Taming unstable inverse problems

Mathematical routes toward high-resolution medical imaging modalities

François Monard

Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2012

ABSTRACT

Taming unstable inverse problems

François Monard

This thesis explores two mathematical routes that make the transition from some severely ill-posed parameter reconstruction problems to better-posed versions of them.

The general introduction starts by defining what we mean by an *inverse problem* and its theoretical analysis. We then provide motivations that come from the field of medical imaging.

The first part consists in the analysis of an inverse problem involving the Boltzmann transport equation, with applications in Optical Tomography. There we investigate the reconstruction of the spatially-dependent part of the scattering kernel, from knowledge of angularly averaged outgoing traces of transport solutions and isotropic boundary sources. We study this problem in the stationary regime first, then in the time-harmonic regime. In particular we show, using techniques from functional analysis and stationary phase, that this inverse problem is severely ill-posed in the former setting, whereas it is mildly ill-posed in the latter. In this case, we deduce that making the measurements depend on modulation frequency allows to improve the stability of reconstructions.

In the second part, we investigate the inverse problem of reconstructing a tensor-valued conductivity (or diffusion) coefficient in a second-order elliptic partial differential equation, from knowledge of internal measurements of *power density* type. This problem finds applications in the medical imaging modalities of Electrical Impedance Tomography and Optical Tomography, and the fact that one considers power densities is justified in practice by assuming a coupling of this physical model with ultrasound waves, a coupling assumption that is characteristic of

so-called *hybrid medical imaging methods*. Starting from the famous Calderón's problem (i.e. the same parameter reconstruction problem from knowledge of boundary fluxes of solutions), and recalling its lack of injectivity and severe instability, we show how going from Dirichlet-to-Neumann data to considering the power density operator leads to reconstruction of the full conductivity tensor via explicit inversion formulas. Moreover, such reconstruction algorithms only require the loss of either zero or one derivative from the power density functionals to the unknown, depending on what part of the tensor one wants to reconstruct. The inversion formulas are worked out with the help of linear algebra and differential geometry, in particular calculus with the Euclidean connection.

The practical pay-off of such theoretical improvements in injectivity and stability is twofold: (i) the lack of injectivity of Calderón's problem, no longer existing when using power density measurements, implies that future medical imaging modalities such as hybrid methods may make anisotropic properties of human tissues more accessible; (ii) the improvements in stability for both problems in transport and conductivity may yield practical improvements in the resolution of images of the reconstructed coefficients.

Contents

Acknowledgments **v**

Introduction **1**

Analysis of inverse problems 2

Motivation: recent challenges in medical imaging 9

Outline of the thesis 11

Research articles by the author 13

I Inverse transport and applications **14**

1 Introduction **15**

1.1 The albedo decomposition 20

1.2 Strategies of inversion 23

1.2.1 Direct inversion 24

1.2.2 Iterative improvement 25

2 Inverse transport and numerical methods **27**

2.1 From severe to mild ill-posedness in inverse transport 27

2.1.1 Inverse stationary transport with angularly averaged measurements . . . 27

2.1.2 Asymptotic behavior of time-harmonic solutions at high frequencies . . . 32

2.1.3 Inverse time-harmonic transport in two dimensions 37

2.2 Numerical methods for two-dimensional transport 42

II Inverse Conductivity/Diffusion from power densities 45

3 Introduction and preliminaries 46

3.1 Introduction 46

3.1.1 Hybrid medical imaging methods 46

3.1.2 Hybrid methods involving ultrasound modulations 53

3.1.3 Outline 60

3.2 Recalls on elliptic theory and Calderón's problem 62

3.2.1 Forward elliptic theory and properties of the power density operator . . . 62

3.2.2 The classical Calderón problem 66

3.3 Derivation of power density internal functionals 72

3.3.1 By acoustic perturbation 72

3.3.2 By thermoelastic effects 76

4 Local resolution techniques 80

4.1 Redefining the problem 81

4.1.1 Reformulation in terms of a frame 81

4.1.2 The orthonormal frame 83

4.2 Local reconstruction formulas in two dimensions 88

4.2.1 Preliminary properties and vector calculus identities 89

4.2.2 Local reconstruction formulas when $\tilde{\gamma}$ is known 89
4.2.3 Reconstruction of the anisotropy with additional functionals 100
4.3 Local reconstruction formulas in dimension $n \geq 3$ 108
4.3.1 Local reconstruction formulas when the anisotropy is known 109
4.3.2 Reconstruction of the anisotropy with additional functionals 130

5 Global reconstructions **141**
5.1 Global reconstruction schemes 141
5.1.1 The admissibility sets $\mathcal{G}_\gamma^m$, $m \geq n$ 143
5.1.2 Patching together local reconstructions of $|\gamma|$ 145
5.1.3 The strongly coupled elliptic approach when anisotropy is known 152
5.2 How to enforce the reconstructibility conditions 159
5.2.1 Properties of the admissibility sets 161
5.2.2 The two-dimensional case 166
5.2.3 The n-dimensional case, $n \geq 3$ 170

Conclusion and perspectives **177**
On inverse transport theory 177
On Ultrasound-modulated techniques 178

References **182**

A Linear algebraic matters **190**
A.1 The Gram-Schmidt process 190
A.2 Fischer's inequality 192
A.3 Establishing relations of linear dependence using inner products 197
A.4 The positive squareroot of a matrix 198

A.4.1 Construction for general n . . . 199
A.4.2 Example: the case $n = 2$ . . . 200

B Push-forwards by diffeomorphisms **202**

C Elements of Riemannian geometry **205**

D On complex geometrical optics solutions **209**

Index **212**

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my adviser Professor Guillaume Bal for five years of outstanding mentorship. Our relationship is always a tremendous benefit both professionally and personally, and I cannot think of a better way these past years could have turned out.

I am grateful to Profs. Marc Spiegelman, Leslie Greengard, Gunther Uhlmann and Vincent Duchêne for serving on my thesis defense committee, and for the fruitful discussion that followed the defense.

I want to personally thank Gunther Uhlmann, Michael Weinstein, David Keyes, Matias Courdurier, Steve McDowall, Ian Langmore, Alexandre Jollivet, Wenjia Jing, Kui Ren, Nick Hoell, Yu Gu, Ningyao Zhang, Chenxi Guo, Will Martin, David Goluskin, Kate Eckerle, Olivier Pinaud, Éric Bonnetier and Faouzi Triki, collaborators and colleagues, with whom I have shared fruitful scientific discussions, lunch, paper-folding or chess games.

My special thanks to Cédric Bellis, Sébastien Imperiale and Lauri Oksanen for very valuable help toward the end of my dissertation.

I am also very thankful to the administrative staff at the Department of Applied Physics and Applied Mathematics at Columbia for their ever-present professionalism, availability and friendliness: Dina Amin, Ria Miranda, Montserrat Fernandez-Pinkley, Christina Rohm and Marlene Arbo.

To some extent, this work is somewhat indissociable from the five years of life in New York that came with it, a rather chaotic environment which, if taken in full, leads to a human being made of the most random capoeira/DJ/origami/climbing/yoga skills. I have come across people who changed me in many ways and I want to acknowledge their indirect contribution to this work: Mestre João Grande and the Capoeira Angola Center, the Low Country Capoeira Angola

Society, the Angoleiros do Brooklyn, FICA New York, Ali Florian, Paul "Metrosonics" Josephs, Ria Miranda, Taagen Swaby, Chike Nwabukwu, Lia Hall, Cedar Mannan, Leola Bermanzohn, Gya Watson, Jamel Brinkley, Isadora Vieira, Jason Venzor, Gabrielle Bromberg, Florent Scarpel and Marie Pellet. If anything, I am a little bit of all of you.

Finally, I would like to thank my family in France, Tunisia and Togo, for being an endless source of energy, inspiration and support.

À mes parents, Armand et Anne-Marie

THIS PAGE INTENTIONALLY LEFT BLANK

Introduction

Inverse problems involving partial differential equations (PDE's) consist in the reconstruction of the constitutive parameter of a given PDE, from knowledge of functionals that depend on these parameters, the solutions of that PDE and possible additional parameters. Such problems arise in every situation where one can only obtain remote information about a given body without penetrating it. Some fields of applications are

Medicine: Medical imaging is the typical application where one cannot use invasive techniques in order to get information inside of a patient's body. Rather, one collects information outside the patient (e.g. radiated energy, external displacement) and attempts to reconstruct internal parameters (e.g. tissue's optical, conductive, elastic properties) or functional information (e.g. metabolism) for monitoring and diagnosis purposes.

Geophysics: In geophysical imaging, one collects seismic measurements at the Earth's surface and aims at reconstructing the density deeper inside, for e.g. prospection purposes.

Environmental sciences: In atmospheric imaging, one may measure light scattered by clouds in order to assess the sizes, concentrations and composition of particles inside the cloud.

Other fields of application include, but are not limited to, finance, material science, nuclear science, plasma physics, etc...

Before going into the details of the motivation of this work, we need to make more precise our mathematical view of an inverse problem, in particular, what the main steps are in analyzing an inverse problem theoretically.

