Helping Prospective Teachers to Understand Children’s Mathematical Thinking

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ABSTRACT

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The primary aim of this study was to investigate the effects of two video-based interventions, one guided, the other non-guided, on pre-service early childhood education teachers’ understanding of students’ mathematical thinking. Five web-based lessons on various topics in children’s mathematical development were created for this study. Each contained a short reading introducing a videotaped clinical interview of a young child performing a mathematical task. The unguided group then watched a 2-minute video, while the guided group watched the same video segmented into short clips and then answered open-ended questions at each break. The main goal was to examine the effectiveness of the use of videotaped clinical interviews in professional development. More specifically, I was interested in the types of experiences offered by the guided and unguided versions, as compared to those of the control group. The results of this study showed that both the guided- and unguided-video experiences were successful in changing the way prospective teachers interpreted children’s mathematical thinking. While the results show it was possible to use videos to improve prospective teachers’ interpretive abilities, it was not possible to improve their ability to apply the interpretations to developing appropriate teaching activities.
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For my parents, who taught me that success will come so long as you do what you love.
Chapter 1: Introduction

The primary aim of this study is to investigate the effects of two video-based interventions on pre-service early childhood education teachers’ beliefs about and understanding of students’ mathematical thinking. Mathematics at the preschool level has recently become a major point of interest for educational research. The need to implement mathematics in early childhood education (ECE) programs stems partly from the concern that American students are not performing as well as they should on nationwide and international comparisons (National Research Council, 2001a). Given the new technologies in our society and the mathematical knowledge required to use them, this growing gap between American students’ abilities and those of children in China, Japan, and Korea is of great concern. Research has shown that the gap is not terribly wide at the age of four (Case et al., 1996; Ginsburg, 1997); however it widens in kindergarten and grows even larger by the fourth grade (US Department of Education & National Center for Education Statistics, 1997).

Although findings suggest young children have great potential for mathematical thinking (Ginsburg & Amit, 2008; National Research Council, 2001b), we know that the level of preparation provided to the typical ECE teacher is far below what is considered optimal for children’s learning and development (Bowman, Donovan, & Burns, 2001). Most early childhood preparation programs require one course in mathematics education (Ertle et al., 2008), and often these courses do not focus on early childhood (Ginsburg, Cannon, Eisenband, & Pappas, 2006). We also know that ECE teachers do not teach much math at all (Pianta, Belsky, Houts, & Morrison, 2007), and if they do teach it, they usually teach it badly (Ginsburg, Lee, & Boyd, 2008). Entering the workforce ill prepared can lead to allowing teachers’ uninformed
beliefs about children’s learning to interfere with instruction (Brown, 2005). In turn, this can lead to teachers misjudging their students’ abilities.

While it is recommended by the National Research Council that teachers of 3- to 5-year-olds hold a BA with a specialization in early childhood education, the 2004 National Institute for Early Education Research Yearbook reported that 23 of the 44 state-financed preschool initiatives profiled required lead teachers to hold a BA only when teaching in a public school (Barnett, Hustedt, Robin, & Schulman, 2004; Ginsburg, 2010). Pre-service ECE teachers rarely receive appropriate mathematics education training even when enrolled in higher education programs (Clements, Copple, & Hyson, 2002; Copley & Padron, 1999; Darling-Hammond, 1997). Perhaps this contributes to their thinking that children are not capable of learning mathematics. According to Brown, “a lack of appropriate knowledge and preparation could cause teacher candidates and experienced teachers to fail to see mathematics as a priority for young children and have less confidence in their ability to teach mathematics effectively” (2005).

The National Council of Teachers of Mathematics and National Association for the Education of Young Children published a joint position statement (2002) illuminating the need for high quality, challenging, and accessible mathematics education for children aged 3 to 6. Within this report recommendations were made regarding the specific domains children should be introduced to: operations, algebra, geometry, measurement, data analysis, and probability. In order for our teachers to disseminate this knowledge, however, they must first be properly trained. In order to train them, as The Rand Mathematics Study Group (2003) suggests, there is a need for research investigating the mathematical knowledge needed for teachers and the means necessary for them to acquire, apply, and use this knowledge.
This dissertation seeks to explore teachers’ expectancies about and theories of children’s learning, and also to examine the effects of two video-based case study methods designed to help teachers improve their understanding. It is well known that weak subject matter knowledge can constrain pedagogical decisions designed to improve children’s learning (McDiarmid, Ball, & Anderson, 1989). But it is also important to note that teachers’ understanding of students can influence teaching in important ways. “Instruction reflects what a teacher believes about how knowledge in his or her discipline is organized, how students’ minds are structured, how learning occurs in those minds, and how instruction comes to affect learning” (Strauss, 1993, p. 287). Strauss (2001) called for the development of a tool to measure aspects of teachers’ professional knowledge about students’ minds, how learning takes place in those minds, and how we can teach to influence that learning. This dissertation study sought to answer that call.

Researchers have used video to help teachers review and improve their own teaching practice (Fuller & Manning, 1973), but it is not often used for learning to observe and interpret children’s behavior. Currently, the majority of studies on video-based methods of teacher development focus on lesson analysis. These studies range in focus from what teachers attend to in the videos (Star & Strickland, 2008), to teachers’ development of observation and reasoning skills (Santagata, Zannoni, & Stigler, 2007), and cross-cultural comparisons of teacher "noticing" about the features of videotaped lessons (Miller & Zhou, 2007).

This study explores the effects of two video-based case study methods on teachers’ knowledge of mathematical development: the guided-video experience and the unguided-video experience. The guided experience walks the participant through a videotaped interview
between an adult and a child. Guidance comes through the video pausing at several points and asking the participant to think about a particular question (e.g. “what does this child know about addition?”). After answering, the participant is then shown three or four sample responses. This is intended to guide them in their thinking about what is happening in the video as they continue to watch and answer questions. What is novel about this approach is that it is similar to having a professor by your side focusing your attention on the important details of the video while actively engaging your own thinking by answering the questions and reading the sample responses. The participants in the unguided experience watch the same video, although without the pauses or the intermittent questions and sample responses. This is intended to mirror the participant’s experience of being given access to the videos without the structure provided by the guidance. The main rationale behind creating these two methods was an interest in examining the effectiveness of the use of videotaped case-studies in professional development. More specifically, I was interested in whether the two versions provided different experiences for the participants, or whether they were equally effective.

I was also interested in the effects of the interventions on participants’ expectancies. Before and after participating in the interventions, participants were presented with a series of written scenarios between a typically developing child and an adult asking him or her to perform various mathematical tasks (e.g. “Find the shape that has four sides. All sides are the same length.”). Participants’ expectancies were measured by asking them to select the ages at which they think children would be ready to start learning and when they would be able to master the particular content. The rationale behind this measure was to see whether the video-based interventions affected participants’ expectancies concerning the development of children’s mathematical thinking.
To gain insight into participants’ observation and understanding of and response to children’s mathematical thinking, guided and unguided participants also answered summative questions at the end of each video lesson. These open-ended responses were analyzed using the Knowledge of Mathematical Development coding scheme. This coding scheme sought to categorize the participants’ responses to questions about observing children’s behavior, making interpretations of children’s behavior, and proposing what to do next to help the child. It was hoped that through this coding scheme stories would emerge about the participants’ understanding of the development of mathematical thinking.
Chapter 2: Background

Teacher Beliefs

Beliefs are generally defined to include attitudes, judgments, or perspectives (Strauss, 1993). Research has found that teachers’ implicit beliefs about subject matter, their students, and their roles and responsibilities significantly influence how they behave in the classroom (Ball & Cohen, 1996). Additionally, Thompson (1984) concludes that “there is strong reason to believe that in mathematics, teachers’ conceptions (their beliefs, views, and preferences) about the subject matter and its teaching play an important role in affecting their effectiveness as the primary mediators between the subject and the learners.” Beliefs also tend to be stable and quite resistant to change (Brousseau, Book, & Byers, 1988).

Given that beliefs held by teachers regulate their perceptions, judgments, and behaviors in the classroom (Bandura, 1997), understanding teacher belief structures is central to improving their preparation and classroom practices (Pajares, 1996). Previous experience is what helps shape our beliefs. Evidence suggests that ECE teachers’ memories of their own schooling impact their motivations, expectations, and values in their classrooms (Hollingsworth, 1989; Nespor, 1987). For example, a recent study found that female teachers’ math anxieties can be an impediment to their own math achievement. More importantly, female students with "math anxious" teachers were more likely to endorse the stereotype that boys are good at math and girls are not by the end of the school year (Beilock, Gunderson, Ramirez, & Levine, 2009). This is an example of a teacher’s belief system and past experience negatively influencing her students’ beliefs.
Just as previous experiences influence belief systems, general views of children’s thinking do so as well. Research has shown that teachers’ mental models of how children’s minds work tend to be very similar, even when the teachers’ subject matter knowledge and years of experience are taken into account (Strauss, Ravid, Magen, & Berliner, 1998). Many teachers view children as being concrete learners and believe that children cannot think or understand abstract ideas (1998). Teachers’ general perception is that they possess “the knowledge” and it is their job to get that into the children’s minds. Also, teachers will have beliefs about how the knowledge can get inside, as well as beliefs about how it can be stored once it is inside (1998). A widely accepted theory is that children’s minds have small, flap-like openings. A “good” teacher is thought of as someone who can break the information into chunks small enough to fit through the openings (Strauss, 1993). Once the information has traveled through these little openings, many believe the child links the new information to previously stored information. Therefore, in the absence of existing knowledge, many teachers think the only way for the child to store new information is through repetition and practice (1993). What is remarkable is that many teachers possess a similar mental model, yet this model for learning is never taught to them: the models are constructed independent of preservice and inservice training.

**Teacher Knowledge**

Very little is known about the nature of elementary teachers’ mathematical knowledge for teaching (Hill, 2010). Most of the research has focused on one teacher, or a small handful of randomly sampled teachers. Using multiple cases to compare U.S. and Chinese teachers, Ma (1999) found that problems requiring conceptual understanding of division of fractions, areas, perimeters, and place value were difficult for the U.S. teachers. Another international study
found striking differences between what U.S. and Chinese elementary school teachers noticed when watching classroom videos (Miller & Zhou, 2007): U.S. teachers paid attention to the videotaped teachers’ personalities, while Chinese teachers paid attention to the mathematical content of the class. Other research showed the relationship between teachers’ mathematical knowledge and the quality of their classroom work (2010). Hill, Blunk, et al. (2008) found strong correlations between 10 teachers’ mathematical knowledge for teaching and mathematical elements of the classroom. These elements included the presence of teacher mathematical errors, the quality of mathematical work, and the depth of teacher interpretation of student behavior.

Shulman (1986) sought to understand how teachers decide what to teach, how to represent it, how to question students about it, and how to deal with misunderstanding. He argued that teachers must understand that something is so, in addition to why it is so, how it can be asserted and how our beliefs in it can be weakened. The particular type of teacher knowledge most relevant to this dissertation study is termed Pedagogical Content Knowledge. This type of knowledge goes beyond knowledge of the subject matter to understanding its teachability. This includes understanding what makes learning something easy or difficult, as well as knowledge of the various strategies that are useful to learners.

Learning to notice in particular ways is part of the development of expertise. Jacobs et al. (2009) define professional noticing of children’s mathematical thinking as a set of three skills: attending to children’s strategies, interpreting children’s understandings, and deciding how to respond based on what the children understand. Children’s strategies are complex. Understanding the details of these strategies provides insight into children’s understanding of mathematical concepts. Similarly, interpreting children’s strategies requires that the teacher’s reasoning take into account the child’s actual strategy and ideally be informed by the research on
mathematical development. Finally, deciding how to respond based on a child’s behavior draws on what the teacher understands about the child’s strategy, as well as their understanding of mathematical development in general. Jacobs found that increased experience with children’s thinking was related to increased engagement with children’s thinking. Their findings suggest that expertise in attending to and interpreting children’s strategies grew with two years of continual professional development and with teaching experience.

This dissertation study explored prospective teachers’ knowledge of mathematical development (KMD). I define KMD as involving three components: knowing that, as well as knowing both how and why. Declarative knowledge encompasses knowing that something is true, without knowing how or why it is true (e.g. “1 comes before 2.”). Procedural knowledge entails knowing how to carry out a mathematical act such as performing addition or subtraction. Conceptual understanding pertains to knowing why something is mathematically true or why we use certain procedures to arrive at an answer. These three types of knowledge sit in a hierarchy, with declarative knowledge at the bottom and conceptual understanding at the top. For the purposes of this dissertation study, a shift in focus from declarative knowledge to procedural or conceptual understanding was considered growth in KMD.

This hierarchy is widely held conception in the world of cognitive development research. For example, Kilpatrick et al. (2001) define the five strands of mathematical proficiency as being conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. They define conceptual understanding as the comprehension of mathematical concepts. Procedural fluency is one’s ability to carry out procedures. Strategic competence is the ability to formulate, represent, and solve mathematical problems. Adaptive reasoning is one’s ability for logical thoughts or justifications about relationships among
concepts and situations. Finally, productive disposition is defined as one’s inclination to see mathematics as sensible and worthwhile. In the KMD coding scheme, conceptual understanding includes strategic competence and adaptive reasoning, while procedural knowledge encompasses procedural fluency only. Declarative knowledge is not included in these strands, perhaps because it is required in order for these strands to exist.

KMD’s relationship to both pedagogical content knowledge and professional noticing is evident, as it sits at the core of both. Research has found that KMD is related to both student achievement (Baumert et al., 2010; Hill, Rowan, & Ball, 2005), and to the quality of mathematical instruction (Hill & Ball, 2008). In particular, Baumert (2010) found that teacher’s pedagogical content knowledge was predictive of student learning gains. KMD entails understanding the mathematical ideas students are learning, as well as an appreciation for the ways in which children learn mathematics. It involves understanding the developmental sequences children go through, as well as an understanding of their strategies and misconceptions.

The Nature of Professional Development

Some studies exploring the development of pedagogical content knowledge have used a professional development method known as Cognitively Guided Instruction (CGI) (Carpenter, Fennema, & Franke, 1996; Carpenter, Fennema, Peterson, & Carey, 1988; Franke, Carpenter, Levi, & Fennema, 2001). Carpenter, et al. (1988) found that teachers’ knowledge of children’s thinking is informal and lacked coherence. Thus they designed CGI to help teachers understand children’s thinking through the development of models of children’s mathematical thinking in various domains. Teachers were not provided with instructional materials. Instead, they developed their own instructional materials and practices through listening to and watching their
students. It is during the struggle to listen and interpret that the teachers come to see themselves as ongoing learners, ultimately leading to improvements in their teaching. Research found that teachers who participated in CGI taught significantly more problem solving, and significantly less number facts and skills (Wilson & Berne, 1999).

