Market Design for Shared Experiences, Affirmative Action, and Information Provision

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Abstract

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In recent years, markets have evolved due to the disruption of digital marketplaces, and the rise of concerns about fairness, accountability and privacy. These changes have introduced new challenges for market designers. In this dissertation, we study the design and optimization of different markets. For each market, we provide a theoretical framework to analyze current solutions. Furthermore, we propose alternative solutions and identify the trade-offs between efficiency and other goals.

In the first part of this dissertation, we study markets where tickets for a shared experience are allocated through a lottery. A group of agents is successful if and only if its members receive enough tickets for everyone. We study the efficiency and fairness of existing lottery mechanisms and propose practical alternatives. If agents must identify the members of their group, a natural solution is the Group Lottery, which orders groups uniformly at random and processes them sequentially. We show that the Group Lottery is approximately fair and approximately efficient. If agents may request multiple tickets without identifying members of their group, the most common mechanism is the Individual Lottery, which orders agents uniformly at random and awards each their request until no tickets remain. This approach can yield arbitrarily unfair and inefficient outcomes. As an alternative, we propose the Weighted Individual Lottery, in which the processing order is biased against agents with large requests. This simple modification makes the Weighted Individual Lottery approximately fair and approximately efficient, and similar to the Group Lottery.
when there are many more agents than tickets.

The second part of the dissertation focuses on markets in which an organization is presented with a set of individuals and must choose which subset to accept. The organization makes a selection based on a priority ranking of individuals as well as other observable characteristics. We propose the outcome based selection rules, which are defined by a collection of feasible selections and a greedy processing algorithm. For these rules, we (i) provide an axiomatic characterization, (ii) show that it chooses the only selection that respects priorities, and (iii) identify several cases where is efficient (choose the feasible selection with the highest value). Finally, we connect these ideas with the Chilean Constitutional Assembly election, and show that the rule that was implemented in practice is an outcome based selection rule.

In the third part of this work, we study digital marketplaces where an online platform maximizes its revenue by influencing consumer buying behavior through the disclosure of information. In this market, consumers need to engage in a costly search process to acquire additional information. We develop a new model that combines a Bayesian persuasion problem with an optimal sequential search framework inspired by Weitzman’s 1979. We characterize the platform’s optimal policy under the assumption that the platform must provide a certain level of disclosure to incentivize the consumer to investigate. The optimal policy uses a binary signal indicating whether the item is a good match for the consumer or not. Additionally, we provide a conjecture on the platform’s optimal policy when the assumption is relaxed and there are only two items. The structure of the optimal policy depends on the consumer’s prior beliefs about the items and how they compare with the value of the outside option. However, in all scenarios, the optimal signals are either binary or uninformative. This conjecture is supported by a numerical analysis performed on a novel formulation based on quadratic programming.
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Chapter 1: Introduction

The field of market design focuses on understanding how the design of marketplaces affects the operations of markets. The main theoretical methodology of this field is mechanism design, which studies how to design institutions to achieve a desired outcome in the presence of strategic participants. In the last decades, the relevance of this field has come from the redesign of prominent markets such as school choice systems (Abdulkadiroğlu and Sönmez 2003), kidney exchange (Roth, Sönmez, and Ünver 2004), medical labor (Roth and Peranson 1999) and others.

In recent years, markets have evolved due to the disruption of digital marketplaces, and the rise of concerns about fairness, accountability and privacy. These changes have introduced new challenges for market designers. For example, the scale of digital marketplaces has introduced new questions such as how the consumers gather information about the products or how the information provided by the platforms affects this process. Moreover, the solutions previously proposed by the market design literature need to adapt and incorporate these new concerns.

In this dissertation, we study the design and optimization of different markets. For each market, we provide a theoretical framework to analyze current solutions. Moreover, we propose alternative solutions and identify the trade-offs between efficiency and other goals.

1.1 Lotteries for Shared Experiences

In Chapter 2, we study a setting where tickets for an experience are allocated by lottery. Each agent belongs to a group, and a group is successful if and only if its members receive enough tickets for everyone. A lottery is efficient if it maximizes the number of agents in successful groups, and fair if it gives every group the same chance of success. We study the efficiency and fairness of existing approaches, and propose practical alternatives.
If agents must identify the members of their group, a natural solution is the *Group Lottery*,
which orders groups uniformly at random and processes them sequentially. We provide tight
bounds on the inefficiency and unfairness of this mechanism, and describe modifications that ob-
tain a fairer allocation.

If agents may request multiple tickets without identifying members of their group, the most
common mechanism is the *Individual Lottery*, which orders agents uniformly at random and
awards each their request until no tickets remain. Because each member of a group may apply
for (and win) tickets, this approach can yield arbitrarily unfair and inefficient outcomes. As an
alternative, we propose the *Weighted Individual Lottery*, in which the processing order is biased
against agents with large requests. Although it is still possible to have multiple winners in a group,
this simple modification makes this event much less likely. As a result, the Weighted Individual
Lottery is approximately fair and approximately efficient, and similar to the Group Lottery when
there are many more agents than tickets.

### 1.2 Explainable Affirmative Action

In Chapter 3, we study Prioritized Selection Problems in which an organization is presented
with a set of individuals, and must choose which subset to accept. The organization makes a
selection based on a priority ranking of individuals as well as other observable characteristics. We
study outcome based selection rules, which are defined by a collection of feasible selections and a
greedy processing algorithm.

Our first contribution is to argue that outcome based selection rules are uniquely explainable.
We support this claim with two characterization results. First, these rules are the only rules that are
monotonic, non-bossy, and lower invariant. Second, given a collection of feasible selections, the
greedy processing rule chooses the only one that respects priorities.

Our second contribution is to connect these ideas with the Chilean Constitutional Assembly
election. In this election, candidates were ranked by the number of votes received, and feasible
selections had to allocate the correct number of seats to each party and ensure gender parity. We
show that the rule that was implemented in practice is the outcome based selection rule associated with these constraints.

The family of rules we advocate uses a greedy algorithm, and thus might not be efficient (choose the feasible selection with the highest value). We identify several cases when the set of feasible selections induces a matroid. In these cases, greedy algorithms are efficient, and thus explainability need not come at the expense of efficiency.

1.3 Information Provision and Consumer Search in Digital Marketplaces

In Chapter 4, we study the information provision problem faced by online platforms in markets where consumers can search for additional information. The platform seeks to maximize its expected revenue by influencing consumer buying behavior through the disclosure of information. Moreover, our model considers a costly search process where consumers can investigate an item to resolve their uncertainty.

We develop a new model to study an information design problem under an optimal sequential search framework. Our model effectively captures the information asymmetry present in online marketplaces, where consumers have limited information about the multiple items available and their attributes. We model the interactions between the online platform and the consumers using a Bayesian persuasion framework. Additionally, our model considers a consumer’s search process inspired by Weitzman’s 1979 optimal sequential search.

We characterize the platform’s optimal information provision policy, under the assumption that the platform must provide a certain level of disclosure to incentivize the consumer to investigate. The optimal policy uses a binary signal for each item, indicating whether the item is a good match for the consumer or not. This solution is appealing due to its simple structure and ease of implementation.

Additionally, we provide a conjecture on the platform’s optimal policy when the assumption described above is relaxed and there are only two items. The structure of the optimal policy depends on the consumer’s prior beliefs about the items and how they compare with the value
of the outside option. Based on the different possibilities, we distinguish between five scenarios. However, in all scenarios, the optimal signals are either binary or uninformative. This conjecture is supported by the results of a numerical analysis where we solved multiple instances of the platform problem. To approach this problem numerically, we provide a novel formulation based on quadratic programming.
Chapter 2: Lotteries for Shared Experiences

2.1 Introduction

2.1.1 Motivation

Although matching models often assume that agents care only about their own allocation, there are many scenarios where people also care about the allocation received by their friends or family members. For example, couples entering residency may wish to be matched to programs in the same region, siblings may wish to attend the same school, and friends may want to share a hiking trip. Practitioners often employ ad-hoc solutions in an effort to accommodate these preferences.

This paper studies a special case of this problem, in which there are multiple copies of a homogeneous good. Each agent belongs to a group, and is successful if and only if members of her group receive enough copies for everyone in the group. Examples of such settings include:

- *American Diversity Visa Lottery*. Each year 55,000 visas are awarded to citizens of eligible countries. Applicants are selected by lottery. Recognizing that families want to stay together, the state department grants visas to eligible family members of selected applicants.\(^1\)

- *Big Sur Marathon*. Many popular marathons limit the number of entrants and use a lottery to select applicants. The Big Sur Marathon uses several lotteries for different populations (i.e. locals, first-timers, and returning runners from previous years). One of these is a “Groups and Couples" lottery which “is open for groups of from 2-15 individuals, each of whom want to run the Big Sur Marathon but only if everyone in the group is chosen." In 2020, 702 spots

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were claimed by 236 successful groups selected from 1296 applicants. ²

• Hiking Permits on Recreation.gov. Many parks use a permit system to limit the number of hikers on popular trails. For example, the permits to hike Half Dome in Yosemite National Park are awarded through a pre-season lottery, as well as daily lotteries.³ To enable applicants to hike with friends and family, each applicant is allowed to apply for up to six permits.

• Discounted Broadway Tickets. Many popular Broadway shows hold lotteries for discounted tickets. While some people may be happy going to a Broadway show alone, most prefer to share the experience with others. Recognizing this fact, theaters typically allow each applicant to request up to two tickets. On the morning of the show, winners are selected and given the opportunity to purchase the number of tickets that they requested.

Inspired by the last application, in the rest of the paper we will refer to a copy of the homogeneous good as a ‘ticket’.

The settings above present several challenges. First and foremost, the designer must prevent individuals from submitting multiple applications. In high-stakes environments such as the diversity visa lottery, this can be accomplished by asking applicants to provide government identification as part of their application. In applications with lower stakes, this is frequently accomplished by tying each application to an e-mail address, phone number, or social media account. The effectiveness of this approach will vary across settings. If the designer is concerned that individuals may be submitting multiple applications, then this concern should be addressed before anything else. In this paper, we assume that the designer has a way to identify each individual, and verify that nobody has submitted duplicate applications.

A second challenge is that designers do not know who belongs to each group. One solution is to ask applicants to identify members of their group in advance. While this is done for the diversity

²More information about the 2020 Big Sur Marathon Drawing is available at https://www.bigsurmarathon.org/random-drawing-results-for-the-2020-big-sur-marathon/
³More information available at https://www.nps.gov/yose/planyourvisit/hdpermits.htm
visa lottery and for affordable housing lotteries, it can be quite cumbersome. It requires additional effort from applicants, which may be wasted if their applications are not selected. In addition, to ensure that applicants do not submit false names, when awarding tickets the designer must verify that the identity of each recipient matches the information on the application form. Perhaps for these reasons, many designers opt for a simpler interface which allows applicants to specify how many tickets they wish to receive, but does not ask them to name who these tickets are for.

Motivated by these observations, we study two types of mechanisms: “direct” mechanisms which ask applicants to identify members of their group, and mechanisms which only ask each applicant to specify a number of tickets requested. In the former case, the most natural approach is to place groups in a uniformly random order, and sequentially allocate tickets until no more remain. This procedure, which we refer to as the \textit{Group Lottery}, is used, for example, to allocate affordable housing in New York City. In the latter case, an analogous procedure is often used: applicants are processed in a uniformly random order, with each applicant given the number of tickets that they requested until no tickets remain. We call this mechanism the \textit{Individual Lottery}, and variants of it are used in all of the applications listed above.\footnote{Recreation.gov goes into great detail about the algorithm used to generate a uniform random order of applicants (https://www.recreation.gov/lottery/how-they-work), while the FAQ for the Diversity Visa Lottery notes, “a married couple may each submit a DV Lottery application and if either is selected, the other would also be entitled to a permanent resident card” (https://www.dv-lottery.us/faq/).}

The objective of this work is to evaluate the Group Lottery and the Individual Lottery, and propose possible improvements.

2.1.2 Goals and concerns of existing approaches

One natural goal is to allocate tickets \textit{efficiently}, which means maximizing the number of tickets that are claimed and used. The fact that organizations use lotteries to allocate tickets suggests that \textit{fairness} is also an important criterion. However, it is not obvious how to define fairness in a setting with unknown groups of different sizes. In this work, we say that a lottery is fair if groups of different sizes have similar chances of success.

We believe that this definition is aligned with the organizers’ objectives. Although organizations typically don’t state their objectives explicitly, they do describe policies and procedures in
ways that hint at their objectives. In several applications of interest, language on the webpage suggests that organizers intended to implement a Group Lottery. Furthermore, email correspondence with the organizer indicates that they aim to give each group an equal chance of success. Below, we present quotes which we believe support our interpretations.

**Quotes showing intention of implementing the Group Lottery**

- **Big Sur Marathon**\(^5\).

  A designated group leader should be the only one to enter the Groups & Couples drawing and establish a group name... You may enter each drawing category only one time. Duplicate names in the same category will be removed before selection.

- **Mount Whitney Lottery**\(^6\).

  Submit only one application to the lottery. Groups or households that submit multiple applications will be rejected... Submit only one application per group or household or you may be disqualified.

- **Half Dome Lottery**\(^7\).

  Applicants may be a permit holder or alternate only once on only one application during the preseason lottery... People applying multiple times as permit holder/alternate permit holder will have all their lottery applications canceled.

**Quotes supporting our notion of fairness**

- **Big Sur Marathon.**

  The chances of being picked did not change based on the size of the group... Since there was no charge to enter the drawing, some folks entered several times or had

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\(^5\)https://www.bigsurmarathon.org/random-drawings-faq/

\(^6\)https://www.fs.usda.gov/detail/inyo/passes-permits/recreation/?cid=stelprdb5150055

\(^7\)https://www.nps.gov/yose/planyourvisit/hdpermits.htm
each team member enter, hoping it would increase their chances (which it did not).

- **Mount Whitney Lottery.**

  We ask groups to only submit one application, as submitting multiple applications would skew the chances for success. Most of the disqualified applications were in this category.

Presumably organizers believed that the Group Lottery is fair and efficient. Our analysis show that this intuition is partly correct as this mechanism is approximately fair and approximately efficient. It is not perfectly fair as when only a few tickets remain, small groups still have a chance of success while large groups do not. Similarly, it is not perfectly efficient as when only a few tickets remain, these tickets may be wasted if the next group to be processed is large.

An additional concern raised by the organizers of the Big Sur Marathon, is that some groups are (rationally) not abiding by the recommendation that only one member enter the lottery. Anecdotally, we see strong evidence of groups with multiple winners in this lottery. Specifically, the lottery winners in 2019 included two teams titled “Taylor’s” (with leaders Molly Taylor and Amber Taylor, respectively), as well as a team titled “What the Hill?” and another titled “What the Hill?!”

The Individual Lottery presents additional issues as each member of a group can submit a separate application. This is arguably unfair, as members of large groups might have a much higher chance of success than individual applicants. In addition, the Individual Lottery may be inefficient. One reason for this is that there is no penalty for submitting a large request, so some individuals may ask for more tickets than their group needs. Even if this does not occur, multiple

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9Applicants are very aware of this. One of the authors received an e-mail from the organizer of a Half Dome trip who noted, “It costs nothing extra to apply for 6 spots. If you do win, you might as well win big!” Meanwhile, a guide about the lottery for the Broadway show Hamilton advises, “You can enter the lottery for either one or two seats.
members of a group might apply and win tickets, resulting in some of these tickets going to waste.

Our first contribution is to quantify the unfairness and inefficiency of these mechanisms. Although neither mechanism is perfectly fair or efficient, there is a large qualitative and quantitative difference between them. Our second contribution is to identify modifications to each algorithm which use the same user interfaces but offer improved fairness and/or efficiency. We elaborate on these contributions below.

2.1.3 Overview of Model and Results

We consider a model with $k$ identical tickets. The set of agents is partitioned into a set of groups, and agents have dichotomous preferences: an agent is successful if and only if members of her group receive enough tickets for everyone in the group. We treat the group structure as private information, unknown to the designer. Because there are only $k$ tickets, there can be at most $k$ successful agents. We define the efficiency of a lottery allocation to be the expected number of successful agents, divided by $k$. If this is at least $\beta$, then the allocation is $\beta$-efficient. A lottery allocation is fair if each agent has the same success probability, and $\beta$-fair if for any pair of agents, the ratio of their success probabilities is at least $\beta$.

Given these definitions, we seek lottery allocations that are both approximately efficient and approximately fair. Although this may be unattainable if groups are large, in many cases group sizes are much smaller than the total number of tickets. We define a family of instances characterized by two parameters, $\kappa$ and $\alpha$. The parameter $\kappa$ bounds the ratio of group size to total number of tickets, while $\alpha$ bounds the supply-demand ratio. For any $\kappa$ and $\alpha$, we provide worst-case performance guarantees in terms of efficiency and fairness.

We first consider a scenario where applicants can identify each member of their group. Here, the mechanism typically used is the Group Lottery. We show in Proposition 1 that this mechanism incentivizes agents to truthfully report their groups. Moreover, Theorem 1 establishes that the Group Lottery is $(1-\kappa)$-efficient and $(1-2\kappa)$-fair. It is not perfectly efficient, as tickets might be

Always enter it for two. A friend you bring to Hamilton will be a friend for life" (https://www.timeout.com/newyork/theater/hamilton-lottery).
wasted if the size of the group being processed exceeds the number of remaining tickets. It is not perfectly fair, since once only a few tickets remain, a large group can no longer be successful, but a small group can. Proposition 17 shows that this guarantee is tight.

Could there be a mechanism with stronger performance guarantees than the Group Lottery? Proposition 3 establishes the limits of what can be achieved. Specifically, it says that there always exists an allocation \( \pi \) that is \((1 - \kappa)\)-efficient and fair, but for any \( \epsilon > 0 \), there are examples where any allocation that is \((1 - \kappa + \epsilon)\)-efficient is not even \( \epsilon \)-fair. To show the existence of the random allocation \( \pi \), we use a generalization of the Birkhoff-von Neumann theorem proved by Nguyen, Peivandi, and Vohra 2016. By awarding groups according to the allocation \( \pi \), we can obtain a mechanism that attains the best possible performance guarantees. Therefore, the \( 2\kappa \) loss in fairness in the Group Lottery can be thought of as the “cost” of using a simple procedure that orders groups uniformly, rather than employing a Birkhoff-von Neumann decomposition to generate the allocation \( \pi \).

In many applications, developing an interface that allows applicants to list their group members may be too cumbersome. This motivates the study of a second scenario, where applicants are only allowed to specify the number of tickets they need. The natural mechanism in this setting is the Individual Lottery. Unfortunately, Theorem 2 establishes that the Individual Lottery may lead to arbitrarily inefficient and unfair outcomes. It is perhaps not surprising that the Individual Lottery will be inefficient if agents request more tickets than needed, or if each agent has a large chance of success. However, we show that the waste due to over-allocation may be severe even if all agents request only their group size and demand far exceeds supply. Furthermore, because the probability of success will be roughly proportional to group size, small groups are at a significant disadvantage.

Can we achieve approximate efficiency and fairness without asking applicants to identify each member of their group? We show that this is possible with a minor modification to the Individual Lottery which gives applicants with larger requests a lower chance of being allocated. This eliminates the incentive to inflate demand, and reduces the possibility of multiple winners from the same
Mechanism | Action Set | Efficiency | Fairness
---|---|---|---
Benchmark | | $1 - \kappa$ | $1$
Group Lottery | $2^N$ | $1 - \kappa$ | $1 - 2\kappa$
Individual Lottery | $\{1, 2, \ldots, k\}$ | $0$ | $0$
Weighted Individual Lottery | $\{1, 2, \ldots, k\}$ | $1 - \kappa - \alpha/2$ | $1 - 2\kappa - \alpha/2$

Table 2.1: Summary of main results: worst-case guarantees for the efficiency and fairness of instances in $I(\kappa, \alpha)$. These guarantees are established in Theorems 1, 2 and 3. Meanwhile, Proposition 3 establishes that the best one can hope for is a mechanism that is $(1 - \kappa)$ efficient and 1-fair.

To make the allocation fair, we choose a particular method for biasing the lottery against large requests: sequentially select individuals with probability inversely proportional to their request. We call this approach the \textit{Weighted Individual Lottery}. In the Weighted Individual Lottery, a group of four individuals who each request four tickets has the same chance of being drawn next as a group of two individuals who each request two tickets. As a result, outcomes are similar to the Group Lottery. We prove that the Weighted Individual Lottery is $(1 - \kappa - \alpha/2)$-efficient and $(1 - 2\kappa - \alpha/2)$-fair (in fact, Theorem 3 establishes slightly stronger guarantees). Notice that these guarantees coincide with those of the Group Lottery when demand far exceeds supply ($\alpha$ is close to $0$).

Our main results are summarized in Table 2.1. Our conclusion is that the Individual Lottery can be arbitrarily unfair and inefficient. These deficiencies can be mostly eliminated by using a Group Lottery. Perhaps more surprisingly, approximate efficiency and fairness can also be achieved while maintaining the Individual Lottery interface, by suitably biasing the lottery against agents with large requests.

2.2 Related work

Our high-level goal of allocating objects efficiently, subject to fairness and incentive compatibility constraints, is shared by numerous papers. The definitions of efficiency, fairness, and incentive compatibility differ significantly across settings, and below, we focus on papers that are closely related to our own.

If the group structure is known to the designer, then our problem simplifies to allocating copies
of a homogeneous item to groups with multi-unit demand. This problem has received significant attention. Benassy 1982 introduces the uniform allocation rule, in which each group requests a number of copies, and receives the minimum of its request and a quantity $q$, which is chosen so that every copy is allocated. Sprumont 1991 and Ching 1992 show that when preferences are single-peaked, this is the unique rule that is Pareto efficient, envy-free, and incentive compatible. Ehlers and Klaus 2003 extend this characterization to randomized allocation mechanisms. Cachon and Lariviere 1999 consider the uniform allocation rule in a setting where groups have decreasing marginal returns from additional items.

In contrast to these papers, we assume that groups have dichotomous preferences, with no value for receiving only a fraction of their request. As a result, uniform allocation would be extremely inefficient. Instead, we propose the Group Lottery, which resembles the “lexicographic" allocation rule from Cachon and Lariviere 1999. Dichotomous preferences have also been used to model preferences in kidney exchange Roth, Sönmez, and Ünver 2005, two-sided matching markets Bogomolnaia and Moulin 2004, and collective choice problems Bogomolnaia, Moulin, Stong, et al. 2005.

The all-or-nothing nature of preferences means that our work is related to the “fair knapsack" problem introduced by Patel, Khan, and Louis 2020, where a planner must choose a subset of groups to allocate, subject to a resource constraint. Groups are placed into categories, and the number of successful groups from each category must fall into specified ranges. Their model is fairly general, and if groups are categorized by size, then ranges can be chosen to make their fairness notion similar to ours. However, they do not quantify the cost of imposing fairness constraints. By contrast, we show that in our setting, fairness can be imposed with little or no cost to efficiency. Furthermore, approximate efficiency and fairness can be achieved in our setting using mechanisms that are much simpler than their dynamic-programming based algorithms.

Closer to our work is that of Nguyen, Peivandi, and Vohra 2016. They consider a setting in which each group has complex preferences over bundles of heterogeneous items, but only wants a small fraction of the total number of items. They find approximately efficient and fair allocations.
using a generalization of the Birkhoff-von Neumann theorem. Although their notion of fairness is different from ours, we use their results to prove Proposition 3. However, our papers have different goals: their work identifies near-optimal but complex allocation rules, while we study the performance of simple mechanisms deployed in practice, and close variants of these mechanisms.

An important difference between all of the aforementioned papers and our own is that we assume that the group structure is unknown to the designer. In theory, this can be solved by asking agents to identify the members of their group (as in the Group Lottery), but in many contexts this may be impractical. Hence, much of our analysis considers a scenario where agents are asked to report only a single integer (interpreted as the size of their group). We show what can be achieved is this setting, through our analysis of the Individual Lottery and Weighted Individual Lottery. We are unaware of any prior work with related results.

We close by highlighting two papers with results that are used in our analysis. Serfling 1974 introduces the martingale when sampling without replacement from a finite population. This martingale is key in the proof of Proposition 2, which establishes bounds on the expected hitting time for the sample sum. This, in turn, is used to establish our fairness result for the Group Lottery. Johnson, Kemp, and Kotz 2005 state a simple bound on the probability that a Poisson random variable deviates from its expectation at least by a given number. We use this result in our analysis of the Weighted Individual Lottery, where we use a Poisson random variable to bound the probability that a group has at least \( r \) members awarded.

### 2.3 The model

#### 2.3.1 Agents, Outcomes, Utilities

A designer must allocate \( k \in \mathbb{N} \) indivisible identical tickets to a set of agents \( \mathcal{N} = \{1, \ldots, n\} \). A feasible allocation is represented by \( x \in \{0, 1, \ldots, k\}^n \) satisfying \( \sum_{i \in \mathcal{N}} x_i \leq k \), where \( x_i \) indicates the number of tickets that agent \( i \) receives. We let \( \mathcal{X} \) be the set of all feasible allocations.

A lottery allocation is a probability distribution \( \pi \) over \( \mathcal{X} \), with \( \pi_x \) denoting the probability of allocation \( x \). Let \( \Delta(\mathcal{X}) \) be the set of all lottery allocations.
The set $\mathcal{N}$ is partitioned into groups according to $\mathcal{G}$: that is, each $G \in \mathcal{G}$ is a subset of $\mathcal{N}$, $\bigcup_{G \in \mathcal{G}} G = \mathcal{N}$, and for each $G, G' \in \mathcal{G}$ either $G = G'$ or $G \cap G' = \emptyset$. Given agent $i \in \mathcal{N}$, we let $G_i \in \mathcal{G}$ be the group containing $i$. Agents are successful if and only if the total number of tickets allocated to the members of their group is at least its cardinality. Formally, each $i \in \mathcal{N}$ is endowed with a utility function $u_i : \mathcal{X} \to \{0, 1\}$ given by

$$u_i(x) = 1 \left\{ \sum_{j \in G_i} x_j \geq |G_i| \right\}. \quad (2.1)$$

We say that agent $i$ is successful under allocation $x$ if $u_i(x) = 1$.

In a slight abuse of notation, we denote the expected utility of agent $i \in \mathcal{N}$ under the lottery allocation $\pi$ by

$$u_i(\pi) = \sum_{x \in \mathcal{X}} \pi_x u_i(x). \quad (2.2)$$

2.3.2 Performance Criteria

We define the expected utilization of a lottery allocation $\pi$ to be

$$U(\pi) = \frac{1}{k} \sum_{i \in \mathcal{N}} u_i(\pi). \quad (2.3)$$

**Definition 1** (Efficiency). Lottery allocation $\pi$ is **efficient** if $U(\pi) = 1$. It is **$\beta$-efficient** if $U(\pi) \geq \beta$.

**Definition 2** (Fairness). Lottery allocation $\pi$ is **fair** if for every $i, i' \in \mathcal{N}$, $u_i(\pi) = u_{i'}(\pi)$. It is **$\beta$-fair** if for every $i, i' \in \mathcal{N}$, $u_i(\pi) \geq \beta u_{i'}(\pi)$.

We believe that under dichotomous preferences our notions of efficiency and fairness seems quite natural. Intuitively, our notion of fairness states that agents in groups of different sizes should have similar expected utilities. This definition is motivated by the evidence from multiple applications presented in Section 2.1.2. Of course, we could also consider other notions of fairness. In Appendix A.6, we adapt several other notions of fairness from the literature and show that our main conclusions hold.
2.3.3 Actions and Equilibria

The designer can identify each agent (and therefore prevent agents from applying multiple times), but does not know the group structure a priori. Therefore, the designer must deploy a mechanism that asks individual agents to take actions. When studying incentives induced by a mechanism, however, we assume that members of a group can coordinate their actions.

Formally an anonymous mechanism consists of an action set \( A \) and an allocation function \( \pi : A^N \rightarrow \Delta(X) \), which specifies a lottery allocation \( \pi(a) \) for each possible action profile \( a \in A^N \).

**Definition 3.** The actions \( a_{G_i} \in A^{G_i} \) are **dominant** for group \( G_i \) if for any actions \( a_{-G_i} \in A^{N \setminus G_i} \),

\[
 a_{G_i} \in \arg \max_{a'_{G_i} \in A^{G_i}} u_i(\pi(a'_{G_i}, a_{-G_i})). \tag{2.4}
\]

The action profile \( a \in A^N \) is a **dominant strategy equilibrium** if for each group \( G \in \mathcal{G} \), actions \( a_G \) are dominant for \( G \).

Note that although actions are taken by individual agents, our definition of dominant strategies allows all group members to simultaneously modify their actions. We believe that this reasonably captures many settings, where group members can coordinate but the mechanism designer has no way to identify groups a priori.

2.4 Results

In general, it might not be feasible to achieve approximate efficiency, even if the group structure is known. Finding an efficient allocation involves solving a knapsack problem where each item represents a group, and the knapsack’s capacity is the total number of tickets. If it is not possible to select a set of groups whose sizes sum up to the number of tickets, then some tickets will always be wasted. This issue can be particularly severe if groups’ sizes are large relative to the total number of tickets. Therefore, one important statistic will be the ratio of the maximum group size to the total number of tickets.
Additionally, one concern with the Individual Lottery is that tickets might be wasted if groups have multiple winners. Intuitively, this is more likely when the number of tickets is close to the number of agents. Therefore, a second important statistic will be the ratio of tickets to agents.

These thoughts motivate us to define a family of instances characterized by two parameters: $I(\kappa, \alpha)$. The parameter $\kappa$ captures the significance of the “knapsack” structure, and $\alpha$ captures the “abundance” of the good. For any $\kappa, \alpha \in (0, 1)$, define the family of instances

$$I(\kappa, \alpha) = \left\{ (n, k, \mathcal{G}) : \frac{\max_{G \in \mathcal{G}} |G| - 1}{k} \leq \kappa, \frac{k}{n} \leq \alpha \right\}. \tag{2.5}$$

Therefore, when analyzing a mechanism we study the worst-case efficiency and fairness guarantees in terms of $\kappa$ and $\alpha$. Ideally, we might hope for a solution that is both approximately fair and approximately efficient. Theorem 1 shows that this is achieved by the Group Lottery, which asks agents to reveal their groups. By contrast, Theorem 2 establishes that the Individual Lottery may lead to arbitrarily inefficient and unfair outcomes. Finally, Theorem 3 establishes that the Weighted Individual Lottery is approximately fair and approximately efficient, and similar to the Group Lottery when there are many more agents than tickets ($\alpha$ is small).

2.4.1 Group Lottery

In this section, we present the Group Lottery ($GL$) and show that it is approximately fair and approximately efficient. In this mechanism, each agent is asked to report a subset of agents, interpreted as their group. We say that a group $S \subseteq N$ is valid if all its members declared the group $S$. Valid groups are placed in a uniformly random order and processed sequentially (agents that are not part of a valid group will not receive tickets). When a group is processed, if enough tickets remain then every member of the group is given one ticket. Otherwise, members of the group receive no tickets and the lottery ends.\footnote{There is a natural variant of this mechanism which skips over large groups when few tickets remain, and gives these tickets to the next group whose request can be accommodated. This variant may be arbitrarily unfair, as can be seen by considering an example with an odd number of tickets, one individual applicant, and many couples. Then the individual applicant is always successful, while the success rate of couples can be made arbitrarily small by increasing the number of couples.}
We now introduce notation that allows us to study this mechanism. For any finite set $E$, we let $S_E$ be the set of finite sequences of elements of $E$, and let $O_E$ be the set of sequences such that each element of $E$ appears exactly once. We refer to an element $\sigma \in O_E$ as an order over $E$, with $\sigma_t \in E$ and $\sigma_{[t]} = \bigcup_{t' \leq t} \sigma_{t'}$ denoting the subset of $E$ that appears in the first $t$ positions of $\sigma$.

Next we provide a formal description of the mechanism. The action set is the power set of $N$. Given an action profile $a$, we call a set of agents $S \subseteq N$ a valid group if for every agent $i \in S$ we have that $a_i = S$. We define a function $\tau$ that will let us to characterize the number of valid groups that obtain their full request. Fix a finite set $E$ and a size function $|\cdot| : E \to \mathbb{N}$. For any $c \in \mathbb{N}$ and $\sigma \in S_E$ satisfying $\sum_t |\sigma_t| \geq c$, define

$$\tau(c, \sigma) = \min \left\{ T \in \mathbb{N} : \sum_{t=1}^{T} |\sigma_t| \geq c \right\}. \tag{2.6}$$

Fix an arbitrary action profile $a$ and let $V$ be the resulting set of valid groups. For any order $\sigma \in O_V$, we let $\tau = \tau(k + 1, \sigma)$ be as in (2.6) where the size of each valid group is its cardinality. Then the number of valid groups that are processed and obtain their full request is $\tau - 1$. We define

$$x_i^{GL}(a, \sigma) = \sum_{j=1}^{\tau - 1} \mathbf{1}\{i \in \sigma_j\}. \tag{2.7}$$

For any $x' \in X$, the allocation function of the Group Lottery is

$$\pi_{x'}^{GL}(a) = \sum_{\sigma \in O_V} \mathbf{1}\{x' = x^{GL}(a, \sigma)\} \frac{1}{|V|!}.$$

Incentives.

In every mechanism that we study, there is one strategy that intuitively corresponds to truthful behavior. We refer to this as the group request strategy. In the Group Lottery, this is the strategy in which each agent declares the members of his or her group.
**Definition 4.** In the Group Lottery, group $G \in \mathcal{G}$ follows the **group request** strategy if $a_i = G$ for all $i \in G$.

**Proposition 1.** *In the Group Lottery, the group request strategy is the only dominant strategy.*

The intuition behind Proposition 1 is as follows. Potential deviations for group $G$ include splitting into two or more groups, or naming somebody outside of the group as a member. We argue that in both cases the group request is weakly better. First, neither approach will decrease the number of other valid groups. Second, if there are at least $|G|$ tickets remaining and a valid group containing a member of $G$ is processed, then under the group request $G$ gets a payoff of 1. This might not be true under the alternatives strategies.

In light of Proposition 1, we will assume that groups follow the group request strategy when analyzing the performance of the Group Lottery.

**Performance.**

We next argue that the Group Lottery is approximately fair and approximately efficient. Of course, the Group Lottery is not perfectly efficient, as it solves a packing problem greedily, resulting in an allocation that does not maximize utilization. Similarly, it is not perfectly fair, as once there are only a few tickets left, small groups still have a chance of being allocated but large groups do not. Thus, in the Group Lottery smaller groups are always weakly better than larger groups. This is formally stated in Lemma 9 located in Appendix A.1.

**Theorem 1** ("GL is Good"). *Fix $\kappa, \alpha \in (0, 1)$. For every instance in $I(\kappa, \alpha)$, the group request equilibrium outcome of the Group Lottery is $(1 - \kappa)$-efficient and $(1 - 2\kappa)$-fair.*

In Proposition 17 (located in Appendix A.1), we construct instances where the fairness of the Group Lottery is arbitrarily close to the guarantee provided in Theorem 1. These instances are fairly natural: groups all have size one or two, and the total number of tickets is odd. These conditions are met by the Hamilton Lottery, which we discuss in Section 2.5.
Proof Sketch of Theorem 1.

The efficiency guarantee is based on the fact that for any order over groups, the number of tickets wasted can be at most \( \max_G |G| - 1 \). Therefore, the tight lower bound on efficiency of the group lottery is \( 1 - \frac{\max_G |G| - 1}{k} \geq 1 - \kappa \).

We now turn to the fairness guarantee. We will show that for any pair of agent \( i, j \),

\[
\frac{u_i(\pi^{GL}(a))}{u_j(\pi^{GL}(a))} \geq (1 - 2\kappa). \tag{2.8}
\]

Because all groups are following the group request strategy, the set of valid groups is \( \mathcal{G} \). Fix an arbitrary agent \( i \). We construct a uniform random order over \( \mathcal{G} \) using Algorithm 4: first generate a uniform random order \( \Sigma^{-i} \) over \( \mathcal{G} \setminus G_i \), and then extend it to \( \mathcal{G} \) by uniformly inserting \( G_i \) in \( \Sigma^{-i} \). Moreover, if groups in \( \mathcal{G} \setminus G_i \) are processed according \( \Sigma^{-i} \), then \( \tau(k - |G_i| + 1, \Sigma^{-i}) \) represents the last step in which at least \( |G_i| \) tickets remains available. Therefore, if \( G_i \) is inserted in the first \( \tau(k - |G_i| + 1, \Sigma^{-i}) \) positions it will get a payoff of 1. This is formalized in the next lemma.

**Lemma 1.** For any instance in \( I(\kappa, \alpha) \) and any agent \( i \), if we let \( a \) be the group request strategy under the Group Lottery and \( \Sigma^{-i} \) be a uniform order over \( \mathcal{G} \setminus G_i \), then

\[
u_i(\pi^{GL}(a)) = \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{m} \leq \frac{k}{n} (1 + \kappa), \tag{2.9}
\]

where \( \tau(k - |G_i| + 1, \Sigma^{-i}) \) is as in (2.6) using the cardinality of each group as the size function.

Lemma 9 in Appendix A.1 states that if two groups are selecting the group request strategy under the Group Lottery, then the utility of the smaller group will be at least the utility of the larger group. Therefore, we assume without loss of generality that \( |G_i| \geq |G_j| \). From Lemma 1, it follows that

\[
\frac{u_i(\pi^{GL}(a))}{u_j(\pi^{GL}(a))} = \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-i})]} \geq \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-i})]]. \tag{2.10}
\]
To complete the proof, we express the denominator on the right hand side as the sum of the numerator and the difference

$$\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-j})] - \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})],$$

which reflects the advantage of the small group $G_j$. We bound this ratio by taking a lower bound on the numerator $\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]$ and an upper bound on the difference $\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-j})] - \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]$. Both bounds follow from the lemma below.

**Proposition 2.** Given a sequence of numbers $\{a_1, \ldots, a_n\}$ such that $a_i \geq 1$, define $\mu = \sum_i a_i/n$ and $\bar{a} = \max a_i$. Let $\sigma$ be an order over $\{1, \ldots, n\}$. For $k \in \{1, \ldots, \sum_i a_i\}$, we let $\tau = \tau(k, \sigma)$ be as in (2.6) where the size of $i$ is $a_i$, that is, $|\sigma_i| = a_{\sigma_i}$. If $\Sigma$ is a uniform random order of $\{1, \ldots, n\}$, then

$$1 + \frac{k - \bar{a}}{\mu} \leq \mathbb{E}[\tau(k, \Sigma)] \leq \frac{k + \bar{a} - 1}{\mu}. \tag{2.11}$$

Furthermore, if $k, k' \in \mathbb{N}$ are such that $k + k' \leq \sum_i a_i$ then

$$\mathbb{E}[\tau(k', \Sigma)] + \mathbb{E}[\tau(k, \Sigma)] \geq \mathbb{E}[\tau(k' + k, \Sigma)]. \tag{2.12}$$

Equation (2.11) establishes that the expected time to reach $k$ is approximately $k$ divided by the average size $\mu$, while (2.12) establishes that hitting times are sub-additive. Both results are well known when the values $a_{\Sigma_i}$ are sampled with replacement from $\{a_1, \ldots, a_n\}$. Proposition 2 establishes the corresponding results when values are sampled *without* replacement. The proof of (2.11) employs a martingale presented in Serfling (1974), while the proof of (2.12) uses a clever coupling argument.\textsuperscript{11} Although both statements are intuitive, we have not seen them proven elsewhere, and we view Proposition 2 as a statement whose importance extends beyond the setting in which we deploy it.

\textsuperscript{11}We thank Matt Weinberg for suggesting the appropriate coupling.
A Fair Group Lottery.

Theorem 1 establishes that the Group Lottery has strong performance guarantees. However, this mechanism is not perfectly fair, as small groups have an advantage over large groups, nor perfectly efficient, as the last few tickets might be wasted. It is natural to ask whether there exists a mechanism that overcomes these issues. Proposition 3 shows that the best we can hope for is a mechanism that is $(1 - \kappa)$-efficient and fair. We then describe a fairer version of the Group Lottery which attains these performance guarantees. We conclude with a discussion of advantages and disadvantages of this fair Group Lottery.

Proposition 3.

1. Fix $\kappa, \alpha \in (0, 1)$. For every instance in $I(\kappa, \alpha)$, there exists a random allocation that is $(1 - \kappa)$-efficient and fair.

2. For any $\epsilon > 0$, there exists $\kappa, \alpha \in (0, 1)$ and an instance in $I(\kappa, \alpha)$ such that no random allocation is $(1 - \kappa + \epsilon)$-efficient and $\epsilon$-fair.

The first statement follows from a result in Nguyen, Peivandi, and Vohra (2016), which implies that any utility vector such that (i) the sum of all agents’ utilities is at most $k - \max_{G \in \mathcal{G}} |G| + 1$, and (ii) members of the same group have identical utility, can be induced by a lottery over feasible allocations. To prove the second part of Proposition 3, we construct an instance where a particular group must be awarded in order to avoid wasting a fraction $\kappa$ of the tickets. Therefore, to improve beyond $(1 - \kappa)$-efficiency it is necessary to allocate that group more frequently. The complete proof of Proposition 3 is located in Appendix A.1.3.

Proposition 3 establishes that the best guarantee we can hope for is a mechanism that is $(1 - \kappa)$-efficient and fair. In fact, this can be achieved by first asking agents to identify their groups (as in the Group Lottery), and then allocating according to the random allocation referred to in the first part of Proposition 3. When using this mechanism, it is dominant for a group to truthfully report its members, as long as it can not influence the total number of tickets awarded. In the following discussion, we refer to this mechanism as the Fair Group Lottery.
To conclude this section, we discuss the trade-offs between the Fair Group Lottery and the Group Lottery. In light of its stronger performance guarantees, one might conclude that the Fair Group Lottery is superior. However, we think that there are several practical reasons to favor the standard Group Lottery. First, the computation of the Fair Group Lottery outcome is not trivial: Nguyen, Peivandi, and Vohra (2016) give two procedures, one which they acknowledge might be “impractical for large markets,” and the other which returns only an approximately fair allocation. By contrast, the Group Lottery is simple to implement in code, and can even be run physically by writing applicants’ names on ping-pong balls or slips of paper. In some settings, a physical implementation that allows applicants to witness the process may increase their level of trust in the system. Even when implemented digitally, the ability to explain the procedure to applicants may provide similar benefits.

A final benefit of the Group Lottery is that it provides natural robustness. Although we assume that the number of tickets is known in advance and that all successful applicants claim their tickets, either assumption may fail to hold in practice. When using the Group Lottery, if additional tickets become available after the initial allocation, they can be allocated by continuing down the list of groups. This intuitive policy preserves the fairness and efficiency guarantees from Theorem 1. By contrast, if tickets allocated by the Fair Group Lottery go unclaimed, there is no obvious “next group” to offer them to, and any approach will likely violate the efficiency and fairness guarantees that this mechanism purports to provide.

