y} \\
& \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \ \leq\ \mathbf{h},\ \forall \mathbf{B} \in \mathcal{U} \\
& \mathbf{x}\ \in\ \mathbb{R}^{n_1} \\
& \mathbf{y}\ \in\ \mathbb{R}_+^{n_2}.
\end{aligned}
\tag{1.2.2}
$$

Note that the second-stage solution $\mathbf{y}$ is static and does not depend on the realization of uncertain $\mathbf{B}$. Both first-stage and second-stage decisions $\mathbf{x}$ and $\mathbf{y}$ are selected before the second-stage uncertain constraint matrix is known and $(\mathbf{x},\mathbf{y})$ is feasible for all $\mathbf{B} \in \mathcal{U}$. An optimal static robust solution to (1.2.2) can be computed efficiently if $\mathcal{U}$ has an efficient separation oracle. In fact, Ben-Tal and Nemirovski [5] give compact formulations for solving (1.2.2) for polyhedral and conic uncertainty sets.

In Chapter 2 and 3, our goal is to compare the performance of an optimal static robust solution with respect to the optimal adjustable robust solution of $\Pi_{\mathsf{AR-pack}}$ (1.2.1). The above models have been considered in the literature. Ben-Tal and Nemirovski [5] show that a static solution is optimal if the uncertainty set $\mathcal{U}$ is *constraint-wise* where each constraint $i = 1,\ldots,m$ can be selected independently from a compact convex set $\mathcal{U}_i$, i.e., $\mathcal{U}$ is a Cartesian product of $\mathcal{U}_i, i = 1,\ldots,m$. However, the authors do not discuss performance of static solutions if the constraint-wise condition on $\mathcal{U}$ is not satisfied. Bertsimas and Goyal [11] consider a general multi-stage convex optimization problem under uncertain constraints and objective functions and show that the performance of the static solution is related to the symmetry of the uncertainty set $\mathcal{U}$. However, the symmetry bound in [11] can be quite loose in many instances. For example, consider the case when $\mathcal{U}$ is constraint-wise where each $\mathcal{U}_i$, $i = 1, \ldots, m$ is a simplex, i.e.,

$$\mathcal{U}_i = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{e}^T \mathbf{x} \leq 1\}.$$

The symmetry of $\mathcal{U}$ is $O(1/n)$ [12] and the results in [11] imply an approximation bound of $\Omega(n)$. While from Ben-Tal and Nemirovski [5], we know that a static solution is optimal.

As static solution has been shown to be optimal for adjustable robust problem with constraint-wise uncertainty sets, it is natural to consider column-wise uncertainty sets, i.e., each column $j \in [n]$ of the uncertain matrix $\mathbf{B}$ belongs to a compact convex set $\mathcal{C}_j \subseteq \mathbb{R}^m_+$ unrelated to other columns

$$\mathcal{U} = \{[\mathbf{b}_1\, \mathbf{b}_2 \ldots \mathbf{b}_n] \mid \mathbf{b}_j \in \mathcal{C}_j, j \in [n]\}. \tag{1.2.3}$$

In fact, the hardness result for $\Pi_{\mathsf{AR-pack}}$ (1.2.1) mentioned earlier is obtained when the uncertainty set is column-wise. In Chapter 3, we focus on such uncertainty sets and show that the static solution provides an $O(\log n)$-approximation for the adjustable robust problem $\Pi_{\mathsf{AR-pack}}$ (1.2.1). Moreover, our results can be generalized to column-wise and constraint-wise uncertainty sets, i.e.,

$$\mathcal{U} = \left\{\mathbf{B} \in \mathbb{R}^{m\times n}_+ \mid \mathbf{B}\mathbf{e}_j \in C_j,\ \forall j \in [n],\ \mathbf{B}^T \mathbf{e}_i \in R_i,\ \forall i \in [m]\right\}.$$

In Chapter 4, we consider a multi-stage adjustable robust linear optimization problem with covering constraints. Specifically, we consider the following problem Π^K_{AR} where $K \in \mathbb{N}_+$ denotes the number of decision stages.

$$
\begin{aligned}
z_{\mathsf{AR}}^K = \; & \max \mathbf{c}_0^T \mathbf{x}_0 + \min_{\mathbf{B}_1 \in \mathcal{U}_1} \left[\max_{\mathbf{x}_1(\mathbf{B}_1)} \mathbf{c}_1^T \mathbf{x}_1(\mathbf{B}_1) + \min_{\mathbf{B}_2 \in \mathcal{U}_2} \left[\max_{\mathbf{x}_2(\mathbf{B}_1, \mathbf{B}_2)} \mathbf{c}_2^T \mathbf{x}_2(\mathbf{B}_1, \mathbf{B}_2) + \ldots \right.\right. \\
& \left.\left.\left. + \min_{\mathbf{B}_K \in \mathcal{U}_K} \left[\max_{\mathbf{x}_K(\mathbf{B}_1, \ldots, \mathbf{B}_K)} \mathbf{c}_K^T \mathbf{x}_K(\mathbf{B}_1, \ldots, \mathbf{B}_K) \right] \right] \right] \\
& \mathbf{A}\mathbf{x}_0 + \mathbf{B}_1 \mathbf{x}_1(\mathbf{B}_1) + \mathbf{B}_2 \mathbf{x}_2(\mathbf{B}_1, \mathbf{B}_2) + \ldots + \mathbf{B}_K \mathbf{x}_K(\mathbf{B}_1, \ldots, \mathbf{B}_K) \; \leq \; \mathbf{h}, \\
& \qquad \forall \mathbf{B}_t \in \mathcal{U}_t, t \in [K] \\
& \mathbf{x}_0, \mathbf{x}_1(\mathbf{B}_1), \ldots, \mathbf{x}_K(\mathbf{B}_1, \ldots, \mathbf{B}_K) \geq \mathbf{0}
\end{aligned}
$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{c}_i \in \mathbb{R}^n$, $\mathbf{h} \in \mathbb{R}_+^m$, and $\mathbf{B}_t \in \mathcal{U}_t \subseteq \mathbb{R}_+^{m \times n}$ be the uncertain constraint coefficient matrix for the t^{th}-stage for all $t \in [K]$. Note that the uncertainty for each stage is independent of the uncertainties for the other stages, i.e., the uncertainty set $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_K$. Iancu et al. [28] consider single dimension multi-stage linear adjustable problem with covering constraints and give optimality conditions for affine policy. Other the other hand, we study the multi-dimensional adjustable robust problem with packing constraints and the performance of static solution as its approximation. In particular, we generalize the result of Ben-Tal and Nemirovski [5] by showing that the static solution is optimal for the multi-stage adjustable robust problem when the uncertainty set for each stage $t \in [K]$ is constraint-wise. We also give an approximation bound on the performance of static solution that is related to the measure of non-convexity introduced in Chapter 2.

1.2.3 Adjustable Robust Covering Problem and Affine Policies

In Chapter 5, we consider a two-stage adjustable robust linear optimization problems with covering constraints and uncertain right-hand-side. In particular, we consider the following model

$\Pi_{\mathsf{AR-cover}}(\mathcal{U})$:

$$
\begin{aligned}
z_{\mathsf{AR-cover}}(\mathcal{U}) = \min\ & \mathbf{c}^T\mathbf{x} + \max_{\mathbf{h}\in\mathcal{U}} \min_{\mathbf{y}(\mathbf{h})} \mathbf{d}^T\mathbf{y}(\mathbf{h}) \\
& \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{h}) \ \geq\ \mathbf{h} \\
& \mathbf{x} \ \in\ \mathbb{R}_+^{n_1} \\
& \mathbf{y}(\mathbf{h}) \ \in\ \mathbb{R}_+^{n_2},
\end{aligned} \tag{1.2.4}
$$

where $\mathbf{A} \in \mathbb{R}^{m\times n_1}, \mathbf{c} \in \mathbb{R}_+^{n_1}, \mathbf{d} \in \mathbb{R}_+^{n_2}, \mathbf{B} \in \mathbb{R}^{m\times n_2}$. The mechanism of such model is the same as that of $\Pi_{\mathsf{AR-pack}}(\mathcal{U})$ (1.2.1) except that the right-hand-side $\mathbf{h}$ is uncertain and belongs to a compact, convex and full-dimensional uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^m$. The choice of $\mathbf{h} \in \mathcal{U}$ is subject to adversary selection, i.e., $\mathbf{h}$ is chosen so that the second-stage cost is maximized. Again, we can assume without loss of generality that $n_1 = n_2 = n$ and $\mathcal{U}$ is down-monotone, i.e., $\mathbf{h} \in \mathcal{U}$ and $\mathbf{0} \leq \hat{\mathbf{h}} \leq \mathbf{h}$ implies that $\hat{\mathbf{h}} \in \mathcal{U}$.

Similar to previous model, we would like to note that the objective coefficients $\mathbf{c}$, $\mathbf{d}$ and the decision variables $\mathbf{x}, \mathbf{y}(\mathbf{B})$ are all non-negative. Moreover, the uncertainty set $\mathcal{U}$ is constrained to be in the positive orthant. Again, this is slightly restrictive but the above model still captures many important applications. For instance, in a demand-supply problem, $\mathbf{h}$ represents the uncertain demand, $\mathbf{A}$ and $\mathbf{B}$ denote the supply-demand adjacency network matrix for the two decision stages, and $\mathbf{c}$ and $\mathbf{d}$ are the corresponding costs for supply. In the framework of $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$ (1.2.4), our goal is to minimize the worst-case total cost over all possible future demand from $\mathcal{U}$. As another example, we can obtain a two-stage set-cover problem by setting $\mathbf{A}$ and $\mathbf{B}$ to the element-set incidence matrix. In fact, many combinatorial optimization problems with uncertain right-hand-side can be modeled using the framework such as facility location and Steiner trees.

Feige et al. [23] show that a two-stage robust set cover under some plausible complexity assumptions is hard to approximate within any factor that is better than $\Omega(\log m/\log\log m)$. This motivates us to consider approximation algorithms for the adjustable robust problem Π_{AR} (1.2.4) for general uncertainty sets. Ben-Tal et al. [3] introduce an affine adjustable solution (also known as affine policy), which assumes an affine relationship between the second-stage variable $\mathbf{y}(\mathbf{h})$ and the uncertain right-hand-side $\mathbf{h}$, i.e., $\mathbf{y}(\mathbf{h}) = \mathbf{P}\mathbf{h} + \mathbf{q}$ for some $\mathbf{P} \in \mathbb{R}^{n\times m}$ and $\mathbf{q} \in \mathbb{R}^m$. Therefore, under affine policy, $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$ (1.2.4) can be formulated as

$$
\begin{aligned}
z_{\mathsf{AR-cover-aff}}(\mathcal{U}) = \min\ & \mathbf{c}^T\mathbf{x} + z \\
& z - \mathbf{d}^T\mathbf{q} \geq \mathbf{d}^T\mathbf{P}\mathbf{h},\ \forall \mathbf{h} \in \mathcal{U} \\
& \mathbf{e}_i^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{q}) \ \geq\ \mathbf{e}_i^T(\mathbf{I} - \mathbf{B}\mathbf{P})\mathbf{h},\ \forall i \in [m], \mathbf{h} \in \mathcal{U} \\
& \mathbf{e}_i^T(\mathbf{P}\mathbf{h} + \mathbf{q}) \ \geq\ 0,\ \forall i \in [m], \mathbf{h} \in \mathcal{U} \\
& \mathbf{x} \ \in\ \mathbb{R}^n_+,
\end{aligned}
$$

which can be solved efficiently provided a separation oracle over $\mathcal{U}$. For general uncertainty sets, Bertsimas and Goyal [10] give a bound of $O(\sqrt{m})$ on the performance of affine policy. Moreover, they show that the bound is tight when the uncertainty set is the intersection of the unit ℓ_2-norm ball and positive orthant. In Chapter 4, we provide a approximation framework that gives a approximation ratio of $m^{1/4}$ for such uncertainty set. Moreover, we generalize our result to general ℓ_p-norm balls with $p > 1$.

1.3 Our Contributions

Although mentioned in a scattered fashion previously, we would like to summarize our main contributions at this point:

- In Chapter 2 and 3, we consider the two-stage adjustable robust linear packing problems $\Pi_{\mathsf{AR-pack}}$ (1.2.1). Our goal is to compare the performance of an optimal static robust solution with respect to the optimal adjustable robust solution.

  Hardness of Approximation. We show that the adjustable robust problem $\Pi_{\mathsf{AR-pack}}$ (1.2.1) is $\Omega(\log n)$ hard to approximate for the case of column-wise uncertainty sets. In other words, there is no polynomial time algorithm that computes an adjustable two-stage solution with worst case objective value within a factor better than $\Omega(\log n)$ of the optimal. Our proof is based on an approximation preserving reduction from the set cover problem [36]. In particular, we show that any instance of set cover problem can be reduced to an instance of the two-stage adjustable robust problem with column-wise sets where each column set is a simplex. For the more general case where the uncertainty set $\mathcal{U}$ and objective coefficients $\mathbf{d}$ are not constrained to be in the non-negative orthant, we show that the two-stage adjustable robust problem is $\Omega(2^{\log^{1-\varepsilon} m})$-hard to approximate for any constant $0 < \varepsilon < 1$ by a reduction from the Label-Cover-Problem [1]. The hardness of approximation results motivate us to find good approximations for the two-stage adjustable robust problem.

  Optimality of static solution. We give a tight characterization of the conditions under which a static solution is optimal for the two-stage adjustable robust problem $\Pi_{\mathsf{AR-pack}}$ (1.2.1). The optimality of static solutions depends on the geometric properties of a transformation of the uncertainty set. In particular, we show that the static solution is optimal if the transformation of $\mathcal{U}$ is convex. If $\mathcal{U}$ is a constraint-wise set, we show that the transformation of $\mathcal{U}$ is convex. Ben-Tal and Nemirovski [5] show that for such $\mathcal{U}$, a static solution is optimal for adjustable robust problem. Therefore, our result extends the result in [5] for the case where $\mathcal{U}$ is contained in the non-negative orthant. We also present other families of uncertainty sets for which the transformation is convex.

This result is quite surprising as the worst-case choice of $\mathbf{B} \in \mathcal{U}$ usually depends on the first-stage solution even if $\mathcal{U}$ is constraint-wise unless $\mathcal{U}$ is a hypercube. For the case of hypercube, each uncertain element can be selected independently from an interval and in that case, the worst-case $\mathbf{B}$ is independent of the first-stage decision. However, a constraint-wise set is a Cartesian product of general convex sets. We show that if the transformation of $\mathcal{U}$ is convex, there is an optimal recourse solution for the worst-case choice of $\mathbf{B} \in \mathcal{U}$ that is feasible for all $\mathbf{B} \in \mathcal{U}$. Furthermore, we show that our result can also be interpreted as the following min-max theorem.

$$\min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}\geq\mathbf{0}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h}\} = \max_{\mathbf{y}\geq\mathbf{0}} \min_{\mathbf{B}\in\mathcal{U}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h}\}.$$

The inner minimization on the max-min problem implies that the solution $\mathbf{y}$ must be feasible for all $\mathbf{B} \in \mathcal{U}$ and therefore, is a static robust solution. We would like to note that the above min-max result does not follow from the general saddle-point theorem [17].

In Chapter 4, we generalize the optimality condition for static solution to a multi-stage problems where the choice of the uncertainty constraint coefficients for each stage is independent of the others. In particular, we show that the static solution is optimal for the multi-stage adjustable robust problem if the uncertainty set for each stage is constraint-wise, thereby generalizing the result of Ben-Tal and Nemirovski [5] to the multi-stage problems. Moreover, we give an approximation bound on the performance of static solution for multi-stage adjustable robust problem that is related to the measure of non-convexity introduced in Chapter 2.

Approximation bounds for the static solution. We give a tight approximation bound on the performance of the optimal static solution for the adjustable robust problem when the transformation of $\mathcal{U}$ is not convex and the static solution is not optimal. We relate the performance of static solutions to a natural measure of non-convexity of the transformation of $\mathcal{U}$. We also present a family of uncertainty sets and instances where we show that the approximation bound is tight, i.e., the ratio of the optimal objective value of the adjustable robust problem (1.2.1) and the optimal objective value of the robust problem (1.2.2) is exactly equal to the bound given by the measure of non-convexity.

We also compare our approximation bounds with the bound in Bertsimas and Goyal [11] where the authors relate the performance of the static solutions with the symmetry of the uncertainty set. We show that our bound is stronger than the symmetry bound in [11]. In particular, for any instance, we can show that our bound is at least as good as the symmetry bound, and in fact in many cases, significantly better. For instance, consider the following uncertainty set

$$\mathcal{U} = \left\{ \mathbf{B} \in \mathbb{R}_+^{m \times n} \;\middle|\; \sum_{i,j} B_{ij} \leq 1 \right\}.$$

In this case, $\mathsf{sym}(\mathcal{U}) = 1/mn$ [12] and the symmetry bound is $\Omega(mn)$. However, we show that a static solution is optimal for the adjustable robust problem (our bound is equal to one).

Models with both constraint and objective uncertainty. We extend our result to two-stage models where both constraint and objective coefficients are uncertain. In particular, we consider a two-stage model where the uncertainty in the second-stage constraint matrix $\mathbf{B}$ is independent of the uncertainty in the second-stage objective $\mathbf{d}$. Therefore, $(\mathbf{B}, \mathbf{d})$ belong to a convex compact uncertainty set $\mathcal{U}$ that is a Cartesian product of the uncertainty set of constraint matrices $\mathcal{U}^B$ and uncertainty set of second-stage objective $\mathcal{U}^d$.

We show that our results for the model with only constraint coefficient uncertainty can also be extended to this case of both constraint and objective uncertainty. In particular, we show that a static solution is optimal if the transformation of $\mathcal{U}^B$ is convex. Furthermore, if the transformation is not convex, then the approximation bound on the performance of the optimal static solution is related to the measure of non-convexity of the transformation of $\mathcal{U}^B$. Surprisingly, the approximation bound or the optimality of a static solution does not depend on the uncertainty set of objectives $\mathcal{U}^d$. If the transformation of $\mathcal{U}^B$ is convex, a static solution is optimal for all convex compact uncertainty sets $\mathcal{U}^d \subseteq \mathbb{R}_+^{n_2}$. We also present a family of examples to show that our bound is tight for this case as well.

We also consider a two-stage adjustable robust model where in addition to the second-stage constraint matrix $\mathbf{B}$ and objective $\mathbf{d}$, the right hand side $\mathbf{h}$ of the constraints is also uncertain and

$$(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U} = \mathcal{U}^{B,h} \times \mathcal{U}^d,$$

where $\mathcal{U}$ is a convex compact set that is a Cartesian product of the uncertainty set for $(\mathbf{B},\mathbf{h})$ and the uncertainty set for $\mathbf{d}$. For this case, we give a sufficient condition for the optimality of a static solution that is related to the convexity of the transformation of uncertainty set $\mathcal{U}^{B,h}$. Note again that the optimality of a static solution does not depend on the uncertainty set of objectives $\mathcal{U}^{d}$. However, our approximation bounds do not extend for this case if the transformation of $\mathcal{U}^{B,h}$ is not convex.

Uniform Approximation Bound for Column-wise and Constraint-wise Uncertainty Sets. In Chapter 3, we focus on column-wise and constraint-wise uncertainty set (1.2.3) and show that in this case, a static solution provides an $O\big(\log n \cdot \min(\log\Gamma, \log(m+n))\big)$-approximation for the two-stage adjustable robust problem where Γ is the maximum possible ratio of the upper bounds of different matrix entries in the uncertainty set (See Section 3.3 for details). Therefore, if Γ is a constant, a static solution gives a $O(\log n)$-approximation for the adjustable robust problem for column-wise and constraint-wise uncertainty sets; thereby, matching the hardness of approximation. This is quite surprising as it shows the static solution is the best possible efficient approximation for the adjustable robust problem in this case. We would like to note that the two-stage adjustable robust optimization problem is $\Omega(\log n)$-hard even for the case when Γ is a constant. Furthermore, when Γ is large, we show that a static solution gives a $O(\log n \cdot \log(m+n))$-approximation for the adjustable robust problem. Therefore, the static solution provides a nearly optimal approximation for the two-stage adjustable robust problem for column-wise and constraint-wise uncertainty sets in general.

We first consider the case when the uncertainty set is column-wise and prove a bound of $O\big(\log n \cdot \min(\log \Gamma, \log(m+n))\big)$ on the adaptivity gap for the adjustable robust problem. Our analysis is based on the structural properties of the optimal adjustable and static robust solutions. In particular, we first show that the worst adaptivity gap is achieved when each column is a simplex. This is based on the property of the optimal static robust solution that it depends only on the hypercube containing the given uncertainty set $\mathcal{U}$ (Soyster [34]). We formalize this in Theorems 3.2.1 and 3.2.2. Furthermore, for the simplex column-wise uncertainty sets, we relate the adjustable robust problem to an appropriate set cover problem and relate the static robust problem to the corresponding LP relaxation in order to obtain the bound on the adaptivity gap.

We extend the analysis to the case when $\mathcal{U}$ is a column-wise and constraint-wise uncertainty set and prove a similar bound on the performance of static solutions. In particular, we show that if a static solution provides an α-approximation for the adjustable robust problem with column-wise uncertainty sets, then a static solution is an α-approximation for the case of column-wise and constraint-wise uncertainty sets. Moreover, we also extend our result to the case where the second-stage objective coefficients are also uncertain and show that the same bound holds when the uncertainty in the objective coefficients does not depend on the column-wise and constraint-wise constraint coefficient uncertainty sets.

Our results confirm the power of static robust solutions for the two-stage adjustable robust problem. In particular, its performance nearly matches the hardness of approximation factor for the adjustable robust problem, which indicates that it is nearly the best approximation possible for the problem. In addition, we would like to note that our approximation bound only compares the optimal objective values of the adjustable and static robust problems. The performance of the static robust solution policy can potentially be better: if $(\mathbf{x}^*, \mathbf{y}^*)$ is an optimal static robust solution, we only implement the first-stage solution $\mathbf{x}^*$ and compute the recourse solution after observing the realization of the uncertain matrix $\mathbf{B}$. Therefore, the objective value of the recourse solution can potentially be better than that of $\mathbf{y}^*$.

- In Chapter 5, we consider the two-stage adjustable robust linear optimization problems with covering constraints and uncertain right-hand-side $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$ (1.2.4). We introduce a new approximation framework for the problem. Our framework is based on choosing an appropriate dominating set $\hat{\mathcal{U}}$ by exploring the geometric structure of $\mathcal{U}$ in order to get better approximation bounds than the affine policy.

  One of the main reasons of intractability of adjustable robust optimization problems is that the number of extreme points of the uncertainty set $\mathcal{U}$ can be large. Our new approach approximates the uncertainty set $\mathcal{U}$ with a "simple" set that is "close" to $\mathcal{U}$ and over which the adjustable problem can be solved efficiently. In particular, for any uncertainty set $\mathcal{U}$, we construct an uncertainty set $\hat{\mathcal{U}}$ with small number of extreme points that dominates $\mathcal{U}$, i.e., for any $\mathbf{h} \in \mathcal{U}$, there exists $\hat{\mathbf{h}} \in \hat{\mathcal{U}}$ such that $\mathbf{h} \leq \hat{\mathbf{h}}$. Therefore, solving the adjustable robust problem over $\hat{\mathcal{U}}$ gives a feasible solution for the adjustable robust problem over $\mathcal{U}$. We show that the approximation bound is related to a geometric factor $\beta_{(\mathcal{U},\hat{\mathcal{U}})}$ that represents the Banach Mazur distance between the sets $\mathcal{U}$ and $\hat{\mathcal{U}}$.

  Using this framework, we provide approximation bounds that are better than the bounds given by an affine policy in [7] for a couple of interesting uncertainty sets. For instance, we provide an approximation solution that gives a $m^{1/4}$-approximation for the two-stage adjustable robust problem $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$ (1.2.4) with hypersphere uncertainty set, while the affine policy has an approximation ratio of $O(\sqrt{m})$. More general, our bound for the p-norm unit ball is $m^{\frac{p-1}{p^2}}$ as opposed to $m^{\frac{1}{p}}$ given by an affine policy.

Chapter 2

A Tight Characterization of the Performance of Static Solutions in Two-stage Adjustable Robust Linear Optimization

2.1 Introduction

In this chapter, we consider a two-stage adjustable robust linear packing problems $\Pi_{\mathsf{AR-pack}}$ (1.2.1) under uncertain constraint coefficients. For the ease of discussion, we denote the problem as Π_{AR} throughout this and next chapter.

Outline. The rest of the chapter is organized as follows: In Section 2.2, we present the hardness of approximation for the two-stage adjustable robust problems. In Section 2.3, we discuss the optimality of static solutions for the two-stage adjustable robust problem under constraint uncertainty and relate it to the convexity of an appropriate transformation of the uncertainty set. In Section 2.4, we introduce a measure of non-convexity for any compact set. Moreover, we present a tight approximation bound for the performance of an optimal static solution for the adjustable robust problem, that is related to the measure of non-convexity of the transformation of the uncertainty set. In Section 2.5, we extend our result to two-stage models where both second-stage constraint and objective are uncertain.

2.2 Hardness of Approximation.

In this section, we show that the two-stage adjustable robust problem Π_{AR} is $\Omega(\log n)$-hard to approximate for column-wise uncertainty sets (1.2.3). In other words, there is no polynomial time algorithm that guarantees an approximation within a factor of $\Omega(\log n)$ of the optimal two-stage adjustable robust solution. We achieve this via an approximation preserving reduction from the set cover problem , which is $\Omega(\log n)$-hard to approximate [36]. In particular, we have the following theorem.

Theorem 2.2.1. *The two-stage adjustable robust problem,* Π_{AR} *as defined in* (1.2.1) *is* $\Omega(\log n)$*-hard to approximate for column-wise uncertainty sets.*

Proof. Consider an instance I of the set cover problem with ground set of elements $S = \{1,\ldots,n\}$ and a family of subsets $\mathcal{S}_1,\ldots,\mathcal{S}_m \subseteq S$. The goal is to find minimum cardinality collection C of subsets $\mathcal{S}_i, i \in [m]$ that covers all $j \in [n]$. We construct an instance I' of the two-stage adjustable robust problem Π_{AR} (1.2.1) with a column-wise uncertainty set $\mathcal{U}$ as follows.

$$\mathbf{c} = \mathbf{0}, \mathbf{A} = \mathbf{0}, \;\; h_i = 1, \; \forall i \in [m], \;\; d_j = 1, \; \forall j \in [n]$$
$$\mathcal{U}_j = \left\{ \mathbf{b} \in [0,1]^m \;\middle|\; \sum_{i=1}^{m} b_i \leq 1, \; b_i = 0, \; \forall i \text{ s.t. } j \notin \mathcal{S}_i \right\}$$
$$\mathcal{U} = \left\{ [\mathbf{b}_1 \; \mathbf{b}_2 \; \ldots \; \mathbf{b}_n] \mid \mathbf{b}_j \in \mathcal{U}_j \right\}.$$

Note that there is a row corresponding to each subset $\mathcal{S}_i$ and a column corresponding to each element j. Moreover, $\mathcal{U}$ is a column-wise uncertainty set. Now,

$$\begin{aligned} z_{\mathsf{AR}} &= \min_{\mathbf{b}_j \in \mathcal{U}_j, j \in [n]} \max_{\mathbf{y} \in \mathbb{R}^n_+} \left\{ \mathbf{e}^T \mathbf{y} \;\middle|\; \sum_{j=1}^{n} y_j \mathbf{b}_j \leq \mathbf{e} \right\} \\ &= \min_{\mathbf{b}_j \in \mathcal{U}_j, j \in [n]} \min_{\mathbf{v} \in \mathbb{R}^m_+} \{ \mathbf{e}^T \mathbf{v} \mid \mathbf{b}_j^T \mathbf{v} \geq 1, \forall j \in [n] \}, \end{aligned}$$

where the second equality follows from taking the dual of the inner maximization problem in the original formulation. Suppose $\hat{\mathbf{v}}, \hat{\mathbf{b}}_j$ for all $j \in [n]$ is a feasible solution for instance I'. Then, we can compute a solution for instance I with cost at most $\mathbf{e}^T \hat{\mathbf{v}}$. To prove this, we show that we can construct an integral solution $\tilde{\mathbf{v}}, \tilde{\mathbf{b}}_j$ for all $j \in [n]$ such that

$$\mathbf{e}^T \tilde{\mathbf{v}} \leq \mathbf{e}^T \hat{\mathbf{v}}.$$

Note that $\hat{\mathbf{b}}_j$ may not necessarily be integral. For each $j \in [n]$, consider a basic optimal solution $\tilde{\mathbf{b}}_j$ where

$$\tilde{\mathbf{b}}_j \in \arg\max \{ \mathbf{b}^T \hat{\mathbf{v}} \mid \mathbf{b} \in \mathcal{U}_j \}.$$

Therefore, $\mathbf{b}_j$ is a vertex of $\mathcal{U}_j$ for any $j \in [n]$, which implies $\tilde{\mathbf{b}}_j = \mathbf{e}_{i_j}$ for some $i_j \in \mathcal{S}_j$. Also,

$$\tilde{\mathbf{b}}_j^T \hat{\mathbf{v}} \geq \hat{\mathbf{b}}_j^T \hat{\mathbf{v}} \geq 1, \ \forall j \in [n].$$

Now, let

$$\tilde{\mathbf{v}} \in \arg\min\{\mathbf{e}^T \mathbf{v} \mid \tilde{\mathbf{b}}_j^T \mathbf{v} \geq 1, \forall j \in [n], \mathbf{v} \geq \mathbf{0}\}.$$

Clearly, $\mathbf{e}^T \tilde{\mathbf{v}} \leq \mathbf{e}^T \hat{\mathbf{v}}$. Also, for all $j \in [n]$, since $\tilde{\mathbf{b}}_j = \mathbf{e}_{i_j}$ for some $i_j \in \mathcal{S}_j$,

$$\tilde{\mathbf{b}}_j^T \tilde{\mathbf{v}} \geq 1 \implies \tilde{v}_{i_j} = 1, \ \forall j \in [n].$$

Therefore, $\tilde{\mathbf{v}} \in \{0,1\}^m$. Let

$$C = \{\mathcal{S}_i \mid \tilde{v}_i = 1\}.$$

Clearly, C covers all the element $j \in [n]$ and $|C| = \mathbf{e}^T \tilde{\mathbf{v}} \leq \mathbf{e}^T \hat{\mathbf{v}}$.

Conversely, consider set cover $C \subseteq \{\mathcal{S}_i, i \in [m]\}$ of instance I. For any $j \in [n]$, there exists $i_j \in [m]$ such that $j \in \mathcal{S}_{i_j}$ and $\mathcal{S}_{i_j} \in C$. Now, we can construct a feasible solution $\overline{\mathbf{v}}, \overline{\mathbf{b}}_j$ for all $j \in [n]$ for z_{AR} as follows.

$$\overline{\mathbf{b}}_j = \mathbf{e}_{i_j}, \ \forall j \in [n]$$

$$\overline{v_i} = \begin{cases} 1 & \text{if } \mathcal{S}_i \in C \\ 0 & \text{otherwise} \end{cases}, \ \forall i \in [m].$$

It is easy to observe that $\overline{\mathbf{b}}_j^T \overline{\mathbf{v}} \geq 1$ for all $j \in [n]$ and $\mathbf{e}^T \overline{\mathbf{v}} = |C|$. $\square$

2.2.1 General Two-stage Adjustable Robust Problem.

If the uncertainty set $\mathcal{U}$ of second-stage constraint matrices and the objective coefficients $\mathbf{d}$ are not constrained to be in the non-negative orthant in Π_{AR}, we can prove a stronger hardness of approximation result. In particular, consider the following general problem $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$:

$$
\begin{aligned}
z_{\mathsf{AR}}^{\mathsf{Gen}} = \; \max \; & \mathbf{c}^T\mathbf{x} + \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}(\mathbf{B})} \mathbf{d}^T\mathbf{y}(\mathbf{B}) \\
& \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}) \; \leq \; \mathbf{h} \\
& \mathbf{y}(\mathbf{B}) \geq \mathbf{0},
\end{aligned}
\tag{2.2.1}
$$

where $\mathcal{U} \subseteq \mathbb{R}^{m\times n}$ is a convex compact column-wise set, $\mathbf{c},\mathbf{d} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$. We show that it is $\Omega(2^{\log^{1-\varepsilon} m})$-hard to approximate for any constant $0 < \varepsilon < 1$.

Theorem 2.2.2. *The adjustable robust problem* $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ (2.2.1) *is* $\Omega(2^{\log^{1-\varepsilon} m})$*-hard to approximate for any constant* $0 < \varepsilon < 1$.

We prove this by an approximation preserving reduction from the Label-Cover-Problem [1]. The proof is presented in Appendix B.1.

2.3 Optimality of Static Solutions

As shown in previous section, the two-stage adjustable robust problem Π_{AR} (1.2.1) is $\Omega(\log n)$-hard even for column-wise uncertainty sets. This motivates us to find efficient approximation algorithms for the problem. In particular, we consider static solution for (1.2.2) as an approximation for the adjustable robust problem. In this section, we present a tight characterization of the conditions under which a static robust solution computed from (1.2.2) is optimal for the adjustable robust problem (1.2.1). We introduce a transformation of the uncertainty set $\mathcal{U}$ and relate the optimality of a static solution to the convexity of the transformation.

An optimal static solution for (1.2.2) can be computed efficiently. Note that a static solution $(\mathbf{x},\mathbf{y})$ is feasible for all $\mathbf{B} \in \mathcal{U}$. To observe that an optimal static robust solution can be computed in polynomial time, consider the separation problem: given a solution $\mathbf{x},\mathbf{y}$, we need to decide whether or not there exists $\mathbf{B} \in \mathcal{U}$ and $j \in \{1,\ldots,m\}$ such that

$$\mathbf{e}_j^T(\mathbf{Ax}+\mathbf{By}) > h_j,$$

and find a separating hyperplane if $(\mathbf{x},\mathbf{y})$ is not feasible. Therefore, by solving m linear optimization problems over $\mathcal{U}$ we can decide whether the given solution is feasible or obtain a separating hyperplane. From the equivalence of the separation and optimization [25], we can compute an optimal static robust solution in polynomial time. In fact, there is a compact linear formulation to compute the optimal static solution for Π_{Rob} for a fairly general class of uncertainty sets [2, 5].

We can easily see that the static solution is a lower bound of the optimal value of the adjustable robust problem. Suppose $(\mathbf{x}^*,\mathbf{y}^*)$ is an optimal solution for Π_{Rob}. Then, $\mathbf{x}=\mathbf{x}^*, \mathbf{y}(\mathbf{B})=\mathbf{y}^*$ for all $\mathbf{B} \in \mathcal{U}$ is feasible for Π_{AR}. Therefore,

$$z_{\mathsf{AR}} \geq z_{\mathsf{Rob}}. \tag{2.3.1}$$

We would like to study the conditions under which $z_{\mathsf{AR}} \leq z_{\mathsf{Rob}}$. Suppose $(\mathbf{x}^*,\mathbf{y}^*(\mathbf{B}))$ for all

$\mathbf{B} \in \mathcal{U}$ is a fully-adjustable optimal solution for Π_{AR}. Then

$$z_{\mathsf{AR}} = \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}(\mathbf{B})} \left\{ \mathbf{d}^T\mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\}$$

$$z_{\mathsf{Rob}} \geq \mathbf{c}^T\mathbf{x}^* + \max_{\mathbf{y}\geq\mathbf{0}} \left\{ \mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U} \right\}.$$

Note that $\mathbf{h} - \mathbf{A}\mathbf{x}^* \geq \mathbf{0}$, since otherwise the second-stage problem becomes infeasible for Π_{AR}. In fact, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Otherwise, it is easy to see that $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$: suppose $(h - Ax^*)_i = 0$ for some i. Since $\mathcal{U}$ is a full-dimensional convex set, we can find $\hat{\mathbf{B}} \in \mathcal{U}$ such that $\hat{B}_{ij} > 0$ for all $j \in [n_2]$. Therefore,

$$\min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}(\mathbf{B})\geq\mathbf{0}} \left\{ \mathbf{d}^T\mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \leq \max_{\mathbf{y}\geq\mathbf{0}} \left\{ \mathbf{d}^T\mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} = 0,$$

which implies $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$. Therefore, we need to study conditions under which

$$\max_{\mathbf{y}\geq\mathbf{0}} \left\{ \mathbf{d}^T\mathbf{y} \,|\, \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U} \right\} \geq \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}\geq\mathbf{0}} \left\{ \mathbf{d}^T\mathbf{y} \,|\, \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\}, \tag{2.3.2}$$

where $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$.