Building instruction based on children’s ways of thinking has been linked to richer classroom environments (Clarke, 2008; Gearhart & Saxe, 2004), as well as related to gains in student achievement (Fennema et al., 1996; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). Knapp and Peterson (1995) found that focusing on student thinking while participating in a CGI professional development program proved to be a valuable mechanism for generative change in teachers. Teachers who learn to focus on the underlying principles of children’s mathematical thinking are able to sustain the positive changes made in professional development. Similarly, Carpenter, et al. (2001) found that those teachers who saw themselves as learners were able to create their own understanding of the development of student thinking. They propose that teachers’ engagement with student thinking generates thinking about the substance and content of their daily work. Their findings suggest that teachers who learn how to learn through children’s thinking continue to learn long after the professional development has ended.

**Effectiveness of Professional Development**

Research has shown that “teaching reforms cannot take place unless teachers’ deeply held beliefs about mathematics and its teaching and learning change” (Ernest, 1989). Strauss (1993) believes that providing teachers with thoughtful professional development begins with understanding their current mental models and then introducing new “learning models” that
reflect advances in our understanding of student learning and cognition. In the same way, we argue that helping children better learn mathematics requires that the teacher understand what children already know and think about the mathematics we want to teach them. Professional development requires more than training in subject matter knowledge or hands-on experience. We must first understand what teachers believe about children’s learning before we can design effective professional development.

There is little disagreement among researchers and policymakers that teachers are the key element in improving U.S. education (Bell, Wilson, Higgins, & McCoach, 2010). Additionally, there is unanimous agreement that we need high-quality, ongoing professional development (American Federation of Teachers, 2002). However, research linking improvements in student learning to professional development has produced mixed results (2010). Yoon, et al. (2007) examined 1,300 studies evaluating the effects of professional development on student achievement and found only nine that met the What Works Clearinghouse evidence standards (2008). Of those nine studies, they found that students of teachers who had an average of 49 hours of in-service professional development had an increase in student achievement in math, science, reading, and language arts. These results are promising, but the fact that only nine out of 1,300 studies met the standards speaks to the need for an improvement in professional development programs and studies evaluating their effectiveness. As Borko (2004) put it, “the professional development currently available to teachers is woefully inadequate.”

Professional consensus has emerged regarding the necessary characteristics of high-quality professional development. These include in-depth, active learning opportunities, a focus on content and how students learn content, an extended duration, the collective participation of groups, and links to high standards (Desimone, Porter, Garet, Yoon, & Buirman, 2002). There is
not much research directly linking these characteristics to higher student achievement or improved teaching, though some studies have found that possessing all of the characteristics can have a positive influence on the teachers’ classroom practice (Birman, Desimone, Garet, & Porter, 2000). Additionally, several studies found that the intensity and duration of the professional development can lead to teacher change (Shields, Marsh, & Adelman, 1998; Weiss, Montgomery, Ridgway, & Bond, 1998). Of particular interest is the research showing that professional development focused on general pedagogy or classroom management is not as helpful as professional development focused on particular math and science content and the ways students learn that content. This is especially true for instruction designed to improve children’s conceptual understanding (Cohen & Hill, 2001; Fennema et al., 1996; Ma, 1999).

Unfortunately, most professional development evaluations use self-reports of learning (e.g. Garet, Porter, & Desimone, 2001), which are not guaranteed to be accurate measures. For example, Desimone, et al. (2002) found that “professional development focused on specific instructional practices increases teachers’ use of those practices” and that the “use of specific features, such as active learning opportunities, increase the effect of the professional development on teachers’ instruction.” These conclusions, however, were drawn on the self-reports of participating teachers, not on observations of those teachers in action. In general there is a paucity of good measures to choose from when one wants to assess teacher learning through professional development (Bell et al., 2010).
A New Approach to Professional development

The Development of Mathematical Thinking Course

In an effort to change the direction of Early Childhood Mathematics Education training, The Development of Mathematical Thinking (DMT) course at Teachers College, Columbia University was developed by Professor Herb Ginsburg. It is provided 1-2 times per year for ECE, Mathematics Education students, and others. The course shows how children understand mathematical content and make use of it in their own worlds, as well as provides a rich offering of the mathematical content preschoolers and kindergartners should be introduced to. The course teaches pre-service ECE teachers not only about the development of mathematical thinking, but how to access that thinking in their students through learning to observe, interpret, and ask probing questions at the right time. One of the methods they learn is to conduct a Clinical Interview (CI). The course is comprised of readings, watching and analyzing videos of adults interviewing children on various mathematical tasks, and learning to understand the development of children’s mathematical thinking, all the while learning how to conduct a CI and subsequently conducting one on their own. Learning how to conduct a CI is woven into the structure of the course. Throughout the course, prospective teachers develop awareness of their personal theories of mathematics learning and their beliefs about children’s math abilities through written essays analyzing CIs, conducting interviews with young children, class discussions and weekly reflections on various course topics. The course has three goals.

The first goal is to create awareness of young children’s capabilities. This entails reflecting on their own theories of how children learn mathematics, building upon those theories based on current research in psychology and education, and then using the new understanding of
how children learn to guide their classroom observations and analyses of their children both in free play and during structured activities (Ginsburg et al., 2005).

A second goal of the course, though no less important than the first, is to learn how to assess knowledge and the development of mathematical thinking in the children. This means the teachers need to learn to conduct clinical interviews to determine what a child needs to learn and whether instruction has been effective (Ginsburg, Jacobs, & Lopez, 1998).

A third goal of the course is to teach the prospective teachers to skillfully select appropriate activities for their students based on new understanding of students’ thinking. This involves better understanding the mathematics being taught to the children (Ma, 1999) and developing a deeper understanding of how the teacher can serve as a guide for the students in delivering instruction (Vygotsky, 1978).

Videotaped interviews of children talking to adults about mathematical tasks play a critical role in the course because of the richness of information that can be gleaned from watching and discussing them. Learning how to watch and how to conduct clinical interviews is the fulcrum of the course.

Clinical Interviewing

Developed by Piaget in the 1920s, the clinical interview is a method used to explore a child’s thinking. At its core, the CI is an opportunity for the adult to test an hypothesis regarding the child’s understanding of a particular concept. It is unique in that no interview is alike as each is composed of “individualized patterns of questioning” (Ginsburg, 1997). The interviewer begins with a protocol, which is based on the original hypothesis, but has the freedom to alter the tasks as the interview progresses. This flexibility comes also from the child’s partial control of
the interview (Ginsburg, 1997). This is something that sits at the core of the concept of CI. These revisions are evidenced by on-the-spot creations of new tasks in response to the child’s verbalizations. For example, a child can point to and say shape names when presented with a prototypical image of the shape. Instead of concluding the child knows his shapes, the researcher then asks the child to find a shape based on its attributes, “please give me a shape with three sides.” In doing this, the teacher is then able to ask the child why he chose a particular shape, which will help determine the limits of the child’s conception of shapes.

During the interview, the adult actively makes, tests and revises hypotheses based on the child’s responses. The child’s behaviors are to be observed carefully and interpreted with caution, as it is easy to lead the child to support your hypothesis instead of letting the child guide you to the essence of his thinking. To avoid this, the interviewer employs probing questions that do not reveal anything about what she might be thinking. Example follow-up questions are, “How did you know that?” or “How did you do that?” These questions are intended to peel back layers of the child’s thinking by forcing him to defend his previously held convictions. Through careful observation of body language, actions, and verbalizations the interviewer should be able to gain insight into the child’s thinking, even if it disproves the interviewer’s original hypothesis. It is here that flexibility plays a role, as with each new insight should come more curiosity regarding the child’s knowledge. In order to be flexible, however, knowledge of how children develop and a strong foundational knowledge of the content being explored are just as essential as being able to interview a child. Without these tools, knowing what to do next can be a challenge.
DMT Course as a Basis for Research

One key feature of the DMT course is its use of videotaped clinical interviews with children performing mathematical tasks. What makes the use of these videos unique is how the professor pauses throughout the viewing in order to ask the class to think about what the child might do next or what the interviewer may ask next. Asking these questions helps to accomplish the three main goals of the course: create awareness of children’s abilities, learn how to assess children’s knowledge and development of mathematical thinking, and skillfully select appropriate activities based on new understandings of student thinking.

The two video-based methods of professional development explored in this dissertation study were built upon the DMT course framework. In an effort to emulate the richness of the course, the guided-video condition was conceived. In this condition, participants view the same clinical interview video as those participants in the unguided-video condition; however the video is segmented into short clips. Following each segment, the participant is asked an interpretive question. This process is very similar to how the professor would present the video during class and moderate class discussion. Participants may see a child state his answer to a question but the video is stopped before the child can explain how he got it. They are then asked to theorize about how the child arrived at his answer. Once an answer has been submitted, participants read sample responses from other students provided by the researchers (something similar to hearing classmate’s thoughts about the paused video). Participants in the unguided-video condition watch the same videos, but without any pauses and without being asked any interpretive questions along the way. The guided-video condition experience is something akin to the DMT course, while the unguided-video condition experience contains the content of the course without the added opportunities to interpret and assess provided by the clinical interview method.
This dissertation is concerned with several issues. The first is whether the guided and unguided-video experiences have different effects on how prospective teachers interpret children’s behavior. Secondly, I was interested in whether there is a difference between the guided- and unguided-video experiences on how prospective teachers use their knowledge of children’s behavior to guide their teaching. Lastly, I was interested in whether the guided- and unguided-video experiences affect expectancies about competence differently than the control group.
Chapter 3: Method

Participants

Sixty-three early and elementary education majors from ten universities participated in this study. None of the participants had taken a course in teaching mathematics, and none had begun student teaching. 57 were female, while 6 were male. Participants were told they were testing new lessons in early childhood mathematics, and were compensated for each lesson they completed (roughly $15/hr). Participants in the control group were paid for two web-based lessons: the pre-test and the post-test. Participants in the two experimental groups were paid for seven web-based lessons: the pre-test, the five video lessons, and the post-test. Ten participants were dropped from the study after failing to complete the lessons on time: five were from the guided-video group and five were from the unguided-video group.

<table>
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<tr>
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<th>Frequency</th>
<th>Percent</th>
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<td>11.1</td>
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</tbody>
</table>

Table 1: A breakdown of the Universities attended by participants.
Procedure

This study took place over the course of four weeks and all data was collected using an online data collection tool, SurveyMonkey.com. Participants were randomly assigned to one of the three conditions: the guided-video group, the unguided-video group, and the control group. All participants completed the pre- and post-tests. Participants in the guided- and unguided-video condition completed two 60-minute web-based lessons per week for four weeks, while those in the control condition completed the pre- and post-tests only with a three-week delay.

The pre- and post-tests included watching a six-minute unsegmented clinical interview on pattern with a five-year old girl and responding to questions intended to assess their understanding of children’s thinking and how to use it to teach: “What does the child know about X?” and “what would you do next in teaching the child?” These videos employed a different mathematical task than the videos used in the interventions, which all concerned number. There were no introductory readings for these tests, nor were the videos segmented. The pre- and post-tests used two roughly similar videos of children engaged in making and talking about patterns with an adult, and were counter-balanced to control for a possible order effect. Therefore, half of the participants viewed Video 1 in the pre-test and Video 2 in the post-test, while the other half viewed Video 2 in the pre-test and Video 1 in post-test (a.k.a. Video Order 1 and Video Order 2).

The pre- and post-tests also included surveys on self-efficacy and teachers’ understanding of student knowledge (TUSK). A collaborator used the results of the self-efficacy survey for independent research, while this dissertation study used the results of the TUSK.

The lessons for the interventions were developed to cover five sub-topics in the development of the mathematical concept of “number” (among the main content areas proposed
by the National Council of Teachers of Mathematics, (2000). The sub-topics follow a developmental sequence in terms of complexity and the age at which children learn them: counting, enumeration, addition, subtraction, and equivalence. Participants in the study started the sequence from a randomized first lesson and continued sequentially until all five lessons were complete to control for an order effect (e.g. 3, 4, 5, 1, 2). The lessons began with a reading that was about two pages in length. The readings were derived from *Children’s Arithmetic* (Ginsburg, 1977) and summarized the topic, as well as helped participants develop a sense of what to look for in the video prior to its viewing (see Appendix A). Once the readings were done, videos of children (roughly 5-years old) undergoing clinical interviews with trained adults were selected to highlight key elements of the development of the specific topic. The videos were about two minutes in length.

**Guided-video condition:** For all five lessons, participants in the guided-video condition read a brief text and watched the same videos as those participants in the unguided-video condition, although the videos were segmented into roughly 15 10-second clips. These points of segmentation were created in consultation with an expert in the fields of cognitive and child development. Following each segment participants were asked questions of the following types (see Appendix A):

- Describe what you observed; interpret the child’s behavior;
- Offer an hypothesis for what the child knows (e.g. “what is the difference in his mind between counting by one’s vs. counting by two’s?”);
- Predict what will happen next;
• Provide further questions to ask or tasks to try;
• Suggest strategies for teaching.

Once answers were submitted on the web, participants could not make any changes. They were then shown a list of sample responses framed as “successful student responses.” The intention was to provide a range of possible interpretations of the video from responses like “his statement makes no sense” to “he thinks counting by two’s is a more powerful method.” These sample responses were also supposed to serve as an opportunity for participants to benchmark their responses against their peers. At the end of each lesson, participants responded to summative questions about what the child knew about the particular topic and what they (the participants) would do next as the child’s teacher. No sample responses were provided for these final questions.

**Unguided-video condition:** For all five lessons, participants in the unguided-video condition read the same short text and watched the same video, but in this condition the video was not segmented. Therefore, participants only responded only to the summative questions and never had the opportunity to read the “successful” student responses. At the conclusion of the lesson, they responded to the same summative questions as the guided-video group: “What does the child know about X?” and “what would you do next in teaching this child?”

**Control condition:** Participants in the control condition completed the pre- and the post-tests only.
Figure 1: Dissertation Study Outline

Dissertation Study Outline

Pre-test

UNGUIDED-VIDEO Condition

TUSK + Self-efficacy Surveys

Pattern video (counterbalanced)

know? do next?

COUNTING VIDEO

CONTROL Condition

TUSK + Self-efficacy Surveys

Pattern video (counterbalanced)

know? do next?

GUIDED-VIDEO Condition

TUSK + Self-efficacy Surveys

Pattern video (counterbalanced)

know? do next?

Post-test

TUSK + Self-efficacy Surveys

Pattern video (counterbalanced)

know? do next?

eval interviewer + most imp thing from study?