For these reasons, we see the Group Lottery as a good practical solution in most cases: the $2\kappa$ loss of fairness identified in Theorem 1 seems a modest price to pay for the benefits outlined above. There is a loose analogy to be drawn between the Fair Group Lottery and the Vickrey auction: although the Vickrey auction is purportedly optimal, practical considerations outside of the standard model prevent it from being widely deployed Ausubel, Milgrom, et al. (2006). Similarly, a Fair Group Lottery is only likely to be used in settings satisfying several specific criteria: the institution running the lottery is both sophisticated and trusted, fairness is a primary concern, and applicants are unlikely to renege.
2.4.2 Individual Lottery

As noted in the introduction, asking for (and verifying) the identity of each participant may prove cumbersome. In this section we consider the widely-used Individual Lottery, in which the action set is $A = \{1, \ldots, k\}$.\footnote{In practice, agents are often limited to asking for $\ell < k$ tickets. We refer to this mechanism as the Individual Lottery with limit $\ell$, and discuss it briefly at the end of the section. Appendix A.2 provides a complete analysis of this mechanism, and demonstrates that like the Individual Lottery without a limit, it can be arbitrarily unfair and inefficient.} Agents are placed in a uniformly random order and processed sequentially. Each agent is given a number of tickets equal to the minimum of their request and the number of remaining tickets.\footnote{As for the Group Lottery, one might imagine using a variant in which agents whose request exceeds the number of remaining tickets are skipped. The negative results in Theorem 2 would still hold when using this variant.}

More formally, given an action profile $a \in A^N$ and an order over agents $\sigma \in O_N$, we let $x_{\sigma}^{IL}(a, \sigma) \in X$ be the feasible allocation generated by the Individual Lottery:

$$x_{\sigma_t}^{IL}(a, \sigma) = \min \left\{ a_{\sigma_t}, \max \left\{ k - \sum_{i \in \sigma_{[t-1]}} a_i, 0 \right\} \right\},$$

for $t \in \{1, \ldots, n\}$. For any $x' \in X$, the allocation function of the Individual Lottery is

$$\pi_{x_t}^{IL}(a) = \frac{1}{n!} \sum_{\sigma \in O_N} 1 \{x' = x_{\sigma}^{IL}(a, \sigma)\}.$$

Incentives.

As in the Group Lottery, we refer to the strategy that corresponds to truthful behavior as the group request strategy. In the Individual Lottery, this is the strategy in which each agent declares his or her group size.

**Definition 5.** In the Individual Lottery, we say that group $G$ follows the group request strategy if $a_i = |G|$ for all $i \in G$.

Our next result establishes that because agents’ requests do not affect the order in which they are processed, each agent should request at least his or her group size.
**Proposition 4.** In the Individual Lottery, the set of actions $a_G$ is dominant for group $G$ if and only if $a_i \geq |G|$ for all $i \in G$.

**Performance.**

Proposition 4 states that it is a dominant strategy to follow the group request strategy, but that there are other dominant strategies in which agents inflate their demand (select $a_i > |G_i|$).\(^{14}\) Our next result implies that the group request equilibrium Pareto dominates any other dominant strategy equilibrium.

**Proposition 5.** Let $i$ be any agent, fix any $a_{-i} \in \{1, 2, \ldots, k\}^{N\backslash\{i\}}$, and let $a'_i > a_i \geq |G_i|$. Then for every agent $j \in N$,

$$u_j(\pi_{IL}(a_i, a_{-i})) \geq u_j(\pi_{IL}(a'_i, a_{-i})).$$

Even when agents request only the number of tickets needed by their group, the outcome will be inefficient if there are multiple winners from the same group. One might expect that this is unlikely if the supply/demand ratio $\alpha$ is small. However, Theorem 2 shows that even in this case, the individual lottery can be arbitrarily unfair and inefficient.

**Theorem 2 (“IL is Bad”).** For any $\alpha, \kappa, \epsilon \in (0, 1)$, there exists an instance in $I(\kappa, \alpha)$ such that any dominant strategy equilibrium outcome of the Individual Lottery is not $\epsilon$-efficient nor $\epsilon$-fair.

**Proof Sketch of Theorem 2.**

We will construct an instance in $I(\kappa, \alpha)$, where the outcome of the Individual Lottery is arbitrarily unfair and arbitrarily inefficient. In this instance, there are $n$ agents and $k = an$ tickets. Furthermore, agents are divided into one large group of size $n^{3/4}$ and $n - n^{3/4}$ groups of size one. If $n$ is large enough, then this instance is in $I(\kappa, \alpha)$ and the following two things happen simultaneously:

\(^{14}\)Our model assumes that agents are indifferent between all allocations that allow all members of their group to receive a ticket. While we believe this to be a reasonable approximation, in practice, groups might follow the group request strategy if each ticket has a cost, or inflate their demand if tickets can be resold on a secondary market.
1. The size of the large group $n^{3/4}$ is small relative to the number of tickets $k = \alpha n$.

2. The fraction of tickets allocated to small groups is insignificant.

Hence, the resulting allocation is unfair as the large group has an advantage over small groups, and inefficient as a vanishing fraction of the agents get most of the tickets.

Formally, let agents $i, j$ be such that $|G_i| = 1$ and $|G_j| = n^{3/4}$. We will start by proving the efficiency guarantee. By Proposition 5 it follows that the group request is the most efficient dominant action profile. Therefore, we assume without loss of generality that this action profile is being selected. The utilization of this system is

$$\frac{n^{3/4}u_j(\pi^{IL}(a))}{k} + \frac{(n-n^{3/4})u_i(\pi^{IL}(a))}{k}.$$  \hspace{1cm} (2.14)

We now argue that both terms in (2.14) can be made arbitrarily small by making $n$ sufficiently large. We begin by studying the first term in (2.14). Using the fact that utilities are upper bounded by 1, it follows that

$$\frac{n^{3/4}u_j(\pi^{IL}(a))}{k} \leq \frac{n^{3/4}}{k} = \frac{1}{\alpha n^{1/4}}.$$  \hspace{1cm}

Hence, to ensure that (1) holds it suffices to have $n$ growing to infinity. We now analyze the second term in (2.14). Because the group request action profile is being selected, this term is equal to the fraction of tickets allocated to small groups. Moreover, we show the following upper bound on utility of agent $i$:

$$u_i(\pi^{IL}(a)) \leq \frac{k}{(n^{3/4})^2} = \frac{\alpha}{n^{1/2}}.$$  \hspace{1cm} (2.15)

The intuition behind this bound is as follows. If we restrict our attention only to agents in $G_i$ and $G_j$, then we know that $i$ will get a payoff 0 unless it is processed after at most $k/n^{3/4} - 1$ members of $G_j$. Because the order over agents is uniformly distributed, this event occurs with probability

$$\frac{k/n^{3/4}}{n^{3/4} + 1} \leq \frac{k}{(n^{3/4})^2}.$$
From the first inequality in (2.15), it follows that

\[
\frac{(n - n^{3/4})u_i(\pi^{IL}(a))}{k} \leq \frac{n}{n^{3/2}} \leq \frac{1}{n^{1/2}}.
\]

Notice that the right hand side goes to 0 as \(n\) grows, so (2) holds.

We now turn to the fairness guarantee. To this end, we use a trivial upper bound on the utility of agent \(j\), based on the fact that the first agent to be processed always obtains a payoff of 1. Thus,

\[
u_j(\pi^{IL}(a)) \geq \frac{n^{3/4}}{n} = n^{-1/4}.
\]

Note that this lower bound is attained when all agents in small groups request \(k\) tickets. Combining the bound above and the second inequality in (2.15), we obtain

\[
\frac{u_i(\pi^{IL}(a))}{u_j(\pi^{IL}(a))} \leq \frac{\alpha n^{1/4}}{n^{1/2}}.
\]

We conclude noting that the right side goes to 0 as \(n\) grows. The full proof of Theorem 2 is located in Appendix A.2.

**Limiting the Number of Tickets Requested.**

In many applications, a variant of the Individual Lottery is used where a limit is imposed on the number of tickets an agent can request. For example, in the Hamilton Lottery agents can request at most 2 tickets, and in the Big Sur Marathon groups can have at most 15 individuals. This motivates us to study the Individual Lottery with limit \(\ell\). Formally, the only difference between this and the standard Individual Lottery is the action set, which is \(A = \{1, \ldots, \ell\}^n\), with the limit \(\ell\) chosen by the designer.

The choice of limit must balance several risks. Imposing a limit of \(\ell\) reduces the risk from inflated demand, but harms groups with more than \(\ell\) members. The latter effect reduces fairness and may also reduce efficiency if there are many large groups. In fact, we show in Proposition 6

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that the Individual Lottery with limit $\ell$ is still arbitrarily unfair and arbitrarily inefficient in the worst case.

**Proposition 6.** For any $\alpha, \kappa, \epsilon \in (0, 1)$ and $\ell \in \mathbb{N}$, there exists an instance in $I(\kappa, \alpha)$ such that, regardless the action profile selected, the outcome of the Individual Lottery with limit $\ell$ is not $\epsilon$-efficient nor $\epsilon$-fair.

The proof of Proposition 6 is in Appendix A.2. In the example considered in the proof, problems stem from the fact that most groups have more than $\ell$ members. However, even if group sizes are upper bounded by $\ell$, the Individual Lottery with limit $\ell$ still performs poorly in the worst case. In particular, Propositions 19 and 20 show that, if no group have more than $\ell$ members, every dominant strategy equilibrium outcome of the Individual Lottery with limit $\ell$ is $1/\ell$-efficient and $1/\ell$-fair. Moreover, these guarantees are tight in the worst case. We give a complete analysis of this variant of the Individual Lottery in Appendix A.2.

### 2.4.3 Weighted Individual Lottery

The example presented in Theorem 2 is an extreme case that we shouldn’t see too often in practice. However, it illustrates the major issues of the Individual Lottery. In this section, we show that minor modifications to the Individual Lottery can yield strong performance guarantees even in these extreme cases.

We study the Weighted Individual Lottery ($IW$), whose only departure from the Individual Lottery is the order in which agents are placed. Instead of using a uniform random order, the Weighted Individual Lottery uses a random order biased against agents with large requests. Theorem 3 shows that the Weighted Individual Lottery is approximately fair and approximately efficient, and similar to the Group Lottery when there are many more agents than tickets.

Formally, each agent selects an action in $\{1, \ldots, k\}$. For each $\sigma \in O_N$, we let random order over agents $\Sigma$ be such that

$$
P(\Sigma = \sigma|a) = \prod_{t=1}^{n} \frac{1/a_{\sigma_t}}{\sum_{i \in N \setminus \sigma[\tau-1]} 1/a_{\sigma_i}}.
\tag{2.18}$$
There are several ways to generate $\Sigma$. This order can be thought of as the result of sequentially sampling agents without replacement, with probability inversely proportional to the number of tickets that they request. One property that motivates the study of the Weighted Individual Lottery is that when agents declare their group size, every group that has not been drawn is equally likely to be drawn next.

Let $\Sigma \in O_N$ be distributed according to (2.18). For any $x' \in X$, the allocation function of the Weighted Individual Lottery is

$$
\pi^{IW}_{x'}(a) = \sum_{\sigma \in O_N} 1 \{x' = x^{IL}(a, \sigma)\} \mathbb{P}(\Sigma = \sigma),
$$

with $x^{IL}$ defined as in (2.13).

We define group request strategy as in Definition 5: each agent $i$ requests $|G_i|$ tickets.

Incentives.

In this section, we will see that under the Weighted Individual Lottery, there are instances where no strategy is dominant for every group. However, we will argue that if demand significantly exceeds supply, then it is reasonable to assume that groups will select the group request strategy.

We first show that for groups of size three or less, the group request is the only dominant strategy.

**Proposition 7.** In the Weighted Individual Lottery, if $G \in G$ is such that $|G| \leq 3$, then the group request is the only dominant strategy for $G$.

The following example shows that for groups of more than three agents, deviating from the group request may be profitable.

**Example 1.** There are $n$ agents and $n - 1$ tickets. There is one group of size 4 and $n - 4$ groups of a single agent. If $n \geq 17$, then the optimal strategy for the large group will depend on the action profile selected by the small groups. If the small groups follow the group request strategy, then
members of the large group benefit from each requesting 2 tickets instead of 4. The analysis of this example is located in Appendix A.3.

In the example above, when \( n \leq 16 \), it is actually optimal for the large group to play the group request. Thus, this deviation is only profitable when \( n \geq 17 \), and the group success probability is bigger than 92%. In general, when agents request fewer tickets than their group size, their chance of being selected increases, but now multiple agents from the group must be drawn in order to achieve success. This should be profitable only if the chance of each agent being drawn is high.

We formalize this intuition in Conjecture 1. Roughly speaking, we conjecture that in scenarios where the success probability of a group is below \( 1 - 1/e \approx 63\% \), the group request strategy maximizes its conditional expected utility. Proposition 8 lends additional support to the conjecture. This proposition establishes that our conjecture holds when restricted to a broad set of strategies.

In order to present our conjecture, we need first to introduce some definitions.

In what follows, we fix an arbitrary group \( G \). Given any action profile \( a \), we generate an order over agents \( \Sigma \) using the following algorithm:

**ALGORITHM 1:**

1. Draw \( \{X_i\}_{i \in N} \) as i.i.d. exponentials, with \( \Pr(X_i > t) = e^{-t} \) for \( t \geq 0 \).

2. Place agents in increasing order of \( a_iX_i \): that is, output \( \Sigma \) such that

\[
a_{\Sigma_1}X_{\Sigma_1} < \cdots < a_{\Sigma_n}X_{\Sigma_n}.
\]

From Proposition 21 in Appendix A.3, it follows that \( \Sigma \) is distributed according to (2.18) conditional on \( a \). We will refer to \( a_iX_i \) as the *score* obtained by agent \( i \). Note that a lower score is better as it increases the chances of getting awarded.

The usual way to study the incentives of group \( G \), is to find a strategy that maximizes its utility given the actions of other agents. Here, we will assume that \( G \) has additional information: the scores of other agents. Thus, we study the problem faced by \( G \) of maximizing its success probability given the actions and scores of everyone else. This problem seems to be high-dimensional.
and very complex, however, we will show that all the information relevant for $G$ can be captured by a sufficient statistic $T$. Define,

$$T = \inf \left\{ t \in \mathbb{R} : \sum_{j \notin G} a_j 1 \{a_j X_j < t\} > k - |G| \right\}.$$  \hfill (2.19)

We show in Lemma 12 located in Appendix A.3, that $G$ gets a utility of 1 if and only if the sum of the requests of its members whose score is lower than $T$ is at least $|G|$. Therefore, we can formulate the problem faced by $G$ as follows:

$$\max \ P(\sum_{i \in G} a_i 1 \{a_i X_i < T\} \geq |G|)$$

subject to $a_i \in \{1, \ldots, k\} \ \forall i \in G.$ \hfill (2.20)

Notice that under the group request strategy, the objective value in (2.20) evaluates to

$$1 - e^{-T}.$$ \hfill (2.21)

This follows because $G$ will get a payoff of 1 if and only if at least one of its members has a score lower than $T$, that is, $\min_{i \in G} \{a_i X_i\} < T$. Moreover, using that $\sum_{i \in G} 1/a_i = 1$, and by the well-known properties of the minimum of exponential random variables, we have that $\min_{i \in G} \{a_i X_i\} \sim Exp(1)$.

Definitions out of the way, we can present our conjecture and the proposition supporting it.

**Conjecture 1.** If $T \leq 1$, then no other strategy yields a higher objective value in (2.20) than the group request.

From equation (2.21), we see that the utility yield by the group request is an increasing function of $T$. Therefore, we can think of $T$ as an indicator of how competitive the market is. Thus, the interpretation of Conjecture 1 is that if the market is moderately competitive ($G$ has success probability below $1 - 1/e \approx 63\%$), then the group request is optimal.

One intuition for this conjecture is that for any strategy, the expected number of tickets received
by each individual is upper-bounded by $T$. Thus, when $T < 1$, the expected number of tickets won by the group is less than the size of the group. In this case, “high variance” strategies that require only one successful outcome should be optimal. While intuitive, this conjecture is closely related to long-standing open problems proposed by Samuels 1966 and Feige 2004, and we have been unable to prove it.

While we haven’t proved the conjecture, we do have a proof that holds for a broad subset of strategies. For $r \in \{0, \ldots, |G| - 1\}$, we define $\mathcal{B}_r \subseteq A_G$ to be the set of strategies for which the sum of any $r$ requests is less than $|G|$, while the sum of any $r + 1$ requests is greater than or equal to $|G|$. Let

\[
\mathcal{B} = \bigcup_{r=0}^{\lfloor G \rfloor - 1} \mathcal{B}_r.
\]  

(2.22)

**Proposition 8.** If $T \leq 1$, then no other strategy in $\mathcal{B}$ yields a higher objective value in (2.20) than the group request.

Note that $\mathcal{B}$ is rich enough such that for any group of size greater than 3, it contains a strategy that is better than the group request for $T$ large enough.

**Proof sketch of Proposition 8.** In this proof, we will say that an agent is awarded if and only if it has a score lower than $T$.

From (2.21), it suffices to show that under any strategy in $\mathcal{B}_r$, the objective value in (2.20) is at most $1 - e^{-T}$. We start by studying a relaxation of the problem defined in (2.20). In this relaxation, the number of times an agent $i$ is awarded follows a Poisson distribution with rate $T/a_i$, and the total number of times $G$ is awarded follows a Poisson distribution with rate $\sum_{i \in G} T/a_i$. Note that if the set of feasible strategies is $\mathcal{B}_r$, then by Lemma 12 in Appendix A.3 it follows that $G$ needs to be awarded at least $r + 1$ times. Finally, using a Poisson tail bound, we show that this event happens with probability at most $1 - e^{-T}$. This bound can only be applied if the expected number of times $G$ is awarded is at most $r + 1$. This follows because $T \leq 1$ and Lemma 13 in Appendix A.3, which establishes that for any $a'_G \in \mathcal{B}_r$, $\sum_{i \in G} 1/a'_i \leq r + 1$. □
Performance.

We now study the performance of the Weighted Individual Lottery, under the assumption that groups are selecting the group request strategy. We think that this assumption is reasonable for two reasons: (i) for groups of size at most three, the group request is the only dominant strategy, and (ii) for larger groups, we conjecture that in scenarios where its success probability is moderate (at most 63%), the group request strategy is optimal. Indeed, in Appendix A.5 we numerically verify Conjecture 1 for groups of size at most 15. We are not aware of any application that permits larger groups, hence, our numerical analysis addresses the incentives of the Weighted Individual Lottery for all practical purposes. The main result of this section is Theorem 3, which establishes that the Weighted Individual Lottery is approximately efficient and fair.

To state these guarantees, we define for any \( x > 0, \)
\[
g(x) = \frac{1 - e^{-x}}{x}. \tag{2.23}
\]

**Theorem 3.** Fix \( \kappa, \alpha \in (0, 1). \) For every instance in \( I(\kappa, \alpha) \), the group request outcome of the Weighted Individual Lottery is \((1 - \kappa)g(\alpha)\)-efficient and \((1 - 2\kappa)g(\alpha)\)-fair.

These guarantees resemble the ones offered for the Group Lottery. Recall that Theorem 1 establishes that the Group Lottery is \((1 - \kappa)\)-efficient and \((1 - 2\kappa)\)-fair. It is not perfectly efficient, as the last tickets might be wasted. Similarly, it is not perfectly fair, as once there are only a few tickets left, small groups still have a chance of being allocated but large groups do not. These issues persist under the Weighted Individual Lottery. In addition, the Weighted Individual Lottery has the additional concern that multiple members of a group may be selected. This explains the multiplicative factor of \( g(\alpha) \) in the theorem statement. Because \( g(\alpha) \geq 1 - \alpha/2 \), when \( \alpha \) is close to 0 the guarantees for the Group Lottery and the Weighted Individual Lottery coincide. Although it is intuitive that a small supply-demand ratio implies a small chance of having groups with multiple winners, the previous section shows that this may not be the case under the standard Individual Lottery.
**Proof Sketch of Theorem 3.**

In order to prove the efficiency and fairness guarantees, we first introduce a new mechanism: the *Group Lottery with Replacement (GR)*. This is a variant of the Group Lottery in which valid groups can be processed more than once. Formally, the set of actions, the set of valid groups $V$, the group request strategy and the allocation rule $x^{GL}$ are defined exactly as in the Group Lottery. However, the allocation function $\pi^{GR}$ is different, in particular, this mechanism processes valid groups according to a sequence of $k$ elements $\Sigma \in S_V$, where $\Sigma_i$ is independently and uniformly sampled with replacement from $V$. Hence, for any $x' \in X$, the allocation function of the Group Lottery is

$$\pi^{GR}_{x'}(a) = \sum_{\sigma \in O_v} 1 \{ x' = x^{GL}(a, \sigma) \} P(\Sigma = \sigma),$$

with $x^{GL}$ defined as in (2.7). Having defined this new mechanism, we now present a lemma that will be key in proving both guarantees. This lemma establishes a dominance relation between the Weighted Individual Lottery, the Group Lottery and the Group Lottery with Replacement, when the group request action profile is being selected. As we will see, every agent prefers the Group Lottery to the Weighted Individual Lottery, and the Weighted Individual Lottery to the Group Lottery with Replacement.

**Lemma 2.** For any instance and any agent $i \in N$, if $a$ denote the corresponding group request action profile for each mechanism below, then

$$u_i(\pi^{GR}(a)) \leq u_i(\pi^{IW}(a)) \leq u_i(\pi^{GL}(a)).$$

(2.24)

The key idea to prove Lemma 2 is that the order or sequence used in each of these mechanism can be generated based on a random sequence of agents $\Sigma'$. Roughly speaking, each order or sequence is generated from $\Sigma'$ as follows:

- Group Lottery with replacement: replace every agent by its group.
- Weighted Individual Lottery: remove every agent that has already appeared in a previous
• Group Lottery: replace every agent by its group, and then remove every group that has already appeared in a previous position.

Note that because in each mechanism the group request strategy is being selected, whenever a group or agent is being processed, it is given a number of tickets equal to the minimum of its group size and the number of remaining tickets.

This implies that, under the Group Lottery with Replacement, a group could be given more tickets than needed because one of its members appeared more than once in the first positions of \( \Sigma' \). This situation is avoided in the Weighted Individual Lottery, hence, making all agents weakly better. Similarly, under the Weighted Individual Lottery, a group could be given more tickets than needed because its members appeared more than once in the first positions of \( \Sigma' \). This situation is avoided in the Group Lottery, hence, making all agents weakly better. The full proof is located in Appendix A.3.2.

We now turn to the efficiency guarantee. From Lemma 2, it follows that for any instance the utilization under the Weighted Individual Lottery is at least the utilization under the Group Request with Replacement. Therefore, it suffices to show that for any instance in \( I(\kappa, \alpha) \), the Group Lottery with Replacement is \((1 - \kappa)g(\alpha)\)-efficient. To this end, we present in Lemma 3 a lower bound on the utility of any agent under the Group Lottery with Replacement.

**Lemma 3.** For any instance in \( I(\kappa, \alpha) \) and any agent \( i \), if we let \( \mathbf{a} \) be the group request under the Group Lottery with Replacement, then

\[
    u_i(\pi^{GR}(\mathbf{a})) \geq \frac{k}{n}(1 - \kappa)g(\alpha).
\]

(2.25)

The proof of Lemma 3 is in Appendix A.3. This lemma immediately give us the desired lower bound on the utilization of the Group Lottery with replacement.

We now show the fairness guarantee. From Lemma 2, we have that for any instance and any
pair of agents $i, j$, 

$$\frac{u_i(\pi_{IW}(a))}{u_j(\pi_{IW}(a))} \geq \frac{u_i(\pi_{GR}(a))}{u_j(\pi_{GL}(a))}. \tag{2.26}$$

Hence, it suffices to show that the ratio on the right hand side is at least $(1 - 2\kappa)g(\alpha)$. In Lemma 1 we proved an upper bound on the utility of an agent under the Group Lottery. Meanwhile, in Lemma 3 we established a lower bound on the utility of an agent under the Group Lottery with Replacement. Combining equation (2.26), Lemma 1 and Lemma 3 yields our fairness factor of $(1 - 2\kappa)g(\alpha)$.

### 2.5 Discussion

We consider a setting where groups of people wish to share an experience that is being allocated by lottery. We study the efficiency and fairness of simple mechanisms in two scenarios: one where agents identify the members of their group, and one where they simply request a number of tickets. In the former case, the Group Lottery is $(1 - \kappa)$-efficient and $(1 - 2\kappa)$-fair. However, its natural and widespread counterpart, the Individual Lottery, suffers from deficiencies that can cause it to be arbitrarily inefficient and unfair. As an alternative, we propose the Weighted Individual Lottery. This mechanism uses the same user interface as the Individual Lottery, and Theorem 3 establishes that it is $(1 - \kappa)g(\alpha)$-efficient and $(1 - 2\kappa)g(\alpha)$-fair.

Although our bounds are based on worst-case scenarios, they can be combined with publicly available data to provide meaningful guarantees. In 2016 the Hamilton Lottery received approximately $n = 10,000$ applications daily for $k = 21$ tickets, with a max group size of $s = 2$.\textsuperscript{15} Hence, in this case $\kappa \leq .05$ and $g(\alpha) \geq .99$. Therefore, by Theorem 1, the Group Lottery outcome is at least 95\% efficient and 90\% fair. Furthermore, Theorem 3 gives nearly identical guarantees for the Weighted Individual Lottery. Meanwhile, the 2020 Big Sur Marathon Groups and Couples lottery received 1296 applications for $k = 702$ tickets, with a maximum group size of 15.\textsuperscript{16} This yields $\kappa = 14/702 \approx .02$, so Theorem 1 implies 98\% efficiency and 96\% fairness for the Group Lot-

\textsuperscript{15}Source: https://www.bustle.com/articles/165707-the-odds-of-winning-the-hamilton-lottery-are-too-depressing-for-words

\textsuperscript{16}Source: https://www.bigsurmarathon.org/random-drawing-results-for-the-2020-big-sur-marathon
tery. Determining $\alpha$ is trickier, as we do not observe the total number of agents $n$. Using the very conservative lower bound $n \geq 1296$ (based on the assumption that every member of every group submitted a separate application) yields $g(\alpha) \geq .77$. Thus, Theorem 3 implies 76% efficiency and 74% fairness for the Weighted Individual Lottery. Its true performance would likely be much better, but accurate estimates would rely on understanding how many groups are currently submitting multiple applications.

Our analysis makes the strong assumption of dichotomous preferences. In practice, the world is more complicated: groups may benefit from extra tickets that can be sold or given to friends, and groups that don’t receive enough tickets for everyone may choose to split up and have a subset attend the event. Despite these considerations, we believe that dichotomous preferences capture the first-order considerations in several markets while maintaining tractability. The mechanisms we have proposed, while imperfect, are practical and offer improvements over the Individual Lottery (which is often the status quo). Furthermore, we conjecture that the efficiency and fairness of the Group Lottery would continue to hold in a model where the utility of members of group $G$ is a more general function of the number of tickets received by $G$, so long as this function is convex and increasing on $\{1, 2, \ldots, |G|\}$, and non-increasing thereafter.

One exciting direction for future work is to adapt these mechanisms to settings with heterogeneous goods. For example, while the daily lottery for Half Dome allocates homogeneous permits using the Individual Lottery, the pre-season lottery allocates 225 permits for each day. Before the hiking season, each applicant enters a number of permits requested (up to a maximum of six), as well as a ranked list of dates that would be feasible. Applicants are then placed in a uniformly random order, and sequentially allocated their most preferred feasible date. This is the natural extension of the Individual Lottery to a setting with with heterogeneous goods, and has many of the same limitations discussed in this paper. It would be interesting to study the performance of (generalizations of) the Group Lottery and Weighted Individual Lottery in this setting.
Chapter 3: Explainable Affirmative Action

3.1 Introduction

We study Prioritized Selection Problems, in which an organization is presented with a set of individuals, and must choose which subset to accept. We assume that the organization has a complete priority ranking of individuals. This ranking may be determined by various factors, including date of application, test scores or other measures of merit, a lottery, or some combination of these. The organization also has objectives or constraints that are not reflected in the priority ranking, but influence the final selection. Examples of such settings include:

- **Chilean Constitutional Assembly Elections.** In 2021, Chile held an election to choose 155 representatives to rewrite its constitution. Candidates were prioritized by votes received. However, the elected representatives from each district had to include an equal number of men and women, and include an appropriate number of members of each party (based on that party’s vote share in the district).

- **US Immigrant Visa Programs.** Both H1B and Diversity Visas are allocated by lottery. However, some H1B visas are reserved for applicants with advanced degrees from US institutions. Furthermore, the Diversity Visa lottery includes upper quotas on the number of individuals selected from each country and region.

- **Affordable Housing Allocation.** People who apply for affordable housing through New York City’s “HousingConnect” website are placed in a uniformly random order. However, half of the units in each building are reserved for residents of the local community district, additional units are set aside for people with disabilities and city employees, and each unit has eligibility criteria based on household size and income.
• **School Assignment.** Specialized high schools in New York City admit students based on a test score. The “Discovery Program” reserves seats at these schools for students from disadvantaged backgrounds (Faenza, Gupta, and Zhang 2022). Priority for university admissions is determined by exam scores in many countries. In India, positions are also reserved for people from disadvantaged castes, people with disabilities, women, and people from the local state (Baswana et al. 2019). In Brazil, positions are reserved for students who attended public high school, are low-income, or belong to a disadvantaged minority group.

• **Civil Service Positions.** In India, priority for government positions is determined by standardized exam scores. Many positions are reserved for applicants from historically marginalized groups, including women and “backwards classes.” (Sönmez and Yenmez 2022; Aygün and Turhan 2022).

We refer to goals unrelated to the priority ranking as “diversity constraints” or “affirmative action objectives.”

It is not obvious how to best combine the priority ranking and affirmative action objectives to reach a final selection. In practice, there are many ad hoc methods for doing so, but these algorithms frequently produce outcomes that can be difficult to explain to participants. For example,

• **New York Affordable Housing:** Applicants from outside the community district are often passed over for lower-ranking applicants from the community district, even if the community preference constraint is not binding and swapping these outcomes is feasible.

• **New York Specialized High Schools:** high scoring students who are eligible for the Discovery Program are sometimes assigned to their second or third choice, while seats in their preferred schools are offered to lower-scoring students (Faenza, Gupta, and Zhang 2022).

• **Brazilian University Admissions:** cutoff scores for applicants claiming multiple privileges are frequently higher than corresponding cutoffs for applicants claiming a subset of those privileges (Aygün and Bó 2021).
• **Indian Civil Service:** Numerous lawsuits document cases where the algorithm selected a woman from a majority group instead of another woman from a backwards class *with a higher exam score* (Sönmez and Yenmez 2022).

Each of these examples features an apparent violation of priority that is difficult to justify on the basis of the diversity constraints. Individually, each example provides an interesting research opportunity, and indeed the papers mentioned above generally provide “solutions” (new algorithms) for the particular context they study. Collectively, however, these examples illustrate the challenge of reasoning about the behavior of complex algorithms.

Our primary contribution is to introduce a family of rules which we argue are uniquely *explainable*. These rules avoid the priority violations described above, ensuring that any violations are necessitated by the diversity constraints. Furthermore, we provide an axiomatic characterization that shows that these are the only rules that respond in “explainable” ways to changes in the priority order. We study when explainable rules can be implemented without losing efficiency, and establish that the algorithm used to elect members of Chile’s recent Constitutional Assembly belongs to our family of rules. We now elaborate on each contribution.

3.1.1 Axiomatizing Explainability

Given a set of individuals $I$, a *selection* is a subset of these individuals, and a *selection rule* $\varphi$ is a function mapping each priority order $\succ$ to the resulting selection $\varphi(\succ)$. We consider three natural properties for a selection rule to satisfy, which constrain how the rule responds to changes in the priority list.

• **Monotonic:** Increasing a selected individual’s priority never causes that individual to become unselected.

• **Non-bossy:** If a change in an individual’s priority does not change that individual’s outcome, then it does not change the final selection.
• **Lower invariant:** Whether an individual is selected depends only on the priorities of higher-priority individuals, and not on those of lower-priority individuals.

Theorem 4 establishes that a selection rule satisfies all three of these properties if and only if it is *outcome-based*. Outcome-based selection rules are defined by a collection of feasible selections \( \mathcal{F} \subseteq \{0, 1\}^I \) along with a greedy processing rule presented in Algorithm 2. This rule considers individuals in priority order, and commits to selecting each individual if it is feasible to do so while honoring the commitments to higher-priority individuals.

We choose the name “outcome-based” selection because we believe that it is much easier for the public and policymakers to form opinions about what *outcomes* are acceptable, rather than the *algorithms* that should be used to find these outcomes. Indeed, in many of the applications above, both priorities and feasible outcomes are clearly specified, but the algorithm for combining these considerations is either unspecified or receives very little attention. For example, Gonczarowski et al. (2019) describe a matching system in which

\[
\text{Each PMA [Pre-Military Academy] defines a set of “populations” that it cares about (e.g., based on gender, religiousness, city of origin, belonging to certain minority groups). For each such population the PMA is allowed to define a maximum quota as well as minimum target. In addition to ranking the candidates...}
\]

They go on to note that this information does not necessarily pin down the PMAs selection (“There are several issues that need be specified before this becomes fully formal”), but that “virtually none of the PMAs showed any desire to dig into these issues.” Relatedly, Pathak, Rees-Jones, and Sönmez (2023) demonstrate experimentally that although changing the processing order for different reserve categories often has significant consequences, people do not understand this fact. To illustrate that the feasible selections \( \mathcal{F} \) and the priority order \( \succ \) do not obviously pin down the final selection, we present the following example.

**Example 2.** *There are four individuals, with \( 1 \succ 2 \succ 3 \succ 4 \). The feasible selections are \( \mathcal{F} = \{\{1, 4\}, \{2, 3\}\} \).*
Selection Rule 2
Hire top candidate, then
Hire top remaining woman.

Selection Rule 1
Hire top woman, then
Hire top remaining candidate.

Selections must have
2 people, at least 1 woman,
and must respect priorities.

<table>
<thead>
<tr>
<th>Algorithmic Description</th>
<th>Outcome Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selection Rule 1</td>
<td>Selections must have 2 people, at least 1 woman, and must respect priorities.</td>
</tr>
<tr>
<td>Selection Rule 2</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.1: In some cases, outcome-based selection rules provide a new way to describe an existing procedure. For example, Theorem 7 establishes that the algorithm used to select members of Chile’s Constitutional Assembly is equivalent to an outcome based rule, despite having a very different description. More straightforwardly, Selection Rule 1 above is equivalent to the outcome-based rule associated with the constraint that selections must contain at least one woman and at most two people. Other rules, however – such as Selection Rule 2 – do not have any outcome-based description.

In this case, it is not clear whether \{1, 4\} or \{2, 3\} should be chosen. However, we argue that \{1, 4\} is easier to explain. We formalize this argument as follows. We say that a feasible selection respects priorities if, for every individual who is not selected, it is impossible to select this individual without displacing some higher-priority individual who is currently selected. Proposition 9 establishes that there is always a unique feasible selection that respects priorities (\{1, 4\} in Example 2), and it can be found by the greedy procedure described in Algorithm 2.

In essence, we can offer individuals 2 and 3 the following explanation for why they were not selected: selecting them would require displacing the higher-priority individual 1. No such simple explanation is available if we choose \{2, 3\}. In that case, we must explain to individual 1 why they were not chosen despite having the top priority. Our explanation will be complicated by the fact that had the bottom of the ranking been slightly different (with individual 4 ahead of individual 3), we presumably would have selected \{1, 4\}. Individual 1 may feel that the precise order of lower-priority agents should be irrelevant, motivating our definition of lower invariance above.

3.1.2 Special Cases: Reserves and Quotas

Having defined our family of outcome-based selection rules and characterized them as uniquely explainable, we turn to their performance. The fact that these rules find a selection using a greedy
algorithm can come at a cost. For example, if the only feasible selections are \{1\} and \( I \setminus \{1\}\) and individual 1 is at the top of the priority order, then the outcome based selection rule will choose \{1\} and reject all others.

Recognizing that this example is somewhat contrived, we consider two ways to define feasible sets which arise in many applications: reserves and quotas. In each case, we identify conditions under which the feasible selections induce a matroid. In these cases, explainability need not come at the expense of efficiency: the selection made by the greedy outcome based rule priority dominates all other feasible selections.

When using reserves, the feasible selections are defined by a set of positions and a compatibility graph \( G \) indicating which individuals may be matched to which positions. We consider several ways to interpret the graph \( G \). Hard reserves require that compatibility constraints are always strictly enforced. Soft reserves allow the selection of incompatible individuals so long as all compatible individuals have already been selected. We formulate two interpretations of this requirement: soft maximum reserves require that feasible selections must be a superset of the individuals matched in some maximal matching \( M \) of \( G \), while soft maximum reserves require that \( M \) be a maximum matching. While maximum reserves have been studied before (see for example Sönmez and Yenmez (2022)), to our knowledge we are the first to define maximal reserves.

Hard reserves interpret compatibility constraints very strictly, while maximum and especially maximal reserves are more permissive and allow higher-priority selections of applicants. Theorem 5 establishes that all three interpretations induce a matroid, implying that there exists a feasible selection that priority dominates all others. Proposition 10 shows that finding this selection can be done in polynomial time for maximum and hard reserves, but is NP-hard for maximal reserves. This latter result is interesting, and stems from the fact that in this context, we do not have an independence oracle to tell us whether a given selection is a subset of a feasible selection.

Quotas, meanwhile, are specified by (i) a finite set of traits \( T \); (ii) for each individual a subset of traits that the individual possesses; and (iii) upper and lower bounds (quotas) on the number of selected individuals with each trait. Proposition 11 establishes that for this family of feasible
selections: (i) determining the existence of a feasible selection is NP-complete, and (ii) even if the set of feasible selections is nonempty, there may be no feasible selection that priority dominates all others. However, Lemma 5 identifies that if the traits form a hierarchy, the resulting feasible sets induce a matroid (and this condition is “necessary” in a maximal domain sense).

3.1.3 Electing the Chilean Constitutional Assembly

In 2021, the country of Chile convened an assembly to propose a new constitution. Members of this assembly were selected by a special election. Each district elected a set of representatives whose parties needed to reflect the district’s vote totals. Furthermore, representatives needed to be equally balanced between male and female. These constraints do not induce a hierarchy, so in general there will not be a “obvious” selection that priority dominates all others. Interestingly, the algorithm used by Chile (which begins by ignoring the gender constraints and then corrects imbalances by replacing the lowest vote-getters from the over-represented gender) turns out to be equivalent to the outcome-based rule associated with the party and gender constraints (Theorem 7). In other words, each candidate who was not selected can be assured that in order to select them while complying with these constraints, a candidate with more votes would have to be rejected.

3.2 Related Work

Our paper contributes to a large and growing literature on assignment and matching problems with distributional constraints. There are too many papers to discuss each one in detail, so we divide our discussion into two clusters: those where individuals belong to disjoint groups, and those where affirmative action targets overlapping groups. Within each cluster, we focus our discussion on the most related work.

3.2.1 Disjoint Types

Most of the early matching work on distributional constraints and affirmative action assumes that individuals can be partitioned into types, and that constraints are specified with respect to the
number of selected individuals of each type. We note that most of these papers adopted uses of the word “reserve” and “quota” which are not perfectly consistent with our own. Specifically, they use “quota” to mean maximum quotas, and “reserve” to mean minimum quotas. In settings where types do not overlap, the distinction between reserves and minimum quotas is of little consequence. However, with overlapping types, we show that reserving spots for certain groups of people actually induces a very different mathematical structure than setting a minimum quota (for example, the number of reserved positions cannot exceed the maximum number of selected individuals, whereas the sum of minimum quotas can).

Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu, Pathak, and Roth (2005) assume students belong to disjoint types, and that there are type-specific maximum quotas. Kojima (2012) considers a setting with only two student types (minority and majority), and shows that implementing upper quotas on majority applicants can end up hurting all minority applicants. Hafalir, Yenmez, and Yıldırım (2013) also consider a model with two types of students, and argue that minority reserves (minimum quotas) result in more efficient outcomes than majority (maximum) quotas. A related model of Ehlers et al. (2014) attempts to achieve diversity goals by imposing both upper and lower quotas on the number of students of each type. Not surprisingly, if these bounds are treated as hard constraints, a good assignment of students to schools may not exist; so these authors interpret these upper and lower bounds as “soft” constraints that should be satisfied “as much as possible” and design an algorithm to find a fair and non-wasteful assignment of students to schools.

More recently, Abdulkadiroğlu and Grigoryan (2021) consider a model with upper and lower constraints on a disjoint set of traits, and show how to construct a feasible selection that priority dominates all others. This is related to our work in Section 3.4, though we work with a more general model in which quotas can apply to overlapping groups of individuals.

Recent work by Pathak, Rees-Jones, and Sönmez (2023) loosely relates to our motivation of explainability. It uses a survey of over 1,000 people to demonstrate that the effects of changing the processing order of different positions are poorly understood. This work helps to reinforce our
belief that people are generally better at understanding and forming opinions about final outcomes, rather than the procedures that generate them.

3.2.2 Overlapping Types

An important shortcoming of these early models is the assumption that agents are partitioned into disjoint types. In practice, affirmative action policies often target groups whose memberships overlap. A number of recent papers consider this case (as do we). Most of these papers focus on a specific application, and describe feasibility constraints, axioms, and algorithms suitable for that domain. For this reason, it can be difficult to translate results from one paper to the next. We instead propose axioms and a greedy algorithm which can be applied regardless of the structure of the feasible sets.

A similar approach is taken by Hafalir et al. (2022). They consider choice rules that determine a final selection based on (i) a set of applicants, and (ii) a priority ranking for all individuals. Fixing the priority ranking, these rules determine a choice function. Fixing the set of applicants, these rules determine a selection rule (as defined in this paper). The family of rules on which they focus is “Diversity Choice Rules.” These rules rely on a function $f$ mapping selections to a score indicating how “diverse” they are. For any set of applicants $I$, their rule defines the feasible sets to be the most diverse subsets of $I$ (that is, $\mathcal{F} = \arg \max_{S \subseteq I} f(S)$), and selects among sets in $\mathcal{F}$ using a greedy processing algorithm. Thus, our outcome-based selection rules can be seen as a special case of their Diversity Choice Rule where the set of applicants $I$ is fixed and known.

Although their paper considers a generalization of our outcome-based selection rules, our central questions are fairly different. We focus on explainability, and show that outcome-based rules are uniquely explainable through two characterization results. They focus primarily on efficiency: that is, what conditions on $f$ ensure that the feasible set $\mathcal{F}$ induces a matroid? While similar in spirit to our investigation in Section 3.4 the conditions they provide are based on discrete concavity notions, making them possibly more general but also much more abstract than those in our Theorems 5 and 6. In addition, they consider the question of when their choice rule is “path
independent” (closely related to substitutability) and thus suitable for use in a two-sided matching market where applicants express preferences. Because we assume a fixed and known set of applicants, this is not a question that we address.

Several other papers present results that are closely related to our findings in Section 3.4. Gonczarowski et al. (2019) study a setting with lower and upper quotas, and identify the importance of having the traits form a hierarchy (which they refer to as ‘laminar’). However, in order for their algorithms to be guaranteed to find a priority dominant selection, their Theorems 1 and 2 require further restrictions on the set of traits with a nonzero lower quota. By contrast, our Lemma 5 needs no such restrictions. When traits do not form a hierarchy, they propose a computationally simple algorithm that does not respect priorities, while we propose finding the selection that respects priorities (which is NP-hard).

A third closely related paper is that of Sönmez and Yenmez (2022). Their analysis of overlapping horizontal reservations under the “one-to-one” convention is essentially equivalent to our soft maximum reserves (their “one to all” convention is equivalent to our minimum quotas, but receives relatively little attention in their paper). They give a two-pass algorithm that is equivalent to our outcome-based selection rule (despite different descriptions), and show that the result priority dominates all other feasible selections. Their characterization results and ours are quite different: whereas they focus on axioms motivated by the Indian Constitution, we attempt to formalize the notion of explainability. This difference leads us in different directions: their vertical reservations are constitutionally mandated, whereas our axioms rule out such reservations (due to their bosiness). For similar reasons, our results are not directly comparable with those of Aygün and Turhan (2022): while their de-reservation policy resembles our soft reserves, their focus is on vertical or “over and above” reserves, which our axioms rule out.

