Minimal number of function evaluations for computing topological degree in two dimensions

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by

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Abstract

A lower bound n_{min} roughly equal $\log_2(\mathrm{diam}(T)/\eta)$, is established for the minimal number of function evaluations necessary to compute the topological degree of every function f in a class F. The class F consists of continuous functions $f = (f_1, f_2)$ defined on a triangle T, $f: T \to \mathbb{R}^2$, such that the minimal distance between zeros of f_1 and zeros of f_2 on the boundary of T is not less than η, $\eta > 0$.

Information is exhibited which permits the computation of the degree for every f in F with at most $2n_{min}$ function evaluations. An algorithm, due to Kearfott, uses this information to compute the degree.

These results lead to tight lower and upper complexity bounds for this problem.

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1. Introduction.

The problem of computing the topological degree of a function has been studied in many recent papers, e.g. [3,4,6,8,9,10,11]. From the topological degree one may ascertain whether there exists a zero of a function inside a domain. Namely, Kronecker's theorem [1,4] states that if the degree is not zero, then there exists at least one zero of a function inside a domain. By computing a sequence of nonzero degrees for domains with decreasing diameters one can ascertain a region with arbitrarily small diameter which includes at least one zero of a function, see [2,3,4,8]. Algorithms proposed in thses papers were tested by their authors on relatively easy examples. They concluded that the degree of an arbitrary continuous function could be computed. It was observed, however, e.g. [3,4,11] that the algorithms may require an arbitrarily large number of function evaluations. In this paper we restrict the class of functions to be able to compute the degree for every element in the restricted class using an a priori bounded number of evaluations.

We consider the class F of continuous functions $f = (f_1, f_2)$ defined on a triangle $T = \Delta T_1T_2T_3$, where the T_i are vertices of T, $f\colon T \to \mathbb{R}^2$. We assume that each of

f_1 and f_2 restricted to the boundary ∂T of T, say $\bar{f}_1$ and $\bar{f}_2$, has at most one zero on each edge of T and that this zero is not a point of local extremum. We also assume that the $f_i(T_j)$ are not zero for every i and j and that the distance between zeros of $\bar{f}_1$ and $\bar{f}_2$ on each edge is not less than η, $0 < \eta < \min_{i \neq j}(\|T_i - T_j\|_2)$. The last assumption with $\eta > 0$ is necessary for the existence of the topological degree.

The information N_n on f consists of n values of f_1 and/or f_2 on ∂T, which are computed adaptively. This form of information is assumed since the topological degree is determined uniquely by the values of f on ∂T, see [5]. In fact we can show that using adaptive evaluations of n arbitrary linear functionals on f one cannot do better than just using function values on ∂T. The proof is technically difficult and is based on the idea presented in [7].

The topological degree is computed by means of an algorithm φ which is a mapping depending on the information, $\varphi: N_n(f) \to I$, where I denotes the set of all integers.

In this paper we solve the following problems:

(i) Find a tight lower bound n_{min} on the minimal number of function evaluations necessary to find the topological degree of f for every f in F using an arbitrary information N_n.

(ii) Exhibit an information N_n^* with n roughly equal $2n_{min}$ which allows to compute the degree for every f in F. This information is used by an algorithm φ^* developed by Kearfott [3] to compute the degree.

We briefly summarize the contents of the paper. In Sect. 2 we define information and algorithm. In Sect. 3 we obtain a formula for n_{min} and in Sect. 4 we exhibit the information N_n^* and algorithm φ^*.

2. Basic definitions -formulation of the problem.

Let $T = \Delta T_1 T_2 T_3$ be a triangle in $\mathbb{R}^2$, where T_i are vertices of T, with the notation $T_{j+3} = T_j$, $\forall j$. Let I be the set of all integers, $\|\cdot\| = \|\cdot\|_2$ the euclidean norm in $\mathbb{R}^2$, and $\theta = (0,0)$. Denote

$$G = \{f: T \to \mathbb{R}^2, f = (f_1,f_2),\ f \text{ continuous}\}.$$

For an arbitrary $f = (f_1,f_2)$, $f \in G$, let $\bar{f}_1 = f_1|_{\partial T}$ and $\bar{f}_2 = f_2|_{\partial T}$ be the restrictions of f_1 and f_2 to the boundary of T. Then for a given η, $0 < \eta < \min\limits_{i\neq j}\|T_i - T_j\|$, define

(2.1) $$F = \{f = (f_1,f_2) \in G\colon f_i(T_j) \neq 0,\ \forall i,j, \text{ each of } \bar{f}_1 \text{ and } \bar{f}_2 \text{ has at most one zero on } [T_j,T_{j+1}], \text{ which is not a point of a local extremum, and } \|\alpha-\beta\| \geq \eta,\ \forall \alpha,\beta \in [T_j,T_{j+1}] \text{ such that } \bar{f}_1(\alpha) = \bar{f}_2(\beta) = 0,\ \forall j\}.$$