Analysis of inverse problems

In mathematical terms, an *inverse problem* (IP) formulates an attempt to invert a functional relation (or *forward mapping*) of the form

$$y = \mathfrak{M}(x), \quad \text{for } x \in \mathfrak{X} \quad \text{and} \quad y \in \mathfrak{Y}, \tag{1}$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ denote functional spaces, typically Banach or Hilbert spaces. $\mathfrak{M}$ stands for the *measurement operator*, x stands for the *parameter* (i.e. the unknown) and y stands for the *data* or *measurement*. When the inverse problem is PDE-based, the relation (1) sometimes mean "y is a functional of the constitutive parameter x of a PDE", which at first comes more under the form of an implicit relation $\mathfrak{F}(x, y) = 0$ rather than (1).

From the theoretical analysis to the practical implementation of inversion algorithm, the analysis of a given inverse problem consists in several tasks, each of which constitutes a domain of expertise of its own. For the sake of brevity we will restrict ourselves to describing what steps the theoretical analysis of a given inverse problem consists in, namely, *injectivity* and *stability* assessments as well as the derivation of *explicit inversions* if possible. We may mention issues of *noise in measurements* and *numerical implementation*, though we refer the reader to e.g. Natterer (2001); Bal (2004); Isakov (2006); Kaipio and Somersalo (2004); Bal (2012d); Scherzer (2011) for more thorough accounts on these topics and other ones such as *statistical inversions*.

Injectivity

The first step in the theoretical analysis of an inverse problem consists in assessing its injectivity. Namely, one must quantify how much information of the parameter has been passed on to the data through the functional mapping $\mathfrak{M}$. In order to prove injectivity, one must satisfy the property

$$\mathfrak{M}(x_1) = \mathfrak{M}(x_2) \quad \Longrightarrow \quad x_1 = x_2, \quad \text{for all } x_1, x_2 \in \mathfrak{X}. \tag{2}$$

If the operator $\mathfrak{M}$ is not injective, the next question is to analyze how well one can characterize this lack of injectivity. In particular, if one is able to find an *equivalence relation* $\sim$ (or *gauge transform*) such that $\mathfrak{M}$ becomes injective over the quotient space $\mathfrak{X}/\sim$, the first thing to do is to replace the initial inverse problem by its injective counterpart $y = \widetilde{\mathfrak{M}}(x), x \in \mathfrak{X}/\sim$ before going to the next step. Heuristically, since there is no way to reconstruct x other than modulo the relation $\sim$, we might as well give up on that information. Examples of non-injective inverse problems where the lack of injectivity has been properly characterized with a gauge transform are the anisotropic two-dimensional Calderón problem (Astala et al., 2005), an inverse problem in transport with varying index of refraction (McDowall et al., 2010), or more recently, the determination of the coefficients of an elliptic PDE from knowledge of its solutions (Bal and Uhlmann, 2012a).

Stability

For a problem where injectivity is proved, a stability statement describes for what topologies the inverse $\mathfrak{M}^{-1}$ is continuous. Stability is therefore a statement of the form

$$\|x_1 - x_2\|_{\mathfrak{X}} \le \omega(\|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{\mathfrak{Y}}), \quad x_1, x_2 \in \mathfrak{X}, \tag{3}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $\omega(0) = 0$ describing the modulus of continuity of $\mathfrak{M}^{-1}$.

As explained in Bal (2012d), the notion of stability is subjective in that it is relative to the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ in which (3) holds. Moreover, any modulus of continuity can be changed into a linear one by adjusting properly the distance function on the space $\mathfrak{Y}$. While all possible stability statements for the same IP express comparable intrinsic qualitative properties of that IP, the correct stability estimate to pick will be the one such that, when taking into account noise in the measurements, that noise must live in the space $\mathfrak{Y}$ somehow, and we will explain in the next section how stability impacts on resolution in practical applications, once noise is known.

Among possible spaces that one may choose for $\mathfrak{X}$ and $\mathfrak{Y}$, if the measurement y is a function, the space $\mathfrak{Y}$ may be as exotic a functional space as one wants: Hölder-continuous spaces $\mathcal{C}^{k,\alpha}(\Omega)$, Sobolev spaces $W^{n,p}(\Omega)$ (including the Hilbert case $H^2(\Omega) := W^{2,p}(\Omega)$), or any weighted version of them, where Ω denotes $\mathbb{R}^n$ or some sub domain of it. The choice of the appropriate functional space often depends on the underlying structure of the mathematical problem. If y is an operator then $\mathfrak{Y}$ is a space of operators. For instance in the Caldeón problem (see Calderón (1980)) where y is the Dirichlet-to-Neumann map, a natural choice for $\mathfrak{Y}$ is $\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega)))$.

While all these spaces may reflect different versions of the mathematical concepts of derivative, regularity and singularity, let us mention as a first rule of thumb that the intrinsic stability of an IP is locally governed by "how regularizing" the forward operator $\mathfrak{M}$ is. Namely, whenever this makes sense at all, the stability degrades with the number of times one needs to "integrate" in order to pass from the input x to the output y. This fact alone allows to classify inverse problems into three categories which differ mostly by their stability properties, which we now describe in more detail.

Well-posed problems: These are the ones where the measurement operator $\mathfrak{M}$ does not regularize, as is the cases for isometries of L^2 such as the Fourier Transform for instance, upon which the medical imaging modality of *magnetic resonance imaging* is based. Another example is that of inverse wave problems (with applications in *echography* and other methods involving a coupling with acoustic waves), as the wave propagator is an isometry of L^2 or some other H^s space. Typical well-posed stability statements are statements of the form (3) with Lipschitz-type continuity moduli $\omega(x) = Cx$, with $\mathfrak{X} = \mathfrak{Y}$ are of equal power on the same scale of functional spaces. As an example, if $\mathfrak{M}$ is the Fourier transform, then we have the stability estimate

$$\|x_2 - x_1\|_{L^2(\mathbb{R}^n)} \leq C\|\mathfrak{M}(x_2) - \mathfrak{M}(x_1)\|_{L^2(\mathbb{R}^n)}. \tag{4}$$

Mildly ill-posed problems: Heuristically, these are the problems where the measurement operator "integrates a finite number of times". Examples of mildly ill-posed stability statements, using the Hilbert scale $\{H^s\}_{s\in\mathbb{R}}$, may be written as

$$\|x_1 - x_2\|_{H^s} \leq C\|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{H^{s+t}}, \tag{5}$$

for some constant C and $t \geq 0$ (the smallest t such that (5) holds may be called the *order* of ill-posedness of the problem). Thus $\mathfrak{M}^{-1}$ is Lipschitz-continuous from H^s to H^{s+t}. In the same space however, the modulus of continuity becomes of Hölder type: making the *a priori* assumption that $x_1 - x_2 \in H^{s+t'}$ with $t' \geq t$, that is, $\|x_1 - x_2\|_{H^{s+t'}} \leq E < \infty$, and using interpolation, we deduce the estimate

$$\|x_1 - x_2\|_{H^{s+t}} \leq \|x_1 - x_2\|_{H^s}^{\alpha}\|x_1 - x_2\|_{H^{s+t'}}^{1-\alpha} \leq CE^{1-\alpha}\|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{H^{s+t}}^{\alpha} \tag{6}$$

where $\alpha = \frac{t'-t}{t'} \in [0,1]$, thus a mildly ill-posed problem may be rewritten with Hölder type moduli of continuity.

Examples of such problems are the inversion of the X-Ray transform[1], ill-posed of order $\frac{1}{2}$, as well as the inversion of the Radon transform[2], ill-posed of order $\frac{n-1}{2}$ in dimension n, see e.g. Natterer (2001). As a toy example, the problem of reconstructing a function $f : [0,1] \to \mathbb{R}$ from knowledge of its N-th antiderivative is clearly mildly ill-posed of order N.

Severely ill-posed problems: When the order (as defined in the previous section) cannot be finite, the ill-posedness is called *severe*. In this case, the operator $\mathfrak{M}$ is a smoothing operator, or loosely speaking, $\mathfrak{M}$ "integrates infinitely many times". In this case, the modulus of continuity that one may derive is of logarithmic type, i.e. one obtains an estimate of the form

$$\|x_1 - x_2\|_{\mathfrak{X}} \leq C|\log \|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{\mathfrak{Y}}|^{-\beta} \quad \text{for some } \beta > 0. \tag{7}$$

In cases where the theory of Fourier integral operators (FIO's, see Hörmander (1971)) helps describing the operator $\mathfrak{M}$, the notion of *singularity*[3] (in particular, the process of losing the singularities of x when applying $\mathfrak{M}$) describes very well what severe ill-posedness consists in. If $\mathfrak{M}$ is a smoothing operator, which implies that $\mathfrak{M}(x)$ contains no singularities no matter how wild x may be (even a distribution), then it is clear that no inequality of the form (5) can be derived even for large t.