Finally, we note that Cembrano et al. (2021) also discuss the Chilean Constitutional Convention, and compare the selection rule used there to many alternatives. However, they do not provide any characterization of this rule. By contrast, we show that this procedure finds the unique selection that respects priorities.
3.3 Explainable Selection Rules

In a selection problem there is a set of applicants $I$, from which a decision maker has to select a subset $S \subseteq I$. There is a strict priority order $\succ$ over $I$. We denote by $P$ the set of all priority orders over $I$. A selection rule is a mapping $\varphi : P \rightarrow 2^I$ that determines a selection for every priority order. We let $\varphi(\succ)$ denote the selection of rule $\varphi$ for priority order $\succ$.

3.3.1 Respecting Priorities

In some cases, there is a collection of feasible selections $F \subseteq 2^I$ which may capture capacity constraints, diversity goals or other restrictions imposed by law or by the designer. In this section, our goal is to define a selection rule that (i) always returns a feasible selection and (ii) adheres to the priority order “as much as possible.” As discussed in Example 2, the correct interpretation of the latter criteria is not obvious.

**Definition 6.** Selection $S'$ **priority dominates** selection $S$ if (and only if) $S' \neq S$ and there exists an injective function $\psi$ from $S$ to $S'$, such that for each $i \in S$, $\psi(i) \succeq i$.

Note in particular that if $S'$ priority dominates $S$, then $|S'| \geq |S|$. One way to think of priority domination is as follows. For $k \leq |S|$, let $i_k^S$ be the $k^{th}$ highest priority individual in $S$, and $i_k^{S'}$ be the $k^{th}$ highest priority individual in $S'$. If $S'$ priority dominates $S$, then for each $k \leq |S|$, we must have $i_k^{S'} \succeq i_k^S$. If $1 \succ 2 \succ 3 \succ 4$, then $\{1, 3\}$ priority dominates both $\{1, 4\}$ and $\{2, 3\}$. However, neither of the selections $\{1, 4\}$ and $\{2, 3\}$ priority dominate the other.

If our goal is to respect the priority order as much as possible, then at a minimum, we should choose a feasible selection that is not priority dominated. However, this does not address the question of which selection to choose when there are multiple undominated feasible selections. To resolve this question, we propose the following criteria.

**Definition 7.** A feasible selection $S \in F$ **respects priorities** if, for each unselected applicant $i \notin S$, it is impossible to select them without displacing a higher-priority applicant. Formally, there is no

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1Our notion of priority domination is equivalent to the Gale domination presented in Gale 1968.
ALGORITHM 2: Outcome Based

**Input:** Set of applicants $\mathcal{I}$, feasible selections $\mathcal{F}$ and priority order $\succ$, with $i_1 \succ i_2 \succ \cdots \succ i_n$.

$S := \emptyset$;

foreach $k = 1, \ldots, n$ do
    if $\exists S' \in \mathcal{F} : S \cup \{i_k\} \subseteq S'$ then
        $S = S \cup \{i_k\}$
    end
end

return $S$

feasible selection $S' \in \mathcal{F}$ such that $\{i\} \cup \{j \in S : j \succ i\} \subseteq S'$.

The motivation for this definition is that it offers a “simple” explanation for each individual’s outcome. Suppose that an individual $i$ complains that they were not selected, while some lower priority individuals were. If we have chosen a selection that respects priorities, then we can tell $i$ that these lower priority individuals were not the reason that $i$ was not selected. Rather, $i$ was not selected because any feasible selection including $i$ would require the displacement of some individual with a higher priority.

Applying this concept to our previous example with $1 \succ 2 \succ 3 \succ 4$ and $\mathcal{F} = \{\{1, 4\}, \{2, 3\}\}$, the selection $\{1, 4\}$ respects priorities, while the selection $\{2, 3\}$ does not. Informally, if we choose $\{2, 3\}$, there is no “simple” explanation for why individual 1 was not selected. The following result establishes that there is always a unique selection that respects priorities.

**Proposition 9.** For any feasible set $\mathcal{F}$ and any priority order $\succ$:

1. There is a unique feasible selection that respects priorities, which we denote $OB_\mathcal{F}(\succ)$.

2. The selection $OB_\mathcal{F}(\succ)$ is not priority dominated by any other feasible selection.

3. If there exists $S^* \in \mathcal{F}$ that priority dominates every other feasible selection, then $OB_\mathcal{F}(\succ) = S^*$.

**Proof.** For part (1), let $OB_\mathcal{F}$ be the selection chosen by the ‘outcome based ’ algorithm (Algorithm 2). Let $S$ be any other feasible selection. Let $i$ be the highest priority individual for which the
selections \( OB_F \) and \( S \) differ. Because \( S \) is feasible and the two selections make identical choices for individuals with higher priority than \( i \), it must be that \( i \in OB_F(\succ) \), and therefore that \( i \notin S \). Therefore, \( S \) does not respect priorities, as \( OB_F(\succ) \) is feasible and selects \( i \) along with all higher-priority individuals selected by \( S \).

For part (2), suppose that there is a feasible selection \( S \) that priority dominates \( OB_F(\succ) \). Let \( i \) be the highest-ranking individual that is in \( S \) but not \( OB_F(\succ) \). By definition, \( S \) and \( OB_F(\succ) \) make identical decisions on individuals with higher priority than \( i \): \( \{ j : j \succ i \} \cap S = \{ j : j \succ i \} \cap OB_F(\succ) \). Therefore, when \( OB_F(\succ) \) considered individual \( i \), \( S \) was a feasible selection containing \( i \) and all higher-priority individuals that had already been selected. Therefore, \( OB_F(\succ) \) should select \( i \), contradicting our assumption that \( S \) priority dominates \( OB_F(\succ) \).

From this, (3) follows immediately. If there exists \( S^* \in F \) that priority dominates every other feasible selection, then if \( OB_F(\succ) \) is not equal to this selection, it is priority dominated by \( S^* \), contradicting (2).

\[ \Box \]

3.3.2 An Axiomatic Characterization: Monotonicity, Non-Bossiness, and Lower Invariance

The previous subsection defined the outcome based rule \( OB_F \) and showed that for any fixed \( F \), it is the only rule that is feasible and respects priorities. Often, policymakers get to choose the feasible selections \( F \). Different choices of \( F \) correspond to different selection rules: we refer to \( (OB_F)_{F \subseteq 2^I} \) as the family of outcome based selection rules. In this section, we give an axiomatic characterization of this family of selection rules.

In practice, selection rules need not be defined through a set of feasible outcomes, but rather by a description of an algorithm for considering and selecting applicants. In such cases, the feasible outcomes are often defined only implicitly. As one example, we consider the case of “over and above” reserves (Pathak, Rees-Jones, and Sönmez 2023).

**Example 3.** There are four individuals, 1, 2, 3, 4. Individuals 1 and 3 belong to a minority group. We can select at most two individuals, and wish to reserve one seat for minorities. We decide to use an “over and above” selection rule \( \varphi^{OA} \). This rule first selects the highest-priority individual,
and then selects the highest priority minority that remains.

Note that the selection rule $\varphi^{OA}$ is defined algorithmically, and does not explicitly identify which selections are feasible. However, for any selection rule $\varphi$, we can define

$$\text{Image}(\varphi) = \{ S \subseteq 2^I : S = \varphi(\succ) \text{ for some } \succ \in P \}$$

to be the set of selections that $\varphi$ might choose. In the case of $\varphi^{OA}$, it is easy to see that this consists of the five selections $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$ and $\{3, 4\}$. Note that $\{2, 4\}$ is never selected, as neither 2 nor 4 is a minority.

We claim that for any choice of $F$ containing $\text{Image}(\varphi^{OA})$, the rule $\varphi^{OA}$ does not respect priorities. To see this, note that when the priority order is $1 \succ 2 \succ 3 \succ 4$, $\varphi^{OA}$ selects $\{1, 3\}$, even though $\{1, 2\}$ is feasible (meaning that it would have been chosen for a different priority order) and priority dominates $\{1, 3\}$.

Thus, while choosing $F$ gives the designer substantial flexibility (as we explore in Section 3.4), restricting to the family of outcome based rules does rule out certain selection rules commonly used in practice. One might therefore ask, why should a designer use a outcome based rule? What advantages do these rules offer?

We have already argued informally that defining these rules using acceptable outputs (rather than algorithms) makes them more transparent. Now, we show that outcome based rules satisfy three desirable properties, and are in fact the only selection rules to satisfy all three properties. To define each of these properties, we introduce new notation. Given a priority order $\succ$ and a subset of individuals $S \subseteq I$, we let $\succ_{\neg S}$ be the restriction of $\succ$ to $I \setminus S$. In other words, for $i, j \in I \setminus S$,

$$i \succ_{\neg S} j \iff i \succ j.$$

When $S = \{i\}$ is a singleton, we write $\succ_{\neg i}$ in place of $\succ_{\neg \{i\}}$.

Our first property is monotonicity, which states that improvements to an individual’s priority should weakly improve their outcome.
Definition 8. A selection rule \( \varphi \) is **monotone** if for any applicant \( i \in I \), and for every \( \succ, \succ' \in P \) such that (i) \( \succ_{-i} = \succ'_{-i} \), and (ii) the priority of applicant \( i \) is higher in \( \succ' \) than in \( \succ \), then

\[
i \in \varphi(\succ) \Rightarrow i \in \varphi(\succ').
\]

The second property is **non-bossiness**, which says that a change to an individual’s priority which does not affect whether that individual is selected should also not affect who else is selected.

Definition 9. A selection rule \( \varphi \) is **non-bossy** if for any applicant \( i \in I \), and for every \( \succ, \succ' \in P \) such that \( \succ_{-i} = \succ'_{-i} \),

\[
\{i\} \cap \varphi(\succ) = \{i\} \cap \varphi(\succ') \Rightarrow \varphi(\succ) = \varphi(\succ').
\]

Our final property is **lower-invariance**, which says that whether an individual is selected should depend only on the priority of higher-priority individuals, and not on the exact priority of lower-priority individuals.

Definition 10. A selection rule \( \varphi \) is **lower invariant** if for any applicant \( i \in I \), and for every \( \succ, \succ' \in P \) such that \( \succ_{-\{j:i \succ j\}} = \succ'_{-\{j:i \succ j\}} \),

\[
\{i\} \cap \varphi(\succ) = \{i\} \cap \varphi(\succ').
\]

We are now ready to present our characterization of the family of outcome based rules.

Theorem 4.

- For any collection \( F \subseteq 2^I \), the selection rule \( OB_F \) is monotone, non-bossy and lower invariant.

- If \( \varphi \) is a selection rule that is monotone, non-bossy and lower invariant, then there exists \( F \subseteq 2^I \) such that \( \varphi(\succ) = OB_F(\succ) \) for every \( \succ \in P \).
We note that $\varphi^{OA}$ is monotonic and lower invariant, but bossy. Appendix B.1.2 proves that our axioms are independent by providing examples of rules that fail to satisfy one axiom while satisfying the other two.

3.4 Special Cases of Interest: Quotas and Reserves

In general, choosing the selection that respects priorities may come at a significant cost. For example, suppose that there are only two feasible selections: $\{1\}$, and $I\backslash\{1\}$. Then if individual 1 is at the top of the priority order, respecting priorities requires that we select this individual, and reject all others. However, this example is somewhat contrived and unlikely to arise in practice.

In this section, we impose additional structure on the feasible sets $F$, in ways intended to capture common real-world constraints. Our over-arching question is, in which cases can $OB_{F}$ be shown to priority dominate all other feasible selections? In these cases, the normative case for using this selection rule is clear.

In essence, $OB_{F}$ greedily selects individuals, so long as doing so would not violate feasibility. Greedy algorithms are well-known to solve optimization problems when the underlying constraints define a matroid.\(^2\) We now reproduce this result in our setting.

**Definition 11.** The feasible selections $F$ induce a matroid if for any $I_1, I_2 \subseteq I$ such that $I_1 \subseteq S_1 \in F$, $I_2 \subseteq S_2 \in F$, and $|I_1| > |I_2|$, there exists $i \in I_1 \backslash I_2$ and $S_3 \in F$ such that $I_2 \cup \{i\} \subseteq S_3$.

**Lemma 4** (Theorem 3 Gale 1968). If $F$ induces a matroid, then for any priority order $\succ$, $OB_{F}(\succ)$ priority dominates any other feasible assignment.

While Lemma 4 gives a complete answer to when $OB_{F}$ will produce a priority dominant selection, verifying whether $F$ induces a matroid can be non-trivial. In addition, in some cases, $F$ may induce a matroid, but $OB_{F}$ may nevertheless be NP-hard to compute. Therefore, below, we consider two common structures for $F$ from practice: Quotas and Reserves.

3.4.1 Reserved Positions

We now study several families of feasible selections which we collectively refer to as reserved positions. These families are described by a bipartite graph $G = (I, P, E)$, where each node of the left side corresponds to an individual $i \in I$, each node on the right side corresponds to a position $p \in P$, and there is an edge $(i, p) \in E \subseteq I \times P$ only if individual $i$ is eligible for position $p$. We refer to $G = (I, P, E)$ as the compatibility graph. A matching $M$ of $G$ is a subset of edges such that every node is incident to at most one edge in $M$. For any matching $M$ of $G$, we denote by $I_M$ the set of applicants that are incident to an edge in $M$:

$$I_M = \{i \in I : \text{exists a position } p \in P \text{ such that } (i, p) \in M\}. \quad (3.1)$$

In some environments, compatibility constraints must be strictly enforced. We model these situations using what we call hard reserves. Formally, given a compatibility graph $G = (I, P, E)$, the feasible selections when using hard reserves are

$$\mathcal{F}^{hard}(G) = \{S \subseteq I : \text{exists a matching } M \text{ of } G \text{ such that } I_M = S\}. \quad (3.2)$$

Under hard reserves, a position may go unfilled if there is no unselected individual who is eligible for it. In many applications, it may be preferable to reduce waste by allowing someone who would otherwise not be eligible for this position to be assigned to it (Aygün and Turhan (2022) refer to this as “de-reserving” this position). To model such cases, we introduce two types of “soft” reserves: maximum reserves and maximal reserves. We define

$$\mathcal{F}^{maximum}(G) = \{S \subseteq I : |S| \leq |P| \text{ and } \exists \text{ a maximum matching } M \text{ of } G \text{ such that } I_M \subseteq S\}. \quad (3.3)$$

$$\mathcal{F}^{maximal}(G) = \{S \subseteq I : |S| \leq |P| \text{ and } \exists \text{ a maximal matching } M \text{ of } G \text{ such that } I_M \subseteq S\}. \quad (3.4)$$
Both of these definitions allow applicants to be matched to positions for which they are not eligible, so long as everyone who is eligible for that position is also selected.

The distinction between hard, maximum, and maximal reserves is illustrated in Figure 3.2. Intuitively, hard constraints interpret compatibility identified by the edges of $G$ very strictly, while maximum and maximal reserves reduce waste and can increase the priority of selected individuals. Depending on the relative importance of compatibility and the priority order, the designer can choose any of these three interpretations. The main result of this section is the following.

**Theorem 5.** For any $G = (I, P, E)$, the feasible selections $\mathcal{F}^{\text{hard}}(G)$, $\mathcal{F}^{\text{maximum}}(G)$ and $\mathcal{F}^{\text{maximal}}(G)$ defined by (3.2), (3.3) and (3.4) each induce a matroid.

From Lemma 4, it follows that in these cases the selection that respects priorities will also priority dominate all other feasible selections. Furthermore, the selection when using maximal reserves will priority dominate the selection when using maximum reserves, which will priority dominate the selection when using hard reserves.

We now comment on connections between this result and others in the matching literature. In the literature of matroids is well-known that Hard reserves induce the “transversal matroid”\(^3\) (see e.g. Example 3.7 in Yokoi 2019).

\[^3\text{https://en.wikipedia.org/wiki/Matroid#Matroids_from_graph_theory.}\]
The horizontal reserves in Sönmez and Yenmez (2022) correspond to what we call maximum reserves: their non-wastefulness condition implies that reserves are soft, and their condition of “maximally complying with reservations” further specifies maximum (rather than maximal) reserves. They provide a “meritorious horizontal choice rule”, and their Theorem 1 establishes that this choice rule priority dominates all other feasible selections. This result is analogous to our conclusion that maximum reserves induce a matroid. While their meritorious choice rule and our outcome based rule $OB_{F_{\text{maximum}}(G)}$ always find the same selection, the descriptions of these algorithms differ substantially. In particular, their algorithm takes two passes: one to fill as many reserved seats as possible, and another to fill any remaining positions with the highest-priority remaining individuals. By contrast, our algorithm is described in a single pass, which selects each individual so long as doing so is feasible. We see advantages to both descriptions. Our description offers participants a simpler and more transparent description of why they were or were not selected. Meanwhile, their algorithm can be more easily converted into functional software.

We are not aware of any paper that explicitly defines a feasible set corresponding to our maximal reserves. One might argue that this is because this definition is “too permissive”: in the example in Figure 3.2, choosing applicants 1 and 2 over applicant 4 goes against the spirit of the reserve policy. However, widely studied and deployed algorithms may select matchings that are feasible under maximal reserves but not maximum reserves. For example, this is the case for slot-specific priority algorithms such as those proposed by Aygün and Bó (2021) for Brazilian University admissions and used in Chile for school assignment (Correa et al. 2021).

In contexts where existing algorithms may find selections that are not feasible according to maximum reserves, it seems natural that instead of using an exogenously specified order to fill positions, one could endeavor to choose the selection that respects priorities with respect to maximal reserves. Perhaps one reason that this has not been done is that the problem of finding this selection is NP hard. By contrast, the priority-respecting assignment can be found in polynomial

4For example, in the example in Figure 3.2, if position $C$ chooses first and $1 \succ 2 \succ 3 \succ 4$, then position $C$ will choose individual 3 and positions $A$ and $B$ will choose individuals 1 and 2, resulting in a selection that is not feasible according to soft maximum reserves.
time for maximum and hard reserves.

**Proposition 10.** The problem of evaluating $OB_{F_{\text{maximal}}(G)}$ is NP-complete.

For any graph $G$, the rules $OB_{F_{\text{maximum}}(G)}$ and $OB_{F_{\text{hard}}(G)}$ can be evaluated in polynomial time.

3.4.2 Quotas

In the second family of constraints we study, there are multiple groups of applicants. For each group, there is an upper and lower limit on the number of its members that can be selected. Formally, there is a finite set of traits $T$. For each trait $t \in T$, there is a set of applicants with the trait $I_t \subseteq I$, and an upper and lower quota $u_t, \ell_t \in \mathbb{N}$ on the number of selected applicants with trait $t$. We use $I_T$ as shorthand for $\{I_t\}_{t \in T}$. Then, a selection is feasible if it satisfies both quotas for every trait. That is,

$$\mathcal{F}(I_T, u, \ell) = \{S \subseteq I : \text{for every } t \in T, \ \ell_t \leq |S \cap I_t| \leq u_t\}. \quad (3.5)$$

In contrast with the reserved positions defined in Section 3.4.1, the problem of determining whether there exists a feasible selection is NP-complete. Furthermore, even when a feasible selection exists, there may be no feasible selection that priority dominates all others.

**Proposition 11.** Determining whether $\mathcal{F}(I_T, u, \ell)$ defined in (3.5) is empty is NP-complete.

If $\mathcal{F}(I_T, u, \ell)$ is non-empty, there may be no feasible selection that priority dominates all others.

Our proof located in Appendix B establishes NP-completeness result even in special cases in which there is just a single global lower quota or a single global upper quota. These proofs reduce from the independent set problem and the set cover problem, respectively. Meanwhile, Example 4 shows that in general, there may be no feasible selection that priority dominates all others.

**Example 4.** Consider an instance with three applicants $I = \{1, 2, 3\}$ and two traits $T = \{A, B\}$. Let $I_A = \{1, 2\}, I_B = \{1, 3\}$ and $u_A = u_B = 1 = \ell_A = \ell_B$. In this instance, there are only two feasible selections: $\mathcal{F}(I_T, u, \ell) = \{\{1\}, \{2, 3\}\}$. If $1 \succ 2 \succ 3$, then neither feasible selection priority dominates the other.
Hierarchies

As Example 4 illustrates, under quotas there might be tension between respecting priorities and other goals, such as maximizing the number of selected individuals. We now provide conditions under which the feasible selection that respects priorities priority dominates all others.

**Definition 12.** The traits \( I_T = \{I_t \}_{t \in T} \) form a hierarchy if for every \( t, t' \in T \), either \( I_t \) and \( I_{t'} \) are disjoint \( (I_t \cap I_{t'} = \emptyset) \) or one contains the other \( (I_t \subset I_{t'} \text{ or } I_{t'} \subset I_t) \).

**Lemma 5** (Lemma 3 Yokoi 2017). If the collection of sets \( I_T = \{I_t \}_{t \in T} \) is a hierarchy, then for any upper and lower quotas \( u, l \in \mathbb{N}_T \) such that the set of feasible selections is non-empty, the feasible sets \( F(I_T, u, l) \) induce a matroid. Therefore, the selection that respects priorities also priority dominates all other feasible selections.

**Theorem 6.** If the collection of sets \( I_T = \{I_t \}_{t \in T} \) is not a hierarchy, then there exist upper and lower quotas \( u, l \in \mathbb{N}_T \) and a priority order \( \succ \) such that no feasible selection priority dominates all others.

Lemma 5 and Theorem 6 show that the hierarchy structure is in some sense “necessary and sufficient” for there to exist a selection that priority dominates all others.

### 3.5 Chilean Constitutional Assembly election

In what follows, we describe the mechanism used in Chile to elect the members of the 2021 Constitutional Assembly. Elections are independent across districts. In addition, there was an election to select 17 members of indigenous communities, which was conducted separately from the district elections. Below, we focus on what happened within each district. Elections were conducted as follows:

1. Candidates are ranked by votes received.

2. Each candidate has a gender (male/female) and a party\(^5\).

---

\(^5\)Strictly speaking, each candidate belongs to a list, and each list is composed of different parties or independent candidates. However, to simplify exposition we use the term party.
3. Based on the total number of votes received by each party, the number of elected candidates \( E_i \) from each party \( i \) is determined. (See Cembrano et al. (2021) for details on how the \( E_i \) are calculated.)

4. Let \( M \) and \( W \) denote, respectively, the number of male and female candidates elected. Then, across parties, we must have gender parity: \(|M - W| \leq 1\).

Notice that once the \( E_i \) are determined, the set of feasible selections can be described with quotas. Specifically, the traits are Male, Female, and the set of Parties. For male and female, the lower quota is \( \lfloor N/2 \rfloor \) and the upper quota is \( \lceil N/2 \rceil \), where \( N \) is the number of seats allocated to the district. For each party, the upper and lower quotas are identically equal to \( E_i \).

Since traits are overlapping there is no hierarchy. Hence, Lemma 5 doesn’t apply and the designer may face a trade-off between explainability and efficiency. To see this, consider a simple case with 2 parties, each of which has won a single seat. Say that candidate 1 received the most votes, followed by candidate 2, and so on. Candidate 1 is a female from Party A, Candidate 2 is a female from Party B, Candidate 3 is a male from Party A, and Candidate 4 is a male from Party B. Then the feasible selections are \{1, 4\} and \{2, 3\}, exactly as in Example 2.

The algorithm used to select candidates in Chile works as follows.

**ALGORITHM 3: Chilean Constitutional Mechanism**

(A) Choose the \( E_i \) highest priority candidates from each party \( i \). Let \( M \) and \( W \) denote the total number of men and women (across all parties) candidates chosen in this step.

(B) While gender balance condition (constraint 4) is not satisfied, take the lowest vote-getter among the “majority” gender electees, and replace them with the highest vote-getter (who is not selected) in the same party of the opposite gender. (If no opposite gender candidate exists in the same party, skip past this party, and find the lowest vote-getter among the majority gender electees belonging to another party)

Note that every execution of step (B) reduces the disparity by 2; moreover, if step (A) chooses more female candidates than male candidates, then step (B) ensures that at least as many female candidates are elected as male candidates. Thus, the “majority” gender and “minority” gender are
determined based only on step (A).

The main result of this section is Theorem 7, which establishes that the rule implemented in Chile corresponds to the outcome based selection rule where feasible selections (i) allocate the correct number of seats to each party, and (ii) satisfy the gender parity constraint. Formally, let $F_{\text{chile}}$ be the collection of feasible sets for the Chilean election problem, that is, every set $S \in F_{\text{chile}}$ satisfies the following two properties: (i) the number of candidates in $S$ from party $i$ is exactly $E_i$; and (ii) the difference between the number of men in $S$ and the number of women in $S$ is at most 1. Additionally, let $\succ$ be the priority order candidates based on the votes received. Hence, the highest priority candidate in $\succ$ is the one who received the most votes and so on.

**Theorem 7.** *The selection made by Algorithm 3 and $OB_{F_{\text{chile}}} (\succ)$ coincide.*

This theorem provides another way of explaining the outcome of the election to disappointed candidates: anybody who was not elected could not have been elected without displacing a candidate who received more votes (or violating the diversity constraints). This fact is not obvious from the original description of the Chilean algorithm. We take the fact that Chile chose this rule – despite the potential downside that it may elect some candidates with very low vote totals – as (admittedly incomplete) evidence that our definition of explainability is salient in practice.
Chapter 4:
Information Provision and Consumer Search in Digital Marketplaces

4.1 Introduction

Online marketplaces like Amazon, Booking.com, eBay, Mercado Libre and Taobao have become increasingly prevalent. In 2023, it is anticipated that a significant 20.8% of all retail purchases will occur online. Meanwhile, global e-commerce market is expected to total $6.3 trillion (Baluch 2023).

The digital nature of these marketplaces allows them to offer a vast array of items across multiple categories. In this context, it becomes impractical for consumers to evaluate all options available. Instead, consumers rely on the information displayed on the platform’s website to shape their perceptions about the products available. Online platforms understand the importance of their website and they carefully design the organization, structure, and prioritization of information.

In this work, we study how online platforms should disclose information about their product to consumers who can acquire additional information. We consider markets where both the platform and the consumers have imperfect information, and model their interactions using a Bayesian-persuasion framework. In our model, the online platform can influence the consumer’s purchasing decision by disclosing information about each item. However, this influence is limited as consumers need to engage in a costly search to acquire further information. In this setting, we study the platform’s information provision policy that maximizes its expected revenue.

4.1.1 Model

We study the interactions between an online platform and a representative consumer using a Bayesian persuasion framework. In each interaction, the consumer is interested in buying at most
one item of a specific category. Moreover, from each item $i$ in this category, the consumer derives a utility $U_i = V_i + E_i$, where $V_i$ represents the predictable component and $E_i$ captures the private component. The information the consumer and the platform have about these components evolves throughout the interaction. Initially, the platform and the consumer have imperfect information about $V_i$ and $E_i$, respectively, this is reflected in our model by letting them be random variables. However, once the consumer arrives at the platform, the platform learns the predictable component $E_i$ based on the consumer’s preferences. This component can be estimated using a prediction algorithm that uses the attributes of item $i$ and relevant consumer information as covariates. Furthermore, the consumer gathers additional information in two different ways: (i) through signals about $V_i$ provided by the platform and (ii) through a costly search process that reveals $U_i$.

Within a given category, the platform offers $n$ items, with each item $i$ having a predetermined revenue $r_i$. The platform’s goal is to maximize its expected revenue by influencing the consumer’s purchase decision. Specifically, the platform provides the consumer with additional information about the predictable component of each item. There is a wide spectrum of disclosure policies. On one end, the no-disclosure policy provides no information about the predictable component of each item. On the other end, the full-disclosure policy provides the value estimated by the platform\(^1\). With this new information in hand, the consumer updates her beliefs about the utility of each item. Then, she proceeds to investigate some items and makes a purchase decision. The mechanism the platform uses to disclose information works as follows:

1. At the beginning of the interaction, the platform chooses, for each item $i$, a signal structure $(S_i, \pi_i)$ where $S_i$ denotes a set of signal realizations and $\pi_i$ denotes a family of distributions $\{\pi_i(\cdot|v_i)\}_{v_i \in \mathcal{V}}$ over $S_i$.

2. The consumer enters the platform, observes the signal structure $(S_i, \pi_i)$ and submits a search query.

\(^1\)An example of the full-disclosure policy is Netflix’s recommendations system, in which the platform shares their estimate of the likelihood that a user will watch a particular title in their catalog. (https://help.netflix.com/en/node/100639)
3. The platform observes the consumer’s identity and learns the predictable component $v_i$ of each item $i$.

4. A randomized signal $s_i$ is generated according to distribution $\pi_i(\cdot|v_i)$.

5. The consumer uses her knowledge of the distribution $\pi_i$ and the value of $s_i$ to update her beliefs about the predictable component.

6. The consumer performs a search for additional information based on her updated beliefs. Finally, the consumer makes a purchase decision.

The last piece of our model is the consumer’s search process, which is built upon Weitzman’s 1979 optimal sequential search. In this search process, the consumer has the option to incur a search cost $c_i$ to investigate item $i$ and learn the realized utility $u_i$. Alternatively, the consumer can stop the search at any point, buying the current best item, which could be an outside option. We assume that the consumer follows the search strategy that maximizes her expected utility, that is, the utility of the selected item minus the search costs. As shown by Weitzman 1979, the consumer’s optimal search strategy has a simple index-based structure.

4.1.2 Summary of Contributions

We develop a new model to study an information design problem under an optimal sequential search framework. Our model effectively captures the information asymmetry present in online marketplaces, where consumers have limited information about the multiple items available and their attributes. In this context, the platform’s decisions concerning the organization, structure, and prioritization of information play a pivotal role as it will shape the consumers’ perceptions of the items. Although previous works have incorporated a consumer’s search process in the context of online platforms (Chu, Nazerzadeh, and Zhang 2020; Derakhshan et al. 2022), most of these studies concentrate on the platform’s item ranking problem and ignore the information provision problem. By contrast, we give more flexibility to the platform which can decide the level of information to disclose for each item.
We characterize the platform’s optimal information provision policy, under Assumption 1 which states that the platform must provide a certain level of disclosure to incentivize the consumer to investigate. The optimal policy uses a binary signal for each item, indicating whether the item is a good match for the consumer or not. This solution is appealing due to its simple structure and ease of implementation.

Additionally, we provide a conjecture on the platform’s optimal policy when Assumption 1 is relaxed and there are only two items. This conjecture is based on the results of a numerical analysis where we solved multiple instances of the platform problem. To approach this problem numerically, we provide a novel formulation based on quadratic programming. Moreover, the structure of the optimal signals depends on the prior beliefs of the items and how they compare with the value of the outside option $u_0$. Based on the different possibilities, we distinguish between five scenarios. However, in all scenarios, the optimal signals are either binary or uninformative.

4.2 Related Literature

4.2.1 Assortment Optimization and Rank Optimization

In the operations research literature, there are several strands of literature studying how a designer/seller can influence the consumer’s purchasing decision. These lines of research differ on the levers the designer has to influence the consumer. Two of them are particularly relevant to our work: assortment optimization and rank optimization.

In assortment optimization, the seller makes a binary decision about each item: to offer it or not. In this family of problems, the seller maximizes her expected profit, while the consumer seeks to maximize her utility through purchasing. A key aspect of these problems is the consumer choice model, which determines the consumer purchasing decision given an assortment. For example, Talluri and Van Ryzin 2004 establishes the optimality of the revenue-ordered assortment when consumers choose according to standard multinomial logit model. Subsequent work extended the analysis to different consumer choice models (Davis, Gallego, and Topaloglu 2014; Li, Rusemichientong, and Topaloglu 2015), and to incorporate constraints on the set offered (Feldman
and Topaloglu 2015; Désir et al. 2020). More recently, Ma 2023 proposes a Bayesian mechanism design approach to study the benefits of randomized mechanisms over traditional deterministic assortments.

In ranking optimization, the designer must choose the order in which items will be displayed to the consumer. Here, top positions are the most valuable as consumers tend to search for their desired items by scanning from the top of the list and proceeding downward. Of course, the impact of the ranking in the consumer decision is ultimately determined by the consumer choice adopted. For example, a series of works consider an MNL choice model where the position affects the item’s intrinsic utility in a predetermined way (Abeliuk et al. 2016; Lei et al. 2022). Closer to our work is Chu, Nazerzadeh, and Zhang 2020, which applies a sequential search model inspired by Weitzman 1979. The authors consider a general objective that includes the consumer’s welfare, the seller’s welfare and the platform revenue. They propose the surplus-ordered ranking mechanism to sell the top positions and provide constant factor guarantees. Derakhshan et al. 2022 proposes a two-stage sequential search model, where the uncertainty about the items is progressively resolved. The authors characterize the consumer’s optimal search and use it to study two objectives: maximize the platform’s market share and maximize the consumer’s welfare.

Our work differs from the literature on assortment optimization and ranking optimization as we considered a more sophisticated mechanism for the platform to influence the consumer’s purchase decision. In particular, the platform shapes the consumer’s perception of each item by choosing the degree of information to disclose regarding the characteristics of the item. However, this influence is limited as consumers can acquire additional information through a costly search process.

4.2.2 Consumer Search Theory

In digital marketplaces, each product category typically contains hundreds or thousands of items, this makes consumers often uncertain about the utility associated with each product. To resolve this uncertainty, consumers can engage in a costly search process to acquire information that will help them refine their beliefs before making a purchase decision. The consumer search
literature studies the trade-off faced by the consumer between paying the search cost and acquiring further information, or avoiding this cost but making a less informed purchase decision. We distinguish between two families of consumer search problems: sequential search problems and simultaneous search problems.

In sequential problems, consumers investigate each product at a time, resolving their utility uncertainty regarding that particular product. The seminal work by Weitzman 1979, proposed an elegant index-based rule for a broad class of optimal sequential search problems. This rule can be characterized by a set of indices called reservation prices, which have an intuitive economic interpretation: it is the hypothetical value that makes the consumer indifferent between investigating or not an item. Morgan and Manning 1985 study fixed-sample-size rules and sequential rules, providing sufficient conditions for the optimality of these rules. Multiple works have extended the literature on sequential search problems, to consider gradual learning. Branco, Sun, and Villas-Boas 2012 presents a tractable model where consumers may acquire additional product information at a cost. They characterize the optimal stopping rules for purchase decisions. Ke, Shen, and Villas-Boas 2016 introduce a model where the set of products is known, and consumers pay to gradually learn about products’ attributes utilities.

In simultaneous search problems, the consumer first screens products and chooses a subset of them that will be further investigated. This subset of products is often referred to as the consideration set. Then the consumer simultaneously investigates all products in the consideration set and purchases the best option. This paradigm is known as the consider-then-choose. Simultaneous search problems have been studied in a wide range of applications. Anderson, De Palma, and Thisse 1992 and Liu and Dukes 2013 study the formation of consideration sets in multi-product firms setting. Jagabathula and Rusmevichientong 2017 study consideration set that only includes products within a price range of the consumer’s willingness to pay.

In our work, consumers resolve the uncertainty regarding the items’ utility by engaging in a costly search process. We model this search process using the optimal sequential search framework proposed by Weitzman 1979. The uniqueness of our model lies in integrating consumers’
search behavior into the information provision problem faced by an online platform. The platform problem is particularly interesting as it has to deal with the substitution effects between items.

4.2.3 Information Design

The last strand of literature relevant to our work pertains to information design or Bayesian persuasion, a field initiated by Kamenica and Gentzkow 2011. This line of research explores how a sender, in our context, the platform, strategically designs and disseminates information to influence the beliefs or decisions of a receiver, in our case, the consumer.

One work closely related to our study is Anderson and Renault 2006, which combines a seller’s information provision problem and a consumer search process. The author shows the optimality of a binary signal structure which indicates to the consumer whether her match value for the item is above a threshold. The main difference with our work is that our model considers multiple items so the platform must take into account the substitution effects between items.

The work by Anderson and Renault 2006 was extended in Lyu 2023 by allowing ex-ante heterogeneous consumers with privately known outside option values. The author shows that the optimal signal structure depends on the outside option value distribution, however, under certain regularity conditions, it can be implemented in an upper-censorship signal. Specifically, consumers with match values below a threshold are fully informed of their matches, otherwise, consumers are only informed that their matches are above the threshold.

Another work studying an information design problem in a sequential consumer search environment is Dogan and Hu 2022. The sequential consumer search model considered in this work is by Wolinsky 1986, in this model, consumers are uncertain about the price and the utility of products. Moreover, consumers can sequentially investigate the items by incurring a search cost, this reveals the product’s price and provides a noisy signal of the product’s utility. In this setting, the authors show the optimality of a conditional unit-elastic demand signal distribution. This structure has a simple structure incorporating an atom that reveals low utilities and a continuum of signals that reveals high utilities. The conditional unit-elastic demand distribution also plays an important
role in Choi, Kim, and Pease 2019. In this work, the authors characterize the consumer-optimal policy in a search good environment.

4.3 Model and Preliminaries

Consider an online platform that offers \( n \) items. We associate with each item \( i \) a predetermined revenue \( r_i \). Without loss of generality, we index items so that

\[
r_1 \geq r_2 \geq \cdots \geq r_n \geq 0.
\]

In this market, each consumer maximizes her utility by purchasing at most one item. Moreover, the consumer’s utility derived from item \( i \) is given by

\[
U_i = V_i + E_i,
\]

(4.1)

where \( V_i \) represents the predictable component and \( E_i \) captures the private component. The predictable component depends on some observable attributes of the item and the consumer’s preferences. By contrast, the private component captures all the unobserved determinants of the utility derived from the item.

We assume that, before the consumer’s arrival to the platform, \( V_i \) and \( E_i \) are unknown to the consumer and the platform. This assumption is motivated by two reasons. First, online platforms typically offer a large number of items, resulting in consumers having limited initial knowledge about most of these items and their attributes. Secondly, the large number of customers of online platforms introduces a level of uncertainty for the platform, as predicting the identity of the next consumer and their preferences becomes a challenging task.

We model the uncertainty on the utility derived from item \( i \), by assuming that \( V_i \) and \( E_i \) are random variables. We follow the convention of denoting random variables by the uppercase letters \( U_i, V_i, \) and \( E_i \), and their realizations by the lowercase letters \( u_i, v_i, \) and \( \varepsilon_i \). Moreover, we assume that \( V_i \) is drawn from a continuous distribution \( \mu_i^0 \) with support \( \mathcal{V} \), and \( E_i \) is drawn from probability
distribution $\kappa_i$. We assume that both components, $V_i$ and $E_i$, are independent. Furthermore, we assume that $\mu_i^0$ and $\kappa_i$ are known to both the consumer and the platform.

4.3.1 Platform’s Information Provision

In our model, initially both the platform and the consumer have the same level of information. However, once the consumer enters a search query, the platform observes her identity and becomes capable of estimating her preferences. Thus, the platform effectively learns, for every item $i$, the predictable component of the utility $v_i$. Furthermore, the platform can influence the consumer’s purchasing decision by disclosing partial information about $v_i$. Therefore, the goal of the platform is to design an information provision policy that maximizes its expected revenue. We remark that the consumer can only learn her total utility $u_i$ after a costly search, which we explain in Section 4.3.2.

We adopt a Bayesian persuasion framework to model the interaction between the platform and the consumer. For every item $i$, the platform chooses a signal structure $(S_i, \pi_i)$, where $S_i$ denotes a set of signal realizations, and $\pi_i$ denotes a family of distributions $\{\pi_i(\cdot|v)\}_{v \in \mathcal{V}}$ over $S_i$. We assume that the platform chooses the signal $(S_i, \pi_i)$ before the arrival of the consumer. Furthermore, the consumer is aware of the chosen signal, allowing her to apply a Bayesian update from the prior $\mu_i^0$ to the posterior

$$
\mu^\pi_i(v_i|s_i) = \frac{\pi_i(s_i|v_i)\mu_i^0(v_i)}{\int_{v_i'} \pi_i(s_i|v_i')d\mu_i^0(v_i')} \quad \text{for every } v_i \in \mathcal{V}.
$$

(4.2)

A key assumption in our model is that each item has its own independent signal structure. This assumption ensures tractability as it simplifies (i) the consumer inference process and (ii) the consumer optimal search. Note that, as shown in (4.2), in order to update her belief about item $i$ the consumer uses only the signal of item $i$. Furthermore, when signals are correlated, the optimal search requires making decisions adaptively based on the realizations observed previously (Chawla et al. 2020), which can result in a computationally intractable process. Thus, an information provision policy $\pi$ corresponds to a collection of independent signal realizations $\{(S_i, \pi_i)\}_{i=1}^n$.
4.3.2 Consumer’s Optimal Search

Even after incorporating the information revealed by the platform, the consumer remains uncertain about the utility derived from each item. Note that at this point, the private component remains unknown to both the platform and the consumer. However, our model allows the consumer to investigate item $i$ by incurring a search cost $c_i$, which allows her to learn the realization of $u_i$. In practice, this investigation involves the consumer navigating to a dedicated subpage for the item. On this subpage, the consumer can access various features such as customer reviews, additional details about the item’s attributes and more. The impact of these search costs on the consumer’s behavior in digital marketplaces has been studied by multiple empirical works (Kim, Albuquerque, and Bronnenberg 2010; Ursu 2018). In the remainder of this section, we present the consumer search process and characterize the optimal search strategy.

The consumer search process in our model is based on the optimal sequential search introduced by Weitzman 1979. In this model, consumers are rational decision-makers seeking to maximize their expected utility, that is, the utility of the selected item minus the search costs. For each item $i$, the consumer has a posterior belief $\mu_i$ that is determined using the Bayesian update (4.2). Moreover, the consumer can stop the search at any point, buying the current best item, which could be an outside option with utility $u_0$. Formally, let $I$ denote the set of items investigated by the consumer. The consumer will buy item

$$i^* \in \arg\max_{i \in I \cup \{0\}} u_i.$$  

We assume that ties are broken in favor of the platform, that is, whenever the consumer is indifferent, she prioritizes the item with the highest revenue. In this scenario, the consumer utility is

$$u_{i^*} - \sum_{i \in I} c_i,$$

and the platform’s revenue is $r_{i^*}$. We assume that both the search costs $c_1, \ldots, c_n$ and the utility of
the outside option $u_0$ are known by the platform and the consumer.

As show by Weitzman 1979, the consumer’s optimal search strategy has a simple index-based structure. Specifically, the author shows that all the relevant information about an item can be compressed into a single sufficient statistic, which he refer to as the reservation price.

**Definition 13 (Reservation price).** For every item $i$ and posterior $\mu \in \Delta(V)$, we define $z_i(\mu)$ to be a solution of the following equation:

$$c_i = \mathbb{E}_{V_i \sim \mu, \mathcal{E}_{i} \sim \kappa_i} [\max \{V_i + \mathcal{E}_i - z, 0\}].$$

(4.3)

The reservation price has an intuitive interpretation: it is the hypothetical reward that makes the consumer indifferent between investigating the item and not investigating it.

**Lemma 6 (Optimal Consumer Search - Weitzman 1979).** The consumer stops the search whenever the maximum utility seen so far exceeds the reservation price of every uninvestigated item. If we let $I$ denote the set of items that have already been investigated, then the stopping criteria correspond to

$$\max_{i \in I \cup \{0\}} \{u_i\} > \max_{i \in I^c} \{z_i(\mu_{\pi_i})\}. \quad (4.4)$$

Otherwise, the consumer continues the search and investigates the item $i^*$ with the highest reservation price among the uninvestigated items, that is,

$$i^* = \arg \max_{i \in I^c} \{z_i(\mu_{\pi_i})\}. \quad (4.5)$$

4.3.3 From Signals to Posteriors

The goal of the platform is to design an information provision policy $\pi^*$ that maximizes its expected revenue. This problem is challenging as we impose no restrictions on the platform’s signals space $(S_i, \pi_i)$, and the consumer’s optimal search depends on the posterior beliefs of every item.
In the remainder of this section, we show to reformulate this problem in terms of distributions over posteriors that are *Bayes-plausible*. To this end, we adapt Proposition 1 from Kamenica and Gentzkow 2011 to our setting. For the sake of notation simplicity, in what follows we omit the dependence on item $i$. Let $\tau$ be a distribution over posteriors $\mu \in \Delta(\mathcal{V})$, that is, $\tau \in \Delta(\Delta(\mathcal{V}))$. We say that a signal structure $(S, \pi)$ induces $\tau$ if

$$Supp(\tau) = \{\mu_s\}_{s \in S},$$

$$\mu_s(\cdot) = \mu^\pi(\cdot|s) \text{ for every } s \in S,$$

$$\tau(\mu_s) = \int_{\mathcal{V}} \pi(s|v)\mu^0(dv) \text{ for every } s \in S.$$

A belief $\mu \in \Delta(\mathcal{V})$ is induced by the signal structure $\pi$, if $\tau$ is induced by $\pi$ and $\tau(\mu) > 0$. Moreover, a distribution of posteriors is Bayes-plausible if the expected posterior probability of each state equals its prior probability:

$$\int_{\Delta(\mathcal{V})} \mu \tau(d\mu) = \mu^0. \quad (4.6)$$

As the following lemma shows, the Bayes-plausible condition is necessary and sufficient for $\tau$ to ensure the existence of a signal structure $(S, \pi)$ that induces it.