Our problem is to find the topological degree, $\deg(f,T,\theta)$, of f relative to T at θ, see [5], for every f in F. To solve this problem we use <u>information</u> N_n and an <u>algorithm</u> φ using N_n. These are defined as in [12]:

Let $f \in F$ and

$$(2.2) \qquad N_n(f) = [f_{i_1}(x_1), f_{i_2}(x_2), \ldots, f_{i_n}(x_n)],$$

where $x_1 \in \partial T$ and $i_1 \in \{1,2\}$ are given a priori,

$$x_j = \tilde{x}_j(f_{i_1}(x_1), \ldots, f_{i_{j-1}}(x_{j-1})),$$

$$i_j = \tilde{i}_j(f_{i_1}(x_1), \ldots, f_{i_{j-1}}(x_{j-1})),$$

and $\tilde{x}_j$ is a transformation, $\tilde{x}_j: \mathbb{R}^{j-1} \to \partial T$, $\tilde{i}_j$ is a transformation, $\tilde{i}_j: \mathbb{R}^{j-1} \to \{1,2\}$, $j = 1, \ldots, n$.

The total number of function evaluations n is called the cardinality of N_n. Let us denote the class of all such information by $\mathcal{N}$.

Knowing N_n we approximate $\deg(f,T,\theta)$ by an algorithm φ, which is an arbitrary mapping

$$(2.3) \qquad \varphi: N_n(F) \to I.$$

By minimal cardinality number n^* we mean the minimal n for which there exists information N_n which allows to determine the degree of any f from F, i.e.,

$$N_n(\tilde{f}) = N_n(f) \Rightarrow \deg(\tilde{f},T,\theta) = \deg(f,T,\theta), \quad \forall \tilde{f}, f \in F.$$

In the paper we solve the following problems:

(2.4): Find a tight lower bound n_{min} on the minimal cardinality number n^*.

(2.5) Find information N_n^* with cardinality close to the n_{min}.

We also present an easily implementable algorithm φ^* using N_n^*, which computes $\deg(f,T,\theta)$ for any f in F. The algorithm φ^* was developed by Kearfott and is based on his Parity Theorem in [3].

We discuss the complexity (minimal cost) of finding the topological degree. Assume that one function evaluation costs c and that arithmetic operation or comparison costs unity. Usually c is much larger than one. The complexity of the algorithm φ^* is equal to the sum of the costs of computing N_n^* and of computing φ^* given N_n^*. If $c \gg 1$, then the complexity of φ^* is roughly equal twice the lower bound of the complexity of solving the problem. Therefore φ^* is an almost optimal complexity algorithm.

3. Minimal cardinality number.

In this section we show how to bound from below the minimal cardinality number. Suppose without loss of generality that

$$\|T_1-T_2\| \le \|T_2-T_3\| \le \|T_1-T_3\|.$$

We prove

Theorem 3.1: For every information N_n in η such that $n < n_{min} = \lfloor \log_2(\|T_2-T_3\|/\eta) \rfloor$, there exist two functions f,g in F such that $N_n(f) = N_n(g)$ and $\deg(f,T,\theta) = 0$, $\deg(g,T,\theta) = -1$. □

Theorem 3.1 says that if the number n of function evaluations is less than n_{min} then for every information N_n in η there exist two functions in F with different degrees and hence we must use at least n_{min} function evaluations to be able to compute degree for every f in F, i.e. that $n^* \ge n_{min}$.

First we prove the following Lemma:

Lemma 3.1: For every n, $N_n \in \eta$, $N_n(f) = [f_{i_1}(x_1), f_{i_2}(x_2), \ldots, f_{i_n}(x_n)]$, see (2.2), $a > 0$, and ϵ, $0 < \epsilon < \|T_2-T_3\|/2^{n+1}$, there exist a function $f_n = (f_{n,1}, f_{n,2})$,

$f_n \in F$, see Fig. 3.1, and intervals $I_{n,1} = [X_{n,1}, X_{n,2}]$, $I_{n,2} = [Y_{n,1}, Y_{n,2}]$, $I_{n,1} \subset [T_1, T_3]$, $I_{n,2} \subset [T_2, T_3]$, $\operatorname{diam}(I_{n,1}) \geq \|T_1 - T_3\|/2^n$, $\operatorname{diam}(I_{n,2}) \geq \|T_2 - T_3\|/2^n$, such that

(i) $x_i \notin [X_{n,1}, X_{n,2}] \cup [Y_{n,1}, Y_{n,2}]$;

(ii) f_n is an arbitrary continuous extension of the function $g = (\bar{f}_{n,1}, \bar{f}_{n,2})$, $g\colon \partial T \to \mathbb{R}^2$, to the triangle T, where $\bar{f}_{n,1}$ and $\bar{f}_{n,2}$ are given by the formulas (3.1) and (3.2). $\square$

$$\bar{f}_{n,1}(x) = \bar{f}_{n,2}(x) = a, \quad x \in [T_1, T_2] \cup [T_1, X_{n,1}] \cup [T_2, Y_{n,1}]$$

$$\bar{f}_{n,1}(x) = \bar{f}_{n,2}(x) = -a, \quad x \in [X_{n,2}, T_3] \cup [Y_{n,2}, T_3],$$