For example, in inverse problems described by PDE's, the reconstruction of an initial condition of a parabolic PDE from knowledge of either (i) its solution at a later time, or (ii) the trace of that solution at the boundary over some time interval, is a severely ill-posed problem. This is

[1] the transform that maps a scalar function to the collection of its line integrals

[2] the transform that maps a scalar function to the collection of its hyperplane integrals

[3] i.e. elements of the *wavefront set*, see e.g. Hörmander (1971)

because the heat propagator has a smoothing kernel. Likewise, IP's involving elliptic PDE's, e.g. reconstruction of internal parameters from boundary measurements such as *Calderón's problem*, are severely ill-posed; in this case, the analysis of elliptic pseudo-differential operators shows that the singularities inside the domain never make it to the boundary, as they actually stay in place. A toy example of a severely ill-posed IP is the reconstruction of a function $f \in L^\infty(\mathbb{R})$ from knowledge of its convolution $f \star g$, with g a fixed integrable, $\mathcal{C}^\infty$-smooth function.

Another class of severely ill-posed problems is that of IP's with *partial data*. Although they may remain injective, their stability is highly degraded, since one might throw away some singularities in the process of restricting the measurements y. In this case, when an input x is such that all of its singularities were thrown away, the measurements that remain are $\mathcal{C}^\infty$-smooth while the input could be a distribution. Again, this results in severe ill-posedness. Such a story happens when one considers the problem of inverting the X-Ray transform (a problem that was initially described as mildly ill-posed) from either (i) limited-angle data, or (ii) exterior data, see e.g. Natterer (2001) for a treatment of these problems.

How stability impacts resolution in practical applications

Now that we have described the possible types of stability that one may encounter, let us explain how the theoretical stability rules the resolution on reconstructed x's in practice. In a way, the stability statements (4), (6) or (7) quantify how errors on measurements translate into errors in the reconstruction. If the error expected on the reconstructed quantity is of the form

$$\|x_1 - x_2\|_{\mathfrak{X}} \leq \varepsilon,$$

then the conditions that this requires on the measurement error $\|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{\mathfrak{Y}}$ is all the more stringent that the IP is ill-posed:

- For a well-posed problem with stability (4), we need to make a measurement error of size $C^{-1}\varepsilon$.
- For a mildly ill-posed problem with α-Hölder stability (6) with $0 < \alpha < 1$, this requires to make an error of size $(C^{-1}\varepsilon)^{\frac{1}{\alpha}}$, which becomes increasingly drastic as α decreases to zero.
- For a severely ill-posed problem with stability (7) and $\beta > 0$, this requires a measurement error of size $\exp(-(C^{-1}\varepsilon)^{\frac{1}{\alpha}})$, which becomes increasingly drastic as β increases.

Conversely, if the measurement error is fixed, the control on the reconstruction error decreases with the lack of stability.

Usually, in order to salvage stability in a ill-posed problem, one needs to restrict the possible values of the unknown, i.e. choose a smaller space $\mathfrak{X}$. This is the process of *adding prior knowledge* to the unknown by assuming to know that some representation of it is sparse, so that the amount of unknown information that describes x is reduced. As an example, if the unknown is assumed to belong to a space of increasingly smoother functions (H^p with p large for instance), we see that estimates (6) and (7) improve. Indeed, in the first case, the Hölder exponent α increases with t', and in the second case, one can also show (see Santacesaria (2011) for instance) that β increases, in turn improving stability.

The drawback of such an assumption on x is that one is giving up on its high-frequency content, which contains the sharp variations and fine details of x. This is because imposing smoothness on x amounts to enforcing strong decay at infinity of its Fourier Transform, thus limiting its high-frequency content, i.e. the size of the details of x. Thus, stability on the low-frequency part of x can be restored by giving up on the high-frequency part of x. In the light of the first paragraph, how much one must give up in order to recover stability is precisely what is dictated by the ill-posedness of the problem. The more ill-posed a problem is, the more frequencies of x one must give up in order to stably recover a blurred-out version of x, i.e. the lower one must set the threshold between the low- and high-frequency parts.

Explicit inversions

While uniqueness and stability may be established without having an explicit expression for the inverse operator $\mathfrak{M}^{-1}$, and numerical inversions do not always require an explicit inversion formula either, research is always active in trying to invert inverse problems explicitly, as they provide additional insight on the interdependence between the unknown x and the measurements y. Whenever one can derive such formulas, injectivity and stability statements are often greatly simplified, as they reduce to the analysis of an explicit operator. These are however not always available, and in certain cases, despite the fact that injectivity and stability may be proven, explicit inversions may or may not exist.

Motivation: recent challenges in medical imaging

Our main application of focus in this manuscript is medical imaging, where the pool of techniques at hand is known to be split between high-contrast modalities such as Optical Tomography, Electrical Impedance Tomography and Elastography, and high-resolution modalities such as Echography, X-Ray Computerized Tomography (CT) and Magnetic Resonance Imaging (MRI). By *contrast* (a.k.a. specificity) we mean a measure of how relevant the quantity to be imaged is in order to discriminate between healthy and unhealthy tissues, while *resolution* is important for an accurate localization of anomalies (e.g. tumors in their early stages). Since all of the afore mentioned techniques are modelled by inverse problems, the discussion from the past section applies, and the resolution available in these techniques may be described in terms of theoretical stability assessments (together with proper models for the noise in the measurements). The contrast, on the other hand, is known by experience.

The high-contrast modalities are crucial for medical diagnosis as they have high tumor specificity, however all these problems involve the reconstruction of the constitutive parameter of an elliptic PDE, which, as we briefly mentioned earlier, is a severely ill-posed problem. As a result, such methods suffer from low resolution. Typically for EIT, the accessible resolutions are of the order of a centimeter, while millimeter-precision is desired in practice. On the other hand, Echography involves an inverse wave problem, MRI involves an inverse Fourier Transform and X-Ray CT involves an inverse X-Ray transform, all of which were previously described as well-posed or mildly ill-posed problems. These methods thus offer good resolution but the parameters they measure do not vary enough from healthy to unhealthy tissues for proper discrimination.

The question that comes to mind is, is this all we can have ? Is this all what physics can offer us ? How can we improve resolution when physics has told us we reached the limit ? Improving the stability of the high-contrast methods so we can add resolution to them is some sort of a challenge of beating mathematical physics at its own game.

Once an inverse problem is defined, its intrinsic stability (or lack thereof) and the limit in resolution that results from it cannot be improved from the theoretical standpoint. In practice, technological advances may reduce noise in measurements and improve reconstructions a bit, but this is not enough to go beyond the limit imposed by the mathematical ill-posedness, an issue that can only be addressed by adjusting the theoretical model. Specifically, one has to change the inverse problem into another one, either by (i) changing the model, (ii) changing the form of the measurement functionals (i.e. make them “richer”, more informative in some sense), or (iii) changing the unknown (i.e. make its range of possible values “smaller” by adding prior information).

Trying to improve stability of ill-posed inverse problem is the underlying goal of all research presented in this manuscript, of which we now give a brief outline.

Outline of the thesis

Part I: Improvement by changing regime. A first strategy is to enrich the measurement data by changing the regime considered (e.g. from stationary to time-harmonic). Indeed, by making the measurement operator depend upon additional variables and provided that they are not redundant with the ones that are already present, it is likely that the data will contain "more" information about the unknown. In particular, if the data contains more singularities of the unknown, it will improve the stability of the corresponding inverse problem in the end. Following this idea, we present in this part an example of an inverse problem involving the *Boltzmann* transport equation, where going from the stationary regime to the time-harmonic regime, everything else being equal, tremendously improves the theoretical stability of the problem of reconstructing the spatial part of the scattering kernel. This work has led to four research articles (Bal et al., 2008; Bal and Monard, 2010; Bal et al., 2011a; Monard and Bal, 2012d) and the present manuscript aims at summarizing them, inscribing them into the bigger picture without including the whole papers as an attempt to provide a digested, more understandable account of this work.

Part II: Improvement by coupling modalities. The second strategy may appear somewhat similar to the first, in that the net result on the inverse problem considered is a qualitative change in the measurements. However, the way it is done is by coupling two physical models together. This idea comes from the fact that the poor resolution of high-contrast medical imaging techniques could be improved by coupling them physically with a high-resolution modality, in the end benefiting from the joint strengths of both techniques. This philosophy has led to the field of so-called *hybrid* (a.k.a. coupled-physics, multi-wave) medical imaging methods, which has received a lot of attention over the past decade and a half and will be described in more detail in Chapter 3.

In this context, the problem considered is the reconstruction of a (conductivity or diffusion) tensor in an elliptic PDE, from internal measurements of *power density* type. This problem results from a coupling model between acoustic waves and either Optical- or Electrical Impedance- Tomography. The main breakthrough here is precisely that the measurements are supported inside the domain rather than at its boundary. Such functionals are described as having a "stabilizing" effect on the inverse problem considered, which means that they will improve the resolution of reconstructions in practice. Generating these internal functionals practically does not mean that the measurement procedure becomes invasive. Rather, they are the result of a two-step inversion process, where one of the steps is a well-posed problem (usually involving waves) that allows to backtrack stably a quantity at the domain's interior from boundary measurements.

This part is longer than the first one, as it tries to cover all aspects of this problem in its greater generality (fully anisotropic tensors and general dimensions), using results that the author has previously established in Bal et al. (2012a); Monard and Bal (2012a,c) with collaborators, but also including work in progress (Monard and Bal, 2012b).