**Lemma 7** (Kamenica and Gentzkow 2011). *Let $\tau$ be a distribution of posteriors. There exists a signal structure $\pi$ that induces $\tau$ if and only if $\tau$ is Bayes-plausible.*

### 4.3.4 Timing of events

Below we enumerate the different stages of an interaction between the consumer and the platform.

1. The platform chooses an information provision policy $\pi$.
2. The consumer arrives to the platform, observes $\pi$ and enters a search query.
3. For each product $i$, nature chooses: (i) $v_i$ according to $\mu_i^0$, (ii) $\epsilon_i$ according to $\kappa_i$ and (iii) a signal $s_i \in S_i$ according to $\pi_i(v_i)$.

4. The consumer observes the signal realizations $s_1, \ldots, s_n$. Given her knowledge of the signal structure $\pi$, she updates her belief from the prior $\mu_i^0$ to the posterior $\mu_i^{\pi_i}(\cdot | s_i)$.

5. The consumer follows the optimal search strategy with respect to $\mu_i^{\pi_i}(\cdot | s_i)$, $\kappa_i$ and the search costs $c_1, \ldots, c_n$. Denoting the set of items investigated by $I$, the consumer chooses item $i^* = \arg\max_{i \in I \cup \{0\}} \{u_i\}$, resulting in a utility of $u_{i^*} - \sum_{i \in I} c_i$, while the platform receives revenue $r_{i^*}$.

4.4 Single-Item Case

In order to gain intuition, we begin our analysis by examining the single-item case. This special case presents two main simplifications: firstly, the consumer’s action space is binary, as she can either choose to investigate or not to investigate the item; secondly, there are no substitution effects between items. Despite these simplifications, solving this special case is valuable as the structure of its solution is similar to that of the general case.

4.4.1 Optimal Signal Structure

We now introduce a family of binary-signal schemes characterized by a threshold $t \in \mathbb{R}$. Intuitively, the platform can use these binary-signal schemes to reveal whether the item is a good match for the consumer or not. The main result of this section (Proposition 12) establishes the optimality of these signals and determines the optimal threshold $t^*$.

**Definition 14.** For every $t \in \mathbb{R}$, the $t$–threshold signal structure reveals whether the realized predictable component $v$ weakly exceeds threshold $t$ or not. Formally, the $t$–threshold signal structure consists of a set of signals $S = \{\text{Low}, \text{High}\}$, and a map $\pi_t : \mathcal{V} \to \Delta(S)$ defined
by

\[
\pi_t(s|v) = \begin{cases} 
1 & \text{if } s = \text{High and } v \geq t, \\
1 & \text{if } s = \text{Low and } v < t, \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that the probability of observing each signal is

\[
\pi_t(\text{High}) = \mu^0(\{v \in \mathcal{V} : v \geq t\}),
\]

\[
\pi_t(\text{Low}) = \mu^0(\{v \in \mathcal{V} : v < t\}).
\]

Moreover, the posteriors induced by the signals correspond to

\[
\mu^H_t(v) = \frac{1\{v \geq t\}\mu^0(v)}{\pi_t(\text{High})},
\]

\[
\mu^L_t(v) = \frac{1\{v < t\}\mu^0(v)}{\pi_t(\text{Low})}.
\]

**Proposition 12.** Let

\[
t^* = \inf \{t \in \mathcal{V} : z(\mu^H_t) \geq u_0\}.  \quad (4.7)
\]

Then, the \(t^*\)-threshold signal structure maximizes the platform’s revenue.

Proposition 12 establishes the optimality of the \(t^*\)-threshold signal scheme. This solution is appealing due to its simple structure and ease of implementation. Note that to implement this solution the platform requires to determine the optimal threshold \(t^*\), which can be done using a binary search, and then compare it to the estimation of the predictable component \(v\).

It is instructive to compare the \(t^*\)-threshold solution with the extreme policies of no-disclosure and full-disclosure. The no-disclosure policy refers to any uninformative signal. Here, we must distinguish between two scenarios:

1. If the prior doesn’t induce search \((z(\mu^0) < u_0)\), then the consumer always takes the outside option and the platform doesn’t generate any revenue.
2. By contrast, if the prior $\mu^0$ induces search ($z(\mu^0) \geq u_0$), then the consumer always investigates the item and purchases it whenever its utility is at least the outside option $u_0$. In this case, the no-disclosure policy coincides with the $t^*$-threshold solution as the optimal threshold is $t^* = -\infty$, hence, it is optimal.

On the other hand, the full-disclosure policy reveals the exact value of $v$. In general, this policy is sub-optimal because it induces the consumer to search for fewer values of $v$ compared to the $t^*$-threshold signal scheme. In particular, the consumer searches only if $z_i(\delta_v) \geq u_0$, where $\delta_v$ is a distribution equals to $v$ with probability one. Because of the randomness of the private component $\mathcal{E}_i$, the consumer might end up not buying the item even if $u_i > u_0$, contingent on the condition that no search is performed, i.e., $z_i(\delta_v) < u_0$. In contrast, the optimal signal structure pools the highest realization of the predictable component together to the largest extent possible while guaranteeing that the consumer searches. This maximizes the chance that the consumer searches and, consequentially, buys the item.

Our contributions in this section are (i) to model the interaction between the platform and the consumer as a persuasion problem with a binary action space, and (ii) to characterize the optimal policy. It is well established that for binary action space problems, the $t$-threshold signal scheme is optimal (see, e.g., Anderson and Renault 2006; Kolotilin 2015; Renault, Solan, and Vieille 2017). However, as far as we know, our work extends previous results. For example, in Kolotilin 2015 the platform’s revenue does not depend on the private component whereas in our case it does.

4.4.2 Proof of Proposition 12

We start by formulating the problem faced by the platform as an optimization problem. Instead of optimizing over the space of all signal structures, we apply Lemma 7 to optimize over the space of distributions over posteriors that are Bayes-plausible. Hence, we need to derive an expression for the revenue induced by a distribution over posteriors $\tau \in \Delta(\Delta(\mathcal{V}))$. Before we proceed, some notation is needed.

For every $v \in \mathcal{V}$, we let $R(v)$ be the platform’s expected revenue conditional on (i) the con-
sumer searching the item and (ii) the realized predictable component being \( v \). Formally,

\[
R(v) = r P_{E \sim k}(v + E \geq u_0).
\] (4.8)

Moreover, we let \( U^1 (U^0) \) be the set of posteriors that (don’t) induce search:

\[
U^1 = \{ \mu \in \Delta(V) : z(\mu) \geq u_0 \}, \tag{4.9}
\]

\[
U^0 = \{ \mu \in \Delta(V) : z(\mu) < u_0 \}. \tag{4.10}
\]

Observe that in the definition of \( U^1 \) in (4.10), we include the equality \( z(\mu) = u_0 \) as we are assuming that ties are broken in favor of the platform.

Therefore, the platform’s expected revenue induced by \( \tau \) corresponds to

\[
\int_{U^1} \int_{\mathcal{V}} R(v) \mu(dv) \tau(d\mu).
\]

All in all, we formulate the problem faced by the platform as

\[
\text{OPT} = \max \int_{U^1} \int_{\mathcal{V}} R(v) \mu(dv) \tau(d\mu)
\]

subject to \( \int_{\Delta(V)} \mu(v) \tau(d\mu) = \mu^0(v) \quad \forall v \in \mathcal{V} \)

\[
\tau \in \Delta(\Delta(V)) \tag{4.11}
\]

**An action-based reformulation**

Inspired by Proposition 1 in Kamenica and Gentzkow 2011, we propose a reformulation of \( \text{OPT} \) based on “straightforward mechanisms.” In this class of mechanisms: (i) the signal produces a “recommended action” for the consumer, (ii) the platform reports the recommendation honestly, and (iii) the consumer takes the action recommended. In our setting, the action space corresponds to \( \mathcal{A} = \{0, 1\} \), where \( a = 1 \) (\( a = 0 \)) represents the consumer (not) investigating the item. After observing recommendation \( a \in \mathcal{A} \), the consumer updates her belief to the posterior \( \mu^a \in \Delta(\mathcal{V}) \).
Besides, the platform must choose a distribution over actions \( \lambda \in \Delta(A) \), where \( \lambda^a \) represents the probability that action \( a \) is recommended. Let \( \ell(v) \) denote the expected difference between the utility derived from the item selected and the value of the outside option, conditional on (i) investigating the item, and (ii) the predictable component of the item being \( v \). Formally,

\[
\ell(v) = \mathbb{E}_{E \sim \kappa} \left[ \max \{ v + E - u_0, 0 \} \right].
\]

(4.12)

Hence, the action-based reformulation corresponds to:

\[
\text{ABR} = \max \lambda^1 \int_V R(v) \mu^1(dv)
\]

subject to

\[
\sum_{a \in A} \lambda^a \mu^a(v) = \mu^0(v) \quad \forall v \in V
\]

\[
\mathbb{E}_{v \sim \mu^1}[\ell(v)] \geq c
\]

\[
\mathbb{E}_{v \sim \mu^0}[\ell(v)] < c
\]

(4.13)

In the formulation above, the first set of constraints \( \sum_{a \in A} \lambda^a \mu^a(v) = \mu^0(v) \) require that the distribution over actions \( \lambda \) satisfies the Bayes-plausible condition. Additionally, constraints \( \mathbb{E}_{v \sim \mu^1}[\ell(v)] \geq c \) and \( \mathbb{E}_{v \sim \mu^0}[\ell(v)] < c \) ensure that the recommended action is optimal for the consumer, we refer to them as the obedience constraints. To see this, apply function

\[
f(z; \mu) = \mathbb{E}_{q \sim \mu, E \sim \kappa} \left[ \max \{ q + E - z, 0 \} \right]
\]

to conditions \( z(\mu^1) \geq u_0 \) and \( z(\mu^0) < u_0 \), respectively.

The \( \text{ABR} \) reformulation is non-linear since it involves the product of variables in both the objective function and the constraints. However, in Section 4.6, we show that \( \text{ABR} \) can be reformulated as a linear problem by introducing decision variables \( \alpha^a = \lambda^a \mu^a \).
Proposition 13. \textit{The action-base reformulation is optimal, that is,}

\[ OPT = ABR. \] (4.14)

The proof of Proposition 13 is located in Appendix C.1.

Now we show the optimality of the \( t^* \)-threshold signal. First, we define a family of solutions for \( ABR \) based on the family of \( t \)-threshold signal structures. Specifically, we associate the actions search (\( a = 1 \)) and no search (\( a = \emptyset \)) with the signals \( \text{High} \) and \( \text{Low} \), respectively. Then, it suffices to show that the \( t^* \)-threshold solution is optimal for \( ABR \).

Definition 15. \textit{For every} \( t \in \mathbb{R} \), \textit{the} \( t \)-\textit{threshold solution is defined as}

\[ \lambda_t = (\lambda^0_t, \lambda^1_t) = (\pi_t(\text{Low}), \pi_t(\text{High})), \] (4.15)

\[ \mu_t = (\mu^0_t, \mu^1_t) = (\mu^\text{Low}_t, \mu^\text{High}_t). \] (4.16)

Observation 1. \textit{A feasible solution} \( (\lambda, \mu) \) \textit{of} \( ABR \) \textit{is a} \( t \)-\textit{threshold solution only if}

\[ \sup\{v \in \mathcal{V} : \mu^0(v) > 0\} < t \leq \inf\{v \in \mathcal{V} : \mu^1(v) > 0\}. \] (4.17)

Proposition 14. \textit{The} \( t^* \)-\textit{threshold solution is optimal for} \( ABR \).

\textit{Proof.} Let \( (\lambda, \mu) \) \textit{be any feasible solution of} \( ABR \) \textit{that is not a} \( t \)-\textit{threshold. From Observation 1, we have}

\[ \underline{v} = \inf\{v \in \mathcal{V} : \mu^1(v) > 0\} < \sup\{v \in \mathcal{V} : \mu^0(v) > 0\} = \bar{v}. \]

For every \( x \in [\underline{v}, \bar{v}] \), we define

\[ f(x) = \mu^0(\{v \in \mathcal{V} : v \in [x, \bar{v}]\}), \] (4.18)

\[ g(x) = \mu^1(\{v \in \mathcal{V} : v \in [\underline{v}, x]\}). \] (4.19)
Notice that $f$ and $g$ are decreasing and increasing functions, respectively. By assumption the prior $\mu^0$ is a continuous distribution, hence, $f$ and $g$ are continuous functions because the measures $\mu^0$ and $\mu^1$ are absolutely continues with respect to $\mu^0$. Therefore,

$$g(\nu) = 0 < \mu^0(\{\nu \in \mathcal{V} : \nu \in [\underline{\nu}, \overline{\nu}]\}) = f(\nu),$$

$$g(\overline{\nu}) = \mu^1(\{\nu \in \mathcal{V} : \nu \in [\underline{\nu}, \overline{\nu}]\}) > 0 = f(\overline{\nu}).$$

Thus, by the intermediate value theorem, there must exist $\hat{\nu} \in [\nu, \overline{\nu}]$ such that

$$f(\hat{\nu}) = g(\hat{\nu}).$$

Consider the $\hat{\nu}$-threshold solution $(\lambda_{\hat{\nu}}, \mu_{\hat{\nu}})$. We claim that $(\lambda_{\hat{\nu}}, \mu_{\hat{\nu}})$ is a feasible solution of $ABR$, and yields a weakly higher objective value than $(\lambda, \mu)$.

We start by showing that $(\lambda_{\hat{\nu}}, \mu_{\hat{\nu}})$ is feasible for $ABR$. From the definition of the $t$-threshold solution, it follows that $(\lambda_{\hat{\nu}}, \mu_{\hat{\nu}})$ satisfies all constraints other than obedience. Hence, it suffices to show that $\mathbb{E}_{\nu \sim \mu_{\hat{\nu}}} \left[ \ell(\nu) \right] \geq c$ and $\mathbb{E}_{\nu \sim \mu^{0}_{\hat{\nu}}} \left[ \ell(\nu) \right] < c$. Because $(\lambda, \mu)$ is feasible for $ABR$, we have that $\mathbb{E}_{\nu \sim \mu^{1}_{\hat{\nu}}} \left[ \ell(\nu) \right] \geq c$. Moreover, from the definition of $\mu^{1}_{\hat{\nu}}$, we obtain

$$\mathbb{E}_{\nu \sim \mu^{1}_{\hat{\nu}}} \left[ \ell(\nu) \right] - \mathbb{E}_{\nu \sim \mu^{1}_{\hat{\nu}}} \left[ \ell(\nu) \right] = \int_{\hat{\nu}}^{\overline{\nu}} \ell(\nu) d\mu^{1}_{\hat{\nu}}(\nu) - \int_{\underline{\nu}}^{\hat{\nu}} \ell(\nu) d\mu^{1}_{\hat{\nu}}(\nu)$$

$$\geq \ell(\hat{\nu}) f(\hat{\nu}) - \ell(\hat{\nu}) g(\hat{\nu}) = 0.$$

The equality follows as posteriors $\mu^{1}_{\hat{\nu}}$ and $\mu^{1}$ only differ in the interval $[\underline{\nu}, \overline{\nu}]$. Specifically, in $[\hat{\nu}, \overline{\nu}]$ distribution $\mu^{1}_{\hat{\nu}}$ combines $\mu^{1}$ and $\mu^{0}$, while in $[\underline{\nu}, \hat{\nu}]$ posterior $\mu^{1}_{\hat{\nu}}$ put no mass. The inequality follows as $\ell(\nu)$ is increasing in $\nu$. 

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Meanwhile, the condition $\mathbb{E}_{v \sim \mu^\emptyset}[\ell(v)] < c$ follows as $\mathbb{E}_{v \sim \mu^\emptyset}[\ell(v)] < c$ and

$$\mathbb{E}_{v \sim \mu^\emptyset}[\ell(v)] - \mathbb{E}_{v \sim \mu^\emptyset}[\ell(v)] = \int^\gamma _\Gamma \ell(v) d\mu^\emptyset(v) - \int^\hat{i} _\Gamma \ell(v) d\mu^1(v) \geq \ell(\hat{i}) f(\hat{i}) - \ell(\hat{i}) g(\hat{i}) = 0.$$

We now show that $(\lambda_\hat{i}, \mu_\hat{i})$ yields a weakly higher objective value than $(\lambda, \mu)$. First, we derive an expression for the objective obtained by the $\hat{i}$-threshold solution. Because only the search action $a = 1$ derives revenue, we obtain:

$$\lambda^1_\hat{i} \int_{\Gamma} R(v) \mu^1_\hat{i}(dv) = \int^\hat{i} _\Gamma R(v) \lambda^1_\hat{i} \mu^1_\hat{i}(dv) = \int^{\hat{i}} _\Gamma R(v) \mu^0_\hat{i}(dv). \tag{4.20}$$

In the second equality, we use (i) the Bayes-plausible condition:

$$\mu^0 = \lambda^0_\emptyset \mu^\emptyset_\hat{i} + \lambda^1_\hat{i} \mu^1_\hat{i},$$

and (ii) the fact that under a $t$-threshold solution, for any $v \geq t, \mu^\emptyset_\hat{i}(v) = 0$. Secondly, we write the objective yielded by $(\lambda, \mu)$:

$$\lambda^1 \int_{\Gamma} R(v) \mu^1_\hat{i}(dv) = \int^\hat{i} _\Gamma R(v) \lambda^1 \mu^1_\hat{i}(dv) + \int^\infty _\Gamma R(v) \lambda^1 \mu^1_\hat{i}(dv)$$

$$= \int^\hat{i} _\Gamma R(v) \lambda^1 \mu^1_\hat{i}(dv) + \int^\infty _\Gamma R(v) \lambda^1 \mu^1_\hat{i}(dv)$$

$$+ \int^\infty _\Gamma R(v) \lambda^0 \mu^0_\hat{i}(dv) - \int^\hat{i} _\Gamma R(v) \lambda^0 \mu^0_\hat{i}(dv) \tag{4.21}$$

We claim that

$$\int^\infty _\Gamma R(v) \lambda^1 \mu^1_\hat{i}(dv) + \int^\hat{i} _\Gamma R(v) \lambda^0 \mu^0_\hat{i}(dv) = \int^\infty _\Gamma R(v) \mu^0_\hat{i}(dv) \tag{4.22}$$

Combining equations (4.20), (4.21) and (4.22), we obtain that $(\lambda_\hat{i}, \mu_\hat{i})$ yields a weakly higher
objective value than \((\lambda, \mu)\):

\[
\lambda^1 \int_{\mathcal{V}} R(v)\mu^1_i (dv) - \lambda^1 \int_{\mathcal{V}} R(v)\mu^1 (dv) = \int_{\mathcal{V}}^{\mathcal{V}'} R(v)\lambda^0 \mu^0 (dv) - \int_{\mathcal{V}}^i R(v)\lambda^1 \mu^1 (dv) \\
\geq R(\hat{t}) \left( f(\hat{t}) - g(\hat{t}) \right) = 0.
\]

The inequality follows as \(R\) is increasing, and by definition of \(\hat{t}\), \(f(\hat{t}) = g(\hat{t})\).

Our claim (4.22) follows as

\[
\int_{\hat{t}}^{\infty} R(v)\lambda^1 \mu^1 (dv) + \int_{\hat{t}}^{\mathcal{V}'} R(v)\lambda^0 \mu^0 (dv) = \int_{\mathcal{V}}^{\infty} R(v)\lambda^1 \mu^1 (dv) + \int_{\hat{t}}^{\mathcal{V}'} R(v)\lambda^1 \mu^1 (dv) + \int_{\hat{t}}^{\mathcal{V}'} R(v)\lambda^0 \mu^0 (dv) \\
= \int_{\mathcal{V}}^{\infty} R(v)\mu^0 (dv) + \int_{\hat{t}}^{\mathcal{V}'} R(v)\mu^0 (dv)
\]

In the second equality, we use (i) the Bayes-plausible condition and (ii) the fact that under the \((\lambda, \mu)\), for any \(v \geq \mathcal{V}'\), \(\mu^0(v) = 0\).

Finally, we show the optimality of the threshold \(t^*\). From equation (4.20), we see that the revenue induced by \((\lambda_i, \mu_i)\) is decreasing as a function of \(\hat{t}\). Furthermore, to maintain feasibility, \(\mu^1_i\) must satisfy \(E_{v \sim \mu^1_i}[\ell(v)] \geq u_0\). \(\square\)

4.5 Multiple-Items Case

We now turn to the multiple-items case. Throughout our analysis, we adhere to Assumption 1, which states that the consumer will be willing to investigate an item, only if the platform discloses information about it. This assumption is motivated by the enormous number of items offered by online platforms, the majority of which are initially unknown to the consumer and are therefore unlikely to be investigated.

**Assumption 1 (No Search Without Disclosure).** For every item \(i\), the consumer’s prior belief \(\mu^0_i\) doesn’t induce search, that is,

\[
z_i(\mu^0_i) < u_0. \tag{4.23}
\]
Theorem 8. Under Assumption 1, it is optimal for the platform to follow the $t_i^*-$threshold signal for every item $i$.

The economic intuition behind Theorem 8 is as follows. First, the $t_i^*-$threshold policy minimizes substitution effects across competing items as the consumer searches them in decreasing order of revenue, and an item is investigated only if the previous ones had a realized utility lower than the outside option. Secondly, this policy maximizes the probability of the consumer making a purchase. Note that the consumer makes no purchase only if each item is either not investigated or has a realized utility lower than the outside option. Because signals are independent across items, it can be shown that these events are also independent across items. Finally, by Proposition 12, it follows that the $t_i^*-$threshold signal minimizes the probability of these events.

Notice that, under the $t_i^*-$threshold policy, all the items that are investigated have a reservation value equal to the outside option $u_0$. Hence, the order in which the consumer investigates them follows from the assumption that ties are broken in favor of the platform. That being said, this assumption can be relaxed by setting a threshold $t'_i$ that induces a reservation price equal to $u_0 + \varepsilon_i$, where $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n$ and $\varepsilon_i$ is arbitrarily small.

4.5.1 Proof of Theorem 8

Fix an arbitrary item $i$. We will show that if the platform is following the $t_j^*-$threshold signal for lower-revenue items $j > i$, then it is optimal to follow the $t_i^*-$threshold signal for item $i$.

For every item $j$, let $\tau_j^*$ be the distribution over posteriors induced by the $t_j^*-$threshold signal structure:

$$
\tau_j^*(\mu) = \begin{cases} 
\pi_t^j(High) & \text{if } \mu = \mu_{t_j}^{High}, \\
\pi_t^j(Low) & \text{if } \mu = \mu_{t_j}^{Low}, \\
0 & \text{otherwise.}
\end{cases}
$$

Additionally, let $\tau_j$ be any distribution over posteriors that is Bayesian-plausible. We define the
policies $\tau$ and $\tau^*$ as follows:

$$
\tau = (\tau_1, \ldots, \tau_{i-1}, \tau_i, \tau^*_{i+1}, \ldots, \tau^*_n),
$$

$$
\tau^* = (\tau_1, \ldots, \tau_{i-1}, \tau^*_i, \tau^*_{i+1}, \ldots, \tau^*_n).
$$

Note that policies $\tau$ and $\tau^*$ coincide for all items but $i$. Moreover, let $R$ and $R^*$ be the platform’s revenue under $\tau$ and $\tau^*$, respectively. Our goal is to prove that

$$
\mathbb{E}[R^*] \geq \mathbb{E}[R]. \tag{4.24}
$$

Let $\mathbf{z}_{-i} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ and $\mathbf{u}_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$ denote the reservation values and utilities of all items other than $i$, respectively. We claim that for every possible realization of $\mathbf{z}_{-i}$ and $\mathbf{u}_{-i}$,

$$
\mathbb{E}[R^*|\mathbf{z}_{-i}, \mathbf{u}_{-i}] \geq \mathbb{E}[R|\mathbf{z}_{-i}, \mathbf{u}_{-i}]. \tag{4.25}
$$

This implies (4.24) by taking expectation over the reservation values and utilities, because the distribution of $\mathbf{z}_{-i}$ and $\mathbf{u}_{-i}$ is the same under both policies.

To prove our claim additional definitions are needed. For every item $j$, let $B_j = \min\{z_j, u_j\}$. Furthermore, let $\sigma$ be a function that associates each pair $(\mathbf{z}_{-i}, \mathbf{u}_{-i})$ to a consumer’s purchase decision. Formally, we define $\sigma : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \rightarrow \{1, \ldots, i - 1, i + 1, \ldots, n\} \cup \{\emptyset\}$ according to:

$$
\sigma(\mathbf{z}_{-i}, \mathbf{u}_{-i}) = \begin{cases} 
    j & \text{if } B_j \geq \max_{j' \neq i} B_{j'} \text{ and } B_j \geq u_0, \\
    \emptyset & \text{if } u_0 > \max_{j' \neq i} B_{j'}.
\end{cases}
$$

We assume that ties are broken in favor of the platform, that is, if $\max_{j' \neq i} B_{j'}$ has multiple maximizers then $\sigma$ selects the one with the highest revenue. Let $z_i$ and $z^*_i$ be the reservation price of item $i$ under $\tau$ and $\tau^*$, respectively. We note that $z_i$ and $z^*_i$ are different random variables as $\tau$ and $\tau^*$ differ for item $i$. 

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We divide our analysis into two cases:

1. If $\sigma(z_i - i, u_i) < i$ (a higher revenue item is purchased), then

$$E[R^*|z_i, u_i] \stackrel{(a)}{=} r_{\sigma(z_i, u_i)} \stackrel{(b)}{=} E[R|z_i, u_i]. \quad (4.26)$$

Equality (a) follows as $B_{\sigma(z_i, u_i)} \geq u_0 \geq z_i^*$. Note that from the definition of $\sigma$ and the assumption $\sigma(z_i, u_i) < i$, we have that $B_{\sigma(z_i, u_i)} \geq u_0$. Moreover, we have that $u_0 \geq z_i^*$ as under the $t_i^*$-threshold signal, reservation prices satisfy $z_i^* \in \{z_i(\mu_i^{Low}), z_i(\mu_i^{High})\}$, which never exceed the value of the outside option:

$$z_i(\mu_i^{Low}) < z_i(\mu_i^{High}) = u_0.$$

Meanwhile, inequality (b) follows as under $\tau$, the consumer can buy either item $i$ or $\sigma(z_i, u_i)$ and the revenue can be equal to $r_i$ which is smaller than $r_{\sigma(z_i, u_i)}$. This could be the case if $B_i > B_{\sigma(z_i, u_i)}$.

2. If $\sigma(z_i, u_i) > i$ or $\sigma(z_i, u_i) = \emptyset$, then

$$E[R^*|z_i, u_i] = p^*_ir_i + (1 - p^*_i)r_{\sigma(z_i, u_i)}, \quad (4.27)$$
$$E[R|z_i, u_i] = p_ir_i + (1 - p_i)r_{\sigma(z_i, u_i)}, \quad (4.28)$$

where $p^*_i$ and $p_i$ correspond to the probability of selling item $i$ under $\tau_i^*$ and $\tau_i$ in the single-item case, respectively. Since $r_i \geq r_{\sigma(z_i, u_i)}$, it suffices to show that $p^*_i \geq p_i$. This follows from Proposition 12.

### 4.6 Beyond “No Search Without Disclosure” - The 2 Items Case

In this section, we relax Assumption 1 and consider scenarios where it might be optimal for the consumer to investigate an item even if the platform decides not to disclose information. In
these scenarios, it may not be possible to make the customer indifferent between all items using the \( t'_i \)–threshold signal so the platform can’t induce any search order. Of course, now the problem faced by the platform becomes more challenging, so we focus on the case with two items \( n = 2 \).

4.6.1 The Platform’s Problem

We start by formulating the platform’s problem as an optimization problem. In this case, an information provision policy corresponds to two independent signal structures: \((S_1, \pi_1)\) and \((S_2, \pi_2)\). As in the single-item case, we will restrict to distributions over posteriors \( \tau_i \in \Delta(\Delta(\mathcal{V})) \) that are Bayes-plausible. Hence, we derive an expression for the revenue generated by \( \tau \).

For every item \( i \) and \( v \in \mathcal{V} \), we let \( R_i(v) \) be the platform’s expected utility conditional on (i) the consumer searching only item \( i \), and (ii) the realized predictable component of item \( i \) being \( v \):

\[
R_i(v) = r_i P_{\mathcal{E}_i \sim \kappa_i}(v + \mathcal{E}_i \geq u_0).
\] (4.29)

For every item \( i \), threshold \( t \geq u_0 \) and vector \( v \in \mathcal{V} \times \mathcal{V} \), we let \( R'_i(v) \) be the platform’s expected utility conditional on (i) the consumer searching first item \( i \), (ii) the reservation price of item \( j \) being \( t \), and (iii) the realized predictable components being \( v \). To formally define \( R'_i(v) \), it is useful to compute the conditional probability that the consumer purchases each item. We start by studying the probability of buying item \( i \). Of course, an item will be purchased only if its utility is at least the value of the outside option \( u_0 \). Additionally, item \( i \) will be purchased if either (i) its utility exceeds the threshold \( t \) or (ii) its utility exceeds the utility of item \( j \). Hence, the probability of buying item \( i \) corresponds to

\[
P_{\mathcal{E}_i \sim \kappa_i, \mathcal{E}_j \sim \kappa_j}(v_i + \mathcal{E}_i \geq \max\{\min\{t, v_j + \mathcal{E}_j\}, u_0\})
\].

We now turn to item \( j \). Again, a necessary condition is that the utility of item \( j \) must be at least the value of the outside option \( u_0 \). Additionally, the search can’t stop after investigating item \( i \) so its utility should be less than the threshold \( t \). Finally, the match value of item \( j \) must exceed the
match value of item \( i \). Putting all together, we obtain that the probability of purchasing item \( j \) is

\[
P_{\mathcal{E}_i \sim \kappa_i, \mathcal{E}_j \sim \kappa_j} (\{ v_i + \mathcal{E}_i < \min\{ t, v_j + \mathcal{E}_j \} \}, \{ v_j + \mathcal{E}_j \geq u_0 \}).
\]

Therefore, we have that

\[
R'_i (v) = r_i P_{\mathcal{E}_i \sim \kappa_i, \mathcal{E}_j \sim \kappa_j} (v_i + \mathcal{E}_i \geq \max \{ \min \{ t, v_j + \mathcal{E}_j \}, u_0 \}) \\
+ r_j P_{\mathcal{E}_i \sim \kappa_i, \mathcal{E}_j \sim \kappa_j} (\{ v_i + \mathcal{E}_i < \min \{ t, v_j + \mathcal{E}_j \} \}, \{ v_j + \mathcal{E}_j \geq u_0 \}).
\]

Moreover, for every threshold \( t \geq u_0 \) we define the following sets:

\[
U^0 = U^0_1 \times U^0_2 = \{ \mu \in \Delta (\mathcal{V}) : z_1 (\mu) < u_0 \} \times \{ \mu \in \Delta (\mathcal{V}) : z_2 (\mu) < u_0 \},
\]

\[
U^i = U^i_1 \times U^i_2 = \{ \mu \in \Delta (\mathcal{V}) : z_i (\mu) \geq u_0 \} \times \{ \mu \in \Delta (\mathcal{V}) : z_j (\mu) < u_0 \},
\]

\[
U^{12t} = U^{12t}_1 \times U^{12t}_2 = \{ \mu \in \Delta (\mathcal{V}) : z_1 (\mu) \geq t \} \times \{ \mu \in \Delta (\mathcal{V}) : z_2 (\mu) = t \},
\]

\[
U^{21t} = U^{21t}_1 \times U^{21t}_2 = \{ \mu \in \Delta (\mathcal{V}) : z_1 (\mu) = t \} \times \{ \mu \in \Delta (\mathcal{V}) : z_2 (\mu) > t \}.
\]

The sets defined above form a partition of the space of posteriors based on the consumer’s action they induced. First, the set \( U^0 \) is associated with the consumer no searching, here the posteriors of both items induce reservation prices lower than the value of the outside option \( \mu_0 \). Secondly, the set \( U^i \) corresponds to the consumer only searching item \( i \), in this case only item \( i \) has a reservation price equal to or higher than \( u_0 \). Third, we have the sets of the form \( U^{ij} \), in which the consumer first searches item \( i \), then searches item 2 only if the match value of item \( i \) is below the threshold \( t \). Notice that in \( U^{ij} \) we condition on the reservation price \( t \) of item \( j \), since this determines the consumer’s action after seeing the utility of item \( i \). By contrast, we only require the reservation price of item \( i \) exceeding \( t \) as this ensures the consumer will search this item first.

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Figure 4.1: Let $A = \{0, 1, 2\} \cup \{12t\}_{t \geq u_0} \cup \{21t\}_{t \geq u_0}$. The collection $\{U^a\}_{a \in A}$ forms a partition of $\Delta(\mathcal{V}) \times \Delta(\mathcal{V})$, that is, for every $a, a' \in A$ such that $a \neq a'$, $U^a \cap U^{a'} = \emptyset$ and $\bigcup_{a \in A} U^a = \Delta(\mathcal{V}) \times \Delta(\mathcal{V})$.

With these definitions out of the way, we proceed to formulate the platform’s problem as

$$
OPT = \max \quad \tau_2(U^1_2) \int_{U^1_2} \int_{\mathcal{V}} R_1(v_1) \mu_1(dv_1) \tau_1(d\mu_1)
+ \tau_1(U^2_1) \int_{U^2_1} \int_{\mathcal{V}} R_2(v_2) \mu_2(dv_2) \tau_2(d\mu_2)
+ \int_{u_0}^{\infty} \int_{U^{12t}} \int_{\mathcal{V} \times \mathcal{V}} R_1^t(v_1, v_2) \mu_1(dv_1) \mu_2(dv_2) \tau_1(d\mu_1) \tau_2(d\mu_2) \, dt
+ \int_{u_0}^{\infty} \int_{U^{21t}} \int_{\mathcal{V} \times \mathcal{V}} R_2^t(v_1, v_2) \mu_1(dv_1) \mu_2(dv_2) \tau_1(d\mu_1) \tau_2(d\mu_2) \, dt
$$

subject to

$$
\int_{\Delta(\mathcal{V})} \mu_i(v_i) \tau_i(d\mu_i) = \mu_i^0(v_i) \quad \forall i \in \{1, 2\}, \, v_i \in \mathcal{V}
\tau_i \in \Delta(\Delta(\mathcal{V})) \quad \forall i \in \{1, 2\}
$$

In the objective function of $OPT$ the first two terms correspond to the revenue obtained by searching only item 1 and item 2, respectively. In these terms, the preceding $\tau_j(U^i_j)$ corresponds to the
probability of item $j$ having a posterior that induces no search, while $R_i(v_i)$ is the platform’s revenue conditional on (i) searching only item $i$ and (ii) the realized predictable component of item $i$ being $v_i$. Moreover, the last two terms on the objective function correspond to the revenue obtained when the consumer searches first item $i$, and then searches item $j$ whenever the realized utility of item $i$ is below the threshold $t$. In these terms, $R'_i(v_1, v_2)$ corresponds to the expected revenue conditional on (i) the consumer following the action previously described, and (ii) the realized private components being $v_1$ and $v_2$.

4.6.2 Two Reformulations Based on Reservation Values

We now propose two reformulations of $OPT$. Our goal is to obtain an optimization problem that can be solved numerically, and then conduct a numerical analysis by solving multiple instances. These reformulations are based on the following observation: under the consumer’s optimal search, posteriors that induced the same reservation price lead to the same decisions, so they can be pooled together without affecting the platform’s revenue. By a similar logic, all posteriors that don’t induce search can also be pooled together. Hence, for every $z \geq u_0$, there is a single posterior $\mu^z_1$ that induces reservation price $z$. Additionally, there is a single posterior $\mu^{<u_0}_i$ which doesn’t induce search. We denote by $\lambda_i$ the distribution over posteriors $\mu^z_i$ chosen by the platform. Hence, we can then reformulate the problem as

$$\begin{align*}
RPR &= \max \lambda_2(<u_0) \int_{[u_0,\infty)} \int_V R_1(v_1) \mu^z_1(dv_1) \lambda_1(\mu^z_1) dz \\
&\quad + \lambda_1(<u_0) \int_{[u_0,\infty)} \int_V R_2(v_2) \mu^z_2(dv_2) \lambda_2(\mu^z_2) dz \\
&\quad + \int_{[u_0,\infty)} \int_{[t,\infty)} \int_V \int_V R'_1(v_1, v_2) \mu^z_1(dv_1) \mu^z_2(dv_2) \lambda_1(z) \lambda_2(t) \lambda_2(z) dz dt \\
&\quad + \int_{[u_0,\infty)} \int_{[t,\infty)} \int_V \int_V R'_2(v_1, v_2) \mu^z_1(dv_1) \mu^z_2(dv_2) \lambda_1(t) \lambda_2(z) \lambda_2(z) dz dt
\end{align*}$$

subject to

$$\begin{align*}
\lambda_i(<u_0) + \int_{[u_0,\infty)} \lambda_i(dz) &= 1 & \forall i \in \{1, 2\} \\
\mu^{<u_0}_i \lambda_i(<u_0) + \int_{[u_0,\infty)} \mu^z_i \lambda_i(dz) &= \mu^0_i & \forall i \in \{1, 2\} \\
\int_V \ell(v_i + u_0 - z) \mu^z_i(dv_i) &= c_i & \forall i \in \{1, 2\}, \ z \geq u_0.
\end{align*}$$

(4.31)
**Proposition 15.** The reservation price reformulation is optimal, that is,

\[
OPT = RPR.
\]  
(4.32)

The proof of Proposition 15 is located in Appendix C.2.

Unfortunately, in the objective function of \(RPR\) there are expressions involving the multiplication of four variables. Thus, we cannot directly solve instances of this problem using an off-the-shelf solver. This motivates the introduction of a new reformulation \((QP)\) with a quadratic objective function. For every item \(i\) and reservation price \(z\), we introduce the variable \(\alpha_i^z = \lambda_i^z \mu_i^z\), resulting in the following formulation:

\[
\begin{align*}
QP = & \max \int_{V} \alpha_2^{z<u_0}(dv_2) \int_{[u_0,\infty)} \int_{V} R_1(v_1) \alpha_1^z(dv_1) dz \\
& + \int_{V} \alpha_1^{z<u_0}(dv_1) \int_{[u_0,\infty)} \int_{V} R_2(v_2) \alpha_2^z(dv_2) dz \\
& + \int_{[u_0,\infty)} \int_{(t,\infty)} \int_{V \times V} R_1(v) \alpha_1^z(dv_1) \alpha_2^z(dv_2) dz dt \\
& + \int_{[u_0,\infty)} \int_{(t,\infty)} \int_{V \times V} R_2(v) \alpha_1^z(dv_1) \alpha_2^z(dv_2) dz dt \\
\text{subject to} \quad & \alpha_i^{z<u_0}(v_i) + \int_{[u_0,\infty)} \alpha_i^z(v_i) dz = \mu_i^0(v_i) \quad \forall i \in \{1, 2\}, v_i \in \mathcal{V} \\
& \int_{V} \ell(v_i + u_0 - z) \alpha_i^z(dv_i) = \int_{V} \alpha_i^z(dv_i) c_i \quad \forall i \in \{1, 2\}, z \geq u_0 \\
& \alpha_i^z(v_i) \geq 0 \quad \forall i \in \{1, 2\}, z \in \{<u_0\} \cup [u_0, \infty), v_i \in \mathcal{V} \end{align*}
\]  
(4.33)

**Proposition 16.** The quadratic programming reformulation is optimal, that is,

\[
QP = RPR.
\]

**Proof.** For each item \(i\), reservation value \(z \in \{<u_0\} \cup [u_0, \infty)\) and predictable component \(v_i \in \mathcal{V}\), we introduce the variable

\[
\alpha_i^{z}(v_i) = \lambda_i^z \mu_i^z(v_i).
\]  
(4.34)
Since $\lambda^z_i$ and $\mu^z_i$ are probability distributions, then

$$\alpha^z_i(v_i) \geq 0.$$ \hspace{1cm} (4.35)

Additionally, from the definition of $\alpha^z_i$, we have that

$$\alpha^{<u_0}_i(v_i) + \int_{u_0}^{\infty} \alpha^z_i(v_i) \, dz = \lambda^z_i \mu^{<u_0}_i(v_i) + \int_{u_0}^{\infty} \lambda^z_i \mu^z_i(v_i) \, dz = \mu^0_i(v_i).$$

The second equality follows since $\lambda, \mu$ are feasible for RPR.

We now turn to the obedience constraints. Observe that

$$\int_{V} \ell(v_i + u_0 - z) \alpha^z_i(dv_i) = \lambda^z_i \int_{V} \ell(q_i + u_0 - z) \mu^z_i(dv_i) = \lambda^z_i c_i$$

$$= \int_{V} \alpha^z_i(dv_i) c_i.$$  

In the last equality we use that $\int_{V} \alpha^z_i(dv_i) = \lambda^z_i$, this follows from the obedience constraint in RPR:

$$\int_{V} \alpha^z_i(dv_i) = \int_{V} \lambda^z_i \mu^z_i(dv_i) = \lambda^z_i \int_{V} \mu^z_i(dv_i) = \lambda^z_i,$$

where the last equality follows since $\mu^z_i$ is a probability distribution so $\int_{V} \mu^z_i(dv_i) = 1$. \hspace{1cm} \Box

In Appendix C.2.2 we include a third reformulation based on integer programming. This alternative reformulation was used to solve instances where the QP formulation was not able to find the optimal solution under an hour time limit.

### 4.6.3 A Conjecture: Optimal Information Provision

We now present a conjecture concerning the platform’s optimal information provision policy for the two items case. This conjecture is based on the results from our numerical analysis, where we implemented the quadratic reformulation (QP) presented in the previous section. Below, we
provide a concise overview of the key findings from our analysis.

The structure of the optimal signals depends on the reservation prices induced by the prior beliefs, their relative order, and how they compare with the value of the outside option $u_0$. Based on the different possibilities, we distinguish between five scenarios.

In all five scenarios we considered, the optimal signals are either binary or uninformative. Specifically, when it comes to item 1, we distinguish between two cases.

- If the prior $\mu_1^0$ induces the consumer to search, then the platform decides not to share further information.
- If the prior $\mu_1^0$ doesn’t induce search, then it is optimal to use a binary signal in which only one signal realization induces the consumer to investigate.

For item 2, we can also distinguish between two scenarios:

- If the prior $\mu_2^0$ doesn’t induce search, then the platform should use the $t_2^*$-threshold signal structure.
- If the prior $\mu_2^0$ induces search, then the optimal signal will be contingent on the reservation price induced by item 1.