2.3.1 One-stage Models

Motivated by (2.3.2), we consider the following one-stage adjustable robust problem $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$:

$$\begin{aligned} z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}) = \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}} \ & \mathbf{d}^T\mathbf{y} \\ \mathbf{B}\mathbf{y} \ & \leq \ \mathbf{h} \\ \mathbf{y} \ & \in \ \mathbb{R}^n_+, \end{aligned} \tag{2.3.3}$$

where $\mathbf{h} \in \mathbb{R}^m_+$ and $\mathbf{h} > \mathbf{0}$, $\mathbf{d} \in \mathbb{R}^n_+$ and $\mathcal{U} \subseteq \mathbb{R}^{m\times n}_+$ is the convex, compact and down-monotone uncertainty set. The corresponding one-stage static robust problem $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ can be formulated as follows:

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) = \max_{\mathbf{y}} \ & \mathbf{d}^T\mathbf{y} \\ \mathbf{B}\mathbf{y} \ & \leq \ \mathbf{h}, \ \forall \mathbf{B}\in\mathcal{U} \\ \mathbf{y} \ & \in \ \mathbb{R}^n_+. \end{aligned} \tag{2.3.4}$$

Consider $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ as defined in (2.3.3). We can write the dual problem of the inner maximization problem.

$$z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}) = \min_{\mathbf{B},\alpha}\{\mathbf{h}^T\alpha \mid \mathbf{B}^T\alpha \geq \mathbf{d}, \mathbf{B}\in\mathcal{U}, \alpha\in\mathbb{R}^m_+\}.$$

Substituting $\lambda = \mathbf{h}^T\alpha$ and $\alpha = \lambda\mu$, we can reformulate $z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ as follows.

$$z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}) = \min_{\lambda,\mathbf{B},\mu} \{\lambda \mid \lambda\mathbf{B}^T\mu \geq \mathbf{d}, \ \mathbf{h}^T\mu = 1, \ \mathbf{B}\in\mathcal{U}, \mu\in\mathbb{R}^m_+\}. \tag{2.3.5}$$

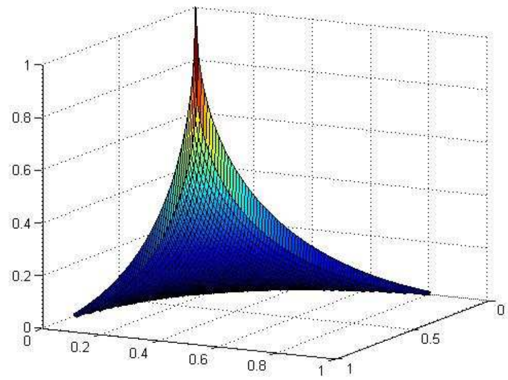


Figure 2.1: The boundary of the set $T(\mathcal{U},\mathbf{e})$ when $n=3$.

2.3.2 Transformation of $\mathcal{U}$

Motivated from the formulation (2.3.5), we define the following transformation $T(\mathcal{U},\mathbf{h})$ of the uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m\times n}$ and $\mathbf{h} > \mathbf{0}$.

$$T(\mathcal{U},\mathbf{h}) = \left\{ \mathbf{B}^T\mu \;\middle|\; \mathbf{h}^T\mu = 1,\, \mathbf{B} \in \mathcal{U},\, \mu \geq \mathbf{0} \right\}. \tag{2.3.6}$$

For instance, if $\mathbf{h}=\mathbf{e}$, then $T(\mathcal{U},\mathbf{e})$ is the set of all convex combinations of rows of $\mathbf{B} \in \mathcal{U}$ for all $\mathbf{B} \in \mathcal{U}$. Note that $T(\mathcal{U},\mathbf{e})$ is not necessarily convex in general. We discuss several examples below to illustrate properties of $T(\mathcal{U},\mathbf{h})$.

Example 1 ($\mathcal{U}$ where $T(\mathcal{U},\mathbf{h})$ is non-convex). Consider the following uncertainty set $\mathcal{U}$:

$$\mathcal{U} = \left\{ \mathbf{B} \in [0,1]^{n\times n} \;\middle|\; B_{ij} = 0,\, \forall i \neq j,\, \sum_{j=1}^{n} B_{jj} \leq 1 \right\}. \tag{2.3.7}$$

$T(\mathcal{U},\mathbf{e})$ is non-convex. Figure 2.1 illustrates $T(\mathcal{U},\mathbf{e})$ when $n=3$. In fact, in Theorem B.4.1, we prove that $T(\mathcal{U},\mathbf{h})$ is non-convex for all $\mathbf{h}>\mathbf{0}$.

On the other hand, in the following two lemmas, we show that $T(\mathcal{U},\mathbf{h})$ can be convex for all $\mathbf{h}>\mathbf{0}$ for some interesting families of examples.

Example 2 (Constraint-wise uncertainty set). Suppose the uncertainty set $\mathcal{U}$ is constraint-wise where each constraint $i\in[m]$ can be selected independently from a compact convex set $\mathcal{U}_i$. In other words, $\mathcal{U}$ is a Cartesian product of $\mathcal{U}_i, i\in[m]$, i.e.,

$$\mathcal{U}=\mathcal{U}_1\times\mathcal{U}_2\times\ldots\times\mathcal{U}_m,$$

then $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h}>\mathbf{0}$. In particular, we have the following lemma.

Lemma 2.3.1. *Suppose the convex compact uncertainty set $\mathcal{U}$ is constraint-wise:*

$$\mathcal{U}=\{\mathbf{B}\mid\mathbf{B}^T\mathbf{e}_j\in\mathcal{U}_j\},$$

where $\mathcal{U}_j$ is a compact convex set in $\mathbb{R}^n_+$. Then $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h}>\mathbf{0}$.

We provide a detailed proof of Lemma 2.3.1 in Appendix B.2. In Ben-Tal and Nemirovski [5], the authors show that a static solution is optimal for the adjustable robust problem if $\mathcal{U}$ is constraint-wise. In later discussion, we show that a static solution is optimal if $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h}>\mathbf{0}$; thereby extending the result in [5] for the case where $\mathcal{U}$ is contained in the non-negative orthant. Note that constraint-wise uncertainty is analogous to independence in distributions for stochastic optimization problems.

Example 3 (Symmetric projections). Suppose the uncertainty set $\mathcal{U}$ has symmetric projections, i.e., the projections of $\mathcal{U}$ onto each of its row sets are the same, then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. In particular, we have the following lemma.

Lemma 2.3.2. *Consider any convex compact uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$. For any $j = 1, \ldots, m$, let*

$$\mathcal{U}_j = \left\{ \mathbf{b} \mid \exists\, \mathbf{B} \in \mathcal{U}, \mathbf{b} = \mathbf{B}^T \mathbf{e}_j \right\}.$$

Suppose $\mathcal{U}$ is such that $\mathcal{U}_i = \mathcal{U}_j$ for all $i, j \in \{1, \ldots, m\}$. Then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$.

We provide a proof of Lemma 2.3.2 in Appendix B.2.

The family of *permutation invariant* sets is an important sub-class of sets with symmetric projections. A set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$ is *permutation invariant* if for any $\mathbf{B} \in \mathcal{U}$ and any permutation σ of $\{1, \ldots, m\}$, the matrix obtained by permuting the rows of $\mathbf{B}$, say $\mathbf{B}^\sigma$ where $B^\sigma_{ij} = B_{\sigma(i)j}$, also belongs to $\mathcal{U}$. For example, consider the following set:

$$\mathcal{U} = \left\{ \mathbf{B} \in \mathbb{R}_+^{m \times n} \;\middle|\; \sum_{i,j} B_{ij} \leq 1 \right\}.$$

It is easy to observe that $\mathcal{U}$ is permutation invariant. Any permutation invariant set $\mathcal{U}$ has symmetric projections since

$$\mathbf{b} \in \mathcal{U}_j \text{ for some } j = 1, \ldots, m \Rightarrow \mathbf{b} \in \mathcal{U}_i,\ \forall i = 1, \ldots, m.$$

Therefore, $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ for any permutation invariant $\mathcal{U}$. However, not all sets with symmetric projections are permutation invariant. For example, consider the following set $\mathcal{U} \subseteq \mathbb{R}_+^{2\times 2}$:

$$\mathcal{U} = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 \leq 1, \\ x_2 + x_3 + x_4 \leq 1, \\ x_i \geq 0, i = 1,2,3,4. \end{array} \right\}.$$

Note that $\mathcal{U}$ has symmetric projections as its projections on both rows are $\{\mathbf{x} \in \mathbb{R}_+^2 \mid \mathbf{e}^T \mathbf{x} \leq 1\}$. However, $\mathcal{U}$ is not permutation invariant as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{U}, \text{ but } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin \mathcal{U}.$$

2.3.3 Main Theorem

Now, we introduce the main theorem which gives a tight characterization of the optimality of the static solution for the two-stage adjustable robust problem.

Theorem 2.3.3 (Optimality of Static Solutions). *Let z_{AR} be the objective of the two-stage adjustable robust problem Π_{AR} defined in* (1.2.1) *and z_{Rob} be that of Π_{Rob} defined in* (1.2.2)*. Then, $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$ if $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. Furthermore, if $T(\mathcal{U},\mathbf{h})$ is not convex for some $\mathbf{h} > \mathbf{0}$, then there exist an instance such that $z_{\mathsf{AR}} > z_{\mathsf{Rob}}$.*

Note that $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$ implies that the optimal static robust solution for Π_{Rob} is also optimal for the adjustable robust problem Π_{AR}. In order to prove Theorem 2.3.3, we first reformulate

$\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ and $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ as optimization problems over $T(\mathcal{U},\mathbf{h})$. From (2.3.5) and the definition of $T(\mathcal{U},\mathbf{h})$, we have the following lemma.

Lemma 2.3.4. *Given $\mathcal{U} \subseteq \mathbb{R}^{m\times n}_+$ and $\mathbf{h} > \mathbf{0}$, the one-stage adjustable robust problem $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ defined in* (2.3.3) *can be formulated as*

$$z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}) = \min_{\lambda,\mathbf{b}} \{\lambda \mid \lambda\mathbf{b} \geq \mathbf{d},\ \mathbf{b} \in T(\mathcal{U},\mathbf{h})\}. \tag{2.3.8}$$

We can also reformulate $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ as an optimization problem over $\mathsf{conv}(T(\mathcal{U},\mathbf{h})$ as follows.

Lemma 2.3.5. *Given $\mathcal{U} \subseteq \mathbb{R}^{m\times n}_+$ and $\mathbf{h} > \mathbf{0}$, the one-stage static robust problem $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ defined in* (2.3.4) *can be formulated as*

$$z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) = \min_{\lambda,\mathbf{b}} \{\lambda \mid \lambda\mathbf{b} \geq \mathbf{d},\ \mathbf{b} \in \mathsf{conv}(T(\mathcal{U},\mathbf{h}))\}. \tag{2.3.9}$$

We provide a detailed proof in Appendix B.3.

Note that the formulations (2.3.8) and (2.3.9) are bilinear optimization problems over $T(\mathcal{U},\mathbf{h})$ and not necessarily convex even if $T(\mathcal{U},\mathbf{h})$ is convex. However, the reformulations provide interesting geometric insights about the relation between the adjustable robust and static robust problems with respect to properties of $\mathcal{U}$. Figure 2.2 illustrates the geometric interpretation of $z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ and $z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ in terms of the formulation in Lemma 2.3.4 and 2.3.5. Now, we are ready to prove Theorem 2.3.3.

Proof of Theorem 2.3.3 Suppose $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. Let $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ be an

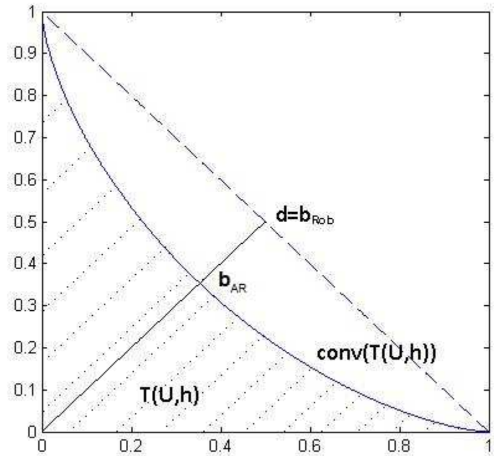


Figure 2.2: A geometric illustration of $z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ and $z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ when $\mathbf{d}=\frac{1}{2}\mathbf{e}$: For $z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$, the optimal solution $\mathbf{b}$ is the point where $\mathbf{d}$ intersects with the boundary of $T(\mathcal{U},\mathbf{h})$, while for $z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$, the optimal solution is $\mathbf{b}=\mathbf{d}$ since $\mathbf{d}\in\mathsf{conv}(T(\mathcal{U},\mathbf{h}))$.

optimal fully-adjustable solution to Π_{AR}. Then

$$\begin{aligned} z_{\mathsf{AR}} &= \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B}\in\mathcal{U}}\max_{\mathbf{y}(\mathbf{B})\geq\mathbf{0}}\left\{\mathbf{d}^T\mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B})\leq\mathbf{h}-\mathbf{A}\mathbf{x}^*\right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}-\mathbf{A}\mathbf{x}^*), \end{aligned}$$

where the second equation follows from (2.3.3). We can assume without loss of generality that $\mathbf{h}-\mathbf{A}\mathbf{x}^*>\mathbf{0}$ as discussed earlier. Now,

$$\begin{aligned} z_{\mathsf{Rob}} &\geq \mathbf{c}^T\mathbf{x}^* + \max_{\mathbf{y}\geq\mathbf{0}}\left\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y}\leq\mathbf{h}-\mathbf{A}\mathbf{x}^*, \forall\mathbf{B}\in\mathcal{U}\right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}-\mathbf{A}\mathbf{x}^*) \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}-\mathbf{A}\mathbf{x}^*) \qquad (2.3.10) \\ &= z_{\mathsf{AR}}, \end{aligned}$$

where the first inequality follows as $\mathbf{x}^*$ is a feasible first-stage solution for the static robust problem. The second equation follows from (2.3.4). Equation (2.3.10) follows from Lemma 2.3.4 and Lemma 2.3.5 and the fact that $T(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*)$ is convex. Also, from (2.3.1) we know that $z_{\mathsf{AR}} \geq z_{\mathsf{Rob}}$ which implies $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$.

Conversely, suppose $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$. For the sake of contradiction, assume $T(\mathcal{U}, \mathbf{h})$ is non-convex for some $\mathbf{h} = \hat{\mathbf{h}}$. Then, there must exist $\hat{\mathbf{b}} \in \mathbb{R}^n_+$ such that $\hat{\mathbf{b}} \notin T(\mathcal{U}, \hat{\mathbf{h}})$ but $\hat{\mathbf{b}} \in \mathsf{conv}(T(\mathcal{U}, \hat{\mathbf{h}}))$. Consider the following instance of Π_{AR} and Π_{Rob}:

$$\mathbf{A} = \mathbf{0},\ \mathbf{c} = \mathbf{0},\ \mathbf{h} = \hat{\mathbf{h}},\ \mathbf{d} = \hat{\mathbf{b}}.$$

Note that in this case, we have $z_{\mathsf{AR}} = z^I_{\mathsf{AR}}(\mathcal{U}, \hat{\mathbf{h}})$ and $z_{\mathsf{Rob}} = z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}})$. Therefore, by our assumption,

$$z^I_{\mathsf{AR}}(\mathcal{U}, \hat{\mathbf{h}}) = z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}}).$$

Since $\hat{\mathbf{b}} \in \mathsf{conv}(T(\mathcal{U}, \hat{\mathbf{h}}))$, $\alpha = 1, \mathbf{b} = \hat{\mathbf{b}}$ is a feasible solution for $z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}})$. Therefore, $z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}}) \leq 1$, which implies $z^I_{\mathsf{AR}}(\mathcal{U}, \hat{\mathbf{h}}) \leq 1$. However, this would further imply that there exists some $\mathbf{b_1} \in T(\mathcal{U}, \hat{\mathbf{h}})$ such that $\mathbf{b_1} \geq \hat{\mathbf{b}}$. Since $\mathcal{U}$ is down-monotone by our assumption, so is $T(\mathcal{U}, \hat{\mathbf{h}})$ (see Appendix A.1). Therefore, $\hat{\mathbf{b}} \in T(\mathcal{U}, \hat{\mathbf{h}})$, which is a contradiction. $\square$

We give examples of families of $\mathcal{U}$ in Lemma 2.3.1 and Lemma 2.3.2, where $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. We would like to note that for a given $\mathbf{h} > \mathbf{0}$, it is not necessarily tractable to decide whether $T(\mathcal{U}, \mathbf{h})$ is convex or not for any arbitrary $\mathcal{U}$.

2.3.4 Min-Max Theorem Interpretation

We can interpret a special case of Theorem 2.3.3 as a min-max theorem. Consider the case where $\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}$, in which we have

$$z_{\mathsf{AR}} = z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h}), z_{\mathsf{Rob}} = z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}).$$

Recall:

$$z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h}) = \min_{B \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T \mathbf{y} \;\middle|\; \mathbf{B}\mathbf{y} \leq \mathbf{h} \right\}.$$

We define the following function for $\mathbf{y} \in \mathbb{R}^n_+, \mathbf{B} \in \mathcal{U} \subseteq \mathbb{R}^{m \times n}_+$:

$$f(\mathbf{y}, \mathbf{B}) = \begin{cases} \mathbf{d}^T \mathbf{y}, \text{ if } \mathbf{B}\mathbf{y} \leq \mathbf{h} \\ -\infty, \text{ otherwise.} \end{cases}$$

Now, we can express $z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h})$ and $z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h})$ as follows:

$$z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h}) = \min_{B \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} f(\mathbf{y}, \mathbf{B})$$

and

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}) = \max_{\mathbf{y} \geq \mathbf{0}} \min_{B \in \mathcal{U}} f(\mathbf{y}, \mathbf{B}).$$

Therefore, from Theorem 2.3.3, we have:

$$\min_{B \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} f(\mathbf{y}, \mathbf{B}) = \max_{\mathbf{y} \geq \mathbf{0}} \min_{B \in \mathcal{U}} f(\mathbf{y}, \mathbf{B}) \tag{2.3.11}$$

if $T(\mathcal{U},\mathbf{h})$ is convex. We would like to note that the min-max equality (2.3.11) does not follow from the general Saddle-Point Theorem [17] since $f(\mathbf{y},\mathbf{B})$ is not a quasi-convex function of $\mathbf{B}$.

2.4 Measure of Non-convexity and Approximation Bound

In this section, we introduce a measure of non-convexity for general down-monotone compact sets in the non-negative orthant and show that the performance of the optimal static solution is related to this measure of non-convexity of the transformation $T(\mathcal{U},\mathbf{h})$ of the uncertainty set $\mathcal{U}$. We also compare our bound with the symmetry bound introduced by Bertsimas and Goyal [11]. In particular, we show that our bound is at least as good as the symmetry bound, and is significantly better in many cases.

Definition 2.4.1. *Given a down-monotone compact set $\mathcal{S} \subseteq \mathbb{R}^n_+$ that is contained in the non-negative orthant, the* measure of non-convexity $\kappa(\mathcal{S})$ *is defined as follows.*

$$\kappa(\mathcal{S}) = \min\{\alpha \mid \mathsf{conv}(\mathcal{S}) \subseteq \alpha\mathcal{S}\}. \tag{2.4.1}$$

For any down-monotone compact set $\mathcal{S} \subseteq \mathbb{R}^n_+$, $\kappa(\mathcal{S})$ is the smallest factor by which $\mathcal{S}$ must be scaled to contain the convex hull of $\mathcal{S}$. If $\mathcal{S}$ is convex, then $\kappa(\mathcal{S}) = 1$. Therefore, if the uncertainty set $\mathcal{U}$ is constraint-wise, then $\kappa(T(\mathcal{U},\mathbf{h})) = 1$ for all $\mathbf{h} > \mathbf{0}$ (Lemma 2.3.1). On the other hand, if $\mathcal{S}$ is non-convex, then $\kappa(\mathcal{S}) > 1$. For instance, consider the following set:

$$\mathcal{S}^n = \left\{ \mathbf{x} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} x_j^{\frac{1}{2}} \leq 1 \right\}$$

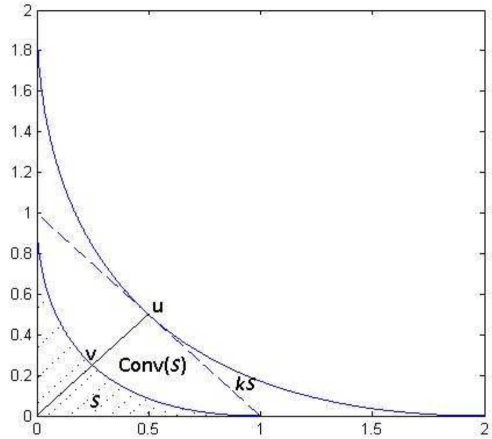


Figure 2.3: A geometric illustration of $\kappa(\mathcal{S})$ when $n=2$: $\mathcal{S}$ is down-monotone and shaded with dot lines, $\mathsf{conv}(\mathcal{S})$ is marked with dashed lines, and the outmost curve is the boundary of $\kappa\cdot\mathcal{S}$. Draw a line from the origin which intersects with the boundary of $\mathcal{S}$ at v and the boundary of $\mathsf{conv}(\mathcal{S})$ at u. $\kappa(\mathcal{S})$ is the largest ratio of such u and v's.

Figure 2.3 illustrates set $\mathcal{S}^n$ for $n=2$ and its measure of non-convexity. We would like to emphasize that given an arbitrary down-monotone compact set $\mathcal{U}$ and $\mathbf{h}>\mathbf{0}$, it is not necessarily tractable to compute $\kappa(T(\mathcal{U},\mathbf{h}))$.

2.4.1 Approximation Bounds

In this section, we relate the performance of the static solution for the two-stage adjustable robust problem to the measure of non-convexity of $T(\mathcal{U},\mathbf{h})$.

Additional Assumption: For the analysis of the performance bound for static solutions, we make two additional assumptions in the model (1.2.1): the first-stage objective coefficients $\mathbf{c}$ and the first-stage decision variables $\mathbf{x}$ in Π_{AR} (1.2.1) are both non-negative. We work with these assumptions for the rest of this and next chapter.

Theorem 2.4.2. *For any down-monotone, compact set* $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$*, let*

$$\rho(\mathcal{U}) = \max\{\kappa(T(\mathcal{U},\mathbf{h})) \mid \mathbf{h} > \mathbf{0}\}.$$

Let z_{AR} *be the optimal value of* Π_{AR} *in* (1.2.1) *and* z_{Rob} *be the optimal value of* Π_{Rob} *in* (1.2.2) *under the additional assumption that* $\mathbf{x} \geq \mathbf{0}$ *and* $\mathbf{c} \geq \mathbf{0}$*. Then,*

$$z_{\mathsf{AR}} \leq \rho(\mathcal{U}) \cdot z_{\mathsf{Rob}}.$$

Furthermore, we can show that the bound is tight.

Proof. Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ is an optimal fully-adjustable solution for Π_{AR}. Based on the discussion in Theorem 2.3.3, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Then

$$\begin{aligned} z_{\mathsf{AR}} &= \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}(\mathbf{B})\geq\mathbf{0}} \left\{\mathbf{d}^T\mathbf{y}(\mathbf{B}) \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*\right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*), \end{aligned}$$

and

$$\begin{aligned} z_{\mathsf{Rob}} &\geq \mathbf{c}^T\mathbf{x}^* + \max_{\mathbf{y}} \left\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*,\ \forall \mathbf{B} \in \mathcal{U}\right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*). \end{aligned} \tag{2.4.2}$$

Let $\hat{\mathbf{h}} = \mathbf{h} - \mathbf{A}\mathbf{x}^*$ and $\kappa = \kappa(T(\mathcal{U},\hat{\mathbf{h}}))$. From Lemmas 2.3.5, we can reformulate $z^I_{\mathsf{Rob}}(\mathcal{U},\hat{\mathbf{h}})$ as follows.

$$z^I_{\mathsf{Rob}}(\mathcal{U},\hat{\mathbf{h}}) = \min_{\mathbf{b}\in\mathsf{conv}(T(\mathcal{U},\hat{\mathbf{h}}))} \{\lambda \mid \lambda\mathbf{b} \geq \mathbf{d}, \lambda \geq 0\}. \tag{2.4.3}$$

Suppose $(\hat{\lambda}, \hat{\mathbf{b}})$ be the minimizer for $z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}})$ in (2.4.3). Therefore,

$$\hat{\mathbf{b}} \in \mathsf{conv}(T(\mathcal{U}, \hat{\mathbf{h}})) \Rightarrow \frac{1}{\kappa} \cdot \hat{\mathbf{b}} \in T(\mathcal{U}, \hat{\mathbf{h}}).$$

Now,

$$\begin{aligned} z^I_{\mathsf{AR}}(\mathcal{U}, \hat{\mathbf{h}}) &= \min_{\mathbf{b} \in T(\mathcal{U}, \hat{\mathbf{h}})} \{\lambda \mid \lambda \mathbf{b} \geq \mathbf{d}, \lambda \geq 0\} \\ &\leq \kappa \cdot \hat{\lambda} \\ &= \kappa \cdot z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}}), \end{aligned} \tag{2.4.4}$$

where the first equation follows from the reformulation of $z^I_{\mathsf{AR}}(\mathcal{U}, \hat{\mathbf{h}})$ in Lemma 2.3.4. The second inequality follows as $(1/\kappa)\hat{\mathbf{b}} \in T(\mathcal{U}, \hat{\mathbf{h}})$ and $\hat{\lambda}\hat{\mathbf{b}} \geq \mathbf{d}$ and the last equality follows as $z^I_{\mathsf{Rob}}(\mathcal{U}, \hat{\mathbf{h}}) = \hat{\lambda}$. Therefore,

$$\begin{aligned} z_{\mathsf{AR}} &= \mathbf{c}^T \mathbf{x}^* + z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq \mathbf{c}^T \mathbf{x}^* + \kappa \cdot z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) && (2.4.5) \\ &\leq \kappa \cdot \left(\mathbf{c}^T \mathbf{x}^* + z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*)\right) \\ &\leq \rho(\mathcal{U}) \cdot z_{\mathsf{Rob}}, && (2.4.6) \end{aligned}$$

where (2.4.5) follows from (2.4.4) and the last inequality follows from (2.4.2) and the fact that $\kappa = \kappa(T(\mathcal{U}, \hat{\mathbf{h}})) \leq \rho(\mathcal{U})$.

Tightness of the bound. We can show that the bound is tight. In particular, given any scalar

$\mu > 1$ and some $n \in \mathbb{Z}_+$, take $\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{d} = \mathbf{e}, \mathbf{h} = \mathbf{e}$ and $\theta = \log_\mu n$. Consider the following uncertainty set:

$$\mathcal{U} = \left\{ \mathbf{B} \in [0,1]^{n\times n} \;\middle|\; B_{ij} = 0,\ \forall i \neq j,\ \sum_{j=1}^{n} B_{jj}^{\theta} \leq 1. \right\}.$$

For Π_{AR}, we have

$$\begin{aligned} z_{\mathsf{AR}} &= \min_{\mathbf{B}} \max_{\mathbf{y}} \left\{ \mathbf{e}^T\mathbf{y} \;\middle|\; B_{jj}y_j \leq 1, j = 1,\ldots,n, \sum_{j=1}^{n} B_{jj}^{\theta} \leq 1 \right\} \\ &= \min_{\mathbf{B}} \left\{ \sum_{j=1}^{n} \frac{1}{B_{jj}} \;\middle|\; \sum_{j=1}^{n} B_{jj}^{\theta} \leq 1 \right\}. \end{aligned}$$

This is a convex problem and by solving the KKT conditions, we have the optimal solution as $B_{jj} = n^{-\frac{1}{\theta}}$ for $j = 1,\ldots,n$. Hence, the optimal value of $z_{\mathsf{AR}} = n \cdot n^{\frac{1}{\theta}} = n^{1+\frac{1}{\theta}}$.

For Π_{Rob}, we have

$$z_{\mathsf{Rob}} = \max_{\mathbf{y}} \left\{ \mathbf{e}^T\mathbf{y} \;\middle|\; B_{jj}y_j \leq 1,\ \forall \mathbf{B} \in \mathcal{U},\ j = 1,\ldots,n. \right\}$$

The constraints essentially enforce $B_{jj}y_j \leq 1$ for all $B_{jj} \leq 1$, $j = 1,\ldots,n$. We only need to consider the extreme case where $B_{jj} = 1$, which yields $y_j = 1$. Therefore, $z_{\mathsf{Rob}} = n$ and

$$\frac{z_{\mathsf{AR}}}{z_{\mathsf{Rob}}} = \frac{n^{1+\frac{1}{\theta}}}{n} = n^{\frac{1}{\theta}} = \mu.$$

In Appendix B.4,we show that $\kappa(T(\mathcal{U},\mathbf{h})) = n^{\frac{1}{\theta}}$ for all $\mathbf{h} > \mathbf{0}$. Therefore, $\rho(\mathcal{U}) = n^{\frac{1}{\theta}} = \mu$ and $z_{\mathsf{AR}} = \rho(\mathcal{U}) \cdot z_{\mathsf{Rob}}$. □

In Theorem 2.4.2, we prove a bound on the optimal objective value z_{AR} of Π_{AR} with respect to the optimal objective value z_{Rob} of Π_{Rob}. Note that this also implies a bound on the performance of the optimal static robust solution for Π_{Rob} for the adjustable robust problem Π_{AR}. Furthermore, in using a static robust solution $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for the two-stage adjustable robust problem, we only implement the first-stage solution $\hat{\mathbf{x}}$ and recompute the optimal second-stage solution $\mathbf{y}(\mathbf{B})$ after the uncertain constraint matrix $\mathbf{B}$ is known. Therefore, the cost of such a solution would only be better than z_{Rob} which is at most $\rho(\mathcal{U}) \cdot z_{\mathsf{AR}}$. We would like to note that given any arbitrary down-monotone uncertainty set $\mathcal{U}$, it is not necessarily tractable to compute $\kappa(T(\mathcal{U},\mathbf{h}))$ or $\rho(\mathcal{U})$. In Table 2.1, we compute $\rho(\mathcal{U})$ for some commonly used uncertainty sets. Moreover, in the following theorem, we show that $\kappa(T(\mathcal{U},\mathbf{h}))$ is at most m for any for any $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ and $\mathbf{h} > \mathbf{0}$.

Theorem 2.4.3. *For any down-monotone convex compact set $\mathcal{U} \in \mathbb{R}_+^{m\times n}$ and $\mathbf{h} > \mathbf{0}$,*

$$\kappa(T(\mathcal{U},\mathbf{h})) \leq m.$$

Proof. Note that

$$T(\mathcal{U},\mathbf{h}) = \left\{ \mathbf{B}^T \mu \;\middle|\; \mathbf{B} \in \mathcal{U}, \mathbf{h}^T \mu = 1, \mu \geq \mathbf{0} \right\}.$$

For all $j = 1, \ldots, m$, let

$$\mathcal{U}_j = \left\{ \left(\frac{1}{h_j}\right) \cdot \mathbf{B}^T \mathbf{e}_j \;\middle|\; \mathbf{B} \in \mathcal{U} \right\}.$$

We can show that

$$\bigcup_{j=1}^{m} \mathcal{U}_j \subseteq T(\mathcal{U},\mathbf{h}) \subseteq \mathsf{conv}\left(\bigcup_{j=1}^{m} \mathcal{U}_j \right). \tag{2.4.7}$$

For any $j = 1, \ldots, m$, consider $\mu = \mathbf{e}_j / h_j$. Then $\mathcal{U}_j = \{\mathbf{B}^T \mu \,|\, \mathbf{B} \in \mathcal{U}\} \subseteq T(\mathcal{U},\mathbf{h})$ for all $j = 1, \ldots, m$.

Consider any $\mathbf{b} \in T(\mathcal{U}, \mathbf{h})$ where

$$\mathbf{b} = \sum_{j=1}^{m} \mathbf{B}^T \mathbf{e}_j \mu_j,$$

for some $\mathbf{B} \in \mathcal{U}$ and $\mu \geq \mathbf{0}$ and $\mathbf{h}^T \mu = 1$. Therefore,

$$\mathbf{b} = \sum_{j=1}^{m} \left(\frac{1}{h_j} \mathbf{B}^T \mathbf{e}_j \right) \cdot (h_j \mu_j) = \sum_{j=1}^{m} \mathbf{b}_j \cdot (h_j \mu_j),$$

where $\mathbf{b}_j \in \mathcal{U}_j$ for all $j \in [m]$ and $h_1\mu_1 + \ldots + h_m\mu_m = 1$ which proves that $\mathbf{b}$ belongs to the convex hull of the union of $\mathcal{U}_j,\ j \in [m]$. From (2.4.7), we have that

$$\mathsf{conv}(T(\mathcal{U}, \mathbf{h})) = \mathsf{conv}\left(\bigcup_{j=1}^{m} \mathcal{U}_j \right).$$

Now consider any $\mathbf{b} \in \mathsf{conv}(T(\mathcal{U}, \mathbf{h}))$. Therefore, $\mathbf{b}$ belongs to the convex hull of union of sets $\mathcal{U}_j$, i.e.,

$$\mathbf{b} = \sum_{j=1}^{m} \mathbf{b}_j \lambda_j,$$

for some $\mathbf{b}_j \in \mathcal{U}_j,\ j = 1, \ldots, m$ and some $\lambda \geq \mathbf{0}$ such that $\lambda_1 + \ldots + \lambda_m = 1$. For all $j = 1, \ldots, m$, let

$$\mathbf{B}_j = h_j \cdot \mathbf{e}_j \mathbf{b}_j^T.$$

Since $\mathbf{b}_j \in \mathcal{U}_j$ and $\mathcal{U}$ is down-monotone, $\mathbf{B}_j \in \mathcal{U}$. Now, let

$$\hat{\mathbf{B}} = \sum_{j=1}^{m} \frac{1}{m} \mathbf{B}_j \ \in \mathcal{U},$$

as $\hat{\mathbf{B}}$ is a convex combination of elements in $\mathcal{U}$. Also, let $\hat{\mu}_j = \lambda_j/h_j$ for all $j = 1,\ldots,m$. Therefore, $\mathbf{h}^T\hat{\mu} = 1$ and $\hat{\mathbf{b}} = \hat{\mathbf{B}}^T\hat{\mu} \in T(\mathcal{U},\mathbf{h})$. Also,

$$\hat{\mathbf{b}} = \frac{1}{m}\cdot\left(\sum_{j=1}\mathbf{B}_j^T\hat{\mu}\right) = \frac{1}{m}\cdot\left(\sum_{j=1}h_j\mathbf{b}_j\mathbf{e}_j^T\hat{\mu}\right) = \frac{1}{m}\sum_{j=1}^{m}\mathbf{b}_j\lambda_j = \frac{1}{m}\cdot\mathbf{b}.$$

□

2.4.2 Comparison with Symmetry Bound [11]

Bertsimas and Goyal [11] consider a general two-stage adjustable robust convex optimization problem with uncertain convex constraints and under mild conditions, show that the performance of a static solution is related to the symmetry of the uncertainty set. In this section, we compare our bound $\rho(\mathcal{U})$ defined in (2.4.1) with the symmetry bound of [11] for the case of two-stage adjustable robust linear optimization problem under uncertain constraints. The notion of symmetry is introduced by Minkowski [30].

Definition 2.4.4. *Given a nonempty convex compact uncertainty set $\mathcal{S} \subseteq \mathbb{R}^m$ and a point $s \in \mathcal{S}$, the symmetry of s with respect to $\mathcal{S}$ is defined as:*

$$\mathsf{sym}(s,\mathcal{S}) := \max\{\alpha \geq 0 \mid \mathbf{s} + \alpha(\mathbf{s} - \hat{\mathbf{s}}) \in \mathcal{S}, \forall \hat{\mathbf{s}} \in \mathcal{S}\}.$$

The symmetry of the set $\mathcal{S}$ is defined as:

$$\mathsf{sym}(\mathcal{S}) := \max\{\mathsf{sym}(s,\mathcal{S}) \mid \mathbf{s} \in \mathcal{S}\}. \tag{2.4.8}$$

The maximizer of (2.4.8) *is called the point of symmetry.*

In Bertsimas and Goyal [11], the authors prove the following bound on the performance of static solution for the two-stage adjustable robust convex optimization with uncertain constraints under some mild conditions.

$$z_{\mathsf{AR}} \leq \left(1 + \frac{1}{\mathsf{sym}(\mathcal{U})}\right) \cdot z_{\mathsf{Rob}}$$

We show that for the case of two-stage adjustable robust linear optimization under uncertain constraints, our approximation bound in 2.4.2 is at least as good as the symmetry bound for all instances.

