Approximation of Linear Operators on a Wiener Space

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Abstract.

We study optimal algorithms and optimal information in an average case model for linear problems in a Wiener space. We show that a linear algorithm is optimal among all algorithms. We illustrate the theory by interpolation, integration and approximation. We prove that adaption does not help.

1. Introduction

In a series of pioneering papers commencing with [4], Larkin studied average case error, mostly for linear problems in a Hilbert space equipped with a Gaussian measure. The average case model was further developed in [8], [13], and [14].

Following the average case model of [13], in this paper we study linear problems in a Wiener space. A Wiener space is a Banach space of continuous functions equipped with a Wiener measure. Linear problems in a Wiener space were first studied in [7], where optimality was considered in the class of linear algorithms. This paper investigates optimality in the class of all algorithms. It also studies optimal information and adaptive information.

We summarize the main contents of this paper.

In section 3 we formulate the problem and recall the concepts of information, algorithm, radius of information, optimal information and optimal algorithm.

We address the problem of interpretation in section 4 and we derive the optimal algorithm, which turns out to be linear, and the radius of information.

Based on the results in section 4, we study the problem of approximation of continuous linear functionals in section 5. We derive the optimal algorithm and the radius of information. As a specific case, we investigate the problem of integration.

In section 6 we study the problem of approximation of bounded linear operators. As a specific case we study the approximation problem.

In section 7 we discuss adaptive information versus nondapative information, and we show that adaption does not help for linear problems in a Wiener space.

2. Wiener Space.

Since the original work by N. Wiener in the 1920's, Wiener measures have received a great deal of attention, because of their usefulness in the applied fields of statistical and quantum mechanics as much as for their intrinsic mathematical interest, see [15], [16], [1] and [2].

In this section we recall the definition of the classical Wiener space and measure; for more detailed discussion, see [3].

Let F_1 denote the set of real-valued continuous functions f in the unit interval $[0,1]$ with $f(0) = 0$. F_1 is a Banach space with the supremum norm $\|f\| = \sup_{0\le t\le 1} |f(t)|$. Let $\mathcal{B}$ be the Borel σ-field of F_1, and let w be a Wiener measure defined on $\mathcal{B}$. Recall that w is uniquely defined by

$$(2.1)\qquad w(\{f \in F_1 : (f(t_1),\ldots,f(t_n)) \in E\}) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} (t_i - t_{i-1})^{-\frac{1}{2}} \int_E \exp\Big[-\frac{1}{2}\sum_{i=1}^{n} \frac{(u_i-u_{i-1})^2}{t_i - t_{i-1}}\Big] \times du_1 \ldots du_n,$$

where $n \geq 1$, $0 = t_0 < t_1 \ldots < t_n \leq 1$, $u_0 = 0$, and E is a Borel set in R^n. Here $du_1 \ldots du_n$ denotes the Lebesgue measure in R^n. The space F_1 with a Wiener measure is called a Wiener space. For a measurable function $G: F_1 \to R$, $\int_{F_1} G(f)w(df)$ is understood as the Lebesgue integral with respect to w. If $G(f) = V(f(t_1),\ldots,f(t_n))$, where $V: R^n \to R$ and $0 < t_1 < \ldots < t_n \leq 1$, then

$$(2.2)\qquad \int_{F_1} G(f)w(df) = \int_{F_1} V(f(t_1),\ldots,f(t_n))w(df)$$

$$= (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} (t_i - t_{i-1})^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} V(u_1,\ldots,u_n)\exp\Big[-\frac{1}{2}\sum_{i=1}^{n} \frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\Big] \times du_1 \ldots du_n,$$

where $t_0 = 0$ and $u_0 = 0$.

In particular, see [3, p. 38], for $G(f) = f(t_1)f(t_2)$, where $0 \leq t_1, t_2 \leq 1$,

$$(2.3) \qquad \int_{F_1} f(t_1)f(t_2)w(df) = \min\{t_1, t_2\}.$$

We need the following

<u>Proposition 2.1</u>: If $s(t)$ is of bounded variation, continuous from the right and $s(0) = 0$, then

$$\text{(i)} \quad \int_{F_1}\int_0^1 f(t)ds(t)\cdot f(\bar{t})w(df) = \int_0^{\bar{t}} tds(t) + \bar{t}\int_{\bar{t}}^1 ds(t),$$

where $0 \leq \bar{t} \leq 1$,

$$\text{(ii)} \quad \int_{F_1}[\int_0^1 f(t)ds(t)]^2 w(df)$$

$$= \int_0^1[\int_0^t uds(u)]ds(t) + \int_0^1[t\int_t^1 ds(u)]ds(t).$$

If $s(t)$ is continuous, then

$$\text{(iii)} \quad \int_{F_1}\int_0^1 f(t)s(t)dt\cdot f(\bar{t})w(df) = \int_0^{\bar{t}} ts(t)dt+\bar{t}\int_{\bar{t}}^1 s(t)dt,$$

where $0 \leq \bar{t} \leq 1$.

$$\text{(iv)} \quad \int_{F_1}[\int_0^1 f(t)s(t)dt]^2 w(df)$$

$$= \int_0^1[s(t)\int_0^t us(u)du]dt + \int_0^1[ts(t)\int_t^1 s(u)du]dt.$$

For the proof, see [5]. ∎

3. Formulation of the Problem.

Let F_1 be a Wiener space, and let F_2 be a separable Hilbert space. Let

$$(3.1) \qquad S: F_1 \to F_2$$

be a continuous linear operator, called a solution operator.

We seek an approximation to $S(f)$ for all $f \in F_1$, given function values of f at n points: $0 < t_1 <\ldots< t_n \le 1$. That is, the information N is defined as $N: F_1 \to R^n$, and

$$(3.2) \qquad N(f) = [f(t_1),\ldots,f(t_n)], \text{ for all } f \in F_1.$$

An approximation to $S(f)$ is provided by $\varphi(N(f))$ where

$$(3.3) \qquad \varphi: N(F_1) \to F_2.$$

We call φ an algorithm using information N. The (global average) error of φ is defined as

$$(3.4) \qquad e(\varphi,N) = \{\int_{F_1} \|S(f)-\varphi(N(f))\|^2 w(df)\}^{1/2}.$$

Let $\Phi(N)$ be the class of all algorithms φ using N for which the error of φ is well defined, i.e., $\|S(\cdot) - \varphi(N(\cdot))\|^2$ is a measurable function. We stress that the assumption about the measurability of $\|S(\cdot) - \varphi(N(\cdot))\|^2$ is not restrictive as is shown in [11].