Although both problems considered rely on a common PDE background (they are motivated by physics), the techniques of resolution used in the two parts are fairly different in spirit. Part I involves a great deal of analysis, in particular stationary phase. Part II on the other hand involves a fair amount of analysis, linear algebra and geometry, as we will see that the problem can be rather naturally treated using geometric tools, in particular calculus using connections on manifolds.

Research articles by the author

Bal, G., Bonnetier, E., Monard, F., and Triki, F. (2012a). Inverse diffusion from knowledge of power densities. *to appear in Inverse Problems and Imaging*. arXiv:1110.4577. 12

Bal, G., Jollivet, A., Langmore, I., and Monard, F. (2011a). Angular average of time-harmonic transport solutions. *Comm. PDE*, 36(6):1044–1070. 11

Bal, G., Langmore, I., and Monard, F. (2008). Inverse transport with isotropic sources and angularly-averaged measurements. *Inverse Probl. Imaging*, 2(1). 11

Bal, G. and Monard, F. (2010). An accurate solver for forward and inverse transport. *Journal of Comp. Phys.*, 229(13). 11

Monard, F. and Bal, G. (2012a). Inverse anisotropic diffusion from power density measurements in two dimensions. *to appear in Inverse Problems*. arXiv:1110.4606. 12

Monard, F. and Bal, G. (2012b). Inverse conductivity from power density functionals, the general case. in preparation. 12

Monard, F. and Bal, G. (2012c). Inverse diffusion problems with redundant internal information. *to appear in Inverse Problems and Imaging*. arXiv:1106.4277. 12

Monard, F. and Bal, G. (2012d). Inverse transport with isotropic time-harmonic sources. *SIAM J. Math. Anal.*, 44(1):134–161. 11

Part I

Inverse transport and applications

Chapter 1

Introduction

The *Boltzmann transport equation* models the linear transport of particles as they travel through a medium and interact with it via absorption and scattering phenomena. In medical imaging, this equation models the transport of photons through human tissues. It also has applications in atmospheric and oceanographic sciences, neutron transport in nuclear reactors, as well as seismology at the level of the Earth's crust, see Bal (2009) for a review on inverse transport and its applications. It is a rich model, posed in phase space, which, depending on the regime, may present very different qualitative properties from the viewpoint of inverse problems. In particular, the regime one considers greatly impacts the stability of the associated inverse problems, as the next examples may show.

- High energy photons (e.g. in X-Ray Computerized Tomography) scatter very little in human tissues, in which case photons travel along straight lines and the information they collect along their path is well-localized along these straight trajectories. From the inverse problems viewpoint, this regime leads to the most explicit and stable inversions, see Choulli and Stefanov (1999, 1996b,a); Bal and Jollivet (2010b, 2008) and their references.

- Low energy photons (e.g. at optical and near infrared wavelengths in the context of Optical Tomography) change direction so often that in nature, the dynamics looks more lke a diffusion process. Such regime is highly-averaging and the information one collects at the boundary is much smoother than the unknown coefficients of the PDE may be. An accurate approximation model for this regime is a parabolic (or, in the stationary case, elliptic) PDE, obtained either as a first non-trivial approximation in spherical harmonics of the Boltzmann equation[1] (Arridge, 1999), or as an asymptotic model for high (and not too forward-peaked) scattering (Bal, 2009). Reconstructing the parameters of the resulting PDE from measurements of outgoing photons at the boundary results in severely ill-posed inverse problems, see e.g. Bal et al. (2008).

- Media with highly-peaked scattering yield solutions of pencil-beam type (i.e. including diffusion in the directions that are transverse to propagation), in the limit becoming Fokker-Planck regime (diffusion in the angular variable), see Leakeas and Larsen (2001).

The transport equation is the theoretical backbone of X-Ray Computerized Tomography (CT), Single Proton Emission CT, Optical Tomography as well as several imaging methods in atmospheric imaging related to assessment problems of e.g. droplet size or densities of aerosols in clouds.

At optical wavelengths, depending on the optical depth of the medium, the transport equation model can be an acceptable model for Optical Tomography (Arridge, 1999), where one wants to reconstruct optical properties of tissues (absorption and scattering coefficients) from outgoing boundary measurements of solutions. The optical properties display high contrast, in the sense that healthy cells and cancerous cells may have very different optical properties; in

[1] more accurate models may be derived by keeping more terms in the spherical harmonic extension, although we do not follow this route further here.

particular, these parameters can reveal crucial biological information such as hypermetabolism and angiogenesis, both of which are well-known indicators of cancer (Wang and Wu, 2007).

Restricting to our purposes and considering $X \subset \mathbb{R}^d$ an open, bounded domain with $\mathcal{C}^1$ boundary, the model for this equation takes the form

$$\begin{aligned} \partial_t u + v \cdot \nabla_x u(x,v) + \sigma(x,v)u(x,v) &= \int_{\mathbb{S}^{d-1}} u(x,v')k(x,v',v)\, dS(v'), \quad \text{on } X \times \mathbb{S}^{d-1}, \\ u|_{\Gamma_-} &= f, \end{aligned} \tag{1.1}$$

where $u(x,v)$ denotes the density of particles about x with velocity v, and the functions $\sigma(x,v)$ and $k(x,v,v')$ are referred to as *absorption* coefficient and *scattering kernel*, respectively (they are also jointly called the *optical parameters*). We have also defined in (1.1)

$$\Gamma_\pm := \{(x,v) \in \partial\Omega \times \mathbb{S}^{d-1}, \pm v \cdot \nu_x > 0\}, \tag{1.2}$$

where ν_x is the outward unit normal at $x \in \partial\Omega$. In the present unit speed case, we define on $\Gamma_\pm$ the following measure

$$d\xi(x,v) = |\nu_x \cdot v|\ d\mu(x)\ dS(v), \tag{1.3}$$

where $d\mu(x)$ and $dS(v)$ denote the surface measures on ∂X and $\mathbb{S}^{d-1}$, respectively. More general settings with non-unit speed are also studied in e.g. Choulli and Stefanov (1999); Dautray and Lions (1993). For $(x,v) \in (X \times \mathbb{S}^{d-1}) \bigcup \Gamma_+ \bigcup \Gamma_-$, let $\tau_\pm(x,v)$ be the distance from x to ∂X traveling in the direction of $\pm v$, and $x_\pm(x,v) = x \pm \tau_\pm(x,v)v$ be the boundary point encountered when we travel from x in the direction of $\pm v$, and $\tau := \tau_+ + \tau_-$.

The term σu accounts for loss of particles in direction v due to absorption and scattering, more precisely σ admits the decomposition

$$\sigma(x,v) = \sigma_a(x,v) + \int_{\mathbb{S}^{d-1}} k(x,v,v')\, dS(v') := \sigma_a(x,v) + \sigma_p(x,v), \tag{1.4}$$

where the first term accounts for particles truly absorbed by the medium and the second accounts for particles that changed direction due to scattering (considering all particles globally, the scattering process neither creates nor annihilates particles). The natural (admissible) setting for such coefficients is

$$\begin{aligned} &0 \le \sigma_a \in L^\infty(X \times \mathbb{S}^{d-1}), \qquad 0 \le k(x,v',\cdot) \in L^1(\mathbb{S}^{d-1}) \text{ for a.e. } (x,v') \in X \times \mathbb{S}^{d-1}, \\ &\text{and } \int_{\mathbb{S}^{d-1}} k(x,v',v)\, dS(v) \in L^\infty(X \times \mathbb{S}^{d-1}). \end{aligned} \tag{1.5}$$

Existence theory of solutions for the PDE (1.1) is established in Choulli and Stefanov (1999); Dautray and Lions (1993) and is stated as follows: under either of the following assumptions:

$$\|\tau\sigma_p\|_{L^\infty(X\times\mathbb{S}^{d-1})} < 1, \tag{1.6}$$

$$\sigma_a(x,v) \ge \sigma_0 > 0, \quad \text{for a.e. } (x,v) \in X \times \mathbb{S}^{d-1}, \tag{1.7}$$

and for $f \in L^1(\Gamma_-, d\xi)$, the solution of (1.1) exists and admits a trace $u|_{\Gamma_+}$ in $L^1(\Gamma_+, d\xi)$. Moreover, the resulting *albedo operator* $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$ defined by

$$\mathcal{A} : f \mapsto \mathcal{A}f = u|_{\Gamma_+}, \quad u \text{ solves (1.1)}, \tag{1.8}$$

is well-defined and bounded.