Counting Reading

know? do next? learn?

Counting Video

Enumeration Reading

know? do next? learn?

Enumeration Video

Addition Reading

know? do next? learn?

Addition Video

Subtraction Reading

know? do next? learn?

Subtraction Video

Equivalence Reading

know? do next? learn?

Equivalence Video

Pattern video (counterbalanced)

know? do next?

Pattern video (counterbalanced)

know? do next?

Pattern video (counterbalanced)

know? do next?

know? do next?

know? do next?

know? do next?

know? do next?

know? do next?

know? do next?
Measures

Teachers Understanding of Student Knowledge (TUSK)

One pretest was the Teachers’ Understanding of Student Knowledge instrument (TUSK), designed to measure teachers’ expectancies concerning their students’ math abilities (see Appendix B). The instrument contains ten scenarios, each of which introduces a particular math activity and provides information about what the child knows about the domain or how the child performs on the task. For example, “Theo can consistently count from 1 to 20. He answers, ‘9’ when asked, ‘what comes after 6?’” The scenarios explore the teachers’ expectancies of their students’ thinking. Participants are asked to identify the ages they think a typically developing child would be ready to start learning and then be able to master the activity (understanding of developmental trajectory/age expectancies). Participants selected one age for each from the following options: <2 2.5 3 3.5 4 4.5 5 5.5 6 >6. (The TUSK was also designed to investigate other aspects of teachers’ understanding, but because of limitations on the time available for pre- and post-tests, only this expectancy part of the TUSK was employed.)

The strategy for constructing the TUSK was to select topics on which I have information about what children know and can do, so that I can use this information to assess teacher knowledge. All but one of the items was selected from the Early Mathematics Assessment System (EMAS), which employs the theme of a bear’s birthday party. Activities at “the party” cover the following domains of 3, 4, and 5-year old mathematics: number, operations, shape, space, and pattern. The assessment items have been tested and validated across a variety of student populations. For my purposes I selected tasks within each domain that had stronger differentiating abilities. The items used in the TUSK are considered representative
measurements of their domains. The Bag It activity was borrowed from the Big Math for Little Kids curriculum because of the variety of possible responses participants could give.

Knowledge of Mathematical Development Coding Scheme

The KMD coding scheme was developed to analyze responses to the summative questions asked at the end of each lesson and also at the end of the pre- and post-test videos: “what does this child know about X?” and “what would you do next in teaching X?” There are three codes: declarative knowledge (DK), procedural knowledge (PK), and conceptual understanding (CU). The Declarative Knowledge code indicates the participant thinks the child knows “that something just is.” One acquires declarative knowledge through memorization or repetition. Possessing declarative knowledge of something does not require understanding why that something is or how it works (see table 2 for examples). The Procedural Knowledge code is applied when participants imply the need to “know how to do” something. The implication is that an action must be carried out on something in order to achieve the present goal. Knowing how to do something (PK) is quite different from knowing what something is (DK). The Conceptual Understanding code is applied to responses that reference the child’s deeper understanding of the principle at hand, or the child’s need for it. Conceptual Understanding is knowing why something is the way it is. Responses coded as conceptual understanding mention “understanding” as a relevant part in the child’s conceptualization of the topic, or in the teaching of the topic, however simply using the word “understand” in a response did not warrant receiving the code. This code was sparsely applied. The greatest number of instances of conceptual understanding in a single response totaled only two. Inter-rater reliability was established for this coding scheme using Cohen’s Kappa (0.91, p<0.001).
### Table 2: KMD coding examples

<table>
<thead>
<tr>
<th>Code</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Declarative Knowledge (DK)</strong></td>
<td>• “She knows that a pattern is something repeating.”</td>
</tr>
<tr>
<td></td>
<td>• “He knows that the equals sign means the same as.”</td>
</tr>
<tr>
<td></td>
<td>• “Rachel knows that 3 and 3 equal 6. She knows this by memorization.”</td>
</tr>
<tr>
<td><strong>Procedural Knowledge (PK)</strong></td>
<td>• “I might try and have her separate the bears into color groups.”</td>
</tr>
<tr>
<td></td>
<td>• “Possibly ask him if he can count using his fingers up to 10.”</td>
</tr>
<tr>
<td></td>
<td>• “She also knew that you can find the answer by counting and using blocks.”</td>
</tr>
<tr>
<td><strong>Conceptual Understanding (CU)</strong></td>
<td>• “I would try to show Vienna that a pattern is more than a trail of colors and is actually made up of repeating families.”</td>
</tr>
<tr>
<td></td>
<td>• “He comes to realize that the equals sign ultimately means the same thing, regardless of the numbers and other symbols that surround it.”</td>
</tr>
<tr>
<td></td>
<td>• “I think that she knows a lot about arithmetic and how to use it to answer subtraction problems. She was able to explain her ideas correctly and quickly without much hesitation.”</td>
</tr>
</tbody>
</table>

### Research Questions

This study was designed to answer the following questions:

1. Is there a difference between the effects of the guided- and unguided-video experiences on prospective teachers’ interpretations of children’s behavior throughout the video-based lessons? To answer this question I used the KMD coding scheme to analyze participants’ responses to “what does this child know?” in their first and fifth lessons completed during the interventions.

2. Is there a difference between the effects of guided- and unguided-video experiences on prospective teachers’ proposed suggestions for what to teach next throughout the video-based lessons? To answer this question I used the KMD coding scheme to analyze
participants’ responses to “what would you do next in teaching?” in their first and fifth lessons completed during the interventions.

3. Do the guided- and unguided-video experiences alter prospective teachers’ age expectancies, more so than the control group? To answer this question I used participants’ responses to the TUSK at the pre-test and the post-test to examine change.

4. Looking at responses to the pre- and post-test videos, do the guided- and unguided-video experiences cause increases in prospective teachers’ perception of conceptual understanding in children’s thinking, more so than the control group? To answer this question, I used the KMD coding scheme to analyze participants’ responses to “what does this child know about pattern?” from the pre-test and the post-test.
Chapter 4: Results

This section is organized by research question. Each question is stated and the hypotheses are presented, followed by the results.

Research Question 1: Is there a difference between the effects of the guided- and unguided-video experiences on prospective teachers’ interpretations of children’s behavior throughout the video-based lessons?

Hypotheses: Instances of conceptual understanding will increase more for those in the guided-video condition than the unguided condition when responding to the question “what does the child know about X?” while instances of declarative knowledge and procedural knowledge will decrease more so for the guided than the unguided condition.

Results: Responses to the first and the last lessons were chosen so as to determine if any growth had taken place over the course of the interventions. To ensure that the pre-test/post-test pattern video order did not have an effect on participants’ responses during the interventions, preliminary analyses included Video Order as a between-subjects factor in a two-way repeated measures ANOVA. There were no significant results, indicating that watching either video during the pre-test could not be shown to impact the way participants responded during the interventions.

Two-way repeated measures ANOVAs were then run for the KMD variables (declarative knowledge, procedural knowledge, and conceptual knowledge) using group as the between-subjects factor to determine if there were any group differences between participants’ responses
during the intervention to the “what does the child know?” question. The analyses used time (from first to fifth lesson) as the within subjects factor. There were no group differences. However, when considering the time factor, instances of declarative knowledge decreased significantly from the first to the fifth lesson, $F(1,39) = 12.26, p < .001$ (table 3 – highlighted numbers are those with significant results). Conversely, instances of conceptual understanding increased significantly from the first to the fifth lesson when not taking group into account, $F(1,39) = 24.24, p < .001$. Procedural knowledge stayed roughly the same for both groups. These results show that both the unguided and the guided versions of the intervention were effective.

Participants were encouraged to think critically about what they were observing and to identify instances of learning and thinking. Thus instances of declarative knowledge decreased, while instances of conceptual understanding increased (figure 2). For example, one participant suggested in her first lesson, “He knows that he needs to use his fingers to count the numbers,” which received a declarative knowledge code, while in her fifth lesson she suggested, “In his thinking he relates ‘makes’ to ‘equal’ because that's the only way he is able to express something being the same,” which received a conceptual understanding code.

<table>
<thead>
<tr>
<th></th>
<th>First Lesson</th>
<th>Fifth Lesson</th>
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</thead>
<tbody>
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<td>Mean</td>
</tr>
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<td></td>
<td></td>
</tr>
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<td>DK</td>
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</tr>
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</tr>
<tr>
<td>CU</td>
<td>20</td>
<td>0.30</td>
</tr>
<tr>
<td>Guided</td>
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<td></td>
</tr>
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<td>DK</td>
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</tr>
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</tr>
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<td>CU</td>
<td>41</td>
<td>0.29</td>
</tr>
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</table>

Table 3: First and fifth lesson responses to “what does this child know?”
Research Question 2: Is there a difference between the guided- and unguided-video experiences on prospective teachers’ proposed suggestions for what to teach next?

Hypotheses: Instances of teaching for conceptual understanding will increase more so for those in the guided condition than the unguided condition when responding to the question, “what would you do next in teaching X?” while instances of teaching for declarative knowledge and procedural knowledge will decrease more so for the guided condition than the unguided condition.

Results: Looking at first and fifth/final lesson completed by the participants during the intervention, preliminary analyses used a two-way repeated measures ANOVA with the pre-test video as the between-subjects factor and time as the within-subjects factor. There was no
significant effect of pre-test/post-test video order on how participants responded to the intervention lessons. This indicates that the pattern video order had no effect on how participants responded to the question, “what would you do next in teaching?” during the interventions.

Continuing the analyses, a two-way repeated measures ANOVA was run using group as the between-subjects factor and time (from first to fifth lesson) as the within-subjects factor on participants’ responses to the intervention. There were no significant group differences. However, when considering the time factor, instances of procedural knowledge increased significantly $F(1, 39) = 12.24, p < .001$ (table 4). Again, these results show that both the unguided and guided versions of the intervention were working. As participants progressed through the lessons, their suggestions of what to teach next became more focused on procedural knowledge (figure 3). It is possible that participants have not learned enough to be able to increase the amount of conceptual understanding they include in their teaching. Skills, however, are something that teachers and non-teachers are likely inclined to teach.

<table>
<thead>
<tr>
<th></th>
<th>First Lesson</th>
<th></th>
<th>Fifth Lesson</th>
<th></th>
</tr>
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<tr>
<td></td>
<td>N</td>
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<tr>
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<td>0.51</td>
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<td>PK</td>
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</tr>
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<td>Guided</td>
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<td></td>
<td></td>
<td></td>
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<td>DK</td>
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<td>41</td>
<td>0.41</td>
<td>0.59</td>
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</table>

Table 4: First and fifth lesson responses to “what would you do next?”
Research Question 3: Do the guided- and unguided-video experiences alter prospective teachers’ age expectancies, more so than the control group?

Hypotheses: Ready to start learning ages will decrease in the post-test for those participants in the unguided and guided groups, while able to master ages will not change. It is hypothesized that the interventions will show participants that children are more capable at younger ages than they would typically think. The goal is not to show that children are able to learn faster (i.e. master the content at a younger age). It is hypothesized that the difference between able to master and ready to start (the required learning time) will increase as the ready to start learning ages decrease and the able to master ages stay the same.

Results: 3 x 2 analyses of variance (ANOVA) were run using group as the between-subjects factor and time as the within-subjects factor on both the ready to start learning and able to
master ages. Separate analyses were run for the ready to start learning and the able to master ages. There were no significant differences among the groups for either of the analyses.

However, when considering the time variable, a paired-samples t-test showed the ready to start learning age decreased significantly from pre- to post-test, $t(62) = 2.67, p < .01$ (table 5). This shows that the group average indicated children were more capable at younger ages at the time of the post-test.

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Range</th>
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<td>22</td>
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<td>0.71</td>
<td>2.55-5.05</td>
<td>22</td>
<td>3.78</td>
<td>0.66</td>
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<td>0.75</td>
<td>3.35-6.10</td>
<td>22</td>
<td>4.82</td>
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<td>0.41</td>
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<td>1.04</td>
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<td>.45-2.20</td>
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<tr>
<td>Learning Time</td>
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<td>0.47</td>
<td>.35-2.10</td>
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<td>1.06</td>
<td>0.48</td>
<td>.20-2.20</td>
</tr>
</tbody>
</table>

Table 5: Pre/Post TUSK responses

To analyze learning time, a difference score was created by subtracting the ready to start learning age from the able to master age. A 3 x 2 analysis of variance (ANOVA) was run using group as the between-subjects factor and time as the within-subjects factor on the difference score from the pre- to the post-test. There were no significant group differences. However, when looking only at the results for the unguided group, a paired-sample t-test showed the
learning time increased significantly from pre- to post-test, t(19) = -2.31, p < .03 (see figure 4).

Table 5 shows that the ready to start learning ages decreased for the unguided group, while the able to master ages barely changed. This indicates that while there is still much to master (hence, the able to master age hardly changed), earlier ages to begin learning are being recognized by the participants (the ready to start learning age decreased), causing an increase in the overall amount of time it takes to learn the content.

![Figure 4: Mean Learning Time from pre- to post-test by group](image)

**Research Question 4:** Looking at responses to the pre- and post-test videos, do the guided- and unguided-video experiences cause increases in prospective teachers’ perception of conceptual understanding in children’s thinking, more so than the control group?

**Hypotheses:** There will be an increase in instances of perceived conceptual understanding in the guided condition’s responses to the questions, “what does this child know about X?” and “what would you do next in teaching X?” more so than in the unguided condition and the control.
Results: Responses to the questions were coded using the KMD coding scheme, which resulted in the data containing count variables. Therefore, Poisson regression was used. As an initial check, the preliminary analysis included the pattern video order as a factor, in addition to the treatment group. Analyses showed that the order in which participants watched the pattern videos in the pre- and post-test had a significant effect on how they responded to the questions.

In response to “what does the child know about pattern?” the Poisson regression indicated that instances of conceptual understanding in the post-test increased on average by 670% for those who had Video Order 1 \( (e^{1.9} = 6.7) \). Further analyses showed there was a significant interaction between Video Order and responses coded as conceptual understanding, \( F(1, 57) = 13.828, p < .001 \). Those who viewed Video 1 first increased conceptual understanding significantly, \( t(30)= -4.491, p < .001 \), while those who viewed Video 2 first decreased, though not significantly (figure 5). This indicates that participants watching one of the videos were identifying the child as having conceptual understanding, while in the other video they were not.

![Figure 5: Interaction between video order and mean number of instances of conceptual understanding in response to “what does this child know about pattern?” at the pre- and post-test](image-url)
Similarly, in response to the same question, there was an interaction between instances of procedural knowledge and Video Order, $F(1, 57) = 4.806, p = .032$. While both viewing orders decreased from pre- to post-test, those who viewed Video 1 second had a significant decrease in instances of procedural knowledge from pre- to post-test, $t(31) = 3.561, p < .001$ (figure 6). Again, this indicated that there was a difference in the type of content each video contained.