Theorem 2.4.5. *Consider uncertainty set* $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$. *Then,*

$$\max\{\kappa(T(\mathcal{U},\mathbf{h})) \mid \mathbf{h} > \mathbf{0}\} \leq \left(1 + \frac{1}{\mathsf{sym}(\mathcal{U})}\right).$$

Proof. For a given $\mathbf{h} > \mathbf{0}$, from the definition of $\kappa(\cdot)$ in (2.4.1), we have

$$\mathsf{conv}(T(\mathcal{U},\mathbf{h})) \subseteq \kappa(T(\mathcal{U},\mathbf{h})) \cdot T(\mathcal{U},\mathbf{h}).$$

Therefore, it is sufficient to show

$$\mathsf{conv}(T(\mathcal{U},\mathbf{h})) \subseteq \left(1 + \frac{1}{\mathsf{sym}(\mathcal{U})}\right) \cdot T(\mathcal{U},\mathbf{h}) \tag{2.4.9}$$

for all $\mathbf{h} > \mathbf{0}$. Let

$$\mathbf{B}_0 = \arg\max\{\mathsf{sym}(\mathbf{B},\mathcal{U}) \mid \mathbf{B} \in \mathcal{U}\}$$

be the point of symmetry. Then, from the result in [12], we have

$$\left(1+\frac{1}{\mathsf{sym}(\mathcal{U})}\right)\cdot \mathbf{B}_0 \geq \mathbf{B}, \forall \mathbf{B} \in \mathcal{U}. \tag{2.4.10}$$

Now, given any $\mathbf{h} > \mathbf{0}$, consider an arbitrary $\mathbf{b} \in \mathsf{conv}(T(\mathcal{U},\mathbf{h}))$. There exists $\mathbf{B}_1,\ldots,\mathbf{B}_K \in \mathcal{U}$ such that

$$\mathbf{b} = \sum_{j=1}^{K} \theta_j \mathbf{B}_j^T \lambda^j,\ \mathbf{h}^T\lambda^j = 1,\ \lambda^j \in \mathbb{R}^m_+,\ j = 1,\ldots,K,\ \mathbf{e}^T\theta = 1,\ \theta \in \mathbb{R}^K_+.$$

From (2.4.10), since $\mathbf{B}_1,\ldots,\mathbf{B}_K \in \mathcal{U}$, we have

$$\begin{aligned}\mathbf{b} &\leq \sum_{j=1}^{K} \theta_j \left(1+\frac{1}{\mathsf{sym}(\mathcal{U})}\right)\mathbf{B}_0^T\lambda^j \\ &= \left(1+\frac{1}{\mathsf{sym}(\mathcal{U})}\right)\mathbf{B}_0^T\left(\sum_{j=1}^{K}\theta_j\lambda^j\right) \in \left(1+\frac{1}{\mathsf{sym}(\mathcal{U})}\right)\cdot T(\mathcal{U},\mathbf{h}).\end{aligned}$$

The last inequality holds because

$$\mathbf{h}^T\left(\sum_{j=1}^{K}\theta_j\lambda^j\right) = \left(\sum_{j=1}^{K}\theta_j\mathbf{h}^T\lambda^j\right) = \mathbf{e}^T\theta = 1.$$

Since $\mathcal{U}$ is down-monotone by assumption, so is $T(\mathcal{U},\mathbf{h})$ (Appendix A.1), and we have

$$\mathbf{b} \in \left(1+\frac{1}{\mathsf{sym}(\mathcal{U})}\right)\cdot T(\mathcal{U},\mathbf{h}).$$

□

Theorem 2.4.5 states that our bound in Theorem 2.4.2 is at least as good as the symmetry bound

Uncertainty set $\mathcal{U}$	$\rho(\mathcal{U})$	Symmetry bound [11]
Constraint-wise set $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_m$	1	$1 + \frac{1}{\min_{1<i<m} \mathsf{sym}(\mathcal{U}_i)}$
Permutation invariant $\mathcal{U}$	1	$1 + \frac{1}{\mathsf{sym}(\mathcal{U})}$
$\{\mathbf{B} : \lvert\lvert\mathbf{B}\rvert\rvert_{\theta_1} \leq 1, \lvert\lvert\mathbf{B}\rvert\rvert_{\theta_2} \leq r\} \subset \mathbb{R}_+^{m\times n}$	1	$1 + r(mn)^{\frac{1}{\theta_1}}$
$\{\mathbf{B} : \lvert\lvert\mathbf{B}\rvert\rvert_1 \leq 1\} \subset \mathbb{R}_+^{m\times n}$	1	$1 + mn$
$\{\mathbf{B} : \lvert\lvert\mathbf{B}\rvert\rvert_\theta \leq 1\} \subset \mathbb{R}_+^{m\times n}$	1	$1 + (mn)^{\frac{1}{\theta}}$
$\{\mathbf{B} : \sum_{j=1}^n B_{jj} \leq 1, B_{ij} = 0,\ \forall i \neq j\} \subset [0,1]^{n\times n}$	n	$1 + n$
$\{\mathbf{B} : \sum_{j=1}^n B_{jj}^\theta \leq 1, B_{ij} = 0,\ \forall i \neq j\} \subset [0,1]^{n\times n}, \theta > 1$	$n^{\frac{1}{\theta}}$	$1 + n^{\frac{1}{\theta}}$

Table 2.1: A comparison between the non-convexity bound and the symmetry bound for various uncertainty sets. All the norms are entry-wise, i.e., $||\mathbf{A}||_p = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p\right)^{1/p}$.

and in many cases significantly better. For instance, consider the following example:

$$\mathcal{U} = \left\{ \mathbf{B} \in [0,1]^{n\times n} \;\middle|\; \sum_{i,j} B_{ij} \leq 1 \right\}.$$

In this case, $\mathcal{U}$ has symmetric projections. Therefore, from Lemma 2.3.2, $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ and

$$\max\{\kappa(T(\mathcal{U}, \mathbf{h})) \mid \mathbf{h} > \mathbf{0}\} = 1.$$

On the other hand, $\mathcal{U}$ is a simplex and $\mathsf{sym}(\mathcal{U}) = \frac{1}{n^2}$ [12]. Therefore,

$$1 + \frac{1}{\mathsf{sym}(\mathcal{U})} = n^2 + 1,$$

which is a significantly worse bound. Table 2.1 compares our bound with the symmetry bound for several interesting uncertainty sets.

2.5 Two-stage Model with Constraint and Objective Uncertainty

In this section, we consider a two-stage adjustable robust linear optimization problem where both constraint and objective coefficients are uncertain. In particular, we consider the following two-stage adjustable robust problem $\Pi_{\mathsf{AR}}^{(B,d)}$.

$$\begin{aligned} z_{\mathsf{AR}}^{(B,d)} = \max\, \mathbf{c}^T\mathbf{x} + \min_{(\mathbf{B},\mathbf{d})\in\mathcal{U}}\ \max_{\mathbf{y}(\mathbf{B},\mathbf{d})}\ & \mathbf{d}^T\mathbf{y}(\mathbf{B},\mathbf{d}) \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B},\mathbf{d}) \ &\leq\ \mathbf{h} \\ \mathbf{x} \ &\in\ \mathbb{R}_+^{n_1} \\ \mathbf{y}(\mathbf{B},\mathbf{d}) \ &\in\ \mathbb{R}_+^{n_2}, \end{aligned} \tag{2.5.1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n_1}, \mathbf{c} \in \mathbb{R}_+^{n_1}$, $\mathbf{h} \in \mathbb{R}_+^m$, and $(\mathbf{B},\mathbf{d})$ are uncertain second-stage constraint matrix and objective that belong to a convex compact uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n_2} \times \mathbb{R}_+^{n_2}$. We consider the case where the uncertainty in constraint matrix $\mathbf{B}$ does not depend on the uncertainty in objective coefficients $\mathbf{d}$. Therefore,

$$\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d,$$

where $\mathcal{U}^B \subseteq \mathbb{R}_+^{m\times n_2}$ is a convex compact uncertainty set of constraint matrices and $\mathcal{U}^d \subseteq \mathbb{R}_+^{n_2}$ is a convex compact uncertainty set of the second-stage objective. As previous sections, we can assume without loss of generality that $\mathcal{U}^B$ is down-monotone.

We formulate the corresponding static robust problem $\Pi^{(B,d)}_{\mathsf{Rob}}$, as follows.

$$\begin{aligned} z^{(B,d)}_{\mathsf{Rob}} = \max_{\mathbf{x},\mathbf{y}} \; \min_{\mathbf{d}\in\mathcal{U}^d} \; & \mathbf{c}^T\mathbf{x} + \mathbf{d}^T\mathbf{y} \\ & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \;\le\; \mathbf{h},\; \forall \mathbf{B} \in \mathcal{U}^B \\ & \mathbf{x} \;\in\; \mathbb{R}^{n_1}_+ \\ & \mathbf{y} \;\in\; \mathbb{R}^{n_2}_+. \end{aligned} \tag{2.5.2}$$

We can compute an optimal static robust solution efficiently. It is easy to see that the separation problem for (2.5.2) can be solved in polynomial time. In fact, we can also give a compact LP formulation to compute an optimal static robust solution similar to (1.2.2). Now, suppose the optimal solution of $\Pi^{(B,d)}_{\mathsf{Rob}}$ is $(\mathbf{x}^*, \mathbf{y}^*)$, then $\mathbf{x} = \mathbf{x}^*, \mathbf{y}(\mathbf{B},\mathbf{d}) = \mathbf{y}^*$ for all $(\mathbf{B},\mathbf{d}) \in \mathcal{U}$ is a feasible solution to $\Pi^{(B,d)}_{\mathsf{AR}}$. Therefore,

$$z^{(B,d)}_{\mathsf{AR}} \ge z^{(B,d)}_{\mathsf{Rob}}. \tag{2.5.3}$$

We prove the following main theorem.

Theorem 2.5.1. *Let $z^{(B,d)}_{\mathsf{AR}}$ be the optimal objective value of $\Pi^{(B,d)}_{\mathsf{AR}}$ in* (2.5.1) *defined over the uncertainty $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$. Let $z^{(B,d)}_{\mathsf{Rob}}$ be the optimal objective value of $\Pi^{(B,d)}_{\mathsf{Rob}}$ in* (2.5.2). *Also, let*

$$\rho(\mathcal{U}^B) = \max_{\mathbf{h}>\mathbf{0}} \kappa(T(\mathcal{U}^B, \mathbf{h})).$$

Then,

$$z^{(B,d)}_{\mathsf{AR}} \le \rho(\mathcal{U}^B) \cdot z^{(B,d)}_{\mathsf{Rob}}.$$

Furthermore, the bound is tight.

If $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$, then $\rho(\mathcal{U}^B) = 1$ and $z_{\mathsf{AR}}^{(B,d)} \leq z_{\mathsf{Rob}}^{(B,d)}$. In this case, Theorem 2.5.1 implies that a static solution is optimal for the adjustable robust problem $\Pi_{\mathsf{AR}}^{(B,d)}$. Therefore, if $\mathcal{U}^B$ is constraint-wise or has symmetric projections then $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ (Lemmas 2.3.1 and 2.3.2). In general, the performance of static solution depends on the worst-case measure of non-convexity of $T(\mathcal{U}^B, \mathbf{h})$ for all $\mathbf{h} > \mathbf{0}$. Surprisingly, the approximation bound for the static solution does not depend on the uncertain set of objectives $\mathcal{U}^d$.

To prove the Theorem 2.5.1, we need to introduce the following one-stage models as in Section 2.3, $\Pi_{\mathsf{AR}}^I(\mathcal{U}, \mathbf{h})$ and $\Pi_{\mathsf{Rob}}^I(\mathcal{U}, \mathbf{h})$.

$$
\begin{aligned}
z_{\mathsf{AR}}^I(\mathcal{U}, \mathbf{h}) = \min_{(\mathbf{B},\mathbf{d}) \in \mathcal{U}} \max_{\mathbf{y}} \ & \mathbf{d}^T \mathbf{y} \\
& \mathbf{B}\mathbf{y} \leq \mathbf{h} \\
& \mathbf{y} \in \mathbb{R}_+^n,
\end{aligned}
\tag{2.5.4}
$$

$$
\begin{aligned}
z_{\mathsf{Rob}}^I(\mathcal{U}, \mathbf{h}) = \max_{\mathbf{y}} \min_{\mathbf{d} \in \mathcal{U}^{\mathbf{d}}} \ & \mathbf{d}^T \mathbf{y} \\
& \mathbf{B}\mathbf{y} \leq \mathbf{h}, \ \forall \mathbf{B} \in \mathcal{U}^{\mathbf{B}} \\
& \mathbf{y} \in \mathbb{R}_+^n.
\end{aligned}
\tag{2.5.5}
$$

where $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$ and $\mathbf{h} > \mathbf{0}$. Similar to Lemma 2.3.4 and Lemma 2.3.5, we can reformulate the above problems as optimization problems over the transformation set $T(\mathcal{U}^B, \mathbf{h})$.

Lemma 2.5.2. *The one-stage adjustable robust problem* $\Pi_{\mathsf{AR}}^I(\mathcal{U}, \mathbf{h})$ *defined in* (2.5.4) *can be writ-*

ten as:

$$z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}) = \min_{\lambda,\mathbf{b},\mathbf{d}}\{\,\lambda \mid \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in T(\mathcal{U}^B,\mathbf{h}), \mathbf{d} \in \mathcal{U}^d\}.$$

Proof. Consider $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$, by writing the dual of its inner maximization problem, we have

$$\begin{aligned} z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}) &= \min_{\mathbf{B},\mathbf{d},\alpha}\{\mathbf{h}^T\alpha \mid \mathbf{B}^T\alpha \geq \mathbf{d}, (\mathbf{B},\mathbf{d}) \in \mathcal{U}, \alpha \in \mathbb{R}^m_+\} \\ &= \min_{\lambda,\mathbf{B},\mathbf{d},\alpha}\left\{\lambda\mathbf{h}^T\left(\frac{\alpha}{\lambda}\right) \,\middle|\, \lambda\mathbf{B}^T\left(\frac{\alpha}{\lambda}\right) \geq \mathbf{d}, \mathbf{h}^T\alpha = \lambda, (\mathbf{B},\mathbf{d}) \in \mathcal{U}, \alpha \in \mathbb{R}^m_+\right\}. \\ &= \min_{\lambda,\mathbf{b},\mathbf{d}}\{\,\lambda \mid \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in T(\mathcal{U}^B,\mathbf{h}), \mathbf{d} \in \mathcal{U}^d\}, \end{aligned}$$

where the last equality holds because $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$. ☐

Lemma 2.5.3. *The one-stage static robust problem* $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ *defined in* (2.5.5) *can be written as:*

$$z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) = \min_{\lambda,\mathbf{b},\mathbf{d}}\{\,\lambda \mid \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}(T(\mathcal{U}^B,\mathbf{h})), \mathbf{d} \in \mathcal{U}^d\}.$$

We provide a detailed proof in Appendix B.5. Now, we are ready to prove Theorem 2.5.1.

Proof of Theorem 2.5.1 Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B},\mathbf{d}), (\mathbf{B},\mathbf{d}) \in \mathcal{U})$ is a fully-adjustable optimal solution for $\Pi^{(B,d)}_{\mathsf{AR}}$. As discussed earlier, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Then,

$$\begin{aligned} z^{(B,d)}_{\mathsf{AR}} &= \mathbf{c}^T\mathbf{x}^* + \min_{(\mathbf{B},\mathbf{d})\in\mathcal{U}} \max_{\mathbf{y}(\mathbf{B},\mathbf{d})\geq\mathbf{0}} \left\{\mathbf{d}^T\mathbf{y}(\mathbf{B},\mathbf{d}) \mid \mathbf{B}\mathbf{y}(\mathbf{B},\mathbf{d}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*\right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*), \end{aligned} \tag{2.5.6}$$

and

$$z_{\mathsf{Rob}}^{(B,d)} \geq \mathbf{c}^T\mathbf{x}^* + \max_{\mathbf{y}\geq\mathbf{0}} \min_{\mathbf{d}\in\mathcal{U}^d} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U}^B \right\}$$
$$= \mathbf{c}^T\mathbf{x}^* + z_{\mathsf{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*). \tag{2.5.7}$$

Let $\hat{\mathbf{h}} = \mathbf{h} - \mathbf{A}\mathbf{x}^*$ and $\kappa = \kappa(T(\mathcal{U}^B, \hat{\mathbf{h}}))$. Suppose $(\hat{\lambda}, \hat{\mathbf{b}}, \hat{\mathbf{d}})$ is an optimal solution for $\Pi_{\mathsf{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}})$.

Therefore,

$$\hat{\mathbf{b}} \in \mathsf{conv}(T(\mathcal{U}^B, \hat{\mathbf{h}})) \;\Rightarrow\; \frac{1}{\kappa} \cdot \hat{\mathbf{b}} \in T(\mathcal{U}^B, \hat{\mathbf{h}}).$$

Also,

$$\hat{\lambda} \cdot \hat{\mathbf{b}} \geq \hat{\mathbf{d}} \;\Rightarrow\; (\kappa\hat{\lambda}) \cdot \left(\frac{1}{\kappa} \hat{\mathbf{b}} \right) \geq \hat{\mathbf{d}},$$

which implies that $(\kappa\hat{\lambda}, \hat{\mathbf{b}}/\kappa, \hat{\mathbf{d}})$ is a feasible solution to $\Pi_{\mathsf{AR}}^I(\mathcal{U}, \hat{\mathbf{h}})$ and

$$z_{\mathsf{AR}}^I(\mathcal{U}, \hat{\mathbf{h}}) \leq \kappa \cdot z_{\mathsf{Rob}}^I(\mathcal{U}, \hat{\mathbf{h}}).$$

From (2.5.6), we have

$$\begin{aligned} z_{\mathsf{AR}}^{(B,d)} &= \mathbf{c}^T\mathbf{x}^* + z_{\mathsf{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq \mathbf{c}^T\mathbf{x}^* + \kappa \cdot z_{\mathsf{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\leq \kappa \cdot (\mathbf{c}^T\mathbf{x}^* + z_{\mathsf{Rob}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*)) \qquad (2.5.8) \\ &\leq \kappa \cdot z_{\mathsf{Rob}}^{(B,d)}, \end{aligned}$$

where (2.5.8) holds because $\kappa \geq 1$, the last inequality holds from (2.5.7).

We can show that the bound is tight using the same family of uncertainty sets of matrices $\mathcal{U}_\theta^B$

in the discussion of Theorem 2.4.2:

$$\mathcal{U}^B_\theta = \left\{ \mathbf{B} \in [0,1]^{n\times n} \;\middle|\; B_{ij} = 0,\ \forall i \neq j,\ \sum_{j=1}^{n} B^{\theta}_{jj} \leq 1 \right\}.$$

Consider the following instance of $\Pi^{(B,d)}_{\mathsf{AR}}$ and $\Pi^{(B,d)}_{\mathsf{Rob}}$:

$$\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{h} = \mathbf{e}, \mathcal{U}^d = \{\mathbf{e}\}.$$

From the discussion in Theorem 2.4.2, the bound in Theorem 2.5.1 is tight. □

Note that surprisingly, the bound only depends on the measure of non-convexity of $\mathcal{U}^B$ and is not related to $\mathcal{U}^d$. Therefore, if $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$, then a static solution is optimal for the adjustable robust problem $\Pi^{(B,d)}_{\mathsf{AR}}$ irrespective of $\mathcal{U}^d$. As a special case where there is no uncertainty in $\mathbf{B}$, i.e., $\mathcal{U}^B = \{\mathbf{B}^0\}$, and the only uncertainty is in $\mathcal{U}^d$, $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ and a static solution is optimal. In fact, the optimality of static solution in this case follows from von Neumann's Min-max theorem [35]. Therefore, we can interpret the result as a generalization of von Neumann's theorem.

General Case when $\mathcal{U}$ is not a Cartesian product. For the general case where the uncertainty set $\mathcal{U}$ of constraint matrices $\mathbf{B}$ and objective coefficients $\mathbf{d}$ is not a Cartesian product of the respective uncertainty sets, our bound of Theorem 2.5.1 does not extend. Consider the following example:

$$\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{h} = \mathbf{e},$$

$$\mathcal{U} = \left\{ (\mathbf{B}, \mathbf{d}) \in \mathbb{R}_+^{n \times n} \times \mathbb{R}_+^n \;\middle|\; \sum_{i=1}^n B_{ii} \le \sum_{j=1}^n d_j \le 1, \mathbf{d} \ge \frac{\varepsilon}{n}\mathbf{e}, B_{ij} = 0, \forall i \neq j \right\}.$$

Now,

$$\begin{aligned} z_{\mathsf{AR}}^{(B,d)} &= \min_{(\mathbf{B},\mathbf{d}) \in \mathcal{U}} \max_{\mathbf{y}} \left\{ \mathbf{d}^T \mathbf{y} \;\middle|\; B_{jj} y_j \le 1, \ \forall j = 1, \ldots, n, \ \mathbf{y} \ge \mathbf{0} \right\} \\ &= \min_{(\mathbf{B},\mathbf{d}) \in \mathcal{U}} \sum_{j=1}^n \frac{d_j}{B_{jj}} \\ &\ge 1, \end{aligned}$$

where the second equation follows from the fact that at optimum of the outer minimization problem, $B_{jj} > 0$ for all $j = 1, \ldots, n$ and $y_j = 1/B_{jj}$ for the inner maximization problem. Otherwise, if $B_{jj} = 0$ for some j, then y_j and $d_j y_j$ are both unbounded as $d_j > \varepsilon/n > 0$. The last equality follows as for any $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$,

$$\sum_{j=1}^n B_{jj} \le \sum_{j=1}^n d_j,$$

which implies that $B_{jj} \le d_j$ for some $j \in [n]$.

For the robust problem $\Pi_{\mathsf{Rob}}^{(B,d)}$, consider any static solution $\mathbf{y} \ge \mathbf{0}$. For all $j = 1, \ldots, n$,

$$B_{jj} y_j \le 1, \ \forall (\mathbf{B}, \mathbf{d}) \in \mathcal{U}.$$

Since there exist $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$ such that $B_{jj} = 1$, $y_j \le 1$ for all $j = 1, \ldots, n$. Moreover, $\mathbf{y} = \mathbf{e}$ is a feasible solution as $B_{jj} \le 1$ for all $(\mathbf{B}, \mathbf{d}) \in \mathcal{U}$ for all $j \in [n]$. Therefore,

$$z_{\mathsf{Rob}}^{(B,d)} = \min_{(\mathbf{B},\mathbf{d}) \in \mathcal{U}} \mathbf{d}^T \mathbf{e} \le \varepsilon,$$

where the second inequality follows by setting $d_j = \varepsilon/n$ for all $j = 1, \ldots, n$. Therefore,

$$z_{\mathsf{AR}}^{(B,d)} \geq \frac{1}{\varepsilon} \cdot z_{\mathsf{Rob}}^{(B,d)},$$

where $\varepsilon > 0$ is arbitrary. Therefore, the performance of the optimal static robust solution as compared to the optimal fully adjustable solution can not be bounded by the measure of non-convexity as in Theorem 2.5.1.

2.5.1 Constraint, Right-hand-side and Objective Uncertainty

In this section, we discuss the case where the right-hand-side, the constraint and the objective coefficients are all uncertain. Consider the following adjustable robust problem $\Pi_{\mathsf{AR}}^{(B,h,d)}$.

$$\begin{aligned} z_{\mathsf{AR}}^{(B,h,d)} = \max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} + & \min_{(\mathbf{B},\mathbf{h},\mathbf{d}) \in \mathcal{U}^{B,h,d}} \max_{\mathbf{y}(\mathbf{B},\mathbf{h},\mathbf{d})} \mathbf{d}^T\mathbf{y}(\mathbf{B},\mathbf{h},\mathbf{d}) \\ & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B},\mathbf{h},\mathbf{d}) \leq \mathbf{h} \\ & \mathbf{x} \in \mathbb{R}_+^{n_1} \\ & \mathbf{y}(\mathbf{B},\mathbf{h},\mathbf{d}) \in \mathbb{R}_+^{n_2}, \end{aligned} \tag{2.5.9}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n_1}, \mathbf{c} \in \mathbb{R}_+^{n_1}$. In this case, $(\mathbf{B},\mathbf{h},\mathbf{d}) \in \mathcal{U}^{B,h,d}$ are uncertain and $\mathcal{U}^{B,h,d} \subseteq \mathbb{R}_+^{m \times n_2} \times \mathbb{R}_+^m \times \mathbb{R}_+^{n_2}$ is convex and compact. We consider the case that the uncertainties in constraint matrix $\mathbf{B}$ and right-hand-side vector $\mathbf{h}$ are independent of the uncertainty in the objective coefficients $\mathbf{d}$, i.e.,

$$\mathcal{U}^{B,h,d} = \mathcal{U}^{B,h} \times \mathcal{U}^d,$$

where $\mathcal{U}^{B,h} \subseteq \mathbb{R}^{m\times(n_2+1)}$ is the convex compact uncertainty set of constraint matrices and right-hand-side vectors, and $\mathcal{U}^d \subseteq \mathbb{R}^{n_2}$ is the convex compact set of the constraint coefficients.

The corresponding static robust version $\Pi_{\mathsf{Rob}}^{(B,h,d)}$, can be formulated as follows.

$$
\begin{aligned}
z_{\mathsf{Rob}}^{(B,h,d)} = \max_{\mathbf{x},\mathbf{y}} \min_{\mathbf{d}\in\mathcal{U}^d} & \ \mathbf{c}^T\mathbf{x}+\mathbf{d}^T\mathbf{y} \\
& \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{y} \ \leq \ \mathbf{h}, \ \forall(\mathbf{B},\mathbf{h}) \in \mathcal{U}^{B,h} \\
& \mathbf{x} \ \in \ \mathbb{R}_+^{n_1} \\
& \mathbf{y} \ \in \ \mathbb{R}_+^{n_2}.
\end{aligned} \tag{2.5.10}
$$

We can compute an optimal solution for (2.5.10) efficiently by solving a compact LP formulation for its separation problem. Now, we study the performance of static solution and show that it is optimal if $\mathcal{U}^{B,h}$ is constraint-wise. In particular, we have the following theorem.

Theorem 2.5.4. *Let $z_{\mathsf{AR}}^{(B,h,d)}$ be the optimal value of $\Pi_{\mathsf{AR}}^{(B,h,d)}$ defined in* (2.5.9) *and $z_{\mathsf{Rob}}^{(B,h,d)}$ be the optimal value of $\Pi_{\mathsf{Rob}}^{(B,h,d)}$ defined in* (2.5.10)*. Suppose $\mathcal{U}^{B,h}$ is constraint-wise, then the static solution is optimal for $\Pi_{\mathsf{AR}}^{(B,h,d)}$, i.e.,*

$$
z_{\mathsf{AR}}^{(B,h,d)} = z_{\mathsf{Rob}}^{(B,h,d)}. \tag{2.5.11}
$$

To prove Theorem 2.5.4, we need to introduce the one-stage models. Consider the one-stage adjustable robust problem $\Pi^I_{\mathsf{AR}}(\mathcal{U}^{B,h,d})$

$$z^I_{\mathsf{AR}}(\mathcal{U}^{B,h,d}) = \min_{(\mathbf{B},\mathbf{h},\mathbf{d})\in\mathcal{U}^{B,h,d}} \max_{\mathbf{y}} \mathbf{d}^T\mathbf{y}$$
$$\mathbf{B}\mathbf{y} \leq \mathbf{h} \qquad (2.5.12)$$
$$\mathbf{y} \in \mathbb{R}^n_+,$$

where $\mathcal{U}^{B,h,d} = \mathcal{U}^{B,h} \times \mathcal{U}^d$. The corresponding one-stage static robust problem $\Pi^I_{\mathsf{Rob}}(\mathcal{U}^{B,h,d})$ can be formulated as follows

$$z^I_{\mathsf{Rob}}(\mathcal{U}^{B,h,d}) = \max_{\mathbf{y}} \min_{\mathbf{d}\in\mathcal{U}^d} \mathbf{d}^T\mathbf{y}$$
$$\mathbf{B}\mathbf{y} \leq \mathbf{h}, \ \forall(\mathbf{B},\mathbf{h}) \in \mathcal{U}^{B,h} \qquad (2.5.13)$$
$$\mathbf{y} \in \mathbb{R}^n_+,$$

We can reformulate these models as optimization problems over $T(\mathcal{U}^{B,h},\mathbf{e})$.

Lemma 2.5.5. *The one-stage adjustable robust problem* $\Pi^I_{\mathsf{AR}}(\mathcal{U}^{B,h,d})$ *defined in* (2.5.12) *can be written as*

$$z^I_{\mathsf{AR}}(\mathcal{U}^{B,h,d}) = \min_{\lambda,\mathbf{b},t,\mathbf{d}} \{\lambda t \mid \lambda\mathbf{b} \geq \mathbf{d}, (\mathbf{b},t) \in T(\mathcal{U}^{B,h},\mathbf{e}), \mathbf{d} \in \mathcal{U}^d\}.$$

We present the proof of Lemma 2.5.5 in Appendix B.6.

Lemma 2.5.6. *The one-stage static-robust problem* $\Pi^I_{\mathsf{Rob}}(\mathcal{U}^{B,h,d})$ *defined in* (2.5.13) *can be written as*

$$z^I_{\mathsf{Rob}}(\mathcal{U}^{B,h,d}) = \min_{\lambda,\mathbf{b},t,\mathbf{d}} \{\lambda t \mid \lambda\mathbf{b} \geq \mathbf{d}, (\mathbf{b},t) \in \mathsf{conv}(T(\mathcal{U}^{B,h},\mathbf{e})), \mathbf{d} \in \mathcal{U}^d\}.$$

We present the proof of Lemma 2.5.6 in Appendix B.6. Now, with the reformulations in Lemma 2.5.5 and Lemma 2.5.6, we are ready to prove Theorem 2.5.4.

Proof of Theorem 2.5.4 Suppose the optimal solution of $\Pi_{\mathsf{Rob}}^{(B,h,d)}$ is $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, then $\mathbf{x} = \tilde{\mathbf{x}}, \mathbf{y}(\mathbf{B}, \mathbf{h}, \mathbf{d}) = \tilde{\mathbf{y}}$ for all $(\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}$ is a feasible solution to $\Pi_{\mathsf{AR}}^{(B,h,d)}$. Therefore,

$$z_{\mathsf{AR}}^{(B,h,d)} \geq z_{\mathsf{Rob}}^{(B,h,d)}. \tag{2.5.14}$$

On the other hand, suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}, \mathbf{h}, \mathbf{d}), (\mathbf{B}, \mathbf{h}, \mathbf{d}) \in \mathcal{U}^{B,h,d})$ is a fully-adjustable optimal solution for $\Pi_{\mathsf{AR}}^{(B,h,d)}$. As discussed earlier, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$ for all $\mathbf{h}$ such that $(\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h}$ for some $\mathbf{B}$. Then,

$$\begin{aligned} z_{\mathsf{AR}}^{(B,h,d)} &= \mathbf{c}^T\mathbf{x}^* + \min_{(\mathbf{B},\mathbf{h},\mathbf{d}) \in \mathcal{U}^{B,h,d}} \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^* \right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z_{\mathsf{AR}}^I(\mathcal{U}^{B,h-Ax^*,d}), \end{aligned} \tag{2.5.15}$$

and

$$\begin{aligned} z_{\mathsf{Rob}}^{(B,h,d)} &\geq \mathbf{c}^T\mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \min_{\mathbf{d} \in \mathcal{U}^d} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h} \right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z_{\mathsf{Rob}}^I(\mathcal{U}^{B,h-Ax^*,d}). \end{aligned} \tag{2.5.16}$$

Since $\mathcal{U}^{B,h}$ is constraint-wise, so is $\mathcal{U}^{B,h-Ax^*}$. From Lemma 2.3.1, $T(\mathcal{U}^{B,h-Ax^*}, \mathbf{e})$ is convex and $T(\mathcal{U}^{B,h-Ax^*}, \mathbf{e}) = \mathsf{conv}(T(\mathcal{U}^{B,h-Ax^*}, \mathbf{e}))$. From Lemma 2.5.5 and Lemma 2.5.6, this implies that

$$z^I_{\mathsf{AR}}(\mathcal{U}^{B,h-Ax^*,d}) = z^I_{\mathsf{Rob}}(\mathcal{U}^{B,h-Ax^*,d}).$$

Therefore, from (2.5.15) and (2.5.16), we have

$$z^{(B,h,d)}_{\mathsf{AR}} \leq z^{(B,h,d)}_{\mathsf{Rob}}.$$

Together with (2.5.14), we have $z^{(B,h,d)}_{\mathsf{AR}} = z^{(B,h,d)}_{\mathsf{Rob}}$. □

We would like to note that we can not extend the approximation bounds similar to Theorem 2.5.1 in this case. In fact, the measure of non-convexity of $T(\mathcal{U}^{B,h}, \mathbf{e})$ is not even well defined in this case since $\mathcal{U}^{B,h}$ is not down-monotone.

2.6 Conclusion

In this chapter, we study the performance of static robust solution as an approximation of two-stage adjustable robust linear optimization problem under uncertain constraints and objective coefficients. We show that the adjustable problem is $\Omega(\log n)$-hard to approximate. In fact, for a more general case where the uncertainty set $\mathcal{U}$ and objective coefficients $\mathbf{d}$ are not constrained in the non-negative orthant, we show that the adjustable robust problem is $\Omega(2^{\log^{1-\varepsilon} m})$-hard to approximate for any constant $0 < \varepsilon < 1$.

We give a tight characterization of the performance of static solution and relate it to the measure of non-convexity of the transformation $T(\mathcal{U}, \cdot)$ of the uncertainty set $\mathcal{U}$. In particular, we show that a static solution is optimal if $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. If $T(\mathcal{U}, \cdot)$ is not convex, the measure of non-convexity of $T(\mathcal{U},\cdot)$ gives a tight bound on the performance of static solutions. For several interesting families of uncertainty sets such as constraint-wise or symmetric projections, we show that $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h} > 0$; thereby, extending the result of Ben-Tal and Nemirovski [5] for the case where $\mathcal{U}$ is contained in the non-negative orthant. Also, our approximation bound is better than the symmetry bound in Bertsimas and Goyal [11].

We also extend our result to models where both constraint and objective coefficients are uncertain. We show that if $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$, where $\mathcal{U}^B$ is the set of uncertain second-stage constraint matrices $\mathbf{B}$ and $\mathcal{U}^d$ is the set of uncertain second-stage objective, then the performance of static solution is related to the measure of non-convexity of $T(\mathcal{U}^B,\cdot)$. In particular, a static solution is optimal if $T(\mathcal{U}^B,\mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. Surprisingly, the performance of static solution does not depend on the uncertainty set $\mathcal{U}^d$. We also present several examples to illustrate such optimality and the tightness of the bound.

Our results develop new geometric intuition about the performance of static robust solutions for adjustable robust problems. The reformulations of the adjustable robust and static robust problems based on the transformation $T(\mathcal{U},\cdot)$ of the uncertainty set $\mathcal{U}$ give us interesting insights about properties of $\mathcal{U}$ where the static robust solution does not perform well. Therefore, our results provide useful guidance in selecting uncertainty sets such that the adjustable robust problem can be well approximated by a static solution.

Chapter 3

The Adaptivity Gap in Two-Stage Robust Linear Optimization under Column-wise and Constraint-wise Uncertain Constraints

3.1 Introduction

In this chapter, we consider the two-stage adjustable robust linear packing problems Π_{AR} (1.2.1) under *column-wise* and *constraint-wise* uncertain constraint coefficients. In the previous chapter, we provide an instance-based tight approximation bound on the performance of static robust solution for Π_{AR}, which is related to a measure of non-convexity of a transformation of the uncertainty set. However, for the following family of uncertainty sets of non-negative diagonal matrices with an upper bound on the ℓ_1-norm of the diagonal vector

$$\mathcal{U} = \left\{ \mathbf{B} \in \mathbb{R}_+^{m\times m} \;\middle|\; B_{ij} = 0,\ \forall i \neq j,\ \sum_{i=1}^{m} B_{ii} \leq 1 \right\},$$

the measure of non-convexity is m. Moreover, it is not necessarily tractable to compute the measure of non-convexity for an arbitrary convex compact set. We would like to note that such (diagonal) uncertainty sets do not arise naturally in practice. For instance, consider the resource allocation problem where the uncertainty set $\mathcal{U}$ represents the set of uncertain resource requirement matrices. A constraint on the diagonal relates requirements of different resources across different demands, which is not a naturally arising relation. This motivates us to study the special class of *column-wise* and *constraint-wise* sets. In particular,

$$\mathcal{U} = \{\mathbf{B} \in \mathbb{R}_+^{m\times n} \mid \mathbf{B}\mathbf{e}_j \in C_j,\ j \in [n], \mathbf{B}^T\mathbf{e}_i \in R_i, i \in [m]\},$$

where $C_j \subseteq \mathbb{R}_+^m$ for all $j \in [n]$ and $R_i \subseteq \mathbb{R}_+^n$ for all $i \in [m]$ are compact, convex and down-monotone sets. We assume that the sets $C_j,\ j \in [n]$ and R_i, $i \in [m]$ are such that linear optimization problems over $\mathcal{U}$ can be solved in time that is polynomial in the encoding length of $\mathcal{U}$. We refer to the above uncertainty set as a column-wise and constraint-wise set since the constraints describing the uncertainty set $\mathcal{U}$ involve entries of only a single column or a single row of the matrix. In the resource allocation problem, this would imply that we can have a constraint on the resource requirements of a particular resource for different demands, and a constraint on resource requirements of different resources for any particular demand.

Outline. In Section 3.2, we present the separation problem of the adjustable robust problem and the corresponding static robust problem. In Sections 3.3 and 3.4, we present the bounds on the adaptivity gap for column-wise uncertainty sets. We extend the analysis to the general case of column-wise and constraint-wise uncertainty sets in Section 3.5. In Section 3.6, we compare our result with the measure of non-convexity bound in previous chapter and extend our bound to the case where the objective coefficients are also uncertain in Section 3.7.

3.2 Adjustable Robust Problem: Separation Problem.

Before proving the adaptivity gap for the general column-wise and constraint-wise uncertainty sets, we first consider the case where the uncertainty set $\mathcal{U}$ is column-wise. Recall that $\mathcal{U}$ being column-wise implies that

$$\mathcal{U} = \{[\mathbf{b}_1\, \mathbf{b}_2 \ldots \mathbf{b}_n] \mid \mathbf{b}_j \in \mathcal{U}_j,\, j \in [n]\},$$

where $\mathcal{U}_j \subseteq \mathbb{R}^m_+$ is a compact, convex, down-monotone set for all $j \in [n]$.