We wish to find an algorithm φ^* from $\Phi(N)$ with the smallest error. Such an algorithm is called an <u>optimal algorithm</u>, and its error is called the <u>radius of information</u>, denoted by

$$(3.5) \qquad r(N) = e(\varphi^*,N) = \inf_{\varphi\in\Phi(N)} e(\varphi,N).$$

An n-th optimal information N^* minimizes the radius of information among all information $\Psi = \{N: N(f)=[f(t_1),\ldots,f(t_n)],\ 0 < t_1 <\ldots< t_n \le 1\}$, i.e.,

$$(3.6) \qquad r(N^*) = \inf_{N\in\Psi} r(N).$$

To verify whether an algorithm is optimal, we need

<u>Lemma 3.1</u>: Given information N, an algorithm $\varphi^* \in \Phi(N)$ is optimal iff

$$(3.7) \qquad \int_{F_1} (S(f) - \varphi^*(N(f)),\varphi(N(f)))w(df) = 0$$

for all $\varphi \in \Phi(N)$. ∎

The proof is similar to that of theorem 4.4 in [13] and is omitted.

From Lemma 3.1, we can easily derive

<u>Corollary 3.1</u>: Given information N, let φ_1^* and φ_2^* be optimal algorithms for the continuous linear solution

operators S_1 and S_2, respectively. Then the algorithm $\varphi^* = \alpha_1\varphi_1^* + \alpha_2\varphi_2^*$ is an optimal algorithm for the solution operator $S = \alpha_1 S_1 + \alpha_2 S_2$, where α_1 and α_2 are arbitrary real numbers. ∎

4. Interpolation.

In this section we study the interpolation problem, that is, we approximate

$$S(f) = f(\bar{t}), \text{ where } 0 \le \bar{t} \le 1,$$

given information

$$(4.1) \qquad N(f) = [f(t_1),\dots,f(t_n)], \text{ where } 0 < t_1 < \dots < t_n \le 1.$$

The solution of the more general problems will follow from the solution of this simple problem. We shall show that there exists an optimal linear algorithm, which is piecewise linear interpolation. The radius of information will also be derived.

We first prove the optimality of piecewise linear interpolation. Let $f_k = f(t_k)$, $k = 1,\dots,n$, and let $f_0 = 0$ and $t_0 = 0$. We have

Theorem 4.1: For the interpolation problem, piecewise linear interpolation is optimal. More specifically, let

(4.2) $\varphi^*(f_1,\dots,f_n)$

$$= \begin{cases} \dfrac{t_{k+1}-\bar t}{t_{k+1}-t_k} f_k + \dfrac{\bar t - t_k}{t_{k+1}-t_k} f_{k+1}, & \text{if } t_k \le \bar t \le t_{k+1}, \text{ for some } k \text{ from } \{0,\dots,n-1\}, \\ f_n, & \text{if } t_n < \bar t \le 1. \end{cases}$$

Then φ^* is an optimal linear algorithm among all algorithms from $\Phi(N)$. ■

Proof: It is obvious that φ^* is optimal if $\bar t = t_k$, for some k from $\{0,\dots,n\}$, since $e(N,\varphi^*) = 0$ for this case. Thus it is sufficient to consider the following two cases: (i) $t_k < \bar t < t_{k+1}$, $k = 0,1,\dots,n-1$; (ii) $t_n < \bar t \le 1$, if $t_n < 1$.

Case (i). By Lemma 3.1, we need only to show that

$$(4.3)\quad I = \int_{F_1} [f(\bar t) - \frac{t_{k+1}-\bar t}{t_{k+1}-t_k} f_k - \frac{\bar t - t_k}{t_{k+1}-t_k} f_{k+1}]\varphi(f_1,\dots,f_n)w(df) = 0$$

for all $\varphi \in \Phi(N)$. Let

$$(4.4)\qquad I = I_1 - I_2 - I_3,$$

where

$$I_1 = \int_{F_1} f(\bar t)\varphi(f_1,\dots,f_n)w(df),$$

$$I_2 = \int_{F_1} \frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} f_k \varphi(f_1,\ldots,f_n) w(df),$$

and

$$I_3 = \int_{F_1} \frac{\bar{t}-t_k}{t_{k+1}-t_k} f_{k+1} \varphi(f_1,\ldots,f_n) w(df).$$

Let $f_{\bar{t}} = f(\bar{t})$. Then from (2.2) we have

$$(4.5)\quad I_2 = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} (t_i - t_{i-1})^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} u_k \varphi(u_1,\ldots,u_n) \times \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\right] du_1\ldots du_n,$$

$$(4.6)\quad I_3 = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} (t_i - t_{i-1})^{-\frac{1}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\bar{t}-t_k}{t_{k+1}-t_k} u_{k+1} \varphi(u_1,\ldots,u_n) \times \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\right] du_1\ldots du_n,$$

where $u_0 = 0$, and

$$(4.7)\quad I_1 = (2\pi)^{-\frac{n+1}{2}} \left\{\left[\prod_{i=1}^{k} (t_i - t_{i-1})^{-\frac{1}{2}}\right] (\bar{t}-t_k)^{-\frac{1}{2}} (t_{k+1}-\bar{t})^{-\frac{1}{2}} \times \left[\prod_{i=k+2}^{n} (t_i-t_{i-1})^{-\frac{1}{2}}\right]\right\}$$

$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} u_{\bar{t}} \varphi(u_1,\ldots,u_n) \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{k} \frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}} + \frac{(u_{\bar{t}}-u_k)^2}{\bar{t}-t_k} + \frac{(u_{k+1}-u_{\bar{t}})^2}{t_{k+1}-\bar{t}} + \sum_{i=k+2}^{n} \frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\right]\right\}$$

$$\times du_1\ldots du_k du_{\bar{t}} du_{k+1}\ldots du_n$$

$$= (2\pi)^{-\frac{n+1}{2}}\left[\prod_{i=1}^{n}(t_i-t_{i-1})^{-\frac{1}{2}}\right]\left[(t_{k+1}-t_k)^{\frac{1}{2}}(\bar{t}-t_k)^{-\frac{1}{2}}(t_{k+1}-\bar{t})^{-\frac{1}{2}}\right]$$

$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(u_1,\ldots,u_n)\exp\left[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\right]\exp\left[\frac{1}{2}\frac{(u_{k+1}-u_k)^2}{t_{k+1}-t_k}\right]$$

$$\times \left\{\int_{-\infty}^{\infty} u_{\bar{t}}\exp\left\{-\frac{1}{2}\left[\frac{(u_{\bar{t}}-u_k)^2}{\bar{t}-t_k}+\frac{(u_{k+1}-u_{\bar{t}})^2}{t_{k+1}-\bar{t}}\right]\right\}du_{\bar{t}}\right\}du_1\ldots du_n.$$