![Figure 6: Interaction between video order and mean number of instances of procedural knowledge in response to “what does this child know about pattern?” at the pre- and post-test](image)

In response to “what would you do next in teaching pattern?” Poisson regression indicated instances of declarative knowledge in the post-test decreased by 14% for those who had Video Order 2 ($e^{-1.98} = .14$). Further analyses showed there was a significant interaction between instances of declarative knowledge and Video Order ($F(1, 57) = 13.256, p = .001$). Instances of declarative knowledge decreased significantly for those who had Video Order 1, $t(30) = 2.516, p = .017$, while it increased significantly for those who had Video Order 2, $t(31) = -2.775, p = .009$ (figure 7). Here, participants were focusing their suggestions for teaching on
declarative knowledge for one child, while they were focusing less on declarative knowledge for the other child.

Figure 7: Interaction between video order and mean number of instances of declarative knowledge in response to “what would you do next in teaching?” at the pre- and post-test

For the same question, Poisson regression indicated that those with Video Order 1 had a 21% decrease in instances of conceptual understanding ($e^{-1.58} = .206$). Additionally, those with Video Order 1 had a 336% increase in instances of conceptual understanding at the post-test, ($e^{1.21} = 3.36$). Further analyses showed there was a significant interaction between instances of conceptual understanding and video order, $F(1, 57) = 29.163$, $p < .001$. Those who had Video Order 1 had a significant increase in instances of conceptual understanding from pre- to post-test, $t(30) = -6.987$, $p < .001$, while those who had Video Order 2 decreased slightly (figure 8). Again, participants’ suggestions for teaching focused more on conceptual understanding for one child than the other.
Figure 8: Interaction between video order and mean number of instances of conceptual understanding in response to “what would you do next in teaching?” at the pre- and post-test.

All of this indicated that there was something different about the content of the videos selected to use for the pre- and post-tests. The two videos were initially chosen due to the fact that the children are carrying out the same task with the interviewer; they are about the same age; they were both girls; and the videos were about the same length. How the children interacted with the interviewer, however, was unique to each video. That difference in interaction with the interviewer is evidenced by the drastically different responses to each video.

Further analyses of the two pattern videos used in the pre- and post-test showed them to have drastically different content. Though the tasks being performed in the two videos are identical and the children are about the same age, the two girls display very different amounts and types of knowledge. Using the KMD coding scheme, the content of the videos was analyzed by two trained independent raters with an inter-rater reliability of .90. Where there was disagreement, the raters discussed their differences and agreed upon a single code (table 6). Video 1 contains more instances of declarative knowledge, while Video 2 contains more...
instances of procedural knowledge and conceptual understanding. Additionally, the child in Video 1 is distracted 15 times by other things happening in the room, while the child in Video 2 is not distracted at all. Thus, Video 1 is considered the Declarative video and Video 2 is the Procedural/Conceptual video.

<table>
<thead>
<tr>
<th></th>
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<tr>
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Table 6: KMD coding analysis of the pre/post-test pattern videos

Given these results, the original research question regarding changes from pre- to post-test must be modified to account for the fact that the data cannot be analyzed in that fashion. The most relevant information will come from analyzing the post-test responses for those participants who viewed the Procedural/Conceptual video at the post-test since it had the richer content of the two videos.

Research Question 4 Redux: Are there any differences among the three groups in prospective teachers’ interpretations of children’s behavior at the post-test?

Hypotheses: Looking only at the post-test responses for viewers of the procedural/conceptual video at post-test, it is hypothesized that those in the guided-video group will have more instances of perceived conceptual understanding in response to “what does this child know about pattern?” than the unguided group and the control. It is also hypothesized that the control group will have more instances of perceived declarative knowledge. Since we have a natural inclination to want to show others “how to do” various procedures (Barnett, 1968), all groups are
expected to have to the same amount of procedural knowledge at the pre- and post-test. Thus, instances of procedural knowledge are not hypothesized to vary among the groups.

**Results:** Looking for differences in perceived declarative knowledge in response to “what does this child know about pattern?” a Poisson regression was run using group as the main factor. There were no significant differences among the three group means. A second Poisson regression was run looking for differences among the groups in perceived procedural knowledge. There were no significant differences among the three group means. Finally, looking for differences in perceived conceptual understanding, a Poisson regression was run using group as main factor and there were no significant differences.

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Table 7: Instances of KMD in post-test responses to “what does this child know about pattern?” for viewers of the procedural/conceptual video

**Research Question 4 Redux:** Are there any differences among the three groups when analyzing prospective teachers’ suggestions for what to teach next at the post-test?

**Hypotheses:** Looking only at the post-test responses for viewers of the procedural/conceptual video at post-test, it is hypothesized that those in the guided-video group will have more instances of conceptual understanding in response to “what would you do next in teaching?” It is
also hypothesized that the control group will have more instances of declarative knowledge. Instances of procedural knowledge are not expected to vary among the groups.

Results: A Poisson regression was run looking for differences among the three groups in instances of declarative knowledge in response to the question “what would you do next in teaching?” There were no significant differences. Looking at instances of procedural knowledge, a Poisson regression was run that had no significant results. Lastly, a Poisson regression was run looking at instances of conceptual understanding and there were no significant results.

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Table 8: Instances of KMD in post-test responses to “what would you do next in teaching?” for viewers of the procedural/conceptual video

The goal of analyzing the post-test responses to the procedural/conceptual video was to determine if there were any differentiating transfer effects from the unguided- and guided-video experiences. Unfortunately, there were none to report. The implications of this are discussed in the next chapter.
Chapter 5: Discussion

This chapter is divided into four sections: a summary and discussion of the key findings, implications of the findings, limitations and future directions for the study, and conclusions.

Summary and Discussion of Key Findings

Intervention

The hope in analyzing participants’ responses using the KMD coding scheme was to measure changes in KMD over time and to determine whether the interventions had the predicted effect. Of particular interest were changes in conceptual understanding when both analyzing a child’s mathematical understanding and deciding on the next steps to take as the child’s teacher.

*What does this child know about X?*

It was hypothesized that participants in the guided-video group (i.e. the segmented video version of the intervention) would have significantly greater increases in perceived instances of conceptual understanding in response to the question “what does this child know about X?” than the unguided-video group. This was not the case. Instead the results showed that both versions of the intervention, the unguided and guided, helped participants to identify more instances of conceptual understanding in their observations of the children over the course of the five video lessons. While it was thought that segmenting the videos and asking additional questions would be beneficial to participants, perhaps that is unnecessary. It is also possible that those participants in the guided group grew fatigued by the amount of questions, and therefore did not show a greater increase in instances of conceptual understanding than the unguided group.
Finally, it is also possible that the number of weeks spent participating in the intervention was not enough to cause a greater change in participants’ responses.

Though not hypothesized to change, instances of declarative knowledge in response to the same question decreased from the first to the fifth lesson. Again, both the unguided and the guided versions of the intervention showed this significant decrease across the interventions. This result supports the goal of the intervention: to improve participants’ interpretations of children’s behavior. The decrease in instances of declarative knowledge coupled with the increase in instances of conceptual knowledge illuminates the shift that participants made: they moved from a focus on facts to a focus on trying to identify what the child knows conceptually about the topic. Additionally, as hypothesized there was no change in perceived procedural knowledge from the first to the fifth lesson.

*What would you do next in teaching?*

It was hypothesized that participants in the guided group would show a significantly greater increase in teaching for conceptual understanding after participating in the intervention than the unguided group. This was not the case. In fact, neither the unguided nor the guided groups showed significant changes in conceptual understanding. One possible explanation is that fatigue had set in for the guided participants, as this was the second to last question out of about 17 answered.

There was, however, a significant increase in instances of procedural knowledge in the participants’ suggestions for what to teach next. This was not hypothesized, but was the case for participants in both groups. It is possible that when thinking about teaching, participants were
prone to think more about particular procedural knowledge the children might need, and not conceptual understanding. One explanation is that these are undergraduate students who have not participated in any student teaching opportunities; therefore their knowledge of what to do next is quite limited. While the effect was not significant, instances of declarative knowledge also decreased for both groups.

Generally speaking, participants’ interpretive abilities improved during the course of their lessons. Their use of this interpretive ability to improve teaching, however, was limited to procedural knowledge.

**Age Expectancies**

In analyzing responses to the age expectancy measure, significant decreases from the pre- to the post-test in the *ready to start* learning age were found. These decreases were across all groups, including the control group. While it was hypothesized that there would be decreases for the two intervention conditions, it was not hypothesized that there would be a decrease for the control group. At the time of the post-test, the control participants had only seen the video from the pre-test. Perhaps this decrease can be attributed to general educational experience, as these were prospective teachers who were early education majors and were taking other childhood education courses (though none were enrolled in an early childhood math course).

For the unguided group only, the time needed to learn (the difference between *able to master* and *ready to start* learning) increased significantly in the post-test. Recall that after completing the TUSK pre-test, these participants viewed the pre-test pattern video and then five unsegmented videos of children carrying out various mathematical tasks during the intervention.
At the end of each lesson, they also responded to questions about what the children in the videos knew and what they would do next in teaching these children. Thus, they spent a significant amount of time watching the clinical interviews and thinking about their content before responding to the TUSK at post-test. Looking at their mean responses, we see that their able to master ages hardly changed, while their ready to start learning ages decreased. This indicates that while feeling there is still much for the children to master (hence, the able to master age barely changed), the unguided participants recognized that math learning can take place at younger ages than they had previously expected (the ready to start learning age decreased), causing an increase in the overall amount of time it takes to learn the content. Therefore, participating in the unguided intervention led to a perceived increase in time needed for learning because participants were recognizing that younger children were capable of being introduced to these concepts, but were not necessarily expecting them to learn faster.

Pre- and Post-Test Pattern Video Analyses

The discovery that the counterbalanced videos used in the pre- and post-test were drastically different in content was unexpected. As mentioned before, they were chosen based on their similarities in tasks, children’s ages and gender, and length. Post-hoc analyses of the video content exposed the extremely different interactions the two children have with the interviewer and their different levels of understanding. This difference, unfortunately, is what prevented any successful pre-post analyses. Therefore, I chose to analyze post-test responses to the procedural/conceptual video only, as it contained the richer content of the two videos, offering the prospective teachers more opportunities to identify the child’s conceptual thinking.

Looking at the control, unguided, and guided groups’ responses to “what does this child know about pattern?” and “what would you do next in teaching?” there were no significant
differences found in perceived instances of the three KMD codes. Though none was significant, some of the results were in the direction of the original hypotheses: the control group had the greatest number of perceived instances of declarative knowledge in response to “what does this child know about pattern?” while the guided group had the greatest number of perceived instances of conceptual understanding.

Unfortunately, there were no statistically significant results showing that the intervention lessons resulted in learning that transferred to the post-test pattern video. We must remember that this was a relatively small-scale study: there were only 32 participants who viewed the procedural/conceptual video at the post-test and the entire study took place over the course of four weeks. Research on successful professional development shows that an extended duration is necessary in order for it to be effective.

Implications of Key Findings

Research has shown that “mathematically stronger teachers can do many tasks beyond simply solving problems in front of students. These tasks include sensibly interpreting and responding to student mathematical productions and producing more conceptually grounded mathematics lessons” (Hill, 2010, p. 514). There is, however, a need for a more detailed description of what teachers’ mathematical knowledge for teaching actually entails—specifically their ability to interpret children’s behavior and then link their interpretation to specific teaching plans. This is especially the case if programs and policies are being created that call for an increased focus on mathematics in early childhood education.
This dissertation study sought to measure aspects of teachers’ professional knowledge about students’ minds and how learning takes place in those minds through the use of a series of videotaped clinical interviews. These interviews show children working with particular mathematical concepts one-on-one with the adult. Not only does the content of the videos provide the viewer with information about the development of mathematical thinking in children, but it also provides them with ideas for what successful pedagogical techniques look like and how effective they can be.

Just as interpreting children’s strategies requires knowledge of those strategies and the research on mathematical development, deciding how to respond based on a child’s behavior exposes what the teacher understands about the child’s strategy, as well as her understanding of mathematical development. Thus attending to children’s behavior, interpreting children’s understandings, and deciding how to respond based on what the children understand are all part of the teacher’s ability to teach for mathematical proficiency. In responding to “what does this child know about X?” and “what would you do next in teaching?” this study attempted to measure this ability. I found that participants of both the unguided- and guided-video experiences displayed a tendency to perceive more conceptual understanding in the children’s behavior by the final intervention lesson. Additionally, those same participants emphasized teaching for procedural knowledge by the final lesson, as opposed to declarative knowledge or conceptual understanding. The results of this study showed that it is possible to use videos to improve prospective teachers’ interpretive abilities, but not their ability to apply the interpretations to teaching next steps.

The implications regarding the use of video for professional development purposes speak to both the importance of the structure of the video, as well as to the video’s content. While it
was hypothesized that the guided-video version of the intervention would be more effective because it emulated the structure of the DMT course (through the pauses, questions, and sample feedback), the findings did not fully support that argument. In fact, both the guided and unguided versions proved successful at increasing participants’ perceived amount of conceptual understanding in the children’s behavior. Going forward, it is important to think about what these results imply about the use of videos used for professional development. How critical is the element of discussion (as emulated by the questions and sample feedback in the guided-video experience) to the learning process? This is something that warrants further exploration with a longer lasting intervention.

Additionally, it is important to think about how the types of video segmentation may impact what the prospective teachers learn from the videos. The decisions on where to place the breaks in the videos to answer the questions were made by an expert in the fields of cognitive and developmental psychology. This expert has worked intimately with the videos used in the lessons, and is therefore very knowledgeable of their content and how best to present them to the viewers. While it could be argued that each viewer may perceive the breaks differently, the strength of the guided-video methodology comes from the fact that these sections of video were expertly selected to represent important aspects of the video content. The issue of where to place the pauses could be clarified through further exploration of classroom pedagogy involving video.

As we saw, the types of mathematical knowledge displayed in a video and the amount and quality of interaction with the adult can greatly impact what prospective teachers interpret the child’s behavior to mean. If the goal is to teach about the importance of knowing what, knowing how, and knowing why in the context of teaching for mathematical proficiency, then the content of the clinical interviews needs to address each of those in a clear and impactful way.
Additionally, if teachers’ expectations influence the tasks they select, the questions they ask, and the encouragement they give (Kilpatrick et al., 2001), then one of the main goals of the videos should be to show how capable young children are so that those expectations can be changed.