3.2.1 The Separation Problem.

In this section, we consider the separation problem for the two-stage adjustable robust problem and a reformulation of the one-stage static robust problem introduced by Soyster [34]. In particular, we have the following epigraph reformulation of Π_{AR}.

$$\begin{aligned} z_{\mathsf{AR}} = \max\ & \mathbf{c}^T\mathbf{x} + z \\ & z \ \leq \ \mathbf{d}^T\mathbf{y}(\mathbf{B}),\ \forall \mathbf{B} \in \mathcal{U} \\ & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B}) \ \leq \ \mathbf{h},\ \forall \mathbf{B} \in \mathcal{U} \\ & \mathbf{x}, \mathbf{y}(\mathbf{B}) \ \geq \ \mathbf{0}. \end{aligned}$$

Consider the following separation problem.

Separation problem: Given $\mathbf{x} \geq \mathbf{0}, z$, decide whether

$$\min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}\} \ \geq \ z, \tag{3.2.1}$$

or give a violating hyperplane by exhibiting $\mathbf{B} \in \mathcal{U}$ such that

$$\max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}\} < z.$$

In Appendix C.1, we show that a γ-approximate algorithm for the separation problem (3.2.1) implies a γ-approximate algorithm for the two-stage adjustable robust problem. Moreover, from previous discussion, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x} > \mathbf{0}$. Therefore, we can rescale $\mathcal{U}$ by $\hat{\mathcal{U}} = [\mathsf{diag}(\mathbf{h} - \mathbf{A}\mathbf{x})]^{-1}\mathcal{U}$ so that the right-hand-side $(\mathbf{h} - \mathbf{A}\mathbf{x})$ is $\mathbf{e}$. Note that $\hat{\mathcal{U}}$ is also a convex, compact, down-monotone and column-wise set. Therefore, we can assume without loss of generality that the right-hand-side is $\mathbf{e}$. In addition, we can interpret the separation problem as the one-stage adjustable robust problem $\Pi^I_{\mathsf{AR}}(\mathcal{U}, \mathbf{e})$ as in (2.3.3). For the ease of notation, we denote it as z^I_{AR}. By taking the dual of the inner maximization problem, we have

$$z^I_{\mathsf{AR}} = \min\{\mathbf{e}^T\mathbf{v} \mid \mathbf{B}^T\mathbf{v} \geq \mathbf{d}, \mathbf{B} \in \mathcal{U}, \mathbf{v} \geq \mathbf{0}\}$$

On the other hand, we consider the corresponding one-stage static robust problem $\Pi^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{e})$ as in (2.3.4).

$$z^I_{\mathsf{Rob}} = \max_{\mathbf{y} \geq \mathbf{0}}\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{e}, \forall \mathbf{B} \in \mathcal{U}\}.$$

We can reformulate z^I_{Rob} as a compact LP using the following result of Soyster [34].

Theorem 3.2.1 (Soyster [34])**.** *Suppose $\mathcal{U} \subseteq \mathbb{R}^{m\times n}_+$ is a compact, convex, and column-wise uncertainty set. Let $\hat{\mathbf{B}} \in \mathbb{R}^{m\times n}$ be such that*

$$\hat{B}_{ij} = \max\{B_{ij} \mid \mathbf{B} \in \mathcal{U}\}, \; \forall i \in [m], j \in [n]. \tag{3.2.2}$$

Then,

$$\max_{\mathbf{y} \geq \mathbf{0}}\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{e}, \forall \mathbf{B} \in \mathcal{U}\} = \max\{\mathbf{d}^T\mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{e}, \mathbf{y} \geq \mathbf{0}\}. \tag{3.2.3}$$

For the sake of completeness, we provide the proof of Theorem 3.2.1 in Appendix C.2. Therefore, we can reformulate z^I_{Rob} as follows.

$$z^I_{\mathsf{Rob}} = \min\{\mathbf{e}^T\mathbf{v} \mid \hat{\mathbf{B}}^T\mathbf{v} \geq \mathbf{d}, \mathbf{v} \geq \mathbf{0}\}, \tag{3.2.4}$$

where $\hat{\mathbf{B}}$ is as defined in (3.2.2).

3.2.2 Worst Case Instances for Adaptivity Gap.

In this section, we show that the adaptivity gap is worst on column-wise uncertainty set when each column set is a simplex. In particular, we prove the following theorem.

Theorem 3.2.2. *Given an arbitrary convex, compact, down-monotone and column-wise uncertainty set* $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ *with* $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_n$*, let* $\hat{\mathbf{B}}$ *be defined as in* (3.2.2)*. For each* $j \in [n]$*, let*

$$\hat{\mathcal{U}}_j = \left\{ \mathbf{b} \in \mathbb{R}_+^m \;\middle|\; \sum_{i=1}^{m} \frac{1}{\hat{B}_{ij}} b_i \leq 1, b_i = 0, \forall i : \hat{B}_{ij} = 0 \right\}, \; \forall j \in [n].$$

and

$$\hat{\mathcal{U}} = \left\{ [\mathbf{b}_1 \; \mathbf{b}_2 \; \ldots \; \mathbf{b}_n] \;\middle|\; \mathbf{b}_j \in \hat{\mathcal{U}}_j, \; \forall j \in [n] \right\}.$$

Let $z_{\mathsf{AR}}(\mathcal{U})$ *(*$z_{\mathsf{AR}}(\hat{\mathcal{U}})$ *respectively) and* $z_{\mathsf{Rob}}(\mathcal{U})$ *(*$z_{\mathsf{Rob}}(\hat{\mathcal{U}})$ *respectively) be the optimal values of the two-stage adjustable robust problem and the static robust problem over uncertainty set* $\mathcal{U}$ *(*$\hat{\mathcal{U}}$ *respectively). Then,*

$$z_{\mathsf{AR}}(\hat{\mathcal{U}}) \geq z_{\mathsf{AR}}(\mathcal{U}) \text{ and } z_{\mathsf{Rob}}(\hat{\mathcal{U}}) = z_{\mathsf{Rob}}(\mathcal{U}).$$

Proof. Given arbitrary $\mathbf{b} \in \hat{\mathcal{U}}_j, j \in [n]$, $\mathbf{b}$ is a convex combination of $\hat{B}_{ij}\mathbf{e}_i, i \in [m]$, which further implies that $\mathbf{b} \in \mathcal{U}_j$. Therefore, $\mathbf{B} \in \hat{\mathcal{U}}$ implies that $\mathbf{B} \in \mathcal{U}$ and we have $\hat{\mathcal{U}} \subseteq \mathcal{U}$. Therefore, any $\mathbf{x}$ that is feasible for $\Pi_{\mathsf{AR}}(\mathcal{U})$ is feasible for $\Pi_{\mathsf{AR}}(\hat{\mathcal{U}})$, and we have $z_{\mathsf{AR}}(\hat{\mathcal{U}}) \geq z_{\mathsf{AR}}(\mathcal{U})$.

Since $\hat{\mathcal{U}} \subseteq \mathcal{U}$, any feasible solution for $\Pi_{\mathsf{Rob}}(\mathcal{U})$ is also feasible for $\Pi_{\mathsf{Rob}}(\hat{\mathcal{U}})$. Therefore, $z_{\mathsf{Rob}}(\hat{\mathcal{U}}) \geq z_{\mathsf{Rob}}(\mathcal{U})$. Conversely, let $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ be the optimal solution of $\Pi_{\mathsf{Rob}}(\hat{\mathcal{U}})$. Noting that $(\hat{\mathbf{x}}, \mathbf{0})$

is a feasible solution for $\Pi_{\mathsf{Rob}}(\mathcal{U})$, we have

$$z_{\mathsf{Rob}}(\mathcal{U}) \geq \mathbf{c}^T \hat{\mathbf{x}} + \max\{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\hat{\mathbf{x}}, \forall \mathbf{B} \in \mathcal{U}\}$$

$$= \mathbf{c}^T \hat{\mathbf{x}} + \max\{\mathbf{d}^T \mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\hat{\mathbf{x}}\},$$

where the last equality follows from Theorem 3.2.1. Furthermore,

$$z_{\mathsf{Rob}}(\hat{\mathcal{U}}) = \mathbf{c}^T \hat{\mathbf{x}} + \max\{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\hat{\mathbf{x}}, \forall \mathbf{B} \in \hat{\mathcal{U}}\}$$

$$= \mathbf{c}^T \hat{\mathbf{x}} + \max\{\mathbf{d}^T \mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\hat{\mathbf{x}}\},$$

where the last equality follows from Theorem 3.2.1 and the fact that $\mathcal{U}$ and $\hat{\mathcal{U}}$ have the same $\hat{\mathbf{B}}$. Therefore, $z_{\mathsf{Rob}}(\mathcal{U}) = z_{\mathsf{Rob}}(\hat{\mathcal{U}})$. □

The above theorem shows that the for column-wise uncertainty sets, the gap between the optimal values of Π_{AR} and Π_{Rob} for a column-wise set is largest when each column set is a simplex. Therefore, to provide the tight bound on the performance of static solutions, we can assume without loss of generality that the column-wise, convex compact uncertainty $\mathcal{U}$ is a Cartesian product of simplices. The worst known instance of Π_{AR} with a column-wise uncertainty set has an adaptivity gap of $\Theta(\log n)$. We present the family of instances below.

Family of Adaptivity Gap Examples. Consider the following instance (I^{LB}) of Π_{AR}:

$$\mathbf{A} = \mathbf{0}, \mathbf{c} = \mathbf{0}, \mathbf{d} = \mathbf{e}, \mathbf{h} = \mathbf{e}, \mathcal{U} = \{[\mathbf{b}_1\, \mathbf{b}_2 \ldots \mathbf{b}_n] \mid \mathbf{b}_j \in \mathcal{U}_j, j \in [n]\}, \qquad (I^{LB})$$

where

$$\mathcal{U}_1 = \left\{ \mathbf{b} \in \mathbb{R}^n_+ \mid 1 \cdot b_1 + 2 \cdot b_2 + \ldots + (n-1) \cdot b_{n-1} + n \cdot b_n \leq 1 \right\},$$

$$\mathcal{U}_2 = \left\{ \mathbf{b} \in \mathbb{R}^n_+ \mid n \cdot b_1 + 1 \cdot b_2 + \ldots + (n-2) \cdot b_{n-1} + (n-1) \cdot b_n \leq 1 \right\},$$

$$\vdots$$

$$\mathcal{U}_n = \left\{ \mathbf{b} \in \mathbb{R}^n_+ \mid 2 \cdot b_1 + 3 \cdot b_2 + \ldots + n \cdot b_{n-1} + 1 \cdot b_n \leq 1 \right\}.$$

Therefore,

$$\mathcal{U}_j = \left\{ \mathbf{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} [(n+i-j+1) \text{ mod } n] \cdot b_i \leq 1 \right\}, \forall j \in [n]$$

where mod is the standard remainder operation and let $(0 \text{ mod } n) = n$. We have the following lemma.

Lemma 3.2.3. *Let z_{AR} be the optimal objective value of the instance (I^{LB}) of Π_{AR} and z_{Rob} be the optimal objective value of the corresponding static robust problem. Then,*

$$z_{\mathsf{AR}} = \Theta(\log n) \cdot z_{\mathsf{Rob}}.$$

We provide the proof in Appendix C.3.

3.3 $O(\log n \cdot \log \Gamma)$ Adaptivity Gap for Column-wise Uncertainty Sets

In this section, we first consider the case of column-wise uncertainty sets and show that a static solution gives a $O(\log n \cdot \log \Gamma)$-approximation for the two-stage adjustable robust problem where

Γ is defined as follows.

$$\begin{aligned} \beta_{\max} &= \max\{\hat{B}_{ij} \mid i \in [m], j \in [n]\} \\ \beta_{\min} &= \min\{\hat{B}_{ij} \mid i \in [m], j \in [n], \hat{B}_{ij} \neq 0\} \\ \Gamma &= 2 \cdot \frac{\beta_{\max}}{\beta_{\min}}, \end{aligned} \tag{3.3.1}$$

where $\hat{\mathbf{B}}$ is defined as in (3.2.2). From Theorem 3.2.2, the worst case adaptivity gap for two-stage adjustable robust problem with column-wise uncertainty sets is achieved when $\mathcal{U}$ is a Cartesian product of simplices. Therefore, to provide a bound on the performance of static solutions, we assume that $\mathcal{U}$ is a Cartesian product of simplices.

3.3.1 One-stage Adjustable and Static Robust Problems

We first compare the one-stage adjustable robust, z^I_{AR} and static robust, z^I_{Rob} problems. Recall,

$$\begin{aligned} z^I_{\mathsf{AR}} &= \min\{\mathbf{e}^T\mathbf{v} \mid \mathbf{B}^T\mathbf{v} \geq \mathbf{d}, \mathbf{B} \in \mathcal{U}, \mathbf{v} \geq \mathbf{0}\} \\ z^I_{\mathsf{Rob}} &= \min\{\mathbf{e}^T\mathbf{v} \mid \hat{\mathbf{B}}^T\mathbf{v} \geq \mathbf{d}, \mathbf{v} \geq \mathbf{0}\}. \end{aligned}$$

Theorem 3.3.1. *Given $\mathbf{d} \in \mathbb{R}^n_+$ and a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}^{m\times n}_+$ that is column-wise with simplex column uncertainty sets $\mathcal{U}_1, \ldots, \mathcal{U}_n$. Let z^I_{AR} be as defined in* (2.3.3)*, and z^I_{Rob} be as defined in* (3.2.4)*. Then*

$$z^I_{\mathsf{AR}} \leq O(\log\Gamma \log n) \cdot z^I_{\mathsf{Rob}}.$$

Our proof exploits the structural properties of the optimal solutions for the adjustable robust and static robust problems. In particular, we relate the one-stage adjustable robust problem to an integer set cover problem and relate the static robust problem to the dual of the corresponding LP relaxation. As earlier, by appropriate rescaling of $\mathcal{U}$, we can assume that the cost $\mathbf{d}$ is $\mathbf{e}$. We can write the one-stage adjustable robust problem as

$$z^I_{\mathsf{AR}} = \min\{\mathbf{e}^T\mathbf{v} \mid \mathbf{v}^T\mathbf{b}^j \geq 1, \mathbf{b}^j \in \mathcal{U}_j, \forall j \in [n], \mathbf{v} \geq \mathbf{0}\}. \tag{3.3.2}$$

and the corresponding static robust problem:

$$\begin{aligned} z^I_{\mathsf{Rob}} &= \max\left\{ \sum_{j=1}^n y_j \;\middle|\; \sum_{j=1}^n \beta^j_i y_j \leq 1,\ \forall i \in [m], \mathbf{y} \geq \mathbf{0} \right\} && (3.3.3)\\ &= \min\{\mathbf{e}^T\mathbf{v} \mid \mathbf{v}^T\beta^j \geq 1, \forall j \in [n], \mathbf{v} \geq \mathbf{0}\}, && (3.3.4) \end{aligned}$$

where

$$\beta^j_i = \hat{B}_{ij},\ \forall i \in [m], j \in [n]. \tag{3.3.5}$$

We first show that there exists an "integral" optimal solution for the one-stage adjustable robust problem (3.3.2).

Lemma 3.3.2. *Consider the one-stage adjustable robust problem* (3.3.2) *where the uncertainty set* $\mathcal{U}$ *is a Cartesian product of simplices* $\mathcal{U}_j$, $j \in [n]$. *Let* $\beta^j, j \in [n]$ *be defined as in* (3.3.5). *Then,* *there exists an optimal solution* $(\bar{\mathbf{v}}, \bar{\mathbf{b}}^j, j \in [n])$ *for* (3.3.2) *such that*

$$\bar{\mathbf{b}}^j = \beta^j_{i_j} \mathbf{e}_{i_j} \text{ for some } i_j \in [m], \ \forall j \in [n]$$

$$\bar{v}_i \in \left\{ 0, 1/\beta^j_i \ \middle| \ j \in [n] \right\}, \ \forall i \in [m].$$

Proof. Suppose this is not the case. Let $(\tilde{\mathbf{v}}, \tilde{\mathbf{b}}^j)$ be an optimal solution for (2.3.3). For all $j \in [n]$, let $\bar{\mathbf{b}}^j$ be an extreme point optimal for

$$\max\{\tilde{\mathbf{v}}^T \mathbf{x} \mid \mathbf{x} \in \mathcal{U}_j\}.$$

Since $\mathcal{U}_j$ is a down-monotone simplex, $\bar{\mathbf{b}}^j = \beta^j_{i_j} \mathbf{e}_{i_j}$ for some $i_j \in [m]$. Note that $\tilde{\mathbf{v}}^T \bar{\mathbf{b}}^j \geq 1$. Therefore, $(\tilde{\mathbf{v}}, \bar{\mathbf{b}}^j, j \in [n])$ is also an optimal solution for (2.3.3). Now, we can reformulate the separation problem as follows.

$$z^I_{\mathsf{AR}} = \min\{\mathbf{e}^T \mathbf{v} \mid \mathbf{v}^T \bar{\mathbf{b}}^j \geq 1, \forall j \in [n]\},$$

where only $\mathbf{v}$ is the decision variable. Let $\bar{\mathbf{v}}$ be an extreme point optimal of the above LP. Then for all $j \in [n]$,

$$\bar{v}_{i_j} \bar{b}^j_{i_j} = v_{i_j} \beta^j_{i_j} \geq 1,$$

as $\bar{\mathbf{b}}^j = \beta^j_{i_j} \mathbf{e}_{i_j}$. Therefore, we have

$$\bar{v}_i \in \left\{ 0, 1/\beta^j_i \ \middle| \ j \in [n] \right\}, \ \forall i \in [m]$$

at optimality. $\square$

From the above lemma, we can reformulate the one-stage adjustable robust problem (2.3.3) as

$$z^I_{\mathsf{AR}} = \min\left\{ \sum_{i=1}^m v_i \;\middle|\; \forall j \in [n], \exists i_j \in [m] \text{ s.t. } v_{i_j}\beta^j_{i_j} \geq 1, \mathbf{v} \geq \mathbf{0} \right\}. \tag{3.3.6}$$

A 0-1 **formulation of** z^I_{AR}**.** We formulate a 0-1 integer program that approximates (3.3.6) within a constant factor. From Lemma 3.3.2, we know that there is an optimal solution $(\mathbf{v}, \mathbf{b}^j, j \in [n])$ for (3.3.6) such that

$$v_i \in \left\{ 0, 1/\beta^j_i \;\middle|\; j \in [n] \right\}, \ \forall i \in [m].$$

Therefore, if $v_i \neq 0$, then

$$\frac{1}{\beta_{\max}} \leq v_i \leq \frac{1}{\beta_{\min}}.$$

To formulate an approximate 0-1 program, we consider discrete values of v_i in multiples of 2 starting from $1/\beta_{\max}$. Denote $T = \lceil \log \Gamma \rceil$ and $\mathcal{T} = \{0, \ldots, T\}$. We consider

$$v_i \in \{0\} \cup \left\{ \frac{2^t}{\beta_{\max}} \;\middle|\; t \in \mathcal{T} \right\}.$$

For any $i \in [m]$, $t \in \mathcal{T}$, let C_{it} denote the set of columns $j \in [n]$ that can be covered by setting $v_i = 2^t/\beta_{\max}$, i.e.,

$$C_{it} = \left\{ j \in [n] \;\middle|\; \frac{2^t}{\beta_{\max}} \cdot \beta^j_i \geq 1 \right\}.$$

Also, for all $i \in [m]$, $t \in \mathcal{T}$, let

$$x_{it} = \begin{cases} 1, \text{ if } v_i = \dfrac{2^t}{\beta_{\max}}, \\ 0, \text{ otherwise,} \end{cases}$$

$$c_t = \frac{2^t}{\beta_{\max}}.$$

Consider the following 0-1 integer program.

$$z_{\mathsf{AR}}^{\mathsf{mod}} = \min \left\{ \sum_{i=1}^{m} \sum_{t=0}^{T} c_t x_{it} \;\middle|\; \sum_{i=1}^{m} \sum_{t \in \mathcal{T}:\, j \in C_{it}} x_{it} \geq 1,\ \forall j \in [n],\ x_{it} \in \{0,1\} \right\}. \tag{3.3.7}$$

In the following lemma, we show that the above integer program approximates z_{AR}^{I} within a constant factor.

Lemma 3.3.3. *The IP problem in* (3.3.7) *is feasible and provides a near-optimal solution for the one-stage adjustable robust problem* z_{AR}^{I} (3.3.6). *In particular, we have*

$$\frac{1}{2} z_{\mathsf{AR}}^{\mathsf{mod}} \leq z_{\mathsf{AR}}^{I} \leq z_{\mathsf{AR}}^{\mathsf{mod}}.$$

Proof. Consider an optimal solution $\mathbf{v}^*$ for z_{AR}^{I} (3.3.6). Note that for all $i \in [m]$, $t \in \mathcal{T}$, let

$$\bar{x}_{it} = \begin{cases} 1, \text{ if } \dfrac{c_t}{2} < v_i^* \leq c_t, \\ 0, \text{ otherwise.} \end{cases}$$

For any $j \in [n]$, there exists $i \in [m]$, $t \in \mathcal{T}$ such that

$$v_i^* \beta_i^j \geq 1.$$

Then, $\bar{\mathbf{x}}$ is a feasible solution to the IP problem (3.3.7) and

$$z_{\mathsf{AR}}^{\mathsf{mod}} \leq \sum_{i=1}^{m} \sum_{t=0}^{T} c_t \bar{x}_{it} \leq 2\mathbf{e}^T \mathbf{v}^* = 2 \cdot z_{\mathsf{AR}}^{I}.$$

Conversely, suppose x_{it}^*, $i \in [m]$, $t \in \mathcal{T}$ is an optimal solution for (3.3.7). We construct a feasible solution $\tilde{\mathbf{v}}$ for (3.3.6) as follows:

$$\tilde{v}_i = \sum_{t \in \mathcal{T}} c_t \cdot x_{it}, \ \forall i \in [m].$$

For each $j \in [n]$, there exists $i \in [m]$ and $t \in \mathcal{T}$ such that $j \in C_{it}$ and $x_{it}^* = 1$. Therefore,

$$v_i \geq c_t = \frac{2^t}{\beta_{\max}},$$

and

$$v_i \beta_i^j \geq \frac{2^t}{\beta_{\max}} \cdot \beta_i^j \geq 1,$$

since $j \in C_{it}$. Therefore, $\tilde{\mathbf{v}}$ is a feasible solution for the one-stage adjustable robust problem (3.3.6) and

$$z_{\mathsf{AR}}^{I} \leq \mathbf{e}^T \tilde{\mathbf{v}} \leq \sum_{i=1}^{m} \sum_{t=0}^{T} c_t x_{it}^* = z_{\mathsf{AR}}^{\mathsf{mod}}.$$

□

Note that (3.3.7) is a 0-1 formulation for the set cover instance problem on ground set of elements $\{1, \ldots, n\}$ and family of subsets C_{it} for all $i \in [m]$, $t \in \mathcal{T}$ where C_{it} has cost c_t. We can formulate the LP relaxation of (3.3.7) as follows.

$$z_{\mathsf{LP}} = \min\left\{ \sum_{i=1}^{m}\sum_{t=0}^{T} c_t x_{it} \;\middle|\; \sum_{i=1}^{m}\sum_{t\in\mathcal{T}:\, j\in C_{it}} x_{it} \geq 1,\ \forall j\in[n],\ x_{it}\geq 0 \right\}. \tag{3.3.8}$$

From [36], we know that the LP relaxation (3.3.8) is an $O(\log n)$-approximation for (3.3.7), i.e.,

$$z_{\mathsf{AR}}^{\mathsf{mod}} \leq O(\log n)\cdot z_{\mathsf{LP}}.$$

Consider the dual of (3.3.8).

$$z_{\mathsf{LP}} \;=\; \max\left\{ \sum_{j=1}^{n} y_j \;\middle|\; \sum_{j\in C_{it}} y_j \leq c_t,\ \forall i\in[m], t\in\mathcal{T},\ y_j\geq 0, \forall j\in[n] \right\} \tag{3.3.9}$$

We relate the dual of (3.3.8) to the one-stage static robust problem (3.2.4) to obtain the desired bound on the adaptivity gap.

Proof of Theorem 3.3.1. From Lemma 3.3.3, it is sufficient to show that

$$z_{\mathsf{LP}} \leq O(\log\Gamma)\cdot z_{\mathsf{Rob}}^{I}.$$

Let $\mathbf{y}^*$ by an optimal solution of (3.3.9). We show that we can construct a feasible solution for (3.3.3) by scaling $\mathbf{y}^*$ by a factor of $O(\log\Gamma)$. For each $i\in[m]$, we have

$$\sum_{j:\beta_i^j\geq\frac{\beta_{\max}}{2^t}} \frac{\beta_{\max}}{2^t} y_j^* \leq 1,\ \forall t\in\mathcal{T}.$$

Sum over all $t \in \mathcal{T}$, we have

$$\sum_{t=0}^{T} \sum_{j:\beta_i^j \geq \frac{\beta_{\max}}{2^t}} \frac{\beta_{\max}}{2^t} y_j^* \leq T+1, \forall i \in [m].$$

Switching the summation, we have

$$\sum_{j=1}^{n} \sum_{t \in \mathcal{T}: \frac{\beta_{\max}}{2^t} \leq \beta_i^j} \frac{\beta_{\max}}{2^t} y_j^* \leq T+1 \leq \log\Gamma + 2, \forall i \in [m]$$

Note that

$$\frac{\beta_{\max}}{2^T} \leq \beta_{\min} \leq \beta_i^j \leq \beta_{\max},$$

which implies

$$\frac{1}{2}\beta_i^j \leq \sum_{t \in \mathcal{T}: \frac{\beta_{\max}}{2^t} \leq \beta_i^j} \frac{\beta_{\max}}{2^t} \leq 2\beta_i^j.$$

Therefore,

$$\hat{y}_j = \frac{1}{2(\log\Gamma + 2)} y_j^*, \ \forall j \in [n]$$

is a feasible solution to the maximization formulation of z_{Rob}^I (3.3.3) and

$$z_{\mathsf{LP}} = \mathbf{e}^T \mathbf{y}^* = O(\log\Gamma) \cdot \mathbf{e}^T \hat{\mathbf{y}} \leq O(\log\Gamma) \cdot z_{\mathsf{Rob}}^I,$$

which completes the proof. □

3.3.2 $O(\log n \cdot \log \Gamma)$ Bound on Adaptivity Gap

Based on the result in Theorem 3.3.1, we show that a static solution gives an $O(\log n \cdot \log \Gamma)$-approximation for the two-stage adjustable robust problem (1.2.1) for column-wise uncertainty sets. In particular, we prove the following theorem.

Theorem 3.3.4. *Let z_{AR} be the objective value of an optimal fully-adjustable solution for the adjustable robust problem Π_{AR}* (1.2.1)*, and z_{Rob} be the optimal objective value of the corresponding static robust problem Π_{Rob}* (1.2.2)*. If $\mathcal{U}$ is a column-wise uncertainty set, then,*

$$z_{AR} \leq O(\log n \cdot \log \Gamma) \cdot z_{Rob}.$$

Proof. Let $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ be an optimal fully-adjustable solution to Π_{AR}. Then,

$$z_{\mathsf{AR}} = \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}} \max_{y(B) \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*\}.$$

From previous chapter, we can assume without loss of generality that $(\mathbf{h} - \mathbf{A}\mathbf{x}^*) > \mathbf{0}$. Let

$$\mathcal{U}^* = [\mathsf{diag}(\mathbf{h} - \mathbf{A}\mathbf{x}^*)]^{-1} \mathcal{U}.$$

Then,

$$z_{\mathsf{AR}} = \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}^*} \max_{y(B) \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y}(\mathbf{B}) \leq \mathbf{e}\}.$$

By writing the dual of the inner maximization problem, we have

$$z_{\mathsf{AR}} = \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B},\mu} \{\mathbf{e}^T \mu \mid \mathbf{B}^T \mu \geq \mathbf{d}, \mathbf{B} \in \mathcal{U}^*, \mu \geq \mathbf{0}\}.$$

On the other hand, since $(\mathbf{x}^*, \mathbf{0})$ is a feasible solution of Π_{Rob}, we have

$$\begin{aligned} z_{\mathsf{Rob}} &\geq \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*, \forall \mathbf{B} \in \mathcal{U}\} \\ &= \mathbf{c}^T \mathbf{x}^* + \max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{e}, \forall \mathbf{B} \in \mathcal{U}^*\}. \end{aligned}$$

Let $\hat{B}$ be defined as in (3.2.2). For $\mathcal{U}^*$, from Theorem 3.2.1, we have

$$\begin{aligned} z_{\mathsf{Rob}} &\geq \mathbf{c}^T \mathbf{x}^* + \max\{\mathbf{d}^T \mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{e}, \mathbf{y} \geq \mathbf{0}\} \\ &= \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{v} \geq \mathbf{0}} \{\mathbf{e}^T \mathbf{v} \mid \hat{\mathbf{B}}^T \mathbf{v} \geq \mathbf{d}\}. \end{aligned}$$

Note that $\mathcal{U}^*$ is compact, convex, down-monotone and column-wise. Therefore, from Theorem 3.3.1, we have

$$\begin{aligned} z_{\mathsf{AR}} &= \mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{B},\mu} \{\mathbf{e}^T \mu \mid \mathbf{B}^T \mu \geq \mathbf{d}, \mathbf{B} \in \mathcal{U}^*, \mu \geq \mathbf{0}\} \\ &\leq \mathbf{c}^T \mathbf{x}^* + O(\log \Gamma \log n) \cdot \min_{\mathbf{v} \geq \mathbf{0}} \{\mathbf{e}^T \mathbf{v} \mid \hat{\mathbf{B}}^T \mathbf{v} \geq \mathbf{d}\} \\ &\leq O(\log \Gamma \log n) \cdot \left(\mathbf{c}^T \mathbf{x}^* + \min_{\mathbf{v} \geq \mathbf{0}} \{\mathbf{e}^T \mathbf{v} \mid \hat{\mathbf{B}}^T \mathbf{v} \geq \mathbf{d}\} \right) \\ &\leq O(\log n \cdot \log \Gamma) \cdot z_{\mathsf{Rob}} \end{aligned}$$

where the second last inequality follows as $\mathbf{c}, \mathbf{x}^* \geq \mathbf{0}$. $\square$

We would like to note that if the ratio between the largest and smallest entries of $\hat{\mathbf{B}}$ is constant, then static solution provides an $O(\log n)$-approximation for the two-stage adjustable robust problem. The two-stage adjustable robust problem is hard to approximate within a factor better than $O(\log n)$ even when the ratio is one. Therefore, quite surprisingly, the performance of the static solution matches the hardness of approximation in this case. Furthermore, in the following section, we show that even when the ratio is large, the static solution still provides a near-optimal approximation for the adjustable robust problem.

3.4 $O(\log n \cdot \log(m+n))$ Bound on Adaptivity Gap

In this section, we show that a static solution provides an $O(\log n \cdot \log(m+n))$-approximation for the two-stage adjustable robust problem Π_{AR} (1.2.1) with column-wise uncertainty sets. Note that this bound on adaptivity gap is uniform across instances and does not depend on Γ. In particular, we have the following theorem.

Theorem 3.4.1. *Let z_{AR} be the objective value of an optimal fully-adjustable solution for Π_{AR}, and z_{Rob} be the optimal objective value of the corresponding static robust problem Π_{Rob} (1.2.2). If $\mathcal{U}$ is a column-wise uncertainty set, then,*

$$z_{AR} \leq O(\log n \cdot \log(m+n)) \cdot z_{Rob}.$$

To prove Theorem 3.4.1, it is sufficient to prove the approximation bound for corresponding one-stage problems since we can extend the bound to the two-stage problem using arguments as in Theorem 3.3.4.

Theorem 3.4.2. *Let* z^I_{AR} *be as defined in* (3.3.6)*, and* z^I_{Rob} *be as defined in* (3.2.4)*. If the uncertainty set* $\mathcal{U}$ *is column-wise, then*

$$z^I_{\mathsf{AR}} \leq O(\log n \cdot \log(m+n)) \cdot z^I_{\mathsf{Rob}}.$$

If Γ is a polynomial in $(m+n)$, the result follows from Theorem 3.3.1 as $\log\Gamma = O(\log(m+n))$. However, if Γ is super-polynomial, we need to handle extreme values of $\hat{B}_{ij}$ differently in order to avoid the dependence on Γ. Let $\mathbf{v}^*$ be an optimal solution for the one-stage adjustable robust problem (3.3.6) and $\theta = ||\mathbf{v}^*||_\infty$. Let

$$J_1 = \left\{ j \in [n] \;\middle|\; \text{there exists } i \in [m] \text{ s.t. } \beta^j_i \geq \frac{2m}{\theta} \right\}$$
$$J_2 = [n] \setminus J_1$$

We show that we can delete the columns in J_1 from z^I_{AR} (3.3.6) (corresponding to the large values of $\hat{B}_{ij}$) such that the modified problem is only within a constant factor of z^I_{AR}. As before, we consider only discrete values of v_i for all $i \in [m]$. Let $T = \lceil \max\{\log m, \log n\} \rceil$ and $\mathcal{T} = \{-T, \ldots, T\}$. For all $i \in [m]$, we consider

$$v_i \in \{0\} \cup \left\{ \frac{\theta}{2^t} \;\middle|\; t \in \mathcal{T} \right\}.$$

Also, for all $i \in [m]$, $t \in \mathcal{T}$, let C_{it} denote the set of columns in $J_2 = [n] \setminus J_1$ that can be covered by setting $v_i = \theta/2^t$, i.e.,

$$C_{it} = \left\{ j \in J_2 \;\middle|\; \beta_i^j \geq \frac{2^t}{\theta} \right\}, \text{and}$$
$$c_t = \frac{\theta}{2^t}.$$

Consider the following 0-1 formulation for the modified one-stage problem.

$$z_{\mathsf{AR}}^{\mathsf{mod}} = \min \left\{ \sum_{i\in[m],t\in\mathcal{T}} c_t x_{it} \;\middle|\; \sum_{i=1}^{m} \sum_{t\in\mathcal{T}:j\in C_{it}} x_{it} \geq 1,\ \forall j \in J_2,\ x_{it} \in \{0,1\} \right\}. \tag{3.4.1}$$

We have the following lemma.

Lemma 3.4.3. *The IP problem in* (3.4.1) *is feasible and provides a near-optimal solution for the one-stage adjustable robust problem* z_{AR}^{I} (3.3.6)*. In particular, we have*

$$\frac{1}{2} z_{\mathsf{AR}}^{\mathsf{mod}} \leq z_{\mathsf{AR}}^{I} \leq 2 z_{\mathsf{AR}}^{\mathsf{mod}}.$$

Proof. Consider an optimal solution $\mathbf{v}^*$ for (3.3.6). We construct a feasible solution for (3.4.1) as follows. Now, for all $i \in [m]$, $t \in \mathcal{T}$, let

$$\bar{x}_{it} = \begin{cases} 1, \text{ if } \dfrac{c_t}{2} < v_i^* \leq c_t \\ 0, \text{ otherwise.} \end{cases}$$

Since $\mathbf{v}^*$ is feasible, $\bar{\mathbf{x}}$ is a feasible solution to the set cover problem (3.4.1) and

$$z_{\mathsf{AR}}^{\mathsf{mod}} \leq \sum_{i=1}^{m} \sum_{t=-T}^{T} c_t \bar{x}_{it} \leq 2\mathbf{e}^T \mathbf{v}^* = 2 z_{\mathsf{AR}}^{I}.$$

Conversely, consider an optimal solution $\mathbf{x}^*$ for the set cover problem (3.4.1). We construct a feasible solution $\tilde{\mathbf{v}}$ for (3.3.6) as follows. For all $i \in [m]$,

$$\tilde{v}_i = \frac{\theta}{2m} + \sum_{t \in \mathcal{T}} c_t x_{it}^*.$$

Note that we add $\theta/2m$ to each v_i in order to handle the constraints for columns in J_1 that are not considered in (3.4.1). For each $j \in J_1$, there exists $i \in [m]$ such that $\beta_i^j \geq 2m/\theta$ and $v_i \beta_i^j \geq 1$. For all $j \in J_2$, there exists $i \in [m]$ and $t \in \{-T, \ldots, T\}$ such that $j \in C_{it}$ and $x_{it}^* = 1$. Therefore, $v_i \geq c_t$ which implies that $v_i \cdot \beta_i^j \geq 1$. Therefore, $\tilde{\mathbf{v}}$ is a feasible solution for the one-stage adjustable robust problem z_{AR}^I (3.3.6). Moreover, we have

$$z_{\mathsf{AR}}^I \leq \mathbf{e}^T \tilde{\mathbf{v}} \leq \left(\frac{\theta}{2} + z_{\mathsf{AR}}^{\mathsf{mod}} \right) \leq \frac{z_{\mathsf{AR}}^I}{2} + z_{\mathsf{AR}}^{\mathsf{mod}} \Rightarrow z_{\mathsf{AR}}^I \leq 2 \cdot z_{\mathsf{AR}}^{\mathsf{mod}},$$

which completes the proof. □

We can formulate the LP relaxation of set cover problem in (3.4.1) as follows.

$$z_{\mathsf{LP}} = \min \left\{ \sum_{i=1}^{m} \sum_{t=-T}^{T} c_{it} x_{it} \;\middle|\; \sum_{i=1}^{m} \sum_{t \in \mathcal{T} : \frac{2^t}{\theta} \leq \beta_i^j} x_{it} \geq 1, \, \forall j \in J_2, \, x_{it} \geq 0 \right\}. \tag{3.4.2}$$

We have

$$z_{\mathsf{AR}}^{\mathsf{mod}} \leq O(\log n) \cdot z_{\mathsf{LP}}.$$

Consider the dual of (3.4.2).

$$z_{\mathsf{LP}} = \max\left\{ \sum_{j\notin J_1} y_j \;\middle|\; \sum_{j\in C_{it}} y_j \leq c_t,\ \forall i\in[m], t\in\mathcal{T},\ y_j\geq 0, \forall j\in J_2 \right\} \tag{3.4.3}$$

We will construct a feasible solution for the one-stage static robust problem (3.3.3) from (3.4.3).