Since

$$\int_{-\infty}^{\infty} u_{\bar{t}}\ \exp\left\{-\frac{1}{2}\left[\frac{(u_{\bar{t}}-u_k)^2}{\bar{t}-t_k}+\frac{(u_{k+1}-u_{\bar{t}})^2}{t_{k+1}-\bar{t}}\right]\right\}du_{\bar{t}}$$

$$= (2\pi)^{\frac{1}{2}}\left[(t_{k+1}-t_k)^{-\frac{1}{2}}(\bar{t}-t_k)^{\frac{1}{2}}(t_{k+1}-\bar{t})^{\frac{1}{2}}\right]\frac{(\bar{t}-t_k)u_{k+1}+(t_{k+1}-\bar{t})u_k}{t_{k+1}-t_k}$$

$$\times \exp\left[-\frac{1}{2}\frac{(u_{k+1}-u_k)^2}{t_{k+1}-t_k}\right],$$

we have

$$(4.8)\quad I_1 = (2\pi)^{-\frac{n+1}{2}}\left[\prod_{i=1}^{n}(t_i-t_{i-1})^{-\frac{1}{2}}\right]\left[(t_{k+1}-t_k)^{\frac{1}{2}}(\bar{t}-t_k)^{-\frac{1}{2}}(t_{k+1}-\bar{t})^{-\frac{1}{2}}\right]$$

$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big\{\varphi(u_1,\ldots,u_n)\exp\left[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\right]$$

$$\times \exp\left[\frac{1}{2}\frac{(u_{k+1}-u_k)^2}{t_{k+1}-t_k}\right](2\pi)^{\frac{1}{2}}\left[(t_{k+1}-t_k)^{-\frac{1}{2}}(\bar{t}-t_k)^{\frac{1}{2}}\right.$$

$$\left.\times(t_{k+1}-\bar{t})^{\frac{1}{2}}\right]\frac{(\bar{t}-t_k)u_{k+1}+(t_{k+1}-\bar{t})u_k}{t_{k+1}-t_k}\exp\left[-\frac{1}{2}\frac{(u_{k+1}-u_k)^2}{t_{k+1}-t_k}\right]\Big\}du_1\ldots du_n$$

$$= (2\pi)^{-\frac{n}{2}}\left[\prod_{i=1}^{n}(t_i-t_{i-1})^{-\frac{1}{2}}\right]\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k}u_k+\frac{\bar{t}-t_k}{t_{k+1}-t_k}u_{k+1}\right]\varphi(u_1,\ldots,u_n)$$

$$\times \exp[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}]du_1\ldots du_n.$$

Comparing (4.8) with (4.5) and (4.6), we have $I = I_1 - I_2 - I_3 = 0$.

Case (ii). By Lemma 3.1, we need only to show that

$$(4.9) \quad J = \int_{F_1}[f(\bar{t})-\varphi^*(f_1,\ldots,f_n)]\varphi(f_1,\ldots,f_n)w(df) = 0$$

$$\text{for all } \varphi \in \Phi(N).$$

From (4.2) we have

$$(4.10) \quad J = \int_{F_1}(f_{\bar{t}}-f_n)\varphi(f_1,\ldots,f_n)w(df)$$

$$= \int_{F_1}f_{\bar{t}}\varphi(f_1,\ldots,f_n)w(df)-\int_{F_1}f_n\varphi(f_1,\ldots,f_n)w(df).$$

We now compute

$$\int_{F_1}f_{\bar{t}}\varphi(f_1,\ldots,f_n)w(df) = (2\pi)^{-\frac{n+1}{2}}[\prod_{i=1}^{n}(t_i-t_{i-1})^{-\frac{1}{2}}](\bar{t}-t_n)^{-\frac{1}{2}}$$

$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}u_{\bar{t}}\varphi(u_1,\ldots,u_n)\exp[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}]$$

$$\times \exp[-\frac{1}{2}\frac{(u_{\bar{t}}-u_n)^2}{\bar{t}-t_n}]du_1\ldots du_n du_{\bar{t}}$$

$$= (2\pi)^{-\frac{n+1}{2}}[\prod_{i=1}^{n}(t_i-t_{i-1})^{-\frac{1}{2}}](\bar{t}-t_n)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(u_1,\ldots,u_n)$$

$$\times \exp[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}]$$

$$\times \{\int_{-\infty}^{\infty} u_{\bar t} \exp[-\frac{1}{2}\frac{(u_{\bar t}-u_n)^2}{\bar t - t_n}] du_{\bar t}\} du_1 \ldots du_n.$$

Since

$$\int_{-\infty}^{\infty} u_{\bar t} \exp[-\frac{1}{2}\frac{(u_{\bar t}-u_n)^2}{\bar t - t_n}] du_{\bar t} = (\bar t - t_n)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} u_n,$$

we have

$$(4.11) \quad \int_{F_1} f_{\bar t}\phi(f_1,\ldots,f_n) w(df)$$

$$= (2\pi)^{-\frac{n+1}{2}} [\prod_{i=1}^{n} (t_i - t_{i-1})^{-\frac{1}{2}}] (\bar t - t_n)^{-\frac{1}{2}}$$

$$\times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\phi(u_1,\ldots,u_n) \exp[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i - t_{i-1}}] (\bar t - t_n)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} u_n\} du_1 \ldots du_n$$

$$= (2\pi)^{-\frac{n}{2}} [\prod_{i=1}^{n} (t_i - t_{i-1})^{-\frac{1}{2}}] \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_n \phi(u_1,\ldots,u_n)$$

$$\times \exp[-\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{t_i - t_{i-1}}] du_1 \ldots du_n$$

$$= \int_{F_1} f_n \phi(f_1,\ldots,f_n) w(df).$$

From (4.11) and (4.10), we have (4.9). This completes the proof. ∎

Recall that the radius of information is the error of the optimal algorithm. From Theorem 4.1, we have

Theorem 4.2: For the interpolation problem, the radius of information is

$$
(4.4)\qquad r(N) = \begin{cases} 0\,, & \text{if } \bar{t} = t_k, \text{ for some } k \text{ from } \{0,\ldots,n\}; \\ \sqrt{\dfrac{(t_{k+1}-\bar{t})(\bar{t}-t_k)}{t_{k+1}-t_k}}\,, & \text{if } t_k < \bar{t} < t_k, \text{ for some } k \text{ from } \{0,\ldots,n-1\}; \\ \sqrt{\bar{t}-t_n}\,, & \text{if } t_n < \bar{t} \le 1. \end{cases}
$$