This study sought to establish a way to gauge not only prospective teachers’ expectancies and beliefs about children’s mathematical development, but also their knowledge of mathematical development. Having such information would provide researchers a way to determine the effectiveness of their professional development efforts. While this study’s attempt was not completely successful, it did establish one way to measure participants’ KMD when responding to open-ended questions during the interventions. This method can inform future research; the constructs on which it was based are in line with others’ recommendations for measuring the success of professional development programs.

**Limitations and Future Directions**

The greatest limitation to this study was the difference found in the pattern videos used for the pre- and the post-test. While they were initially chosen due to their similarity in mathematical task, children’s age, gender, and video length, there was a qualitative difference in the types of mathematical thinking exhibited by the children. While one video was full of instances of conceptual understanding and procedural knowledge, the other was full of instances of declarative knowledge. These differences made it impossible to analyze changes in the way participants responded to “what does this child know about pattern?” before and after the intervention. Future studies would need to use clinical interviews with equivalent amounts and kinds of KMD exhibited by the children.
Recognizing that there are differences in the amount and quality of mathematical knowledge exhibited by children during clinical interviews, future research could also explore how teachers respond to this variability in children. In particular, would there be a difference in suggestions for what to teach based on a child’s displayed mathematical knowledge? Research has already found that prospective teachers’ beliefs about their students’ abilities can interfere with their pedagogical decision making processes. Does the teacher of a child who only exhibits declarative knowledge focus on declarative knowledge when teaching that child, instead of encouraging deeper understanding through concepts? Or vice versa, does the teacher of the child who exhibits strong conceptual understanding focus only on conceptual understanding when teaching that child? Further exploration through the use of clinical interviews portraying a wide variety of children’s abilities could help answer these questions.

Additionally, just as student motivation plays a role in the outcomes of internet-based learning (Artino, 2008; Saadé, He, & Kira, 2007), so too might teachers’ motivation for web-based professional development. While web-based professional development has its advantages (convenience for the teacher, ease of use, and access to streaming content from the internet), there are disadvantages, most importantly the lack of face-time with other participants and the experts running the program. Collaboration and communication among participants is an important element to many existing professional development programs. Ultimately, motivation to use an internet-based program warrants continued exploration.

The guided-video experience could be improved upon in many ways. To begin, I would have participants watch a short clinical interview wherein the child’s response to the adult’s question visually exposes the child’s thinking (e.g. the child is asked how high he can count and to answer this, he stretches his hands above his head and starts to count from 1 on his fingers).
Seeing this video opens the viewer’s eyes to how insightful a conversation with a young child can be, which I believe would prepare them for the guided lessons they are about to complete. Once they have seen this video, they would then proceed to the guided lessons. However, rather than have them answer questions at each pause, I could include a video of an expert discussing what we just watched in the segment. Thus, instead of providing participants with sample student responses that vary from good to bad without any indication of which one is which, an expert would provide them with a clear and concise interpretation of the video content. This might help blend elements of a classroom experience (i.e. the professor talking to the class) into the web-based experience. Additionally, outside of a research context and in the context of a professional development program, I would create a forum for discussion among the participants so that everyone could share in the experience together. Ultimately, the guided experience needs to translate many elements of a high-quality classroom environment into its web-based format, which is no easy feat.

Conclusions

NCTM and NAEYC’s joint position statement (2002) illuminated the pressing need for high quality, challenging, and accessible mathematics education for young children. Sadly, the current state of teacher training in mathematics is dismal, both for pre-service and in-service teachers: many students continue to be taught by teachers with shaky grasps on mathematics (Kilpatrick et al., 2001). Making matters worse, research has shown that entering the workforce ill prepared can lead to a situation in which teachers’ uninformed beliefs about children’s learning result in teachers misjudging their students’ abilities, leading to misguided instruction
(Brown, 2005). This is an unfortunate reality in classrooms today. Thus, something needs to change and it starts with how we train our teachers to understand their students’ thinking.

Because there is a need for research exploring the mathematical knowledge needed for teachers and the means necessary for them to acquire, apply, and use this knowledge (Bowman et al., 2001), this dissertation study sought to investigate the impact of two versions of a video-based professional development program, as well as explore a proposed model for prospective teachers’ knowledge of mathematical development. Thinking back to the Development of Mathematical Thinking course, which served as the basis for the guided and unguided intervention models, it is important to keep in mind the three main goals: to create awareness of young children’s capabilities, to learn how to assess knowledge and the development of mathematical thinking in children, and finally to teach how to skillfully select appropriate activities based on their understanding of the children’s thinking. Each of these is a critical component to a teacher’s ability to successfully teach all of her students. Without knowledge of how children think, teachers will continue to fail to see the importance of mathematics in early childhood education. And, without knowledge of how teachers think about their students and their teaching, researchers will fail to create effective professional development programs. As Shulman put it, “Those who can, do. Those who understand, teach” (1986). It is our job to make sure that this understanding is both comprehensive and generative for pre-service and in-service teachers alike.
References


Appendix A

Learning to Count

Saying the number words “One, two, three…” or even “One thousand fifty six, one thousand fifty seven, one thousand fifty eight…” seems to us so easy and trivial that we fail to appreciate how complex is the task of learning them in the first place. For children, learning number words that they frequently hear poses several difficulties. It involves both memory and creativity; the child must first memorize the small basic numbers, and this having been accomplished, can then learn rules for generating larger and even novel numbers.

One of the first aspects of number children try to learn is the ordered counting words, “One, two, three….” Virtually all children engage in this learning process and typically encounter two distinct problems. First, there are a lot of numbers to learn. Second, they must be said in a certain way and not others. Children solve the first problem—the apparently endless number of number words—by limiting what they try to learn at any one time. Thus, they usually struggle with one through five before they try six through ten. They carefully regulate their own learning and attempt only a little more than they can master at any given time. The second problem is more difficult. Why is it wrong to say “one, three, two” but right to say “one, two, three?” Gradually, children resolve this difficulty too and with practice learn the sequence of number words slowly. They begin to see a pattern in the apparent chaos of number words. They perceive that numbers are like a song: the number words involve sequence of sounds in the same order all the time. Parents try to make the task simpler by imposing on it some rhyme: “One, two, buckle my shoe; three, four, close the door…” This is intended as a device to reduce the load on brute memory and thereby make the numbers easier to learn. Young children make a great discovery when they learn that “two” always comes after “one” and “three” always after “two.” Unfortunately the beginning of the sequence—in English, the first 12 or so numbers—is completely arbitrary. There is no rational basis for predicting what comes after a certain number. Children have a lot to memorize, and it’s not until they are about three years old that they learn to say the beginning numbers in the proper sequence (Baldwin & Stecher, 1925).

After a period of time, they discover that the numbers after about 13 contain an underlying pattern. Using it, children develop a few simple rules by which to generate the numbers up to about 100. We can ourselves get some appreciation for the young child’s difficulty by performing a little experiment on our knowledge of Jack and Jill. “Let us ask a friend to test us in this fashion: What word comes before down? What word comes after his? Go through the verse saying only every second word. Say only every third word. Say the whole verse backward.” The task is quite hard, and it is the same kind of task children have to do with “one, two, three.”

Children also acquire some beliefs about the number words. At the age of three and a half, Josh demonstrated that he could count to 12 without error. The interviewer then asked Josh to listen to him (the adult) count, and indicate if any mistakes were made. The opportunity to correct an adult is no doubt novel for a young child, and so Josh listened with intense interest.

Interviewer (I): Listen to me now and tell me if I make a mistake. One, two, three, red.
Josh (J), interrupting with a laugh: No, red was color.

I: OK, I’ll start again. One, two, three, four, five, seven.

J, laughing again: No. You’re wrong!

I: Sorry. One, two, three, four, five, six, seven, eight, nine, ten, nine, eight.

J: No, you made a mistake. After ten comes something else.

I: OK. I’ll do it right this time. Nine, ten.

J: No. That’s wrong. You have to start with “one.”

So Josh not only learned to say the number words in the conventional order, but also believed that number words are different from color words; that you cannot skip numbers; that numbers cannot be said in backwards order; and that you must always start counting with the number “one.” Children do not learn simply to execute behavior; they also develop theories about their behavior.

A few general principles about learning to count:

1. Children search for meaning. Children try to make sense of the world by looking for an underlying pattern, for a deeper meaning. Sometimes it does not exist, as in the case of the first 12 or so numbers. But often it does: the larger numbers display clear underlying regularities. Having searched these out, children use them to develop rules for producing the larger numbers themselves. The exploitation of underlying patterns makes intellectual work easier and more efficient, and allows children to avoid the drudgery of rote memorization.

2. Errors are meaningful and informative. Children’s errors usually make sense. Often they provide insight into what children are really trying to do. Thus, they say “twenty-ten” because they are trying to capture the underlying structure of the counting numbers, not because they weren’t thinking.

3. Children can use different learning strategies depending on environmental circumstances. When there is no underlying pattern, children memorize the numbers in a rote fashion. When the pattern is there to be found, they often exploit it. Some learning is done by rote; some is meaningful. Children do not learn in only one way.

4. Children can learn in a wide range of circumstances. Children learn a great deal about numbers outside of school, without instruction or special help; indeed their parents are often unaware that they are trying to learn numbers. Also, they manage to learn even though their experience is often confusing and unplanned. For example, children may hear an adult counting by twos or by fives before they have mastered counting by ones. Nevertheless, they manage to learn.
COUNTING

This lesson will focus on the topic of counting. You will watch a series of short video clips and answer questions about them. After you submit your responses, you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Lateek, age 4. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

Clip 1 (00:32-00:53)

1. **Watch the video closely two times. What stood out to you in this clip?**

2. **What does Lateek know about counting?**

Here are some comments from other students:

- He counts using his fingers.
- He appears to know that each number goes with a different finger.
- He skips numbers 7 through 10.
- He picks up at 11 and counts 11, 12, 13, 14, and 16, omitting 15.
- He counts most numbers without hesitating, although he hesitates before 14.
- He appears confident in his counting abilities.

Clip 2 (00:48-00:59)

3. **Lateek skipped the number 15 when he counted. Why didn't the interviewer correct him?**

Here are some comments from other students:

- The interviewer was most likely demonstrating a particular teaching approach.
- Perhaps she is planning to continue to test his knowledge in other ways, thus she avoids correcting and teaching him at this time.
- She may want to facilitate a positive learning environment, free of corrections at this point.

4. **Do you think the interviewer should have said "good job" or not? What would you have done?**

Here are some comments from other students:
• Saying "good job" is a way for the interviewer to build rapport with the student and provide encouragement.
• It is important to facilitate a neutral learning environment where the child is not feeling evaluated on a continuum of good or bad.

5. What do you think of her question, "Can I start counting?"

Here are some comments from other students:

• By asking the child permission to count, perhaps she intends to create a rapport in which the child perceives he is on equal footing with the adult.
• She is giving him the chance to say "no," which could be a potential problem if she wants to continue the interaction.
• Another way to phrase this question might be as a statement: "Now it's my turn to count."

Clip 3 (00:54-01:16)

6. What does the interviewer do here? Why? What would you have done?

Here are some comments from other students:

• She counts without using her fingers, but skips the numbers between 6 and 11 as he did, perhaps to see if he would accept this version as correct.
• She includes 15, which he omitted.
• One thing teachers can learn from this interaction is that it's OK to make mistakes in front of children.
• Mistakes can provide room for interesting discussion about the mistakes and possibly reveal a lot about a child's reasoning.

7. How does Lateek react? What does it mean?

Here are some comments from other students:

• He seems to count along with the interviewer, sometimes mouthing the words that go with the numbers.
• When she jumps from 6 to 11 his face changes expression. Perhaps he recognized that something was missing, or perhaps he is getting ready to express the bigger numbers?
8. Now we are going to focus on what the interviewer did. What do you think about the question she asks? What would you have asked?

Here are some comments from other students:

- She asks, "Is that good?" to invite him to react to what she did, without giving away her own opinion.
- By asking for his opinion, she is trying to get him to respond to her answer and encourage discussion about his ideas.
- She is putting him on equal footing with her.
- I may have asked, "Was that right or did you notice anything wrong" to initiate a discussion about the missing numbers.

9. What did Lateek mean that you count "one 16"?

Here are some comments from other students:

- He said that you should count 16 only once.
- He is instructing her to count each number only once and may have thought she said 16 twice.
- Perhaps he does not know 15 is a number in the counting sequence and thinks 15 and 16 are the same number when he listens to her count.

10. Why did the interviewer accept his answer? What do you think would have happened if she had challenged him? Would you have approached this situation differently, and if so, how?

Here are some comments from other students:

- She is verifying his comment and perhaps giving him a chance to correct her.
- Had she disagreed with him, she may have learned more about what he does or doesn't know about 15.
- Note that she interpreted his statement to mean that she said "16" twice.
11. What is the interviewer trying to find out? What does Lateek know about counting?

Here are some comments from other students:

- The interviewer is trying to figure out if Lateek recognizes that he skipped number 15.
- By his correction, it appears he thought it was OK to skip those numbers or still does not realize the mistake in his counting.
- Lateek knows that each number occurs only once in the counting sequence.

Clip 5 (01:27-01:49)

12. What do you think of the interviewer's questions? Why does she ask if it's OK to use her fingers?

Here are some comments from other students:

- The questions involve him in what she is about to do and make him feel like he's in charge.
- The interviewer is exploring his competence and his reactions to her counting.
- She asks to use her fingers perhaps to see if he might express any ideas about using fingers to count.

Clip 6 (01:43-02:01)

13. What is the most important thing that happened here? How do you explain it?

Here are some comments from other students:

- At the number 6, he took over the counting from the interviewer, and counted correctly from 7 to 10.
- It turns out he may actually know his numbers through 10, but just needed a little help to get past 6, or needed someone else's fingers to guide him.
- He may have just remembered these numbers after several trials.

14. In what ways, if any, did you see the interviewer help him with 7?

Here are some comments from other students:

- She did not actually help him with 7.
• Starting with 6, she gradually stopped counting and let him take over.
• At first she is simply modeling counting, but as he starts to participate, she drops out.
• Watching her fingers (instead of his own) may have helped him.

15. What did Lateek do this time? Why do you think he was able to do this?

Here are some comments from other students:

• He was able to count to 16 successfully, missing only 15.
• Practice seems to have helped him gain confidence in counting to 10.
• He stopped trying to use his fingers after 10, which may have helped him count to 16—although he continues to skip 15, perhaps as a duplicate of 16 and still not aware that 15 and 16 are different numbers, or that 15 even exists as a number.

16. What did the interviewer do? Why do you think she did this?

Here are some comments from other students:

• She counted along with him, but said the number words only after he did, probably to avoid guiding him too much.
• Her technique is a good idea for teachers because the student feels supported while the teacher is not giving away the answer.