(3.1)
$$\bar{f}_{n,1}(x) = \begin{cases} a & x \in [Y_{n,1}, Y_2] \cup [X_{n,1}, X_2], \\ a - 2a/\varepsilon\|x - X_2\| & x \in [X_2, X_{n,2}], \\ a - 2a/\varepsilon\|x - Y_2\| & x \in [Y_2, Y_{n,2}], \end{cases}$$

where

$$X_2 = X_{n,2} - \varepsilon(T_3 - T_1)/\|T_3 - T_1\|$$

$$Y_2 = Y_{n,2} - \varepsilon(T_3 - T_2)/\|T_3 - T_2\|$$

(3.2)
$$\bar{f}_{n,2}(x) = \begin{cases} -a & x \in [X_1, X_{n,2}] \cup [Y_1, Y_{n,2}], \\ a - 2a/\varepsilon\|x - X_{n,1}\| & x \in [X_{n,1}, X_1], \\ a - 2a/\varepsilon\|x - Y_{n,1}\| & x \in [Y_{n,1}, Y_1], \end{cases}$$

where

$$X_1 = X_{n,1} + \epsilon(T_3-T_1)/\|T_3-T_1\|,$$

$$Y_1 = Y_{n,1} + \epsilon(T_3-T_2)/\|T_3-T_2\|.$$

We remark that Lemma 3.1 implies that $f_{n,1}(x_i) = f_{n,2}(x_i)$, $i = 1,2,\ldots,n$, and that the distances between zeros of $\bar{f}_{n,1}, \bar{f}_{n,2}$ on $[T_1,T_3]$ say $\alpha_1(\bar{f}_{n,1})$, $\alpha_1(\bar{f}_{n,2})$ and on $[T_2,T_3]$ say $\alpha_2(\bar{f}_{n,1})$, $\alpha_2(\bar{f}_{n,2})$ are

$$\|\alpha_1(\bar{f}_{n,1}) - \alpha_1(\bar{f}_{n,2})\| \geq \|T_1-T_3\|/2^n - \epsilon$$

$$\|\alpha_2(\bar{f}_{n,1}) - \alpha_2(\bar{f}_{n,2})\| \geq \|T_2-T_3\|/2^n - \epsilon.$$