$\blacksquare$

<u>Proof:</u> It is obvious that $r(N) = 0$ if $\bar{t} = t_k$ for some k from $\{0,\ldots,n\}$. Suppose therefore that $t_k < \bar{t} < t_{k+1}$ for some $k = 0,1,\ldots,n-1$. Then

$$
r(N)^2 = e(N,\varphi^*)^2 = \int_{F_1} [f(\bar{t}) - \frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} f_k - \frac{\bar{t}-t_k}{t_{k+1}-t_k} f_{k+1}]^2 w(df)
$$

$$
= \int_{F_1} [f(\bar{t})]^2 w(df) + \left(\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k}\right)^2 \int_{F_1} [f(t_k)]^2 w(df)
$$

$$
+ \left(\frac{\bar{t}-t_k}{t_{k+1}-t_k}\right)^2 \int_{F_1} [f(t_{k+1})]^2 w(df)
$$

$$
- 2\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} \int_{F_1} f(\bar{t}) f(t_k) w(df) - 2\frac{\bar{t}-t_k}{t_{k+1}-t_k} \int_{F_1} f(\bar{t}) f(t_{k+1}) w(df)
$$

$$
+ 2\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} \, \frac{\bar{t}-t_k}{t_{k+1}-t_k} \int_{F_1} f(t_k) f(t_{k+1}) w(df)
$$

$$
= \bar{t} + \left(\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k}\right)^2 t_k + \left(\frac{\bar{t}-t_k}{t_{k+1}-t_k}\right)^2 t_{k+1} - 2\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} - 2\frac{\bar{t}-t_k}{t_{k+1}-t_k}\bar{t}
$$

$$
+ 2\frac{t_{k+1}-\bar{t}}{t_{k+1}-t_k} \, \frac{\bar{t}-t_k}{t_{k+1}-t_k} \cdot t_k = \frac{(t_{k+1}-\bar{t})(\bar{t}-t_k)}{t_{k+1}-t_k}.
$$

So

$$
r(N) = \sqrt{\frac{(t_{k+1}-\bar{t})(\bar{t}-t_k)}{t_{k+1}-t_k}}.
$$

Finally, suppose that $t_n < t \leq 1$, then

$$r(N)^2 = e(N,\varphi^*)^2 = \int_{F_1} [f(\bar{t})-f_n]^2 w(df) = \int_{F_1} [f(\bar{t})]^2 w(df)$$

$$- 2\int_{F_1} f(\bar{t}) f(t_n) w(df) + \int_{F_1} [f(t_n)]^2 w(df) = \bar{t}-t_n.$$

So

$$r(N) = \sqrt{\bar{t}-t_n},$$

which completes the proof. ∎

5. Approximation of Continuous Linear Functionals.

In this section, we consider the optimal algorithm and the radius of information for a solution operator S, which is a continuous linear functional. The problem of integration is considered as a specific case.

Since F_1 is a subspace of the space $C[0,1]$, S has a continuous linear extension to C. Therefore, by the Riesz representation Theorem,

$$(5.1) \qquad S(f) = \int_0^1 f(t)d\, s(t),$$

where s is of bounded variation, continuous from the right, and $s(0) = 0$.

Given information as in (4.1), we have

Theorem 5.1: For the solution operator S of the form (5.1),

$$(5.2) \qquad \varphi^*(f_1,\ldots,f_n) = \sum_{i=1}^{n} \beta_i f_i$$

is an optimal linear algorithm among all algorithms from $\Phi(N)$, where

$$f_i = f(t_i), \quad i = 1,\ldots,n.$$

$$(5.3) \qquad \beta_i = \frac{1}{t_{i+1}-t_i}[t_{i+1}\int_{t_i}^{t_{i+1}} ds(t) - \int_{t_i}^{t_{i+1}} t\,ds(t)]$$

$$- \frac{1}{t_i-t_{i-1}}[t_{i-1}\int_{t_{i-1}}^{t_i} ds(t) - \int_{t_{i-1}}^{t_i} t\,ds(t)],$$

$i = 1,\ldots,n-1$, and

$$\beta_n = \int_{t_n}^{1} ds(t) + \frac{1}{t_n-t_{n-1}}[\int_{t_{n-1}}^{t_n} t\,ds(t) - t_{n-1}\int_{t_{n-1}}^{t_n} ds(t)]. \quad \blacksquare$$

<u>Proof:</u> For $0 = t_0 < t_1 <\ldots< t_n \le t_{n+1} = 1$, let $\Delta^{(i)} = \frac{1}{m}(t_i-t_{i-1})$, and let $t_j^{(i)} = t_{i-1} + j\Delta^{(i)}$, $j = 0,1,\ldots,m$; $i = 1,\ldots,n+1$. By the definition of Riemann-Stieltjes integral we have

$$\int_0^1 f(t)ds(t) = \lim_{m\to\infty} \sum_{i=1}^{n+1} \sum_{j=0}^{m-1} f(t_j^{(i)})[s(t_{j+1}^{(i)})-s(t_j^{(i)})].$$

We use the solution of the interpolation problem for each $t_j^{(i)}$ to solve our problem. By Theorem 4.1, we have

$$\int_{F_1} \{f(t_j^{(i)})-[\frac{t_i-t_j^{(i)}}{t_i-t_{i-1}} f(t_{i-1}) + \frac{t_j^{(i)}-t_{i-1}}{t_i-t_{i-1}} f(t_i)]\}\varphi(f_1,\ldots,f_n)w(df)$$

$$= 0,$$

and

$$\int_{F_1} [f(t_j^{(n+1)})-f(t_n)]\varphi(f_1,\ldots,f_n)w(df) = 0,$$

for all $\varphi \in \Phi(N)$ and $i = 1,\ldots,n$, $j = 0,1,\ldots,m$.