**Conjecture 2.** For every item $i$, let $z_i$ be the reservation price induced by the prior belief, that is, $z_i = z_i(\mu_i^0)$. Furthermore, consider a binary signal structure with signals low and high. We denote this signal by a pair $(z_{\text{low}}, z_{\text{high}})$, where $z_{\text{low}}$ and $z_{\text{high}}$ correspond to the reservation price induced by signal low and high, respectively. Given a constant $c \in \mathbb{R}$, we let $z_{\text{low}} = <c$ denote any reservation that is smaller than $c$. Similarly, we let $z_{\text{high}} = >c$ denote any reservation that is greater than $c$.

The optimal information provision policy is given by
<table>
<thead>
<tr>
<th>Case</th>
<th>Item 1</th>
<th>Item 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1, z_2 &lt; u_0$</td>
<td>$(&lt;u_0,u_0)$</td>
<td>$(&lt;u_0,u_0)$</td>
</tr>
<tr>
<td>$z_1 &lt; u_0 \leq z_2$</td>
<td>$(&lt;z_2,z_2)$</td>
<td>no-disclosure</td>
</tr>
<tr>
<td>$z_2 &lt; u_0 \leq z_1$</td>
<td>no-disclosure</td>
<td>$(&lt;u_0,u_0)$</td>
</tr>
<tr>
<td>$u_0 \leq z_1 &lt; z_2$</td>
<td>no-disclosure</td>
<td>$(z_1,z_1)$</td>
</tr>
<tr>
<td>$u_0 \leq z_2 \leq z_1$</td>
<td>no-disclosure</td>
<td>no-disclosure</td>
</tr>
</tbody>
</table>
References


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Feige, Uriel (2004). “On sums of independent random variables with unbounded variance, and estimating the average degree in a graph”. In: *Proceedings of the thirty-sixth annual ACM symposium on theory of computing (STOC)*.


Correa, Jose et al. (2021). “School choice in Chile”. In: Operations Research.


Derakhshan, Mahsa et al. (2022). “Product ranking on online platforms”. In: Management Science 68.6, pp. 4024–4041.


Abeliuk, Andrés et al. (2016). “Assortment optimization under a multinomial logit model with position bias and social influence”. In: 4OR 14, pp. 57–75.

Lei, Yanzhe et al. (2022). “Joint product framing (display, ranking, pricing) and order fulfillment under the multinomial logit model for e-commerce retailers”. In: Manufacturing & Service Operations Management 24.3, pp. 1529–1546.


Appendix A: Chapter 2

A.1 Group Lottery

ALGORITHM 4:

Input: A finite set \( E \)
Output: A random order \( \Sigma \in O_E \)

Choose \( S \subset E \). Generate

1. a uniform random order \( \Sigma^S \in O_S \),
2. a uniform random order \( \Sigma^- \in O_{E \setminus S} \),
3. a uniform random subset \( P \subset \{1, \ldots, |E|\} \) with \( |P| = |S| \).

Generate \( \Sigma \) from \( \Sigma^S, \Sigma^-, P \) placing elements of \( S \) in positions \( P \), maintaining the order of elements of \( S \) as given by \( \Sigma^S \) and the order of elements of \( E \setminus S \) as given by \( \Sigma^- \).

Lemma 8. For any finite set \( E \), Algorithm 4 generates a uniform random order \( \Sigma \in O_E \): for each order \( \sigma \in O_E \), \( P(\Sigma = \sigma) = 1/|E|! \).

Proof of Lemma 8. Fix an order \( \sigma \in O_E \). Let \( \sigma_S \) and \( \sigma^- \) be the restriction of \( \sigma \) to \( S \) and \( E \setminus S \), respectively. For any \( i \in E \), we let \( p_i \) be the position of \( i \) in \( \sigma \), that is, \( \sigma_{p_i} = i \). We define \( P_S = \cup_{i \in S} \{p_i\} \). In order to end with the order \( \sigma \), it must be that:

- The order \( \Sigma \) generated in step i. is equal to \( \sigma_S \), which occurs with probability \( P(\Sigma = \sigma_S) = 1/|S|! \).
- The order \( \Sigma^- \) generated in step ii. is equal to \( \sigma^- \), which occurs with probability \( P(\Sigma^- = \sigma^-) = 1/(|E| - |S|)! \).
- The random subset \( P \) generated in step iii. is equal to \( P_S \), which occurs with probability \( P(P = P_S) = \prod_{j=0}^{|S|-1} (|S| - j)/(|E| - j) \).
Hence, the probability that the algorithm generates the order $\sigma$ is

$$\frac{1}{|S|!} \frac{1}{E - |S|!} \prod_{j=0}^{|S| - 1} \frac{|S| - j}{|E| - j} = \frac{1}{|E|!}. \quad \Box$$

A.1.1 Incentives

**Proof of Proposition 1.** Fix an arbitrary agent $i \in N$. Let $a$ be an action profile such that $a_{G_i}$ is the group request strategy and $a_{-G_i}$ is arbitrary. Let $a'$ denote an alternative strategy profile in which $a'_j = a_j$ for $j \notin G_i$. Let $V$ be the set of valid groups according to $a$, $V'$ the set of valid groups according to $a'$, and $V^-$ be the set of valid groups not containing any members of $G_i$:

$$V = \{ S \subset N : a_j = S \forall j \in S \}.$$

$$V' = \{ S \subset N : a'_j = S \forall j \in S \}.$$

$$V^- = \{ S \subset N \setminus G_j : a_j = S \forall j \in S \}.$$

Note that $V^- \subseteq V \cap V'$, and that agents in $G_i$ do not influence $V^-$. We generate uniform random orders $\Sigma$ and $\Sigma'$ over $V$ and $V'$ (respectively) using Algorithm 4: we first generate a uniform random order $\Sigma^-$ over $V^-$, and then extend this to obtain $\Sigma$ and $\Sigma'$. We will prove that for any realization of $\Sigma^-$,

$$\mathbb{E}[u_i(x^{GL}(a, \Sigma))|\Sigma^-] \geq \mathbb{E}[u_i(x^{GL}(a', \Sigma'))|\Sigma^-]. \quad (A.1)$$

Because agents in $G_i$ cannot influence $\Sigma^-$, it follows immediately that the unconditional expected utility of agent $i$ is also higher under the group request strategy.

If $\sum_{S \in V^-} |S| \leq k - |G_i|$, then the group request strategy guarantees that all members of $G_i$ will receive a ticket, so there is nothing to prove. Otherwise, let

$$\tau(\Sigma^-) = \tau(k - |G_i| + 1, \Sigma^-),$$
be as defined in (2.6) where the size function is the cardinality of the valid group declared by the agent, that is, $|\sigma_i| = |a_{\sigma_i}|$. Intuitively, $\tau$ is the first point at which the number of remaining tickets would be less than $|G_i|$, when processing valid groups in $V^-$ according to order $\Sigma^-$. Because agents in $G_i$ follow the group request strategy under $a$, we have $V = V^\prime \cup G_i$. Members of $G_i$ get a payoff of 1 if and only if $G_i$ is in the first $\tau(\Sigma^-)$ positions of $\Sigma$. Therefore,

$$
\mathbb{E}[\mu_i(x_{GL}(a, \Sigma))|\Sigma^-] = \frac{\tau(\Sigma^-)}{1 + |V^-|}.
$$

We now turn to the action profile $a'$. Because the Group Lottery gives at most one ticket to each agent, $i$ gets a payoff of 1 if and only if all members of $G_i$ receive a ticket. This is not possible unless (i) every agent in $G_i$ is included in a valid group in $V^\prime$, and (ii) in the order $\Sigma'$, all valid groups in $V^\prime \setminus V^-$ appear before group $S = \Sigma_{\tau(\Sigma^-)}^-$. According to the algorithm, the conditional probability of (ii) given $\Sigma^-$ is at most

$$
\frac{\tau(\Sigma^-) - 1}{2 + |V^-|} \cdot \frac{\tau(\Sigma^-)}{1 + |V^-|},
$$

which is smaller than the right side of (A.2), implying that group $G_i$ has not benefited from its deviation.

Next, we show that any other strategy is not dominant. Let $j \notin G_i$ and $\tilde{a}$ denote an action profile such that $j \in \tilde{a}_i$, $i \notin \tilde{a}_j$ and the remaining actions $\tilde{a}_{-\{i,j\}}$ are arbitrary. Under $\tilde{a}$ agent $i$ is not in a valid group then it is not award and group $G_i$ get a payoff of 0. This is strictly less than the payoff under a group request, which is greater than the probability of $G_i$ being the first group to be processed. Therefore, we can restrict to strategies $\hat{a}$ where $\hat{a}_{i'} \subset G_i$ for any $i' \in G_i$. Furthermore, $G_i$ will have a positive expected payoff only if under $\hat{a}$ its members are divided into two or more valid groups. Let actions $\hat{a}_{\sim G_i}$ be such that $\hat{a}_j = N \setminus G_i$ for any $j \in N \setminus G_i$, and $\hat{V}$ be the set of valid groups according to $\hat{a}$,

$$
\hat{V} = \{S \subset N : \hat{a}_j = S \forall j \in S\}.
$$
Observe that $|\hat{V}| \geq 3$. By assumption $n > k$, so $G_i$ will get a payoff of 1 if and only if valid group $N \setminus G_i$ is the last valid group to be processed. This event occurs with probability

$$\frac{(|\hat{V}| - 1)!}{|\hat{V}|!} = \frac{1}{|\hat{V}|}.$$ 

This is strictly smaller than $1/2$ the expected utility when $G_i$ select a group request. \hfill \Box

A.1.2 Performance

**Lemma 9.** Fix any instance and any pair of agents $i, j \in N$. Let $a$ be an action profile under the Group Lottery such that $G_i$ and $G_j$ select the group request strategy. If $|G_i| \geq |G_j|$, then

$$u_i(\pi^{GL}(a)) \leq u_j(\pi^{GL}(a)). \quad (A.3)$$

**Proof of Lemma 9.** Let $V$ be the set of valid groups given $a$. Observe that by assumption $G_i, G_j \in V$. We define $S_i, S_j$ to be the set of orders over $V$ that guarantee a payoff of 1 to group $G_i$ and $G_j$, respectively. It suffices to show that

$$|S_j| \geq |S_i|.$$ 

To prove this, we will construct an injective map $f : S_i \to S_j$. We let $f$ be the map that only swap the positions of $G_i$ and $G_j$, keeping all the remaining positions unchanged. Clearly $f$ is injective. Thus, all that remains to show is that for any $\sigma \in S_i$, $f(\sigma) \in S_j$. Fix $\sigma \in S_i$. If $\sigma \in S_j$, then there are enough tickets to satisfy both groups and $f(\sigma) \in S_i \cap S_j \subseteq S_j$. If $\sigma \notin S_j$, then then the number of tickets remaining before $G_j$ is processed under $f(\sigma)$ is the same as the number of tickets remaining before $G_i$ is processed under $\sigma$. Because $\sigma \in S_i$, this number is at least $|G_i|$, which by assumption is at least $|G_j|$, so $f(\sigma) \in S_j$. \hfill \Box

**Fact 1.** For any $a, b, c \in \mathbb{R}$ such that $a \leq b$ and $c \geq 0$, then

$$\frac{a}{b} \leq \frac{a + c}{b + c}. \quad (A.4)$$
Proof of Proposition 2. First, we will prove the right inequality of (2.11). If \( k \geq \sum a_i - \bar{a} + 1 \), then our upper bound is at least \( n \) and immediately holds. Hence, without loss of generality we can assume \( k \leq \sum a_i - \bar{a} \). We let \( S_t = S_t(\Sigma) \) be the sum of the first \( t \) numbers according to \( \Sigma \), that is,

\[
S_t = \sum_{i=1}^{t} a_{\Sigma_i}.
\]

We define

\[
Z_t^* = \frac{S_t - t\mu}{n - t}.
\]

As mentioned in Serfling (1974), the sequence \( Z_1^*, \ldots, Z_{n-1}^* \) is a forward martingale. Furthermore, \( k \leq \sum a_i - \bar{a} \) implies \( P(\tau \leq n - 1) = 1 \) so \( \tau \) is bounded and \( Z_\tau^* \) is well defined. Hence, we can apply Doob’s optional stopping theorem to obtain

\[
\mathbb{E} \left[ \frac{S_\tau - \tau \mu}{n - \tau} \right] = \mathbb{E} [Z_\tau^*] = \mathbb{E} [Z_1^*] = 0.
\]

(A.7)

From the definition of \( \tau \), we get

\[
\mathbb{E} \left[ \frac{S_\tau}{n - \tau} \right] \leq (k + \bar{a} - 1) \mathbb{E} \left[ \frac{1}{n - \tau} \right].
\]

(A.8)

We claim that

\[
\mathbb{E} [\tau] \mathbb{E} \left[ \frac{1}{n - \tau} \right] \leq \mathbb{E} \left[ \frac{\tau}{n - \tau} \right].
\]

(A.9)

For any \( x < n \), define

\[
f(x) = (x - \mathbb{E}[\tau]) \left( \frac{1}{n - x} - \frac{1}{n - \mathbb{E}[\tau]} \right).
\]

Note that \( f(x) \geq 0 \) for all \( x \), so \( \mathbb{E} [f(\tau)] \geq 0 \). Thus,

\[
0 \leq \mathbb{E} [f(\tau)] = \mathbb{E} \left[ (\tau - \mathbb{E} [\tau]) \left( \frac{1}{n - \tau} - \frac{1}{n - \mathbb{E} [\tau]} \right) \right] = \mathbb{E} \left[ \frac{\tau - \mathbb{E} [\tau]}{n - \tau} \right].
\]

Combining equations (A.7), (A.8) and (A.9) yields our desired result.
Now, we prove the left inequality of (2.11). If $k \leq \bar{a}$, then our lower bound is at most $1$ and immediately holds, then without loss of generality we can assume $k \geq \bar{a}+1$. To construct $\Sigma$ we will generate a random order $\Sigma'$ and iterate through it backwards, that is, $\Sigma_t = \Sigma'_n-t+1$ for $t = 1, \ldots, n$. We claim that for every $\Sigma$,

$$\tau(k, \Sigma) + \tau(\sum_i a_i - k + 1, \Sigma') = n + 1. \quad (A.10)$$

It suffices to show that

$$\tau(\Sigma_t, a_t-1, \Sigma') = \sum_{i=1}^n a_{\Sigma_t} = \sum_{i=1}^n a_{\Sigma_t}.$$

From the definition of $\tau$, we have that

$$\sum_{t=\tau(k, \Sigma)+1}^n a_{\Sigma_t} = \sum_{i=1}^{\tau(k, \Sigma)} a_i - \sum_{t=1}^{\tau(k, \Sigma)} a_{\Sigma_t} \leq \sum_{i=1}^n a_i - k.$$

Similarly,

$$\sum_{t=\tau(k, \Sigma)}^{\tau(k, \Sigma)-1} a_{\Sigma_t} = \sum_{i=1}^{\tau(k, \Sigma)-1} a_i - \sum_{t=1}^{\tau(k, \Sigma)-1} a_{\Sigma_t} \geq \sum_{i=1}^n a_i - k + 1.$$

Applying the upper bound in (2.11) to (A.10) immediately implies

$$\mathbb{E}[\tau(k, \Sigma)] = n + 1 - \mathbb{E}[\tau(\sum_i a_i - k + 1, \Sigma')] \geq n + 1 - \frac{\sum_i a_i - k + \bar{a}}{\mu} = 1 + \frac{k - \bar{a}}{\mu}.$$

We now turn to equation (2.12). For any order $\sigma$, we define $h(\sigma)$ to be the order that: (i) is identical to $\sigma$ from position $\tau(k+k', \sigma)$ until the end, and (ii) flip the ordering of all elements from position 1 to $\tau(k+k', \sigma) - 1$. More precisely,

$$h(\sigma)_t = \begin{cases} 
\sigma_t & \text{if } t \geq \tau(k+k', \sigma), \\
\sigma_{\tau(k+k', \sigma)-t} & \text{if } t < \tau(k+k', \sigma). 
\end{cases} \quad (A.12)$$
We claim that
\[ \tau(k + k', \sigma) = \tau(k + k', h(\sigma)). \]

This implies that \( h(h(\sigma)) = \sigma \), which further implies that \( h \) is a bijective map. This follows from two observations: (i) the elements in the first \( \tau(k + k', \sigma) - 1 \) positions are the same in both orders, and do not sum to \( k + k' \). Additionally, (ii) the agents in the first \( \tau(k + k', \sigma) \) positions are also the same in both orders, and they do sum to \( k + k' \).

We will show that for any \( \sigma \in O_{[n]} \),
\[ \tau(k, \sigma) + \tau(k', h(\sigma)) \geq \tau(k + k', \sigma). \] \hspace{1cm} (A.13)

This implies our result as
\[
\mathbb{E} [\tau(k, \Sigma) + \tau(k', h(\Sigma))] = \frac{1}{n!} \left( \sum_{\sigma \in O_{[n]}} \tau(k, \sigma) + \tau(k', \sigma) \right)
\]
\[ = \frac{1}{n!} \left( \sum_{\sigma \in O_{[n]}} \tau(k, \sigma) + \tau(k', h(\sigma)) \right)
\]
\[ \geq \frac{1}{n!} \left( \sum_{\sigma \in O_{[n]}} \tau(k + k', \sigma) \right)
\]
\[ = \mathbb{E} [\tau(k + k', \Sigma)]. \]

The second equality follows as \( h \) is a bijective map, and the inequality follows from (A.13). Thus all that remains is to show (A.13). From the definition of \( \tau \), we have
\[
\sum_{t=1}^{\tau(k, \sigma)} a_{\sigma_t} \geq k,
\]
\[
\sum_{t=1}^{\tau(k + k', \sigma) - 1} a_{\sigma_t} < k + k'.
\]
Implying that
\[ \sum_{t=\tau(k,\sigma)+1}^{\tau(k+k',\sigma)-1} a_{\sigma_t} < k'. \] (A.14)

Moreover, from the definition of \( h \) in (A.12) it follows that
\[ \{ h(\sigma) \}_{1} \ldots h(\sigma)_{\tau(k+k',\sigma)-\tau(k,\sigma)-1} = \{ \sigma_{\tau(k+k',\sigma)-1} \ldots \sigma_{\tau(k,\sigma)+1} \}. \] (A.15)

Combining (A.14) and (A.15) yields
\[ \sum_{t=1}^{\tau(k+k',\sigma)-\tau(k,\sigma)-1} a_{h(\sigma)_t} < k'. \]

This implies (A.13) as by definition \( \tau \) is integral and
\[ \tau(k', h(\sigma)) > \tau(k+k', \sigma) - \tau(k, \sigma) - 1. \]

\[ \square \]

**Proof of Lemma 1.** Because all groups are playing the group request, the set of valid groups is \( G \).

In what follows we fix an arbitrary agent \( i \in N \). We construct a random order \( \Sigma \) over \( G \) using Algorithm 4: we generate an order \( \Sigma^{-i} \) over \( G \setminus G_i \) and then extend it to \( G \). By Lemma 8, the resulting order \( \Sigma \) is uniformly distributed. We let \( \tau^{-i} = \tau(k - |G_i| + 1, \Sigma^{-i}) \) be the number of positions in \( \Sigma \) that ensure \( G_i \) a payoff of 1 given \( \Sigma^{-i} \). Note that \( \tau^{-i} \) is well defined as \( k < n \) implies \( k - |G_i| + 1 \leq n - |G_i| = \sum_i |\Sigma^{-i}_i| \). Moreover, if \( G_i \) is in the first \( \tau^{-i} \) positions of \( \Sigma \), then it gets a payoff of 1 as the number of remaining tickets before it is processed is at least
\[ k - \sum_{t=1}^{\tau^{-i}-1} |\Sigma^{-i}_t| \geq k - (k - |G_i|) = |G_i|. \]

On the other hand, if \( G_i \) is in the last \( m - \tau^{-i} \) positions of \( \Sigma \), then it gets a payoff of 0 because the
number of remaining tickets when $G_i$ is processed is at most

$$k - \sum_{i=1}^{\tau_i} |\Sigma_i^{i}| \leq k - (k - |G_i| + 1) = |G_i| - 1.$$ 

Therefore,

$$u_i(x^{GL}(a, \Sigma)) = \mathbb{E}[\mathbb{E}[u_i(x^{GL}(a, \Sigma))|\Sigma_i^{i}]] = \frac{\mathbb{E}[\tau_i]}{m}. \quad (A.16)$$

By Proposition 2 equation (2.11), we have

$$\mathbb{E}[\tau_i] \leq \frac{k - G_i + \max_G |G|}{(n - |G_i|)/(m - 1)}. \quad (A.17)$$

This and equation (A.16) yields

$$u_i(\pi^{GL}(a)) \leq \left(\frac{k - G_i + \max_G |G|}{n - |G_i|}\right) \left(\frac{m - 1}{m}\right). \quad (A.18)$$

Because there is a group of size $|G_i|$ and the remaining $n - |G_i|$ agents can be in at most $n - |G_i|$ groups of size 1, we have

$$m \leq n - |G_i| + 1. \quad (A.19)$$

This implies

$$\frac{m - 1}{m} \leq \frac{n - |G_i|}{n - |G_i| + 1}. \quad (A.20)$$

From (A.18) and (A.20) it follows that

$$u_i(\pi^{GL}(a)) \leq \frac{k - G_i + \max_G |G|}{n - |G_i| + 1}. \quad (A.21)$$

Applying Fact 1 with $a = k + \max_G |G| - |G_i|$, $b = n + 1 - |G_i|$ and $c = |G_i| - 1$, we get

$$\frac{k + \max_G |G| - |G_i|}{n - |G_i| + 1} \leq \frac{k + \max_G |G| - 1}{n} \leq \frac{k}{n} (1 + \kappa). \quad (A.22)$$

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Note that to apply Fact 1 we need $a \leq b$, we can assume this without loss of generality. Otherwise, $a > b$ or equivalently

$$k + \max_G |G| > n + 1. \quad (A.23)$$

We claim that if the inequality above holds then $\frac{k}{n} (1 + \kappa) > 1$, hence, our bound is vacuous. This can be seen by noting that

$$1 + \kappa \geq \frac{k + \max_G |G| - 1}{k} > \frac{n}{k}. \quad (A.24)$$

The last inequality follows by (A.23).

□

Proof of Theorem 1. Fix $\alpha, \kappa \in (0, 1)$ and an arbitrary instance in $I(\kappa, \alpha)$. In what follows, we let $a$ denote the group request action profile under the Group Lottery. We define $s$ to be the maximum group size, that is, $s = \max_{G \in G} |G|$.

We begin with the efficiency guarantee. For any order $\sigma \in O_G$,

$$U(x^{GL}(a, \sigma)) \geq 1 - \frac{s - 1}{k} \geq 1 - \kappa, \quad (A.25)$$

where the second inequality follows as our instance is in $I(\kappa, \alpha)$. This is fairly trivial: if we let

$$\tau(\sigma) = \tau(k + 1, \sigma),$$

be as defined in (2.6) where the size of a group is its number of elements. Then $U(x^{GL}(a, \sigma))$ is exactly $\frac{1}{k} \sum_{j=1}^{\tau(\sigma, k+1)-1} |\sigma_j|$, which is at least $\frac{1}{k} (k - (s - 1))$, because adding one more group (of size at most $s$) brings the sum above $k$. From (A.25), it immediately follows that if $\Sigma$ is a random order on $G$, then

$$\mathbb{E}[U(x^{GL}(a, \Sigma))] \geq 1 - \kappa.$$

We now show that in this setting the outcome is $(1 - 2\kappa)$-fair. Our goal is to show that for any pair of agents $i, j \in N$,

$$\frac{u_i(\pi^{GL}(a))}{u_j(\pi^{GL}(a))} \geq 1 - 2\kappa. \quad (A.26)$$
By Lemma 9, we can assume without loss of generality that

$$|G_i| = \max_{G \in \mathcal{G}} |G| \text{ and } |G_j| = \min_{G \in \mathcal{G}} |G|. \quad (A.27)$$

We let $\mu^{-i}$ be the average group size in $\mathcal{G} \setminus G_i$, more precisely,

$$\mu^{-i} = \frac{n - |G_i|}{m - 1}. \quad (A.28)$$

We claim that

$$\frac{u_i(\pi^{GL}(a))}{u_j(\pi^{GL}(a))} \geq \frac{k - 2|G_i| + 1 + \mu^{-i}}{k - |G_j| + \mu^{-i}}. \quad (A.29)$$

This implies our result as

$$\frac{k - 2|G_i| + 1 + \mu^{-i}}{k - |G_j| + \mu^{-i}} \geq \frac{k - 2|G_i| + 1 + |G_j|}{k} \geq 1 - 2\kappa.$$

In the first inequality, we apply Fact 1 with $a = k - 2|G_i| + 1 + |G_j|$, $b = k$ and $c = \mu^{-i} - |G_j|$. The last inequality follows from the definition of $I(\kappa, \alpha)$ in (2.5). Remember that for any instance in $I(\kappa, \alpha)$, we have

$$\frac{|G_i| - 1}{k} = \max_{G \in \mathcal{G}} \frac{|G| - 1}{k} \leq \kappa.$$

We now turn to the proof of equation (A.29). Let $\Sigma^{-i}$ be a uniform order on $\mathcal{G} \setminus G_i$. Applying Lemma 1 to agents $i, j$, it follows that

$$\frac{u_i(x^{GL}(a, \Sigma))}{u_j(x^{GL}(a, \Sigma))} = \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-j})]} \quad (A.30)$$

By Proposition 2 equation (2.12), we have

$$\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-j})] \leq \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-j})] + \mathbb{E}[\tau(|G_i| - |G_j|, \Sigma^{-j})]. \quad (A.31)$$
We claim that for any constant \( c \in \mathbb{N} \), such that \( c \leq n - |G_i| = \sum_i |\Sigma^{-i}| \),

\[
\mathbb{E}[\tau(c, \Sigma^{-i})] \leq \mathbb{E}[\tau(c, \Sigma^{-j})].
\] (A.32)

We now show (A.32). We generate \( \Sigma^{-i} \) using \( \Sigma^{-j} \) in the following way:

\[
\Sigma^{-i}(\Sigma^{-j}) = \begin{cases} 
\Sigma^{-j} & \text{if } \Sigma^{-j} \neq G_i, \\
G_j & \text{otherwise.}
\end{cases}
\]

Note that by construction for any \( \Sigma^{-i} \), \( \tau(c, \Sigma^{-i}(\Sigma^{-j})) \geq \tau(c, \Sigma^{-j}) \). This establishes (A.32). Applying (A.32) twice we get

\[
\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-j})] + \mathbb{E}[\tau(|G_i| - |G_j|, \Sigma^{-j})] \leq \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})] + \mathbb{E}[\tau(|G_i| - |G_j|, \Sigma^{-i})].
\] (A.33)

Then by (A.31) and (A.33),

\[
\frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-i})]} \geq \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-j})]}{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-j})] + \mathbb{E}[\tau(|G_i| - |G_j|, \Sigma^{-j})]}.
\] (A.34)

By Proposition 2 equation (2.11), we have

\[
\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})] \geq \frac{k - 2|G_i| + 1 + \mu^{-i}}{\mu^{-i}},
\] (A.35)

\[
\mathbb{E}[\tau(|G_i| - |G_j|, \Sigma^{-i})] \leq \frac{2|G_i| - |G_j| - 1}{\mu^{-i}}.
\] (A.36)

From (A.35) and (A.36), we have

\[
\frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-j})] + \mathbb{E}[\tau(|G_i| - |G_j|, \Sigma^{-i})]} \geq \frac{k - 2|G_i| + 1 + \mu^{-i}}{\frac{k - 2|G_i| + 1 + \mu^{-i} + 2|G_i| - |G_j| - 1}{k - |G_j| + \mu^{-i}}.}
\]
Tightness

**Proposition 17.** For any $\epsilon > 0$ and $\alpha \in (0, 1)$, there exists $\kappa \in (0, 1)$ and an instance in $I(\kappa, \alpha)$ such that the group request equilibrium outcome of the Group Lottery is not $1 - (2 - \epsilon)\kappa$ fair.

**Proof of Proposition 17.** We consider an instances in which there is one “single person” (group of size one), and the remaining groups are “couples” (groups of size two). Fix $r \in \mathbb{N}$ such that

$$\frac{1}{r} < \epsilon,$$

and set $k = 2r - 1$ and $\kappa = 1/k$. Let $m$ be the total number of groups, and note that for all sufficiently large $m$, the instance will be in $I(\kappa, \alpha)$.

The single person is successful if among the first $r$ groups (which occurs with probability $\frac{r}{m}$), while each couple is successful if among the first $r - 1$ couples (which occurs with probability $\frac{r-1}{m-1}$).

Thus, as $m \to \infty$, the ratio of the utility each couple to the utility of the single person converges to

$$\frac{r-1}{r} < 1 - \frac{2 - \epsilon}{2r - 1} = 1 - (2 - \epsilon)\kappa,$$

where the first equality follows from (A.37). □

A.1.3 Proof of Proposition 3

We divide the proof into two parts: Lemma 10 establishes that for any instance in $I(\kappa, \alpha)$, there exists a random allocation that is $(1 - \kappa)$-efficient and fair. Lemma 11 shows that improving beyond $(1 - \kappa)$-efficiency may require abandoning even approximate fairness.

**Lemma 10.** Fix $\kappa, \alpha \in (0, 1)$. For every instance in $I(\kappa, \alpha)$, there exist a random allocation that is $(1 - \kappa)$-efficient and fair.
Proof of Lemma 10. We claim that there exists of a random allocation $\pi^*$ such that for every agent $i \in N$,

$$u_i(\pi^*) = u^* = \frac{k - \max_{G \in G} |G| + 1}{n}.$$ \hspace{1cm} (A.38)

It immediately follows that $\pi^*$ is fair as the utility of every agent is equal to $u^*$. Moreover, the utilization in this system is

$$U(\pi^*) = \frac{nu^*}{k} = 1 - \frac{\max_{G} |G| - 1}{k} \geq 1 - \kappa.$$

The inequality follows because our instance is in $I(\kappa, \alpha)$.

All that remains is to prove our claim. To this end, we will apply Theorem 2.1 in Nguyen, Peivandi, and Vohra (2016) which establishes that any utility vector such that (i) the sum of all agents’ utilities is at most $k - \max_{G \in G} |G| + 1$, and (ii) members of each group receive the same utility, can be induced by a lottery over feasible allocations. Before formally presenting this result some definitions are needed.

A group allocation is represented by $x \in \{0, \ldots, k\}^m$ satisfying $\sum_{G \in G} x_G \leq k$, where $x_G$ represents the number of tickers assigned to group $G$. For simplicity, we restrict to allocations such that for every group $G$, $x_G \in \{0, |G|\}$. Notice that a group $G$ is successful if and only if $x_G = |G|$. A random group allocation corresponds to a distribution $\pi$ over the set of group allocations. A group utility vector $u \in [0, 1]^m$ associates to each group $G$ a utility $u_G$.

Theorem 2.1 in Nguyen, Peivandi, and Vohra (2016) establishes that if a group utility vector $u'$ satisfies

$$\sum_{G \in G} |G|u'_G \leq k - \max_{G \in G} |G| + 1,$$ \hspace{1cm} (A.39)

then it can be induced by a random group allocation, that is, there exists a random allocation $\pi$ such that

$$u_i(\pi) = u'_G$$ for every $G \in G, i \in G$.

Observe that the expected number of tickets awarded to each group is equal to the sum of the
utilities of its members. Therefore, condition (A.39) can be interpreted as an upper bound on
the total expected number of tickets awarded, which depends on the maximum number of tickets
demanded by a single group.

Hence, the existence of a random allocation \( \pi^* \) that yields (A.38) follows by letting \( \hat{u} \) be the
utility vector that gives each group a utility of \( u^* \). Note that \( \hat{u} \) satisfies condition (A.39) as
\[
\sum_{i \in N} \hat{u}_i = nu^* = k - \max_{G \in G} |G| + 1.
\]

\[\square\]

**Lemma 11.** For any \( \epsilon > 0 \), there exists \( \alpha, \kappa \in (0, 1) \) and an instance in \( I(\kappa, \alpha) \) such that no
random allocation is \( \epsilon \)-fair and \( (1 - \kappa + \epsilon) \)-efficient.

**Proof of Lemma 11.** Fix \( \epsilon > 0 \) and let \( m, s, r \in \mathbb{N} \). We consider an instance with \( k = rs - 1 \) tickets,
one group of size \( s - 1 \) and \( m - 1 \) groups of size \( s \). Let \( i \) be a member of the group of size \( s - 1 \).
Let \( \pi \) be any random allocation. Because at most \( r - 1 \) of the large groups can be satisfied in any
deterministic allocation, we have that
\[
sm(m - 1) \min_{j \notin G_i} \{u_j(\pi)\} \leq \sum_{j \notin G_i} u_j(\pi) \leq s(r - 1). \tag{A.40}
\]
If the allocation is \( \epsilon \)-fair, then it must be the case that
\[
\epsilon u_i(\pi) \leq \min_{j \notin G_i} \{u_j(\pi)\} \leq \frac{r - 1}{m - 1}, \tag{A.41}
\]
where the right inequality follows from (A.40).

Observe that if \( G_i \) is successful then there will be no tickets wasted; otherwise, there will be
\( s - 1 \) tickets wasted. Hence, the utilization under \( \pi \) is
\[
1 - (1 - u_i(\pi)) \frac{s - 1}{k} \leq 1 - \left(1 - \frac{r - 1}{\epsilon(m - 1)}\right) \frac{s - 1}{k}. \tag{A.42}
\]
If we choose \( r, m \) such that \((r - 1)/(m - 1) < \varepsilon^2 k/(s - 1)\), and define \( \kappa = (s - 1)/k \) and \( \alpha = k/n \), then our instance is in \( I(\kappa, \alpha) \), and (A.42) implies that utilization is strictly smaller than \( 1 - \kappa + \varepsilon \).

\[\square\]

### A.2 Individual Lottery

#### A.2.1 Incentives

**Proof of Proposition 4.** This is a direct consequence of Proposition 18. Notice that for any given instance with \( k \) tickets, the Individual Lottery is equivalent to the Individual Lottery with limit \( \ell = k \). Therefore, \( r = \lceil |G|/k \rceil = 1 \) and our result follows. \[\square\]

**Proof of Proposition 5.** Consider any agent \( i \). We let \( a_{-i} \in A_{-i} \) be an arbitrary set of actions and \( a'_i > a_i \geq |G_i| \). We begin by showing that for any order over agents \( \sigma \), the conditional expected utility of \( i \) is the same under both strategies. We consider two possible cases. First, the number of tickets remaining before agent \( i \) is processed is \( a_i \) or less. Then, under both strategies the allocation of every agent is the same and the payoff of \( G_i \) coincide. Second, the number of tickets remaining before agent \( i \) is processed is greater than \( a_i \). Then, under both strategies agent \( i \) obtains at least \( |G_i| \) tickets and the group gets a payoff of 1.

Now, we will show that the utility of every group \( G \neq G_i \) is weakly better under \( a_i \). It suffices to show that for any order \( \sigma \in O_N \),

\[
x_f^IL((a_i, a_{-i}), \sigma) \geq x_f^IL((a'_i, a_{-i}), \sigma) \text{ for every agent } j \neq i.
\]

(A.43)

Because this holds for any order \( \sigma \), and the random order over agents used in the Individual Lottery is uniformly distributed, this implies our result. Let \( T = T(\sigma) \) be the position of agent \( i \) in \( \sigma \), that is, \( T = \{ t \in \{1, \ldots, n\} : \sigma_t = i \} \). The allocation of agents \( \sigma_1, \ldots, \sigma_{T-1} \) is not affected by the action of \( i \), then (A.43) holds. A smaller request can only lead to a smaller allocation, hence the allocation of agent \( i \) is weakly smaller under \( a_i \). Therefore, the allocation of agents \( \sigma_{T+1}, \ldots, \sigma_n \) is weakly greater under \( a_i \) as the only difference is due to agent \( i \). \[\square\]
A.2.2 Performance

**Proof of Theorem 2.** Consider an instance with \( n = rs \) agents divided into one large group of size \( s \) and \( s(r-1) \) small groups of size one. Besides, the number of tickets is \( k = \lfloor ar \rfloor s \). Observe that for any \( s, r \in \mathbb{N}, \)

\[
\frac{k}{n} = \frac{\lfloor ar \rfloor s}{rs} \leq \alpha.
\]

Thus, if \( (s-1)/k \leq \kappa \) then this instance will be in \( I(\kappa, \alpha) \). We claim that a sufficient condition for this is

\[ r \geq \frac{\kappa + 1}{\alpha \kappa}. \tag{A.44} \]

This can be seen as

\[
\frac{s-1}{k} = \frac{s-1}{\lfloor ar \rfloor s} \leq \frac{1}{\alpha r - 1} \leq \kappa.
\]

In the first inequality we use that for any \( x, \lfloor x \rfloor \geq x - 1 \) and \( (s-1)/s \leq 1 \). The last inequality follows from (A.44). For ease of exposition, in the proof sketch of Theorem 2 we consider \( n \) agents, a large group of size \( s = n^{3/4} \) and \( k = an \) tickets.

Let agents \( i, j \) be such that \( |G_i| = 1 \) and \( |G_j| = s \). We claim that for any dominant strategy equilibrium \( a \),

\[
u_i(\pi^{LL}(a)) \leq \frac{k}{s^2} \leq \frac{\alpha r}{s}. \tag{A.45}
\]

This bound is key to prove both guarantees. We start by proving the efficiency result. The expected utilization in this system is:

\[
\frac{1}{k} \sum_{i' \in N} u_i(\pi^{LL}(a)) = \frac{s}{k} \left( (r-1)u_i(\pi^{IL}(a)) + u_j(\pi^{IL}(a)) \right)
\]

\[
\leq \frac{r-1}{s} + \frac{s}{k}
\]

\[
= \frac{r-1}{s} + \frac{1}{\lfloor ar \rfloor}.
\]

In the inequality we use that \( u_j(\pi^{IL}(a)) \leq 1 \) and the first inequality in (A.45). Hence, if we choose
\( s = s(r) \) such that \( r/s(r) \to 0 \) as \( r \) grows, then the right side goes to 0 as we take the limit.

We now turn to the fairness guarantee. Because the first agent to be processed always get a payoff of 1, we get that

\[
u_j(\pi^{IL}(a)) \geq \frac{s}{n} = \frac{1}{r}.
\]

Note that this lower bound is independent of \( s \), and is tight when all agents in small groups request \( k \) tickets.

Using this and the second inequality in (A.45), we obtain

\[
\frac{u_i(\pi^{IL}(a))}{u_j(\pi^{IL}(a))} \leq \frac{\alpha r^2}{s}.
\]

Therefore, if we choose \( s = s(r) \) such that \( r^2/s(r) \to 0 \) as \( r \) grows, then again the right side goes to 0 as we take the limit.

All that remains is to prove (A.45). We let \( \Sigma \) be a random order over agents. To generate \( \Sigma \) we use Algorithm 4 from Lemma 8: set \( S = G_i \cup G_j \), and independently generate (i) a uniform random order \( \Sigma^S \) over \( S \), (ii) a uniform random order \( \Sigma^- \) over \( N \setminus S \), and (iii) uniform random positions \( P \subseteq \{1, \ldots, |N|\} \) where agents in \( S \) will be placed. By Lemma 8, the resulting order \( \Sigma \) is uniformly distributed. Note that \( i \) will get a payoff 0 unless it appears in the first \( k/s \) positions of \( \Sigma^S \). Because \( \Sigma^S \) is uniformly distributed this event occurs with probability

\[
\frac{k/s}{s + 1} \leq \frac{k}{s^2}.
\]

This implies (A.45) and concludes our proof. \( \square \)

A.2.3 Extension

**Proposition 18.** In the Individual Lottery with limit \( \ell \), the set of actions \( a_G \) is dominant for group \( G \) if and only if \( \sum_{i \in S} a_i \geq |G| \) for all \( S \subseteq G \) such that \( |S| = \lceil |G|/\ell \rceil \).

**Proof of Proposition 18.** Fix an arbitrary agent \( i \). Let \( a_{-G_i} \in A_{-G_i} \) be an arbitrary action profile
for agents not in $G_i$. We let $r = \lceil |G_i|/\ell \rceil$ be the minimum number of members of $G_i$ that must be awarded in the Individual Lottery with limit $\ell$ in order for $G_i$ to get a payoff of 1. Let $a_{G_i} \in A_{G_i}$ be any action profile such that $\sum_{j \in S} a_j \geq |G_i|$ for all $S \subseteq G_i$ such that $|S| = r$.

First, we show that for any order $\sigma \in O_N$ the utility of agent $i$ is maximized under $a_{G_i}$, that is,

$$u_i(x^{IL}((a_{G_i}, a_{-G_i}), \sigma)) \geq u_i(x^{IL}((a'_{G_i}, a_{-G_i}), \sigma)),$$

for every $a'_{G_i} \in A_{G_i}$. (A.46)

Because this holds for any order $\sigma$, the expected utility of group $G_i$ will also be maximized by $a_{G_i}$.

Let $T = T(\sigma)$ be the position of the $r^{th}$ member of $G_i$ under $\sigma$:

$$T(\sigma) = \min \{ t \in \{1, \ldots, n\} : |\sigma[t] \cap G_i| = r \}.$$

If $\sum_{j \in \sigma[T \setminus G_i]} a_j > k - |G_i|$, then for any action profile selected by group $G_i$ its payoff is 0.

If $\sum_{j \in \sigma[T \setminus G_i]} a_j \leq k - |G_i|$, then under action profile $a_{G_i}$ group $G_i$ receives a payoff of 1.

Hence, $a_{G_i}$ maximizes the payoff of agent $i$ for each $\sigma$.

Second, let $\hat{a}_{G_i}$ be such that $\sum_{j \in S} \hat{a}_j < |G_i|$ for some $S \subseteq G_i$ with $|S| = r$. We will show that there exists an order $\hat{\sigma} \in O_N$ such that

$$1 = u_i(x^{IL}((a_{G_i}, a_{-G_i}), \hat{\sigma})) > u_i(x^{IL}((\hat{a}_{G_i}, a_{-G_i}), \hat{\sigma})) = 0.$$

This combined with (A.46) implies that $\hat{a}_{G_i}$ is dominated by $a_{G_i}$. We construct $\hat{\sigma}$ in the following way:

- Agents in $S$ are arbitrary placed in the first $r$ positions of $\hat{\sigma}$.

- Agents in $N \setminus G_i$ are arbitrary placed in positions $r + 1, \ldots, r + n - |G_i|$ of $\hat{\sigma}$.

- Agents in $G_i \setminus S$ are arbitrary placed in the last $|G_i| - r$ positions of $\hat{\sigma}$.

We begin by proving that if $a_{G_i}$ is selected then $G_i$ received at least $|G_i|$ tickets, implying that
\( u_i(x^{IL}((a_{G_i}, a_{-G_i}), \hat{\sigma})) = 1 \). To see this note that the number of tickets received by \( G_i \) is

\[
\sum_{j \in G} x_j^{IL}((a_{G_i}, a_{-G_i}), \hat{\sigma}) \geq \sum_{j \in S} x_j^{IL}((a_{G_i}, a_{-G_i}), \hat{\sigma}) \geq \min\{k, \sum_{j \in S} a_j\} \geq |G_i|.
\]

The last inequality follows as \( k \geq |G_i| \) and \( \sum_{j \in S} a_j \geq |G_i| \). On the other hand, we show that when \( \hat{a}_{G_i} \) is selected then \( G_i \) received strictly less than \( |G_i| \) tickets and

\[
u_i(x^{IL}((\hat{a}_{G_i}, a_{-G_i}), \hat{\sigma})) = 0.
\]

For the sake of contradiction, suppose that \( |G_i| \) received at least \( |G_i| \) tickets, then

\[
\sum_{j \in N} x_j^{IL}((a_{G_i}, \hat{a}_{-G_i}), \hat{\sigma}) = \sum_{j \in G_i} x_j^{IL}((a_{G_i}, \hat{a}_{-G_i}), \hat{\sigma}) + \sum_{j \in N \setminus G_i} x_j^{IL}((a_{G_i}, \hat{a}_{-G_i}), \hat{\sigma}) \geq |G_i| + \sum_{j \in N \setminus G_i} 1 = n.
\]

A contradiction, as \( k < n \). Note that \( G_i \) will get \( |G_i| \) or more tickets only if agent in position \( r + n - |G_i| + 1 \) is awarded, this implies that all agents in the first \( r + n - |G_i| \) must also be awarded.

\[ \square \]

**Proof of Proposition 6.** Consider a sequence of instances with \( n \to \infty \) and a constant \( k \) number of tickets. In each instance, there is one group of size 1 and the remaining groups have size \( \ell + 1 \). We let \( i \) be such that \( |G_i| = 1 \) and \( j \) be such that \( |G_j| = \ell + 1 \). We let \( \Sigma \) be a uniform order over agents.