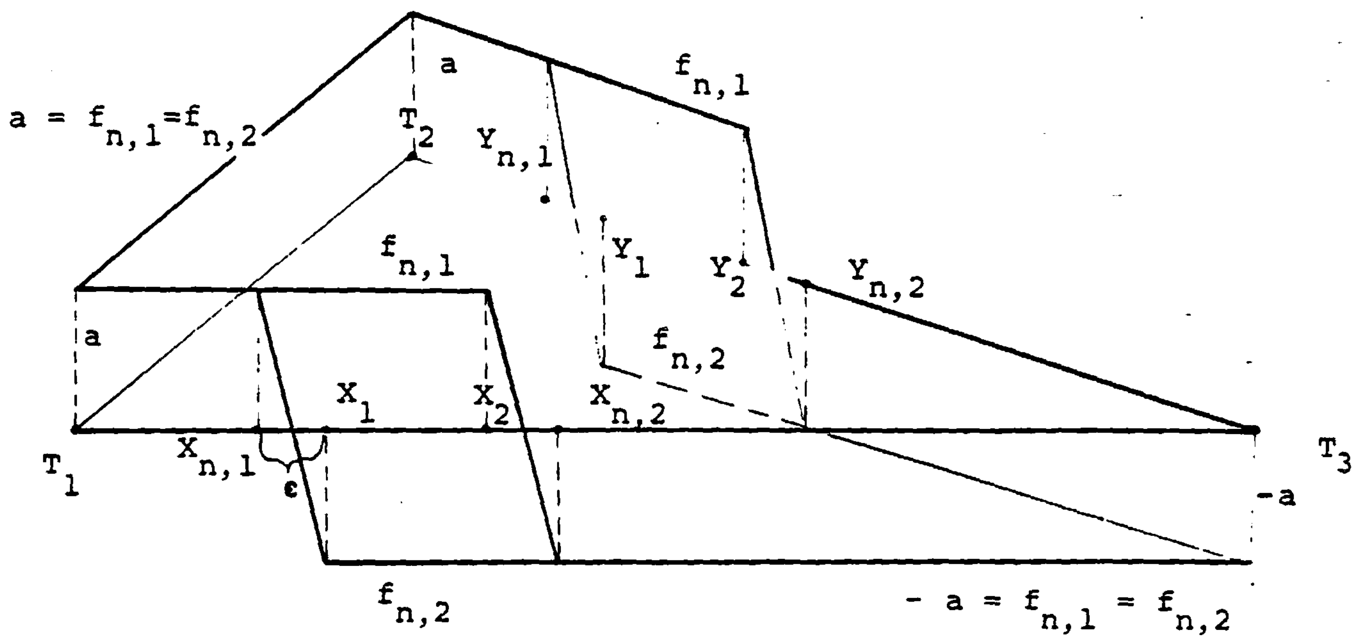

Fig. 3.1

Proof: The proof is by induction (compare [7]). Let $n = 1$. Suppose without loss of generality, that $x_1 \in P = [T_1,T_2] \cup [T_1,M_1] \cup [T_2,M_2]$ where $M_1 = (T_1+T_3)/2$ and $M_2 = (T_2+T_3)/2$. Denote also $e_1 = (T_3-T_1)/\|T_3-T_1\|$ and $e_2 = (T_3-T_2)/\|T_3-T_2\|$. Then define

$$\bar{f}_{1,1}(x) = \bar{f}_{1,2}(x) = a \text{ for } x \in P$$

and

$$\bar{f}_{1,1}(x) = \begin{cases} a & x \in [M_1,X_2] \cup [M_2,Y_2] \,, \\ a - \frac{2a}{\epsilon}\|x-X_2\| & x \in [X_2,T_3] \,, \\ a - \frac{2a}{\epsilon}\|x-Y_2\| & x \in [Y_2,T_3] \,, \end{cases}$$

where

$$X_2 = T_3 - \epsilon e_1 \quad ,$$

$$Y_2 = T_3 - \epsilon e_2 \quad ,$$

and

$$\bar{f}_{1,2}(x) = \begin{cases} -a & x \in [X_1,T_3] \cup [Y_1,T_3] \,, \\ a - \frac{2a}{\epsilon}\|x-M_1\| & x \in [M_1,X_1] \,, \\ a - \frac{2a}{\epsilon}\|x-M_2\| & x \in [M_2,Y_1] \,, \end{cases}$$

where

$$X_1 = M_1 + \epsilon e_1 \quad ,$$

$$Y_1 = M_2 + \epsilon e_2 .$$

Then

$$X_{1,1} = M_1, \quad X_{1,2} = T_3, \quad Y_{1,1} = M_2, \quad Y_{1,2} = T_3,$$

$$\operatorname{diam}(I_{1,1}) = \frac{\|T_1 - T_3\|}{2}, \qquad \operatorname{diam}(I_{1,2}) = \frac{\|T_2 - T_3\|}{2},$$

and $\bar{f}_{1,1}(x_1) = \bar{f}_{1,2}(x_1)$. Taking f_1 as an arbitrary extension of $(\bar{f}_{1,1}, \bar{f}_{1,2})$ to the whole of T completes the proof for $n = 1$.

Assume now that lemma holds for n, (Fg. 3.1). If $x_{n+1} = \tilde{x}_{n+1}(N_n(f_n))$ does not belong to $[X_{n,1}, X_{n,2}] \cup [Y_{n,1}, Y_{n,2}]$ then the function $f_{n+1} = f_n$ satisfies Lemma 3.1. Therefore suppose without loss of generality that $x_{n+1} \in [X_{n,1}, X_{n,2}]$, and define

$$\bar{f}_{n+1,1}(x) = \bar{f}_{n+1,2}(x) = \bar{f}_{n,2}(x) \text{ for } x \in \partial T - [X_{n,1}, X_{n,2}]$$

and

$$\bar{f}_{n+1,2}(x) = \begin{cases} \bar{f}_{n,2}(x) & \Big\} \text{ if } x_{n+1} \in [M, X_{n,2}], \\ \left.\begin{array}{ll} a & x \in [M_{n,1}, M], \\ a - \frac{2a}{\epsilon}\|x - M\| & x \in [M, M+\epsilon e_1], \\ -a & x \in [M+\epsilon e_1, X_{n,2}], \end{array}\right\} & \text{otherwise} \end{cases}$$

where $M = (X_{n,1} + X_{n,2})/2$ and

$$\bar{f}_{n+1,1}(x) = \begin{cases} \left.\begin{array}{ll} a & x \in [X_{n,1}, M-\varepsilon e_1] , \\ a - \frac{2a}{\varepsilon}\|x-M+\varepsilon e_1\| & x \in [M-\varepsilon e_1, M] , \\ -a & x \in [M, X_{n,2}] , \end{array}\right\} \text{if } x_{n+1} \in [M, X_{n,2}], \\ \left.\bar{f}_{n,1}(x) \qquad\qquad\qquad\qquad\qquad \right\} \text{otherwise.} \end{cases}$$