Thus

$$\int_{F_1} \{ \sum_{i=1}^{n+1} \sum_{j=0}^{m-1} f(t_j^{(i)})[s(t_{j+1}^{(i)})-s(t_j^{(i)})] - \sum_{i=1}^{n} \sum_{j=0}^{m-1} [\frac{t_i-t_j^{(i)}}{t_i-t_{i-1}} f(t_{i-1}) + \frac{t_j^{(i)}-t_{i-1}}{t_i-t_{i-1}} f(t_i)] \times [s(t_{j+1}^{(i)})-s(t_j^{(i)})] - \sum_{j=0}^{m-1} f(t_n)[s(t_{j+1}^{(n+1)})-s(t_j^{(n+1)})]\} \times \varphi(f_1,\ldots,f_n)w(df) = 0,$$

and so

$$\text{(5.4)} \quad \lim_{m\to\infty} \int_{F_1} \{ \sum_{i=1}^{n+1} \sum_{j=0}^{m-1} f(t_j^{(i)})[s(t_{j+1}^{(i)})-s(t_j^{(i)})] - \sum_{i=1}^{n} \sum_{j=0}^{m-1} [\frac{t_i-t_j^{(i)}}{t_i-t_{i-1}} f(t_{i-1}) + \frac{t_j^{(i)}-t_{i-1}}{t_i - t_{i-1}} f(t_i)] \times [s(t_{j+1}^{(i)})-s(t_j^{(i)})] - \sum_{j=0}^{m-1} f(t_n)[s(t_{j+1}^{(n+1)})-s(t_j^{(n+1)})]\} \times \varphi(f_1,\ldots,f_n)w(df) = 0.$$

From the definition of Riemann-Stieltjes integral we have

$$(5.5)\qquad \lim_{m\to\infty}\int_{F_1}\sum_{j=0}^{m-1} f(t_j^{(i)})[s(t_{j+1}^{(i)})-s(t_j^{(i)})]\varphi(f_1,\ldots,f_n)w(df)$$

$$= \int_{t_{i-1}}^{t_i}[\int_{F_1} f(t)\varphi(f_1,\ldots,f_n)w(df)]ds(t),\ i = 1,\ldots,n+1.$$

We shall show that

$$(5.6)\qquad \int_{t_{i-1}}^{t_i}[\int_{F_1} f(t)\varphi(f_1,\ldots,f_n)w(df)]ds(t)$$

$$= \int_{F_1}[\int_{t_{i-1}}^{t_i} f(t)\varphi(f_1,\ldots,f_n)ds(t)]w(df),$$

and the proof will be completed, since from (5.4), (5.5) and (5.6) we have

$$\int_{F_1}[\int_0^1 f(t)ds(t)-\varphi^*(f_1,\ldots,f_n)]\varphi(f_1,\ldots,f_n)w(df)$$

$$= \int_{F_1}[\int_0^1 f(t)ds(t) - \sum_{i=1}^{n} \beta_i f_i]\varphi(f_1,\ldots,f_n)w(df)$$

$$= \int_{F_1}\{\int_0^1 f(t)ds(t)-\{\sum_{i=1}^{n-1}\{\frac{1}{t_{i+1}-t_i}[t_{i+1}\int_{t_i}^{t_{i+1}}ds(t)-\int_{t_i}^{t_{i+1}}tds(t)]$$

$$-\frac{1}{t_i-t_{i-1}}[t_{i-1}\int_{t_{i-1}}^{t_i} ds(t)-\int_{t_{i-1}}^{t_i} tds(t)]\}f(t_i)$$

$$+\{\int_{t_n}^1 ds(t)-\frac{1}{t_n-t_{n-1}}[\int_{t_{n-1}}^{t_n} tds(t)-t_{n-1}\int_{t_{n-1}}^{t_n} ds(t)]\}f(t_n)\}\}$$

$$\times\ \varphi(f_1,\ldots,f_n)w(df)$$

$$= \int_{F_1}\{\sum_{i=1}^{n+1}\int_{t_{i-1}}^{t_i} f(t)ds(t)-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}[\frac{t_i-t}{t_i-t_{i-1}}f(t_{i-1})$$

$$+ \frac{t-t_{i-1}}{t_i-t_{i-1}} f(t_i)]ds(t) - \int_{t_n}^{1} f(t_n)ds(t)\}\varphi(f_1,\ldots,f_n)w(df)$$

$$= \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} \{\int_{F_1} f(t)\varphi(f_1,\ldots,f_n)w(df)\}ds(t)$$

$$- \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \{\int_{F_1} [\frac{t_i-t}{t_i-t_{i-1}} f(t_{i-1}) + \frac{t-t_{i-1}}{t_i-t_{i-1}} f(t_i)]$$

$$\times \varphi(f_1,\ldots,f_n)w(df)\}ds(t) - \int_{t_n}^{1}\{\int_{F_1} f(t_n)\varphi(f_1,\ldots,f_n)w(df)\}ds(t)$$

$$= \lim_{m\to\infty} \int_{F_1} \{ \sum_{i=1}^{n+1} \sum_{j=0}^{m-1} f(t_j^{(i)})[s(t_{j+1}^{(i)}) - s(t_j^{(i)})]$$

$$- \sum_{i=1}^{n} \sum_{j=0}^{m-1} [\frac{t_i-t_j^{(i)}}{t_i-t_{i-1}} f(t_{i-1}) + \frac{t_j^{(i)}-t_{i-1}}{t_i-t_{i-1}} f(t_i)][s(t_{j+1}^{(i)})-s(t_j^{(i)})]$$

$$- \sum_{j=0}^{m-1} f(t_n)[s(t_{j+1}^{(n+1)})-s(t_j^{(n+1)})]\}\varphi(f_1,\ldots,f_n)w(df) = 0,$$

i.e.,

$$\int_{F_1} [\int_0^1 f(t)ds(t) - \sum_{i=1}^{n} \beta_i f_i]\varphi(f_1,\ldots,f_n)w(df) = 0 \text{ for all}$$

$$\varphi \in \mathfrak{F}(N),$$

where β_i's are given in (5.3).

We now derive (5.6).