Proof of Theorem 3.4.2. From Lemma 3.4.3, it is sufficient to show that

$$z_{\mathsf{LP}} \leq O(\log(m+n)) \cdot z^{I}_{\mathsf{Rob}}.$$

Let $\mathbf{y}^*$ by an optimal solution of (3.4.3). We construct a feasible solution for (3.3.3) by scaling $\mathbf{y}^*$ by a factor of $O(\log(m+n))$. For $t = 0$, we have

$$\sum_{j\notin J_1:\beta_i^j\geq\frac{1}{\theta}} \frac{1}{\theta} y_j^* \leq 1, \forall i\in[m].$$

Let $\mathbf{v}^*$ be an optimal solution for (3.3.6). From Lemma 3.3.2, for each $j\in[n]$, there exist $i\in[m]$ such that

$$\beta_i^j v_i^* \geq 1 \Rightarrow \beta_i^j \geq \frac{1}{v_i^*} \geq \frac{1}{\theta}.$$

Therefore, for each $j\in J_2$, we have $y_j^*\leq\theta$. Since $\mathbf{y}^*$ is an optimal solution of (3.4.3), we have

$$\sum_{j\notin J_1:\beta_i^j\geq\frac{2^t}{\theta}} \frac{2^t}{\theta} y_j^* \leq 1,\ \forall t\in\mathcal{T}.$$

Sum over all $t \in \mathcal{T}$, we have

$$\sum_{t\in\mathcal{T}} \sum_{j\notin J_1:\beta_i^j \geq \frac{2^t}{\theta}} \frac{2^t}{\theta} y_j^* \leq 2T+1, \forall i \in [m].$$

Switching the summation, we have

$$\sum_{j\notin J_1} \sum_{t\in\mathcal{T}:\frac{2^t}{\theta}\leq\beta_i^j} \frac{2^t}{\theta} y_j^* \leq 2T+1, \forall i \in [m]$$

Note that if $\beta_i^j \geq 1/n\theta$ and $j \notin J_1$, then

$$\frac{1}{2}\beta_i^j \leq \sum_{t:\frac{2^t}{\theta}\leq\beta_i^j} \frac{2^t}{\theta} \leq 2\beta_i^j.$$

Let

$$\hat{y}_j = \begin{cases} \frac{1}{4T+3} y_j^*, & \text{if } j \in J_2 \\ 0, & \text{if } j \in J_1 \end{cases}$$

For any $i \in [m]$, we have

$$\begin{aligned} \sum_{j=1}^n \beta_i^j \hat{y}_j &= \sum_{j\in J_1} \beta_i^j \hat{y}_j + \frac{1}{4T+3}\left(\sum_{j\notin J_1:\beta_i^j<1/n\theta} \beta_i^j y_j^* + \sum_{j\notin J_1:\beta_i^j\geq 1/n\theta} \beta_i^j y_j^* \right) \\ &\leq 0 + \frac{1}{4T+3}\left(1 + 2\sum_{j=1}^n \sum_{t:\frac{2^t}{\theta}\beta_i^j} \frac{2^t}{\theta} y_j^*\right) \\ &\leq 1 \end{aligned}$$

Therefore, $\hat{\mathbf{y}}$ is a feasible solution to the dual of z^I_{Rob} (3.3.3). Note that $T = O(\log(m+n))$. Therefore, we have

$$z_{\mathsf{LP}} = \mathbf{e}^T\mathbf{y}^* = O(\log(m+n)) \cdot \mathbf{e}^T\hat{\mathbf{y}} \leq O(\log(m+n)) \cdot z^I_{\mathsf{Rob}},$$

which completes the proof. □

From Theorems 3.3.4 and 3.4.1, we have the following corollary.

Corollary 3.4.4. *Let z_{AR} be the objective value of an optimal fully-adjustable solution for the adjustable robust problem Π_{AR} (1.2.1), and z_{Rob} be the optimal objective value of the corresponding static robust problem Π_{Rob} (1.2.2). If $\mathcal{U}$ is a column-wise uncertainty set, then,*

$$z_{AR} \leq O(\log n \cdot \min(\log\Gamma, \log(m+n))) \cdot z_{Rob}.$$

3.5 Column-wise and Constraint-wise Uncertainty Sets.

In this section, we consider the general case where the uncertainty set is the intersection of column-wise and constraint-wise sets. Recall that a column-wise and constraint-wise uncertainty set $\mathcal{U}$ implies that

$$\mathcal{U} = \left\{\mathbf{B} \in \mathbb{R}^{m\times n}_+ \mid \mathbf{B}\mathbf{e}_j \in C_j,\ \forall j \in [n],\ \mathbf{B}^T\mathbf{e}_i \in R_i,\ \forall i \in [m]\right\}, \tag{3.5.1}$$

where $C_j \subseteq \mathbb{R}^m_+$ for all $j \in [n]$ and $R_i \subseteq \mathbb{R}^n_+$ for all $i \in [m]$ are compact, convex and down-monotone sets. We refer to the above uncertainty set as a column-wise and constraint-wise set since the constraints on the uncertainty set $\mathcal{U}$ are either over the columns or the rows of the matrix. As mentioned previously, we assume that optimization problems with linear objective over $\mathcal{U}$ can be solved in polynomial time in the encoding length of $\mathcal{U}$.

We show that a static solution provides an $O(\log n \cdot \min(\log\Gamma, \log(m+n)))$-approximation for the two-stage adjustable robust problem Π_{AR} for the above column-wise and constraint-wise uncertainty set where Γ is defined in (3.3.1). In particular, we have the following theorem.

Theorem 3.5.1. *Consider a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise and constraint-wise as in* (3.5.1). *Let $z_{\mathsf{AR}}(\mathcal{U})$ and $z_{\mathsf{Rob}}(\mathcal{U})$ be the optimal values of the two-stage adjustable robust problem $\Pi_{\mathsf{AR}}(\mathcal{U})$* (1.2.1) *and the static robust problem $\Pi_{\mathsf{Rob}}(\mathcal{U})$* (1.2.2) *over uncertainty set $\mathcal{U}$, respectively. Then,*

$$z_{\mathsf{AR}}(\mathcal{U}) \leq O(\log n \cdot \min(\log\Gamma, \log(m+n))) \cdot z_{\mathsf{Rob}}(\mathcal{U}).$$

Our proof is based on a transformation of the static robust problem into a equivalent formulation over a constraint-wise uncertainty set. In particular, we construct the constraint-wise uncertainty set as follows. For each $i \in [m]$, let

$$\tilde{\mathcal{R}}_i = \{\mathbf{B}^T \mathbf{e}_i \mid \mathbf{B} \in \mathcal{U}\}, \tag{3.5.2}$$

i.e., $\tilde{\mathcal{R}}_i$ is the projection of the uncertainty set $\mathcal{U}$ for the i^{th} row. Let

$$\tilde{\mathcal{U}} = \tilde{\mathcal{R}}_1 \times \tilde{\mathcal{R}}_2 \times \ldots \times \tilde{\mathcal{R}}_m, \tag{3.5.3}$$

i.e., a Cartesian product of $\tilde{\mathcal{R}}_i, i \in [m]$. Note that for any $\mathbf{B} \in \tilde{\mathcal{U}}$, the constraints corresponding to row-sets $R_1, \ldots, R_m$ are satisfied. However, the constraints corresponding to column-sets $C_1, \ldots, C_n$ may not be satisfied. We have the following lemma.

Lemma 3.5.2. *Given a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise and constraint-wise and any $\mu \in [0,1]^m$ such that $\mathbf{e}^T\mu = 1$, let $\tilde{\mathcal{U}}$ be defined as* (3.5.3)*. Then, for any $\mathbf{B} \in \tilde{\mathcal{U}}$, we have*

$$\mathsf{diag}(\mu)\mathbf{B} \in \mathcal{U}.$$

Proof. Noting that $\mathbf{B}^T\mathbf{e}_i \in \tilde{\mathcal{R}}_i$ and $\mathsf{diag}(\mathbf{e}_i)\mathbf{B}$ has the i^{th} row as $\mathbf{B}^T\mathbf{e}_i$ and other rows as $\mathbf{0}$, we have $\mathsf{diag}(\mathbf{e}_i)\mathbf{B} \in \mathcal{U}$ since $\mathcal{U}$ is down-monotone. Moreover, μ is convex multiplier,

$$\mathsf{diag}(\mu)\mathbf{B} = \sum_{i=1}^{m} \mu_i \mathsf{diag}(\mathbf{e}_i)\mathbf{B}$$

and $\mathcal{U}$ is convex, we have $\mathsf{diag}(\mu)\mathbf{B} \in \mathcal{U}$. □

In the following lemma, we show that the static robust problem has the same optimal objective value for uncertainty sets $\mathcal{U}$ and $\tilde{\mathcal{U}}$.

Lemma 3.5.3. *Given a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise and constraint-wise, let $\tilde{\mathcal{U}}$ be defined as in* (3.5.3)*. Let $z_{\mathsf{Rob}}(\mathcal{U})$ and $z_{\mathsf{Rob}}(\tilde{\mathcal{U}})$ be the optimal values of the static adjustable robust problem Π_{Rob}* (1.2.2) *over uncertainty set $\mathcal{U}$ and $\tilde{\mathcal{U}}$, respectively. Then*

$$z_{\mathsf{Rob}}(\mathcal{U}) = z_{\mathsf{Rob}}(\tilde{\mathcal{U}}).$$

Proof. For any $\mathbf{B} \in \mathcal{U}$, we have $\mathbf{B}^T\mathbf{e}_i \in \tilde{\mathcal{R}}_i$ for all $i \in [m]$, which implies that $\mathbf{B} \in \tilde{\mathcal{U}}$ since $\tilde{\mathcal{U}}$ is constraint-wise. Therefore, $\mathcal{U} \subseteq \tilde{\mathcal{U}}$ and any solution that is feasible for $\Pi_{\mathsf{Rob}}(\tilde{\mathcal{U}})$ must be feasible for $\Pi_{\mathsf{Rob}}(\mathcal{U})$. Therefore,

$$z_{\mathsf{Rob}}(\tilde{\mathcal{U}}) \leq z_{\mathsf{Rob}}(\mathcal{U}).$$

Conversely, suppose $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is an optimal solution for $\Pi_{\mathsf{Rob}}(\mathcal{U})$. We show that it is feasible for $\Pi_{\mathsf{Rob}}(\tilde{\mathcal{U}})$. For the sake of contradiction, assume that there exists a $\tilde{\mathbf{B}} \in \tilde{\mathcal{U}}$ such that

$$(\tilde{\mathbf{B}}\hat{\mathbf{y}})_i > h_i - (\mathbf{A}\hat{\mathbf{x}})_i \text{ for some } i \in [m] \Rightarrow (\mathsf{diag}(\mathbf{e}_i)\tilde{\mathbf{B}}\hat{\mathbf{y}})_i > h_i - (\mathbf{A}\hat{\mathbf{x}})_i.$$

However, from Lemma 3.5.2, $\mathsf{diag}(\mathbf{e}_i)\tilde{\mathbf{B}} \in \mathcal{U}$, which contradicts the assumption that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible for $\Pi_{\mathsf{Rob}}(\mathcal{U})$. Therefore, $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible for $\Pi_{\mathsf{Rob}}(\tilde{\mathcal{U}})$ and $z_{\mathsf{Rob}}(\mathcal{U}) \leq z_{\mathsf{Rob}}(\tilde{\mathcal{U}})$. □

From Ben-Tal and Nemirovski [5] and previous chapter, we know that

$$z_{\mathsf{Rob}}(\tilde{\mathcal{U}}) = z_{\mathsf{AR}}(\tilde{\mathcal{U}}),$$

since $\tilde{\mathcal{U}}$ is a constraint-wise uncertainty set and a static solution is optimal for the adjustable robust problem. Therefore, to prove Theorem 3.5.1, it is now sufficient to show

$$z_{\mathsf{AR}}(\mathcal{U}) \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{\mathsf{AR}}(\tilde{\mathcal{U}}).$$

Proof of Theorem 3.5.1 Let $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ be an optimal fully-adjustable solution to $\Pi_{\mathsf{AR}}(\mathcal{U})$. Therefore,

$$z_{\mathsf{AR}}(\mathcal{U}) = \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B} \in \mathcal{U}} \max\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}^*,\ \mathbf{y} \geq \mathbf{0}\}.$$

As discussed in previous chapter, we can assume without loss of generality $(\mathbf{h}-\mathbf{A}\mathbf{x}^*) > \mathbf{0}$. Therefore, we can rescale $\mathcal{U}$ and $\tilde{\mathcal{U}}$ as

$$\mathcal{S} = [\mathsf{diag}(\mathbf{h}-\mathbf{A}\mathbf{x}^*)]^{-1}\mathcal{U}, \text{ and } \tilde{\mathcal{S}} = [\mathsf{diag}(\mathbf{h}-\mathbf{A}\mathbf{x}^*)]^{-1}\tilde{\mathcal{U}}.$$

Note that $\tilde{\mathcal{S}}$ is the Cartesian product of the row projections of $\mathcal{S}$. For any $\mathcal{H} \subseteq \mathbb{R}_+^{m\times n}$, let

$$z^I_{\mathsf{AR}}(\mathcal{H}) = \min\{\mathbf{e}^T\mathbf{v} \mid \mathbf{B}^T\mathbf{v} \geq \mathbf{d}, \mathbf{B} \in \mathcal{H}, \mathbf{v} \geq \mathbf{0}\}.$$

Now,

$$\begin{aligned} z_{\mathsf{AR}}(\mathcal{U}) &= \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B}\in\mathcal{S}} \max\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{e},\ \mathbf{y} \geq \mathbf{0}\} \\ &= \mathbf{c}^T\mathbf{x}^* + \min\{\mathbf{e}^T\mathbf{v} \mid \mathbf{B}^T\mathbf{v} \geq \mathbf{d}, \mathbf{B} \in \mathcal{S},\ \mathbf{v} \geq \mathbf{0}\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{AR}}(\mathcal{S}), \end{aligned}$$

where the second equation follows by taking the dual of the inner maximization problem. Also,

$$\begin{aligned} z_{\mathsf{AR}}(\tilde{\mathcal{U}}) &\geq \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B}\in\tilde{\mathcal{U}}} \max\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h}-\mathbf{A}\mathbf{x}^*,\ \mathbf{y} \geq \mathbf{0}\} \\ &= \mathbf{c}^T\mathbf{x}^* + z^I_{\mathsf{AR}}(\tilde{\mathcal{S}}). \end{aligned}$$

Therefore, to complete the proof, it is sufficient to show that

$$z^I_{\mathsf{AR}}(\mathcal{S}) \leq O(\log n \cdot \min(\log\Gamma, \log(m+n))) \cdot z^I_{\mathsf{AR}}(\tilde{\mathcal{S}}). \tag{3.5.4}$$

Let $\tilde{\mathbf{B}} \in \tilde{\mathcal{S}}$ be the minimizer of $z^I_{\mathsf{AR}}(\tilde{\mathcal{S}})$. We construct a simplex column-wise uncertainty set,

$\mathcal{H} \subseteq \mathbb{R}_+^{m\times n}$ where each simplex column set, $H_j \subseteq \mathbb{R}_+^m,\ j \in [n]$ is defined from $\tilde{\mathbf{B}}$ as follows.

$$H_j = \mathsf{conv}\left(\{\mathbf{0}\} \bigcup \{\tilde{B}_{ij}\mathbf{e}_i \mid i = 1,\ldots,m\}\right).$$

and

$$\mathcal{H} = \{[\mathbf{b}_1 \cdots \mathbf{b}_n] \mid \mathbf{b}_j \in H_j, \forall j \in [n]\}.$$

We would like to note that $\mathcal{H} \subseteq \mathcal{S}$: For any $\mathbf{b} \in H_j, j \in [n]$, we have $\mathbf{b} \le \mathsf{diag}(\mu)\tilde{\mathbf{B}}\mathbf{e}_j$ for some convex multiplier μ. From Lemma 3.5.2, $\mathsf{diag}(\mu)\tilde{\mathbf{B}} \in \mathcal{S}$, which indicates that $H_j \subseteq [\mathsf{diag}(\mathbf{h} - \mathbf{A}\mathbf{x})]^{-1}C_j$. Moreover, $\tilde{\mathbf{B}}$ satisfies the row constraints of $\mathcal{S}$ and $\mathbf{e}_i^T\mathbf{B} \le \mathbf{e}_i^T\tilde{\mathbf{B}}$ for any $\mathbf{B} \in \mathcal{H}, i \in [m]$. Therefore, $\mathcal{H} \subseteq \mathcal{S}$ and

$$z_{\mathsf{AR}}^I(\mathcal{S}) \le z_{\mathsf{AR}}^I(\mathcal{H}) \le O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{\mathsf{Rob}}^I(\mathcal{H}) \tag{3.5.5}$$

where the second inequality follows from Theorems 3.3.1 and 3.4.2. Note that $\tilde{\mathbf{B}}$ is the entry-wise maximum matrix over $\mathcal{H}$ as defined in (3.2.2). Therefore,

$$z_{\mathsf{Rob}}^I(\mathcal{H}) = \min\,\{\mathbf{e}^T\mathbf{v} \mid \tilde{\mathbf{B}}^T\mathbf{v} \ge \mathbf{d}\} = z_{\mathsf{AR}}^I(\tilde{\mathcal{S}}),$$

where the first equality follows from Theorem 3.2.1 and the second equality follows from the fact that $\tilde{\mathbf{B}} \in \tilde{\mathcal{S}}$ is a minimizer for $z_{\mathsf{AR}}^I(\tilde{\mathcal{S}})$. Therefore, from (3.5.5), we have $z_{\mathsf{AR}}^I(\mathcal{S}) \le O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{\mathsf{AR}}^I(\tilde{\mathcal{S}})$. □

3.6 Comparison with Measure of Non-convexity Bound

In this section, we compare our bound with the measure of non-convexity bound introduced in the previous chapter. We show that our bound provides an upper bound on the measure of non-convexity for column-wise and constraint-wise uncertainty sets. In particular, we have the following theorem.

Theorem 3.6.1. *Given a convex, compact and down-monotone uncertainty set* $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ *that is column-wise and constraint-wise as in* (3.5.1) *and* $\mathbf{h} > \mathbf{0}$*, let* $T(\mathcal{U},\mathbf{h})$ *and* $\kappa(T(\mathcal{U},\mathbf{h}))$ *be defined as in* (2.3.6) *and* (2.4.1)*, respectively. Then,*

$$\kappa(T(\mathcal{U},\mathbf{h})) \le O(\log n \cdot \min(\log\Gamma, \log(m+n))).$$

Proof. Let $\alpha = \log n \cdot \min(\log\Gamma, \log(m+n))$. Let $\tilde{\mathcal{R}}_i, i \in [m]$ be defined as in (3.5.2). From the proof of Theorem 2.4.3, we have

$$\mathsf{conv}(T(\mathcal{U},\mathbf{h})) = \mathsf{conv}\left(\bigcup_{i=1}^{m} \frac{1}{h_i}\cdot\tilde{\mathcal{R}}_i\right).$$

Given any $\mathbf{d} \in \mathsf{conv}(T(\mathcal{U},\mathbf{h}))$, we have

$$\mathbf{d} = \sum_{i=1}^{m} \frac{\lambda_i}{h_i}\tilde{\mathbf{b}}_i$$

where $\tilde{\mathbf{b}}_i \in \tilde{\mathcal{R}}_i, i \in [m]$, $\lambda \ge \mathbf{0}$ and $\mathbf{e}^T\lambda = 1$. For all $i \in [m]$, let $\mathbf{B}_i = \mathbf{e}_i\tilde{\mathbf{b}}_i^T$. Since $\mathcal{U}$ is down-monotone,

$\mathbf{B}_i \in \mathcal{U}$. Let

$$\tilde{\mathbf{B}} = [\mathsf{diag}(\mathbf{h})]^{-1} \sum_{i=1}^{m} \mathbf{B}_i.$$

Therefore, $\tilde{\mathbf{B}}^T \lambda = \mathbf{d}$. We construct a simplex column-wise uncertainty set $\mathcal{H} \subseteq \mathbb{R}_+^{m\times n}$ using $\tilde{\mathbf{B}}$ similar to the proof of Theorem 3.5.1. Let

$$\mathcal{H} = \{[\mathbf{b}_1 \cdots \mathbf{b}_n] \mid \mathbf{b}_j \in H_j, \forall j \in [n]\}$$

where

$$H_j = \mathsf{conv}\left(\{\mathbf{0}\} \bigcup \left\{ \tilde{B}_{ij}\mathbf{e}_i \;\middle|\; i = 1, \ldots, m \right\}\right)$$

for all $j \in [n]$. Note that $H_j \subseteq [\mathsf{diag}(\mathbf{h})]^{-1} C_j$, which implies that $\mathcal{H} \subseteq [\mathsf{diag}(\mathbf{h})]^{-1}\mathcal{U}$. From Theorem 3.2.1, we know that

$$z^I_{\mathsf{Rob}}(\mathcal{H}) = \min\{\mathbf{e}^T\mathbf{v} \mid \tilde{\mathbf{B}}^T\mathbf{v} \geq \mathbf{d}, \mathbf{v} \geq \mathbf{0}\},$$

and λ is a feasible solution for $z_{\mathsf{Rob}}(\mathcal{H})$. Therefore, $z^I_{\mathsf{Rob}}(\mathcal{H}) \leq \mathbf{e}^T\lambda = 1$. Furthermore,

$$z^I_{\mathsf{AR}}([\mathsf{diag}(\mathbf{h})]^{-1}\mathcal{U}) \leq z^I_{\mathsf{AR}}(\mathcal{H}) \leq O(\alpha) \cdot z^I_{\mathsf{Rob}}(\mathcal{H}) \leq O(\alpha),$$

where the first inequality follows as $\mathcal{H} \subseteq [\mathsf{diag}(\mathbf{h})]^{-1}\mathcal{U}$ and the second inequality follows from Theorems 3.3.1 and 3.4.2. Therefore, there exists $(\mathbf{v}^*, \mathbf{B}^*)$ such that

$$(\mathbf{B}^*)^T\mathbf{v}^* \geq \mathbf{d},\ \mathbf{B}^* \in [\mathsf{diag}(\mathbf{h})]^{-1}\mathcal{U},\ \text{and } \mathbf{e}^T\mathbf{v}^* \leq O(\alpha).$$

Now, let

$$\mathbf{Q} = \mathsf{diag}(\mathbf{h})\mathbf{B}^* \text{ and } \mu = \frac{1}{\mathbf{e}^T\mathbf{v}^*}[\mathsf{diag}(\mathbf{h})]^{-1}\mathbf{v}^*.$$

Then, $\mathbf{Q} \in \mathcal{U}$ and $\mathbf{h}^T\mu = 1$, which implies that $\mathbf{Q}^T\mu \in T(\mathcal{U},\mathbf{h})$. Note that

$$\mathbf{Q}^T\mu = \frac{1}{\mathbf{e}^T\mathbf{v}^*}(\mathbf{B}^*)^T\mathbf{v}^* \geq \frac{1}{O(\alpha)}\mathbf{d}.$$

Since $\mathcal{U}$ is down-monotone, so is $T(\mathcal{U},\mathbf{h})$. Therefore, for $\mathbf{d} \in \mathsf{conv}(T(\mathcal{U},\mathbf{h}))$, we have

$$\frac{1}{O(\alpha)}\mathbf{d} \in T(\mathcal{U},\mathbf{h}),$$

which implies that $\kappa(T(\mathcal{U},\mathbf{h})) \leq O(\log n \cdot \min(\log\Gamma, \log(m+n)))$. □

3.7 Adaptivity Gap under Constraint and Objective Uncertainty.

In this section, we show that our result can be generalized to the case where both constraint and objective coefficients are uncertain. In particular, we consider the two-stage adjustable robust problem $\Pi_{\mathsf{AR}}^{(B,d)}$ as in (2.5.1).

$$\begin{aligned} z_{\mathsf{AR}}^{(B,d)} = \max\, \mathbf{c}^T\mathbf{x} + \min_{(\mathbf{B},\mathbf{d})\in\mathcal{U}}\ & \max_{\mathbf{y}(\mathbf{B},\mathbf{d})} \mathbf{d}^T\mathbf{y}(\mathbf{B},\mathbf{d}) \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{B},\mathbf{d}) \ & \leq \ \mathbf{h} \\ \mathbf{x} \ \in \ \mathbb{R}^n_+,\ \mathbf{y}(\mathbf{B},\mathbf{d}) \ & \in \ \mathbb{R}^n_+ \end{aligned}$$

We consider the case where the uncertainty in constraint matrix $\mathbf{B}$ is column-wise and constraint-wise and does not depend on the uncertainty in objective coefficients $\mathbf{d}$. Therefore,

$$\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d,$$

where $\mathcal{U}^B \subseteq \mathbb{R}_+^{m\times n}$ is a convex compact uncertainty set of constraint matrices that is column-wise and constraint-wise, and $\mathcal{U}^d \subseteq \mathbb{R}_+^n$ is a convex compact uncertainty set of the second-stage objective. Consider the corresponding static robust problem $\Pi_{\mathsf{Rob}}^{(B,d)}$ as in (2.5.2).

$$\begin{aligned} z_{\mathsf{Rob}}^{(B,d)} = \max_{\mathbf{x},\mathbf{y}} \min_{\mathbf{d}\in\mathcal{U}^d} \ & \mathbf{c}^T\mathbf{x} + \mathbf{d}^T\mathbf{y} \\ & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \ \leq \ \mathbf{h}, \ \forall \mathbf{B} \in \mathcal{U}^B \\ & \mathbf{x}, \mathbf{y} \ \in \ \mathbb{R}_+^n. \end{aligned}$$

We prove the following theorem.

Theorem 3.7.1. *Let $z_{\mathsf{AR}}^{(B,d)}$ be the optimal objective value of $\Pi_{\mathsf{AR}}^{(B,d)}$ in* (2.5.1) *defined over the uncertainty $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$, where $\mathcal{U}^B \subseteq \mathbb{R}_+^{m\times n}$ is a convex compact uncertainty set of constraint matrices that is column-wise and constraint-wise, and $\mathcal{U}^d \subseteq \mathbb{R}_+^n$ is a convex compact uncertainty set of the second-stage objective. Let $z_{\mathsf{Rob}}^{(B,d)}$ be the optimal objective value of $\Pi_{\mathsf{Rob}}^{(B,d)}$ in* (2.5.2). *Then,*

$$z_{\mathsf{AR}}^{(B,d)} \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{\mathsf{Rob}}^{(B,d)}.$$

Proof. From Theorem 2.5.1, we have

$$z_{\mathsf{AR}}^{(B,d)} \leq \max\{\kappa(T(\mathcal{U},\mathbf{h})) \mid \mathbf{h} > \mathbf{0}\} \cdot z_{\mathsf{Rob}}^{(B,d)}.$$

From Theorem 3.6.1, we have

$$\max\{\kappa(T(\mathcal{U},\mathbf{h})) \mid \mathbf{h} > \mathbf{0}\} \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))),$$

which completes the proof. □

3.8 Computational Study

In this section, we perform a computational study on the performance of static solutions as an approximation for the two-stage adjustable robust problem Π_{AR} (1.2.1) with column-wise uncertainty sets. From Theorem 3.2.2, we focus on uncertainty sets that are Cartesian products of simplices because they give the worst performance of static solutions. From Theorem 3.2.1, we can compute an optimal one-stage static solution for Π_{Rob}^I (3.2.4) as a single LP. On the other hand, it is NP-hard to compute an optimal solution for the one-stage adjustable robust problem Π_{AR}^I. However, we can consider the set cover formulation of Π_{AR}^I (3.3.6) and solve the integer programming formulation using Gurobi. In particular, given $\hat{\mathbf{B}} \in \mathbb{R}_+^{m\times n}$ as defined in (3.2.2), we consider the following instance of the adjustable robust problem:

$$\mathbf{c} = \mathbf{0}, \mathbf{A} = \mathbf{0}, \;\; h_i = 1, \; \forall i \in [m], \;\; d_j = 1, \; \forall j \in [n]$$

$$\hat{\mathcal{U}}_j = \left\{ \mathbf{b} \in \mathbb{R}^m_+ \;\middle|\; \sum_{i=1}^{m} \frac{1}{\hat{B}_{ij}} b_i \leq 1, b_i = 0, \forall i : \hat{B}_{ij} = 0 \right\}, \; \forall j \in [n].$$

$$\mathcal{U} = \left\{ [\mathbf{b}_1 \; \mathbf{b}_2 \; \ldots \; \mathbf{b}_n] \mid \mathbf{b}_j \in \mathcal{U}_j \right\}.$$

We solve the following IP problem for Π^I_{AR} (3.3.6)

$$z^I_{\mathsf{AR}} = \min\{\mathbf{e}^T\mathbf{v} \mid v_i \hat{B}_{ij} \geq z_{ij}, v_i \geq 0, z_{ij} \in \{0,1\} \forall i \in [m], j \in [n]\}$$

and the LP for Π^I_{Rob} (3.2.4)

$$z^I_{\mathsf{Rob}} = \max\{\mathbf{e}^T\mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{e}, \mathbf{y} \geq \mathbf{0}\}.$$

For given m and n, we sample $\hat{\mathbf{B}}$ under single-sided i.i.d. standard normal distribution for 1000 times, i.e., $\hat{B}_{ij}$ is the absolute value of a independent $\mathcal{N}(0,1)$ random variable for all $i \in [m], j \in [n]$.

Table 3.1 records the worst gap and average gap between the optimal values for different choices of m and n. Note that neither the worst-instance nor the average adaptivity gap follows a strictly increasing pattern when m increases. We conjecture that the upper bound for the adaptivity gap should be $O(\log n)$ instead of $O(\log(m+n)\log n)$. In our analysis, the term $\log(m+n)$ comes from capping Γ, the ratio between that largest element and the smallest element of $\hat{\mathbf{B}}$. Therefore, in Table 3.2, we consider the case where $\Gamma > mn$ to see if this is reflected in computation.

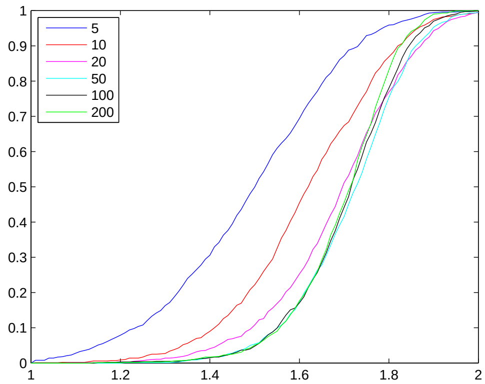
n	m	Worst Gap	Average Gap
10	5	1.9836	1.4897
	10	2.0345	1.6180
	20	2.0752	1.6867
	50	2.0304	1.7140
	100	2.0162	1.7056
	200	1.9708	1.6985
20	5	2.2038	1.6550
	10	2.3890	1.8829
	20	2.3580	2.0125
	50	2.3461	2.0757
	100	2.3338	2.0801
	200	2.2988	2.0699
50	5	2.4140	1.8237
	10	2.8773	2.2516
	20	2.8858	2.5006
	50	2.9403	2.6478

Table 3.1: Computational study for all samples.

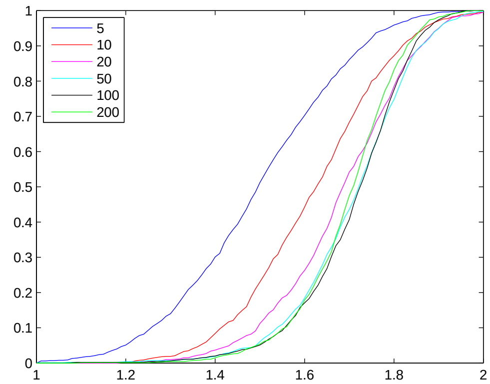
We plot the the percentage of instances versus thresholds in Figure 3.1. In all figure, the x-axis is the threshold for the adaptivity gap, and the y-axis is the percentage of instances where the gap is less than the threshold. As shown in the figures, there is almost no visible difference when we restrict $\Gamma > mn$. However, there is a significant change in the percentage when we change n as shown in the figures.

From our observation from the computational study, we conjecture that the upper bound for the approximation ratio is $O(\log n)$ instead of $O(\log(m+n)\log n)$, where the term $\log(m+n)$ is resulted from our analysis. Table 3.3 compares the worst-case and average adaptivity gaps when $m = 10$.

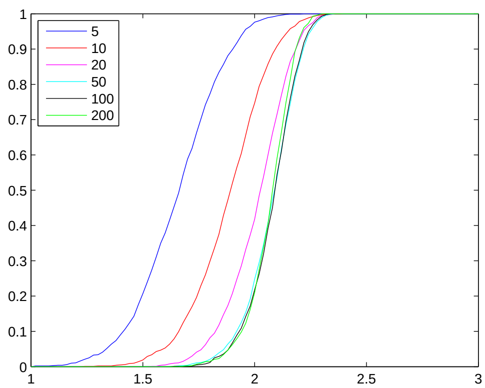
We plot the gaps with respect to the 10-based logarithm of m in Figure 3.2. Note that the curves follow similar trends. This is in conformity with our conjecture that the adaptivity gap for



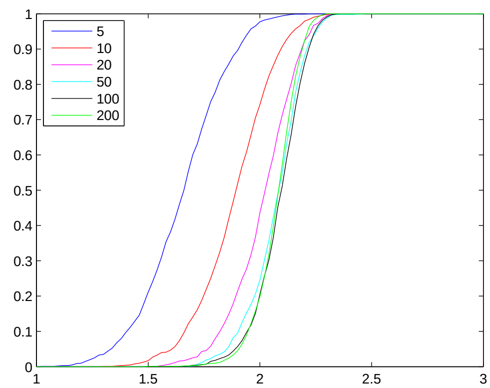
(a) All samples for $n = 10$



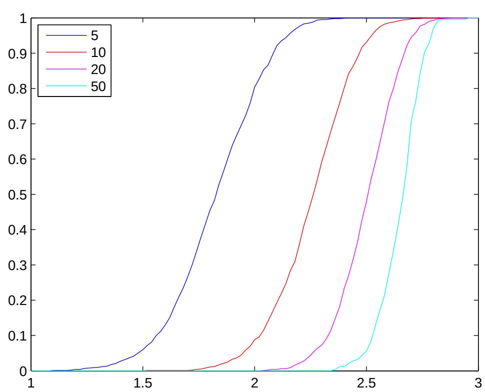
(b) Samples where $\Gamma > mn$ for $n = 10$



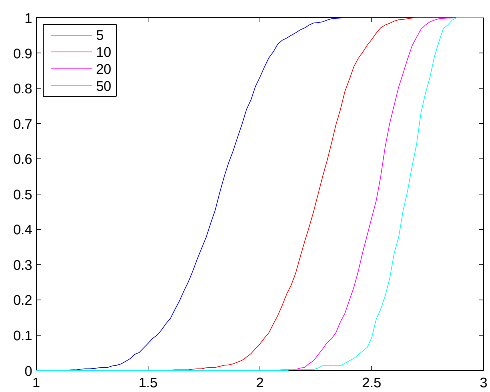
(c) All samples for $n = 20$



(d) Samples where $\Gamma > mn$ for $n = 20$



(e) All samples for $n = 50$



(f) Samples where $\Gamma > mn$ for $n = 50$

Figure 3.1: Plots of percentage of instances versus threshold.

n	m	Worst Gap	Average Gap
10	5	1.9713	1.4954
	10	2.1560	1.6207
	20	2.0510	1.6815
	50	2.0519	1.7066
	100	1.9748	1.7086
	200	1.9654	1.6973
20	5	1.9713	1.4954
	10	2.1560	1.6207
	20	2.0510	1.6815
	50	2.0519	1.7066
	100	1.9748	1.7086
	200	1.9654	1.6973
50	5	2.3652	1.8096
	10	2.6806	2.2487
	20	2.8729	2.5090
	50	2.9193	2.6467

Table 3.2: Computational study when $\Gamma > mn$.

m	n	Worst Gap	Average Gap
10	10	2.0345	1.6180
	20	2.3890	1.8829
	50	2.8773	2.2516
	100	2.9968	2.5082
	200	3.1934	2.7480
	500	3.4894	3.0224

Table 3.3: Computational study when $m = 10$.

column-wise and constraint-wise uncertainty set should be $O(\log n)$. It is an interesting question to close the gap between the upper and lower bounds on the performance of static solution.