First, note that regardless of the action profile \( a \), \( i \) gets utility 1 if among the first \( \lceil k/\ell \rceil \) agents in \( \Sigma \), and gets utility 0 if after the first \( k \) agents. Because \( \Sigma \) is drawn uniformly at random from \( O_N \), we have

\[
\frac{\lceil k/\ell \rceil}{n} \leq u_i(\pi^{IL}(a)) \leq \frac{k}{n}.
\]  

(A.47)

Second, because \( \ell < |G_j| \), at least two agents from group \( G_j \) must be awarded in order for the group to get utility 1. Furthermore, any agent not among the first \( k \) will certainly not receive any tickets. Therefore, group \( G_j \) gets utility 1 only if some pair of agents from \( G_j \) are both among the first \( k \) agents. For any pair of agents, the chance that both are among the first \( k \) agents is \( \binom{n-2}{k-2}/\binom{n}{k} \).

Applying a union bound, we see that

\[
u_j(\pi^{IL}(a)) \leq \frac{\binom{n-2}{k-2}/\binom{n}{k}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)} \frac{\ell + 1}{2}.
\]  

(A.48)
Combining the upper bounds derived in (A.47) and (A.48), we bound the overall efficiency as follow

\[
\frac{1}{k} \sum_{i' \in \mathbb{N}} u_{i'}(\pi^{IL}(a)) \leq \frac{1}{k} \left( \frac{k}{n} + (n-1) \frac{k(k-1)}{n(n-1)} \left( \frac{\ell + 1}{2} \right) \right) = \frac{2 + (k-1)(\ell + 1)\ell}{2n},
\]

which approaches zero as \(n\) grows.

Furthermore, (A.47) and (A.48) imply that

\[
\frac{u_j(\pi^{IL}(a))}{u_i(\pi^{IL}(a))} \leq \frac{k(k-1)}{n(n-1)} \frac{(\ell + 1)}{k/\ell} = \frac{k(k-1)}{[k/\ell](n-1)} \frac{(\ell + 1)}{2},
\]

which also approaches zero as \(n\) grows. \(\square\)

**Proposition 19.** For any \(\ell \in \mathbb{N}\) and any instance such \(\max_{G \in \mathcal{G}} |G| \leq \ell\), every dominant strategy equilibrium outcome of the Individual Lottery with limit \(\ell\) is \(1/\ell\)-efficient.

**Proof of Proposition 19.** We let \(\Sigma\) be a uniform order over agents. We claim that if \(a_{G_i}\) is dominant for \(G_i\), then

\[
\mathbb{E}[u_i(x^{IL}(a, \Sigma))] \geq \frac{k}{\ell n}. \tag{A.49}
\]

From this, it follows that if \(a\) is such that all agents follow a dominant strategy, then

\[
\mathbb{E}[U(x^{IL}(a, \Sigma))] = \frac{1}{k} \sum_{i \in \mathbb{N}} \mathbb{E}[u_i(x^{IL}(a, \Sigma))] \geq \frac{1}{\ell}.
\]

We now prove (A.49). If \(|G_i| = 1\), then no matter the reports of others, \(i\) succeeds if in the first \([k/\ell]\) positions, which occurs with probability \(\frac{[k/\ell]}{n} \geq \frac{k}{\ell n}\).

Otherwise, because \(\max_{G \in \mathcal{G}} |G| \leq \ell\) and agents in \(G_i\) follow a dominant strategy, \(i\) succeeds if any agent from \(G_i\) is in the first \([k/\ell]\) positions.

If \(k/\ell < 2\), then this occurs with probability \(\frac{|G_i|}{n} \geq \frac{k}{\ell n}\). Thus, we turn to the case with \(\min(|G_i|, k/\ell) \geq 2\). Fix two agents in \(G_i\). The chance that at least one of them is in the first
\[ \left\lfloor \frac{k}{\ell} \right\rfloor \text{ positions is} \]
\[
\frac{2\left\lfloor \frac{k}{\ell} \right\rfloor}{n} - \left( \frac{n-2}{n} \right)^2 \left( \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \right)^2 
\geq \frac{k}{\ell} + 1 - \left( \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \right)^2 
= \frac{k}{\ell} - 1 + \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \left( 1 - \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \right). 
\]

All that remains is to establish that
\[
\frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \left( 1 - \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \right) \geq \frac{1}{n}. 
\]

This holds because \( \left\lfloor \frac{k}{\ell} \right\rfloor \geq 2 \) by assumption, and \( 1 - \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{n} \geq 1 - 1/\ell \geq 1/2. \)

**Proposition 20.** For any \( \ell \in \mathbb{N} \) and any instance such that \( \max_{G \in G} |G| \leq \ell \), every dominant strategy equilibrium outcome of the Individual Lottery with limit \( \ell \) is \( 1/\ell \)-fair.

**Proof of Proposition 20.** We construct a random order over agents \( \Sigma \) using Algorithm 4: set \( S = G_i \cup G_j \), and independently generate (i) a uniform random order \( \Sigma^S \) over \( S \), (ii) a uniform random order \( \Sigma^- \) over \( N \setminus S \), and (iii) uniform random positions \( P \subseteq \{1, \ldots, |N|\} \) where agents in \( S \) will be placed. By Lemma 8, the resulting order \( \Sigma \) is uniformly distributed.

Without loss of generality, we assume
\[
\ell \geq |G_i| \geq |G_j|. \tag{A.50}
\]

We let
\[
\tau_i(\Sigma^-) = \tau(k - |G_i| + 1, \Sigma^-), \tag{A.51}
\]
\[
\tau_j(\Sigma^-) = \tau(k - |G_j| + 1, \Sigma^-) - \tau(k - |G_i| + 1, \Sigma^-), \tag{A.52}
\]
be as defined in (2.6) where the size of each agent is its request, that is, \( |\sigma_i| = a_{\sigma_i} \). Note that by
definition,

\[ 1 \leq \tau_i(\Sigma^-), \quad \text{and} \quad \tau_j(\Sigma^-) \leq |G_i| - |G_j|. \quad (A.53) \]

In addition, for \( s \in \{1, \ldots, |S|\} \) let \( T_s(P) \) be the \( s^{th} \) smallest value in \( P \), so \( T_1(P) \) denotes the first position of \( \Sigma \) containing a member of \( G_i \cup G_j \). Note that

\[
P(T_1(P) = t) = \left( \frac{|G_i| + |G_j|}{n} \right) \left( \frac{n-t}{|G_i|+|G_j|-1} \right),
\]

which is decreasing in \( t \). From this, it follows that for any \( \Sigma^- \),

\[
\frac{P(T_1 \leq \tau_i + \tau_j|\Sigma^-)}{P(T_1 \leq \tau_i|\Sigma^-)} \leq \frac{\tau_i + \tau_j}{\tau_i} \leq 1 + |G_i| - |G_j|, \quad (A.54)
\]

where the second inequality comes from (A.53). Our final definition is to let

\[
A_i = \{ \Sigma_1^S \in G_i \}, \quad A_j = \{ \Sigma_1^S \in G_j \}, \quad (A.55)
\]

and note that

\[
P(A_i) = \frac{|G_i|}{|G_i| + |G_j|} = 1 - P(A_j). \quad (A.56)
\]

Definitions out of the way, we proceed with the proof. Note that

\[
\frac{u_j(\pi^{IL}(a))}{u_i(\pi^{IL}(a))} = \frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-]} \leq \max_{\sigma^-} \frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^- = \sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^- = \sigma^-]}.
\]

Therefore, to establish \( 1/\ell \) fairness, it suffices to show that for every \( \Sigma^- \),

\[
1 \leq \frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-]} \leq \ell. \quad (A.57)
\]

We claim that

\[
P(A_i)P(T_1 \leq \tau_i|\Sigma^-) \leq \mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-] \leq P(T_1 \leq \tau_i|\Sigma^-). \quad (A.58)
\]
The left inequality follows because whenever \( T_1 \leq \tau_i \) and a member of \( G_i \) comes before all members of \( G_j \), group \( G_i \) gets a payoff of 1. The right inequality follows because the definition of \( \tau_i \) ensures that \( G_i \) can get a payoff of one only if a member of \( G_i \) is in the first \( \tau_i \) positions of \( \Sigma \). By analogous reasoning, we have

\[
P(A_j)P(T_1 \leq \tau_i|\Sigma^-) \leq \mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-] \leq P(T_1 \leq \tau_i + \tau_j|\Sigma^-) - P(A_j)P(\tau_i < T_1 \leq \tau_i + \tau_j|\Sigma^-),
\]

where the right inequality follows because in order for group \( G_j \) to get utility one, we must have \( T_1 \leq \tau_i + \tau_j \), and if \( T_1 \in (\tau_i, \tau_i + \tau_j] \), then a member of \( G_j \) must appear before all members of \( G_i \).

We now prove the upper-bound in (A.57). Combining (A.58) and (A.59), we see that

\[
\frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-]} \leq \frac{P(T_1 \leq \tau_i + \tau_j|\Sigma^-) - P(A_j)P(\tau_i < T_1 \leq \tau_i + \tau_j|\Sigma^-)}{P(A_j)P(T_1 \leq \tau_i|\Sigma^-)}
= \frac{P(T_1 \leq \tau_i + \tau_j|\Sigma^-)(1 - P(A_j)) + P(A_j)P(T_1 \leq \tau_i|\Sigma^-)}{P(A_j)P(T_1 \leq \tau_i|\Sigma^-)}
\]

\[
\leq \frac{|G_j|P(T_1 \leq \tau_i + \tau_j|\Sigma^-)}{|G_i|P(T_1 \leq \tau_i|\Sigma^-)} + 1
\]

\[
\leq \frac{|G_j| - |G_j|^2}{|G_i|} + |G_j| + 1
\]

\[
\leq \ell. \quad (A.60)
\]

The second inequality uses (A.54). The final inequality follows because if \( |G_j| = \ell \), then \( |G_i| = \ell \) by (A.50), and thus the expression is equal to 2;\(^1\) if \( |G_j| < \ell \), then the expression is at most \( \ell \) because \( \frac{|G_j| - |G_j|^2}{|G_i|} \leq 0 \). Meanwhile, (A.58) and (A.59) also imply that

\[
\frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-]} \geq P(A_j) = \frac{|G_j|}{|G_i| + |G_j|}.
\]

If \( |G_i| < \ell \) or \( |G_j| > 1 \), the ratio on the right is at least \( 1/\ell \), and the proof is complete. Thus, all that remains is to show that the lower bound in (A.57) holds when \( |G_i| = \ell \) and \( |G_j| = 1 \).

\(^1\)We assume \( \ell \geq 2 \) because if \( \ell = 1 \) and all groups have size one, the individual lottery simply selects \( k \) agents uniformly at random, and is perfectly fair.
Our analysis will condition on both $\Sigma^-$ and $P$. We note that

$$\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-, P] > 0 \iff T_i(P) \leq \tau_i(\Sigma^-).$$

Therefore,

$$\frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-]} = \frac{\mathbb{E}_P[\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-, P]]}{\mathbb{E}_P[\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-, P]]} \geq \min_{P: T_i(P) \leq \tau_i(\Sigma^-)} \mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-, P].$$

We will show that the quantity on the right is at least $1/\ell$. To do this, we let

$$\tau_{ij}(\Sigma^-) = \tau(k - |G_j| - |G_i| + 1, \Sigma^-),$$

be as defined in (2.6) with size function $|\sigma| = a_{\sigma_i}$. If $T_2(P) \leq \tau_{ij}(\Sigma^-)$, then because each agent requests at most $\ell = |G_i|$ tickets, agent $j$ will receive utility of one if first or second in $\Sigma^S$. Thus,

$$\frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-]} \geq \mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-, P] \geq \frac{2}{\ell + 1} \geq \frac{1}{\ell}.$$

Meanwhile, if $T_2(P) > \tau_{ij}(\Sigma^-)$, then each group gets utility 1 only if one of its members is first in $\Sigma^S$. In this case,

$$\frac{\mathbb{E}[u_j(x(a, \Sigma))|\Sigma^-, P]}{\mathbb{E}[u_i(x(a, \Sigma))|\Sigma^-, P]} = \frac{\mathbb{P}(A_j)}{\mathbb{P}(A_i)} = \frac{1}{\ell}.$$

\hfill \Box

A.3 Weighted Individual Lottery

A.3.1 Incentives

**Proposition 21.** Algorithm 1 generates a random order $\Sigma \in O_N$ distributed according to (2.18) conditional on $a$.

**Proof of Proposition 21.** Fix any order $\sigma$ over $N$. Let $Y_j = a_j X_j$. It follows that $\mathbb{P}(Y_j > t) = \ldots$
\(e^{-t/a_j}\), so each \(Y_j\) is distributed as an exponential random variable with mean \(a_j\). Moreover, the \(Y_j\) are independent. Let \(\Sigma\) be the order generated by the algorithm. We have that

\[
P(\Sigma_1 = j) = P(Y_j = \min_{i \in \mathcal{N}} Y_i) = \frac{1/a_j}{\sum_{i \in \mathcal{N}} 1/a_i},
\]

where the second equality follows from well-known properties of the minimum of exponential random variables.\(^2\) Furthermore, the definition of \(\Sigma\) and the memoryless property of exponential random variables imply that for \(t \in \{1, \ldots, n\}\) and \(j \notin \Sigma_{[t-1]}\),

\[
P(\Sigma_t = j|\Sigma_{[t-1]}) = P(Y_j = \min_{i \in \mathcal{N}\setminus\Sigma_{[t-1]}} Y_i) = \frac{1/a_j}{\sum_{i \in \mathcal{N}\setminus\Sigma_{[t-1]}} 1/a_i}.
\]

This implies that

\[
\Pr(\Sigma = \sigma) = \prod_{t=1}^{n} P(\Sigma_t = \sigma_t|\Sigma_{[t-1]} = \sigma_{[t-1]}) = \prod_{t=1}^{n} \frac{1/a_{\sigma_t}}{\sum_{i \in \mathcal{N}\setminus\sigma_{[t-1]}} 1/a_i},
\]

as claimed. \(\square\)

**Proof of Proposition 7.** We start by proving that agents have no incentives to request more tickets than their group size. Formally, if we let \(i \in G\), \(a_i = |G|\) and \(a'_i > |G|\) then for every action profile \(a_{-i} \in A_{-i}\),

\[
u_i(\pi^{IW}(a_i, a_{-i})) \geq u_i(\pi^{IW}(a'_i, a_{-i})).
\]

This follows because the set of orders over agents in which \(G\) get a payoff of 1 is the same under both strategies, and by reducing its request agent \(i\) improves her probability of being drawn early.

We now show that if group \(G\) is such that \(|G| \leq 3\), then selecting group request \(a_G\) is dominant for \(G\). Given an action profile \(a \in A\), we generate a random order over agents \(\Sigma\) using the Algorithm 1: we draw iid exponential random variables \(X_i\) for each agent \(i\), and sort agents in increasing order according to \(a_iX_i\). From Proposition 21, it follows that \(\Sigma\) is distributed according

\(^2\)See e.g. https://en.wikipedia.org/wiki/Exponential_distribution.
to (2.18) conditional on \(a\). Let \(T\) be as in Definition (2.19), intuitively \(T\) is the score threshold that some members of \(G\) must clear in order to ensure the group a payoff of 1. Furthermore, when \(G\) is selecting the group request strategy, it will get a payoff of 1 if and only if at least one of its members has a score lower than \(T\), that is, \(\min_{i \in G} \{a_i X_i\} < T\). Because \(\min_{i \in G} \{a_i X_i\} \sim \text{Exp}(1)\), it follows that for \(i \in G\),

\[
\mathbb{E}[u_i(\pi^{IW}(a_G, a_{-G}))|T] = \mathbb{P}(\min_{i \in G} \{a_i X_i\} < T) = 1 - e^{-T}.
\]  

(A.63)

Because \(T\) is independent of the strategy followed by \(G\), it suffices to show that for any deviation \(a_G'\) the conditional expected utility of \(G\) given \(T\) is less than or equal the right side of (A.63).

We have already established that it is never beneficial for agents to request more tickets than their group size. Hence, without loss of generality we assume that each member of \(G\) will request at most \(|G|\) tickets.

If \(|G| = 1\), then the group request is the only feasible strategy so it is dominant.

If \(|G| = 2\), then the only deviations we need to consider are \(a_G' = (1, 2), (1, 1)\). The first strategy is dominated by the group request, because the allocation of the member requesting 1 ticket is irrelevant for the outcome of group \(G\). Under the second strategy, \(G\) gets a payoff of 1 if and only if both members have a score lower than \(T\). In particular, agent \(i \in G\) must have a score lower than \(T\). This happens with probability

\[
\mathbb{P}(a_i' X_i < T) = \mathbb{P}(X_i < T) = 1 - e^{-T}.
\]

Note that the quantity above coincides with (A.63), implying that the utility of \(G\) when selecting \(a_G' = (1, 1)\) is at most its utility under the group request strategy.

If \(|G| = 3\), there are 27 feasible strategies (26 deviations from the group request), but by symmetry we only need to evaluate 9 of them:

\[
a_G' = (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3), (1, 3, 3), (2, 2, 2), (2, 2, 3), (2, 3, 3).
\]
We argue now that the group request dominates all strategies above in which there is at least one agent requesting 1 ticket. Note that under any of these strategies, \( G \) will get a payoff of 1 only if the remaining 2 members are awarded two or more tickets. From the case \(|G| = 2\), we know that the probability of this event is at most the right hand side of (A.63). This implies that the group request strategy dominates all these deviations.

There are only 3 strategies remaining: \( a'_G = (3, 3, 2), (3, 2, 2), (2, 2, 2) \). The first strategy is dominated by the group request, because the allocation of the member requesting 2 tickets is irrelevant for the outcome of group \( G \). The second strategy is dominated by \( (3, 2, 1) \). This follows because the set of orders over agents in which \( G \) get a payoff of 1 is the same under both strategies, and by reducing its request the last agent improves her probability of being drawn early. A similar argument shows that the last strategy is dominated by \( (1, 2, 2) \). □

**Analysis of Example 1.** Let \( i \) be a member of the large group. We let \( a = (a_G, a_{-G}) \) denote the group request action profile, and \( a'_G \) denote the strategy where all members of \( G_i \) request 2 tickets. We will show that for \( n \geq 17 \),

\[
\pi^{IW}(a'_G, a_{-G}) \geq \pi^{IW}(a_G, a_{-G}).
\]

Let \( m = n - 3 \) be the number of groups. We claim that

\[
u_i(\pi^{IW}(a_G, a_{-G})) = 1 - \frac{1}{m}, \tag{A.64}
\]

\[
u_i(\pi^{IW}(a'_G, a_{-G})) = 1 - \sum_{i=1}^{m} \left( \frac{m-i}{m+2-i} \right) \left( \frac{2}{m+2-i} \right) \left( \prod_{i=1}^{m-1} \frac{m-i}{m+3/2-i} \right). \tag{A.65}
\]

This implies our result as for \( m \geq 14 \) the expression in (A.65) exceeds the expression in (A.64).

First, we will show (A.64). Because \( G_i \) is selecting the group request strategy, it will get a payoff of 0 if and only if all agents from small groups are processed before its members. This event happens with probability

\[
\prod_{i=1}^{m-1} \frac{m-i}{m+1-i} = \frac{1}{m}.
\]
Secondly, we show (A.65). If all members of $G_i$ are requesting 2 tickets, then $G_i$ will get a payoff of 0 if and only if three of its members are processed after all agents in small groups. Moreover, the probability that at step $t = 1, \ldots, m$ a member of $G_i$ is processed for the first time is

$$\left( \prod_{i=1}^{t-1} \frac{m-i}{m+2-i} \right) \left( \frac{2}{m+2-t} \right). \quad (A.66)$$

In the expression above we used that $\sum_{i \in G} 1/a_i' = 2$, and that at the beginning of step $i \leq t$, there are $m + 2 - i$ agents in small groups that have not been processed yet. Note that if $t = 1$, then the expression above reduces to the probability of processing a member of the large group at the first step, that is, $2/(m+1)$.

Finally, the probability that all the remaining agents in small groups are processed before the three remaining members of the large group is

$$\prod_{i=t}^{m-1} \frac{m-i}{m+3/2-i}. \quad (A.67)$$

Here we are using that if $j \in G$ was processed at step $t$, then $\sum_{i \notin G \setminus \{j\}} 1/a_i' = 3/2$. Note that if $t = m$, then the expression above is 1.

Multiplying (A.66) by (A.67) and summing all possible values of $t$ yields (A.65). \hfill \Box

**Proof of Proposition 8**

**Lemma 12.** Group $G$ gets a payoff of 1 if and only if

$$\sum_{i \in G} a_i \mathbf{1}(a_iX_i < T) \geq |G|. \quad (A.68)$$

**Proof of Lemma 12.** First, suppose that (A.68) holds. From the definition of $T$ in (2.19), it follows that at most $k - |G|$ tickets are allocated to agents not in $G$ who have a score lower than $T$. Furthermore, as (A.68) holds it must be the case that the sum of the requests of agents in $G$ who have a score lower than $T$ is at least $|G|$. Therefore, group $G$ is awarded $|G|$ or more tickets.
Conversely, suppose that (A.68) does not hold. We will consider two cases:

(i) Only agents with score lower than $T$ are awarded.

(ii) There are agents with score $T$ or higher that are awarded.

Assume first that (i) holds. Then as (A.68) doesn’t hold individuals in $G$ must receive fewer than $|G|$ tickets.

Assume now that (ii) holds. From the definition of $T$ in (2.19), it follows that individuals not in $G$ must receive strictly more than $k - |G|$ tickets. This implies that individuals in $G$ must receive fewer than $|G|$ tickets.

\[\square\]

**Lemma 13.** Fix an arbitrary group $G$. Let $r \in \{0, \ldots, |G| - 1\}$. For every strategy $a_G \in B_r$, it follows that

\[
\sum_{i \in G} \frac{1}{a_i} \leq r + 1. \tag{A.69}
\]

**Proof of Lemma 13.** For simplicity, we shall assume that $G = \{1, \ldots, s\}$ and $a_1 \leq a_2 \leq \cdots \leq a_s$.

Thus, a strategy $a_G \in \mathbb{N}^s$ is in $B_r$ if and only if

\[
\sum_{i=1}^{r+1} a_i \geq s \quad \text{and} \quad \sum_{i=s-r+1}^{s} a_i \leq s - 1. \tag{A.70}
\]

Consider the following optimization problem:

\[
\begin{aligned}
\max & \quad \sum_{i=1}^{s} 1/a_i \\
\text{subject to} & \quad \sum_{i=1}^{r+1} a_i \geq s \\
& \quad \sum_{i=s-r+1}^{s} a_i \leq s - 1. \\
& \quad 1 \leq a_1 \leq \cdots \leq a_s
\end{aligned} \tag{A.71}
\]

Note that from (A.70) it follows that every strategy $a_G \in B_r$ is a feasible solution for this problem. Therefore, to prove (A.69) it suffices to show that the optimal value of this problem is at most $r + 1$. 

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We start by proving that an optimal solution $\mathbf{a}_G^*$ of (A.71) must satisfy

$$a_j^* = s - \sum_{i=1}^{r} a_i^* \text{ for every } j = r + 1, \ldots, s. \quad (A.72)$$

Suppose $\mathbf{a} \in B_r$ is such that $a_j > s + \sum_{i=1}^{r} a_i$ for some $j \in \{r + 1, \ldots, s\}$. If we replace $a_j$ by $a_j' = s - \sum_{i=1}^{r} a_i$ then we increase the objective value as $1/a_j < 1/a_j'$. Moreover, $\mathbf{a}'$ will still be in $B_r$ as $\sum_{i=1}^{r+1} a_i' = \sum_{i=1}^{r} a_i + a_j' = s$ and $\sum_{i=r+1}^{s} a_i' \leq \sum_{i=r+1}^{s} a_i \leq s - 1$. The last inequality follows as $\mathbf{a} \in B_r$.

It follows from (A.72) that we can incorporate in (A.71) the constraints

$$a_{r+1} = \cdots = a_s = s - \sum_{i=1}^{r} a_i,$$

without decreasing the optimal value. Moreover, if we remove the constraint $\sum_{i=s-r+1}^{s} a_i \leq s - 1$ then the optimal value will be higher or the same. By including both modifications we obtain the following relaxation of (A.71):

$$\begin{align*}
\max & \quad \sum_{i=1}^{r} \frac{1}{a_i} + \frac{(s-r)}{a_{r+1}} \\
\text{subject to} & \quad \sum_{i=1}^{r+1} a_i = s \\
& \quad 1 \leq a_1 \leq \cdots \leq a_{r+1} \quad (A.73)
\end{align*}$$

Clearly an optimum of (A.73) exists as the objective function is continuous and the feasible set is non-empty and compact. Moreover, we are maximizing a convex function on a convex set then there exists a globally optimal solution that is an extreme point of the feasible set. The extreme points of the feasible set are $\mathbf{a}^0, \ldots, \mathbf{a}^r$, where

$$1 = a_1^j = \cdots = a_j^j, \quad \frac{s-j}{r+1-j} = a_{j+1}^j = \cdots = a_{r+1}^j.$$ 

Furthermore, the objective value evaluated at any extreme point is equal to $r + 1$. To see this note
that objective value at $a^j$ is
\[ j + (s - j) \left( \frac{r + 1 - j}{s - j} \right) = r + 1. \]

Therefore, the optimal value of (A.73) is $r + 1$. Because (A.73) is a relaxation of (A.71), it follows that the optimal value of (A.71) is at most $r + 1$. □

Proof of Proposition 8. Let $r \in \{0, \ldots, s - 1\}$. We formulate the problem of finding the strategy in $B_r$ that maximizes the expected payoff of $G$ given the threshold $T$ as a programming problem. From Lemma 12 and since we are considering only strategies in $B_r$, it follows that group $G$ will get a payoff of 1 if and only if there are $r + 1$ or more agents with a score lower than $T$. For each agent $i \in G$, we let $B_i$ be a random variable that indicates if the score of agent $i$ is lower than $T$, more precisely, $B_i = 1(a_i X_i < T)$. Observe that given $T$ and any action $a_i$, because $X_i \sim \text{Exp}(1)$ then $B_i \sim \text{Bernoulli}(1 - e^{-T/a_i})$. Hence, our formulation is

\[
\begin{align*}
\max & \quad \mathbb{P}(\sum_{i \in G} B_i \geq r + 1) \\
\text{subject to} & \quad B_i \sim \text{Bernoulli}(1 - e^{-T/a_i}) \quad \forall i \in G \\
& \quad a_G \in B_r
\end{align*}
\]  

(A.74)

Let $Z_i$ be the Poisson random variable of rate $T/a_i$. Note that $Z_i$ first-order stochastically dominates $B_i$. Hence, the following problem is a relaxation of (A.74).

\[
\begin{align*}
\max & \quad \mathbb{P}(\sum_{i \in G} Z_i \geq r + 1) \\
\text{subject to} & \quad Z_i \sim \text{Poisson}(T/a_i) \quad \forall i \in G \\
& \quad a_G \in B_r
\end{align*}
\]  

(A.75)

Using that the sum of independent Poisson random variables is Poisson-distributed, we have that $\sum_{i \in G} Z_i \sim \text{Poisson}(\sum_i T/a_i)$. Moreover, if $X \sim \text{Poisson}(\lambda)$ then Johnson, Kemp, and Kotz (2005) state the following bound:

\[
\mathbb{P}(X \geq x) \leq 1 - e^{-\lambda/x}, \quad x \geq \lambda.
\]  

(A.76)
If \( T \leq 1 \), then Lemma 13 implies

\[
\sum_{i \in G} \frac{T}{a_i} \leq \sum_{i \in G} \frac{1}{a_i} \leq r + 1. \tag{A.77}
\]

Therefore, we can apply (A.76) to obtain

\[
P(\sum_{i \in G} Z_i \geq r + 1) \leq 1 - e^{-(\sum_i T/a_i)/(r+1)} \leq 1 - e^{-T}. \tag{A.78}
\]

The last inequality follows from Lemma 13.

From (2.21) we have that \( 1 - e^{-T} \) correspond to the utility of \( G \) under the group request strategy. This implies our result as the optimal value of the relaxation (A.75) is at most the utility under the group request strategy.

\[\square\]

A.3.2 Performance

**Proof of Theorem 3**

*Proof of Theorem 3.* In this proof, whenever we study a mechanism we assume that the action profile selected \( a \) is its corresponding group request strategy.

We start by proving the efficiency guarantee. From Proposition 2, we have that for any instance the utilization under the Weighted Individual Lottery is at least the utilization under the Group Request with Replacement. This can be seen as

\[
U(\pi^{IW}(a)) = \frac{\sum_{i \in N} u_i(\pi^{IW}(a))}{k} \geq \frac{\sum_{i \in N} u_i(\pi^{GR}(a))}{k} = U(\pi^{GR}(a)) \tag{A.79}
\]

The inequality follows from (2.24). Therefore, it suffices to show that for any instance in \( I(\kappa, \alpha) \), the Group Lottery with Replacement is \( (1-\kappa)g(\alpha) \)-efficient. This follows immediately by Lemma 3:

\[
U(\pi^{GR}(a)) = \frac{\sum_{i \in N} u_i(\pi^{GR}(a))}{k} \geq \frac{n \left( \frac{k}{n} (1-\kappa) g(\alpha) \right)}{k} = (1-\kappa) g(\alpha). \tag{A.80}
\]
Now we turn to the fairness guarantee. From Proposition 2, we have that for any instance and any pair of agents $i, j$,

$$
\frac{u_i(\pi^{IW}(a))}{u_j(\pi^{IW}(a))} \geq \frac{u_i(\pi^{GR}(a))}{u_j(\pi^{GL}(a))}.
$$

(A.81)

Moreover, combining Lemma 3 and Lemma 1 yields

$$
\frac{u_i(\pi^{GR}(a))}{u_i(\pi^{GL}(a))} \geq \frac{k}{n}(1 - \kappa)g(\alpha) \geq (1 - \kappa)^2 g(\alpha) \geq (1 - 2\kappa)g(\alpha).
$$

(A.82)

The second inequality follows from the fact that for any $x \geq 0$,

$$
\frac{1}{1 + x} \geq 1 - x.
$$

□

Proof of Lemma 2

Let $S_N$ be the set of finite sequences of agents and draw the random sequence $\Sigma \in S_N$ by letting $\Sigma_t$ be iid with

$$
P(\Sigma_t = i) = \frac{1/|G_i|}{\sum_{j \in N} 1/|G_j|}.
$$

(A.83)

stopping once all agents have been drawn at least once, that is, for each $i \in N$ there exists $t$ such that $i = \Sigma_t$. This occurs with probability one, implying that this procedure generates a valid distribution over $S_N$.

Define $\sigma^{GR} : S_N \rightarrow S_G$ by

$$
\sigma^{GR}_j(\Sigma) = G_{\Sigma_j}.
$$

(A.84)

Define $\sigma^{IW} : S_N \rightarrow O_N$ by

$$
T_{j}^{IW}(\Sigma) = \min\{t \in \mathbb{N} : |\Sigma_{[t]}| = j\},
$$

$$
\sigma^{IW}_j(\Sigma) = \Sigma_{T_{j}^{IW}(\Sigma)}.
$$

(A.85)
Note that for each $\Sigma \in \mathcal{S}$ and each $t \in \mathbb{N}$, $\sigma^{GR}_{[t]}(\Sigma) \subseteq \mathcal{G}$. Define $\sigma^{GL} : \mathcal{S} \rightarrow \mathcal{O}_{\mathcal{G}}$ by

$$
T^G_L(j) = \min\{t \in \mathbb{N} : |\sigma^{GR}_{[t]}(\Sigma)| = j\}.
$$

$$
\sigma^G_L(j)(\Sigma) = \sigma^{GR}_{T^G_L(j)(\Sigma)}(\Sigma).
$$

(A.86)

**Proposition 22.** Let $\sigma^{GR}, \sigma^{IW}, \sigma^{GL}$ be as in (A.84), (A.85), (A.86). If $\Sigma$ is drawn according to (A.83), then

- $\sigma^{GR}(\Sigma)$ a sequence of $k$ elements in $\mathcal{G}$, where each element is independently and uniformly sampled with replacement from $\mathcal{G}$.

- $\sigma^{IW}(\Sigma)$ is an order over $\mathcal{N}$ distributed as in (2.18) given a group request action profile.

- $\sigma^{GL}(\Sigma)$ is a uniform order over $\mathcal{G}$.

**Proof of Proposition 22.** From the definition of $\sigma^{IW}(\Sigma)$, we know that it skips every agent in $\Sigma$ that has already appeared. Hence, we are sequentially sampling agents without replacement, with probability inversely proportional to the size of its groups. Therefore, it correspond to an order over agents distributed according to (2.18) when each agent is requesting its group size.

From (A.83) it follows that for each $G \in \mathcal{G}$ and each $t$, $\mathbb{P}(\Sigma \in G) = 1/|\mathcal{G}|$. That is, the marginal distribution over groups is uniform. It immediately follows from the definition of $\sigma^{GLR}(\Sigma)$ that it is sampling groups uniformly at random with replacement. Moreover, from the definition of $\sigma^{GL}(\Sigma)$ we know that it skip every agent in $\Sigma$ whose group has already appeared. Therefore, we are sampling groups uniformly at random without replacement, generating a uniform order over $\mathcal{G}$. □

**Proof of Lemma 2.** Let $\Sigma$ be drawn according to (A.83), and $\sigma^{GR}, \sigma^{IW}, \sigma^{GL}$ be as in (A.84),
From Proposition 22 it follows that

\[ u_i(x^{GR}(a)) = \mathbb{E}[u_i(x^{GL}(a, \sigma^{GR}(\Sigma)))], \]
\[ u_i(x^{IW}(a)) = \mathbb{E}[u_i(x^{IW}(a, \sigma^{IW}(\Sigma)))], \]
\[ u_i(x^{GL}(a)) = \mathbb{E}[u_i(x^{GL}(a, \sigma^{GL}(\Sigma)))]. \]

Therefore, it suffices to show that for any realization of \( \Sigma \),

\[ u_i(x^{GL}(a, \sigma^{GR}(\Sigma))) \leq u_i(x^{IW}(a, \sigma^{IW}(\Sigma))) \leq u_i(x^{GL}(a, \sigma^{GL}(\Sigma))). \] (A.87)

Observe that given \( \Sigma \), each of the utilities above is either 0 or 1. Hence, to prove (A.87) we will show that: (i) if the utility of \( i \) under the Group Lottery with Replacement is 1 then its utility under the Weighted Individual Lottery is also 1, and (ii) if the utility of agent \( i \) under the Weighted Individual Lottery is 1 then its utility under the Group Lottery is also 1. Because agents are playing the group request strategy, whenever a group or agent is being processed, it is given a number of tickets equal to the minimum of its group size and the number of remaining tickets.

If the utility of \( i \) under the Group Lottery with Replacement is 1, then the number of tickets allocated before \( G_i \) is processed is at most \( k - |G_i| \). Formally, if we let \( t \) be the first time at which a member of \( G_i \) appears in \( \Sigma \), then

\[ \sum_{j=1}^{t-1} |G_{\Sigma_j}| \leq k - |G_i|. \] (A.88)

In the left hand side, we use that the sequence of groups \( \sigma^{GR}(\Sigma) \) is determined by replacing each agent in \( \Sigma \) by its group. In contrast, in the Weighted Individual Lottery, the order over agents \( \sigma^{IW}(\Sigma) \) is constructed by skipping all agents in \( \Sigma \) that have already appeared. Hence, in this mechanism the number of tickets allocated before \( \Sigma_t \) appears in \( \sigma^{IW}(\Sigma) \) is the same or lower than the left hand side of (A.88). This implies that the utility of \( i \) under the Weighted Individual Lottery is also 1.
Meanwhile, suppose that the utility of $i$ under the Weighted Individual Lottery is 1, then the number of tickets allocated before $\Sigma_i$ appears in $\sigma^{IW}(\Sigma)$ is at most $k - |G_i|$. In the Group Lottery, the order over groups $\sigma^{GL}(\Sigma)$ is constructed by replacing each agent in $\Sigma$ by its group, and skipping all groups that have already appeared. Note that each time $\sigma^{IW}(\Sigma)$ skips an agent $\Sigma_j$, $\sigma^{GL}(\Sigma)$ also skips the group $G_{\Sigma_j}$. Therefore, the number of tickets allocated before $G_{\Sigma_i}$ appears in $\sigma^{GL}(\Sigma)$ is the same or lower than the number of tickets allocated before $\Sigma_i$ appears in $\sigma^{IW}(\Sigma)$. Implying that the utility of agent $i$ under the Group Lottery is also 1. □

**Proposition 23.** Given a set $V$ of $m$ elements and a natural number $k$, the following algorithm generates a sequence $\Sigma$ of $k$ elements where $\Sigma_i$ is independent and uniformly sampled from $V$:

1. Select an element $G$ of $V$.

2. Generate a sequence of $k$ elements $\Sigma^-$, where $\Sigma^-_i$ is independently and uniformly draw from $V \setminus G$.

3. Generate a sequence of $k$ independent binary random variables $X$, where $X_i \sim Bernoulli(1/m)$.

4. For $t = 1, \ldots, k$ set

$$
\Sigma_t = \begin{cases} 
G & \text{if } X_t = 1, \\
\Sigma^-_t & \text{otherwise}.
\end{cases}
$$

**Proof of Proposition 23.** Observe that $\Sigma_t$ depends only on $X_t$ and $\Sigma^-_t$, hence, for any $t' \neq t$, $\Sigma_t$ is independent of $\Sigma_{t'}$. Furthermore, for any $G' \in V$, $P(\Sigma_t = G') = 1/m$. □

**Proof of Lemma 3.** In this proof, we assume that the action profile selected $a$ is the group request strategy, hence, the set of valid groups is $G$. Fix an arbitrary group $G_i$. To generate the sequence $\Sigma \in S_G$ we use the algorithm from Proposition 23, that is, generate a sequence $\Sigma^-$ where $\Sigma^-_t$ is independently and uniformly sample from $G \setminus G_i$ and then extend it to $\Sigma$. Let $S_n = \sum_{t=1}^n |\Sigma^-_t|$. We let $\tau = \tau(k - |G_i| + 1, \Sigma^-)$ be as defined in (2.6) where the size function is the cardinality of each valid group. Intuitively, $\tau$ is the number of positions in $\Sigma$ that ensures a payoff of 1 to $G_i$ given
\( \Sigma^- \). Note that if \( G_i \) is in the first \( \tau \) positions of \( \Sigma \), then the number of tickets awarded before it is processed is at most \( k - |G_i| \). On the other hand, if it is processed after \( \tau \) groups this number is at least \( k - |G_i| + 1 \). Therefore, we get

\[
u_i(\pi^{GR}(a)) = \mathbb{E}[\mathbb{E}[u_i(\pi^{GR}(a))|\Sigma^-]] = \mathbb{E}\left[1 - \left(1 - \frac{1}{m}\right)^\tau\right]. \tag{A.89}
\]

Thus, to prove equation (2.25) it suffices to show

\[
\mathbb{E}\left[1 - \left(1 - \frac{1}{m}\right)^\tau\right] \geq \frac{k}{n} (1 - \kappa) g(\alpha). \tag{A.90}
\]

We let \( m_j \) be the number of groups of size \( j \) in \( G \setminus G_i \), more precisely,

\[
m_j = \sum_{G \in G \setminus G_i} \mathbf{1}\{|G| = j\}.
\]

From this definition, it immediately follows that

\[
m - 1 = \sum_{j \geq 1} m_j, \tag{A.91}
\]

\[
\sum_{j \geq 1} m_j j = n - |G_i|. \tag{A.92}
\]

Define

\[
\phi(\theta) = \mathbb{E}[e^{j|\Sigma^-|\theta}] = \sum_{j \geq 1} \left(\frac{m_j}{m - 1}\right) e^{j\theta}, \tag{A.93}
\]

We let \( F = \{F_n\}_{n \in \mathbb{N}} \) be the filtration generated by \( \Sigma^- \). For any \( \theta \in \mathbb{R} \), we define the following martingale w.r.t. \( F_n \):

\[
\frac{e^{\theta S_n}}{\phi(\theta)^n}. \tag{A.94}
\]

This expression is adapted with respect to \( F_n \), it is bounded as \( |\Sigma^-| \leq \max_G |G| \) and, as shown
below, it satisfies the martingale property:

\[
E \left[ \frac{e^{\theta S_n}}{\phi(\theta)n} | F_{n-1} \right] = \frac{e^{\theta S_{n-1}}}{\phi(\theta)n-1} E \left[ \frac{e^{\theta |\Delta_{n-1}|}}{\phi(\theta)} | F_{n-1} \right] = \frac{e^{\theta S_{n-1}}}{\phi(\theta)n-1} E \left[ \frac{e^{\theta |\Delta_{n-1}|}}{\phi(\theta)} \right] = \frac{e^{\theta S_{n-1}}}{\phi(\theta)n-1}.
\]

Clearly \( \tau \) is a stopping time w.r.t. \( F \), moreover, it is almost surely bounded because \( |\Sigma^-| \geq 1 \) implies that \( P(\tau \leq k - |G_i| + 1) = 1 \). Applying Doob’s optional stopping theorem, we get

\[
1 = E \left[ \frac{e^{\theta S_1}}{\phi(\theta)} \right] = E \left[ \frac{e^{\theta S_\tau}}{\phi(\theta)^\tau} \right].
\]  

(A.95)

Moreover, if we restrict to \( \theta > 0 \) and use that the definition of \( \tau \) implies

\[
S_\tau \geq k - |G_i| + 1,
\]

we obtain

\[
E \left[ \frac{e^{\theta S_\tau}}{\phi(\theta)^\tau} \right] \geq e^{\theta(k-|G_i|+1)} E \left[ \phi(\theta)^{-\tau} \right].
\]  

(A.96)

Combining equations (A.95) and (A.96) yields

\[
e^{-\theta(k-|G_i|+1)} \geq E \left[ \phi(\theta)^{-\tau} \right].
\]  

(A.97)

To prove equation (A.90), we need an upper bound on \( E \left[ \left( 1 - \frac{1}{m} \right)^\tau \right] \). Thus, we let \( \theta^* \) be the unique solution of

\[
\phi(\theta) = \left( 1 - \frac{1}{m} \right)^{-1} = \frac{m}{m-1}.
\]  

(A.98)

The existence and uniqueness of \( \theta^* \) is guaranteed because \( \phi(\theta) \) is increasing and continuous, \( \phi(0) = 1 \) and for \( \theta \geq 0 \), \( \phi(\theta) \geq e^\theta \) hence \( \phi(\log(\frac{m}{m-1})) \geq \frac{m}{m-1} \). Then equation (A.97) evaluates to

\[
e^{-\theta^*(k-|G_i|+1)} \geq E \left[ \phi(\theta^*)^{-\tau} \right] = E \left[ \left( 1 - \frac{1}{m} \right)^\tau \right].
\]
This implies
\[ \mathbb{E}\left[1 - \left(1 - \frac{1}{m}\right)^T\right] \geq 1 - e^{-\theta^*(k - |G_i| + 1)} = \theta^*(k - |G_i| + 1)g(\theta^*(k - |G_i| + 1)). \quad (A.99) \]

The expression above is an increasing function of \( \theta^* \). Hence, if \( \theta^* \geq 1/n \) then (A.90) holds as
\[ \theta^*(k - |G_i| + 1)g(\theta^*(k - |G_i| + 1)) \geq \frac{k - |G_i| + 1}{n} g\left(\frac{k - |G_i| + 1}{n}\right) \quad (A.100) \]
\[ \geq \frac{k - \max_G |G| + 1}{n} g\left(\frac{k - |G_i| + 1}{n}\right), \quad (A.101) \]

and since \( g \) is a decreasing function we have
\[ g\left(\frac{k - |G_i| + 1}{n}\right) \geq g\left(\frac{k}{n}\right) \geq g(\alpha). \quad (A.102) \]

Thus, we assume \( \theta^* < 1/n \). Again, because \( g \) is a decreasing function it follows that
\[ g(\theta^*(k - |G_i| + 1)) \geq g\left(\frac{k - |G_i| + 1}{n}\right) \geq g\left(\frac{k}{n}\right) \geq g(\alpha). \quad (A.103) \]

Therefore, it suffices to show that
\[ \theta^*(k - |G_i| + 1) \geq \frac{k}{n}(1 - \kappa). \quad (A.104) \]

From the definition of \( \theta^* \), we get
\[ \frac{m}{m - 1} = \phi(\theta^*) = \sum_{j \geq 1} \frac{m_j e^{j\theta^*}}{m - 1} \leq \frac{1}{m - 1} \sum_{j \geq 1} \frac{m_j}{1 - j\theta^*}. \quad (A.105) \]

In the inequality we use that for any \( x < 1, \ e^x \leq 1/(1 - x) \). Observe that
\[ j\theta^* < j/n \leq \max_G |G|/n < 1. \]
The first inequality follows by our assumption $\theta^* < 1/n$, the second and third as

$$j \leq \max_G |G| \leq k < n.$$ 

If we multiple both sides of \((A.105)\) by \((m - 1)\) and subtract \((m - 1)\) we obtain

$$1 \leq \sum_{j \geq 1} \frac{m_j}{1 - j\theta^*} - (m - 1) = \sum_{j \geq 1} \frac{m_j}{1 - j\theta^*} - \sum_{j \geq 1} m_j = \sum_{j \geq 1} \frac{m_j j \theta^*}{1 - j\theta^*}.$$ 

The first equality follows from equation \((A.91)\). Besides,

$$\sum_{j \geq 1} \frac{m_j j \theta^*}{1 - j\theta^*} \leq \sum_{j \geq 1} \frac{m_j j \theta^*}{1 - \max_G |G|\theta^*} = \frac{(n - |G_i|)\theta^*}{1 - \max_G |G|\theta^*}.$$ 

In the first inequality we use that $j \leq \max_G |G|$. The equality follows from equation \((A.92)\). Combining both expressions above yields

$$1 - \max_G |G|\theta^* \leq (n - |G_i|)\theta^*. \quad \text{(A.106)}$$

Rearranging, we have

$$n\theta^* \geq 1 - (\max_G |G| - |G_i|)\theta^* > 1 - (\max_G |G| - |G_i|)/n, \quad \text{(A.107)}$$

where the second inequality follows by the assumption $\theta^* < 1/n$. Substituting this last inequality into the left hand side of \((A.104)\), we have

$$\theta^* (k - |G_i| + 1) \geq \frac{k}{n} \left( \frac{|G_i| - 1}{k} \right) \left( 1 - \frac{\max_G |G| - |G_i|}{n} \right) \quad \text{(A.108)}$$

$$\geq \frac{k}{n} \left( 1 - \frac{|G_i| - 1}{k} - \frac{\max_G |G| - |G_i|}{n} \right). \quad \text{(A.109)}$$

The expression at the right hand side is decreasing in $|G_i|$, hence, minimized at $|G_i| = \max_G |G|$. 