The linear transport term $v \cdot \nabla_x u$ in Equation (1.1) allows to propagate singularities through the bounded spatial domain Ω. This feature makes (1.1) a good candidate for modelling some stable inverse problems. Depending on the regime (high scattering leading to diffusion, highly-peaked scattering leading to diffusion in the angular variable (Leakeas and Larsen, 2001)), one can also lose stability, and this model is rich enough to monitor properly the loss of singularities. While the problems of reconstructing the absorption $\sigma(x,v)$ and the scattering $k(x,v,v')$ from the full albedo operator $\mathcal{A}$ are well-understood and display good stability (Choulli and Stefanov, 1999, 1996b; Stefanov and Uhlmann, 2003; Bal and Jollivet, 2010b, 2008), there is active research in taking into account angular averaging and noise in the boundary measurements (Bal, 2008; Langmore and McDowall, 2008; Langmore, 2008; Bal and Jollivet, 2009, 2010a; Bal et al., 2011a, 2008; Monard and Bal, 2012d), a more realistic setting where the present work sits in.

The main loss of information in the inverse problems considered next comes from the loss of angular resolution, both in the sources (we will consider isotropic sources rather than directional) and in the measurements (we will consider angularly averaged measurements). In this case, there is no hope to reconstruct all of the optical coefficients, and we restrict our reconstruction wishes to the sole spatial part of the scattering kernel. Namely, we write the scattering kernel $k(x,v,v')$ under the form $k(x)\phi(v \cdot v')$, where the phase function ϕ is known and satisfies $\int_{\mathbb{S}^{d-1}} \phi(v' \cdot v)\ dS(v) = 1$. Under this assumption, and assuming the attenuation σ to be known as well, we consider the inverse problem of reconstructing k from the following set of measurements

$$\mathcal{M}[k](f,x) = \int_{v\cdot\nu_x>0} \mathcal{A}f(x,v)|v\cdot\nu_x|\ dS(v), \quad x\in\partial\Omega, \quad f\in L^1(\partial X), \tag{1.9}$$

in particular, the boundary input f is *isotropic*, i.e. $f(x,v) = f(x)$. This inverse problem will be analyzed in two regimes of interest:

Stationary regime: setting the $\partial_t u$ term to 0 in (1.1). This presupposes that all quantities are time-independent in the first place.

Time-harmonic regime: Fourier-transforming equation (1.1) in time and considering frequency-modulated boundary sources of the form $f^\omega(x,v,t) = f(x,v)e^{i\omega t}$ for fixed ω, we obtain the stationary version of (1.1) with σ replaced by $\sigma + i\omega$, in which the now complex-valued function u represents the quantity $\hat{u}(x,v,\omega) := \int_{\mathbb{R}} u(x,v,t)e^{-i\omega t}\, dt$.

1.1 The albedo decomposition

We now recall some general results about forward transport that are necessary for subsequent analyses. Let $X \subset \mathbb{R}^d, d \geq 2$ be an open convex bounded domain with $\mathcal{C}^1$ boundary ∂X and diameter $\mathrm{diam}(X) > 0$. The time-harmonic (or, when $\omega = 0$, stationary) transport equation with isotropic boundary sources reads

$$\begin{aligned} v \cdot \nabla u + (\sigma(x,v) + i\omega)u &= \int_{\mathbb{S}^{d-1}} k(x,v',v)u(x,v')\, dS(v'), \quad (x,v) \in X \times \mathbb{S}^{d-1}, \\ u(x,v) &= f(x), \quad (x,v) \in \Gamma_-, \end{aligned} \tag{1.10}$$

where $\omega \geq 0$ and $f(x,v) = f(x) \in L^1(\partial X)$. As it is done in many settings, we integrate (1.10) along the direction v. We obtain that u is a solution of the following integro-differential equation

$$(I - \mathcal{K}_\omega)u = J_\omega f, \tag{1.11}$$

where we have defined, for $\phi \in L^1(X \times \mathbb{S}^{d-1})$ and $\psi \in L^1(\partial X)$

$$\mathcal{K}_\omega \phi(x,v) := \int_0^{\tau_-(x,v)} e^{-i\omega t} E(x-tv,x) \int_{\S^{n-1}} k(x-tv,v',v)\phi(x-tv,v')dv'dt, \tag{1.12}$$

$$J_\omega \psi(x,v) := e^{-i\omega\tau_-(x,v)} E(x_-(x,v),x)\psi(x_-(x,v)), \quad \text{and} \tag{1.13}$$

$$E(x,x') := \exp\left(-\int_0^{|x-x'|} \sigma(x+\widehat{x'-x}s, \widehat{x'-x})\, ds \right). \tag{1.14}$$

When $\omega = 0$, the operators $\mathcal{K} \equiv \mathcal{K}_0$ and $J \equiv J_0$ are positive operators, well-defined and continuous in $\mathcal{L}(L^1(X \times \mathbb{S}^{d-1}))$ and $\mathcal{L}(L^1(\partial X), L^1(X \times \mathbb{S}^{d-1}))$, respectively. Moreover, under condition (1.6) or (1.7), the operator $\mathcal{K}$ becomes a contraction (Dautray and Lions, 1993; Mokhtar-Kharroubi, 1997). Adding a frequency ω only gives an oscillating phase to the integrals and we have, in a straightforward manner, $\|\mathcal{K}_\omega\| \le \|\mathcal{K}\|$ and $\|J_\omega\| \le \|J\|$, where the operator norms are taken in the aforementioned spaces.

Whenever $\mathcal{K}_\omega$ is a contraction, equation (1.11) may be solved via the Neumann series

$$u = \sum_{m=0}^{\infty} \mathcal{K}_\omega^m J_\omega f = J_\omega f + \mathcal{K}_\omega J_\omega f + (I - \mathcal{K}_\omega)^{-1} \mathcal{K}_\omega^2 J_\omega f, \tag{1.15}$$

where the term of order m accounts for particles that scattered exactly m times inside X before arriving at the current point (x, v). As in the second right-hand side of (1.15), we gather the tail of the series into one term for scattering events of order two and up. Plugging this decomposition in the expression of the data (1.9), we obtain the following decomposition

$$\mathcal{M}^\omega[k](f,x) = \mathcal{M}_0(f,x) + \mathcal{M}_1 k(f,x) + \mathcal{M}_{2+}[k](f,x), \quad \text{where} \tag{1.16}$$

$$\mathcal{M}_0^\omega(f,x) := \int_{v\cdot\nu_x>0} J_\omega f(x,v)|\nu_x \cdot v| \; dS(v), \tag{1.17}$$

$$\mathcal{M}_1^\omega k(f,x) := \int_{v\cdot\nu_x>0} \mathcal{K}_\omega J_\omega f(x,v)|\nu_x \cdot v| \; dS(v), \tag{1.18}$$

$$\mathcal{M}_{2+}^\omega[k](f,x) := \int_{v\cdot\nu_x>0} (I-\mathcal{K}_\omega)^{-1}\mathcal{K}_\omega^2 J_\omega f(x,v)|\nu_x \cdot v| \; dS(v), \tag{1.19}$$

where the notation employed stresses the fact that the *ballistic part* $\mathcal{M}_0^\omega$ does not depend on k, the *single-scattering part* $\mathcal{M}_1^\omega k$ depends linearly on k, and the *multiple-scattering part* $\mathcal{M}_{2+}^\omega[k]$ is non-linear in k.

In some cases, it will be useful to consider the Schwartz kernel of the measurement operator rather than the operator $\mathcal{M}^\omega[k](f,x)$ itself, in which case, we will write equivalently $\mathcal{M}^\omega[k](x_0,x) := \mathcal{M}^\omega[k](\delta_{x_0},x)$ for $(x_0,x) \in (\partial X)^2$. By δ_{x_0} we mean the delta distribution that satisfies for each smooth function ψ defined at the boundary:

$$\int_{\partial X} \delta_{x_0}(x)\psi(x) \; d\mu(x) = \psi(x_0), \tag{1.20}$$

where $d\mu(x)$ is the intrinsic measure at the boundary. In practice, $\mathcal{M}^\omega[k](x_0,x)$ would be what is measured by an angularly averaging captor located at x while the medium is being probed by an isotropic point source located at x_0. The measurements may also be considered via their moments of the form

$$\mathcal{M}^\omega[k](f,g) := \int_{\partial X} \mathcal{M}^\omega[k](f,x) g(x) \; d\mu(x), \quad g \in L^\infty(\partial X). \tag{1.21}$$

Hoping that no confusion occurs, we will equivalently use the three notations defined above, that is $\mathcal{M}^\omega[k](f,x)$, $\mathcal{M}^\omega[k](x_0,x) := \mathcal{M}^\omega[k](\delta_{x_0},x)$ and $\mathcal{M}^\omega[k](f,g)$. Lastly, since σ is assumed to be known in this context, we may right away assume that the ballistic part is known and may be removed from the measurements. We therefore define the data operator $\mathcal{D}^\omega[k]$ by

$$\mathcal{D}^\omega[k] := \mathcal{M}^\omega[k] - \mathcal{M}_0^\omega = \mathcal{M}_1^\omega k + \mathcal{M}_{2+}^\omega[k], \tag{1.22}$$

whose arguments may either be of the form (f,x), (x_0,x) or (f,g) depending on the context.