17. How would you interpret what he did here? What would you do next?

Here are some comments from other students:

• He probably interpreted her expression about number to mean "height" rather than higher numbers.
• This is a humorous example of how a child's egocentrism leads them to interpret a question from their own point of view.
• After his response, I would clarify the question by asking, "How high can you count without using your fingers?"
18. What does Lateek know about counting?

Here are some comments from other students:

- At first, it appears Lateek does not know how to count correctly to 10 or 16, but with repeated attempts, he demonstrates more competence.
- Children's performance can be variable at this age and with repetition and instructors guidance, they may reveal knowledge or skills not previously seen.

19. What does this video tell you about assessment?

Here are some comments from other students:

- It's important to be patient and give the child multiple opportunities to express him or herself.
- Sometimes a poor performance doesn't mean poor knowledge.

Clip 10 (00:00-03:01)

20. Now watch the video clip again in its entirety. What do you think this child really knows about counting? Explain how you know.

21. What would you do next in teaching? You can watch the video again.

22. What did you learn from this exercise? How did your thinking change during the lesson?

Congratulations, you have completed the lesson on counting. Good job!
Enumeration: Learning to Count Things

For several years, young children engage in the pleasant struggle of learning to attach the numbers they can already say to the objects they can see. They begin by learning to count small numbers of objects. Then they must repeat the learning process with larger numbers. The typical preschool child has no difficulty in counting very small collections, but when a collection’s number surpasses a certain value (which varies with age), the child’s enumeration falls apart. For example, Vicky, a 4 year old, counted two objects accurately, and then four objects accurately, but when the two and the four were combined she enumerated in a very unsystematic fashion and got ten for an answer. Indeed, her behavior appeared so incompetent when she was counting the larger collection that she seemed like a different child from the one who counted the small collections with ease and accuracy. When larger numbers were involved, she seemed not to understand enumeration at all.

Vicky is not atypical. Young preschoolers achieve sufficient mastery of these concepts and skills to enumerate small sets with considerable success. But they make mistakes in enumerating larger sets, mainly because they are not skilled at considering things once and only once. This is hard for them because they lack a systematic plan for keeping track of things and therefore rely on rote memory, which soon becomes overburdened. Young children’s enumeration of “small” sets is often quite accurate, whereas their counting of “large” sets involves sloppiness and inconsistency. The first time children count a large set, they may get one answer; the next time, another. They do not know which answer is right, or may even think that both are. At 4-6, Samantha was presented with a collection of candies randomly arranged on a table. Her job was to count the candies. A very precocious girl, she had no difficulty in saying the number words; indeed, she could easily reach 100. Yet in counting the candies, she made many errors. On one try, she would get 23; on another, 24; on yet another, 22. Which was right? She had no idea. Her procedure was to point to each candy in its original location; she did not bother to push any candies aside after counting them. Because of this, she forgot which were counted and which were not. She counted several candies twice and several not at all, and as a result got different results each time. This inconsistency did not disturb her in the slightest. Children sometimes believe the same collection can be characterized by two or more numbers: yes, it has 14, and it also has 15!

Why are there so many errors on a task that to us seems trivially easy? One reason is that young children seem to forget which items have been touched and which have not. They lack a systematic plan for making sure that each picture has been considered once and only once. Using a haphazard procedure, they touch the circle first, then the triangle, then the trapezoid, and so on. Doing this puts a great strain on children’s memories; to be correct, they must remember exactly which items have been counted and which have not. For example, on the fourth choice, they must remember that they have already touched the circle, triangle, and trapezoid. Since there is so much to keep in mind at the same time, they frequently forget and therefore count some things twice and some not at all. Young children are not very methodical or organized. They do not make it easy for themselves to consider things once and only once.

To count things accurately, children must learn how to do at least the following:

1. To realize that you can count anything, objects or even ideas. You can count peas or elephants; you can even count the number of candies you ate yesterday or images of unicorns.
2. To say the number words in their proper order. You cannot count things accurately if you do not know the number words. You will be wrong if you say, “One, three, four, seven…”

3. To count each member of the relevant collection once and only once. That is, you must not count one object two times or forget to count an object.

4. To match up each number word with each thing. The word “one” has to go with this object and “two” with that object. “One” cannot go with both. Each object must be assigned one and only one number word.

We will see that children master many of these principles at a very young age, and can apply them successfully to relatively small collections. Problems occur mainly when larger numbers are involved.

In summary, young children spend several years learning how to attach numbers to things. Counting is a complex activity requiring several concepts and skills. The child must realize that anything can be counted, objects or even ideas; must have the ability to say the number words in their proper order; must be able to count each member of the relevant collection once and only once; and must be able to match up each number word with each thing. Young preschoolers achieve sufficient mastery of these concepts and skills to enumerate small sets with considerable success. But they make mistakes in enumerating larger sets, mainly because they are not skilled at considering things once and only once. This is hard for them since they lack a systematic plan for keeping track of things and hence rely on rote memory, which soon becomes overburdened. With development, children overcome these difficulties, first with smaller numbers and then with larger ones. They develop several useful strategies for counting. They learn to push objects aside as they count, thereby reducing the strain on memory and making it easier to count things once and only once. Perhaps partly as a result of repeated counting, they learn to perceive small numbers directly and no longer need to count them. Then they learn shortcuts for counting: instead of considering individual objects, children group them, counting for example by twos or using arithmetic operations to get the totals. Gradually children learn to enumerate absent or imaginary objects, often by enumerating various substitutes for real objects: images, fingers, or written symbols.
This lesson will focus on the topic of enumeration. You will watch a series of short video clips and answer questions about them. After you submit your responses, you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Harry, age 4. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

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Clip 1: 00:01-00:05

1. **The interviewer asks, “Can you count how many red bears there are?”** These three highlighted words/phrases might play an important role in the child's interpretation of this question. What is one of the possible meaning of each?

Here are some possible interpretations of the highlighted text:

- The word “can” may connote either a request or a question about whether the child is able, and may not elicit the response intended by the interviewer (if the child answers "yes" or "no").
- The word “count” most likely refers to the process of enumeration.
- The phrase “how many” refers to cardinality and figuring out the total value of the set.

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2. **How do you think the child will count the bears? What method(s) might he use to get the answer?**

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Clip 2: 00:06-00:15

3. **He was clearly trying to count. What else did you see?**

Here is what a student saw in this clip:
• He started out counting, "1, 2 ..." and then paused for a moment. He decided to start again, and after "2" he proceeded directly to "3 ... 4, 5, 6." He clearly did not count two objects in the group.

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Clip 2: 00:06-00:15

4. Now watch the clip again. What do you make of the above interpretation? Were there any important things you saw this time that you didn't see before?

Here is what an "expert" saw in this clip:

• Here are a few things that I saw that might be important. Harry started out by counting "1, 2," and then paused for a moment. He decided to start again, and after "2" he proceeded to count "3 ... 4, 5, 6." He touched each object in order, and he said the numbers correctly from 1 to 6. At the end, he said two numbers that were out of place, one smaller than the total and one larger. He is very careful about touching each object and counting it.

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5. At the end, he said "3" and then "10." Why do you think he did this?

Here is how an "expert" interpreted this clip:

• Harry may have been confused by the arrangement of the bears, as they were not laid out in any order. As he moved on after the second bear, he may have drawn a conclusion as to an order in which he would count the bears, and therefore he started over. When he reached the last bear, he must have been guessing at a number, as he did point to the bear, but maybe didn't know the number after "6."

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Clip 2: 00:06-00:15

6. What does this clip say about Harry's enumeration skills? What about his understanding of number? Watch the clip again if you like.

7. At this point, what would you do next? What task would you use and/or what question would you ask?
In summary, one could say that Harry is intent on listing each bear once and only once, and he seems to have a good grasp of the sequence of numbers 1 through 6. He may or may not understand that the last number counted tells how many there are altogether.

As for a follow-up task, one could ask him to count again and be more careful, give him a smaller number of bears to count, or have him line them up for more systematic counting.

Clip 3: 00:18-00:22

8. What did the interviewer do before asking another question? Why?

Here is how an "expert" interpreted this clip:

- The interviewer cleared away the bears that Harry had been counting and removed two, presumably to test something about the hypothesis that smaller numbers are easier to count, and would therefore give a better measure of Harry's competence with enumeration than the larger set would.

The factors to consider here are; the quantity of objects used (why 5?), and the phrasing of the question to get the total number versus just counting:

- Can you count these for me?
- How many are there?
- Can you count to find out how many there are?

Clip 4: 00:19-00:27

9. What did you see?

Harry points to each bear as he counts, 1 to 5. He has one-to-one correspondence and is consistent and accurate with the number sequence up to 5. He says five with some authority.
10. What does this clip say about Harry's enumeration skills? What does he understand about number?

He seemed to enumerate OK when the number was smaller, even though the objects were in a haphazard arrangement.

Clip 5: 00:28-00:47

11. Why do you think Harry proposes putting the bears in a line?

To organize them, which makes a simpler task of one-to-one correspondence while counting.

http://www.columbia.edu/ccnmtl/draft/video/vital_nsf/video/enumeration/06.mov

Clip 6: 00:48-01:02

12. Why did the interviewer ask the question, "How many bears did you have before ... when you counted them?"

To confirm whether Harry remembered the last number counted. If he did, then it is possible to determine whether he knows that it indicates cardinality; if he doesn't, then any response is ambiguous.

Clip 7: 01:03-01:12
13. Why did the interviewer rearrange the bears?

To see if Harry understood cardinality -- that is, that the last number indicates the value of the set as a whole -- and therefore could answer without counting. This is also sometimes referred to as conservation of number.

Clip 8: 01:13-01:22

14. Why did the interviewer cover the bears?

To see whether Harry did not need to count each time he is asked how many. If he understands cardinality, he does not need to count again and could rely on the last number counted if he can remember it.

Clip 9: 01:22-1:38

15. Why did the interviewer ask this question? What is another way to find out what Harry knows?

The interviewer may have been trying to determine whether Harry did in fact remember the last number counted and would use it as the cardinal number.

Clip 10: 00:00-01:38
16. Now watch the video clip again in its entirety. What do you think this child really knows about counting and number? Explain how you know.

17. What would you do next in teaching? You can watch the video again.

18. What did you learn from this exercise? How did your thinking change during the lesson?

Congratulations, you have completed the lesson on enumeration. Good job!
Simple Arithmetic: Learning to Calculate

Children’s initial approach to addition and subtraction is concrete. At the age of 3 or so, children cannot deal with addition and subtraction problems unless they have concrete objects immediately before them. Such problems typically take the following form. The child is presented with a toy rabbit that has two toy carrots, right in front of it. Another toy rabbit is introduced that has three carrots, again clearly visible. The child is told how many carrots each rabbit has and asked how many carrots the rabbits have “all together.”

There are of course many possible variations on these problems. The story might involve, for example, a rabbit that has three carrots and then gets two more. The child can then be asked, “How many does he have now?” But in all cases, the key feature is that the problem involves objects the child can see and simple instructions using everyday words like “all together” and “take away,” not “addition” and “subtraction.”

Given problems like these, very young children—2 and 3 year olds—may not even interpret the problem correctly. They apparently do not realize that the answer can be obtained by combining the sets (literally, or by counting), and therefore do not attempt to count them. By about 4 years of age, some children are able to interpret the problem correctly and achieve a solution by counting the objects one by one, although they are not always successful. If there are three objects plus two more to be counted, the child begins by counting the first set—“one, two, three”—and then continues with the second—“four, five.” This technique is called “counting all.”

For the young child, subtraction also involves a process of counting objects. Suppose the problem is: “Johnny had five candies and lost two of them. How many does he have left?” In this case, the child actually removes two of the five candies and simply counts the remainder to get the result.

The development of early calculation has several interesting features. One is that there is a thin line between the child’s counting and addition (or subtraction). For the young child, addition is simply an extension of counting objects (enumeration). From the child’s point of view, adding is the counting of things in two (or more) sets combined. That is, if you want to know how many objects are in some sets, count all of the objects in all of the sets. This may seem very simple and very concrete, and it is. But this approach to addition is quite legitimate: it is an informal expression of the interpretation of addition as the union of sets. Subtraction too is tied to counting: Subtracting is the counting of what is left after things have been removed from a set.

A second interesting point is that over time the young child spontaneously invents strategies like “counting on,” which are more efficient and easier to use than the “counting all” strategies employed at the outset. Note that this development often occurs without the benefit of instruction or adult intervention. Indeed, most adults are not even aware that the child is constructing any strategies for addition. Intellectual development in the natural environment (but not necessarily in school) is characterized by a trend towards mental efficiency and economy. On their own, children tend to develop more and more effective mental procedures.

A third point is that the child’s early calculation suffers from severe limits. For one thing, it is often not very accurate. The child may know generally what to do but frequently makes mistakes in
implementation, so that answers are wrong. For another thing, the child does not yet fully understand the relations between addition and subtraction. At 4 years of age, Andy was given a series of concrete addition and subtraction problems. The first problem involved adding four candies to a set of three. Using a counting all method, he easily determined that there were seven candies all together. Now what happens when you take away three candies? He took away three and counted the results, getting four again. He did not spontaneously make the inference that if you put back what you took away you will get the amount that you started with. Then the interviewer repeated the process. What happens if you take away three? Again Andy counted. And what happens if you put back three? Again Andy counted to get the result.

Why does this happen? The preschool child seems to be struggling so hard to execute each of the operations separately that he does not examine the relations between them. If his attention is exclusively centered on doing correct addition, he cannot easily see how it relates to subtraction. And if he does not see that addition undoes the subtraction, then he has to compute each result separately each time. The young child must do unnecessary work because, mired in the operational details, he has missed the larger picture. Only as the child becomes more and more skillful at the operations of addition and subtraction, does he gradually learn to perceive the relations between them. In this case, efficient performance seems to be a prerequisite to basic understanding.

Suppose that children are asked to add imaginary objects: how many are three Martians and two Martians? Now children cannot simply count objects that they can see in front of them. Instead they have to represent the imaginary objects. They have to use something—fingers, or blocks, or mental images, or tallies—to stand for the imaginary objects. The children may then perform the arithmetic operations on the representations, such as by counting all. That is, since it is not possible to work with imaginary objects directly, the child must create substitutes or surrogates for them, which may then be manipulated in various ways.