Then $\bar{f}_{n+1,1}(x_i) = \bar{f}_{n+1,2}(x_i)$, $\forall i = 1,2,\ldots,n+1$, and

$$I_{n+1,2} = I_{n+1,1},$$

$$I_{n+1,1} = \begin{cases} [X_{n,1}, M] & \text{if } x_{n+1} \in [M, X_{n,2}], \\ [M, X_{n,2}] & \text{otherwise.} \end{cases}$$

Therefore

$$\operatorname{diam}(I_{n+1,2}) \geq \|T_2-T_3\|/2^{n+1},$$

$$\operatorname{diam}(I_{n+1,1}) \geq \|T_1-T_3\|/2^{n+1},$$

and the function f_{n+1} defined as an arbitrary extension of $(\bar{f}_{n+1,1}, \bar{f}_{n+1,2})$ to the whole T satisfies Lemma 3.1 for $n+1$. This completes the proof. □

Proof of Theorem 3.1: Take arbitrary information $N_n \in \mathcal{N}$ with $n < n_{min}$ and consider ε, $0 < \varepsilon < \min(\|T_2-T_3\|/2^{n+1}, \|T_2-T_3\|/2^n - \eta)$. Note that $\|T_2-T_3\|/2^n - \eta$ is positive, since $\eta < \dfrac{\|T_2-T_3\|}{2^n}$ for $n < n_{min}$, and therefore ϵ is well defined. Let $f_n = (f_{n,1}, f_{n,2})$ be a function from Lemma 3.1, such that

$$f_{n,1}(x) = \begin{cases} >0 & x \in \mathrm{Int}(T_1, T_2, \alpha_2(f_{n,1}), \alpha_1(f_{n,1})), \\ 0 & x \in [\alpha_1(f_{n,1}), \alpha_2(f_{n,1})], \\ <0 & x \in \mathrm{Int}(\Delta\ \alpha_1(f_{n,1}), \alpha_2(f_{n,1}), T_3), \end{cases}$$

$$f_{n,2}(x) = \begin{cases} >0 & x \in \mathrm{Int}(T_1, T_2, \alpha_2(f_{n,2}), \alpha_1(f_{n,2})), \\ 0 & x \in [\alpha_1(f_{n,2}), \alpha_2(f_{n,2})], \\ <0 & x \in \mathrm{Int}(\Delta\ \alpha_1(f_{n,2}), \alpha_2(f_{n,2}), T_3). \end{cases}$$

Since $\epsilon < \|T_2 - T_3\|/2^n - \eta$ then

$$\|\alpha_2(\bar{f}_{n,2}) - \alpha_2(\bar{f}_{n,1})\| \geq \frac{\|T_2 - T_3\|}{2^n} - \epsilon \geq \eta$$

$$\|\alpha_1(\bar{f}_{n,2}) - \alpha_1(\bar{f}_{n,1})\| \geq \frac{\|T_1 - T_3\|}{2^n} - \epsilon \geq \eta.$$

Each of $\bar{f}_{n,1}$, $\bar{f}_{n,2}$ has exactly one zero on $[T_1, T_3]$ and $[T_2, T_3]$. These properties imply that f_n belongs to F. Observe however that f_n does not have a zero in T. Kronecker's Theorem [1,5] yields that

$$\deg(f_n, T, \theta) = 0.$$

Now we define a second function. Let

$$\bar{f}_{n,3}(x) = \begin{cases} \bar{f}_{n,1}(x) & x \in [T_1,T_2] \cup [T_2,T_3] , \\ \bar{f}_{n,2}(x) & x \in [T_1,T_3] , \end{cases}$$

and

$$\bar{f}_{n,4}(x) = \begin{cases} \bar{f}_{n,1}(x) & x \in [T_1,T_2] \cup [T_1,T_3] , \\ \bar{f}_{n,2}(x) & x \in [T_2 T_3] . \end{cases}$$

It is obvious that

$$(3.3) \qquad \bar{f}_{n,3}(x_i) = \bar{f}_{n,4}(x_i) = \bar{f}_{n,1}(x_i) = \bar{f}_{n,2}(x_i), \qquad i = 1,2,\ldots,n.$$

Define f_3 and f_4 to be any continuous extension of $\bar{f}_{n,3}, \bar{f}_{n,4}$ into T such that

$$f_3(x) = \begin{cases} >0 & x \in \mathrm{Int}(T_1, T_2, \alpha_2(f_{n,1}), \alpha_1(f_{n,2})) , \\ 0 & x \in [\alpha_1(f_{n,2}), \alpha_2(f_{n,1})] , \\ <0 & x \in \mathrm{Int}(\Delta\ \alpha_1(f_{n,2}), \alpha_2(f_{n,1}), T_3) , \end{cases}$$

and

$$f_4(x) = \begin{cases} >0 & x \in \mathrm{Int}(T_1, T_2, \alpha_2(f_{n,2}), \alpha_1(f_{n,1})) . \\ 0 & x \in [\alpha_1(f_{n,1}), \alpha_2(f_{n,2}]) , \\ <0 & x \in \mathrm{Int}(\Delta\ \alpha_1(f_{n,1}), \alpha_2(f_{n,2}), T_3) . \end{cases}$$

