**A Useful Lemma**

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The following Lemma is extremely useful in many limiting distribution arguments, especially those involving martingales or martingale differences.

Consider the matrix $X$ which may be either **nonstochastic or stochastic**; we need to show that if it obeys $X'X/T \rightarrow M_{xx} > 0$, where convergence is either in the form of an ordinary limit (OL), or convergence in probability, or convergence a.c., then $(\max_{t \leq T} x_{t \cdot} x_{t \cdot}')/T \rightarrow 0$, in the same mode of convergence. More precisely, we have

**Lemma.** Consider the sequence $\{x_{t \cdot} : t \geq 1\}$, which may be either

i. one of nonstochastic vectors lying in a space $\mathcal{X}$;

ii. or one of random vectors defined on the probability space $(\Omega, \mathcal{A}, \mathrm{P})$.

If

$$\frac{X'X}{T} \overset{OL,\ P,\ or\ a.c.}{\longrightarrow} M_{xx},$$

i.e., either as an ordinary limit, or as convergence in probability, or as convergence a.c., where $M_{xx}$ is a well defined nonsingular (positive definite) matrix, the following statements are true:

i. $\mathrm{tr}(X'X/T) \overset{\mathrm{OL,\ P,\ or\ a.c.}}{\longrightarrow} \mathrm{tr} M_{xx}$;

ii. $\mathrm{tr} X'X \overset{\mathrm{OL,\ P,\ or\ a.c.}}{\longrightarrow} +\infty$;

iii. $(\max_{t \leq T} x_{t \cdot} x_{t \cdot}' / T) \overset{\mathrm{OL,\ P,\ or\ a.c.}}{\longrightarrow} 0$.

Proof: The proof of i. is obvious; in fact, since the trace can serve as a norm for symmetric matrices, i. implies convergence in norm; but if the norm converges, all elements converge, so that i. implies the premise, and that is true for all three modes of convergence.

The proof of ii. is equally obvious, since $\mathrm{tr}X'X = T\mathrm{tr}(X'X/T)$.

To prove iii. note that, putting $D_T = \sum_{t=1}^{T} x_{t\cdot}x_{t\cdot}'$, we have

$$\frac{D_T}{T} - \frac{D_{T-1}}{T-1}\frac{T-1}{T} = \frac{x_{T\cdot}x_{T\cdot}'}{T} \overset{\text{OL P or a.c.}}{\longrightarrow} 0. \tag{1}$$

We first examine convergence as an ordinary limit; thus, suppose Eq. (1) holds but

$$\lim_{T\to\infty} \frac{\max_{t\leq T} x_{t\cdot}x_{t\cdot}'}{T} = c > 0;$$

then there exists a (sub)sequence $\{T_i : i \geq 1\}$, such that

$$\frac{\max_{t\leq T_i} x_{t\cdot}x_{t\cdot}'}{T_i} \geq c - \frac{1}{q} > 0$$

for an arbitrary integer $q$; this implies the existence of a further subsequence

$$\frac{x_{t_i\cdot}x_{t_i\cdot}'}{t_i} \geq c - \frac{1}{q} > 0$$

and, consequently, that

$$\lim_{i\to\infty} \frac{x_{t_i\cdot}x_{t_i\cdot}'}{t_i} = c > 0,$$

which is a contradiction of Eq. (1). Thus,

$$\lim_{T\to\infty} \frac{\max_{t\leq T} x_{t\cdot}x_{t\cdot}'}{T} = 0,$$

which proves iii. for nonstochastic sequences. For stochastic sequences, define the sets, for arbitrary integer $r$,

$$A_{t,r} = \{\omega : x_{t\cdot}x_{t\cdot}' > \frac{t}{r}\} = \{\omega : \frac{x_{t\cdot}x_{t\cdot}'}{T} > \frac{t}{Tr},\ \ t \leq T\}, \tag{2}$$

and note that $A_{tT} \subset A_{t,r}$, where

$$A_{tT} = \{\omega : \frac{x_{t\cdot}x_{t\cdot}'}{T} > \frac{1}{r},\ \ t \leq T\}.$$

From Eq. (1), it follows that if the convergence therein is **convergence in probability**,

$$\lim_{T\to\infty} \mathcal{P}(A_{t,r}) = 0; \tag{3}$$

if the convergence therein is **convergence a.c.**, and $A_r^* = \limsup_{t\to\infty} A_{t,r}$ ,

$$\mathcal{P}(A_r^*) = 0. \tag{4}$$

Next, consider the sets

$$B_{T,r} = \{\omega : \frac{\max_{t\leq T} x_{t\cdot}x_{t\cdot}^{'}}{T} > \frac{1}{r}\}; \tag{5}$$

we shall complete the proof of iii. if show that when in Eq. (1) we have convergence in the probability, or convergence a.c., then

$$\lim_{T\to\infty} \mathcal{P}(B_{T,r}) = 0, \quad \mathcal{P}(B_r^*) = 0, \quad \text{respectively,} \tag{6}$$

for arbitrary $r$ , where $B_r^* = \limsup_{T\to\infty} B_{T,r}$ . Thus, suppose we have convergence a.c., and $\mathcal{P}(B^*) = \phi > 0$ . This means that $\omega \in B_r^*$ implies $\omega \in B_{T,r}$ , **for infinitely many values of the index**, $T$ , i.e., that the "event" $\omega \in B_{T,r}$ , *i.o.*, occurs **with positive probability**. Consequently, there exists a subsequence, say $\{T_i : i \geq 1\}$ , for which $\omega \in B_r^*$ implies $\omega \in B_{T_i,r}$ and, moreover,that $\lim_{i\to\infty} B_{T_i,r} = B_r^*$ . Let $\frac{\max_{t\leq T_i} x_{t\cdot}x_{t\cdot}^{'}}{T_i} = \frac{x_{t_i\cdot}x_{t_i\cdot}^{'}}{T_i}$ . Since

$$A_{t_i,r} \supset A_{t_iT_i} \supset B_{T_i,r}, \quad i \geq 1, \tag{7}$$

it follows that $\lim_{i\to\infty} \mathcal{P}(A_{t_i,r}) \geq \phi > 0$ , which is a contradiction of Eq. (4); consequently, $\mathcal{P}(B_r^*) = 0$ , as required.

Next, suppose in Eq. (1) we have convergence in probability, and

$$\lim_{T\to\infty} \mathcal{P}(B_{T,r}) = \phi > 0;$$

then there exists a subsequence $\{T_i : i \geq 1\}$ , for which

$$\mathcal{P}(B_{T_i,r}) > \phi - \frac{1}{q},$$

for arbitrary integer, $q$ . By Eq. (7), there exists a corresponding subsequence $\{A_{t_i,r} : i \geq 1\}$ , for which

$$\lim_{i\to\infty} \mathcal{P}(A_{t_i,r}) \geq \phi > 0,$$

which is a contradiction of Eq. (3); thus, $\lim_{T\to\infty} \mathcal{P}(B_{T,r}) = 0$ , as required.

q.e.d.

**Corollary.** Under the conditions of the Lemma

$$\frac{\max_{t \le T} x'_{t\cdot} x_{t\cdot}}{T} \overset{\text{OL P or a.c.}}{\longrightarrow} = 0,$$

according as $(X'X/T) \overset{\text{OL P or a.c.}}{\longrightarrow} = 0$, respectively.

Proof: We note that $||x'_{t\cdot} x_{t\cdot}|| = x_{t\cdot} x'_{t\cdot}$. Thus the Corollary follows from iii. of the Lemma.

q.e.d.