Summary

In the natural environment, children are frequently faced with problems that adults can solve by means of written mathematics. Lacking this, children gradually develop from their intuition and their counting skills a practical arithmetic. This involves at least two steps: the interpretation of problems and the implementation of a solution. Given concrete objects to add, 2 and 3 year olds may not interpret the problem correctly: they may not even realize that it is necessary to combine the objects in some way. By 4 years of age, the spirit is willing but the flesh is weak: many children realize that things must be combined but are not always good at doing the necessary counting. By 5 years of age, concrete problems present little difficulty and children even invent shortcuts (like counting on from the larger number) to make the calculations easier.

Children also develop proficiency in dealing with imaginary objects. At first, the attempted solution is to represent the imaginary objects by real things and then count them up. Another solution is finger counting. Cross-cultural research shows that counting methods are extremely widespread, and history shows that they have taken elaborate and effective forms; indeed, finger counting was once considered the mark of an educated man. Eventually, children may bypass the use of concrete substitutes and carry out counting operations on the mental level. They also learn to use elementary methods of written symbolism, like tallying, although the use of written numbers presents special difficulties.
Children are quite comfortable with their self-developed practical arithmetic. Such procedures are extremely widespread, and can be put to good use.

**Principles**

1. Before entrance to school, almost all children engage in arithmetic problem solving of an elementary nature. In the natural environment, children develop informal ways of dealing with problems of addition, subtraction, and perhaps multiplication. It is not true that schooling introduces children to arithmetic; their practical, non-written arithmetic originates earlier.

2. Counting forms the core of children’s practical arithmetic. Children add and subtract by counting, sometimes on their fingers. Their practical arithmetic depends on counting as its basic computational technique.

3. Written work presents special difficulties. Children find it hard to represent on paper collections of objects and especially events taking place over time, like adding and subtracting. Children have difficulty in using even so elementary a device as the tally. It is easier for them to count objects or even mental images than to work on paper.
This lesson will focus on the topic of simple arithmetic. You will watch a series of short video clips and answer questions about them. After you submit your responses, you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a girl named Rachel, age 6. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

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**Clip 1**

1. **What did you observe about Rachel's response to the question?**

2. **What was the method she used to get the answer?**

3. **Can you think of any other ways she might have solved the problem?**

Here are some hypotheses offered by other students. You may agree or disagree with what they wrote.

- "She knows how to do mental subtraction and mental addition."
- "She makes a quick calculation in her head to solve the subtraction word problem."
- "She can imagine these pictures in her head and that is how she was able to get the answer."
- "She may already be familiar with basic subtraction facts, to the point where giving answers is rote."
- "She was just repeating the number that she last heard the interviewer say."
- "She was using her fingers in order to reach a solution. I noticed that one of her fingers moved slightly."

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**Clip 1 again**

4. **Three students proposed the following hypotheses to explain what Rachel did.**

**Student A:** "Rachel most likely has her number facts memorized. She probably learned the subtraction fact '6 - 3 = 3' in school. This question is easy to her because she has easily retrieved it from her memory. There was no counting involved in her mind."

**Student B:** "[deleted "Obviously,"] Rachel had the carrots pictured in her mind. She mentally moved three of the carrots away from the group of six that she was picturing, and then counted out the carrots that were left. Of course, she did this rather quickly because her response was fast, but I could tell that she 'saw' them in her head because her eyes flickered a bit. She was also doing some of the visual calculating..."
as Prof. Ginsburg was finishing the question; in this way, she was predicting the question he was about to ask.

**Student C:** "Rachel probably has '3 + 3 = 6' memorized, but not the related subtraction fact. However, she was able to reason about this problem enough to do the reversal in her mind, and came to the conclusion that '6 - 3 = 3'. Therefore, she had a related problem memorized that allowed her to come to the answer very quickly, with the help of a little logical thinking."

Do you think that these hypotheses are plausible, based on the evidence they used? Why or why not?

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5. If you were the interviewer, what would you do next?

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Clip 2

6. Now what did you observe about Rachel and how she got the answer to this problem?

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Clip 3

7. Does this support your previous hypothesis about Rachel's thinking and understanding? Please explain.

Here is an example of a successful response to this question: "After watching the video clip I learned that Rachel was doing something more complicated than I initially thought. She didn't simply memorize or count; instead, she reasoned to get the answer by using the inverse. In summary, she knew 3 + 3 = 6, and she also knew the inverse, 6 - 3 = 3." So it turns out she didn't get her answer by memorizing or counting!

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Clip 4

8. How do you think Rachel got her answer?

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Clip 5

9. What do you think Rachel meant by "I did the same thing"? What would you ask next?

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Clip 6

10. Describe and evaluate Rachel's solution.
Rachel did not correctly remember the number fact 4 + 3. Nevertheless, her reasoning was very sophisticated; she used the same inverse principle as before.

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**Clip 6**

11. What would you do next if you wanted to help her learn the way to get the answer?

12. What do you think Rachel will do next?

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**Clip 7**

13. What was the interviewer trying to accomplish by giving Rachel the blocks?

14. How did she learn from this?

Rather than correct Rachel, the interviewer suggests another method for solving the problem that enables Rachel to discover her error. One effective teaching strategy is to place Rachel in a situation in which she will discover the contradiction between what she said at the beginning and the results of her block counting. This approach seemed to work; Rachel saw the contradiction between the count and her previous answer. She hesitated but finally did accept the right answer.

---

**Clip 8**

15. Now watch the video clip again in its entirety. What do you think this child really knows about simple arithmetic? Explain how you know.

16. What would you do next in teaching? You can watch the video again.

17. What did you learn from this exercise? How did your thinking change during the lesson?
Congratulations, you have completed the lesson on simple arithmetic. Good job!
Subtraction: Strategies and Common Errors

Many of the same strategic and developmental trends described for children’s addition also apply to children’s subtraction. Early on in development, children count to solve simple subtraction problems and rely on the use of manipulatives and fingers to help them represent the problem and to help them keep track of the counting. With experience and maturation, children are better able to mentally keep track of the counting process and thus abandon the use of manipulatives and fingers for verbal counting. Children also rely on their knowledge of addition facts to solve subtraction problems. Here, a child might solve $8 - 3$ by retrieving $5 + 3 = 8$. The most sophisticated problem-solving strategy that can be used to solve subtraction problems involves decomposing the problems into a series of smaller problems.

Simple Subtraction

Many 4- and 5-year-old children can solve formally presented subtraction problems; for instance, “If you had three cookies and gave one to your brother, how many would you have left?” There are three common subtraction procedures that involve the use of manipulatives. The first of these procedures is called separating from. For this problem, the child first gets three blocks to represent the three cookies, removes one block, and then counts or simply states the number of remaining blocks. The second procedure, adding on, involves stating with a number of blocks stated by the subtrahend (the smaller number) and then adding the number of block until the value of the minuend (the larger number) is reached. The number of blocks added to the subtrahend represents the answer. So, for this example, the child would place one block in front of him- or herself and add two more blocks while counting “2, 3.” Because two blocks were added to the first block, the answer would be 2. The final procedure involves matching in a one-to-one fashion the number of blocks represented by the minuend and the subtrahend. The number of unmatched blocks represents the answer. For this example, the child would have one row consisting of a single block aligned with a second row consisting of three blocks. The two unmatched blocks would represent the answer. The use of these different procedures varies with how the problem is presented to the child.

Most 5- and 6-year-olds typically use counting to solve simple subtraction problems. As with addition, counting is sometimes done with the aid of fingers and sometimes done without fingers. Again, finger counting allows the child to represent the numbers to be manipulated and to keep track of the subtraction process and is more likely to be used to solve problems with larger numbers, such as $7 - 3$, than for problems with smaller numbers, such as $3 - 1$. For solving subtraction problems, counting—whether on fingers or verbally—can involve one of two procedures, counting up and counting down. Counting down involves counting backward from the minuend a number of times represented by the value of the subtrahend. To solve the problem $7 - 3$, the child would count, “6, 5, 4; the answer is 4.” If the child cannot keep track of how many values have been counted while counting backward, then she or he will first lift seven fingers and then fold down three fingers in succession. He or she might count backward, “6, 5, 4,” while folding down the fingers or first fold them down and then count the remaining fingers, “1, 2, 3, 4.”

Complex Subtraction

When first learning to solve multicolumn subtraction problems, such as $17 - 3$ or $48 - 27$, children rely on the knowledge and strategies developed for solving simple subtraction problems. In
particular, children count and refer to related addition problems. For example, to solve $17 - 3$, the child might count down, “16, 15, 14; the answer is 14.” Children also use the addition-reference strategy, if they have the complementary addition fact memorized. Moreover, with the introduction of these more complex problems, children also begin to use a problem-solving rule: the delete-10s rule. Deleting 10s involves a type of decomposition in which children treated the 10s value separately from the 1s value. For example, on $15 - 3$, they might explain their answer by saying, “5 – 3 = 2, and you put back the 1, so 12.”

Children also use a columnar-processing strategy, in which the units-column information is processed first, followed by the tens-column information. To solve $48 - 27$, the child might first retrieve, or count, to get the answer to $8 - 7$ and then process in a similar manner the $4 - 2$ in the tens column.

In keeping with the research on children’s addition, the decomposition strategy is also used to solve subtraction problems. There are two common decomposition strategies, and both are based on the base-10 structure of the number system. The first is called the down-over-the-ten method and is used to solve problems with minuends greater than 10, such as $14 - 6$. Here, 10 is first subtracted from the minuend, $14 - 10$; the difference, 4, is then subtracted from the subtrahend, $6 - 4$; and this difference, 4, is then subtracted from 10 to yield the answer, $10 - 2 = 8$.

The second decomposition procedure is called the take-from-the-ten method. Here, the first operation involves subtracting the subtrahend from 10. So, for the problem $14 - 6$, the first operation would involve $10 - 6$. The child then notes the difference, 4 in this example, and then subtracts 10 from the minuend, $14 - 10$. The child then notes the difference, 4 in this example, and then subtracts 10 from the minuend, $14 - 10$. Finally, the two provisional differences are added together to give the answer, $4 + 4 = 8$.

Summary

The same types of errors that were described for children’s addition are also evident in children’s subtraction. Counting errors are often due to losing track of the counting process, which often results in under- or over-counting. Another common counting error involves starting the counting procedure at the wrong value. For instance, to solve $9 - 7$, the child counts, “7, 8, 9; the answer is 3” rather than “8, 9; the answer is 2.” The child includes the number representing the value of the subtrahend as part of the counting-up process. As with addition, retrieval errors in subtraction are often the result of operation confusions, for instance, retrieving 16 to solve $8 - 8$. Errors in complex subtraction typically result from the inappropriate use of a subtraction procedure. The procedure is often appropriate for some subtraction problems, but is indiscriminately applied to problems where it is not appropriate. These types of errors reflect a general lack of understanding of place value and the base-10 structure of the number system.
SUBTRACTION

This lesson will focus on the topic of subtraction. You will watch a series of short video clips and answer questions about them. After you submit your responses, in some cases you will be shown examples of other students’ successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Ricardo, age 8. He is looking at this 100s chart. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

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Clip 1: 12:09 - 12:57

1. What does Ricardo know about the 100s chart?

Here are some comments from other students:

- He knows that adding 10 to a number on the chart makes it the number directly below it.
- He knows that this rule is valid anywhere on the chart.
- He seems to know that counting can help when adding a number that is slightly more or less than 10, such as counting up one from 10 when adding 11.

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Clip 2: 15:38 - 16:07

2. A little later, the interviewer introduces a subtraction problem. Watch this video closely. What are some of the things you notice about his approach?

Here are some observations and hypotheses from other students:

- He seems to think about the number chart as he is listening to the question.
- I noticed he first touched his fingers to a higher number than either of the numbers in the problem.
- He appears to be counting by two's.
- He mistakenly adds instead of subtracts.
- He might have thought he needed to subtract 5.
3. What did you notice about the interviewer's method? How would you evaluate it?

Here are some comments from other students:

- She repeated the question many times.
- She touched each number in the question.
- She didn't ask him why he touched the numbers he did.
- She didn't modify her question to respond to his behavior, or clarify its meaning.

4. As the interviewer, what might you have done differently? What would you have done next? You can watch the video again.

Clip 3: 16:00 - 16:21

5. Now watch this video. How does Ricardo solve the problem?

Here are some observations and hypotheses from other students:

- He appears to count back by 1's.
- He touches each number on the chart as he counts back.
- He counts up to 15 as he touches the numbers going back, and then assesses how many he has left.
- He verifies that there are 10 squares remaining after he counts down 15 squares.
- He touches the 1 perhaps because he didn't know which number to touch next.

6. What are some other ways he could have solved it? You can watch the video again.

Here are some comments from other students:

- If he had used the rule he already knew from addition, he would have realized there are 10 squares between 25 and 15.
- He could have used the number chart to find 25 and 15 and count the difference.
- He might have counted back by 10's.
- He could have subtracted in his head.
• He could have used manipulatives.

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Clip 4: 16:20 - 16:52

7. What does he mean by "count down from back"?

Here are some comments from other students:

• He might have confused his words.
• He appears to know to move down or back, not forward on the number line.
• It's his way of saying "count backwards."

---

8. What is the difference in his mind about counting by one's vs. counting by two's? You can watch the video again.

Here are some comments from other students:

• He thinks counting by two's is a more powerful method.
• He's aware that there's a risk of skipping a number, but he associates this with counting by one's rather than two's.
• He's confused about the difference between counting by one's and by two's.
• His statement makes no sense.

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9. What would you have asked Ricardo? You can watch the video again.

Here are some comments from other students:

• What did you mean by "count down from back"?
• Why is counting by two's better than counting by one's?
• Why might 11 be the right answer?
• Is there a way you can check whether 10 or 11 is the right answer?
10. How does Ricardo get this answer? You can watch the video again.

Here are some comments from other students:

- He counts back by two's, and when he gets to 11, he stops and says his answer: 11.
- The number chart may be confusing him since he would have to move across the page to go from 11 to 10.
- It appears he answers 11 because it is the last number he touches, and he does not realize he must refer to the quantity that's left at the end to get his answer.
- Using an even counting strategy when subtracting an odd amount is a faulty strategy that is confusing him.
- He is silent when he works, which makes it hard to know what he is thinking.

11. Now watch the video again, at half-speed (sound removed). Do you change your interpretation?

Here are some comments from other students:

- It appears that he did know to "fix" his even counting strategy by counting 25 by itself (apparently having "thrown away" 26) before beginning to count by two's.
- It is more clear that he did not know to refer to the remaining quantity at the end, rather than the number he landed on with his fingers.

12. What would you have asked Ricardo at this point? You can watch the video again.

Here are some comments from other students:

- I would have asked him to show me how he solved the problem or encouraged him to repeat what he had done and talk me through it as he solved it.
- I would have asked him why he put fingers on 26 as well as 25 before he began counting by two's.
- I would have asked him why he changed his mind about the answer.
- I might have told him that his answer before was 10 and instead of saying "you changed your
mind," I would asked him if there was a way he could check to be sure it was 10 or 11.