This implies that $g_n = (f_3, f_4)$ belongs to F. Observe that g_n has exactly one zero α in T,

$$\alpha = [\alpha_1(\bar{f}_{n,1}),\alpha_2(\bar{f}_{n,2})] \cap [\alpha_1(\bar{f}_{n,2}),\alpha_2(\bar{f}_{n,1})].$$

One can easily check that the topological degree of g_n is $\deg(g_n,T,\theta) = -1$ (by using for example the Parity Theorem of Kearfott [3]). Equation (3.3) and the definition of f_n and g_n imply that

$$N_n(g_n) = N_n(f_n)$$

which finally completes the proof. $\square$

4. Optimality results.

In this section we find information N_n^* with $n^* \leq n \leq 2n_{min}$ (n roughly equal $2n_{min}$) and exhibit an almost optimal complexity algorithm φ^* using N_n^*, which requires only arithmetic operations and comparisons. The information N_n^* consists of evaluations of function values of points $x_i \in \partial T$, $i = 1,2,\ldots,n$, which yield a sufficient refinement of ∂T relative to sign of f. Here sufficient refinement is defined as follows, see [3,4,10,11].

<u>Definition 4.1</u>: If $f \in F$, then ∂T is sufficiently refined relative to sign of f iff ∂T is decomposed as an oriented (see [3]) union of intervals $I_1,\ldots,I_k$ with the properties:

(i) $\mathrm{Int}(I_j) \cap \mathrm{Int}(I_i) = \emptyset$, $i \neq j$.

(ii) For every I_j one of f_1, f_2 does not vanish on I_j, say f_{i_1}, and then $f_{i_2}(a_j) \cdot f_{i_2}(b_j) \neq 0$ where $I_j = [a_j, b_j]$ and $\{i_1, i_2\} = \{1,2\}$. $\square$

Knowing the information N_n^* we can compute the degree by using the algorithm φ^* based on the Parity Theorem of Kearfott [3]. This is described as follows: Let $I_j = [e_{j,1}, e_{j,2}]$, $I_j = I_j(f)$, $f \in F$, $j = 1,2,\ldots,m(f)$ form a sufficient refinement of ∂T relative to $\mathrm{sign}(f)$. Then define the sign matrices of f:

$$R(I_j,f) = [\mathrm{sgn}(f_k(e_{j,i}))]_{i,k=1,2}$$

where i is the row, k is the column and $\mathrm{sgn}(x) = \{1 \text{ if } x \geq 0 \text{ and } 0 \text{ if } x < 0\}$. The parity of $R(I_j,f)$, $\mathrm{Par}(R(I_j,f)$ is given by:

$$\mathrm{Par}(R(I_j,f)) = \begin{cases} 1 & \text{if } R(I_j,f) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \\ -1 & \text{if } R(I_j,f) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \\ 0 & \text{otherwise}. \end{cases}$$

Define φ^* by

$$\text{(4.1)} \qquad \varphi^*(N_n^*(f) = \Sigma_{j=1}^{m(f)} \mathrm{Par}(R(I_j),f).$$

Then the Parity Theorem states that

$$\text{(4.2)} \qquad \varphi^*(N_n^*(f)) = \deg(f,T,\theta), \qquad \forall f \in F.$$

We remark that the implementation of φ^* requires at most 20 comparisons and 4 additions, since a sufficient refinement in our class F consists of at most 5 intervals (see below). We now proceed to the construction of the information N_n^*.

Take an arbitrary f in F and compute f at the vertices of T.

If $\mathrm{sgn}(f_i(T_j)) = \text{const.}$ for one of the f_i, $i = 1,2$, then the decomposition

$$\partial T = [T_1 T_2] \cup [T_2 T_3] \cup [T_3 T_1]$$

forms a sufficient refinement of ∂T relative to sign of f. The Parity Theorem implies then that $\deg(f,T,\theta) = 0$.

Assume now that none of f_1, f_2 has a constant sign at all vertices, i.e., that the following equations are satisfied:

$$\text{(4.3)} \qquad \mathrm{sign}(f_1(T_i)) = \mathrm{sign}(f_1(T_{i+1})) = -\mathrm{sign}(f_1(T_{i+2}))$$

$$\text{(4.4)} \qquad \mathrm{sign}(f_2(T_j)) = \mathrm{sign}(f_2(T_{j+1})) = -\mathrm{sign}(f_2(T_{j+2}))$$

where

$i,j \in \{1,2,3\}$ and $\mathrm{sign}(x) = \{-1 \text{ if } x < 0,\ 0 \text{ if } x = 0,\ 1 \text{ if } x > 0\}$.