Let $G(t,f) = f(t)\varphi(f_1,\ldots,f_n)$ and let $f_t = f(t)$ for $t_{i-1} < t < t_i$. Then

$$\int_{t_{i-1}}^{t_i} G(t,f)ds(t) = \varphi(f_1,\ldots,f_n)\int_{t_{i-1}}^{t_i} f(t)ds(t).$$

Since $\varphi(f_1,\ldots,f_n) \in L_2(F_1,w)$ and $\int_{t_{i-1}}^{t_i} f(t)ds(t) \in L_2(F_1,w)$,

$$(5.7) \qquad \int_{t_{i-1}}^{t_i} G(t,f)ds(t) \in L_1(F_1,w).$$

On the other hand, since $t_{i-1} < t < t_i$,

$$\int_{F_1} G(t,f)w(df) = \int_{F_1} f_t\varphi(f_1,\ldots,f_n)w(df) = (2\pi)^{-\frac{n+1}{2}}$$

$$\times \Big[\prod_{j\neq i}(t_j-t_{j-1})^{-\frac{1}{2}}\Big](t_i-t)^{-\frac{1}{2}}(t-t_{i-1})^{-\frac{1}{2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}u_t\varphi(u_1,\ldots,u_n)$$

$$\times \exp\Big[-\frac{1}{2}\sum_{j\neq i}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big]\exp\Big[-\frac{1}{2}\frac{(u_i-u_t)^2}{t_i-t}-\frac{1}{2}\frac{(u_t-u_{i-1})^2}{t-t_{i-1}}\Big]$$

$$\times\ du_1\ldots du_{i-1}du_t du_i\ldots du_n$$

$$= (2\pi)^{-\frac{n+1}{2}}\Big[\prod_{j\neq i}(t_j-t_{j-1})^{-\frac{1}{2}}\Big](t_i-t)^{-\frac{1}{2}}(t-t_{i-1})^{-\frac{1}{2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(u_1,\ldots,u_n)$$

$$\times \exp\Big[-\frac{1}{2}\sum_{j\neq i}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big]\Big[\int_{-\infty}^{\infty}u_t\ \exp\Big[-\frac{1}{2}\frac{(u_i-u_t)^2}{t_i-t}$$

$$-\frac{1}{2}\frac{(u_t-u_{i-1})^2}{t-t_{i-1}}\Big]du_t\Big]du_1\ldots du_n.$$

Since

$$\int_{-\infty}^{\infty} u_t\ \exp\Big[-\frac{1}{2}\frac{(u_i-u_t)^2}{t_i-t}-\frac{1}{2}\frac{(u_t-u_{i-1})^2}{t-t_{i-1}}\Big]du_t$$

$$= (2\pi)^{\frac{1}{2}}\Big[(t_i-t_{i-1})^{-\frac{1}{2}}(t-t_{i-1})^{\frac{1}{2}}(t_i-t)^{\frac{1}{2}}\Big]\frac{(t-t_{i-1})u_i+(t_i-t)u_{i-1}}{t_i-t_{i-1}}$$

$$\times \exp\Big[-\frac{1}{2}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\Big],$$

(5.8) $\int_{F_1} G(t,f)w(df)$

$$= (2\pi)^{-\frac{n}{2}}\prod_{j=1}^{n}(t_j-t_{j-1})^{-\frac{1}{2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{(t-t_{i-1})u_i+(t_i-t)u_{i-1}}{t_i-t_{i-1}}$$

$$\times\, \varphi(u_1,\ldots,u_n)\exp[-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}]du_1\ldots du_n$$

$$= (2\pi)^{-\frac{n}{2}}\prod_{j=1}^{n}(t_j-t_{j-1})^{-\frac{1}{2}}[\frac{t-t_{i-1}}{t_i-t_{i-1}}\int_{F_1} f_i\varphi(f_1,\ldots,f_n)w(df)$$

$$+\frac{t_i-t}{t_i-t_{i-1}}\int_{F_1} f_{i-1}\varphi(f_1,\ldots,f_n)w(df)].$$

So $\int_{F_1} G(t,f)w(df)$ is integrable with respect to $s(t)$, and (5.6) follows from this fact, (5.7) and Fubini's theorem. ∎

From Theorem 5.1 and proposition 2.1, we can easily derive

<u>Theorem 5.2</u>: The radius of information $N(f) = [f(t_1),\ldots,f(t_n)]$, $0 < t_1 <\ldots< t_n \le 1$, for the solution operator S as in (5.1) is

$$(5.9)\qquad r(N) = [\int_0^1[\int_0^t u\,ds(u)]ds(t) + \int_0^1[t\int_t^1 ds(u)]ds(t)$$

$$+\sum_{i=1}^{n}\beta_i^2 t_i - 2\sum_{i=1}^{n}\beta_i[\int_0^{t_i} t\,ds(t)+t_i\int_{t_i}^1 ds(t)] + 2\sum_{1\le i<j\le n}\beta_i\beta_j t_i]^{\frac{1}{2}},$$ ∎

where β_i's are given in (5.3).

For the more specific solution operator

$$(5.10)\qquad S(f) = \int_0^1 f(t)s(t)dt, \text{ where } s(t) \text{ is continuous,}$$

we have

Theorem 5.3:

$$(5.11)\qquad \varphi^*(f_1,\ldots,f_n) = \sum_{i=1}^{n} \beta_i f_i$$

is an optimal linear algorithm among all algorithms from $\Phi(N)$, where

$$(5.12)\qquad \beta_i = \frac{1}{t_{i+1}-t_i}[t_{i+1}\int_{t_i}^{t_{i+1}} s(t)dt-\int_{t_i}^{t_{i+1}} ts(t)dt]$$

$$-\frac{1}{t_i-t_{i-1}}[t_{i-1}\int_{t_{i-1}}^{t_i} s(t)dt-\int_{t_{i-1}}^{t_i} ts(t)dt],$$

$$i = 1,2,\ldots,n-1,$$

and

$$\beta_n = \int_{t_n}^{1} s(t)dt + \frac{1}{t_n-t_{n-1}}[\int_{t_{n-1}}^{t_n} ts(t)dt-t_{n-1}\int_{t_{n-1}}^{t_n} s(t)dt].$$

The radius of information is

$$(5.13)\qquad r(N) = \{\int_0^1[s(t)\int_0^t us(u)du]dt + \int_0^1[ts(t)\int_t^1 s(u)du]dt$$

$$+\sum_{i=1}^{n}\beta_i^2 t_i-2\sum_{i=1}^{n}\beta_i[\int_0^{t_i} ts(t)dt+t_i\int_{t_i}^{1} s(t)dt$$

$$+ 2\sum_{1\le i<j\le n}\beta_i\beta_j t_i]^{\frac{1}{2}}. \qquad \blacksquare$$

We finish this section by considering the integration problem, i.e., we consider the solution operator

$$(5.14) \qquad S(f) = \int_0^1 f(t)dt,$$

which is a specific case of (5.10) when $s(t) \equiv 1$. From Theorem 5.3, we easily get