3.9 Conclusion.

In this chapter, we study the adaptivity gap in two-stage adjustable robust linear optimization problem under column-wise and constraint-wise uncertainty sets. As shown in the previous chapter, the

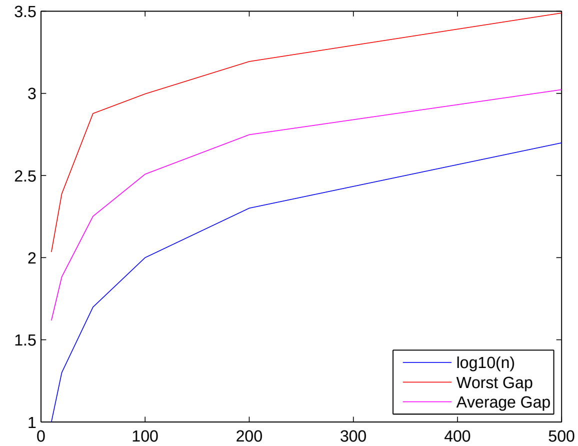


Figure 3.2: Plots of worst-case and average adaptivity gap when $m = 10$.

adjustable problem is $\Omega(\log n)$-hard to approximate in this case. On the positive side, we show that a static solution is an $O(\log n \cdot \min(\log \Gamma, \log(m+n)))$-approximation for the adjustable robust problem when the uncertainty set is column-wise and constraint-wise. Therefore, if Γ (maximum ratio between upper bounds of uncertain constraint coefficients) is a constant, the static solution provides an $O(\log n)$-approximation which matches the hardness of approximation in this case. If Γ is large, the static solution is a $O(\log n \cdot \log(m+n))$-approximation which is a near-optimal approximation for the adjustable robust problem under constraint uncertainty. Moreover, our bound can be extended to the case where the objective coefficients are also uncertain and the uncertainty is unrelated to the column-wise and constraint-wise constraint uncertainty set. Surprisingly, although widely perceived as highly conservative, the static solution provides good approximation for many uncertainty sets. In fact, El Housni and Goyal [21] show that for general uncertainty sets, there is no piecewise static policy with polynomial number of pieces that gives an approximation bound for the two-stage adjustable robust problem that is better than $O(m^{1-\varepsilon})$ for any $\varepsilon > 0$, while we show that static solution provides a m-approximation for the problem. Our result confirms the power of static solution in two-stage adjustable robust linear optimization problem under uncertain constraint and objective coefficients.

Chapter 4

Characterization of the Optimality Condition of Static Solution in Multi-Stage Robust Optimization Problems

4.1 Introduction

In this section, we consider extensions to multi-stage adjustable robust linear optimization problem with uncertain packing constraints where uncertainty is revealed in stages. In each period, the decision maker needs to make decision in face of adversarial future uncertainty. Multi-stage problems are intractable in general. In fact, Dyer and Stougie [19] show that the problem is PSPACE-hard. Therefore, it is natural to consider efficient approximation algorithms for the problem. In this section, we extend our previous result by considering the performance of static solution for multi-stage adjust robust problem. In particular, we consider the following problem Π_{AR}^{L} where $L \in \mathbb{N}_+$

denotes the number of decision stages.

$$\begin{aligned} z_{\mathsf{AR}}^L = \ & \max \mathbf{c}^T\mathbf{x} + \min_{\mathbf{B}_1 \in \mathcal{U}_1} \left[\max_{\mathbf{y}_1(\mathbf{B}_1)} \mathbf{c}_1^T \mathbf{y}_1(\mathbf{B}_1) + \min_{\mathbf{B}_2 \in \mathcal{U}_2} \left[\max_{\mathbf{y}_2(\mathbf{B}_1,\mathbf{B}_2)} \mathbf{d}_2^T \mathbf{y}_2(\mathbf{B}_1,\mathbf{B}_2) + \ldots \right.\right. \\ & \left.\left.\left. + \min_{\mathbf{B}_L \in \mathcal{U}_L} \left[\max_{\mathbf{y}_L(\mathbf{B}_1,\ldots,\mathbf{B}_L)} \mathbf{d}_L^T \mathbf{y}_L(\mathbf{B}_1,\ldots,\mathbf{B}_L) \right]\right]\right] \\ & \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{y}_1(\mathbf{B}_1) + \mathbf{B}_2\mathbf{y}_2(\mathbf{B}_1,\mathbf{B}_2) + \ldots + \mathbf{B}_L\mathbf{y}_L(\mathbf{B}_1,\ldots,\mathbf{B}_L) \ \le \ \mathbf{h}, \\ & \forall \mathbf{B}_t \in \mathcal{U}_t, t \in [L] \\ & \mathbf{x}, \mathbf{y}_1(\mathbf{B}_1), \ldots, \mathbf{y}_L(\mathbf{B}_1,\ldots,\mathbf{B}_L) \ge \mathbf{0} \end{aligned} \tag{4.1.1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{d}_i \in \mathbb{R}^n$, $\mathbf{h} \in \mathbb{R}^m_+$, and $\mathbf{B}_t \in \mathcal{U}_t \subseteq \mathbb{R}^{m\times n}_+$ be the uncertain constraint coefficient matrix for the t^{th}-stage for all $t \in [L]$. In particular, we consider the case where the uncertainty for each stage is unrelated of the uncertainties for the other stages, i.e., the uncertainty set $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_L$. The corresponding static robust problem Π^L_{Rob} can be formulated as follows.

$$\begin{aligned} z_{\mathsf{Rob}}^L = \ & \max \mathbf{c}^T\mathbf{x} + \mathbf{d}_1^T\mathbf{y}_1 + \ldots + \mathbf{d}_L^T\mathbf{y}_L \\ & \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{y}_1 + \mathbf{B}_2\mathbf{y}_2 + \ldots + \mathbf{B}_L\mathbf{y}_L \ \le \ \mathbf{h}, \forall \mathbf{B}_t \in \mathcal{U}_t, t \in [L] \\ & \mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_L \ge \mathbf{0}. \end{aligned} \tag{4.1.2}$$

As in previous sections, we can assume without loss of generality that $\mathcal{U}_t$ is down-monotone for all $t \in [L]$.

4.2 Main Theorem

We have the following main theorem.

Theorem 4.2.1. *Let z^L_{AR} be the optimal objective value of Π^L_{AR} in* (4.1.1) *defined over the uncertainty $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_L$. Let z^L_{Rob} be the optimal objective value of Π^L_{Rob} in* (4.1.2)*. Let $\rho(\cdot)$ be defined as in Theorem 2.4.2, i.e.,*

$$\rho(\mathcal{U}) = \max\{\kappa(T(\mathcal{U},\mathbf{h})) \mid \mathbf{h} > \mathbf{0}\},$$

where $\kappa(\cdot)$ is the measure of non-convexity as defined in (2.4.1)*. Then,*

$$z^L_{\mathsf{Rob}} \le z^L_{\mathsf{AR}} \le \rho(\mathcal{U}) \cdot z^L_{\mathsf{Rob}}.$$

Proof. It is easy to see that $z^L_{\mathsf{AR}} \ge z^L_{\mathsf{Rob}}$: Let $(\mathbf{x}^*, \mathbf{y}^*_1, \mathbf{y}^*_2, \ldots, \mathbf{y}^*_L)$ be an optimal solution for Π^L_{Rob}. Since $\mathcal{U}_t, t \in [L]$ are independent of each other, this implies that $\mathbf{x} = \mathbf{x}^*, \mathbf{y}_1(\mathbf{B}_1) = \mathbf{y}^*_1$, $\mathbf{y}_2(\mathbf{B}_1, \mathbf{B}_2) = \mathbf{y}^*_2$, …, $\mathbf{y}_L(\mathbf{B}_1, \ldots, \mathbf{B}_L) = \mathbf{y}^*_L$ is a feasible solution for the adjustable robust problem Π^L_{AR} (4.1.1). Therefore $z^L_{\mathsf{AR}} \ge z^L_{\mathsf{Rob}}$ for all $L \in \mathbb{N}_+$.

On the other hand, consider the following problem Π_{mod}:

$$\begin{aligned} z_{\mathsf{mod}} = \; & \max \mathbf{c}^T\mathbf{x} + \min_{(\mathbf{B}_1,\ldots,\mathbf{B}_L) \in \mathcal{U}} \; \max_{\mathbf{y}_1(\mathbf{B}_1,\ldots,\mathbf{B}_L),\ldots,\mathbf{y}_L\mathbf{B}_1,\ldots,\mathbf{B}_L)} \sum_{t=1}^{L} \mathbf{d}_t^T \mathbf{y}_t(\mathbf{B}_1,\ldots,\mathbf{B}_L) \\ & \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{y}_1(\mathbf{B}_1,\ldots,\mathbf{B}_L) + \mathbf{B}_2\mathbf{y}_2(\mathbf{B}_1,\ldots,\mathbf{B}_L) + \ldots + \mathbf{B}_L\mathbf{y}_L(\mathbf{B}_1,\ldots,\mathbf{B}_L) \;\le\; \mathbf{h}, \\ & \qquad \forall (\mathbf{B}_1,\ldots,\mathbf{B}_L) \in \mathcal{U} \\ & \mathbf{x}, \mathbf{y}_1(\mathbf{B}_1,\ldots,\mathbf{B}_L),\ldots,\mathbf{y}_L(\mathbf{B}_1,\ldots,\mathbf{B}_L) \ge \mathbf{0} \end{aligned} \tag{4.2.1}$$

Note the in Π_{mod}, the variables $(\mathbf{y}_1, \ldots, \mathbf{y}_L)$ are chosen with full knowledge of the uncertain constraint coefficient matrices $\mathbf{B}_1, \ldots, \mathbf{B}_L$. Therefore, any solution feasible for Π^L_{AR} is also feasible for Π_{mod}, and we have $z^L_{\mathsf{AR}} \le z_{\mathsf{mod}}$. Moreover, Π_{mod} is essentially a two-stage adjustable robust problem with the second-stage uncertainty set $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_L$. Note the the static robust problem for Π_{mod} is exactly Π^L_{Rob}. From Theorem 2.4.2, we have $z_{\mathsf{mod}} \le \rho(\mathcal{U}) \cdot z^L_{\mathsf{Rob}}$. Therefore, $z^L_{\mathsf{AR}} \le \rho(\mathcal{U}) \cdot z^L_{\mathsf{Rob}}$. □

Theorem 4.2.1 is a generalization of our result for two-stage adjust adjustable robust problems. Note that if $\mathcal{U}_t$ are all constraint-wise or all symmetric projections, then $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. Therefore, we have the following Lemma.

Lemma 4.2.2. *Let z^L_{AR} be the optimal objective value of Π^L_{AR} in* (4.1.1) *defined over the uncertainty $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_L$. Let z^L_{Rob} be the optimal objective value of Π^L_{Rob} in* (4.1.2)*. Then, $z^L_{\mathsf{Rob}} = z^L_{\mathsf{AR}}$ if for all $t \in [L]$,*

1. *$\mathcal{U}_t$ is constraint-wise as defined in Lemma 2.3.1, or*

2. *$\mathcal{U}_t$ is symmetric projection as defined in Lemma 2.3.2.*

Proof. Note that the choice of $\mathcal{U}_t$ for each stage $t \in [L]$ is unrelated of the choice of the others. If $\mathcal{U}_t$ are all constraint-wise for $t \in [L]$, so is $\mathcal{U}$. Similar argument holds for the case where $\mathcal{U}_t$ are all symmetric projections. Therefore, $T(\mathcal{U}, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ and from Theorem 4.2.1, $z^L_{\mathsf{Rob}} = z^L_{\mathsf{AR}}$. □

We would like to note that even if $T(\mathcal{U}_t, \mathbf{h})$ is convex for all $t \in [L]$, $T(\mathcal{U}, \mathbf{h})$ may not be convex. Consider the following example:

Example 1 ($T(\mathcal{U}_t, \mathbf{h})$ is convex but not $T(\mathcal{U}, \mathbf{h})$). Consider the following instance of input parameters:

$$\mathbf{c} = \mathbf{0}, \mathbf{A} = \mathbf{0}, \mathbf{h} = \mathbf{d}_1 = \mathbf{d}_2 = [1;1], \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2,$$
$$\mathcal{U}_1 = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| \begin{array}{l} x_1 + x_2 + 2x_3 + 2x_4 \leq 1, \\ x_i \geq 0, i = 1,2,3,4. \end{array} \right\}, \tag{4.2.2}$$
$$\mathcal{U}_2 = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| \begin{array}{l} 2x_1 + 2x_2 + x_3 + x_4 \leq 1, \\ x_i \geq 0, i = 1,2,3,4. \end{array} \right\}$$

We can reformulate the static solution as follows:

$$\begin{aligned} z_{\mathsf{Rob}} = \max\ & y_{11} + y_{12} + y_{21} + y_{22} \\ & \max(y_{11}, y_{12}) + \frac{1}{2}\max(y_{21}, y_{22}) \leq 1, \\ & \frac{1}{2}\max(y_{11}, y_{12}) + \max(y_{21}, y_{22}) \leq 1. \end{aligned}$$

Note that by symmetry, the optimal is achieved when $y_{11} = y_{12} = y_{21} = y_{22} = 2/3$. Therefore, $z_{\mathsf{Rob}} = 8/3$.

For the adjustable robust problem, we consider a special class of solution where

$$y_{11}(\mathbf{B}_1) = y_{12}(\mathbf{B}_1) = y_1(\mathbf{B}_1), y_{21}(\mathbf{B}_1, \mathbf{B}_2) = y_{22}(\mathbf{B}_1, \mathbf{B}_2) = y_2(\mathbf{B}_1, \mathbf{B}_2),$$
$$\mathbf{B}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \ \mathbf{B}_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

Then,

$$\hat{z}_{\mathsf{AR}} = \min_{\mathbf{B}_1} \max_{y_1(\mathbf{B}_1)} 2y_1(\mathbf{B}_1) + \min_{\mathbf{B}_2} \max_{y_2(\mathbf{B}_1,\mathbf{B}_2)} 2y_2(\mathbf{B}_1,\mathbf{B}_2)$$

$$(a_1+b_1)y_1(\mathbf{B}_1) + (a_2+b_2)y_2(\mathbf{B}_1,\mathbf{B}_2) \leq 1,$$

$$(c_1+d_1)y_1(\mathbf{B}_1) + (c_2+d_2)y_2(\mathbf{B}_1,\mathbf{B}_2) \leq 1.$$

For the ease of notation, let

$$\varepsilon_1 = c_1+d_1, \varepsilon_2 = a_2+b_2, y_1 = y_1(\mathbf{B}_1), y_2 = y_2(\mathbf{B}_1,\mathbf{B}_2).$$

Then

$$\begin{aligned} \hat{z}_{\mathsf{AR}} = \min_{\varepsilon_1} \max_{y_1} 2y_1 + \min_{\varepsilon_2} \max_{y_2} 2y_2 & \\ (1-2\varepsilon_1)y_1 + \varepsilon_2 y_2 \leq 1, & \quad (4.2.3) \\ \varepsilon_1 y_1 + (1-\varepsilon_2)y_2 \leq 1. & \end{aligned}$$

Then, we have the following lemma.

Lemma 4.2.3. *Let* z_{AR} *be the optimal objective value of the problem* (4.1.1) *with input parameters as in* (4.2.2) *and* $\hat{z}_{\mathsf{AR}}$ *be the optimal objective value of* (4.2.3)*. Then*

$$z_{\mathsf{AR}} \geq \hat{z}_{\mathsf{AR}} \geq \frac{17}{6}.$$

Proof. The first inequality holds because $\hat{z}_{\mathsf{AR}}$ only consider a special class of solutions to z_{AR}. Now, we discuss the solution of (4.2.3) by categorize on the possible values of ε_1 over $0 \leq \varepsilon_1 \leq 1/2$.

1. If $\varepsilon_1 \in [1/3, 1/2]$, we can set $y_1 = 1/\varepsilon_1 \geq 2$, which implies that $z_{\mathsf{AR}} \geq 4$.

2. If $\varepsilon_1 \in [1/4, 1/3]$, we can set $y_1 = 1/(1-2\varepsilon_1) \geq 2$. which implies that $z_{\mathsf{AR}} \geq 4$.

3. If $\varepsilon_1 \in [0, 1/4]$, we set

$$y_1 = \frac{1}{2(1-2\varepsilon_1)},\, y_2 = \min\left\{\frac{1}{2\varepsilon_2}, \frac{1}{1-2\varepsilon_2}\left(1 - \frac{\varepsilon_1}{2(1-2\varepsilon_1)}\right)\right\}.$$

Therefore,

$$\hat{z}_{\mathsf{AR}} \geq \frac{1}{1-2\varepsilon_1} + \frac{2}{1-2\varepsilon_2} - \frac{\varepsilon_1}{(1-2\varepsilon_1)(1-2\varepsilon_2)}$$

Now, consider the problem

$$\hat{z} = \min\left\{\frac{1}{1-2x} + \frac{1}{1-2y} + \frac{1}{1-2y}\left(1 - \frac{x}{1-2x}\right) \,\middle|\, x \leq \frac{1}{4}, 0 \leq y \leq \frac{1}{2}\right\}.$$

We further discuss on the values of x.

(a) If $x \leq 1/8$, then

$$\begin{aligned}
\hat{z} &= \min\left\{\frac{1}{1-2x} + \frac{1}{1-2y} + \frac{1}{1-2y}\left(1 - \frac{x}{1-2x}\right) \,\middle|\, x \leq \frac{1}{8}, 0 \leq y \leq \frac{1}{2}\right\} \\
&\geq \min_{x \leq 1/8} \frac{1}{1-2x} + \min_{y \leq 1/2}\left\{\frac{1}{1-2y}\left(1 + \min_{x \leq 1/8}\frac{1-3x}{1-2x}\right)\right\} \\
&\geq \min_{x \leq 1/8} \frac{1}{1-2x} + \min_{y \leq 1/2}\left\{\frac{1}{1-2y}\frac{11}{6}\right\} \\
&= \frac{17}{6}.
\end{aligned}$$

(b) If $1/8 \leq x \leq 1/4$, then

$$
\begin{aligned}
\hat{z} &= \min\left\{ \frac{1}{1-2x} + \frac{1}{1-2y} + \frac{1}{1-2y}\left(1 - \frac{x}{1-2x}\right) \,\middle|\, \frac{1}{8} \leq x \leq \frac{1}{4}, 0 \leq y \leq \frac{1}{2} \right\} \\
&\geq \min_{1/8 \leq x \leq 1/4} \frac{1}{1-2x} + \min_{y \leq 1/2} \left\{ \frac{1}{1-2y}\left(1 + \min_{1/8 \leq x \leq 1/4} \frac{1-3x}{1-2x}\right) \right\} \\
&\geq \min_{x \leq 1/8} \frac{1}{1-2x} + \min_{y \leq 1/2} \left\{ \frac{1}{1-2y} \frac{3}{2} \right\} \\
&= \frac{17}{6}.
\end{aligned}
$$

Therefore, $\hat{z}_{\mathsf{AR}} \geq \hat{z} = 17/6$.

From the discussions above, we can see that $\hat{z}_{\mathsf{AR}} \geq 17/6 > 8/3 = z_{\mathsf{Rob}}$. □

Note that the projection of $\mathcal{U}_1$ onto each row is a scaling of the other, and the same holds for $\mathcal{U}_2$. We can see that $T(\mathcal{U}_t, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$ and $t = 1, 2$. However, the static solution is sub-optimal for the multi-stage adjustable robust problem from our previous discussion. Therefore, our previous optimality condition for the static solution as in Theorem 2.3.3 can not be generalized to the multi-stage problems.

4.3 Approximation Bound on the Performance of Static Solution

In this section, we show that for a multi-stage Cartesian uncertainty set $\mathcal{U}$, $\rho(\mathcal{U})$ is at most $L \cdot \max\{\rho(\mathcal{U}_t) \mid t \in [L]\}$, where L is the number of stages. In particular, we prove the following lemma.

Lemma 4.3.1. *Given a L-stage uncertainty set* $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_L$*, let* $\rho(\cdot)$ *be defined as in Theorem 2.4.2. Then, we have*

$$\rho(\mathcal{U}) \leq L \cdot \max\{\rho(\mathcal{U}_t) \mid t \in [L]\}$$

Proof. Given an arbitrary $\mathbf{h} > \mathbf{0}$, consider $\mathbf{b} \in \mathsf{conv}(T(\mathcal{U}, \mathbf{h}))$. We can write

$$\mathbf{b}^T = [\mathbf{b}_1^T \; \mathbf{b}_2^T \; \ldots \; \mathbf{b}_L^T]$$

where $\mathbf{b}_t \in \mathsf{conv}(T(\mathcal{U}_t, \mathbf{h}))$ since $\mathcal{U}$ is a Cartesian product of $\mathcal{U}_t$, $t \in [L]$. From the definition of $\rho(\cdot)$, this implies that

$$\frac{\mathbf{b}^T}{\max\{\rho(\mathcal{U}_t) \mid t \in [L]\}} \leq \left[\frac{\mathbf{b}_1^T}{\rho(\mathcal{U}_1)} \; \cdots \; \frac{\mathbf{b}_L^T}{\rho(\mathcal{U}_L)}\right] = [\mu_1^T \mathbf{B}_1 \; \ldots \; \mu_L^T \mathbf{B}_L]$$

where $\mathbf{h}^T \mu_t = 1, \mu \geq \mathbf{0}, \mathbf{B}_t \in \mathcal{U}_t$ for all $t \in [L]$. Now, let

$$\mathbf{B} = [\mathbf{B}_1 \; \mathbf{B}_2 \ldots \mathbf{B}_L], \mu = \frac{1}{L} \sum_{t=1}^{L} \mu_t.$$

Since $\mathcal{U}$ is a Cartesian product of $\mathcal{U}_t$, $t \in [L]$, we have

$$\mathbf{B} \in \mathcal{U}, \mathbf{h}^T \mu = 1, \mu \geq \mathbf{0}.$$

Therefore, $\mathbf{B}^T\mu \in T(\mathcal{U},\mathbf{h})$. Note that

$$L \cdot \mu^T \mathbf{B} = \sum_{t=1}^{L} \mu_t^T [\mathbf{B}_1\ \mathbf{B}_2 \ldots \mathbf{B}_L] \geq [\mu_1^T \mathbf{B}_1\ \ldots\ \mu_L^T \mathbf{B}_L],$$

we have

$$\frac{\mathbf{b}^T}{L \cdot \max\{\rho(\mathcal{U}_t) \mid t \in [L]\}} \in T(\mathcal{U},\mathbf{h}),$$

thereby complete our proof. □

Therefore, if the uncertainty set $\mathcal{U}$ is a mixture of constraint-wise or symmetric projection uncertainty sets, then the adaptivity gap is bounded by L. Moreover, from our previous result in Theorem 2.4.3, $\rho(\mathcal{U}_t)$ is at most m. Therefore, for a multi-stage adjustable robust problem, the performance of static solution is bounded by Lm.

Chapter 5

Generalized Decision Rule Approximation for Two-Stage Robust Linear Optimization

5.1 Introduction

In this chapter, we consider the two-stage adjustable robust linear optimization problem with covering constraints and uncertain right-hand-side $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$ (1.2.4). In Feige et al. [23], the authors consider a two-stage set cover problem where the size of the second-stage demanded is capped by integer k. They show that the problem is $\Omega \log m/\log\log m$-hard to approximate, and a LP-rounding algorithm gives a $O(\log m \log n)$-approximation. Bertsimas and Goyal [10] consider the general formulation (1.2.1) and show that the affine policy gives an $O(\sqrt{m})$-approximation. Moreover, they show that the bound is tight when the uncertainty set is the intersection of the unit ℓ_2-norm ball and positive orthant. This motivates us to find efficient algorithms to improve this approximation ratio. In particular, we introduce a new framework to approximate $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$.

For the ease of discussion, we denote $\Pi_{\mathsf{AR-cover}}(\mathcal{U})$ as $\Pi_{\mathsf{AR}}(\mathcal{U})$ throughout this chapter. Note that we add the uncertainty set $\mathcal{U}$ as an input to the problem because our new framework depends on computing the optimal two-stage adjustable robust solution on an extended set.

Outline. In Section 5.2, we present the new framework for approximating the two-stage adjustable robust problem (1.2.4). Based on this framework, we provide approximation bounds for $\Pi_{\mathsf{AR}}(\mathcal{U})$ (1.2.4) with unit ℓ_2-norm ball and ℓ_p-norm ball uncertainty sets in Section 5.3.

5.2 A New Approximation Framework via Dominating Uncertainty Set

In this section, we present a new framework to approximate the two-stage adjustable robust problem (1.2.4). Our policy is based on approximating the boundary points of the uncertainty set $\mathcal{U}$ with a simple set. In particular, we construct a set $\hat{\mathcal{U}}$ that *dominates* the uncertainty set $\mathcal{U}$. Moreover, we require that the two-stage adjustable robust problem (1.2.4) can be efficiently solved over $\hat{\mathcal{U}}$. We first define some geometric properties for the uncertainty set $\mathcal{U}$.

Definition 5.2.1. (Domination) *Given uncertainty set $\mathcal{U} \subseteq \mathbb{R}^m_+$, $\hat{\mathcal{U}} \subseteq \mathbb{R}^m_+$ dominates $\mathcal{U}$ if for all $\mathbf{h} \in \mathcal{U}$, there exists $\hat{\mathbf{h}} \in \hat{\mathcal{U}}$ such that $\hat{\mathbf{h}} \geq \mathbf{h}$.*

Definition 5.2.2. (Scaling factor) *Given a full-dimensional uncertainty set $\mathcal{U} \subseteq \mathbb{R}^m_+$ and $\hat{\mathcal{U}} \subseteq \mathbb{R}^m_+$ that dominates $\mathcal{U}$. We define the scaling factor $\beta_{(\mathcal{U},\hat{\mathcal{U}})}$ of $(\mathcal{U}, \hat{\mathcal{U}})$ as the smallest scalar such that $\hat{\mathcal{U}} \subseteq \beta_{(\mathcal{U},\hat{\mathcal{U}})} \cdot \mathcal{U}$, i.e.*

$$\beta_{(\mathcal{U},\hat{\mathcal{U}})} = \min\{\alpha > 0 \mid \hat{\mathcal{U}} \subseteq \alpha \cdot \mathcal{U}\}.$$

For the sake of simplicity, we denote the scaling factor of $(\mathcal{U}, \hat{\mathcal{U}})$ by β throughout this chapter. Note that this scaling factor always exists because $\mathcal{U}$ is full-dimensional. Moreover, it is greater than one because of the assumption of domination. The following theorem shows that solving the adjustable problem over the set $\hat{\mathcal{U}}$ gives an approximation to the two-stage adjustable robust problem (1.2.4) within a factor β.

Theorem 5.2.3. *Given a convex, compact and down-monotone uncertainty set $\mathcal{U}$ and $\hat{\mathcal{U}} \subseteq \mathbb{R}^m_+$ dominates $\mathcal{U} \subseteq \mathbb{R}^m_+$, let β be the scaling factor of $(\mathcal{U}, \hat{\mathcal{U}})$. Moreover, let $z_{\mathsf{AR}}(\mathcal{U})$ and $z_{\mathsf{AR}}(\hat{\mathcal{U}})$ be the optimal values for* (1.2.4) *on $\mathcal{U}$ and $\hat{\mathcal{U}}$, respectively. Then,*

$$z_{\mathsf{AR}}(\mathcal{U}) \leq z_{\mathsf{AR}}(\hat{\mathcal{U}}) \leq \beta \cdot z_{\mathsf{AR}}(\mathcal{U}).$$

Proof. Let $(\hat{\mathbf{x}}, \hat{\mathbf{y}}(\hat{\mathbf{h}}), \hat{\mathbf{h}} \in \hat{\mathcal{U}})$ be an optimal solution for $z_{\mathsf{AR}}(\hat{\mathcal{U}})$. For each $\mathbf{h} \in \mathcal{U}$, let $\tilde{\mathbf{y}}(\mathbf{h}) = \hat{\mathbf{y}}(\hat{\mathbf{h}})$ where $\hat{\mathbf{h}} \in \hat{\mathcal{U}}$ dominates $\mathbf{h}$. Therefore, for any $\mathbf{h} \in \mathcal{U}$,

$$\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\tilde{\mathbf{y}}(\mathbf{h}) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{y}}(\hat{\mathbf{h}}) \geq \hat{\mathbf{h}} \geq \mathbf{h},$$

i.e., $(\hat{\mathbf{x}}, \tilde{\mathbf{y}}(\mathbf{h}), \mathbf{h} \in \mathcal{U})$ is a feasible solution for $z_{\mathsf{AR}}(\mathcal{U})$. Therefore,

$$z_{\mathsf{AR}}(\mathcal{U}) \leq \mathbf{c}^T\hat{\mathbf{x}} + \max_{\mathbf{h} \in \mathcal{U}} \mathbf{d}^T\tilde{\mathbf{y}}(\mathbf{h}) \leq \mathbf{c}^T\hat{\mathbf{x}} + \max_{\hat{\mathbf{h}} \in \hat{\mathcal{U}}} \mathbf{d}^T\hat{\mathbf{y}}(\hat{\mathbf{h}}) = z_{\mathsf{AR}}(\hat{\mathcal{U}}).$$

Conversely, let $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{h}), \mathbf{h} \in \mathcal{U})$ be an optimal solution of $z_{\mathsf{AR}}(\mathcal{U})$. Then, for any $\hat{\mathbf{h}} \in \hat{\mathcal{U}}$, since

$\hat{\mathbf{h}}/\beta \in \mathcal{U}$, we have

$$\mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{y}^*\left(\frac{\hat{\mathbf{h}}}{\beta}\right) \geq \frac{\hat{\mathbf{h}}}{\beta},$$

Therefore, $(\beta\mathbf{x}^*, \beta\mathbf{y}^*\left(\frac{\hat{\mathbf{h}}}{\beta}\right), \hat{\mathbf{h}} \in \mathcal{U})$ is feasible for $\Pi_{\mathsf{AR}}(\hat{\mathcal{U}})$. Therefore,

$$z_{\mathsf{AR}}(\hat{\mathcal{U}}) \leq \mathbf{c}^T\beta\mathbf{x}^* + \max_{\hat{\mathbf{h}}\in\hat{\mathcal{U}}} \mathbf{d}^T\beta\mathbf{y}^*\left(\frac{\hat{\mathbf{h}}}{\beta}\right) \leq \beta\cdot\left(\mathbf{c}^T\mathbf{x}^* + \max_{\mathbf{h}\in\mathcal{U}} \mathbf{d}^T\mathbf{y}^*(\mathbf{h})\right) = \beta\cdot z_{\mathsf{AR}}(\mathcal{U}).$$

□

Theorem 5.2.3 provides a new framework for approximating the two-stage adjustable robust problem $\Pi_{\mathsf{AR}}(\mathcal{U})$ (1.2.4). Note that we require that $\hat{\mathcal{U}}$ dominates $\mathcal{U}$ and $\Pi_{\mathsf{AR}}(\hat{\mathcal{U}})$ can be efficiently solved over $\hat{\mathcal{U}}$. In fact, the latter is satisfied if the number of extreme points of $\hat{\mathcal{U}}$ is small (typically polynomial of m). Therefore, we choose $\hat{\mathcal{U}}$ to be a simplex in our framework. The adjustable problem is easy to solve over a simplex as it can be reduced to a single LP problem. In particular, given simplex uncertainty set

$$\mathcal{U} = \mathsf{conv}\left(\nu_1, \nu_2, \ldots, \nu_{m+1}\right),$$

we can formulate the two-stage adjustable robust problem $\Pi_{\mathsf{AR}}(\mathcal{U})$ as the following LP.

$$\begin{aligned} z_{\mathsf{AR}}(\mathcal{U}) = \min\ & \mathbf{c}^T\mathbf{x} + z \\ & z \geq \mathbf{d}^T\mathbf{y}_i,\ \forall i \in [m+1] \\ & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}_i \geq \nu_i,\ \forall i \in [m+1] \\ & \mathbf{x} \in \mathbb{R}^n_+,\ \mathbf{y}_i \in \mathbb{R}^n_+. \end{aligned}$$

Affine policy is another tractable approach to solve $\Pi_{\mathsf{AR}}(\mathcal{U})$ (1.2.4) when $\mathcal{U}$ is simplex. As mentioned earlier, it is optimal for simplex uncertainty sets. However, for general convex uncertainty sets, its performance can be as bad as $O(\sqrt{m})$. Our goal is to study new approximation framework to improve this ratio. In particular, we would like to find a simplex $\hat{\mathcal{U}}$ that dominates $\mathcal{U}$ such that $\beta = \Omega(m^{\frac{1}{2}-\varepsilon})$ for some $\varepsilon > 0$, thereby give a good approximation for $\Pi_{\mathsf{AR}}(\mathcal{U})$. In the following sections, we provide improved approximation bounds for $\Pi_{\mathsf{AR}}(\mathcal{U})$ with several interesting families of uncertainty sets given by this framework.

5.3 Examples of Improved Approximation Bounds

In this section, we present the approximation bounds for two interesting family of uncertainty sets. In particular, our bounds are better than the results of Bertsimas and Bidkhori [7]. Similar to previous chapters, we can assume without lost of generality that $\mathcal{U} \subseteq [0,1]^n$ by scaling. In particular, we have $\mathbf{e}_j \in \mathcal{U}$ for all $j \in [m]$.

Permutation Invariant Sets. We first consider permutation invariant sets. Recall that an uncertainty set $\mathcal{U}$ is permutation invariant if $\mathbf{x} \in \mathcal{U}$ implies that for any permutation τ of $[m]$, $\mathbf{x}^\tau \in \mathcal{U}$ where $x_i^\tau = x_{\tau(i)}$. We define $\gamma(\mathcal{U})$ where

$$\gamma(\mathcal{U})\mathbf{e} = \arg\max\{\mathbf{e}^T\mathbf{x} \mid \mathbf{x} \in \mathcal{U}\}.$$

Now, consider the simplex

$$\hat{\mathcal{U}} = \beta \cdot \mathsf{conv}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m, \gamma(\mathcal{U}) \cdot \mathbf{e}).$$

Let β be the scaling factor of $\mathcal{U}$ and $\hat{\mathcal{U}}$. By definition, $\beta\hat{\mathcal{U}}$ dominates $\mathcal{U}$. Therefore, solving the two-stage adjustable robust problem Π_{AR} (1.2.4) over $\hat{\mathcal{U}}$ gives an β-approximation to Π_{AR} over $\mathcal{U}$.

Note that β may not be efficiently computable given arbitrary permutation invariant set. In the following examples, we explore several interesting family of uncertainty sets and compute their corresponding β's.

Lemma 5.3.1. (Hypersphere) *Consider* $\mathcal{U} = \{\mathbf{h} \in \mathbb{R}^m_+ \mid ||\mathbf{h}||_2 \leq 1\}$ *as in* (1.1.1). *Then, Theorem 5.2.3 holds with*

$$\hat{\mathcal{U}} = \beta \cdot \mathsf{conv}\left(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m, \frac{1}{\sqrt{m}}\mathbf{e}\right)$$

with $\beta = m^{\frac{1}{4}}$.

Proof. To prove that $\beta\hat{\mathcal{U}}$ dominates $\mathcal{U}$, it is sufficient to show that the boundary of $\mathcal{U}$ is dominated. Consider $\mathbf{h}$ such that $||\mathbf{h}||_2 = 1$. Let $\alpha_i = \frac{h_i^2}{2}$ for $i \in [m]$ and $\alpha_{m+1} = \frac{1}{2}$ be the convex multipliers for $\hat{\mathcal{U}}$. Then, we have $\mathbf{e}^T\alpha = 1$ and for all $i \in [m]$,

$$\beta\hat{h}_i = \beta\left(\alpha_i + \frac{1}{\sqrt{m}}\alpha_{m+1}\right) = \beta\left(\frac{h_i^2}{2} + \frac{1}{2\sqrt{m}}\right) \geq \beta \cdot 2\sqrt{\frac{h_i^2}{4\sqrt{m}}} = h_i.$$

Therefore, $\hat{\mathbf{h}} \in \hat{\mathcal{U}}$ and $\mathcal{U}$ is dominated by $\beta\hat{\mathcal{U}}$. $\square$

Lemma 5.3.2. **(ℓ_p-Norm Ball)** *Consider* $\mathcal{U} = \left\{\mathbf{h} \in \mathbb{R}^m_+ \mid ||\mathbf{h}||_p \leq 1\right\}$ *where* $p \in \mathbb{N}_+$. *Then,*

$$\hat{\mathcal{U}} = \beta \cdot \mathsf{conv}\left(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m, m^{-\frac{1}{p}}\mathbf{e}\right),$$

where $\beta = m^{\frac{p-1}{p^2}}$.

Proof. Similar to the previous proof, it is sufficient to show that the boundary of $\mathcal{U}$ is dominated by $\beta\hat{\mathcal{U}}$. Consider $\mathbf{h} \in \mathcal{U}$ such that $||\mathbf{h}||_p = 1$. Let $\alpha_i = \frac{h_i^p}{p}$ for $i \in [m]$ and $\alpha_{m+1} = \frac{p-1}{p}$ be the convex multipliers for $\hat{\mathcal{U}}$. Then, $\mathbf{e}^T\alpha = 1$ and for all $i \in [m]$,

$$\beta\hat{h}_i = \beta\left(\alpha_i + \frac{1}{\sqrt{m}}\alpha_{m+1}\right) = \beta\left(\frac{h_i^p}{p} + \frac{p-1}{p}m^{-\frac{1}{p}}\right) \geq \beta\left(h_i^p\right)^{\frac{1}{p}}\left(m^{-\frac{1}{p}}\right)^{\frac{p-1}{p}} = h_i,$$

where the inequality follows from the AM-GM inequality. ☐

Lemma 5.3.2 is a generalization for Lemma 5.3.1. In fact, we recover the result in Lemma 5.3.1 for $p = 2$. Bertsimas and Bidkhori [7] show that an affine policy on the uncertainty set $\mathcal{U}$ provides a $m^{\frac{1}{p}}$-approximation for the two-stage adjustable robust problem Π_{AR}. However, by considering a dominating set $\hat{\mathcal{U}}$, we can provide a better approximation ratio without significantly increasing the computational complexity. It would be interesting to consider such approximation framework for other uncertainty sets.