13. One student wrote,

After watching the video clip I learned that Ricardo was not able to solve a problem this complicated. He appeared to understand that subtraction was taking away, but as he was holding all the information for the problem in his head, he lost sight of what the question was. Perhaps there were too many pieces for him to hold in his memory. Perhaps the strategy of counting by two's is confusing him as well. The number chart can be useful here if he were to count back by one's, not two's.

Do you agree with this interpretation? Why or why not? You can watch the video again.

Clip 6: 12:09 - 12:57; 15:38 - 17:15

14. Now watch the video in its entirety. What does Ricardo know about subtraction?

Here are some comments from other students:

- He knows how to relate a spoken number sentence to a number chart.
- He appears to know subtraction means reducing the number, and he knows to take the second number from the first number.
- He knows the strategy of counting by two's or one's can be used when subtracting.
- He makes the common mistake of considering as the answer the last number counted back, rather than the number remaining.

15. What would you do next in teaching? You can watch the video again.

16. What did you learn from this exercise? How did your thinking change during the lesson?
Congratulations, you have completed the lesson on subtraction. Good job!
Equivalence: The Meaning of Written Symbols

In the first few grades of school, children are taught the conventional symbols +, −, = and written numbers like 13 in the context of simple problems like 8 + 5 = 13. Usually the instruction seems to go pretty well: children are proficient at solving problems like 3 + 4 = □ in workbooks. Consequently, we usually conclude that they have learned what they were taught and now understand the elementary symbols much as we do. But this is not necessarily the case. Children construct their own meanings for the symbols—meanings often strikingly different from the adult’s. Consider first + and =.

Young preschoolers have difficulty using writing to represent the events of addition and subtraction. For example, preschool children who are shown a situation in which a toy is joined by two or more may understood the situation well, but they are likely to be unable to use written means—for example, tallies—to represent the act of addition.

How would older children handle this situation? It seems reasonable to think that they should have no trouble since in school they have learned the conventional symbols for adding and subtracting. Thus, shown two frogs joined by one more, they should simply write 2 + 1 = 3. Yet, this is not necessarily so. For example, in one study, children in the first few grades of school were shown toy bricks added to others and asked to use paper and pencil to show what had happened. Under these circumstances, children used some ingenious methods to represent adding and subtracting. For example, one five-year-old drew a hand adding bricks to the original pile. But not one of the children use the conventional + and – symbols to describe the situation even though these were taught in school. As the author of the study writes, "It seems that the whole notion of representing these transformations [addition and subtraction] on paper is something which children find very hard to grasp."

Suppose that now we reverse the situation in this way: The child is first shown symbols like 4 + 3 and then asked to show what these mean in terms of the toy bricks. A simple solution might involve putting four bricks on the table, then adding three more. The point of this task is to see whether the child can translate from arithmetic symbolism to concrete situations. We already know that the child has trouble picturing events in terms of arithmetic symbols. Can he or she describe the symbols in terms of events?

This problem seems so obvious that it is difficult to believe the results. Children from 6 to 10 years of age in a British school experienced considerable difficulty with this task, even though they were exposed almost every day to the correct use of the symbols involved. What can we conclude from this? Apparently, what the children learn in school is of limited generality. Children assume that the symbols + and – are to be used in connection with written arithmetic without seeing an application beyond that domain. The meaning of the symbols is restricted to the context in which they were learned.

Other observations show that children interpret the symbols in distinctive ways. Suppose the child is shown some simple mathematical sentences and asked what the symbols mean. Kenneth, a first grader (about 6 or 7 years), was shown the mathematical sentence 2 + 4 = □.
I: You read that [+], will you? What does that sign say?
K: Plus.
I: What does it tell you?
K: It tells you to add this [2] and this [4].
I: OK. What can you tell me about that [=] symbol?
K: Equals. That means, like this 2 plus this 4 equals 6. There has to be an equal there.

Kenneth interpreted + and = in terms of actions to be performed. So do other children. Presented with the same problem, Evelyn, also a first grader, maintained that 2 + 4 = \[\_\] means "to put number 6 in the box." She said that 2 and 4 are numbers, but 2 + 4 is not a number. A second grader, Donna, said that in 3 + 4 = \[\_\], "the equals sign means what it adds up to." Another first grader, Tammy, said that the = sign means that "you're coming to the end." The children's understanding of symbols refers to actions—calculational operations. The form a + b = \[\_\] means that you do something with a + b to get the answer, namely, the sum.

This, of course, is one legitimate interpretation. In fact, most often when children are presented with sentences of that kind, the two numbers are supposed to be added up. Yet, the interpretation is limited and can lead to trouble:

I: How do you think you would read this \[\_\] = 3 + 4?\]
K: ... Blank equals 3 plus 4.
I: OK. What can you say about that, anything?
K: It's backwards! [He changed it to a 4 + 3 = \[\_\].] You can't go, 7 equals 3 plus 4.

Given the same problem, Tammy also changed it because, "it's backward," and asked the interviewer, "Do you read backwards?"

So one consequence of the child's interpretation in terms of action is that he finds it difficult to read legitimate sentences that do not directly reflect the order of his calculations. The child first does 4 and then 3 to get 7, but that is not what the sentence says, so the sentence must be wrong.

The child's interpretation also leads him to distort some sentences. Kenneth was asked to read \[\_\] = 2 + 5. He said, in effect, that he had to switch around the + and = signs, because they were in the wrong places. This results in "trying to add up to five," namely, \[\_\] + 2 = 5. Later, Kenneth was shown 3 = 5.

I: What can you say about that?
K: Cross that line out. [Kenneth wrote over the = sign to change it to 3 + 5.]
I: Can I write this [3 = 3]? Does it make sense?
K: Nope. Now you could fix that by going like this. [He changed it to 0 + 3 = 3.]

When Charles was shown 3 = 5, he counted on his fingers and wrote 3 = 85. "Five ... there's no plus." That makes it wrong. I'll put a plus in the middle." He wrote 3 + 85.
If the child always interprets symbols in terms of actions on numbers, he cannot make sense of sentences that express relationships, like \(3 = 3\) or \(4 = 4\).

What is the reason for children's tendency to interpret symbols in terms of actions? Part of the reason seems to be that their arithmetic is based on actions, particularly actions of counting. For young children, arithmetic is the activities of adding and taking away, of counting backwards and forwards. Given this orientation, children interpret symbols in terms of action. Furthermore, their action interpretation may simply reflect the ordinary demands of the classroom. Most often, sentences do ask children to perform a calculation; if so, why should they interpret them otherwise? In classrooms that do not stress addition as action but instead promote a relational view (that \(4 + 3\) is the same as or another way of saying \(7\)), children eventually learn to take a different approach, seeing that other interpretations are also possible. But this learning comes hard, because children's natural tendency is to see arithmetic in terms of action.
EQUIVALENCE

This lesson will focus on the topic of equivalence, and symbols in general. You will watch a series of short video clips and answer questions about them. After you submit your responses, in some cases you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Jordan, age 8. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

The lesson begins with an extended conversation about fractions. The interviewer has Jordan use blocks to demonstrate various groupings. Toward the end of this task, Jordan spontaneously introduces a written symbol.

Clip 1: 38:21-38:27

1. You have blocks and paper and pencil. Give an overview about how you would test Jordan's knowledge of the equals sign. Provide four questions you'd like to ask, in sequence.

Clip 2: 39:26-39:36

2. What would you ask Jordan now?

   - I would ask him what "equals" means.
   - I would ask him to show me what equals means using the blocks around the symbol.
   - I would ask him what a "sign" or "symbol" means and if he knows of other symbols.

Clip 3: 39:35-39:56

3. What is he describing here?

   - He's describing equivalence.
   - He's saying that two numbers are equal if they are the same.
   - He's using 12 and 12 as an example of two numbers that are equal.
   - He's saying the relationship between the sets is equal because they are the same amount of elements.
4. Do you think the interviewer should have helped him with his verbal explanation of the concept of equals? You can watch the video again.

- Yes, his language was not quite adequate for the idea, and she could have helped him clarify.
- I would have asked him to elaborate on the sentence, "Those two together are the same number."

Clip 4: 39:52-39:59

5. What do you think of the interviewer's question and gesture at the end of this video clip?

- It's good that she confirms what he's saying.
- I thought she could have asked him to explain further.
- She's affirming his answer and leaving her question open-ended to see what he knows; it's child-centered.
- Her gesture mirrored what he did, to help build rapport by showing that she was following him and could repeat what he was doing and saying.

Clip 5: 39:57-40:01

6. What does Jordan mean by that? What does it imply about his thinking?

- Could mean that he's flexible in his thinking.
- For him, symbols are arbitrary and then don't mean anything to him.
- Trivial generalization -- could do it with four blocks instead of six.
- Could be an interesting new set of ideas.

Clip 6: 39:59-40:41

7. What is the distinction Jordan is making?

- He calls his equation a "number sentence."
- He knows the symbol for addition and that it has a function in the number sentence.
- He says that "equals" is interchangeable with "makes."
- He believes that having both the addition and equals signs in a number sentence indicates "makes."
- He may consider the addition and equal signs to be different from numbers. The symbols tell you what to do and the numbers are the objects used for the process.
Clip 7: 40:41-40:53

8. What would you ask Jordan here?

- Why is it the same symbol as before?
- Why doesn't it just mean the two sides are the same?
- Does equal have more than one meaning?
- Didn't you say before that the addition sign was important?

Clip 8: 40:47 - 40:59

9. What do you think he will say?

- No, I think equals means "makes."
- Yes, I was wrong, equals means "the same as."
- Yes, but in this case it means "makes," and in the other one, "equals."
- It means "the same as" only after you make the two sides the same.

Clip 9: 40:54-41:06

10. When he maintains that the equals sign does not mean "the same as" in this instance," what would you ask or do next?

- I would ask, "Why not?"
- I would have him show me using the blocks or on the paper.
- I would talk to him about how the two types of equations were different.

Clip 10: 41:08-41:20

11. What is the interviewer trying to accomplish here, and how does she go about doing it?

- She wants him to see that the equals sign can mean either "the same as" or "makes."
- She first establishes equivalence, and then she separates the blocks a little.
- She's showing him the link between addition and equivalence by demonstrating the process.
12. What do you think happened?

- It appears that he saw the interviewer's simpler equation and realized that the two sides were equivalent.
- I think the interviewer led him to see what it was he was trying to explain.

13. Do you think that Jordan really understands that the equals sign means "the same as" even in a number sentence? You can watch the video again.

- It's hard to say, because he agrees with her before she finishes explaining, and he doesn't really look at the blocks.
- He probably does understand because he agrees so readily. He also looked back at the blocks indicating $3 + 3$.

14. Would you say that the interviewer was "teaching" him? If so, how was she teaching him? You can watch the video again.

15. Now watch the interview clip in its entirety. What do you conclude about his understanding of the equals sign?

16. What would you do next in teaching? You can watch the video again.

17. What did you learn from this exercise? How did your thinking change?
Congratulations, you have completed the lesson on equivalence. Good job!
Appendix B

Teacher Understanding of Student Knowledge Survey
(TUSK)

The following pages contain real-life scenarios between an adult and a typically developing 4 year-old child. Please answer each question. Your participation is greatly appreciated. —Thank you!

Genevieve Hartman

Teachers College, Columbia University
1. Susan is told she is going to play the “shape hunt game.” Her teacher holds up the page of shapes below and says, “I’m looking for a shape that has 4 sides. Two sides are short and two sides are long. Please find that shape for me.”

```
\[ \text{Shapes: Square, Hexagon, Triangle, Pentagon, Trapezoid} \]
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Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this content.
2. Isaac can count out-loud without error from 1 through 10. His teacher shows him the picture below and asks him “Which side has more? Are there more of these (pointing to the bananas) or more of these (pointing to the apples)?”

Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this content.

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3. Theo can consistently count out-loud from 1 through 20. When asked, “What number comes after 6?” He answers “9.”

Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this problem.

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4. Anna’s teacher tells her to listen very carefully because she’s going need to figure out the answer to a problem in her head and she will not be able to use anything to help her. “If you have three cookies, and I give you two more, how many cookies do you have altogether?”

Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this content.

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5. Elizabeth’s teacher says, “Do you see these animals here? Look, cow, pig, rooster, goat (pointing to each as she says its name).” She then asks Elizabeth to point to the barn. Once Elizabeth does this correctly, her teacher asks her to point to the third animal in line.

Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this content.

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6. A teacher is working with a group of six children. She has a bin containing several small items within reach of everyone. She also has six Ziploc bags, labeled 0 through 5. Each child is given one bag and directed to fill it with the number of objects matching the numeral on their bag. They filled their bags as follows:
Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this content.

7. Catherine’s teacher shows her the following pattern and says, “See this? How does it go?” Catherine correctly says, “blue, orange, blue, orange.” Her teacher says, “That’s right! It goes blue, orange, blue, orange.”

Catherine is then shown the following pattern and told, “I am going to hide my pattern and you will try to make the same one as mine with your cards. Look at it carefully so you know how it goes.” Her teacher
scatters 6 car and 6 airplane cards on the table in front of her and says, “Use these cards here (points to the jumbled cards) to make the same pattern as mine.” She waits about 5 seconds then hides the pattern.

These are the cards that Catherine put out.

Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this problem.

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8. Lilian is told she’s going to play the “present game.” Her teacher holds up the card below and says to her, “Let’s make believe you have 5 presents (pointing to the 5) and I have 3 presents (pointing to the 3). Which is more, 5 presents (pointing to the 5) or 3 presents (pointing to the 3)?”

Place an O beneath the age you believe a typically developing child would be ready to start learning this content, and an X beneath the age you believe the same child would master this content.
9. James’s teacher tells him they are going to play a game called the hidden chips game. “I’ll show you a mat with some chips on it. You count them. Then, I’ll hide the chips and you tell me how many I’m hiding.” His teacher puts out three chips on her mat and asks him to count them. James correctly counts the three chips below.

His teacher then covers the chips with a blank piece of paper so he can longer see them. She then asks James how many chips she’s hiding and he quickly says, “three.”
James’s teacher then removes the blank paper and says, “If I did this (rearranges the chips) how many would there be now? Don’t count them, just tell me.”

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10. Benjamin is sitting next to his teacher at a table. They both have blank 4x4 grids in front of them with several square tiles scattered on the table. His teacher says, “I’m going to make some things on my board, and I want you to make the same things on yours. Watch me very closely so you can make yours just like mine.” She slowly places four squares vertically on her grid. Benjamin places four squares at the top of his grid, horizontally.

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