<u>Theorem 5.4</u>: Given information $N(f) = [f(t_1),\ldots,f(t_n)]$, $0 < t_1 <\ldots< t_n \leq 1$. For the integration problem,

$$(5.15) \qquad \varphi^*(f_1,\ldots,f_n) = \sum_{i=1}^{n-1} \frac{t_{i+1}-t_{i-1}}{2} f_i + \left(1-\frac{t_n+t_{n-1}}{2}\right) f_n$$

is the optimal linear algorithm among all algorithms from $\Phi(N)$, and the radius of information is

$$(5.16) \qquad r(N) = \Big\{\frac{1}{3} + \frac{1}{2}\, t_{n+1} t_n^2 - t_{n+1} t_n + \frac{1}{4} t_{n+1}^2 t_n + \frac{1}{4}\sum_{i=1}^{n} (t_i t_{i-1}^2 - t_i^2 t_{i-1})\Big\}^{\frac{1}{2}}. \qquad \blacksquare$$

We now find the n-th optimal information N^* for the integration problem. From (5.16) we have

$$\frac{\partial r^2(N^*)}{\partial t_i} = \frac{1}{4}(2t_{i+1}t_i - t_{i+1}^2 + t_{i-1}^2 - 2t_i t_{i-1}) = 0, \quad i = 1,\ldots,n-1,$$

so

$$t_{i+1}^* - t_i^* = t_i^* - t_{i-1}^*.$$

Let $t^*_{i+1}-t^*_i = t$. Then $t^*_i = it$. $i = 1,\ldots,n$, $t^*_{n+1} = 2-nt$, and

$$r(N^*)^2 = \frac{1}{3} - \frac{1}{12}n(4n^2-1)t^3 + n^2t^2 - nt.$$

Since

$$\frac{\partial r(N^*)^2}{\partial t} = -\frac{1}{4}n(4n^2-1)t^2 + 2n^2t-n = 0,$$

$$t = \frac{2}{2n+1}.$$

We summarize the above in

<u>Theorem 5.5</u>: For the integration problem, the n-th optimal information is $N^*(f) = [f(t^*_1),\ldots,f(t^*_n)]$, where

(5.17) $$t^*_i = \frac{2i}{2n+1}, \quad i = 1,\ldots,n.$$

The radius of information is

(5.18) $$r(N^*) = \frac{1}{\sqrt{3}(2n+1)}.$$

The optimal linear algorithm using this optimal information is

(5.19) $$\phi^*(f^*_1,\ldots,f^*_n) = \frac{2}{2n+1}\sum_{i=1}^{n} f\left(\frac{2i}{2n+1}\right). \quad \blacksquare$$

5. Approximation of Bounded Linear Operators.

In this section we study the approximation of bounded linear solution operators from a Wiener space F_1 to a separable Hilbert space F_2.

Let $\{e_1,\ldots,e_n,\ldots\}$ be an orthonormal basis in F_2. Then $S(f) = \Sigma_{j=1}^{\infty}(S(f),e_j)e_j = \Sigma_{j=1}^{\infty} S_j(f)e_j$, where $S_j(f) = (S(f),e_j)$, $j = 1,2,\ldots$, is a continuous linear functional on F_1. We denote a continuous linear extension of S_j to C by the same S_j, and we have

$$(6.1) \qquad S_j(f) = \int_0^1 f(t)ds_j(t),$$

where s_j is of bounded variation, continuous from the right, and $s_j(0) = 0$, $j = 1,2,\ldots$. It is straightforward to verify

Theorem 6.1: Given information $N(f) = [f(t_1),\ldots,f(t_n)]$, $0 < t_1 <\ldots< t_n \leq 1$, there exists a linear algorithm φ^*, optimal among all algorithms $\Phi(N)$, which is

$$(6.2) \qquad \varphi^*(N(f)) = \sum_{j=1}^{\infty} \varphi_j^*(N(f))e_j,$$

where φ_j^* are the optimal algorithms for the solution operator S_j, i.e.,

$$\varphi_j^*(f_1,\ldots,f_n) = \sum_{i=1}^{n} \beta_{ij}f_i,$$

$$\beta_{ij} = \frac{1}{t_{i+1}-t_i}[t_{i+1}\int_{t_i}^{t_{i+1}} ds_j(t) - \int_{t_i}^{t_{i+1}} t\,ds_j(t)]$$

$$- \frac{1}{t_i-t_{i-1}}[t_{i-1}\int_{t_{i-1}}^{t_i} ds_j(t) - \int_{t_{i-1}}^{t_i} t\,ds_j(t)],\ i = 1,\dots,n-1,$$

$$\beta_{nj} = \int_{t_n}^{1} ds_j(t) + \frac{1}{t_n-t_{n-1}}[\int_{t_{n-1}}^{t_n} t\,ds_j(t) - t_{n-1}\int_{t_{n-1}}^{t_n} ds_j(t)],$$

s_j is given in (6.1), $j = 1,2,\dots$.

The radius of information is

$$\text{(6.3)}\qquad r(N) = \{\sum_{j=1}^{\infty}\{\int_0^1[\int_0^t u\,ds_j(u)]ds_j(t) + \int_0^1[t\int_t^1 ds_j(t)]ds_j(t)$$

$$+ \sum_{i=1}^{n}\beta_{ij}^2 t_i - 2\sum_{i=1}^{n}\beta_{ij}[\int_0^{t_i} t\,ds_j(t) + t_i\int_{t_i}^1 ds_j(t)]$$

$$+ 2\sum_{1\le i<k\le n}\beta_{ij}\beta_{kj}t_i\}\}^{\frac{1}{2}}.\qquad\blacksquare$$

We now consider approximation of f in L_2-norm, that is, we have the solution operator $S:F_1 \to F_2$, where $S(f) = f$, and $F_2 = \{f: \|f\|_2 = \{\int_0^1 [f(t)]^2\,dt\}^{\frac{1}{2}}\}$. Applying Theorem 6.1 we conclude

<u>Theorem 6.2</u>: Given information $N(f) = [f(t_1),\dots,f(t_n)]$, $0 < t_1 < \dots < t_n \le 1$, for the problem of approximation, the optimal algorithm is

$$(6.4)\qquad \varphi^*(N(f))(u) = \begin{cases} \dfrac{t_{k+1}-u}{t_{k+1}-t_k} f_k + \dfrac{u-t_k}{t_{k+1}-t_k} f_{k+1}, & \text{if } t_k \le u \le t_{k+1},\ k=0,\dots,n-1, \\ f_n, & \text{if } t_n < u \le 1, \end{cases}$$

and the radius of information is

$$(6.5)\qquad r(N) = \{\frac{1}{6}\sum_{k=0}^{n-1}(t_{k+1}-t_k)^2 + \frac{1}{2}(1-t_n)^2\}^{1/2}. \qquad \blacksquare$$

The optimal information N^* can be derived from $\dfrac{\partial r(N^*)^2}{\partial t_k} = 0$, $k = 1,\dots,n$. This yields

<u>Theorem 6.3</u>: For the problem of approximation, the optimal information is $N^*(f) = [f(t_1^*),\dots,f(t_n^*)]$, where

$$t_k^* = \frac{3k}{3n+1},\quad k = 1,\dots,n.$$

The radius of information is

$$(6.6)\qquad r(N^*) = \frac{1}{\sqrt{2(3n+1)}}. \qquad \blacksquare$$

7. Adaption Does not Help.

In previous sections, we only studied nonadaptive information, i.e., information which is in the following class:

(7.1) $\Psi^{non} = \{N^{non}: N^{non}(f) = [f(t_1),\dots,f(t_n)]$, where the points $0 < t_1 < \dots < t_n \le 1$ are given simultaneously$\}$.