5.4 Conclusion

In this Chapter, we consider the two-stage adjustable robust linear optimization problems with covering constraints and uncertain right-hand-side. We introduce a new framework for approximating such problem based on choosing an appropriate dominating set for the uncertainty set. The choice of the dominating set explores the geometric structure of the uncertainty set and gives better approximation bounds than the affine policy for a couple of interesting class of uncertainty sets. In particular, our approximation framework provides a $m^{1/4}$-approximation for the unit hypersphere while the affine policy gives an $O(\sqrt{m})$-approximation. More generally, for general unit ℓ_p-norm balls, our framework gives a $m^{\frac{p-1}{p^2}}$-approximation as opposed to $m^{\frac{1}{p}}$ given by an affine policy.

Chapter 6

Conclusions

In this thesis, we consider adjustable robust linear optimization problems in both packing and covering formulations with constraint and right-hand-side uncertainty, respectively. Such problems arise naturally in real-world applications such as resource allocation and machine scheduling. However, computing an optimal solution for adjustable robust problem is intractable. In fact, we show that for a column-wise constraint uncertainty set, the two-stage packing problem is $\Omega(\log n)$-hard to approximate. For a more general case where the uncertainty set $\mathcal{U}$ and objective coefficients $\mathbf{d}$ are not constrained in the non-negative orthant, we show that the adjustable robust problem is $\Omega(2^{\log^{1-\epsilon} m})$-hard to approximate for any constant $0 < \epsilon < 1$. In addition, Feige et al. [22] show that the covering problem is $\Omega(\log m / \log\log m)$-hard to approximate. This motivates us to study approximation algorithm for the problem.

In Chapter 2 and 3, we consider the two-stage robust packing problem with uncertain constraint coefficients and study the performance of static robust solution as its approximation. We first give a tight characterization of the performance of static solution and relate it to the measure of non-convexity of the transformation $T(\mathcal{U},\cdot)$ of the uncertainty set $\mathcal{U}$. In particular, we show that a static solution is optimal if $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$. For several interesting families of uncertainty sets such as constraint-wise or symmetric projections, we show that $T(\mathcal{U},\mathbf{h})$ is convex for all $\mathbf{h} > 0$; thereby generalize the result of Ben-Tal and Nemirovski [5] for the case where $\mathcal{U}$ is contained in the non-negative orthant. In Chapter 4, we generalize the result to a multi-stage problem where the choice of the uncertain coefficient matrix for each stage is independent of the others. We show that a static solution is optimal for the multi-stage adjustable robust problem if $\mathcal{U}_t$ is constraint-wise for each stage $t \in [K]$. Moreover, we also give an approximation bound on the performance of static solutions that is related to the measure of non-convexity of the transformation of the Cartesian product of the uncertainty sets for each stage.

When $T(\mathcal{U},\cdot)$ is not convex, We show that the measure of non-convexity of $T(\mathcal{U},\cdot)$ gives a tight bound on the performance of static solutions. Our approximation bound is better than the symmetry bound in Bertsimas and Goyal [11]. However, the bound is instanced-based and may not be efficiently computable. Moreover, for a family of diagonal uncertainty sets, the bound can be as large as m. Therefore, we consider column-wise and constraint-wise uncertainty sets, which are more natural in real-world applications. For such uncertainty sets, we show that a static solution is an $O(\log n \cdot \min(\log\Gamma, \log(m+n)))$-approximation for the adjustable robust problem. Therefore, if Γ (maximum ratio between upper bounds of uncertain constraint coefficients) is a constant, the static solution provides an $O(\log n)$-approximation which matches the hardness of approximation in this case. If Γ is large, the static solution is a $O(\log n \cdot \log(m+n))$-approximation which is a near-optimal approximation for the adjustable robust problem under constraint uncertainty. From our computational study, we conjecture the upper bound of the approximation bound is $O(\log n)$

instead of $O(\log n \cdot \log(m+n))$ and it is an interesting open question to close the gap between the upper and lower bounds.

We extend our results to models where both constraint and objective coefficients are uncertain. We show that if $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$, where $\mathcal{U}^B$ is the set of uncertain second-stage constraint matrices $\mathbf{B}$ and $\mathcal{U}^d$ is the set of uncertain second-stage objective, then the performance of static solution is related to the measure of non-convexity of $T(\mathcal{U}^B, \cdot)$. In particular, a static solution is optimal if $T(\mathcal{U}^B, \mathbf{h})$ is convex for all $\mathbf{h} > \mathbf{0}$; it also provides a $O(\log n \cdot \min(\log \Gamma, \log(m+n)))$-approximation if $\mathcal{U}^B$ is column-wise and constraint-wise. Surprisingly, the performance of static solution does not depend on the uncertainty set $\mathcal{U}^d$. We also present several examples to illustrate such optimality and the tightness of the bound.

Piecewise static solution is an interesting generalization of static solution and is perceived as more general. However, in a recent result by El Housni and Goyal [21], the authors show that in general there is no piecewise static policy with a polynomial number of pieces that has a significantly better performance than an optimal static solution. Our results further confirm the power of static solution in two-stage adjustable robust linear optimization problem under uncertain constraint and objective coefficients. Moreover, our results develop new geometric intuition about the performance of static robust solutions for adjustable robust problems. The reformulations of the adjustable robust and static robust problems based on the transformation $T(\mathcal{U}, \cdot)$ of the uncertainty set $\mathcal{U}$ give us interesting insights about properties of $\mathcal{U}$ where the static robust solution does not perform well. Therefore, our results provide useful guidance in selecting uncertainty sets such that the adjustable robust problem can be well approximated by a static solution.

In Chapter 5, we consider the two-stage adjustable robust linear optimization problems with covering constraints and uncertain right-hand-side. Bertsimas and Bidkhori [7] show that for uncertainty set $\mathcal{U}$ that is an intersection of positive orthant and ℓ_p-norm ball, an affine policy on $\mathcal{U}$ provides a $m^{\frac{1}{p}}$-approximation for the problem. We consider a new approximation framework that is based on choosing an appropriate dominating set for the uncertainty set. In particular, we exploit the geometric structure of the dominating set such that solving the adjustable robust problem over the set gives a better performance than affine policy over the original set. Our approximation framework provides a $m^{1/4}$-approximation for the unit hypersphere while the affine policy gives an $O(\sqrt{m})$-approximation. More generally, for general unit ℓ_p-norm balls, our framework gives a $m^{\frac{p-1}{p^2}}$-approximation as opposed to $m^{\frac{1}{p}}$ given by an affine policy.

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Appendix A

Appendix of Chapter 1

A.1 Down-monotone Uncertainty Sets

In this section, we show that in $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ defined in (2.3.3) and $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ defined in (2.3.4), we can assume $\mathcal{U}$ to be *down-monotone* without loss of generality, where down-monotone is defined as follows.

Definition A.1.1. *A set $\mathcal{S} \subseteq \mathbb{R}^n_+$ is* down-monotone *if $\mathbf{s} \in \mathcal{S}, \mathbf{t} \in \mathbb{R}^n_+$ and $\mathbf{t} \leq \mathbf{s}$ implies $\mathbf{t} \in \mathcal{S}$.*

Given $\mathcal{S} \subseteq \mathbb{R}^n_+$, we can construct the down-hull of S, denoted by $\mathcal{S}^{\downarrow}$ as follows.

$$\mathcal{S}^{\downarrow} = \{\mathbf{t} \in \mathbb{R}^n_+ \mid \exists \mathbf{s} \in \mathcal{S} : \mathbf{t} \leq \mathbf{s}\}. \tag{A.1}$$

We would like to emphasize that the down hull of a non-negative uncertainty set is still constrained in the non-negative orthant. Given uncertainty set $\mathcal{U} \in \mathbb{R}^{m\times n}_+$ and $\mathbf{h} > \mathbf{0}$, if $\mathcal{U}$ is down-monotone, then $\mathcal{U}^{\downarrow} = \mathcal{U}$. Therefore, $\Pi^I_{\mathsf{AR}}(\mathcal{U}^{\downarrow},\mathbf{h})$ is essentially the same problem with $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ and we have

$z^I_{\mathsf{AR}}(\mathcal{U}^{\downarrow},\mathbf{h}) = z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$. Similar arguments applies for $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ and $z^I_{\mathsf{Rob}}(\mathcal{U}^{\downarrow},\mathbf{h}) = z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$. On the other hand, if $\mathcal{U}$ is not down-monotone, then $\mathcal{U} \subsetneq \mathcal{U}^{\downarrow}$. Then, we prove the following lemma.

Lemma A.1.2. *Given uncertainty set* $\mathcal{U} \in \mathbb{R}^{m\times n}_+$ *and* $\mathbf{h} > \mathbf{0}$*, let* $z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ *be the optimal value of* $\Pi^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$ *defined in* (2.3.3)*,* $z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ *be the optimal value of* $\Pi^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$ *defined in* (2.3.4)*. Suppose* $\mathcal{U}$ *is not down-monotone, let* $\mathcal{U}^{\downarrow}$ *be defined as in* (A.1)*. Then,*

$$z^I_{\mathsf{AR}}(\mathcal{U}^{\downarrow},\mathbf{h}) = z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h}),\ z^I_{\mathsf{Rob}}(\mathcal{U}^{\downarrow},\mathbf{h}) = z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}).$$

Proof. Consider an arbitrary $\mathbf{X} \in \mathcal{U}^{\downarrow}$ and $\mathbf{X} \notin \mathcal{U}$, i.e., $\mathbf{X} \in \mathcal{U}^{\downarrow}\setminus\mathcal{U}$. From (A.1), there exists $\mathbf{B} \in \mathcal{U}$ such that $\mathbf{X} \le \mathbf{B}$. Since $\mathbf{B},\mathbf{X}$ and $\mathbf{y}$ are all non-negative, any $\mathbf{y} \in \mathbb{R}^n_+$ such that $\mathbf{B}\mathbf{y} \le \mathbf{h}$ satisfies $\mathbf{X}\mathbf{y} \le \mathbf{h}$. Therefore,

$$\max\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \le \mathbf{h}, \mathbf{y} \in \mathbb{R}^n_+\} \le \max\{\mathbf{d}^T\mathbf{y} \mid \mathbf{X}\mathbf{y} \le \mathbf{h}, \mathbf{y} \in \mathbb{R}^n_+\}.$$

Take minimum over all $\mathbf{B} \in \mathcal{U}$ on the left side, we have

$$\min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \le \mathbf{h}, \mathbf{y} \in \mathbb{R}^n_+\} \le \max_{\mathbf{y}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{X}\mathbf{y} \le \mathbf{h}, \mathbf{y} \in \mathbb{R}^n_+\}.$$

Since $\mathbf{X}$ is arbitrarily chosen in $\mathcal{U}^{\downarrow}\setminus\mathcal{U}$, we can take minimum of all $\mathbf{X} \in \mathcal{U}^{\downarrow}\setminus\mathcal{U}$ on the right side

$$\min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \le \mathbf{h}, \mathbf{y} \in \mathbb{R}^n_+\} \le \min_{\mathbf{X}\in\mathcal{U}^{\downarrow}\setminus\mathcal{U}} \max_{\mathbf{y}} \{\mathbf{d}^T\mathbf{y} \mid \mathbf{X}\mathbf{y} \le \mathbf{h}, \mathbf{y} \in \mathbb{R}^n_+\}.$$

Therefore, the minimizer of the outer problem of $\Pi^I_{\mathsf{AR}}(\mathcal{U}^{\downarrow},\mathbf{h})$ is in $\mathcal{U}$, which implies

$$\min_{\mathbf{B}\in\mathcal{U}}\max_{\mathbf{y}}\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y}\leq\mathbf{h},\mathbf{y}\in\mathbb{R}^n_+\} = \min_{\mathbf{X}\in\mathcal{U}^{\downarrow}}\max_{\mathbf{y}}\{\mathbf{d}^T\mathbf{y} \mid \mathbf{X}\mathbf{y}\leq\mathbf{h},\mathbf{y}\in\mathbb{R}^n_+\}.$$

As a result, we have $z^I_{\mathsf{AR}}(\mathcal{U}^{\downarrow},\mathbf{h}) = z^I_{\mathsf{AR}}(\mathcal{U},\mathbf{h})$.

Similarly, any $\mathbf{y}\in\mathbb{R}^n_+$ satisfies $\mathbf{B}\mathbf{y}\leq\mathbf{h}$ for all $\mathbf{B}\in\mathcal{U}$ is guaranteed to be feasible to $\mathbf{X}\mathbf{y}\leq\mathbf{h}$ for all $\mathbf{X}\in\mathcal{U}^{\downarrow}\backslash\mathcal{U}$. Therefore, we conclude that $z^I_{\mathsf{Rob}}(\mathcal{U}^{\downarrow},\mathbf{h}) = z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h})$. □

Therefore, we can assume without loss of generality that $\mathcal{U}$ is down-monotone in (2.3.3) and (2.3.4). Now, we generalize the result for the two-stage problems $\Pi_{\mathsf{AR-pack}}$ in (1.2.1) and Π_{Rob} in (1.2.2). Consider the following adjustable robust problem $\Pi^{\downarrow}_{\mathsf{AR}}$

$$\begin{aligned} z^{\downarrow}_{\mathsf{AR}} = \max\ & \mathbf{c}^T\mathbf{x} + \min_{\mathbf{B}\in\mathcal{U}^{\downarrow}}\max_{\mathbf{y}(\mathbf{B})}\mathbf{d}^T\mathbf{y}(\mathbf{B}) \\ & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{y}(\mathbf{B}) \ \leq\ \mathbf{h} \\ & \mathbf{x} \ \in\ \mathbb{R}^{n_1} \\ & \mathbf{y}(\mathbf{B}) \ \in\ \mathbb{R}^{n_2}_+, \end{aligned} \tag{A.2}$$

and the corresponding two-stage static robust problem $\Pi^{\downarrow}_{\mathsf{Rob}}$

$$\begin{aligned} z^{\downarrow}_{\mathsf{Rob}} = \max\ & \mathbf{c}^T\mathbf{x} + \mathbf{d}^T\mathbf{y} \\ & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{y} \ \leq\ \mathbf{h},\ \forall\mathbf{B}\in\mathcal{U}^{\downarrow} \\ & \mathbf{x} \ \in\ \mathbb{R}^{n_1} \\ & \mathbf{y} \ \in\ \mathbb{R}^{n_2}_+. \end{aligned} \tag{A.3}$$

Again, given uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m\times n_2}$, if $\mathcal{U}$ is down-monotone, then $\mathcal{U}^{\downarrow} = \mathcal{U}$. Therefore, $\Pi_{\mathsf{AR}}^{\downarrow}$ is essentially the same problem with Π_{AR} and we have $z_{\mathsf{AR}}^{\downarrow} = z_{\mathsf{AR}}$. Similarly, $z_{\mathsf{Rob}}^{\downarrow} = z_{\mathsf{Rob}}$. For the case where $\mathcal{U}$ is not down-monotone, we prove the following lemma:

Lemma A.1.3. *Given uncertainty set* $\mathcal{U} \in \mathbb{R}_+^{m\times n_2}$ *and* $\mathbf{h} \in \mathbb{R}^m$, *let* z_{AR} *and* z_{Rob} *be the optimal values of* $\Pi_{\mathsf{AR-pack}}$ *defined in* (1.2.1) *and* Π_{Rob} *defined in* (1.2.2), *respectively. Suppose* $\mathcal{U}$ *is not down-monotone, let* $\mathcal{U}^{\downarrow}$ *be defined as in* (A.1). *Let* $z_{\mathsf{AR}}^{\downarrow}$ *and* $z_{\mathsf{Rob}}^{\downarrow}$ *be the optimal values of* $\Pi_{\mathsf{AR}}^{\downarrow}$ *defined in* (A.2) *and* $\Pi_{\mathsf{Rob}}^{\downarrow}$ *defined in* (A.3), *respectively. Then,*

$$z_{\mathsf{AR}}^{\downarrow} = z_{\mathsf{AR}}, z_{\mathsf{Rob}}^{\downarrow} = z_{\mathsf{Rob}}.$$

Proof. Suppose $(\mathbf{x}^*, \mathbf{y}^*(\mathbf{B}), \mathbf{B} \in \mathcal{U}^{\downarrow})$ is an optimal solution of $\Pi_{\mathsf{AR}}^{\downarrow}$. Based on the discussion in Theorem 2.3.3, we can assume without loss of generality that $\mathbf{h} - \mathbf{A}\mathbf{x}^* > \mathbf{0}$. Then,

$$\begin{aligned} z_{\mathsf{AR}}^{\downarrow} &= \mathbf{c}^T\mathbf{x}^* + \min_{\mathbf{B}\in\mathcal{U}^{\downarrow}} \max_{\mathbf{y}\in\mathbb{R}_+^{n_2}} \left\{\mathbf{d}^T\mathbf{y} \mid \mathbf{B}\mathbf{y} \le \mathbf{h} - \mathbf{A}\mathbf{x}^*\right\} \\ &= \mathbf{c}^T\mathbf{x}^* + z_{\mathsf{AR}}^I(\mathcal{U}^{\downarrow}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &= \mathbf{c}^T\mathbf{x}^* + z_{\mathsf{AR}}^I(\mathcal{U}, \mathbf{h} - \mathbf{A}\mathbf{x}^*) \\ &\le z_{\mathsf{AR}}. \end{aligned}$$

The second equation holds from Lemma A.1.2, and the last inequality holds because $\mathbf{x} = \mathbf{x}^*$ is a feasible first-stage solution for Π_{AR}. Therefore, $z_{\mathsf{AR}}^{\downarrow} \le z_{\mathsf{AR}}$.

Conversely, suppose $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}(\mathbf{B}), \mathbf{B} \in \mathcal{U})$ is the optimal solution for Π_{AR}. Again, we can assume without loss of generality that $\mathbf{h}-\mathbf{A}\tilde{\mathbf{x}} > \mathbf{0}$. Using similar arguments, we have

$$\begin{aligned} z_{\mathsf{AR}} &= \mathbf{c}^T\tilde{\mathbf{x}} + \min_{\mathbf{B}\in\mathcal{U}} \max_{\mathbf{y}\in\mathbb{R}_+^{n_2}} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{B}\mathbf{y} \leq \mathbf{h}-\mathbf{A}\tilde{\mathbf{x}} \right\} \\ &= \mathbf{c}^T\tilde{\mathbf{x}} + z_{\mathsf{AR}}^I(\mathcal{U}, \mathbf{h}-\mathbf{A}\tilde{\mathbf{x}}) \\ &= \mathbf{c}^T\tilde{\mathbf{x}} + z_{\mathsf{AR}}^I(\mathcal{U}^{\downarrow}, \mathbf{h}-\mathbf{A}\tilde{\mathbf{x}}) \\ &\leq z_{\mathsf{AR}}^{\downarrow}. \end{aligned}$$

The last inequality holds because $\mathbf{x} = \tilde{\mathbf{x}}$ is a feasible first-stage solution for $z_{\mathsf{AR}}^{\downarrow}$. Therefore, in both cases, we have $z_{\mathsf{AR}} \leq z_{\mathsf{AR}}^{\downarrow}$. Together with previous result, we have $z_{\mathsf{AR}}^{\downarrow} = z_{\mathsf{AR}}$. In the same way, we can show that $z_{\mathsf{Rob}}^{\downarrow} = z_{\mathsf{Rob}}$, we omit it here. □

Lemma A.1.4. *Given a down-monotone set* $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$, *let* $T(\mathcal{U}, \mathbf{h})$ *be defined as in* (2.3.6), *then* $T(\mathcal{U}, \mathbf{h})$ *is down-monotone for all* $\mathbf{h} > \mathbf{0}$.

Proof. Consider an arbitrary $\mathbf{h} > \mathbf{0}$ and $\mathbf{y} \in T(\mathcal{U}, \mathbf{h}) \subseteq \mathbb{R}_+^n$ such that

$$\mathbf{y} = \mathbf{B}^T\lambda, \mathbf{h}^T\lambda = 1, \lambda \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}.$$

Then, for any $\mathbf{z} \in \mathbb{R}_+^n$ such that $\mathbf{z} \leq \mathbf{y}$, set

$$\hat{B}_{ij} = \frac{z_j}{y_j} B_{ij}, i = 1, \ldots, m,\; j = 1, \ldots, n.$$

Clearly, $\hat{\mathbf{B}} \leq \mathbf{B}$ since $\mathbf{z} \leq \mathbf{y}$. Therefore, $\hat{\mathbf{B}} \in \mathcal{U}$ from the assumption that $\mathcal{U}$ is down-monotone.

Then,

$$\mathbf{z} = \hat{\mathbf{B}}^T \lambda, \mathbf{h}^T \lambda = 1, \lambda \geq \mathbf{0}, \hat{\mathbf{B}} \in \mathcal{U},$$

which implies $\mathbf{z} \in T(\mathcal{U}, \mathbf{h})$. □

Appendix B

Appendix of Chapter 2

B.1 Proof of Theorem 2.2.2.

In this section, we show that the general two-stage adjustable robust problem $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ (2.2.1) is $\Omega(2^{\log^{1-\varepsilon} m})$-hard to approximate for any constant $0 < \varepsilon < 1$. We prove this by an approximation preserving reduction from the Label-Cover-Problem. The reduction is similar in spirit to the reduction from the set cover problem to the two-stage adjustable robust problem.

Label-Cover-Problem: We are given a finite set V ($|V| = m$), a family of subset $\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ of V and graph $G = (V, E)$. Let H be a supergraph with vertices $\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ and edges F where $(\mathcal{V}_i, \mathcal{V}_j) \in F$ if there exists $(k,l) \in E$ such that $k \in \mathcal{V}_i, l \in \mathcal{V}_j$. The goal is to find the smallest cardinality set $C \subseteq V$ such that F is covered, i.e., for each $(\mathcal{V}_i, \mathcal{V}_j) \in F$, there exists $k \in \mathcal{V}_i \cap C, l \in \mathcal{V}_j \cap C$ such that $(k,l) \in E$.

The label cover problem is $\Omega(2^{\log^{1-\varepsilon} m})$-hard to approximate for any constant $0 < \varepsilon < 1$, i.e., there is no polynomial time approximation algorithm that give an $O(2^{\log^{1-\varepsilon} m})$-approximation for any constant $0 < \varepsilon < 1$ unless $\mathbf{NP} \subseteq \mathbf{DTIME}(m^{\mathrm{polylog}(m)})$ [1].

Proof of Theorem 2.2.2 Consider an instance I of Label-Cover-Problem with ground elements V ($|V| = m$), graph $G = (V, E)$, a family of subset of V: $(\mathcal{V}_1, \dots, \mathcal{V}_K)$ and a supergraph $H = (\{\mathcal{V}_1, \dots, \mathcal{V}_K\}, F)$ where $|F| = n$. We construct the following instance I' of the general adjustable robust problem $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ (2.2.1):

$$\mathbf{A} = \mathbf{0},\ \mathbf{c} = \mathbf{0},\ \mathbf{d} = \begin{pmatrix} \mathbf{e} \\ \\ -\mathbf{e} \end{pmatrix} \in \mathbb{R}^{n+m},\ \mathbf{h} = \mathbf{e} \in \mathbb{R}^m,\ \mathcal{U} = \{[\mathbf{B}\ \ -\mathbf{I}_m] \mid \mathbf{B} \in \mathcal{U}_F\}$$

where $d_1 = d_2 = \ldots = d_n = 1$, $\mathbf{I}_m$ is the m-dimensional identity matrix and each column set of $\mathcal{U}_F \subseteq \mathbb{R}_+^{m \times n}$ corresponds to an edge $(\mathcal{V}_i, \mathcal{V}_j) \in F$ with

$$\mathcal{U}_{(\mathcal{V}_i, \mathcal{V}_j)} = \mathsf{conv}\left(\{\mathbf{0}\} \bigcup \left\{ \frac{1}{2}(\mathbf{e}_k + \mathbf{e}_l) \;\middle|\; (k, l) \in E, k \in \mathcal{V}_i, l \in \mathcal{V}_j \right\}\right) \subseteq \mathbb{R}_+^m.$$

Therefore, $\mathcal{U}$ is column-wise with column sets $\mathcal{U}_{(\mathcal{V}_i, \mathcal{V}_j)}, \forall (\mathcal{V}_i, \mathcal{V}_j) \in F$ and $\mathcal{U}_j, j \in [m]$ where $\mathcal{U}_j = \{-\mathbf{e}_j\}$, i.e., there is no uncertainty in $\mathcal{U}_j$. The instance I' of $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ can be formulated as

$$\begin{aligned} z_{\mathsf{AR}}^{\mathsf{Gen}} &= \min_{\mathbf{B} \in \mathcal{U}_F} \max_{\mathbf{y} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0}} \{\mathbf{e}^T \mathbf{y} - \mathbf{e}^T \mathbf{z} \mid \mathbf{B}\mathbf{y} - \mathbf{z} \leq \mathbf{e}, \mathbf{y} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0}\} \\ &= \min_{\mathbf{b}_{(\mathcal{V}_i, \mathcal{V}_j)} \in \mathcal{U}_{(\mathcal{V}_i, \mathcal{V}_j)}} \max_{\mathbf{y} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0}} \left\{ \mathbf{e}^T \mathbf{y} - \mathbf{e}^T \mathbf{z} \;\middle|\; \sum_{(\mathcal{V}_i, \mathcal{V}_j) \in F} y_{(\mathcal{V}_i, \mathcal{V}_j)} \mathbf{b}_{(\mathcal{V}_i, \mathcal{V}_j)} - \mathbf{z} \leq \mathbf{e}, \mathbf{y} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0} \right\}. \end{aligned}$$

Suppose $(\hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{b}}_{(\mathcal{V}_i, \mathcal{V}_j)}, (\mathcal{V}_i, \mathcal{V}_j) \in F)$ is a feasible solution for instance I'. Then, we can compute a label cover of instance I with cardinality at most $\mathbf{e}^T \hat{\mathbf{y}} - \mathbf{e}^T \hat{\mathbf{z}}$. From strong duality, there exists an optimal solution $\hat{\mu}$ for

$$\min\{\mathbf{e}^T\mu \mid \hat{\mathbf{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\mu \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F, \mu \in [0,1]^m\}$$

and $\mathbf{e}^T\hat{\mu} = \mathbf{e}^T\hat{\mathbf{y}} - \mathbf{e}^T\hat{\mathbf{z}}$. For each $(\mathcal{V}_i,\mathcal{V}_j) \in F$, consider a basic optimal solution $(\tilde{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}, (\mathcal{V}_i,\mathcal{V}_j) \in F)$ where

$$\tilde{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} \in \arg\max\{\mathbf{b}^T\hat{\mu} \mid \mathbf{b} \in \mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)}\}.$$

Therefore, $\tilde{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}$ is a vertex of $\mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)}$ for each $(\mathcal{V}_i,\mathcal{V}_j) \in F$, which implies that $\tilde{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} = \frac{1}{2}(\mathbf{e}_{k_i} + \mathbf{e}_{l_j})$ for some $(k_i,l_j) \in E$ and $k_i \in \mathcal{V}_i, l_j \in \mathcal{V}_j$. Also, $\tilde{\mathbf{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\hat{\mu} \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F$. Now, let $\tilde{\mu}$ the optimal solution of the following LP:

$$\min\{\mathbf{e}^T\mu \mid \tilde{\mathbf{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\mu \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F, \mathbf{0} \leq \mu \leq \mathbf{e}\}.$$

Clearly, $\mathbf{e}^T\tilde{\mu} \leq \mathbf{e}^T\hat{\mu}$. Also, since $\tilde{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} = \frac{1}{2}(\mathbf{e}_{k_i} + \mathbf{e}_{l_j})$ and $\tilde{\mathbf{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\tilde{\mu} \geq 1$, $\tilde{\mu}_{k_i} = \tilde{\mu}_{l_j} = 1$. Therefore, $\tilde{\mu} \in \{0,1\}^m$. Let

$$C = \{j \mid \tilde{\mu}_j = 1\}.$$

Clearly, C is a valid label cover for F and $|C| = \mathbf{e}^T\tilde{\mu} \leq \mathbf{e}^T\hat{\mu} = \mathbf{e}^T\hat{\mathbf{y}} - \mathbf{e}^T\hat{\mathbf{z}}$.

Conversely, given a label cover C of instance I, for any $j \in [m]$, let $\bar{\mu}_j = 1$ if $j \in C$ and zero otherwise. This implies that $\mathbf{e}^T\bar{\mu} = |C|$. For any $(\mathcal{V}_i,\mathcal{V}_j) \in F$, let $\bar{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} = \frac{1}{2}(\mathbf{e}_{k_i} + \mathbf{e}_{l_j})$ where $k_i \in \mathcal{V}_i \cap C, l_j \in \mathcal{V}_j \cap C$ such that $(k_i,l_j) \in E$. Then, let μ' be an optimal solution for the following

LP

$$\min\{\mathbf{e}^T\mu \mid \overline{\mathbf{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\mu \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F, \mathbf{0} \leq \mu \leq \mathbf{e}\}.$$

Then, $\mathbf{e}^T\mu' \leq \mathbf{e}^T\bar{\mu}$ as $\bar{\mu}$ is feasible for the above LP. From strong duality, there exists $\bar{\mathbf{y}} \in \mathbb{R}^n_+$ and $\bar{\mathbf{z}} \in \mathbb{R}^m_+$ such that $(\bar{\mathbf{y}}, \bar{\mathbf{z}}, \overline{\mathbf{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}, (\mathcal{V}_i,\mathcal{V}_j) \in F)$ is a feasible solution for instance I' of $\Pi^{\mathsf{Gen}}_{\mathsf{AR}}$ with cost $\mathbf{e}^T\bar{\mathbf{y}} - \mathbf{e}^T\bar{\mathbf{z}} = \mathbf{e}^T\mu' \leq \mathbf{e}^T\bar{\mu} = |C|$. □

B.2 Proofs of Lemmas 2.3.1 and 2.3.2

Proof of Lemma 2.3.1 Consider any $\mathbf{v}_1, \mathbf{v}_2 \in T(\mathcal{U}, \mathbf{h})$. Therefore, for $j = 1,2$,

$$\mathbf{v}_j = \mathbf{B}_j^T\lambda^j, \mathbf{h}^T\lambda^j = 1, \lambda^j \geq \mathbf{0}, \mathbf{B}_j \in \mathcal{U}.$$

For any arbitrary $\alpha \in [0,1]$, let $\mu_i = \alpha\lambda_i^1 + (1-\alpha)\lambda_i^2$ and $\mathbf{b}_i^j = \mathbf{B}_j^T\mathbf{e}_i$ for $i = 1,\ldots,m$. Then,

$$\begin{aligned}
\alpha\mathbf{v}_1 + (1-\alpha)\mathbf{v}_2 &= \sum_{i=1}^m \left(\alpha\lambda_i^1\mathbf{b}_i^1 + (1-\alpha)\lambda_i^2\mathbf{b}_i^2\right)\\
&= \sum_{i=1}^m \mu_i\left(\frac{\alpha\lambda_i^1}{\mu_i}\mathbf{b}_i^1 + \frac{(1-\alpha)\lambda_i^2}{\mu_i}\mathbf{b}_i^2\right)\\
&= \sum_{i=1}^m \mu_i \cdot \hat{\mathbf{b}}_i\\
&= \hat{\mathbf{B}}^T\mu,
\end{aligned}$$

where $\hat{\mathbf{b}}_i \in \mathcal{U}_i$ since $\hat{\mathbf{b}}_i$ is a convex combination of $\mathbf{b}_i^1$ and $\mathbf{b}_i^2$ for all $i = 1,\ldots,m$ and $\mathcal{U}_i$ is convex.

Also, note that $\hat{\mathbf{B}} \in \mathcal{U}$ (since $\mathcal{U}$ is constraint-wise) and $\mathbf{h}^T \mu = \alpha \mathbf{h}^T \lambda^1 + (1-\alpha)\mathbf{h}^T \lambda^2 = 1$, we have

$$\alpha \mathbf{v}_1 + (1-\alpha)\mathbf{v}_2 \in T(\mathcal{U}, \mathbf{h}).$$

Therefore, $T(\mathcal{U}, \mathbf{h})$ is convex. □

Proof of Lemma 2.3.2 Note that in (2.3.6), $\mathbf{h}^T \mu = 1$, which implies $\mu_j \leq \frac{1}{h_j}$ for $j = 1, \ldots, m$. We assume without loss of generality that $h_1 \leq h_j$ for $j = 2, \ldots, m$. Note that $\mathcal{U}$ is down-monotone, so is $\mathcal{U}_j$, $j = 1, \ldots, m$. Therefore, for $j = 2, \ldots, m$, we have

$$\frac{1}{h_j}\mathcal{U}_j \subseteq \frac{1}{h_1}\mathcal{U}_1 \subseteq T(\mathcal{U}, \mathbf{h})$$

where the second set inequality holds because we can take $\mu = \frac{\mathbf{e}_1}{h_1}$ in (2.3.6). Note that $\mathcal{U}_1$ is convex, so is $\frac{1}{h_1}\mathcal{U}_1$. Now, consider an arbitrary $\mathbf{v} \in T(\mathcal{U}, \mathbf{h})$ such that

$$\mathbf{v} = \mathbf{B}^T \lambda, \mathbf{h}^T \lambda = 1, \lambda \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}.$$

Let $\mathbf{b}_j = \mathbf{B}^T \mathbf{e}_j$, we have

$$\begin{aligned}
\mathbf{v} &= \sum_{j=1}^{m} \lambda_j \mathbf{b}_j \\
&= \sum_{j=1}^{m} \lambda_j h_j \cdot \frac{1}{h_j}\mathbf{b}_j \\
&= \frac{1}{h_1}\hat{\mathbf{b}},
\end{aligned}$$

where $\hat{\mathbf{b}} \in \mathcal{U}_1$. The last equation holds because $\mathbf{h}^T\lambda = 1$ and $\frac{1}{h_j}\mathcal{U}_j \subseteq \frac{1}{h_1}\mathcal{U}_1$. Therefore,

$$T(\mathcal{U},\mathbf{h}) = \frac{1}{h_1}\mathcal{U}_1,$$

which is convex. □

B.3 Proof of Lemma 2.3.5

For each $j \in [m]$, let

$$\mathcal{U}_j = \left\{ \frac{1}{h_j} \cdot \mathbf{B}^T \mathbf{e}_j \;\middle|\; \mathbf{B} \in \mathcal{U} \right\}.$$

Then,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) &= \max_{\mathbf{y}} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{B}\mathbf{y} \le \mathbf{h},\ \forall \mathbf{B} \in \mathcal{U}, \mathbf{y} \in \mathbb{R}^n_+ \right\} \\ &= \max_{\mathbf{y}} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{b}_j^T\mathbf{y} \le 1,\ \forall \mathbf{b}_j \in \mathcal{U}_j, j \in [m], \mathbf{y} \in \mathbb{R}^n_+ \right\} \end{aligned}$$

Consider a feasible solution $\mathbf{y}$, we have

$$\begin{aligned} & \mathbf{b}_j^T\mathbf{y} \le 1,\ \forall\, \mathbf{b}_j \in \mathcal{U}_j,\ j \in [m] \\ \Leftrightarrow\quad & \mathbf{b}^T\mathbf{y} \le 1,\ \forall\, \mathbf{b} \in \textstyle\bigcup_{j=1}^m \mathcal{U}_j \\ \Leftrightarrow\quad & \mathbf{b}^T\mathbf{y} \le 1,\ \forall\, \mathbf{b} \in \mathsf{conv}\left(\textstyle\bigcup_{j=1}^m \mathcal{U}_j\right) \end{aligned}$$

where the last inference follows from the fact that if $\mathbf{b}_1^T\mathbf{y} \le 1$ and $\mathbf{b}_2^T\mathbf{y} \le 1$, then

$$(\alpha\mathbf{b}_1 + (1-\alpha)\mathbf{b}_2)^T\mathbf{y} = \alpha\mathbf{b}_1^T\mathbf{y} + (1-\alpha)\mathbf{b}_2^T\mathbf{y} \le 1,$$

for all $0 \leq \alpha \leq 1$. In Theorem 2.4.3, we show that

$$\mathsf{conv}(T(\mathcal{U},\mathbf{h})) = \mathsf{conv}\left(\bigcup_{j=1}^{m} \mathcal{U}_j\right).$$

Therefore,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) &= \max_{\mathbf{y}} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{b}^T\mathbf{y} \leq 1,\ \forall \mathbf{b} \in \mathsf{conv}(T(\mathcal{U},\mathbf{h}), \mathbf{y} \in \mathbb{R}^n_+ \right\} \\ &= \max_{\mathbf{y}} \left\{ \mathbf{d}^T\mathbf{y} \;\middle|\; \mathbf{y} \in (\mathsf{conv}(T(\mathcal{U},\mathbf{h}))^\circ \bigcap \mathbb{R}^n_+ \right\} \end{aligned}$$

where $\mathcal{S}^\circ$ is the polar set of $\mathcal{S}$. Note that the last maximization problem can be viewed as the support function of the set

$$\mathcal{C} = (\mathsf{conv}(T(\mathcal{U},\mathbf{h}))^\circ \bigcap \mathbb{R}^n_+.$$

Therefore, we can reformulate it as the Minkowski functional over the polar $\mathcal{C}^\circ$ as follows (see Proposition 3.2.5 in Chapter 5 of [26]).