Two cases are possible:

(4.5) If $i \neq j$ then f_1 has a constant sign on $[T_i, T_{i+1}]$ and f_2 has a constant sign on $[T_j, T_{j+1}]$. Therefore to obtain a sufficient refinement of ∂T we need to subdivide only $T_k T_{k+1}$, where $\{i,j,k\} = \{1,2,3\}$.

(4.6) If $i = j$ then f_1 and f_2 have constant sign on $[T_i,T_{i+1}]$ and both change sign on the intervals $[T_{i+1},T_{i+2}]$ and $[T_{i-1},T_i]$. Therefore to obtain a sufficient refinement of ∂T we need to subdivide both $[T_{i+1},T_{i+2}]$ and $[T_k,T_{k+1}]$, where $\{i,j,k\} = \{1,2,3\}$.

Consider first the case (4.5). We will see below that to find a sufficient refinement of ∂T it is necessary to find a point $z \in [\alpha(f_1),\alpha(f_2)]$, where recall that $\alpha(f_1)(\alpha(f_2))$ is the zero of $f_1(f_2)$ on $[T_k,T_{k+1}]$.

To do this we use the bisection method to locate the zero $\alpha(f_1)$ to within $\delta < \eta$, see Fig. 4.1.

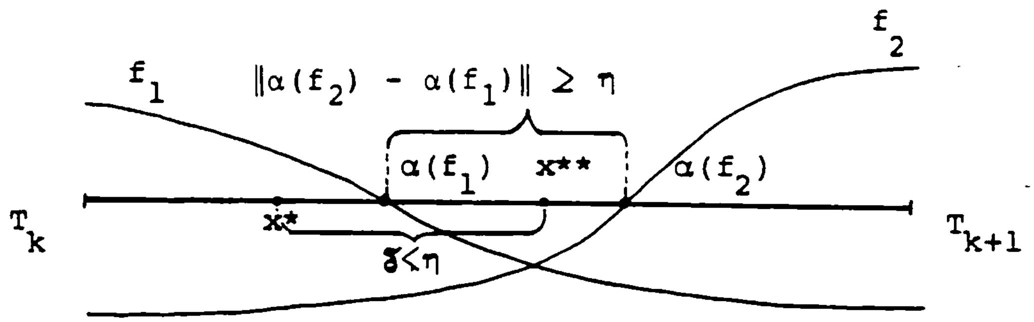

Fig. 4.1

Therefore we compute n times f_1, where n is the smallest integer such that $\|T_{k+1}-T_k\|/2^n < \eta$, i.e.,

$$n = \left\lceil \log_2 \frac{\|T_{k+1} - T_k\|}{\eta} \right\rceil + 1.$$

In this way we obtain the interval $I = [x^*, x^{**}]$, $\operatorname{diam}(I) = \delta < \eta$, $\|x^* - T_k\| < \|x^{**} - T_{k+1}\|$, such that $\alpha(f_1) \in I$. Then $\alpha(f_2)$ does not belong to I, since $\|\alpha(f_2) - \alpha(f_1)\| \geq \eta$. We need only to compute $f_2(x^{**})$ to find which point from x^*, x^{**} is in $E = [\alpha(f_1), \alpha(f_2)]$. Namely,

$$(4.7) \quad \text{If } \left.\begin{array}{l} f_1(T_k) > 0 \\ f_2(T_{k+1}) > 0 \end{array}\right\} \text{ then } \left\{\begin{array}{lll} \text{if } f_2(x^{**}) < 0 & \text{then} & x^{**} \in E, \\ \text{if } f_2(x^{**}) > 0 & \text{then} & x^* \in E, \end{array}\right.$$

$$(4.8) \quad \text{If } \left.\begin{array}{l} f_1(T_k) > 0 \\ f_2(T_{k+1}) < 0 \end{array}\right\} \text{ then } \left\{\begin{array}{lll} \text{if } f_2(x^{**}) < 0 & \text{then} & x^* \in E, \\ \text{if } f_2(x^{**}) > 0 & \text{then} & x^{**} \in E, \end{array}\right.$$

$$(4.9) \quad \text{If } \left.\begin{array}{l} f_1(T_k) < 0 \\ f_2(T_{k+1}) > 0 \end{array}\right\} \text{ then } \left\{\begin{array}{lll} \text{if } f_2(x^{**}) < 0 & \text{then} & x^{**} \in E, \\ \text{if } f_2(x^{**}) > 0 & \text{then} & x^* \in E, \end{array}\right.$$

$$(4.10) \quad \text{If } \left.\begin{array}{l} f_1(T_k) < 0 \\ f_2(T_{k+1}) < 0 \end{array}\right\} \text{ then } \left\{\begin{array}{lll} \text{if } f_2(x^{**}) < 0 & \text{then} & x^* \in E, \\ \text{if } f_2(x^{**}) > 0 & \text{then} & x^{**} \in E. \end{array}\right.$$

By checking (4.7)-(4.10) we find a point z in E, $z = x^*$ or $z = x^{**}$. Observe that the point z subdivides $[T_k, T_{k+1}]$ in such a way that one of f_1, f_2 has a constant sign on $[T_k, z]$ and one of f_1, f_2 has a constant sign on