If the i-th point t_i depends on the previously computed function values, then we have adaptive information, the class of which we denote by

(7.2) $\Psi^{a} = \{N^{a}: N^{a}(f) = [f(t_1),\dots,f(t_n)]$, where $t_i = t_i(f(t_1),\dots,f(t_{i-1}))$ is measurable in R^{i-1}, $i = 1,\dots,n\}$.

The structure of adaptive information is much richer than that of nonadaptive information. Therefore one might hope that adaptive information can be much more powerful than nonadaptive information. As a matter of fact, since $\Psi^{non} \subset \Psi^{a}$,

$$\inf_{N^a \in \Psi^a} r(N^a) \leq \inf_{N^{non} \in \Psi^{non}} r(N^{non}). \tag{7.3}$$

Is it true that the inequality in (7.3) is strict? It turns out that the answer is negative for many cases. For approximation of linear operators in a separable Hilbert space equipped with an orthogonally invariant measure, it is proved in [8] and [14] that adaption does not help. Similar result holds for the worst case, see [9] and [10]. We have

<u>Theorem 7.1</u>: Let S be a continuous linear solution operator from a Wiener space to a separable Hilbert space. Then adaption does not help, i.e.,

$$\inf_{N^a \in \Psi^a} r(N^a) = \inf_{N^{non} \in \Psi^{non}} r(N^{non}). \tag{7.4}$$ ∎

We provide a sketch of the proof, and for a complete one, see [5].

We consider the following class of adaptive information

$$\Psi_1^a = \{\tilde{N}^a : \tilde{N}^a(f) = [\tilde{y}_1, \cdots, \tilde{y}_n], \tag{7.5}$$

$$\tilde{y}_i = \begin{cases} \dfrac{f(\tilde{t}_i)-f(\tilde{t}_{i-1})}{\sqrt{|\tilde{t}_i-\tilde{t}_{i-1}|}}, & \tilde{t}_i \neq \tilde{t}_{i-1} \\ 0 \qquad\qquad , & \tilde{t}_i = \tilde{t}_{i-1}. \end{cases}$$

where $\tilde{t}_i = \tilde{t}_i(\tilde{y}_1,\ldots,\tilde{y}_{i-1})$ is measurable in R^{i-1}, $i = 1,\ldots,n\}$, and the class of nonadaptive information

(7.6) $$\Psi_1^{non} = \{\tilde{N}^{non}:\tilde{N}^{non}(f) = [\tilde{y}_1,\ldots,\tilde{y}_n],$$

$$\tilde{y}_i = \frac{f(\tilde{t}_i)-f(\tilde{t}_{i-1})}{\sqrt{|\tilde{t}_i-\tilde{t}_{i-1}|}}, \ i = 1,\ldots,n.$$

$$0 = \tilde{t}_0 < \tilde{t}_1 <\ldots< \tilde{t}_n \le 1\}.$$

We prove the following inequality

(7.7) $$\inf_{N^{non}\in\Psi^{non}} r(N^{non}) \le \inf_{\tilde{N}^{non}\in\Psi_1^{non}} r(\tilde{N}^{non})$$

$$\le \inf_{\tilde{N}^{a}\in\Psi_1^{a}} r(\tilde{N}^{a}) \le \inf_{N^{a}\in\Psi^{a}} r(N^{a}),$$

and (7.4) follows directly from (7.3) and (7.6).

We decompose the Wiener measure as follows. For each $\tilde{N}^{non} \in \Psi_1^{non}$, let

$$w_1(A|\tilde{N}^{non}) = w((\tilde{N}^{non})^{-1}(A)) \text{ for all Borel set A in } R^n.$$

Then $w_1(\cdot|\tilde{N}^{non})$ is a probability measure in R^n, and for almost all $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_n) \in R^n$, there exists a unique probability measure $w_2(\cdot|\tilde{y})$ concentrated on $V(\tilde{N}^{non},\tilde{y}) = \{f: \tilde{N}^{non}(f) = \tilde{y}\}$, such that

$$(7.8) \quad w(B) = \int_{R^n} w_2(B \cap V(\tilde{N}^{non},\tilde{y})|\tilde{y})\, w_1(d\tilde{y}) \text{ for all } B \in \mathcal{B}.$$

See [6 Th. 8.1, and 11] for details.

For $\tilde{y} \in R^n$, we define the <u>local radius of information</u> $\tilde{N}^{non}$ as

$$(7.9) \quad r(\tilde{N}^{non},\tilde{y}) = \{\inf_{g \in F_2} \int_{V(\tilde{N}^{non},\tilde{y})} \|S(f)-g\|^2 w_2(df|\tilde{y})\}^{\frac{1}{2}}.$$

It is proved in [11] that $r(\tilde{N}^{non},\tilde{y})$ is w_1-integrable, and

$$(7.10) \quad r(\tilde{N}^{non})^2 = \int_{R^n} r(\tilde{N}^{non},\tilde{y})^2 w_1(d\tilde{y}).$$

We have

<u>Lemma 7.2</u>: Given information $\tilde{N}^{non} \in \Psi_1^{non}$, the local radius of information $r(\tilde{N}^{non},y)$ equals the global radius of information $r(\tilde{N}^{non})$. ∎

From Lemma 7.4, we have

$$(7.14)\qquad \inf_{\tilde{N}^a \in \Psi_1^a} r(\tilde{N}^a) \leq \inf_{N^a \in \Psi^a} r(N^a).$$

Similarly, we can prove

$$(7.15)\qquad \inf_{N^{non} \in \Psi^{non}} r(N^{non}) \leq \inf_{\tilde{N}^{non} \in \Psi_1^{non}} r(\tilde{N}^{non}).$$

The inequality (7.7) follows from (7.15), (7.12) and (7.14).

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