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) &= \min_{\lambda} \left\{ \lambda \;\middle|\; \mathbf{d} \in \lambda \left((\mathsf{conv}(T(\mathcal{U},\mathbf{h}))^\circ \bigcap \mathbb{R}^n_+ \right)^\circ \right\} \\ &= \min_{\lambda} \left\{ \lambda \;\middle|\; \mathbf{d} \in \lambda \left(\mathsf{conv}(T(\mathcal{U},\mathbf{h}) \bigcup \mathbb{R}^n_- \right) \right\} \end{aligned}$$

where the second equation follows as

$$\left(\mathcal{S}_1 \bigcap \mathcal{S}_2\right)^\circ = \mathcal{S}_1^\circ \bigcup \mathcal{S}_2^\circ, \text{ and } (\mathcal{S}^\circ)^\circ = \mathcal{S},$$

and $\left(\mathbb{R}^n_+\right)^\circ = \mathbb{R}^n_-$. Since $\mathbf{d} \in \mathbb{R}^n_+$, we have

$$
\begin{aligned}
z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) &= \min_{\lambda}\{\lambda \mid \mathbf{d} \in \lambda \mathsf{conv}(T(\mathcal{U},\mathbf{h})\} \\
&= \min_{\lambda}\{\lambda \mid \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}(T(\mathcal{U},\mathbf{h})\}
\end{aligned}
$$

which completes the proof. □

B.4 Tight Example for Measure of Non-convexity Bound

Theorem B.4.1. *Consider the following uncertainty set, $\mathcal{U}^\theta$,*

$$
\mathcal{U}^\theta = \left\{ \mathbf{B} \in [0,1]^{n\times n} \;\middle|\; B_{ij} = 0,\ \forall i \neq j,\ \sum_{j=1}^{n} B^\theta_{jj} \leq 1 \right\}.
$$

with $\theta > 1$. Then,

1. *$T(\mathcal{U}^\theta,\mathbf{h})$ can be written as:*

$$
T(\mathcal{U}^\theta,\mathbf{h}) = \left\{ \mathbf{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} \left(\frac{b_j}{h_j}\right)^{\frac{\theta}{\theta+1}} \leq 1 \right\} \tag{B.1}
$$

2. *The convex hull of $T(\mathcal{U}^\theta,\mathbf{h})$ can be written as:*

$$
\mathsf{conv}(T(\mathcal{U}^\theta,\mathbf{h})) = \left\{ \mathbf{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} \frac{b_j}{h_j} \leq 1 \right\}. \tag{B.2}
$$

3. *$T(\mathcal{U}^\theta,\mathbf{h})$ is non-convex for all $\mathbf{h} > \mathbf{0}$.*

4. $\kappa(T(\mathcal{U}^\theta,\mathbf{h})) = n^{\frac{1}{\theta}}$ *for all* $\mathbf{h} > \mathbf{0}$*.*

Proof. 1. For given $\mathbf{h} > \mathbf{0}$ and $\mathbf{b} \in T(\mathcal{U}^\theta,\mathbf{h})$, we have

$$\mathbf{b} = \mathbf{B}^T\mu, \mathbf{h}^T\mu = 1, \mu \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}^\theta.$$

Let $\lambda_i = h_i\mu_i$ for $i = 1,\ldots,n$. Therefore, $\mathbf{e}^T\lambda = 1$ and

$$\mathbf{b} = \mathbf{B}^T(\mathsf{diag}(\mathbf{h}))^{-1}\lambda = (\mathsf{diag}(\mathbf{h}))^{-1}\mathbf{B}^T\lambda,$$

where $\mathsf{diag}(\mathbf{h}) \in \mathbb{R}^{n\times n}$ denotes the matrix with diagonal entries being $h_i, i \in [n]$ and off-diagonal entries being zero. The second equality above follows as $\mathbf{B}$ is diagonal. Therefore, $(\mathsf{diag}(\mathbf{h}))\mathbf{b} \in T(\mathcal{U}^\theta,\mathbf{e})$. Using a similar argument, we can show that $\mathbf{b} \in T(\mathcal{U}^\theta,\mathbf{e})$ implies that $(\mathsf{diag}(\mathbf{h}))^{-1}\mathbf{b} \in T(\mathcal{U}^\theta,\mathbf{h})$. Therefore, $T(\mathcal{U}^\theta,\mathbf{h}) = \mathsf{diag}(\mathbf{h}))^{-1}T(\mathcal{U}^\theta,\mathbf{e})$ and it is sufficient to show:

$$T(\mathcal{U}^\theta,\mathbf{e}) = \mathcal{A} := \left\{ \mathbf{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} b_j^{\frac{\theta}{\theta+1}} \leq 1 \right\}.$$

Consider any $\mathbf{b} \in \partial\mathcal{A}$, i.e., $\mathbf{b} \in \mathbb{R}^n_+$ such that

$$\sum_{j=1}^{n} b_j^{\frac{\theta}{\theta+1}} = 1.$$

Set

$$\lambda_j = b_j^{\frac{\theta}{\theta+1}}, x_j = b_j^{\frac{1}{\theta+1}}.$$

Then,

$$\lambda_j x_j = b_j, \mathbf{e}^T\lambda = 1, \sum_{j=1}^{n} {x_j}^{\theta} = 1,$$

which implies $\mathbf{b} \in T(\mathcal{U}^{\theta}, \mathbf{e})$. Since both $\mathcal{A}$ and $T(\mathcal{U}^{\theta}, \mathbf{e})$ are down-monotone, $\mathcal{A} \subseteq T(\mathcal{U}^{\theta}, \mathbf{e})$.

Conversely, consider the following problem:

$$\max_{\lambda, \mathbf{x} \geq \mathbf{0}} \left\{ \sum_{i=1}^{n} (\lambda_j x_j)^{\frac{\theta}{\theta+1}} \;\middle|\; \mathbf{e}^T\lambda = 1, \sum_{j=1}^{n} x_j^{\theta} \leq 1. \right\}$$

From Holder's Inequality, we have

$$\sum_{i=1}^{n} (\lambda_j x_j)^{\frac{\theta}{\theta+1}} \leq (\mathbf{e}^T\lambda)^{\frac{\theta}{\theta+1}} \cdot \left(\sum_{j=1}^{n} x_j^{\theta} \right)^{\frac{1}{\theta+1}} \leq 1.$$

Therefore, for any $\mathbf{b} \in T(\mathcal{U}^{\theta}, \mathbf{e})$, we have

$$\sum_{j=1}^{n} b_j^{\frac{\theta}{\theta+1}} \leq 1,$$

which implies $\mathbf{b} \in \mathcal{A}$. Therefore, $T(\mathcal{U}^{\theta}, \mathbf{e}) \subseteq \mathcal{A}$.

2. Similarly, it is sufficient to show

$$\mathsf{conv}(T(\mathcal{U}^{\theta}, \mathbf{e})) = \mathcal{B} := \left\{ \mathbf{b} \in \mathbb{R}_+^n \;\middle|\; \sum_{j=1}^{n} b_j \leq 1 \right\}.$$

From (B.1), we see that $\mathbf{e}_j \in T(\mathcal{U}^{\theta}, \mathbf{e})$. For any $\mathbf{b} \in \partial\mathcal{B}$, by taking $\lambda = \mathbf{b}$ as the convex multiplier, we have

$$\mathbf{b} = \sum_{j=1}^{n} b_j \mathbf{e}_j.$$

Therefore, $\partial\mathcal{B} \subseteq \mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{e}))$. Since both $\mathcal{B}$ and $\mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{e}))$ are down-monotone, we have $\mathcal{B} \subseteq \mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{e}))$.

Conversely, consider the following problem:

$$\max_{\mathbf{b}\geq\mathbf{0}} \left\{ \mathbf{e}^T\mathbf{b} \;\middle|\; \sum_{j=1}^{n} b_j^{\frac{\theta}{1+\theta}} \leq 1 \right\} = \max_{\mathbf{a}\geq\mathbf{0}} \left\{ \sum_{j=1}^{n} a_j^{\frac{1+\theta}{\theta}} \;\middle|\; \mathbf{e}^T\mathbf{a} \leq 1 \right\}$$

Note that

$$f(\mathbf{x}) = \sum_{j=1}^{n} x_j^{\frac{1+\theta}{\theta}}$$

is a convex function. Therefore,

$$\sum_{j=1}^{n} a_j^{\frac{1+\theta}{\theta}} \leq (\mathbf{e}^T\mathbf{a})^{\frac{1+\theta}{\theta}} \leq 1.$$

Therefore, for any $\mathbf{b} \in T(\mathcal{U}^\theta, \mathbf{e})$, we have $\mathbf{b} \in \mathcal{B}$. Since $\mathcal{B}$ is convex, $\mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{e})) \subseteq \mathcal{B}$.

3. From (B.1) and (B.2), it is easy to see that $\frac{1}{n}\mathbf{h} \in \mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{h}))$, but $\frac{1}{n}\mathbf{h} \notin T(\mathcal{U}^\theta, \mathbf{h})$. Therefore, $T(\mathcal{U}^\theta, \mathbf{h})$ is non-convex for all $\mathbf{h} > \mathbf{0}$.

4. Now, we compute $\kappa(\mathcal{U}^\theta, \mathbf{h})$. Recall that

$$\kappa(\mathcal{U}^\theta, \mathbf{h}) = \min\{\alpha \mid \mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{h})) \subseteq \alpha T(\mathcal{U}^\theta, \mathbf{h})\} = \min\{\alpha \mid \frac{1}{\alpha}\mathsf{conv}(T(\mathcal{U}^\theta, \mathbf{h})) \subseteq T(\mathcal{U}^\theta, \mathbf{h})\}.$$

From (B.2) and scaling, we can observe that it is equivalent to find the largest α such that the hyperplane

$$\left\{ \mathbf{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{j=1}^{n} \frac{b_j}{h_j} = \frac{1}{\alpha} \right\}$$

intersects with the positive boundary of $T(\mathcal{U}^\theta, \mathbf{h})$. Therefore, we formulate the following problem:

$$\begin{aligned}
(\kappa(\mathcal{U}^\theta, \mathbf{h}))^{-1} &= \min_{\mathbf{b} \geq \mathbf{0}} \left\{ \sum_{j=1}^{n} \frac{b_j}{h_j} \;\middle|\; \sum_{j=1}^{n} \left(\frac{b_j}{h_j}\right)^{\frac{\theta}{1+\theta}} = 1 \right\} \\
&= \min_{\mathbf{a} \geq \mathbf{0}} \left\{ \sum_{j=1}^{n} a_j^{\frac{1+\theta}{\theta}} \;\middle|\; \sum_{j=1}^{n} a_j = 1 \right\}
\end{aligned}$$

By solving KKT conditions for the convex problem above, the optimal solution is $\mathbf{a} = \frac{1}{n} \cdot \mathbf{e}$. Therefore, we have

$$\kappa(\mathcal{U}^\theta, \mathbf{h}) = (n \cdot n^{-\frac{1+\theta}{\theta}})^{-1} = n^{\frac{1}{\theta}}.$$

□

B.5 Proof of Lemma 2.5.3

We first introduce some notations. Let

$$\tilde{\mathcal{U}}^B = \left\{ [\mathbf{B}\;\; \mathbf{0}] \in \mathbb{R}_+^{m \times (n+1)} \;\middle|\; \mathbf{B} \in \mathcal{U}^B \right\} \text{ and } \tilde{\mathcal{U}}^d = \left\{ \begin{pmatrix} -\mathbf{d} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \mathbf{d} \in \mathcal{U}^d \right\}.$$

For each $j \in [m]$, let

$$\mathcal{U}_j = \left\{ \frac{1}{h_j} \cdot \mathbf{B}^T \mathbf{e}_j \;\middle|\; \mathbf{B} \in \mathcal{U} \right\} \text{ and } \tilde{\mathcal{U}}_j = \left\{ \begin{pmatrix} \mathbf{b} \\ \\ 0 \end{pmatrix} \in \mathbb{R}_+^{n+1} \;\middle|\; \mathbf{b} \in \mathcal{U}_j \right\}.$$

Lastly, let

$$\tilde{\mathbf{h}} = \begin{pmatrix} \mathbf{h} \\ \\ 0 \end{pmatrix}.$$

It is easy to see that

$$T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) = \left\{ \begin{pmatrix} \mathbf{b} \\ \\ 0 \end{pmatrix} \in \mathbb{R}_+^{n+1} \;\middle|\; \mathbf{b} \in T(\mathcal{U}, \mathbf{h}) \right\}.$$

Then,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}) &= \max_{\mathbf{y}, \mu} \left\{ \mu \;\middle|\; \mu \leq \mathbf{d}^T \mathbf{y}, \forall \mathbf{d} \in \mathcal{U}^d,\ \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall \mathbf{B} \in \mathcal{U}^B,\ \mathbf{y} \in \mathbb{R}_+^n \right\} \\ &= \max_{\mathbf{y}, \mu} \left\{ \mu \;\middle|\; -\mathbf{d}^T \mathbf{y} + \mu + 1 \leq 1, \forall \mathbf{d} \in \mathcal{U}^d,\ \mathbf{b}_j^T \mathbf{y} \leq 1,\ \forall \mathbf{b}_j \in \mathcal{U}_j, j \in [m], \mathbf{y} \in \mathbb{R}_+^n \right\}. \end{aligned}$$

Now, let

$$\mathbf{v} = \begin{pmatrix} \mathbf{y} \\ \\ \mu + 1 \end{pmatrix} \in \mathbb{R}_+^{n+1},$$

we have

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}) = \max_{\mathbf{v}} \left\{ \mathbf{e}_{n+1}^T \mathbf{v} - 1 \;\middle|\; \mathbf{d}^T \mathbf{v} \leq 1, \forall \mathbf{d} \in \tilde{\mathcal{U}}^d,\ \mathbf{b}^T \mathbf{v} \leq 1, \mathbf{b} \in T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}),\ \mathbf{v} \in \mathbb{R}_+^{n+1} \right\}$$

where $\mathbf{e}_{n+1} \in \mathbb{R}_+^{n+1}$ is the unit vector for the $(n+1)$-th coordinate. Following the revised proof of Lemma 4, we can write

$$\begin{aligned} z_{\mathsf{Rob}}^I(\mathcal{U},\mathbf{h}) &= \max_{\mathbf{v}} \left\{ \mathbf{e}_{n+1}^T \mathbf{v} \;\middle|\; \mathbf{v} \in \left(\mathsf{conv}\left(\mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right) \bigcup \tilde{\mathcal{U}}^d \right) \right)^{\circ} \bigcap \mathbb{R}_+^{n+1} \right\} - 1 \\ &= \min_{\gamma} \left\{ \gamma \;\middle|\; \mathbf{e}_{n+1} \in \gamma \left(\mathsf{conv}\left(\mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right) \bigcup \tilde{\mathcal{U}}^d \right) \bigcup \mathbb{R}_-^{n+1} \right) \right\} - 1. \end{aligned}$$

Note that $\mathbf{e}_{n+1} \in \mathbb{R}_+^{n+1}$. Therefore,

$$\begin{aligned} z_{\mathsf{Rob}}^I(\mathcal{U},\mathbf{h}) &= \min_{\gamma} \left\{ \gamma \;\middle|\; \mathbf{e}_{n+1} \in \gamma \mathsf{conv}\left(\mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right) \bigcup \tilde{\mathcal{U}}^d \right) \right\} - 1 \\ &= \min_{\gamma, \alpha \in [0,1]} \left\{ \gamma - 1 \;\middle|\; \gamma \mathbf{z} \geq \mathbf{e}_{n+1}, \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \right\} \\ &= \min_{\lambda, \alpha \in [0,1]} \left\{ \lambda \;\middle|\; (1+\lambda)\mathbf{z} \geq \mathbf{e}_{n+1}, \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \right\}. \end{aligned}$$

Note that

$$\begin{aligned} & (1+\lambda)\mathbf{z} \geq e_{n+1}, \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow \quad & (1+\lambda) z_{n+1} \geq 1,\ z_i \geq 0, \forall i \in [n],\ \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}) \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow \quad & (1+\lambda)\alpha \geq 1,\ (1-\alpha)\mathbf{b} - \alpha \mathbf{d} \geq \mathbf{0}, \mathbf{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \mathbf{h}) \right), \mathbf{d} \in \mathcal{U}^d \end{aligned}$$

where the last step of induction holds because $b_{n+1} = 0$ for all $\mathbf{b} \in \mathsf{conv}(T(\tilde{\mathcal{U}}^B, \tilde{\mathbf{h}}))$ and $d_{n+1} = 1$ for all $\mathbf{d} \in \tilde{\mathcal{U}}^d$. Therefore,

$$
\begin{aligned}
z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}) &= \min_{\lambda,\alpha} \left\{ \lambda \;\middle|\; (1+\lambda)\alpha \geq 1,\ (1-\alpha)\mathbf{b} - \alpha\mathbf{d} \geq \mathbf{0}, \mathbf{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \mathbf{h})\right), \mathbf{d} \in \mathcal{U}^d \right\} \\
&= \min_{\lambda,\alpha} \left\{ \lambda \;\middle|\; \lambda \geq \frac{1}{\alpha} - 1,\ \left(\frac{1}{\alpha} - 1\right)\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \mathbf{h})\right), \mathbf{d} \in \mathcal{U}^d \right\} \\
&= \min_{\lambda} \left\{ \lambda \;\middle|\; \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(T(\mathcal{U}^B, \mathbf{h})\right), \mathbf{d} \in \mathcal{U}^d \right\}.
\end{aligned}
$$

which completes the proof. □

B.6 Proofs of Lemmas 2.5.5 and 2.5.6

Proof of Lemma 2.5.5 We can write the dual of the inner problem of (2.5.12):

$$
\begin{aligned}
z^{(B,h,d)}_{\mathsf{AR}} &= \min_{(\mathbf{B},\mathbf{h},\mathbf{d}) \in \mathcal{U}^{B,h,d}, \alpha \in \mathbb{R}^m_+} \left\{ \mathbf{h}^T\alpha \;\middle|\; \mathbf{B}^T\alpha \geq \mathbf{d} \right\} \\
&= \min_{(\mathbf{B},\mathbf{h}) \in \mathcal{U}^{B,h}, \mathbf{d} \in \mathcal{U}^d, \alpha \in \mathbb{R}^m_+, \lambda} \left\{ \lambda\mathbf{h}^T\left(\frac{\alpha}{\lambda}\right) \;\middle|\; \lambda\mathbf{B}^T\left(\frac{\alpha}{\lambda}\right) \geq \mathbf{d}, \mathbf{h}^T\alpha = \lambda \right\} \\
&= \min_{(\mathbf{b},t) \in T(\mathcal{U}^{B,h}, \mathbf{e}), \mathbf{d} \in \mathcal{U}^d, \lambda} \left\{ \lambda t \;\middle|\; \lambda\mathbf{b} \geq \mathbf{d} \right\},
\end{aligned}
$$

where the second equality holds because $\mathcal{U}^{B,h,d} = \mathcal{U}^{B,h} \times \mathcal{U}^d$. □

Proof of Lemma 2.5.6 We first introduce some notations. Let

$$
\tilde{\mathcal{U}}^{B,h} = \left\{ [\mathsf{diag}^{-1}(\mathbf{h})\mathbf{B}\ \ \mathbf{0}] \in \mathbb{R}^{m\times(n+1)}_+ \;\middle|\; (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h} \right\} \text{ and } \tilde{\mathcal{U}}^d = \left\{ \begin{pmatrix} -\mathbf{d} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \mathbf{d} \in \mathcal{U}^d \right\}.
$$

For each $j \in [m]$, let

$$\mathcal{U}_j = \left\{ \begin{pmatrix} \mathbf{B}^T \mathbf{e}_j \\ \mathbf{h}^T \mathbf{e}_j \end{pmatrix} \middle| (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h} \right\} \subseteq \mathbb{R}^{n+1}, \text{ and } \tilde{\mathcal{U}}_j = \left\{ \mathbf{B}^T \mathbf{e}_j \mid \mathbf{B} \in \tilde{\mathcal{U}}^{B,h} \right\} \subseteq \mathbb{R}^{n+1}.$$

Note that for each $\tilde{\mathcal{U}}_j$, $\tilde{\mathcal{U}}_j$ normalizes any vector $\mathbf{b} \in \mathcal{U}_j$ so that the last component is one, then replace it with zero. This is very similar to the perspective function (See page 39 in [18]), which indicates that $\tilde{\mathcal{U}}_j$ is convex provided that $\mathcal{U}_j$ is convex. Then,

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}) = \max_{\mathbf{y}, z} \left\{ z \;\middle|\; z \leq \mathbf{d}^T \mathbf{y}, \forall \mathbf{d} \in \mathcal{U}^d,\ \mathbf{B}\mathbf{y} \leq \mathbf{h}, \forall (\mathbf{B}, \mathbf{h}) \in \mathcal{U}^{B,h},\ \mathbf{y} \in \mathbb{R}^n_+ \right\}.$$

Similar to the previous proof, by setting

$$\mathbf{v} = \begin{pmatrix} \mathbf{y} \\ z+1 \end{pmatrix} \in \mathbb{R}^{n+1}_+,$$

we have

$$z^I_{\mathsf{Rob}}(\mathcal{U}, \mathbf{h}) = \max_{\mathbf{v}} \left\{ \mathbf{e}^T_{n+1} \mathbf{v} - 1 \;\middle|\; \mathbf{d}^T \mathbf{v} \leq 1, \forall \mathbf{d} \in \tilde{\mathcal{U}}^d,\ \mathbf{b}^T_j \mathbf{v} \leq 1, \mathbf{b}_j \in \tilde{\mathcal{U}}_j, j \in [m],\ \mathbf{v} \in \mathbb{R}^{n+1}_+ \right\}.$$

where $\mathbf{e}_{n+1} \in \mathbb{R}_+^{n+1}$ is the unit vector for the $(n+1)$-th coordinate. Following the revised proof of Lemma 4, we can write

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) &= \max_{\mathbf{v}} \left\{ \mathbf{e}_{n+1}^T \mathbf{v} \;\middle|\; \mathbf{v} \in \left(\mathsf{conv}\left(\mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right) \bigcup \tilde{\mathcal{U}}^d \right) \right)^\circ \bigcap \mathbb{R}_+^{n+1} \right\} - 1 \\ &= \min_{\gamma} \left\{ \gamma \;\middle|\; \mathbf{e}_{n+1} \in \gamma \Big(\mathsf{conv}\left(\mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right) \bigcup \tilde{\mathcal{U}}^d \right) \bigcup \mathbb{R}_-^{n+1} \Big) \right\} - 1. \end{aligned}$$

Note that $\mathbf{e}_{n+1} \in \mathbb{R}_+^{n+1}$. Therefore,

$$\begin{aligned} z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) &= \min_{\gamma} \left\{ \gamma \;\middle|\; \mathbf{e}_{n+1} \in \gamma \mathsf{conv}\left(\mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right) \bigcup \tilde{\mathcal{U}}^d \right) \right\} - 1 \\ &= \min_{\gamma, \alpha \in [0,1]} \left\{ \gamma \;\middle|\; \gamma \mathbf{z} \geq \mathbf{e}_{n+1}, \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \right\} - 1. \end{aligned}$$

Note that

$$\begin{aligned} & \gamma \mathbf{z} \geq e_{n+1}, \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow \quad & \gamma z_{n+1} \geq 1,\; z_i \geq 0, \forall i \in [n],\; \mathbf{z} = (1-\alpha)\mathbf{b} + \alpha \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right), \mathbf{d} \in \tilde{\mathcal{U}}^d \\ \Leftrightarrow \quad & \gamma \alpha \geq 1,\; (1-\alpha)\mathbf{b} - \alpha \mathbf{d} \geq \mathbf{0}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right), \mathbf{d} \in \mathcal{U}^d \end{aligned}$$

where the last statement holds because $b_{n+1} = 0$ for all $\mathbf{b} \in \mathsf{conv}(\cup_{j=1}^m \tilde{\mathcal{U}}_j)$ and $d_{n+1} = 1$ for all $\mathbf{d} \in \tilde{\mathcal{U}}^d$. Therefore,

$$z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) = \min_{\gamma,\alpha} \left\{ \gamma - 1 \;\middle|\; \gamma \geq \frac{1}{\alpha},\; \frac{1-\alpha}{\alpha}\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j \right), \mathbf{d} \in \mathcal{U}^d \right\}$$

Substitute by $\lambda = 1/\alpha - 1 \geq 0$, we have

$$z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) = \min_{\gamma,\lambda}\left\{\gamma - 1 \;\middle|\; \gamma - 1 \geq \lambda,\ \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \mathbf{d} \in \mathcal{U}^d\right\}$$
$$= \min_{\lambda}\left\{\lambda \;\middle|\; \lambda\mathbf{b} \geq \mathbf{d}, \mathbf{b} \in \mathsf{conv}\left(\cup_{j=1}^m \tilde{\mathcal{U}}_j\right), \mathbf{d} \in \mathcal{U}^d\right\}$$
$$= \min_{\lambda}\left\{\lambda \;\middle|\; \lambda\sum_{j=1}^m \mu_j \frac{\mathbf{b}_j}{h_j} \geq \mathbf{d}, (\mathbf{b}_j, h_j) \in \mathcal{U}_j, \mathbf{e}^T\mu = 1, \mu \geq \mathbf{0}, \mathbf{d} \in \mathcal{U}^d\right\}$$

For each $j \in [m]$, let

$$\theta_j = \frac{\mu_j/h_j}{\sum_{i=1}^m \mu_i/h_i}.$$

Note that

$$\mu_j = \frac{\theta_j h_j}{\sum_{j=1}^m \theta_j h_j}.$$

Then,

$$z^I_{\mathsf{Rob}}(\mathcal{U},\mathbf{h}) = \min_{\lambda}\left\{\lambda \;\middle|\; \frac{\lambda}{\sum_{j=1}^m \theta_j h_j}\cdot\sum_{j=1}^m \theta_j \mathbf{b}_j \geq \mathbf{d}, (\mathbf{b}_j, h_j) \in \mathcal{U}_j, \mathbf{e}^T\theta = 1, \theta \geq \mathbf{0}, \mathbf{d} \in \mathcal{U}^d\right\}$$
$$= \min_{\lambda}\left\{\lambda \;\middle|\; \frac{\lambda}{t}\cdot\mathbf{b} \geq \mathbf{d}, (\mathbf{b}, t) \in \mathsf{conv}\left(T(\mathcal{U}^{B,h}, \mathbf{e})\right), \mathbf{d} \in \mathcal{U}^d\right\}$$
$$= \min_{\lambda}\left\{\lambda t \;\middle|\; \lambda\mathbf{b} \geq \mathbf{d}, (\mathbf{b}, t) \in \mathsf{conv}\left(T(\mathcal{U}^{B,h}, \mathbf{e})\right), \mathbf{d} \in \mathcal{U}^d\right\}.$$

which completes the proof. □

Appendix C

Appendix of Chapter 3

C.1 Approximate Separation to Optimization.

For any $\mathbf{x} \in \mathbb{R}^n_+$, let

$$Q^*(\mathbf{x}) = \min_{\mathbf{B} \in \mathcal{U}} \max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}\}.$$

We show that if we can approximate the separation problem, we can also approximate Π_{AR}. Let $\mathcal{A}$ be a γ-approximate algorithm for the separation problem (3.2.1), i.e., $\mathcal{A}$ computes a γ-approximation algorithm for the min-max problem in (3.2.1). For any $\mathbf{x} \in \mathbb{R}^n_+$, let $\mathbf{B}^{\mathcal{A}}(\mathbf{x})$ denote the matrix returned by $\mathcal{A}$ and let

$$Q^{\mathcal{A}}(\mathbf{x}) = \max_{\mathbf{y} \geq \mathbf{0}} \{\mathbf{d}^T \mathbf{y} \mid \mathbf{B}^{\mathcal{A}}(\mathbf{x})\mathbf{y} \leq \mathbf{h} - \mathbf{A}\mathbf{x}\}.$$

Therefore, the approximate separation based on Algorihm $\mathcal{A}$ is as follows: for any $(\mathbf{x}, z)$, return feasible if $Q^{\mathcal{A}}(\mathbf{x}) \geq z$. Otherwise give a violating hyperplane corresponding to $\mathbf{B}^{\mathcal{A}}(\mathbf{x})$. Now, we prove the following theorem.

Theorem C.1.1. *Suppose we have an Algorithm $\mathcal{A}$ that is a γ-approximation for the separation problem* (3.2.1). *Then we can compute a γ-approximation for the two-stage adjustable robust problem* Π_{AR} (1.2.1).

Proof. Since $\mathcal{A}$ is a γ-approximation to the min-max problem in (3.2.1), for any $\mathbf{x} \in \mathbb{R}^n_+$,

$$Q^*(\mathbf{x}) \leq Q^{\mathcal{A}}(\mathbf{x}) \leq \gamma \cdot Q^*(\mathbf{x}).$$

Let $(\mathbf{x}^*, z^*)$ be an optimal solution for Π_{AR} and let

$$\mathsf{OPT} = \mathbf{c}^T \mathbf{x}^* + z^*.$$

Consider the optimization algorithm based on the approximate separation algorithm $\mathcal{A}$ and suppose it returns the solution $(\hat{\mathbf{x}}, \hat{z})$. Note that $(\mathbf{x}^*, z^*)$ is feasible according to the approximate separation algorithm $\mathcal{A}$ as $Q^{\mathcal{A}}(\mathbf{x}^*) \geq Q^*(\mathbf{x}^*) = z^*$. Therefore,

$$\mathbf{c}^T \hat{\mathbf{x}} + \hat{z} \geq \mathbf{c}^T \mathbf{x}^* + z^*. \tag{C.1}$$

Note that $\hat{z}$ is an approximation for the worst case second-stage objective value when the first stage solution is $\hat{\mathbf{x}}$. The true objective value for the first stage solution $\hat{\mathbf{x}}$ is given by

$$\begin{aligned} \mathbf{c}^T\hat{\mathbf{x}} + Q^*(\hat{\mathbf{x}}) &\geq \mathbf{c}^T\hat{\mathbf{x}} + \frac{1}{\gamma}Q^{\mathcal{A}}(\hat{\mathbf{x}}) \\ &\geq \mathbf{c}^T\hat{\mathbf{x}} + \frac{1}{\gamma}\hat{z} \\ &\geq \frac{1}{\gamma}(\mathbf{c}^T\hat{\mathbf{x}} + \hat{z}) \\ &\geq \frac{1}{\gamma}\mathsf{OPT}, \end{aligned} \tag{C.2}$$

where the first inequality follows as $\mathcal{A}$ is a γ-approximation and $Q^{\mathcal{A}}(\hat{\mathbf{x}}) \leq \gamma \cdot Q^*(\hat{\mathbf{x}})$. Inequality (C.2) follows as $(\hat{\mathbf{x}}, \hat{z})$ is feasible according to $\mathcal{A}$ and therefore, $\hat{z} \leq Q^{\mathcal{A}}(\hat{\mathbf{x}})$ and the last inequality follows from (C.1). Therefore, the optimization problem based on algorithm $\mathcal{A}$ computes a γ-approximation for Π_{AR}. □

C.2 Proof of Theorem 3.2.1

Let $\mathbf{y}^*$ be such that $\hat{\mathbf{B}}\mathbf{y}^* \leq \mathbf{h}$. For any $\mathbf{B} \in \mathcal{U}$, we have $\mathbf{B} \leq \hat{\mathbf{B}}$ component-wise by construction. Note that $\mathbf{y}^* \geq \mathbf{0}$, this implies $\mathbf{B}\mathbf{y}^* \leq \hat{\mathbf{B}}\mathbf{y}^* \leq \mathbf{h}$ for all $\mathbf{B} \in \mathcal{U}$.

Conversely, suppose $\tilde{\mathbf{y}}$ satisfies $\mathbf{B}\tilde{\mathbf{y}} \leq \mathbf{h}$ for all $\mathbf{B} \in \mathcal{U}$. For each $i \in [m]$, note that $\mathsf{diag}(\mathbf{e}_i)\hat{\mathbf{B}} \in \mathcal{U}$ by construction. Therefore, $\mathbf{e}_i^T\hat{\mathbf{B}}\tilde{\mathbf{y}} \leq h_i$ for all $i \in [m]$, which implies that $\hat{\mathbf{B}}\tilde{\mathbf{y}} \leq \mathbf{h}$.

C.3 Proof of Lemma 3.2.3.

Let

$$\hat{B}_{ij} = \frac{1}{(n+i-j+1) \bmod m}.$$

From Theorem 3.2.1, Π_{Rob} is equivalent to

$$z_{\mathsf{Rob}} = \max\{\mathbf{e}^T\mathbf{y} \mid \hat{\mathbf{B}}\mathbf{y} \leq \mathbf{e}, \mathbf{y} \geq \mathbf{0}\}.$$

The dual problem is

$$z_{\mathsf{Rob}} = \min\{\mathbf{e}^T\mathbf{z} \mid \hat{\mathbf{B}}^T\mathbf{z} \geq \mathbf{e}, \mathbf{z} \geq \mathbf{0}\}.$$

Let

$$s = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n).$$

It is easy to observe that $\frac{1}{s}\mathbf{e}$ is a feasible solution for both the primal and the dual formulations of z_{Rob}. Moreover, they have the same objective value. Therefore,

$$z_{\mathsf{Rob}} = \frac{n}{s}.$$

On the other hand, for each $j \in [n]$, denote

$$\mathcal{U}_j = \left\{ \mathbf{b} \in \mathbb{R}_+^n \;\middle|\; \sum_{i=1}^{n} [(n+i-j+1) \bmod n] \cdot b_i \leq 1 \right\}.$$

By writing the dual of the inner maximization problem of Π_{AR}, we have

$$
\begin{aligned}
z_{\mathsf{AR}} &= \min\{\mathbf{e}^T\alpha \mid \mathbf{B}^T\alpha \geq \mathbf{e}, \alpha \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}\} \\
&= \min\{\lambda \mid \lambda\mathbf{B}^T\mu \geq \mathbf{e}, \mathbf{e}^T\mu = 1, \mu \geq \mathbf{0}, \mathbf{B} \in \mathcal{U}\} \\
&= \min\left\{ \frac{1}{\theta} \;\middle|\; \mathbf{b}_j^T\mu \geq \theta, \mathbf{b}_j \in \mathcal{U}_j, \mathbf{e}^T\mu = 1, \mu \geq \mathbf{0} \right\}.
\end{aligned}
$$

Therefore, we just need to solve

$$
\frac{1}{z_{\mathsf{AR}}} = \max\{\theta \mid \mathbf{b}_j^T\mu \geq \theta, \mathbf{b}_j \in \mathcal{U}_j, \mathbf{e}^T\mu = 1, \mu \geq \mathbf{0}\} \tag{C.1}
$$

Suppose $(\hat{\theta}, \hat{\mu}, \hat{\mathbf{b}}_j, j \in [m])$ is an optimal solution for (C.1). For each $j \in [n]$, consider a basic optimal solution $\tilde{\mathbf{b}}_j$ of the following LP:

$$
\tilde{\mathbf{b}}_j \in \arg\max\{\mathbf{b}^T\hat{\mu} \mid \mathbf{b} \in \mathcal{U}_j\}.
$$

Therefore, $\tilde{\mathbf{b}}_j$ is a vertex of $\mathcal{U}_j$, which implies that $\tilde{\mathbf{b}}_j = \hat{B}_{i_j j}\mathbf{e}_{i_j}$ for some $i_j \in [n]$ and $\tilde{\mathbf{b}}_j^T\hat{\mu} \geq \hat{\theta}$. For each $i \in [n]$, let $\mathcal{S}_i = \{j \mid i_j = i\}$. We have $\sum_{i=1}^n |\mathcal{S}_i| = n$. For each $i \in [n]$ such that $\mathcal{S}_i \neq \emptyset$, $\hat{B}_{ij}$ can only take values in $\{1, 1/2, \ldots, 1/n\}$ for $j \in \mathcal{S}_i$. Moreover, $\hat{B}_{ij} \neq \hat{B}_{ik}$ for $j \neq k$. Therefore, there exists $l_i \in \mathcal{S}_i$ such that

$$
\hat{B}_{il_i} \leq \frac{1}{|\mathcal{S}_i|}, \text{ and } \tilde{\mathbf{b}}_{l_i}^T\hat{\mu} = \hat{B}_{il_i}\hat{\mu}_i \geq \hat{\theta}.
$$

We have

$$1 = \sum_{i:\mathcal{S}_i \neq \emptyset} \hat{\mu}_i \geq \sum_{i:\mathcal{S}_i \neq \emptyset} \frac{\hat{\theta}}{\hat{B}_{il_i}} \geq \sum_{i:\mathcal{S}_i \neq \emptyset} \hat{\theta}|\mathcal{S}_i| = \hat{\theta} n.$$

Therefore, $\hat{\theta} \leq \frac{1}{n}$, which implies that $z_{\mathsf{AR}} \geq n$.

On the other hand, it is easy to observe that $z_{\mathsf{AR}} \leq n$: $\mathbf{b}_j = \mathbf{e}_j$, $\mu = 1/n \cdot \mathbf{e}$ and $\theta = 1/n$ is a feasible solution for (C.1). Therefore,

$$z_{\mathsf{AR}} = n = \sum_{i=1}^{n} \frac{1}{i} \cdot z_{\mathsf{Rob}} = \Theta(\log n) \cdot z_{\mathsf{Rob}},$$

which completes the proof.