The inverse problem we consider is thus the following

Problem 1.1.1. *Does the measurement operator $\mathcal{M}^\omega[k]$ (or, equivalently under knowledge of σ, the data operator $\mathcal{D}^\omega[k]$) determine the function k uniquely ? If so, with what stability ?*

Incentive for inversion: In the *transport regime* where scattering is assumed to be small, the Neumann series decays rapidly and most of the solution is contained in the first few terms, especially in the ballistic and single scattering parts. The ballistic part should be the overwhelming and most singular part, and contains information about the X-Ray transform of σ. Assuming that one knows σ (either exactly or after reconstructing via a mildly ill-posed inversion), we may assume that the ballistic part is known. Then, in the following sections, the idea is to find a reconstruction algorithm for k from all or part of the single scattering measurements, and to show that the error remainder $R[k]$ is small in some sense. If one can find a setting where the error operator is a contractive operator of k, then one can further improve the reconstruction of k via an appropriate iterative scheme. We now make these statements more precise.

1.2 Strategies of inversion

In the subsequent sections, we will come across the same systematic approach to reconstructions, this is why we deemed worthwhile describing it now, though in a rather loose manner. Assume that one want to reconstruct an unknown k from data $\mathcal{D}$ fixed, which admits the following decomposition

$$\mathcal{D} = Tk + \mathcal{R}[k], \tag{1.23}$$

where T is linear in k, of whom we have an exact or approximate inverse, and where $\mathcal{R}$ stands for the (possibly non-linear) remainder operator, i.e. the part of the measurements upon which the inversion is not based. Call $T^{-1,b}$ a *regularized inverse* for T, where b is a parameter such that the operators $T^{-1,b}$ converge pointwise to T^{-1} on the range of T as $b \to \infty$. In other words, the product $T^{-1,b}T$ is an approximation of identity. Applying $T^{-1,b}$ to (1.23), we obtain

$$T^{-1,b}\mathcal{D} = T^{-1,b}Tk + T^{-1,b}\mathcal{R}[k] = k_b + \varepsilon(k,b), \tag{1.24}$$

where k_b is a regularized version of k and $\varepsilon(k,b) := T^{-1,b}\mathcal{R}[k]$ is the error operator.

1.2.1 Direct inversion

If one is able to bound the error properly like $\|\varepsilon(k,b)\| \le \varepsilon_0(b)$, then we conclude that the strategy of applying $T^{-1,b}$ to the data $\mathcal{D}$ recovers the regularized k_b up to an error of size $\varepsilon_0(b)$, as is the conclusion of e.g. (Bal et al., 2008, Theorem 2.1) and (Monard and Bal, 2012d, Theorem 2.2). Note that the growth of ε_0 in terms of b gives us information about the stability of the inversion:

- if ε_0 is uniformly bounded in b, the inversion is *well-posed* (Lipschitz-stable),
- if ε_0 grows polynomially in b, the inversion is *mildly ill-posed* (Hölder-stable),
- if ε_0 grows exponentially in b, the inversion is *severely ill-posed* (logarithmically stable).

1.2.2 Iterative improvement

If now the error operator can be made a c_ε-contraction (with $c_\varepsilon < 1$) in some Banach space $\mathcal{B}$, that is,

$$\|\varepsilon(k,b) - \varepsilon(k',b)\|_{\mathcal{B}} \leq c_\varepsilon(b)\|k - k'\|_{\mathcal{B}}, \quad k, k' \in \mathcal{B}, \tag{1.25}$$

and assuming that $T^{-1,b}\mathcal{D} \in \mathcal{B}$ and $\mathcal{B}$ is stable by the mapping $k \mapsto T^{-1,b}\mathcal{D} - \varepsilon(k,b)$, then the following iterative scheme

$$k_0 := T^{-1,b}\mathcal{D}, \qquad k_{n+1} = T^{-1,b}\mathcal{D} - \varepsilon(k_n, b), \quad n \geq 0, \tag{1.26}$$

converges by virtue of the Banach Fixed Point theorem, to an element $k^\star$ that satisfies the equation

$$T^{-1,b}\mathcal{D} = k^\star + \varepsilon(k^\star, b). \tag{1.27}$$

Taking the difference of (1.24) (satisfied by the true k) and (1.27) (satisfied by the computed $k^\star$), we obtain the relation

$$k_b - k^\star = \varepsilon(k^\star, b) - \varepsilon(k, b) = (\varepsilon(k^\star, b) - \varepsilon(k_b, b)) + (\varepsilon(k_b, b) - \varepsilon(k, b)).$$

Taking norms and bounding differences of errors appropriately, we can deduce the error estimate

$$\|k_b - k^\star\|_{\mathcal{B}} \leq \frac{c_\varepsilon}{1 - c_\varepsilon}\|k - k_b\|_{\mathcal{B}}. \tag{1.28}$$

The term $k - k_b$ is a measure of the high-frequency content that was not reconstructed. This last estimates shows that if the high-frequency content of the unknown k is small, then the iterative scheme (1.26) improves the reconstruction of k_b by "washing out" the artifacts that were initially caused by the error operator. Such an approach appears in (Bal et al., 2008, Theorem 2.2) and (Monard and Bal, 2012d, Theorem 2.4). This approach by considering contractive error operators and implementing an iterative scheme to improve reconstructions has been used in several contexts in inverse problems, for instance in Bal and Tamasan (2007); Stefanov and Uhlmann (2009, 2011); Qian et al. (2011). It is also the bread and butter of microlocal analysis-based inversions, where one can easily find a parametrix to elliptic pseudo-differential operators, so that the error remainder is a compact operator in the unknown variable. This usually reduces the problem to a Fredholm alternative, i.e. partial invertibility up to finite-dimensional compatibility conditions. In such cases, although the set of non-invertible cases is "of measure zero" in the space of parameters, and unless invertibility can be proved by other techniques, we can only guarantee theoretical invertibility under rather restrictive smallness assumptions which often make the error operator a *contractive mapping*. In this case, the inversion is actually explicit and is done via a Neumann series.

Outline of the next chapter: In the next chapter, we will study Problem 1.1.1 in two regimes. We will first analyze the inverse problem in the stationary case in Section 2.1.1. Going to the time-harmonic regime, we will then provide in Section 2.1.2 asymptotic expansions on time-harmonic solutions of (1.1) which will motivate an inversion formula for k via an inverse X-Ray transform. On to the inversion, we will show results in Section 2.1.3 on the stability of an inversion scheme for k in a two-dimensional setting. Finally, Section 2.2 will summarize some work by the author and G. Bal on numerical methods for solving the transport equation accurately in the transport regime.

Chapter 2

Analysis of inverse transport problems and numerical methods

2.1 Transition from severe to mild ill-posedness in an inverse transport problem

In the next sections, we provide answers to Problem 1.1.1 in two regimes: stationary and high-frequency time-harmonic. Further, we demonstrate how the introduction of a frequency parameter is highly beneficial for the stability of Problem 1.1.1 as we go from severe ill-posedness to mild ill-posedness.

2.1.1 Inverse stationary transport with angularly averaged measurements

Guillaume Bal, Ian Langmore and I first analysed in Bal et al. (2008) Problem 1.1.1 in the stationary regime using isotropic boundary sources (i.e. $f(x, v) \equiv f(x)$ for $(x, v) \in \Gamma_-$) and angularly averaged measurements, assuming that σ is known and $\phi \equiv 1$ in (1.1). Since we are in the stationary case $\omega = 0$, we will drop the ^{0} exponent on the measurements for convenience.

Analysis of the single scattering part $\mathcal{M}_1 k$

Using the form $\mathcal{D}[k](f,g) = \mathcal{M}_1 k(f,g) + \mathcal{M}_{2+}[k](f,g)$ of the measurements, we first show that, using changes of variables from $\mathbb{S}^{d-1}$ to ∂X (Bal et al., 2008, Prop. 3.1), the single scattering component may be rewritten as a linear operator of k of the form

$$\mathcal{M}_1 k(f,g) = \int_X k(x_1) Af(x_1) Ag(x_1)\ dx_1, \tag{2.1}$$

where the operator A is given by

$$Af(x) = \int_{\partial X} f(x_0) \frac{E(x_0,x)|\nu_{x_0}\cdot(x_0-x)|}{|x_0-x|^n}\ d\mu(x_0), \quad x \in X, \tag{2.2}$$

which we rewrite as

$$Af = A_0 f + (A - A_0) f, \quad A_0 f(x) := \int_{\partial X} f(x_0) \frac{|\nu_{x_0}\cdot(x_0-x)|}{|x_0-x|^n}\ d\mu(x_0). \tag{2.3}$$

The operator A_0 is the so-called *double layer potential*, and $A = A_0$ in the absence of attenuation (i.e. $\sigma = 0$). Looking for an inversion for k from the single scattering term (2.1), we looked for particular boundary weights f and g such that the products $AfAg$ would be dense in $L^2(X)$, an approach very similar in spirit to Calderón's in Calderón (1980) when proving injectivity of the linearized inverse conductivity problem. Moreover, assuming $\|\sigma\|_\infty := \|\sigma\|_{L^\infty(X)}$ to be small, the operator A is close to the *double layer potential*, a boundary integral operator which constructs functions that are *harmonic* inside the domain of interest, from dipole densities at the boundary. Still following Calderón's idea, a family of harmonic functions whose products are dense in $L^2(X)$ is that of *Complex Geometrical Optics solutions*, i.e. harmonic solutions of the form

$$u_{\xi,\eta} : \mathbb{R}^n \ni x \mapsto \exp\left(\frac{-i}{2}(\xi + i\eta)\cdot x\right), \quad \text{where} \quad \xi, \eta \in \mathbb{R}^n, \quad |\xi| = |\eta|, \quad \xi\cdot\eta = 0. \tag{2.4}$$