$[z,T_{k+1}]$. Therefore the decomposition

$$\partial T = [T_k,z] \cup [z,T_{k+1}] \cup [T_i,T_{i+1}] \cup [T_{j,j+1}] \quad \text{if} \quad i = k+1$$

or

$$\partial T = [T_k,z] \cup [z,T_{k+1}] \cup [T_j,T_{j+1}] \cup [T_i,T_{i+1}] \text{ if } \quad j = k+1$$

forms a sufficient refinement of ∂T relative to sign of f, which means that the information N^*_{n+7} allows to determine the degree, where N^*_{n+7} is given by

$$(4.11) \quad N^*_{n+7}(f) = [f_1(T_1),f_2(T_1),f_1(T_2),f_2(T_2),f_1(T_3),f_2(T_3), \\ f_1(x_1),\ldots,f_1(x_n),f_2(x^{**})]$$

and $x_1 = (T_k+T_{k+1})/2$, x_i, $i = 2,\ldots,n.$, are defined by the bisection method applied to the function f_1 on $[T_k,T_{k+1}]$,

$$x^{**} = \begin{cases} x_n & \text{if} \quad \|x_{n-1}-T_k\| < \|x_n-T_k\| \\ x_{n-1} & \text{otherwise,} \end{cases}$$

and

$$n = \left\lceil \log_2 \frac{\|T_k-T_{k+1}\|}{\eta} \right\rceil + 1.$$

We now consider the case (4.6). To construct a sufficient refinement we need to apply the procedure from the case (4.5) to both intervals $[T_{i+1},T_{i+2}]$ and $[T_{i-1},T_i]$. Therefore an information which allows to determine the degree is given by

$$(4.12)\quad N^*_{n_1+n_2+8}(f) = [f_1(T_1), f_2(T_1), f_1(T_2), f_2(T_2), f_1(T_3), f_2(T_3), f_1(x_1), \ldots, f_1(x_{n_1}), f_2(x^{**}), f_1(y_1), \ldots, f_1(y_{n_2}), f_2(y^{**})],$$

where

$$x_1 = (T_{i+1}+T_{i+2})/2, \quad y_1 = (T_{i-1}+T_i)/2,$$

x_i (y_j) are defined by the bisection method applied to f_1 on $[T_{i+1}, T_{i+2}]$ ($[T_{i-1}, T_i]$),

$$n_1 = \left\lceil \log_2 \frac{\|T_i - T_{i+1}\|}{\eta} \right\rceil + 1,$$

$$n_2 = \left\lceil \log_2 \frac{\|T_{i-1} - T_i\|}{\eta} \right\rceil + 1,$$

and

$$x^{**} = \begin{cases} x_n & \text{if } \|x_{n-1} - T_{i+1}\| < \|x_n - T_{i+1}\|, \\ x_{n-1} & \text{otherwise,} \end{cases}$$

$$y^{**} = \begin{cases} y_n & \text{if } \|y_{n-1} - T_{i-1}\| < \|y_n - T_{i-1}\|, \\ y_{n-1} & \text{otherwise.} \end{cases}$$

By checking (4.7)-(4.10) for $k = i-1$ ($k = i+1$) we find a point z_1 in $[T_{i+1}, T_{i+2}]$ (z_2 in $[T_{i-1}, T_i]$) such that $z_1 \in [\alpha_1(f_1), \alpha_1(f_2)]$ ($z_2 \in [\alpha_2(f_1), \alpha_2(f_2)]$). Then the sufficient refinement of ∂T relative to sign of f is given by

$$\partial T = [T_i, T_{i+1}] \cup [T_{i+1}, z_1] \cup [z_1, T_{i+2}] \cup [T_{i-1}, z_2]$$

$$\cup\ [z_2, T_i].$$

Observe that for the "worst" mapping f in F the information N_n^* consists of $n = 8 + n_1 + n_2$ function evaluations, where

$$n_1 = \left\lceil \log_2 \frac{\|T_2 - T_3\|}{\eta} \right\rceil + 1, \qquad n_2 = \left\lceil \log_2 \frac{\|T_1 - T_3\|}{\eta} \right\rceil + 1.$$

Since $n_{min} = \left\lfloor \log_2 \frac{\|T_2 - T_3\|}{\eta} \right\rfloor$, then

$$2n_{min} + 10 \le n \le 2n_{min} + 12 + \left\lceil \log_2 \frac{\|T_1 - T_3\|}{\|T_2 - T_3\|} \right\rceil . \tag{4.13}$$

We summarize (4.2) and (4.13) in

Corollary 4.1: The algorithm φ^* using N_n^* computes the topological degree for every f in F. The information N_n^* has almost minimal cardinality, since $n = 2n_{min}(1 + o(1))$ as $\eta \to 0$. □

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