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Substituting \( |G_i| = \max_G |G| \) above yields

\[
\frac{k}{n} \left( 1 - \frac{\max_G |G| - 1}{k} \right) \geq \frac{k}{n} (1 - \kappa).
\]  \hspace{1cm} (A.110)

The inequality follows as our instance is in \( I(\kappa, \alpha) \), hence

\[
\frac{\max_G |G| - 1}{k} \leq \kappa.
\]  \hspace{1cm} (A.111)

\[\square\]

**Tightness**

**Proposition 24.** For any \( \alpha, \kappa \in (0, 1) \) and \( \epsilon > 0 \), there exists an instance in \( I(\kappa, \alpha) \) such that the utilization of the group request equilibrium outcome of the Weighted Individual Lottery is less than \( g(\alpha) + \epsilon \).

**Proof of Proposition 24.** Fix \( \alpha, \kappa \in (0, 1) \) and \( \epsilon > 0 \). For any instance \( I \), we let \( U(I) \) be the utilization of the group request equilibrium outcome under the Weighted Individual Lottery. We will construct a sequence of instances \( \{I_\eta\} \) such that for any \( \eta \in \mathbb{N} \),

\[
I_\eta \in I(\kappa, \alpha) \text{ and } \lim_{\eta \to \infty} U(I_\eta) \to g(\alpha).
\]

In \( I_\eta \), there are \( n_\eta = m_\eta s_\eta \) agents divided in \( m_\eta \) groups of size \( s_\eta \), and \( k_\eta = \alpha m_\eta s_\eta \) tickets. We define \( \{m_\eta\}, \{s_\eta\} \) to be increasing sequences of natural numbers that satisfy three conditions:

1. Each instance has an integer number of tickets, that is, \( \{m_\eta\} \) must be such that

\[
\alpha m_\eta \in \mathbb{N}.
\]  \hspace{1cm} (A.112)
2. Each instance is in \( I(\kappa, \alpha) \), i.e.,

\[
\frac{k_\eta}{n_\eta} \leq \alpha, \quad \frac{s_\eta - 1}{k_\eta} \leq \kappa.
\]

The first condition holds immediately as

\[
\frac{k_\eta}{n_\eta} = \frac{\alpha m_\eta s_\eta}{m_\eta s_\eta} = \alpha.
\]

To ensure the second condition, we will define \( m_1 \) to be such that

\[
\alpha m_1 \geq \kappa^{-1}.
\] (A.113)

Observe that

\[
\frac{s_\eta - 1}{k_\eta} = \frac{s_\eta - 1}{\alpha m_\eta s_\eta} \leq \frac{1}{\alpha m_\eta} \leq \frac{1}{\alpha m_1} \leq \kappa.
\]

The second inequality follows as \( \{m_\eta\} \) is increasing, and the third by condition (A.113).

3. Both sequences grow at a similar rate, more precisely, there exists a positive constant \( c \) such that

\[
\frac{m_\eta}{s_\eta} \leq c.
\] (A.114)

We will define both sequences explicitly when \( \alpha \) is rational. In this case, there exists \( p, q \in \mathbb{N} \) such that \( \alpha = p/q \). Then, we can let

\[
m_\eta = 2\eta q \lceil \kappa^{-1} \rceil, \quad s_\eta = \eta q \lceil \kappa^{-1} \rceil.
\]

It’s easy to see that conditions (A.112), (A.113) and (A.114) holds, in the third condition \( c = 2 \). If \( \alpha \) is irrational, then we can choose a rational number \( \alpha^* \leq \alpha \) that is arbitrarily close to \( \alpha \). We let the number of tickets be \( k_\eta = \alpha^* m_\eta s_\eta \) and define \( m_\eta \) in the same way as before but with respect to
\( \alpha^* \).

Under the Weighted Individual Lottery, we will draw \( \alpha m_\eta \) agents that get a full allocation. In this context, the utility of each agent is

\[
u_i(\pi_i^{lw}(a)) = 1 - \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right).
\] (A.115)

Therefore, the utilization of this system corresponds to

\[
U(I_\eta) = \frac{n_\eta}{k_\eta} \left(1 - \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right)\right) = \frac{1}{\alpha} \left(1 - \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right)\right).
\] (A.116)

We claim that

\[
\lim_{\eta \to \infty} U(I_\eta) \to g(\alpha).
\]

Observe that

\[
\liminf_{\eta \to \infty} \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right) \geq \lim_{\eta \to \infty} \left(1 - \frac{1}{m_\eta}\right)^{\alpha m_\eta} = e^{-\alpha},
\]

\[
\limsup_{\eta \to \infty} \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right) \leq \lim_{\eta \to \infty} \left(1 - \frac{1}{m_\eta - \alpha c}\right)^{\alpha m_\eta - 1} \leq \lim_{\eta \to \infty} \left(1 - \frac{1}{m_\eta - \alpha c}\right)^{\alpha m_\eta - 1} = e^{-\alpha}.
\]

The last inequality follows by condition (A.114). \( \square \)

### A.4 More Complex Utility Functions

Our baseline model assumes dichotomous preferences: each member of \( G \) gets utility 1 if the group receives enough tickets for everyone, and utility 0 otherwise. In reality, groups might derive some utility from receiving a smaller number of tickets. In this appendix, we consider more general
utility functions of the form

\[ u_i(x) = f \left( \sum_{j \in G_i} x_j, |G_i| \right). \]  

(A.117)

**Assumption 2 (Sub-Linear).** For any \( w, s \in \mathbb{N} \), we have \( f(w, s) \in [0, w/s] \) and \( f(s, s) = 1 \).

Assumption 2 is a minimal requirement to capture the complementarities that arise in the settings we study. If \( f(w, s) > w/s \) for \( w \in [1, s) \), then the designer prefers to give groups only a fraction of their request, in order to give tickets to more groups. In this case, the Group Lottery is not an appropriate mechanism. Even the case \( f(w, s) = w/s \) is quite extreme: it captures the case where members of the group share tickets equally, and derive no positive externalities from attending together.

We start by showing in Proposition 25 that our definition of approximate efficiency continues to “make sense” as no allocation can achieve efficiency greater than 1.

**Proposition 25.** If utilities are given by (A.117) for some \( f \) satisfying Assumption 2, then any lottery allocation is at most \( 1 \)-efficient.

**Proof of Proposition 25.** It suffices to show that for every feasible allocation \( x \in X \),

\[ \sum_{i \in N} u_i(x) \leq k. \]  

(A.118)

Observe that (A.117) implies utilities are equal for all members of a group. Therefore,

\[ \sum_{i \in N} u_i(x) = \sum_{i \in N} f \left( \sum_{j \in G_i} x_j, |G_i| \right) = \sum_{G_i \in \mathcal{G}} |G_i| f \left( \sum_{j \in G_i} x_j, |G_i| \right). \]  

(A.119)

Moreover, from Assumption 2 and the feasibility of allocation \( x \), we get

\[ \sum_{G_i \in \mathcal{G}} |G_i| f \left( \sum_{j \in G_i} x_j, |G_i| \right) \leq \sum_{G_i \in \mathcal{G}} |G_i| \left( \frac{\sum_{j \in G_i} x_j}{|G_i|} \right) = \sum_{i \in N} x_i \leq k. \]  

(A.120)

Combining equations (A.119) and (A.120) yields (A.118). \qed
A.4.1 Group Lottery

If groups have utility functions given by (A.117) which satisfy Assumption 2, we argue that the Group Lottery remains an attractive option. It is easy to see that if groups follow the group request strategy, the guarantees of Theorem 1 continue to hold. This is because when groups follow the group request strategy, the number of tickets received by $G$ is either $|G|$ or 0, from which Assumption 2 implies that the utility of each member is either 1 or 0. However, one might ask whether groups still have an incentive to follow the group request strategy. After all, if a group of size 4 derives some utility from receiving 2 tickets, then perhaps it will choose to apply as two separate groups of size 2, to increase the probability of receiving something. Example 5 confirms the intuition that under sub-linear utilities the group request is not dominant.

Example 5 (Group Request is not Dominant). Suppose utilities are given by (A.117) for $f(w, s) = \frac{w}{s},$ and consider an instance with one couple and $n - 2$ groups of size one. If the couple follows the group request, the expected number of tickets they receive is $(2k - 1)/(n - 1)$. If each member of the couple applies as an individual they receive $2k/n$ tickets in expectation. If $2k < n$, then this results in higher utility than the group request strategy.

Proposition 26 addresses this concern and establishes the following: If groups avoid dominated strategies and play an equilibrium of the induced game, then the final outcome is approximately efficient and fair.

Lemma 14. Suppose that utilities are given by (A.117) for some $f$ satisfying Assumption 2. In the Group Lottery, any strategy that lists members of other groups is dominated.

Proof. Let $a$ be an action profile such that $a_i \not\in G_i$. Let $a'$ be the action profile defined as follows

$$a'_j = \begin{cases} a_j \cap G_i & \text{if } j \in a_i \cap G_i, \\ a_j & \text{if } j \not\in a_i \cap G_i. \end{cases}$$

We claim that strategy $a'_{G_i}$ dominates $a_{G_i}$. 

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Let $V$ be the set of valid groups according to $\mathbf{a}$, $V'$ the set of valid groups according to $\mathbf{a}'$, and $V^-$ be the set of valid groups excluding agents in $a_i$. Notice that agents in $a_i$ do not influence $V^-$. We generate uniform random orders $\Sigma$ and $\Sigma'$ over $V$ and $V'$ (respectively) using Algorithm 4: we first generate a uniform random order $\Sigma^-$ over $V^-$, and then extend this to obtain $\Sigma$ and $\Sigma'$. We will prove that for any realization of $\Sigma^-$,

$$\mathbb{E}[u_i(x^{GL}(\mathbf{a}', \Sigma'))|\Sigma^-] \geq \mathbb{E}[u_i(x^{GL}(\mathbf{a}, \Sigma))|\Sigma^-].$$ (A.121)

Because agents in $a_i$ cannot influence $\Sigma^-$, it follows immediately that the unconditional expected utility of agent $i$ is also higher under $\mathbf{a}'$.

Let $P$ be the position determined by Algorithm 4 in which $a_i$ and $a_i'$ are inserted in $\Sigma$ and $\Sigma'$, respectively. Notice that the allocation of valid groups in the $P - 1$ initial positions of $\Sigma^-$ is not affected by the allocation of $a_i$ and $a_i'$. Moreover, since $|a_i'| < |a_i|$ groups in positions $P$ and later in $\Sigma^-$ are weakly better under $\mathbf{a}'$ than $\mathbf{a}$. Hence, it suffices to show that whenever $a_i$ is awarded $a_i'$ is awarded too. Notice that valid group $a_i$ is awarded if and only if the number of agents in groups in the $P - 1$ initial position of $\Sigma^-$ is at most $k - |a_i|$. To conclude, note that by definition $|a_i'| < |a_i|$. \hfill \Box

**Proposition 26.** Suppose that utilities are given by (A.117) for some $f$ satisfying Assumption 2. Then for every $\alpha, \kappa \in (0, 1)$ and every instance in $I(\kappa, \alpha)$, any equilibrium in undominated strategies of the Group Lottery is $(1 - 2\kappa)$-efficient and $(1 - 3\kappa)$-fair.

**Proof of Proposition 26.** Fix an arbitrary instance. Let $\mathbf{a}$ be any equilibrium in undominated strategies. We claim that for every agent $i \in \mathcal{N}$,

$$\frac{k}{n}(1 - 2\kappa) \leq u_i(\pi^{GL}(\mathbf{a})) \leq \frac{k}{n}(1 + \kappa).$$ (A.122)

These bounds imply both performance guarantees. The efficiency guarantee follows from the lower

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bound:
\[ U(\pi^{GL}(a)) = \frac{\sum_{i \in N} u_i(\pi^{GL}(a))}{k} \geq \frac{\sum_{i \in N} k(1 - 2\kappa) / n}{k} = 1 - 2\kappa. \]

Meanwhile, the fairness guarantee follows from both bounds in (A.122): for every pair of agents \( i, j \in N \),
\[ \frac{u_i(\pi^{GL}(a))}{u_j(\pi^{GL}(a))} \geq \frac{1 - 2\kappa}{1 + \kappa} \geq 1 - \frac{3\kappa}{1 + \kappa} \geq 1 - 3\kappa. \]

All that remains is to prove our claim. We start by making an useful observation. Let \( V \) the set of valid groups according to \( a \). Lemma 14 implies that
\[ \max_{S \in V} |S| \leq \max_{G' \in G} |G'|. \quad (A.123) \]

We now turn to the lower bound in (A.122). We will prove that the utility derived from the group request strategy is at least \((1 - 2\kappa)k/n\). Since this is always a feasible strategy and \( a \) an equilibrium then the lower bound holds. Fix a group \( G \), and suppose that the members of \( N \setminus G \) form \( m' - 1 \) valid groups, with maximum size \( s \). If \( G \) follows the group request strategy, then Lemma 1 implies that for \( i \in G \), expected utility is
\[ \mathbb{E}[\tau(k - |G| + 1, \sigma^-)] / m'. \]

Applying the lower bound in (2.11) of Proposition 2, we conclude that for \( i \in G \),
\[ \frac{\mathbb{E}[\tau(k - |G| + 1, \sigma^-)]}{m'} \geq \frac{1}{m'} \left( 1 + \frac{k - |G| + 1 - s}{n/m'} \right) \geq \frac{k - |G| + 2 - s}{n} \geq \frac{k}{n} \left( 1 - 2 \left( \frac{\max_{G' \in G} |G'| - 1}{k} \right) \right) \geq \frac{k}{n} (1 - 2\kappa). \]

In the second inequality we use the fact that \( n \geq m' \), the third inequality follows as equation (A.123)
implies that \( s \leq \max_{G' \in G} |G'| \), and the last inequality follows as \( \kappa \geq (\max_{G' \in G} |G'| - 1)/k \). To conclude, we show the upper bound in (A.122). Assumption 2 implies that the utility of a group is upper bounded by the expected number of tickets awarded to its members divided by its cardinality. Hence, it suffices to show that the expected number of tickets awarded to members of \( G \) is at most

\[
|G| \frac{k}{n} (1 + \kappa). 
\]  
(A.124)

In fact, we will show a stronger result. Let \( S \in V \) be any valid group, then the expected number of tickets received by \( S \) is at most

\[
|S| \frac{k}{n} (1 + \kappa). 
\]  
(A.125)

This and the fact that under the Group Lottery there is no over allocation implies (A.124). Observe that \( V \) has at most \( n \) agents. Moreover, from equation (A.123) we know the size of a valid group can not exceed \( \max_{G' \in G} |G'| \). Therefore, we can apply Lemma 1 to an instance where the set of groups is \( V \), to conclude that the success probability of \( S \) is at most \( k(1 + \kappa)/n \) which implies (A.125).

\[\square\]

A.4.2 Individual Lottery

No modifications are needed for the efficiency analysis. Note that the utility of single individuals remains the same so the upper bound in (A.45) holds \( u_i(\pi^\text{IL}(a)) \leq k/s^2 \). Notice that the upper bound in (A.45) assumes that members of the large group are requesting at least \( s \) tickets. Moreover, for the large group we simply used the fact that utilities are at most 1 (Assumption 2).

For the fairness guarantee, the lower bound on the utility of the large group still holds as we are using the probability of a member being drawn first. For the single individuals the same upper bound in (A.45) is used.

A.5 Incentives of Weighted Individual Lottery - Numerical Analysis

In the incentives analysis of the Weighted Individual Lottery located in Section 2.4.3, we show that groups may not have a dominant strategy. However, we conjecture that if the market is mod-
erately competitive, then the group request is optimal.

More precisely, Lemma 12 shows that the decision problem facing a group which knows the strategies and lottery numbers of other participants can be summarized by a single statistic $T$ defined in (2.19). This statistic captures the level of competition in the market: the group request strategy succeeds with probability $1 - e^{-T}$. Conjecture 1 states that the group request strategy is optimal if $T \leq 1$ (the success probability is below 63%).

Here, we numerically verify Conjecture 1 for groups of size at most 15. Groups of size up to 15 are permitted in the Big Sur marathon lottery, we are not aware of any application that permits larger groups. Thus, our numerical analysis addresses the incentives of the Weighted Individual Lottery for all practical purposes.

Our verification conducts an exhaustive search over possible strategies. It is easy to see that no member of group $G$ should request more than $|G|$ tickets. Hence, we can restrict to a finite number of possible strategy profiles. We further restrict to groups of size at least 4 since Proposition 7 establishes that the group request is dominant for smaller groups. For each group size $4, \ldots, 15$, Figure A.1 includes a plot showing the success probability of each possible strategy, for any $T \in [0, 1]$. In all cases, the group request is optimal.

The number of strategies to evaluate increases exponentially with the group size: for group size 15 we have to evaluate more than 77 million strategies. To reduce the computational work, we calculate success probabilities with dynamic programming.

For any fixed $T \in [0, 1]$, we construct a massive table, with each row representing an action profile $a \in \mathcal{A} = \bigcup_{j=1}^{15} \{1, \ldots, 15\}^j$, and each column representing a ticket target $\ell \in \{0, \ldots, 15\}$. The entries $R(a, \ell)$ denote the probability of receiving at least $\ell$ tickets when following strategy $a$. The base case is that $R(a, 0) = 1$ for all $a$. In general, we compute $R(a, \ell)$ by conditioning on the outcome of one agent $i$. The probability that agent $i$ succeeds is $1 - e^{-T/a_i}$. Hence, if we let $a^{-i}$ denote the action profile after removing agent $i$, the success probabilities satisfy the following
recursion:

\[ R(a, \ell) = R(a^{-i}, \max\{0, \ell - a_i\}) (1 - e^{-T/a_i}) + R(a^{-i}, \ell) e^{-T/a_i}, \quad (A.126) \]

We compute the values of \( R(a, \ell) \) in lexicographical order to ensure that the value of \( R(a^{-i}, \ell) \) and \( R(a^{-i}, \max\{0, \ell - a_i\}) \) have already been computed when evaluating equation (A.126). The expected utility of \( G \) under any action profile \( a \in \{1, \ldots, 15\}^{\mid G\mid} \) is given by \( R(a, \mid G\mid) \).

### A.6 Alternative Definitions of Fairness

Our notion of fairness in Definition 2 state that agents in groups of different sizes should have similar expected utilities. We believe that this notion captures the objectives of the organizers of applications we consider in this work. Section 2.1.2 presents evidence supporting our interpretation from the policies and procedures of these applications.

Of course, there are alternative notions of fairness that one might consider. Three that arise in other contexts are equal treatment of equals, envy-free and egalitarian fairness. Below we present the natural analog of these in our setting, and discuss their relation to our definition. In particular, we show that (i) equal treatment of equals is weaker than our definition and is satisfied by all the mechanisms we study, and (ii) our positive results for the group lottery and weighted individual lottery, and our negative results for the individual lottery, continue to hold for envy-freeness and egalitarian fairness.

**Definition 16.** Lottery allocation \( \pi \) satisfies equal treatment of equals if for every pair of agents \( i, j \) such that \( |G_i| = |G_j| \), we have \( u_i(\pi) = u_j(\pi) \).

This is clearly weaker than our fairness definition, and is an easy property to satisfy. In particular, the group request outcomes of the three mechanisms we study all satisfy equal treatment of equals.
A.6.1 Alternative Definition of Fairness: Group Envy-Free

To define envy-freeness, we introduce additional notation. For any \( x \in X \), we let \( N_G(x) \) be the number of tickets allocated to members of \( G \). For any \( \pi \in \Delta(X) \), let \( N_G(\pi) \) be a random variable representing this number. In a slight abuse of notation, let \( u_G(N) = \mathbb{P}(N \geq |G|) \) be the expected utility of \( G \) when the number of tickets received by members of \( G \) is equal to the random variable \( N \).

**Definition 17.** Lottery allocation \( \pi \) is **group-envy-free** if no group envies the allocation of another: \( u_G(N_G(\pi)) \geq u_G(N_{G'}(\pi)) \) for all \( G, G' \in G \). It is **\( \beta \)-group-envy-free** if \( u_G(N_G(\pi)) \geq \beta u_G(N_{G'}(\pi)) \) for all \( G, G' \in G \).

Example 6 shows that this notion is neither stronger nor weaker than our fairness definition. However, Proposition 27 establishes that the conclusions of Theorems 1, 2 and 3 also hold for this new fairness notion.

**Example 6.** Suppose that there is a group of size 1 and another of size 2. The group of size 1 gets one ticket with probability \( \epsilon \), and otherwise gets zero tickets. The group of size 2 gets two tickets with probability \( \epsilon \) and otherwise gets one ticket. This is fair (according to our definition) but not even approximately group-envy-free. Conversely, if both groups get one ticket with probability \( \epsilon \) and zero tickets otherwise, then the allocation is group-envy-free but not fair.

**Proposition 27.**

1. The group request outcome of the Group Lottery is group-envy-free.

2. For any \( \alpha, \kappa, \epsilon \in (0, 1) \), there exists an instance in \( I(\kappa, \alpha) \) such that any dominant strategy equilibrium outcome of the Individual Lottery is not \( \epsilon \)-group-envy-free.

3. Fix \( \kappa, \alpha \in (0, 1) \). For every instance in \( I(\kappa, \alpha) \), the group request outcome of the Weighted Individual Lottery is \((1 - 2\kappa)g(\alpha)\)-group-envy-free.
Proof of Proposition 27. We begin by showing statement 1. Because under the Group Lottery there is no over-allocation, it follows that no group envies the allocation of a smaller group. Meanwhile, Lemma 9 establishes that, under the Group Lottery, smaller groups are always weakly more likely to receive any tickets. Because the group lottery ensures that any groups that receive tickets receive enough for the entire group, this implies that no group envies the allocation of a larger group.

We now turn to statement 2. The argument is analogous to the one used in the proof of Theorem 2 located in Appendix A.2, but we include it for completeness. Consider an instance with \( k = \lfloor ar \rfloor s \) tickets, one large group of size \( s \) and \( s(r - 1) \) small groups of size one. As shown in the proof of Theorem 2, if \( (s - 1)/k \leq \kappa \) then this instance will be in \( I(\kappa, \alpha) \). Let \( a \) be any dominant strategy equilibrium under the Individual Lottery. We let \( G, G' \in G \) be such that \( |G| = s \) and \( |G'| = 1 \). From the definition of \( u_G(N) \) and since \( |G| > |G'| \), we have that

\[
 u_{G'}(N) = \mathbb{P}(N \geq |G'|) \geq \mathbb{P}(N \geq |G|) = u_G(N).
\]

Therefore,

\[
 \frac{u_{G'}(N_G'(\pi_{IL}(a)))}{u_{G'}(N_G(\pi_{IL}(a)))} \leq \frac{u_G(N_G(\pi_{IL}(a)))}{u_G(N_G(\pi_{IL}(a)))} \leq \frac{ar^2}{s}.
\]

The second inequality follows from the proof of Theorem 2. Statement 2 follows as we can make the right hand side arbitrarily small: choose any \( s = s(r) \) that satisfies \( r^2/s(r) \to 0 \) and let \( r \) grow.

To conclude, we prove statement 3. First, we bound the envy that a group can have towards smaller groups. Let \( G, G' \in G \) be such that \( |G| > |G'| \). From (A.127) and Theorem 3, we have

\[
 \frac{u_G(N_G(\pi_{IW}(a)))}{u_G(N_G(\pi_{IW}(a)))} \geq \frac{u_G(N_G(\pi_{IW}(a)))}{u_G(N_G(\pi_{IW}(a)))} \geq (1 - 2\kappa)g(\alpha).
\]

Second, we bound the envy that a group can have towards larger groups. Since groups are following the group request strategy, any time a group gets a positive number of tickets, it gets enough for
the entire group. Therefore,

\[ u_G'(N_G(\pi^{IW}(a))) = P(N_G(\pi^{IW}(a)) > 0) = u_G(N_G(\pi^{IW}(a))). \]

This and Theorem 3 yields

\[ \frac{u_G'(N_G'(\pi^{IW}(a)))}{u_G'(N_G(\pi^{IW}(a)))} = \frac{u_G'(N_G'(\pi^{IW}(a)))}{u_G(N_G(\pi^{IW}(a)))} \geq (1 - 2\kappa)g(\alpha). \]

A.6.2 Alternative Definition of Fairness: Egalitarian Fairness

The last notion of fairness we consider focuses on the lowest chance of success among groups:

\[ \min_{i \in N} u_i(\pi). \]

**Definition 18.** Lottery allocation \( \pi \) is \( \beta \)-egalitarian fair if \( \min_{i \in N} u_i(\pi) \geq \beta \times k/n. \)

That is, an allocation is approximately egalitarian fair if every group has a “reasonable” chance of success. (Note that no allocation can guarantee every group a utility above \( k/n \), so egalitarian fairness is always between 0 and 1.) We adopt the name “egalitarian fairness” because the quantity \( \min_{i \in N} u_i(\pi) \) is known in the literature as the “egalitarian welfare” of the allocation \( \pi. \)

We show that for this alternative definition, variants of our main conclusions in Theorems 1, 2 and 3 continue to hold: (i) the Group Lottery is egalitarian fair, (ii) the Individual Lottery can be arbitrarily egalitarian unfair, and (iii) the Weighted Individual Lottery is approximately egalitarian fair. In fact, this does not rely on the specific mechanisms that we study: it turns out that any mechanism that is approximately efficient and fair (according to our definition) is also approximately egalitarian fair.

**Proposition 28.**

1. If \( \pi \) is \( \beta \)-efficient and \( \delta \)-fair, then \( \pi \) is \( \beta\delta \)-egalitarian fair.
2. If $\pi$ is $\beta$-egalitarian fair, then it is $\beta$-efficient.

Proof of Proposition 28. Statement 1 follows from the following chain of inequalities, where the first uses $\beta$-efficiency and the second uses $\delta$-fairness:

\[
\beta k \leq \sum_{i' \in N} u_{i'}(\pi) \leq \sum_{i' \in N} \frac{\min_{i \in N} u_i(\pi)}{\delta} = \frac{n}{\delta} \min_{i \in N} u_i(\pi).
\]

Statement 2 follows directly from the definition of egalitarian fairness:

\[
\sum_{i \in N} u_i(\pi) \geq n \min_{i \in N} u_i(\pi) \geq n(\beta k / n) = \beta k.
\]

\square

Statement 1 of Proposition 28 means that positive results for fairness and efficiency directly imply positive results for egalitarian fairness. In particular, Theorems 1 and 3 and Proposition 3 imply the following.

**Corollary 1.** Fix $\kappa, \alpha \in (0, 1)$. For every instance in $I(\kappa, \alpha)$,

- The group request outcome of the Group Lottery is $(1 - 3\kappa)$-egalitarian fair.

- The group request outcome of the Weighted Individual Lottery is $(1 - 3\kappa - \alpha)$-egalitarian fair.

- There exists a random allocation that is $(1 - \kappa)$-egalitarian fair.

Meanwhile, statement 2 of Proposition 28 means that negative results for efficiency directly imply negative results for egalitarian fairness. In particular, our Theorem 2 and Proposition 3 imply the following.

**Corollary 2.**

- For any $\alpha, \kappa, \epsilon \in (0, 1)$, there exists an instance in $I(\kappa, \alpha)$ such that any dominant strategy equilibrium outcome of the Individual Lottery is not $\epsilon$-egalitarian fair.
For any \( \alpha, \epsilon > 0 \), there exists \( \kappa \in (0,1) \) and an instance in \( I(\alpha, \kappa) \) such that no random allocation is \( (1 - \kappa + \epsilon) \)-egalitarian fair.

We also comment that we can provide even stronger guarantees than those resulting from the “black box” reduction of Proposition 28.

**Proposition 29.** Fix \( \kappa, \alpha \in (0,1) \). For every instance in \( I(\kappa, \alpha) \),

- The group request outcome of the Group Lottery is \( (1 - 2\kappa) \)-egalitarian fair.
- The group request outcome of the Weighted Individual Lottery is \( (1 - \kappa)g(\alpha) \)-egalitarian fair.

**Proof of Proposition 29.** Fix \( \kappa, \alpha \in (0,1) \) and an arbitrary instance in \( I(\kappa, \alpha) \). In what follows, we let \( a \) denote the group request strategy for each mechanism below.

We start by showing the performance guarantee for the Weighted Individual Lottery. From Lemmas 2 and 3, we have that

\[
 u_i(\pi^{IW}(a)) \geq u_i(\pi^{GR}(a)) \geq \frac{k}{m} (1 - \kappa)g(\alpha). \tag{A.128}
\]

Hence, our result follows by adding the inequality above over all individuals.

We now turn to the performance guarantee for the Group Lottery. Lemma 2 and Proposition 3 imply that

\[
 u_i(\pi^{GL}(a)) = \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})] \geq \frac{k - 2s + 1}{m\mu^-} + \frac{1}{m}, \tag{A.129}
\]

where \( s = \max_{G \in \mathcal{G}} |G| \) and \( \mu^- = (n - |G_i|)/(m - 1) \). Algebraic manipulations show that

\[
 \frac{k - 2s + 1}{m\mu^-} + \frac{1}{m} = \frac{k - 2s + 1}{m} \frac{\mu^-}{\mu^-} + \frac{1}{m} \\
 \geq \frac{k - 2s + 1}{n} + \frac{1}{m} \geq \frac{k}{n} \left( 1 - \frac{2(s - 1)}{k} + \frac{\mu - 1}{k} \right) \\
 \geq \frac{k}{n} \left( 1 - \frac{2(s - 1)}{k} \right) \geq \frac{k}{n} (1 - 2\kappa). 
\]

\[\square\]
A.7 Naive and Sophisticated Play in the Individual Lottery

As noted in Section 2.1.2, organizers often use the interface of the Individual Lottery (each applicant requests a certain number of tickets without identifying the members of their group), but include language stating that only one member of each group should apply.

If all groups followed this instruction, the mechanism would be equivalent to the group lottery, and there would be no need for a change. However, there is ample evidence that in some cases, multiple members of a group enter the lottery. Our theoretical analysis assumes that every member of every group applies, and shows that in this case, the Individual Lottery may perform quite poorly, while the Weighted Individual Lottery performs much better.

It is likely that the reality lies somewhere in between our assumption and the organizers’ intent: some groups are “compliant” and follow the instructions to enter only once, while others are “sophisticated” and maximize their chances by having each member apply. It is natural to wonder whether the guarantees we provide continue to hold in this case.

The short answer is that they do not.

A.7.1 Worst Case Analysis of with a Mix of Sophisticated and Naive Groups

For fairness, it is clear that if there are two groups of equal size, and one group complies with the instruction to enter only once while the other has each member apply, the latter group will have a significant advantage over the former. Thus, neither the Individual Lottery nor the Weighted Individual Lottery will offer a non-trivial fairness guarantee.

For efficiency, the question is more subtle, but it turns out that it is possible to construct a variant of the bad example used in the proof of Theorem 2. Specifically, suppose that there are $n^{1/3}$ groups of size $n^{2/3}$. Most of these groups follow the instructions and have only one person enter. One group is sophisticated and has every member enter. Because all lottery entrants request the same number of tickets, the Individual Lottery and Weighted Individual Lottery are equivalent on this example. If $k = O(n)$, most tickets will be won by the one sophisticated group, leading to
vanishingly small utilization under either mechanism.

While this example is extreme, utilization may be poor even when groups have modest size. Figure A.2 shows that with 20 groups of size 10 and 100 tickets, if between 2 and 7 of these groups are sophisticated, utilization is below 70%. This figure illustrates that utilization need not be monotonic in the number of sophisticated groups. When one group switches from being naive to sophisticated, utilization may drop because it is now possible for this group to win more tickets than it needs. However, a countervailing effect of this change is that other sophisticated groups become less likely to win too many tickets. Which of these effects dominates depends on the number of sophisticated groups.

A.7.2 Simulations Inspired by Data from the Big Sur Lottery

Above, we argued that neither the Individual Lottery nor the Weighted Individual Lottery offer good worst-case guarantees when there is a mix of naive and sophisticated groups. However, one might also be curious about their performance on more realistic examples. To explore this question, we conducted simulations inspired by data from the Big Sur Lottery, in which approximately 700 spots were awarded to 236 winners selected from 1296 applicants. We were able to obtain the number of spots requested by each selected applicant, which ranged from 2 (requested by 149 successful applicants) to the maximum of 15 (requested by 1 successful applicant). Because the requests of unsuccessful applicants were not available, we conducted simulations with 700 tickets, and 1296 groups whose sizes are drawn iid from the empirical distribution of requests of selected applicants. That is, in our simulations, the expected fraction of groups with size 2 is $\frac{149}{236}$, while the expected fraction of groups with size 15 is $\frac{1}{236}$.

In our simulations, we assume that each group is either “sophisticated” (has every member apply) or “compliant” (has only one member apply), and that the probability of sophistication is independent of group size. We vary the probability of sophistication from 0 to 1 in increments of 0.1, and conduct 4,000 trials for each case. The average utilization of both mechanisms is shown in Figure A.3.
From this preliminary investigation, we conclude the following. First, even when a relatively small fraction (10%) of groups take advantage of the opportunity to apply multiple times, the performance of the Individual Lottery suffers. Second, for any level of sophistication, the Weighted Individual Lottery results in higher utilization than the Individual Lottery. In fact, for this example, the performance of these mechanisms does not heavily depend on the number of sophisticated groups: the best performance of the Individual Lottery across scenarios with some sophisticated groups is worse than the worst performance of the Weighted Individual Lottery. This shows that even though our worst-case guarantees for the Weighted Individual Lottery do not hold, this approach may still offer a significant improvement upon the Individual Lottery.
Figure A.1: Numerical verification of Conjecture 1 for groups of size up to 15.

For each group size, we plot one line for each possible strategy. The $x$ axis corresponds to the threshold $T \in [0, 1]$ (smaller $T$ means the market is more competitive). The $y$ axis represents the ratio of the success probability of the chosen strategy to the success probability of the group request. Solid lines represent strategies in $B$ (for which we have already proven the conjecture). Dashed lines represent other strategies. All lines remain strictly below 1, indicating that the group request is optimal.
Figure A.2: If some groups have every member apply while others have only one member apply, the guarantees for the Weighted Individual Lottery proved in Theorem 3 may no longer hold. In this example with groups of size 10, when only a few groups are sophisticated, these sophisticated groups win a large fraction of the tickets, resulting in low utilization for both the Individual Lottery and the Weighted Individual Lottery.

Figure A.3: In simulations based on statistics from the Big Sur Marathon Groups and Couples Drawing, the expected utilization of the Individual Lottery (red) and the Weighted Individual Lottery (blue) do not depend heavily on the fraction of groups who are sophisticated (have all members apply), so long as this fraction is at least 0.1. For all sophistication levels, the Weighted Individual Lottery results in notably higher utilization than the Individual Lottery.
Appendix B: Chapter 3

B.1 Proofs From Section 3.3

B.1.1 Proof of Theorem 4

We divide the proof into two parts: Lemma 15 establishes that for any collection of feasible selections \( \mathcal{F} \subseteq 2^I \), the rule \( OB_{\mathcal{F}} \) is monotonic, non-bossy and lower invariant. Lemma 16 shows that if a selection rule satisfies these three properties, then it must be equivalent to \( OB_{\mathcal{F}} \) for some suitably chosen \( \mathcal{F} \).

**Lemma 15.** For any collection \( \mathcal{F} \subseteq 2^I \), the selection rule \( OB_{\mathcal{F}} \) is monotone, non-bossy and lower invariant.

**Proof of Lemma 15.** In what follows, we fix an arbitrary collection of feasible selections \( \mathcal{F} \subseteq 2^I \) and an arbitrary applicant \( i \in I \).

We begin by showing that \( OB_{\mathcal{F}} \) is lower invariant. Let \( \succ, \succ' \in \mathcal{P} \) be such that \( \succ_{\{j; j\succ i\}} = \succ'_{\{j; j\succ' i\}} \).

We must prove that

\[
\{i\} \cap OB_{\mathcal{F}}(\succ) = \{i\} \cap OB_{\mathcal{F}}(\succ').
\]

(B.1)

In order to generate \( OB_{\mathcal{F}}(\succ) \) and \( OB_{\mathcal{F}}(\succ') \), we use Algorithm 2 taking as input \((\mathcal{F}, \succ)\) and \((\mathcal{F}, \succ')\), respectively. Since (i) Algorithm 2 iterates over each applicant in decreasing order of priorities, and (ii) before applicant \( i \) orders \( \succ \) and \( \succ' \) are identical, then both executions of Algorithm 2 coincide in every iteration up to applicant \( i \). In particular, in both executions the decision made over applicant \( i \) will be the same and (B.1) holds.

We will now prove that \( OB_{\mathcal{F}} \) is monotone. For any priority order \( \succ \in \mathcal{P} \) and applicant \( i \in I \), we denote the ranking of applicant \( i \) in \( \succ \) as \( rank_{\succ}(i) \). It suffices to show that for any \( \succ, \succ' \in \mathcal{P} \) such
that $\succ_{-i} = \succ'_{-i}$ and $\text{rank}_{\succ} (i) = \text{rank}_{\succ'} (i) + 1$, if $i \in OB_F (\succ)$ then $i \in OB_F (\succ')$. We claim that

$$\{i\} \cup \{j \in OB_F (\succ') : j \succ' i\} \subseteq OB_F (\succ). \quad \text{(B.2)}$$

From (B.2) and the fact that $OB_F (\succ')$ respects priorities, it follows that $i \in OB_F (\succ')$. We now prove our claim (B.2). Let $j^* \in I$ be the individual that precedes $i$ in the order $\succ$, that is, $\text{rank}_{\succ} (i) = \text{rank}_{\succ} (j^*) + 1$. Because $\succ_{-i} = \succ'_{-i}$ and $\text{rank}_{\succ'} (i) = \text{rank}_{\succ'} (j^*)$, it follows that

$$\succ (j; j^*) = \succ' (j; j^*).$$

Hence, from the fact that $OB_F$ is lower invariant, we have that

$$\{j \in OB_F (\succ') : j \succ' i\} = \{j \in OB_F (\succ) : j \succ j^*\}. \quad \text{(B.3)}$$

Moreover, since $j^* \succ i$ it follows that

$$\{j \in OB_F (\succ) : j \succ j^*\} \subseteq \{j \in OB_F (\succ) : j \succ i\} \subseteq OB_F (\succ). \quad \text{(B.4)}$$

Combining (B.3), (B.4) and the assumption that $i \in OB_F (\succ)$ yields (B.2).

Finally, we show that $OB_F$ is non-bossy. The proof is by induction on the number of agents.

For single-applicants problems, the rule $OB_F$ is trivially non-bossy.

Assuming we have proved non-bossiness of $OB_F$ for instances with fewer than $n$ agents. Consider an instance with $n$ agents and suppose we are given orderings $\succ$ and $\succ'$ that agree on the relative ordering of every pair of applicants not involving applicant $i$, that is, $\succ_{-i} = \succ'_{-i}$. Our goal is to establish that if applicant $i$’s outcome is the same for the two orderings $\succ$ and $\succ'$, then the rule $OB_F$ selects the same element of $F$ for the two orderings as well, that is, if $\{i\} \cap OB_F (\succ) = \{i\} \cap OB_F (\succ')$ then $OB_F (\succ) = OB_F (\succ')$.

Suppose there is an applicant $k$ who is not part of any feasible selection. We can omit applicant
\( k \) from the problem and obtain a smaller instance for which the result follows by induction. Thus we may assume that each applicant is part of some member of \( \mathcal{F} \). We consider the following cases:

(1) Applicant \( j \neq i \) is in the top position in both \( \succ \) and \( \succ' \): Let \( \mathcal{G} \) be the elements of \( \mathcal{F} \) that includes applicant \( j \); remove applicant \( j \) from the problem to get an instance involving \((n-1)\) applicants; by the induction hypothesis, the rule \( \text{OB}_G \) is non-bossy on the reduced problem, and the result follows.