If one can find $f_{\xi,\eta}$ and $g_{\xi,\eta}$ such that $Af_{\xi,\eta} = u_{\xi,\eta}$ and $Ag_{\xi,\eta} = u_{\xi,-\eta}$, the single scattering becomes

$$\mathcal{M}_1 k(f_{\xi,\eta}, g_{\xi,\eta}) = \int_X k(x_1) e^{-i\xi\cdot x_1}\ dx_1 = \hat{k}(\xi),$$

where the hat denotes a Fourier Transform. If such an equality could be obtained, then k could be reconstructed from single scattering measurements $\mathcal{M}_1$ via the formula

$$k(x_1) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix_1\cdot\xi}\mathcal{M}_1 k(f_{\xi,\eta(\xi)}, g_{\xi,\eta(\xi)})\ d\xi, \quad \eta(\xi)\cdot\xi = 0, \quad |\eta(\xi)| = |\xi|.$$

In Bal et al. (2008), we prove an approximate version of this statement under the assumption that $\|\sigma\|_\infty$ is small, so that most of the inversion is built through the double layer potential, up to errors that are controlled by $\|\sigma\|_\infty$. We proceed as follows: following results in Ammari and Kang (2004); Folland (1995) on double-layer potentials, we construct in (Bal et al., 2008, Lemma 3.1) a continuous operator $A_0^\dagger : H^{\frac{1}{2}}(X) \to L^2(\partial X)$ such that $A_0 A_0^\dagger u = u|_{\partial X}$ for all harmonic functions $u \in H^{\frac{1}{2}}(X)$ (in particular for the functions $u_{\xi,\eta}$ defined in (2.4)). Therefore, defining $f_{\xi,\eta} = A_0^\dagger u_{\xi,\eta}$ and $g_{\xi,\eta} = A_0^\dagger u_{\xi,-\eta}$, and using the identity

$$AfAg = A_0 f A_0 g + (A - A_0) f A_0 g + Af(A - A_0)g,$$

we obtain

$$\begin{aligned} \mathcal{M}_1 k(f_{\xi,\eta}, g_{\xi,\eta}) &= \hat{k}(\xi) + \varepsilon(\sigma, k, \xi), \qquad \text{where} \\ \varepsilon(\sigma, k, \xi) &:= \int_X k(x_1)[(A - A_0) f_{\xi,\eta} A_0 g_{\xi,\eta} + A f_{\xi,\eta} (A - A_0) g_{\xi,\eta}] \, dx_1. \end{aligned} \tag{2.5}$$

The error ε is shown in Bal et al. (2008) to be bounded by

$$|\varepsilon(\sigma, k, \xi)| \leq C_1 \|\sigma\|_\infty \|k\|_\infty e^{C_2|\xi|}, \tag{2.6}$$

where the constants C_1, C_2 only depend on X. The exponential term in $|\xi|$ is due to the fact that the CGO solutions $u_{\xi,\eta}$ used oscillate along direction ξ while blowing up exponentially in direction η. This is a very bad sign for stability, as it indicates that errors magnify exponentially with the magnitude of the frequency (equivalently, the fineness of details) at which we want to reconstruct k.

Analysis of the multiple scattering $\mathcal{M}_{2+}[k]$

After analyzing the single scattering, we show that the nonlinear term $\mathcal{M}_{2+}[k]$ coming from the tail of the scattering series can be considered as small error provided that $\|k\|_\infty$ is small. This is because each term of order $m \geq 2$ in the scattering series (1.15) is bounded in magnitude by $(C\|k\|_\infty)^m$ with C a fixed finite constant. Therefore the tail of the series for $m \geq 2$, provided that $\|k\|_\infty$ is small enough, may be controlled by $(C\|k\|_\infty)^2(1 - C\|k\|_\infty)^{-1}$. Since the measurements are generated by the exponentially growing boundary terms $f_{\xi,\eta}$ and $g_{\xi,\eta}$, the error due to $\mathcal{M}_{2+}[k](f_{\xi,\eta}, g_{\xi,\eta})$ is also affected by a coefficient $e^{C_2|\xi|}$, and we get, keeping only the relevant quantities

$$|\mathcal{M}_{2+}[k](f_{\xi,\eta}, g_{\xi,\eta}) - \mathcal{M}_{2+}[k'](f_{\xi,\eta}, g_{\xi,\eta})| \leq C\|k - k'\|_\infty e^{C_2|\xi|}, \tag{2.7}$$

where the constant C depends on $\max(\|k\|_\infty, \|k'\|_\infty)$.

Regularization and reconstruction

Now that the error is characterized by estimates (2.6)-(2.7), we move on to the reconstruction strategies as presented in Section 1.2. Putting together the previous estimates, we obtain that

$$\begin{gathered}\mathcal{D}[k](f_{\xi,\eta(\xi)}, g_{\xi,\eta(\xi)}) = \hat{k}(\xi) + \varepsilon'(\sigma, k, \xi), \quad \text{where} \\ |\varepsilon'(\sigma,k,\xi) - \varepsilon'(\sigma,k',\xi)| \le C\|k-k'\|_\infty(\|\sigma\|_\infty + M)e^{C_2|\xi|}, \quad M = \max(\|k\|_\infty, \|k'\|_\infty).\end{gathered} \tag{2.8}$$

On to the reconstruction of k, since the error blows up for large $|\xi|$, we must *regularize* by preprocessing the data at high frequencies before inverting. To this end, we define in Bal et al. (2008) a positive function $\chi(\xi)$ such that $\chi(\cdot)e^{C_2|\cdot|}$ is integrable with respect to $\xi \in \mathbb{R}^n$, and define the projection of k onto low-frequencies as

$$P_\chi k(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{k}(\xi)\chi(\xi)e^{i\xi\cdot x}\, d\xi.$$

Applying the algorithm on the right-hand side to (2.8), we obtain

$$\begin{gathered}(2\pi)^{-n}\int_{\mathbb{R}^n} \mathcal{D}[k](f_{\xi,\eta(\xi)}, g_{\xi,\eta(\xi)})\chi(\xi)e^{i\xi\cdot x}\, d\xi = P_\chi k(x) + \varepsilon''(\sigma,k,\chi,x), \\ \|\varepsilon''(\sigma,k,\chi) - \varepsilon''(\sigma,k',\chi)\|_\infty \le C\|\chi(\cdot)e^{C_2|\cdot|}\|_{L^1(\mathbb{R}^n)}\|k-k'\|_\infty(\|\sigma\|_\infty + M),\end{gathered} \tag{2.9}$$

where $M = \max(\|k\|_\infty, \|k'\|_\infty)$. From the previous estimate, we now state the main two theorems of Bal et al. (2008), which correspond to both inversion strategies presented in section 1.2.

Theorem 2.1.1 (Theorem 2.1 in Bal et al. (2008), direct inversion)**.** *If $\|k\|_\infty$ is small enough, there exists a constant $C_X > 0$ such that for all χ such that $\chi(\cdot)e^{C_2|\cdot|}$ is integrable, the measure-*

ments $\{\mathcal{M}[k](f,g), f,g \in L^1(\partial X)\}$ *determine* $P_\chi k$ *up to an error bounded in* L^∞ *by*

$$C_X \|\chi(\cdot)e^{C_2|\cdot|}\|_{L^1(\mathbb{R}^d)} \|k\|_\infty (\|\sigma\|_\infty + \|k\|_\infty).$$

Typically, if χ is of the form $\chi(\xi) = 1_{|\xi|\le b}(\xi)$ (cutoff in frequency with bandwidth b), then we have

$$\|\chi(\cdot)e^{C_2|\cdot|}\|_{L^1(\mathbb{R}^d)} = P_{d-1}(C_2 b)e^{C_2 b},$$

with P_{d-1} a polynomial of degree $d-1$, hence the severe ill-posedness.

Now following the iterative improvement route, we want to make the error operator a contraction of k in some appropriate space. Judging by estimate (2.9), this space may be chosen to be a small enough ball in L^∞ under the assumption that $\|\sigma\|_\infty$ is small enough.

Theorem 2.1.2 (Theorem 2.2 in Bal et al. (2008), iterative improvement)**.** *Given* χ *such that* $\chi(\cdot)e^{C_2|\cdot|}$ *is integrable,* $0 < c_1 < 1$ *and* σ *such that* $\|\sigma\|_\infty$ *is small enough, there exists* $\varepsilon > 0$ *such that for* $\|k\|_\infty < \varepsilon$*, the measurements* $\{\mathcal{M}[k](f,g), f,g \in L^1(\partial X)\}$ *determine* $P_\chi k$ *up to an error, bounded in* L^∞ *by*

$$\frac{c_1}{1-c_1}\|(I-P_\chi)k\|_{L^\infty(X)}.$$

2.1.2 Asymptotic behavior of time-harmonic transport solutions at high frequencies

As we have seen in the previous section, the reconstruction of the spatial scattering function k is a severely ill-posed problem in the stationary setting. We will now see that a way to reintroduce singularities into the measurements is to go to the time-harmonic regime at high frequencies. To this end, Guillaume Bal, Ian Langmore, Alexandre Jollivet and I analyzed in Bal et al. (2011a) the behavior of time-harmonic solutions of (1.1) for high frequencies ω using stationary phase-based asymptotic expansions. Note that these expansions can be made without making the simplification $k(x,v,v') = k(x)\phi(v\cdot v')$, although we will keep that notation here for consistency.