(2) Applicant \( j \) is in the top position in \( \succ \) and applicant \( i \) is in the top position in \( \succ' \): We start with two observations: (i) the top two applicants in \( \succ' \) are \( i \) and \( j \); and (ii) applicant \( i \) must be selected in \( \succ' \) and applicant \( j \) must be selected in \( \succ \). For a violation of non-bossiness to occur, applicant \( i \) must be selected in \( \succ \) as well. This means there is a member of \( \mathcal{F} \) that contains both \( i \) and \( j \). We claim that the selection for the ordering \( \succ'' \), obtained from \( \succ' \) by swapping the top two applicants, is the same as for the ordering \( \succ' \). Now, note that applicant \( j \) is the top agent in both \( \succ \) and \( \succ'' \); by (1) above, we can infer non-bossiness in this case.

All that remains is to prove our claim. Consider two priority orders \( \succ, \succ' \in \mathcal{P} \) in which the top two applicants are swapped: in the ordering \( \succ \), the first applicant is \( i \) and the second is \( j \), whereas in the ordering \( \succ' \), the first applicant is \( j \) and the second is \( i \). If there is a member of \( \mathcal{F} \) that includes both \( i \) and \( j \), then both applicants are selected in \( \text{OB}_{\mathcal{F}}(\succ) \) and \( \text{OB}_{\mathcal{F}}(\succ') \) as both selections respects priorities. Then it’s easy to see that \( \text{OB}_{\mathcal{F}}(\succ) = \text{OB}_{\mathcal{F}}(\succ') \).

\[ \square \]

**Lemma 16.** Let \( \varphi \) be a selection rule that is monotone, non-bossy and lower invariant. Then there exists \( \mathcal{F} \subseteq 2^I \) such that
\[ \varphi(\succ) = \text{OB}_{\mathcal{F}}(\succ) \text{ for any } \succ \in \mathcal{P}. \]  

**(B.5)**

**Proof.** Let
\[ \mathcal{F} = \{ \varphi(\succ) : \text{ for every } \succ \in \mathcal{P} \}. \]  

**(B.6)**
We will show that $OB_{\mathcal{F}}$ replicates the behavior of $\varphi$, that is, (B.5) holds. For the sake of contradiction, suppose that there exists an order $\succ$ that induces different selections under $\varphi$ and $OB_{\mathcal{F}}$. Formally,

$$\varphi(\succ) \neq OB_{\mathcal{F}}(\succ).$$

Let $i \in I$ be the highest priority applicant for which $\varphi(\succ)$ and $OB_{\mathcal{F}}(\succ)$ differ. Clearly, applicant $i$ must be selected under $OB_{\mathcal{F}}(\succ)$, otherwise, $OB_{\mathcal{F}}(\succ)$ would not respect priorities. From (B.6), there must exists an order $\succ'$ such that

$$\{i\} \cup \{j \in \varphi(\succ) : j \succ i\} \subseteq \varphi(\succ').$$

(B.7)

We claim that there exists an order $\succ''$ that satisfies the following conditions:

1) Identical to $\succ$ in the initial positions up to applicant $i$, and

2) Induces the same selection as $\succ'$, that is, $\varphi(\succ'') = \varphi(\succ').$

From (B.7) and condition 2), it follows that individual $i$ is selected in $\varphi(\succ'')$. This contradicts the assumption that $\varphi$ is lower invariant, notice that orders $\succ$ and $\succ''$ coincide up to individual $i$, but individual $i$ is only selected under $\succ''$.

In order to prove the existence of order $\succ''$, we provide an algorithm that starts with order $\succ'$, and sequentially improves the priority of a single applicant, until conditions 1) and 2) are satisfied:

- Start from order $\succ'$. Let $r$ be the position of applicant $i$ in $\succ$.

- For $k = 1, \ldots, r$,
  
  - Let $j$ be the applicant in the $k$-th position in $\succ$.
  
  - Move applicant $j$ to the $k$-th position, while maintaining the relative order of all the remaining applicants.

Observe that by construction, at the end of step $k$, the first $k$ positions of the order generated and $\succ$ coincide. Hence, condition 1) immediately holds.
We now prove that, at each step, the overall selection remains unchanged. This implies condition 2) above. Fix an arbitrary step \( k \), and let \( j \) be the individual whose priority is improved. We claim that the outcome of individual \( j \) remains unchanged. Because \( \varphi \) is non-bossy and the relative order of all remaining individuals is preserved, our claim implies that the overall selection must remain unchanged. All that remains is to prove our claim. From the observation above and the assumption that \( \varphi \) is lower-invariant, the outcome of \( j \) will be the same as in \( \varphi(\succ) \). If \( j \) is selected in \( \varphi(\succ) \), from (B.7) it must also be selected in \( \varphi(\succ') \) so its outcome remains unchanged. If \( j \) is not selected in \( \varphi(\succ) \), then in \( \varphi(\succ') \) it must not be selected either. This follows as we are improving the priority of individual \( j \) and \( \varphi \) is monotone, hence, individual \( j \) must be weakly better. \( \square \)

B.1.2 Examples Demonstrating Independence of Our Three Axioms

**Example 7** (Monotone, lower-invariant but bossy). Consider a setting where there are multiple minority groups that are disjoint, that is, each applicant can be a member of at most one minority. There are two types of seats: (i) unreserved seats that can be allocated to any applicant, and (ii) for each minority, there are reserved seats that can be allocated only to members of the minority. Let \( \varphi \) be the selection rule that, for any priority order \( \succ \), generates a selection according to the following procedure:

1. Unreserved seats are allocated based on the priority order \( \succ \).
2. For each of the (mutually exclusive) minorities, positions that are set aside are allocated to the remaining members of the minority again based on the priority order \( \succ \).

We start by showing that no selection rule \( OB_{F} \) can replicate the behavior of \( \varphi \). Consider an instance with three applicants \( I = \{1, 2, 3\} \), one open seat and one seat reserved for members of the minority group \( M = \{2, 3\} \). Let

\[
1 \succ 2 \succ 3 \text{ and } 2 \succ' 1 \succ' 3.
\]
From the definition of $\varphi$, we must have that $OB_{F}(\succ) = \{1, 2\}$ so $\{1, 2\} \in F$. In addition, it should be that $OB_{F}(\succ') = \{2, 3\}$. However, selection $\{2, 3\}$ doesn’t respects priorities $\succ'$ as $1 \succ' 3$ and $\{1, 2\} \in F$.

From the instance above, we can verify that $\varphi$ is bossy. Notice that under both orders (i) applicant 2 is selected, and (ii) applicant 1 is preferred to applicant 3. However, the overall selections differ.

**Example 8** (Monotone, non-bossy but not lower-invariant). Consider an instance with three applicants $I = \{1, 2, 3\}$. We define the selection rule $\varphi$ as follows:

1. If $3$ is not the worst priority applicant, then select applicant 3.

2. Otherwise, select applicants 1 and 2.

We begin by showing that no selection rule $OB_{F}$ can replicate the behavior of $\varphi$. Let

$$1 \succ 2 \succ 3 \text{ and } 1 \succ' 3 \succ' 2.$$ 

By 2. we have that $OB_{F}(\succ) = \{1, 2\}$ so $\{1, 2\} \in F$. In addition, by 1. we have that $OB_{F}(\succ') = \{3\}$. However, selection $\{3\}$ doesn’t respects priorities $\succ'$ as $1 \succ' 3$ and $\{1, 2\} \in F$.

It is easy to see that $\varphi$ is non-bossy. Notice that the two feasible selections are disjoint. Hence, if the outcome of an applicant remains unchanged, then the overall selection must be the same.

We now show that $\varphi$ is monotone. On one hand, if the priority of applicant 3 is improved, then she will be selected and weakly better. On the other hand, applicants $\{1, 2\}$ are only selected when they are in the first two positions. In this situation, the only improvement feasible is to increase the priority of the applicant in the second position. Of course, this maintains the overall selection and this agent is weakly better. Finally, we see that $\varphi$ is not lower-invariant. Notice that applicant 1 has the top priority in $\succ$ and $\succ'$. However, $\varphi$ only selects applicant 1 under $\succ$.

**Example 9** (Non-bossy, lower-invariant but not monotone). Consider an instance with two applicants $I = \{1, 2\}$. Let the selection rule $\varphi$ be defined as follows:

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1. If applicant 1 has the top priority, select all applicants.

2. Otherwise, select nobody.

We start by showing that no selection rule $OB_{\mathcal{F}}$ can replicate the behavior of $\varphi$. Let

$$1 \succ 2 \text{ and } 2 \succ' 1.$$

From 1. above, we must have that $OB_{\mathcal{F}}(\succ) = \{1, 2\}$ so $\{1, 2\} \in \mathcal{F}$. In addition, it should be that $OB_{\mathcal{F}}(\succ') = \emptyset$. Of course, selecting no applicants does not respects priorities $\succ'$ as applicant 2 has the first priority and $\{1, 2\} \in \mathcal{F}$.

From the instance above, we see that $\varphi$ is not monotone. Notice that the priority of applicant 2 is better in $\succ'$ than in $\succ$ but it is only selected under $\succ$.

Clearly, $\varphi$ is non-bossy. Note that $\varphi$ either selects all applicants or nobody. Hence, if the status of an applicant remains unchanged so does the overall selection.

Finally, we show that $\varphi$ is lower-invariant. This follows from the fact that the selection is determined by the applicant who has the top priority.

**B.2 Proofs from Section 3.4**

Proof of Lemma 4. For the sake of contradiction, suppose that there exists a priority order $\succ$ and a feasible selection $S$, such that $S$ is not priority dominated by $OB_{\mathcal{F}}(\succ)$. Since $OB_{\mathcal{F}}(\succ)$ respects priorities and doesn’t dominates $S$, we must have that $|OB_{\mathcal{F}}(\succ)| = |S|$. For each selection $S$ and integer $k \in \{1, \ldots, |S|\}$, we let $S^k \subseteq S$ be the subset containing the $k$ highest priority individuals in $S$, that is,

$$S_k = \{i^S_1, \ldots i^S_k\}.$$

Let $k^* \in \{1, \ldots, |S|\}$ be the highest index for which

$$i^S_{k^*} \succ OB_{\mathcal{F}}(\succ).$$
The existence of \( k^* \) is guaranteed as \( S \) is not dominated by \( OB_F(\succ) \). Consider \( OB_F(\succ)^{k^*-1} \) and \( S^{k^*} \). Because \( F \) induces a matroid, there must exist \( j \in S^{k^*} \setminus OB_F(\succ)^{k^*-1} \) such that \( OB_F(\succ)^{k^*-1} \cup \{j\} \in F \). Then \( OB_F(\succ) \) doesn’t respect priorities as \( j \notin OB_F(\succ) \) but can be selected without displacing any higher priority individual in \( OB_F(\succ) \). \( \square \)

B.2.1 Reserved Positions

Proof of Theorem 5. We start by considering the case of hard reserves. Here, our result follows immediately by noticing that \( F^{hard} \) induces the transversal matroid.

We now turn to maximum and maximal reserves. Let \( S \) and \( S' \) be two feasible selections of maximum cardinality, that is, \( |S| = |P| = |S'| \). It suffices to show that for any \( i \in S \setminus S' \) there is a \( j \in S' \setminus S \) such that \( S \setminus \{i\} \cup \{j\} \) is a feasible selection.

Suppose there is a maximal matching \( M \) of \( G \) such that \( I_M \subseteq S \) and such that \( i \notin I_M \). In this case, agent \( i \) is a “free agent” in the sense that \( i \) is given a position that they are not eligible for; observe that \( i \) can be replaced by any agent \( j \in S' \setminus S \) to obtain another feasible selection.

Thus we may assume that \( i \in I_M \) for every maximal matching \( M \) of \( G \) with \( I_M \subseteq S \).

Fix any such \( M \). Let \( p \) be the position assigned to \( i \) under matching \( M \). Starting with node \( i \), build the following alternating tree:

1. Start with the edge \((i, p)\).

2. Add all edges of the graph connecting the agents in \( S' \) to \( p \); Let \( W \) be the set of agents added in this step. Note that \( W \) is non-empty as otherwise \( S' \) cannot be a superset of a maximal matching (some neighbor of \( p \) can be added).

3. If for some \( j \in W, j \notin S \), then replacing \( i \) with \( j \) results in a feasible selection.

4. Every member of \( W \cap S \) should be matched to a position in \( M \): otherwise, we can find a maximal matching in which \( i \) is not matched to \( p \). So the tree can be extended with the corresponding matching edges.
Repeating steps (2) – (4) and noting that the process has to end, and it can only end in step (3), observe that we eventually find a $j$ such that replacing $i$ with $j$ results in a feasible selection. □

Soft Reserves

Proof of Proposition 10. We start by proving the hardness result for maximal reserves. We give a reduction from the problem “minimum maximal matching” (MMM) of deciding whether a bipartite graph $G$ admits a maximal matching of size at most $k$. This problem was shown to be NP-complete by Yannakakis and Gavril 1980. Consider an instance of MMM, that is, a bipartite graph $G = (U, V, E)$ and a positive integer $k$. We generate an instance $I(G)$ of the selection problem under maximal reserves as follows. We set $P = V$ and $I = I^1 \cup I^2$, where $I^1 = U$ and $I^2$ consists of $|V|$ individuals who are not connected to any position. Let the priority order $\succ$ place all individuals in $I^2$ ahead of all individuals in $I^1$. Then $G$ admits a maximal matching of size at most $k$ if and only if it there is a selection in $F_{\text{maximal}}(G)$ containing the first $|V| - k$ individuals from $\succ$.

We now prove the positive result for maximum reserves. Consider the compatibility graph $G = G(I, P, E)$. By the Gallai-Edmonds (actually Dulmage and Mendelsohn 1958) decomposition, the set of applicants $I$ can be partitioned into three parts $I_o, I_e$ and $I_u$ and similarly, the positions can be partitioned into three parts $P_u, P_e$, and $P_o$ such that every maximum cardinality matching in $G$ has the following structure:

- There is a perfect matching of the agents in $I_e$ with the positions in $P_e$.
- All the positions in $P_o$ are assigned to agents in $I_u$, and it is possible to find a matching that omits any specified agent in $I_u$.
- All the agents in $I_o$ are assigned to some position in $P_u$, and it is possible to find a matching that omits any specified position in $P_u$.

In particular, the agents in $I_e$ and $I_o$ are matched in all maximum cardinality matchings, and the agents do not consume any position in $P_o$; and all positions in $P_o$ are allocated to some subset
of agents in $I_u$.

Moreover, this decomposition can be found efficiently using, for example, an algorithm to find a maximum cardinality matching.

Given this result, a priority dominant selection under maximum reserves can be computed as follows: match all agents in $I_e$ and $I_o$; find a maximum-weight matching of the positions in $P_o$ to the agents in $I_u$; finally, pick, in decreasing order of priority, as many of the unassigned agents from $I_u$ as the number of unassigned positions in $P_u$.

B.2.2 Quotas

**Proof of Proposition 11.** This follows from Lemma 17 and/or Lemma 18, each of which proves the result for a special case of the general quotas problem. □

**Lemma 17.** Suppose there is a trait $0 \in T$, such that $I_0 = I$ and $\ell_0 = q$. Additionally, for every other trait $t \in T \setminus \{0\}$, $\ell_t = 0$. If the collection of feasible selections is defined as in (3.5), then

- determining whether there exists a feasible selection is NP-complete, and

- the selection that respects priorities might not priority dominate all other feasible selections.

**Proof of Lemma 17.** We first show that determining whether there exists a feasible selection is NP-complete, via a reduction from the Independent Set Problem. An instance of the Independent Set Problem is given by a graph $G = (V, E)$ and an integer $k$. An independent set of graph $G$ is a set $S$ of vertices such that no two vertices in $S$ are adjacent. The goal is to determine whether there exists an independent set containing at least $k$ elements. Given an instance of the Independent Set Problem, we construct an instance of the Selection Problem with upper quotas as follows: create one individual for each vertex, and one trait (with only two individuals) for each edge. Let each trait have an upper quota of 1, and let the overall lower quota be $k$. Then for any selection, the upper quotas are satisfied if and only if the set of vertices associated with the selected individuals is independent. Therefore, deciding whether there exists a feasible selection is equivalent to determining whether there is an independent set of size at least $k$.  

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We conclude by noting that Example 4 gives an instance satisfying the conditions of Lemma 17 in which the selection that respects priorities does not priority dominate all other feasible selections. □

**Lemma 18.** Suppose there is a trait $0 \in T$, such that $I_0 = \emptyset$ and $u_0 = q$. Additionally, for every other trait $t \in T \setminus \{0\}$, $u_t = |I|$. If the collection of feasible selections is defined as in (3.5), then

- determining whether there exists a feasible selection is NP-complete, and

- the selection that respects priorities might not priority dominate all other feasible selections.

**Proof of Lemma 18.** First, we show that determining whether there exists a feasible selection is NP-complete, using a reduction from the Set Cover Problem. An instance of the Set Cover Problem is given by a ground set $\mathcal{U}$, a collection $\mathcal{S}$ of subsets of $\mathcal{U}$, and an integer $k$. A cover for $\mathcal{U}$ is a subcollection $C \subseteq \mathcal{S}$ of sets whose union is $\mathcal{U}$. The objective is to determine whether there exists a set cover $C \subseteq \mathcal{S}$ for $\mathcal{U}$ containing at most $k$ elements. Given an instance of the Set Cover Problem, we construct an instance of the Selection Problem with lower quotas as follows: create one trait for each element in $\mathcal{U}$, and one individual for each set in $\mathcal{S}$ (having traits identified by the set). There is a lower quota of 1 for each trait, and an overall upper quota equal to $k$. Then a selection of individuals satisfies all minimum quotas if and only if the corresponding subsets cover $\mathcal{U}$. Therefore, determining whether there exists a feasible selection is equivalent to determining whether there is a set cover with at most $k$ elements.

Secondly, we give an instance where the selection that respects priorities does not priority dominate all other feasible selections. Consider an instance with four individuals $1, 2, 3, 4$, and three traits $T = \{0, A, B\}$. Let $I_A = \{2, 4\}$, $I_B = \{3, 4\}$, $\ell_A = 1 = \ell_B$ and $u_0 = 2$. Suppose that $1 \succ 2 \succ 3 \succ 4$. Then the feasible selection that respects priorities is $\{1, 4\}$. However, this selection doesn’t priority dominate selection $\{2, 3\}$. □

**Proof of Theorem 6.** Suppose that $I_T$ does not form a hierarchy. Then there exist traits $t_1$ and $t_2$ and individuals $i_1, i_2, i_3$ such that $i_1 \in I_{t_1} \setminus I_{t_2}, i_2 \in I_{t_2} \setminus I_{t_1}$, and $i_3 \in I_{t_1} \cap I_{t_2}$. 169
Set all lower quotas to be zero, set an upper quota of 1 for each of traits \( t_1 \) and \( t_2 \), and a (vacuous) upper quota of \( |I| \) for all other traits. Let \( 3 \succ 2 \succ 1 \) be the three highest-ranked individuals according to \( \succ \). Then the outcome based rule selects individual 3, along with everyone who does not have traits \( t_1 \) or \( t_2 \). However, it is also feasible to select individuals 1 and 2 along with everyone who does not have traits \( t_1 \) or \( t_2 \). This selection is not priority dominated by the selection of the outcome based rule. By Proposition 9, it follows that there is no feasible selection that priority dominates all others. \( \square \)

B.3 Proofs From Section 3.5

Proof of Theorem 7. In what follows, it will be convenient to relax the gender disparity requirement and allow for the gap between the number of male and female electees to be at most \( K \) for some \( K \geq 1 \). (The rule in the Chilean election requires \( K \) to be exactly 1.) Note that the specification of the rule in Chile applies just as well to any \( K \).

Let \( \mathcal{F}_K \) be the collection of feasible sets for the Chilean election problem under the condition that the gender gap is required to be at most \( K \). That is, every set \( S \in \mathcal{F}_K \) satisfies the following two properties: (i) the number of candidates in \( S \) from party \( i \) is exactly \( E_i \); and (ii) the difference between the number of men in \( S \) and the number of women in \( S \) is at most \( K \).

Suppose there are \( N \) candidates in all. Clearly,

\[
\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \ldots \subseteq \mathcal{F}_{N-1} \subseteq \mathcal{F}_N.
\]

We assume that \( \mathcal{F}_1 \) is nonempty so that there is always a feasible selection regardless of the gender disparity requirement.

We claim that the outcome of this procedure is a selection that respects priorities. Thus, the algorithm used in Chile can be seen as the outcome based rule applied to \( \mathcal{F}_K \).

We do this by showing that Algorithm 3 maintains the following invariant throughout its execution: suppose candidate \( j \) belonging to party \( i \) is not selected:
(a) if \( j \) is a minority gender candidate, then \( E_i \) higher priority candidates from party \( i \) are selected

(b) if \( j \) is a majority gender candidate, either \( E_i \) higher priority candidates from party \( i \) are selected, or the following conditions are both true:

(i) every lower priority candidate belonging to party \( i \) who is selected is a minority gender candidate; and

(ii) if a lower priority candidate of the majority gender is selected, every subsequent (i.e. lower priority) minority gender candidate from the same party is also selected.

First, we argue that any selection respecting the stated invariant respects priorities. Consider any candidate \( j \) who is not selected, and suppose \( j \) belongs to party \( i \). If \( E_i \) candidates of higher priority are already selected, then clearly \( j \) cannot be included in a feasible selection while respecting priorities. Thus, the only possibility is that \( j \) is a majority gender candidate, and at least one lower-priority candidate from party \( i \) is selected. Thus, \( j \) must have been replaced by a candidate \( \ell \) of the minority gender in step (B) of the algorithm to satisfy the gender disparity constraint. To include \( j \) and still satisfy both the party quota constraint and the gender disparity constraint, at least one lower-priority candidate from party \( i \) must be unselected, but every such candidate is a minority candidate. Thus, the only way to reduce the gender disparity is to unselect a lower-priority majority gender candidate of some party (necessarily other than party \( i \)) and add a minority gender candidate (of the same party). By, property (ii) no such choice exists.

Secondly, we prove that Algorithm 3 maintains the invariant. This is established by induction on \( K \): For \( K \) large enough, this is clear: The top \( E_i \) candidates from each party \( i \) are chosen, and each candidate that is not chosen satisfies (a) and (b) in the claim above: in fact (a) and (b) are true for all unselected candidates.

Suppose the claim is true for \( K \geq k \). We want to show that claim remains true for \( K = k - 1 \). Let \( S_k \) be the selection found by this algorithm when the gender gap is required to be at most \( k \). If \( S_k \in \mathcal{F}_{k-1} \), then the claim is clearly true (as \( \mathcal{F}_{k-1} \subseteq \mathcal{F}_k \).)

So the only non-trivial case to consider is when \( S_k \not\in \mathcal{F}_{k-1} \): that is the gender gap in \( S_k \) is
exactly \( k \). Suppose \( S_k \) includes more men than women. This implies that at least one of the men selected in \( S_k \) must not be selected and at least one woman not selected in \( S_k \) must be selected. Suppose the algorithm in step (B) replaces man \( m \) with woman \( w \). The choice made in step (B) of the algorithm is such that the invariant is maintained: this is immediate for all the candidates with a higher priority than \( m \) and all the candidates with a lower priority than \( w \); because a majority candidate is *unselected* and a minority candidate is *selected*, the only candidate for whom (b) needs to be verified is candidate \( m \), and both \((i)\) and \((ii)\) are easily verified. \( \square \)
Appendix C: Chapter 4

C.1 The Single-Item Case

C.1.1 Proof of Proposition 13

We start by showing that

\[ ABR \geq OPT. \]

Consider the following map from \( \tau \in \Delta(\Delta(V)) \) to \( (\lambda(\tau), \mu(\tau)) \in \Delta(A) \times \Delta(V) \times \Delta(V) \):

\[
\lambda(\tau) = (\lambda^0(\tau), \lambda^1(\tau)) = (\tau(U^0), \tau(U^1)),
\]

\[
\mu(\tau) = (\mu^0(\tau), \mu^1(\tau)) = \left( \frac{\int_{U^0} \mu d\tau(\mu)}{\lambda^0(\tau)}, \frac{\int_{U^1} \mu d\tau(\mu)}{\lambda^1(\tau)} \right).
\]

We will show that if \( \tau \) is feasible for OPT, then \( (\lambda(\tau), \mu(\tau)) \) is feasible for ABR. For notational convenience, in what follows we omit the dependency on \( \tau \).

First, we show that \( \lambda \in \Delta(A) \), and \( \mu^0, \mu^1 \in \Delta(V) \). From the definition of \( \lambda \) and \( \mu \) in (C.1) and (C.2), respectively, it follows that both \( \lambda \) and \( \mu^a \) are non-negative. Moreover,

\[ \lambda^0 + \lambda^1 = \tau(U^0) + \tau(U^1) = \tau(\Delta(V)) = 1. \]

The first equality follows from the definition of \( \lambda \) in (C.1). The second equality follows as

\[ U^0 \cap U^1 = \emptyset \text{ and } U^0 \cup U^1 = \Delta(V). \] (C.3)
Furthermore, we have that
\[
\int_{\mathcal{V}} \mu^a(dv) = \int_{\mathcal{V}} \int_{U^a} \mu(dv) \tau(d\mu) = \int_{U^a} \left( \int_{\mathcal{V}} \mu(dv) \right) \tau(d\mu) = \frac{\tau(U^a)}{\lambda^a} = 1.
\]

The first and last equality follow from the definitions of \( \lambda \) and \( \mu \) in (C.1) and (C.2), respectively. The second equality follows from Tonelli’s theorem. The third equality follows as \( \mu \in \Delta(\mathcal{V}) \) so \( \int_{\mathcal{V}} \mu(dv) = 1 \).

We now show that, for every \( v \in \mathcal{V} \), \( \sum_{a \in A} \lambda^a \mu^a(v) = \mu^0(v) \). From the definition of \( \mu \) in (C.2), we have that
\[
\lambda^0 \mu^0(v) + \lambda^1 \mu^1(v) = \lambda^0 \left( \int_{U^0} \mu \tau(d\mu) \right) + \lambda^1 \left( \int_{U^1} \mu \tau(d\mu) \right)
\]
\[
= \int_{\Delta(\mathcal{V})} \mu(v) \tau(d\mu) = \mu^0(v).
\]

The second equality follows from (C.3), and the third follows as \( \tau \) is Bayes-plausible.

We now turn to the last constraint remaining: \( \mathbb{E}_{v \sim \mu^1}[\ell(v)] \geq c \). From the definition of \( \mu^1 \) in (C.2), we have that
\[
\mathbb{E}_{v \sim \mu^1}[\ell(v)] = \int_{\mathcal{V}} \ell(v) \mu^1(dv)
\]
\[
= \int_{\mathcal{V}} \int_{U^1} \ell(v) \mu(dv) \tau(d\mu) = \frac{\int_{U^1} \ell(v) \mu(dv) \tau(d\mu)}{\lambda^1(\tau)}
\]
\[
= \frac{\int_{U^1} \mathbb{E}_{v \sim \mu}[\ell(v)] \tau(d\mu)}{\lambda^1(\tau)}\geq \frac{\int_{U^1} c \tau(d\mu)}{\lambda^1(\tau)} = c.
\]

The third equality follows from Tonelli’s theorem. The inequality follows as, for every \( \mu \in U^1 \),
\( z(\mu) \geq u_0 \) which implies \( \mathbb{E}_{v-\mu}[\ell(v)] \geq c \). To see this, define

\[
f(z; \mu) = \mathbb{E}_{v-\mu,E-\kappa}[\max\{v + E - z, 0\}].
\]  

(C.4)

Notice that \( f \) is decreasing in \( z \). By applying \( f \) to both sides of \( z(\mu) \geq u_0 \) we obtain

\[
c = f(z(\mu), \mu) \leq f(u_0, \mu) = \mathbb{E}_{v-\mu,E-\kappa}[\max\{v + E - u_0, 0\}] = \mathbb{E}_{v-\mu}[\ell(v)].
\]

To conclude, we show that the objective value induced by \( \tau \) in OPT and \((\lambda, \mu)\) in ABR coincide:

\[
\lambda^1 \int_{V} R(v)\mu^1(dv) = \int_{V} \int_{U^1} R(v)\mu(dv)\tau(d\mu) = \int_{U^1} \int_{V} R(v)\mu(dv)\tau(d\mu).
\]

The first equality follows from the definitions of \( \lambda \) and \( \mu \) in (C.1) and (C.2), respectively. The second equality follows from Tonelli’s theorem.

Now we will show that

\[
ABR \leq OPT.
\]

Consider the following map from \((\lambda, \mu) \in \Delta(A) \times \Delta(V)^{|A|}\) to \(\tau(\lambda, \mu) \in \Delta(\Delta(V))\):

\[
\tau(\mu) = \begin{cases} 
\lambda^a & \text{if } \mu = \mu^a, \\
0 & \text{otherwise}.
\end{cases}
\]  

(C.5)

We will show that if \((\lambda, \mu)\) is feasible for ABR, then \(\tau(\lambda, \mu)\) is feasible for OPT. Notice that it suffices to show that \(\tau(\lambda, \mu)\) is Bayesian-plausible. For every \(v \in V\), we have that

\[
\int_{\Delta(V)} \mu(v)\tau(d\mu) = \sum_{a \in A} \mu^a(v)\lambda^a = \mu^0(v).
\]

The first equality follows from the definition of \(\tau(\lambda, \mu)\) in (C.5). The second equality follows as
\((\lambda, \mu)\) is feasible for \(ABR\).

Finally, we show that the objective value induced by \(\tau(\lambda, \mu)\) in \(OPT\) is at least the objective value induced by \((\lambda, \mu)\) in \(ABR\). Since revenues are non-negative and \(\mu^1 \in U^1\), we have that

\[
\lambda^1 \int_{\mathcal{V}} R(v) \mu^1(dv) \leq \int_{U^1} \int_{\mathcal{V}} R(v) \mu(dv) \tau(d\mu).
\]

### C.2 The Two Items Case

#### C.2.1 Proof of Proposition 15

Consider the following map from \(\tau_i\) to \((\lambda_i^{< u_0}, \mu_i^{< u_0}) \cup \{(\lambda_i^z, \mu_i^z)\}_{z \in [u_0, \infty)}\). Let

\[
U_i^{< u_0} = \{\mu \in \Delta(\mathcal{V}) : z_i(\mu) < u_0\}, \quad (C.6)
\]

\[
\lambda_i^{< u_0} = \tau_i(U_i^{< u_0}), \quad (C.7)
\]

\[
\mu_i^{< u_0} = \frac{\int_{U_i^{< u_0}} \mu_i \tau_i(d\mu_i)}{\lambda_i^{< u_0}}. \quad (C.8)
\]

Similarly, for every \(z \geq u_0\), we define

\[
U_i^z = \{\mu \in \Delta(\mathcal{V}) : z_i(\mu) = z\}, \quad (C.9)
\]

\[
\lambda_i^z = \tau_i(U_i^z), \quad (C.10)
\]

\[
\mu_i^z = \frac{\int_{U_i^z} \mu_i \tau_i(d\mu_i)}{\lambda_i^z}. \quad (C.11)
\]

First, we will show that if \(\tau\) is feasible for \(OPT\), then \((\lambda, \mu)\) if feasible for \(RPR\). We start by showing that \(\lambda_i^{< u_0} + \int_{u_0}^{\infty} \lambda_i(\mu_i^z)dz = 1\):

\[
\lambda_i^{< u_0} + \int_{u_0}^{\infty} \lambda_i(\mu_i^z)dz = \tau_i(U_i^{< u_0}) + \int_{u_0}^{\infty} \tau_i(U_i^z) = \tau_i(\Delta(\mathcal{V})) = 1.
\]
The first equality follows from the definition of \( \lambda \) in (C.7) and (C.10). The second equality follows as \( U_i^{\leq u_0} \cup \{ U_i^z \}_{z \in [u_0, \infty)} \) forms a partition of \( \Delta(V) \).

Now we show that \( \lambda_i^{\leq u_0} \mu_i^{\leq u_0} + \int_{[u_0, \infty)} \mu_i^z \lambda_i(\mu_i^z) \, dz = \mu_i^0 \):

\[
\lambda_i^{\leq u_0} \mu_i^{\leq u_0} + \int_{[u_0, \infty)} \mu_i^z \lambda_i(\mu_i^z) \, dz = \int_{U_i^{\leq u_0}} \mu_i \tau_i(\,d\mu_i) + \int_{u_0}^{\infty} \int_{U_i^z} \mu_i \tau_i(\,d\mu_i) \, dz
= \int_{\Delta(V)} \mu_i \tau_i(\,d\mu_i) = \mu_i^0.
\]

The first equality follows from the definitions of \( \lambda \) and \( \mu \) in (C.7), (C.10) and (C.8), (C.11), respectively. The second equality follows as \( U_i^{\leq u_0} \cup \{ U_i^z \}_{z \in [u_0, \infty)} \) forms a partition of \( \Delta(V) \). The last equality follows as \( \tau_i \) is Bayesian-plausible.

The last set of constraints ensures that for every \( i \in N \) and \( z \geq u_0 \),

\[
\int_{\mathcal{V}} \ell(v_i + u_0 - z) \mu_i^z(\,dv_i) = c_i.
\]

From the definitions of \( \mu_i^z \) in (C.11), it follows that

\[
\int_{\mathcal{V}} \ell(v_i + u_0 - z) \mu_i^z(\,dv_i) = \frac{\int_{\mathcal{V}} \int_{U_i^z} \ell(v_i + u_0 - z) \mu_i(\,dv_i) \tau_i(\,d\mu_i)}{\lambda_i^z}
= \frac{\int_{U_i^z} \int_{\mathcal{V}} \ell(v_i + u_0 - z) \mu_i(\,dv_i) \tau_i(\,d\mu_i)}{\lambda_i^z}
= \frac{\int_{U_i^z} c_i \tau_i(\,d\mu_i)}{\lambda_i^z} = c_i.
\]

The second equality follows from Tonelli’s theorem. In the third equality we use that for every \( \mu_i \) in \( U_i^z \), \( z_i(\mu_i) = z \) or equivalently \( \mathbb{E}_{v \sim \mu_i}[\ell(v + u_0 - z)] = c_i \).

To conclude, we show that the objective value induced by \((\lambda, \mu)\) in RPR and the value induced by \( \tau \) in OPT coincide. First, we compare the terms in the objective function associated to the
search of a single item. From the definition of $\lambda$, $U_i^{<u_0}$ and $U_i^j$, it follows that

$$
\lambda_i^{<u_0} = \tau_i(U_i^{<u_0}) = \tau_i(U_i^j).
$$

Hence, it suffices to show that

$$
\int_{[u_0, \infty)} \lambda_i^z \int_{V} R_i(v_i) \mu_i^z(dv_i)dz = \int_{U_i^j} \int_{V} R_i(v_i) \mu_i(dv_i)\tau_i(d\mu_i).
$$

From the definitions of $\lambda$ and $\mu$, it follows that

$$
\int_{[u_0, \infty)} \lambda_i^z \int_{V} R_i(v_i) \mu_i^z(dv_i)dz = \int_{U_i^j} \int_{V} R_i(v_i) \mu_i(dv_i)\tau_i(d\mu_i)dz
$$

$$
= \int_{[u_0, \infty)} \int_{U_i^j} \int_{V} R_i(v_i) \mu_i(dv_i)\tau_i(d\mu_i)dz
$$

$$
= \int_{U_i^j} \int_{V} R_i(v_i) \mu_i(dv_i)\tau_i(d\mu_i).
$$

The second equality follows from Tonelli’s theorem. In the last equality we use that \{U_i^z\} \in [u_0, \infty) forms a partition of $U_i^j$:

$$
U_i^j = \{\mu \in \Delta(V) : z_i(\mu) \geq u_0\} = \cup_{z \in [u_0, \infty)} \{\mu \in \Delta(V) : z_i(\mu) = z\} = \cup_{z \in [u_0, \infty)} U_i^z.
$$

Finally, we compare the terms in the objective function associated to the search of both items. We must show that

$$
\lambda_i^l \int_{[t, \infty]} \lambda_1^z \int_{V_2 \in V} \int_{V_1 \in V} R_i^l(v_1, v_2) \mu_1^z(dv_1) \mu_2^z(dv_2)dz = 
\int_{U_1^j} \int_{V \times V} R_i^l(v_1, v_2) \mu_1(dv_1) \mu_2(dv_2) \tau_1(d\mu_1) \tau_2(d\mu_2).
$$

(C.13)
From the definitions of \( \lambda \) and \( \mu \), it follows that

\[
\lambda^2 \int_{[t, \infty)} \lambda \int_{v_2 \in V} \int_{v_1 \in V} R_1(v_1, v_2) \mu_1(dv_1) \mu_2(dv_2) dz =
\int_{[t, \infty)} \int_{v_2 \in V} \int_{v_1 \in V} R_1(v_1, v_2) \mu_1(dv_1) \tau_1(d \mu_1) \mu_2(dv_2) \tau_2(d \mu_2) dz =
\int_{t}^{\infty} \int_{U_1}^{U_2} \left( \int_{v_2 \in V} \int_{v_1 \in V} R_1(v_1, v_2) \mu_1(dv_1) \mu_2(dv_2) \tau_1(d \mu_1) \tau_2(d \mu_2) \right) dz =
\int_{U_1^{12t}} \int_{v_2 \in V} \int_{v_1 \in V} R_1(v_1, v_2) \mu_1(dv_1) \mu_2(dv_2) \tau_1(d \mu_1) \tau_2(d \mu_2).
\]

The second equality follows from Tonelli’s theorem. In the last equality, we use that

\[
U^{12t} = \{ \mu \in \Delta(V) : \ z_1(\mu) \geq t \} \times \{ \mu \in \Delta(V) : \ z_2(\mu) = t \}
\]

\[
= \bigcup_{z \in [t, \infty)} U_1^{z} \times U_2^{t}.
\]

C.2.2 Integer programming formulation

We now propose an integer programming reformulation of the quadratic programming problem (QP) presented in (4.33). For simplicity, we will assume that the set of values of the predictable component is finite, that is, \( |V| < \infty \).

Since all the constraints in (4.33) are linear, every feasible solution satisfies constraint qualification. Hence, the optimal solutions satisfy the KKT conditions. In order to introduce these conditions, we associate the dual variables \( \beta_i(v_i) \), \( \gamma_i^z \) and \( \omega_i^z(v_i) \) to the Bayesian-plausible constraints, obedience constraints and non-negative constraints, respectively. For the sake of notation, we let

\[
b_i^z(v_i) = \ell(v_i + u_0 - z) - c_i.
\]

Hence, for every item \( i \) and \( t \geq u_0 \), we can rewrite the obedience constraint as

\[
\sum_{v_i \in V} b_i^z(v_i) \alpha_i^t(v_i) = 0.
\]
Similarly, we let the matrix $A$ be such that $\sum_z (\alpha_1^z)\!^T A \!^z (\alpha_2^z)$ correspond to the objective function in (4.33).

Hence, the Lagrangian functions can be written as

$$L(\alpha, \beta, \gamma, \omega) = \sum_z (\alpha_1^z)^T A \!^z (\alpha_2^z) + \sum_i \sum_{v_i} \beta_i(v_i) \sum_z (\alpha_i^z(v_i) - \mu_i^0(v_i)) + \sum_i \sum_z \gamma_i z_i \sum_{v_i} \alpha_i^z(v_i) b_i^z(v_i) + \sum_i \sum_z \omega_i^z(v_i) \alpha_i^z(v_i).$$

The stationary conditions correspond to:

$$\nabla_{\alpha_1^z} L = A \!^z \alpha_2 + \beta_1 + \gamma_1 b_1^z + \omega_1^z = 0 \quad \forall z \geq u_0,$nabla_{\alpha_2^z} L = (A \!^z)^T \alpha_1^z + \beta_2 + \gamma_2 b_2^z + \omega_2^z = 0 \quad \forall z \geq u_0.$$

The primal feasibility conditions are:

$$\alpha_i^{<u_0}(v_i) + \int_{u_0}^{\infty} \alpha_i^z(v_i) \, dz = \mu_i^0(v_i) \quad \forall i, v_i \in \mathcal{V}$$

$$\sum_{v_i \in \mathcal{V}} \ell(v_i + u_0 - t) \alpha_i^t(v_i) = \sum_{v_i \in \mathcal{V}} \alpha_i^t(v_i)c_i \quad \forall i, t \geq u_0$$

$$\alpha_i^t(v_i) \geq 0 \quad \forall i, t \geq u_0, v_i \in \mathcal{V}.$$

The dual feasibility conditions are:

$$\omega_i^z(v_i) \geq 0 \quad \forall i, t \geq u_0, v_i \in \mathcal{V}. \quad (C.14)$$

The complementary slackness conditions are: for every item $i$ and reservation price $z$,

$$(\alpha_i^z)^T b_i^z = 0 \text{ and } (\alpha_i^z)^T \omega_i^z = 0. \quad (C.15)$$
We claim that under an optimal solution, we have that
\[
\sum_{\tilde{z}} (\alpha_1^\tilde{z})^T A^z \alpha_2^\tilde{z} = -(\mu_1^0)^T \beta_1 = -(\mu_2^0)^T \beta_2.
\] (C.16)

All in all, we can reformulate the QP in (4.33) as follows:

\[
- \min_{\tilde{v}_1 \in V} \sum_{\tilde{v}_1 \in V} \mu_1^0(\tilde{v}_1) \beta_1(\tilde{v}_1)
\]

subject to
\[
\begin{align*}
\alpha_i^{z,u_0}(v_i) + \int_{u_0}^{\infty} \alpha_i^z(v_i) dz &= \mu_i^0(v_i) \quad \forall i, v_i \in V \\
\int_{V} \ell(v_i + u_0 - z) \alpha_i^z(dv_i) &= \int_{V} \alpha_i^z(dv_i)c_i \quad \forall i, z \geq u_0 \\
A^z \alpha_2^z + \beta_1(v_1) + \gamma_1^z b_1^z(v_1) + \omega_1^z(v_1) &= 0 \quad \forall v_1 \in V, z \geq u_0 \\
(A^z)^T \alpha_1^z + \beta_2(v_2) + \gamma_2^z b_2^z(v_2) + \omega_2^z(v_2) &= 0 \quad \forall v_2 \in V, z \geq u_0 \\
\alpha_i^z(v_i) &\leq y_i^z(v_i) \quad \forall i, z \geq u_0, v_i \in V \\
\omega_i^z(v_i) &\leq M (1 - y_i^z(v_i)) \quad \forall i, z \geq u_0, v_i \in V \\
\omega_i^z(v_i) &\geq 0 \quad \forall i, z \geq u_0, v_i \in V \\
\alpha_i^z(v_i) &\geq 0 \quad \forall i, z \geq u_0, v_i \in V \\
y_i^z(v_i) &\in \{0, 1\} \quad \forall i, z \geq u_0, v_i \in V
\end{align*}
\] (C.17)

All that remains is to prove our claim (C.16). By summing and multiplying,

\[
0 = (\alpha_1^z)^T \nabla_{\alpha_i^z} L = (\alpha_1^z)^T A^z \alpha_2^z + (\alpha_1^z)^T \beta_1 + \gamma_1^z (\alpha_1^z)^T b_1^z + (\alpha_1^z)^T \omega_1^z.
\] (C.18)

From the complementary slackness conditions, it follows that, for every \( z \),

\[
(\alpha_1^z)^T b_1^z = 0 \text{ and } (\alpha_1^z)^T \omega_1^z = 0.
\]
\[
0 = \sum_z (\alpha_1^z)^T \nabla_{\alpha_1^z} L \\
= \sum_z (\alpha_1^z)^T A^z \alpha_2^z + (\sum z \alpha_1^z)^T \beta_1 \\
= \sum_z (\alpha_1^z)^T A^z \alpha_2^z + (\mu_1^0)^T \beta_1
\]

The last equality follows from the Bayesian-plausible constraint.

An analogous argument shows that

\[
0 = \sum_z (\alpha_1^z)^T A^z \alpha_2^z + (\mu_2^0)^T \beta_2.
\]