In this context we use the representation $\mathcal{D}^\omega[k](x_0,x) \equiv \mathcal{D}^\omega[k](\delta_{x_0},x)$ of the data for $(x_0,x) \in \partial X$. A crucial assumption here is that the scattering kernel be supported away from the domain's boundary, since otherwise the effect of scattering would overwhelm the leading-order asymptotic term of the data $\mathcal{D}^\omega[k]$ inside the domain. We write this hypothesis as

Hypothesis 2.1.3. *Assume for further purposes that*

$$D := \mathit{dist}\ (\partial X, \mathit{supp}(k)) > 0.$$

Under this assumption, we first show in Bal et al. (2011a) that in the time-harmonic regime, the destructive interferences in the oscillating integrals are such that the $\mathcal{L}^1 := \mathcal{L}(L^1(X\times\mathbb{S}^1))$-norm of the scattering operator $\mathcal{K}^2_\omega$ decays in ω, all other parameters being equal. In particular, if we assume σ and k to be of class $\mathcal{C}^1$ without smallness or positivity assumption of the form (1.6) or (1.7), then $\mathcal{K}^2_\omega$ will become a contraction for ω large enough and the series (1.15) will be valid. Specifically, we have the following behaviors depending on dimension:

Lemma 2.1.4 (Lemma 3.1 in Bal et al. (2011a))**.** *Let $(\sigma,k) \in \mathcal{C}^1(\overline{X}) \times \mathcal{C}^1(X)$ and assume hypothesis 2.1.3. Then there is a constant C such that for every $\omega \ge 2$ the following estimates hold*

$$\|\mathcal{K}^2_\omega\|_{\mathcal{L}^1} \le \begin{cases} C(\|\sigma\|_{\mathcal{C}^1}, \|k\|_{\mathcal{C}^1})\ \omega^{-\frac{1}{2}}\log\omega, & d=2, \\ C(\|\sigma\|_{\mathcal{C}^1}, \|k\|_{\mathcal{C}^1})\ \omega^{-1}(\log\omega)^2, & d=3, \\ C(\|\sigma\|_{\mathcal{C}^1}, \|k\|_{\mathcal{C}^1})\ \omega^{-1}\log\omega, & d\ge 4. \end{cases} \tag{2.10}$$

Lemma 2.1.4 is especially useful in order to bound the remainder of the scattering series $\mathcal{M}_{2+}^{\omega}[k](x_0, x)$.

Analysis of the single scattering

In the time-harmonic regime and after changing variables, the single scattering takes the form

$$\begin{aligned}
\mathcal{M}_1^{\omega} k(x_0, x) &= \int_X k(x_1) e^{i\omega\varphi(x_0,x_1,x)} E(x_0, x_1, x) c(x_0, x_1, x)\, dx_1, \quad (x_0, x) \in (\partial X)^2, \\
c(x_0, x_1, x) &:= (|x - x_1||x_1 - x_0|)^{-d+1} |\nu_{x_0} \cdot \widehat{x - x_0}||\nu_x \cdot \widehat{x_c - x}| \phi(\widehat{x_1 - x_0} \cdot \widehat{x - x_1}), \\
\varphi(x_0, x_1, x) &:= -|x_1 - x_0| - |x_1 - x|.
\end{aligned} \tag{2.11}$$

It is thus an oscillatory integral, where the phase function φ describes the path followed by a ray going from x_0, scattering at x_1 and being measured at x. φ is stationary precisely for all points of the segment $[x_0, x]$, therefore each of these critical points is degenerate since the phase function is constant and equal to $|x - x_0|$ for any x_1 belonging to $[x_0, x]$ (so the second derivative of the phase cannot be non-zero along that direction). In order to control the degeneracy properly, we reparameterize $x_1(u, \mathbf{v}) = ue_0 + \mathbf{v}$ with $e_0 := |x - x_0|^{-1}(x - x_0)$ and $\mathbf{v} \cdot e_0 = 0$, and notice that the phase function has nondegenerate Hessian in the $d-1$-dimensional variable $\mathbf{v}$. Thus we expect a leading order behavior for $\mathcal{M}_1^{\omega} k$ of size $\omega^{-\frac{d-1}{2}}$, with a remainder of smaller order in ω, provided that the optical coefficients have enough regularity for the stationary phase to apply. Denoting $\lceil \cdot \rceil$ for the ceiling part, we have the following theorem:

Theorem 2.1.5 (Theorem 2.2 in Bal et al. (2011a), single scattering estimate)**.** *Assuming that k, σ are of class $\mathcal{C}^{\lceil \frac{d+3}{2} \rceil}$, there exists a constant C such that the following decomposition holds for* *every $(x_0, x) \in (\partial X)^2$ with $x_0 \neq x$ (denote $d_0 := |x - x_0|$ and $e_0 = \widehat{x - x_0}$) and for every $\omega > 0$*

$$\begin{aligned}\mathcal{M}_1^\omega k(x_0, x) = e^{-i\omega d_0} \left(\frac{2\pi}{d_0\omega}\right)^{\frac{d-1}{2}} e^{-i(d-1)\frac{\pi}{4}} E(x_0, x)|\nu_{x_0} \cdot e_0||\nu_x \cdot e_0|\phi(1) \\ \times \int_0^{d_0} \frac{k(x_0 + ue_0)}{(u(d_0 - u))^{\frac{d-1}{2}}}\ du + R^\omega(x_0, x),\end{aligned} \tag{2.12}$$

where the remainder R^ω belongs to $L^\infty(\partial X \times \partial X)$ and satisfies

$$\|R^\omega\|_\infty \leq C\left(\|k\|_{\mathcal{C}^{\lceil \frac{d+3}{2} \rceil}}, \|\sigma\|_{\mathcal{C}^{\lceil \frac{d+3}{2} \rceil}}\right)\omega^{-\frac{d+1}{2}}.$$

The theorem above shows that the leading order of the single scattering data is proportional to a weighted ray transform of k. While the weight inside the ray transform may depend on the line of integration in general geometries (and therefore is less systematic to invert), it turns out that when the domain is a centered ball of radius $r > 0$, we have that

$$\int_0^{d_0} \frac{k(x_0 + ue_0)}{(u(d_0 - u))^{\frac{d-1}{2}}}\ du = P[k\rho](L_{x_0,x}),$$

where P stands for the standard X-Ray transform, $L_{x_0,x}$ denotes the line joining x_0 to x and $\rho(x) = (r^2 - |x|^2)^{-\frac{d-1}{2}}$. Note that the blow-up of the weight function ρ at the boundary justifies again the necessity of Hypothesis (2.1.3).

Analysis of the multiple scattering

We now show that the multiple scattering decays faster than the leading term of the decomposition (2.12). In Bal et al. (2011a), we derive the following result:

Theorem 2.1.6 (Theorem 2.5 in Bal et al. (2011a), multiple scattering estimate)**.** *Let σ, k be of class $\mathcal{C}^{\lceil \frac{d+1}{2} \rceil}$ over $\overline{X}$ and assume Hypothesis 2.1.3. Then there exists a frequency $\omega_0 \geq 2$ and* *a constant C such that for every $\omega \geq \omega_0$ the multiple scattering $\mathcal{M}^{\omega}_{2+}[k] \in L^\infty(\partial X \times \partial X)$ and satisfies the estimate*

$$\|\mathcal{M}^{\omega}_{2+}\|_\infty \leq \begin{cases} C\omega^{-1}, & n = 2, \\ C\omega^{-2}\log\omega, & n = 3, \\ C\omega^{-\frac{d+1}{2}}, & n \geq 4. \end{cases} \tag{2.13}$$

The proof of theorem 2.1.6 is based on the following key facts:

- The estimate (2.13) remains true if one replaces $\mathcal{M}^{\omega}_{2+}$ by any term of fixed scattering order $m \geq 2$. This is done, again, by repeated stationary phase analyses.
- For $m \geq 2$ large enough, the tail of the scattering series (1.15) starting at m decays faster in ω than the decay rates in (2.13). This is done by using the estimates on the operator $\mathcal{K}^2_\omega$ from lemma 2.1.4.
- summing up, the data $\mathcal{M}^{\omega}_{2+}$ may be decomposed into a finite sum of terms, each of which satisfy estimate (2.13), and a tail that decays even faster, hence the result.

Perspectives of reconstruction

The expansions for large ω presented above show that the data $\mathcal{D}^\omega[k]$ exhibits the following decomposition

$$\begin{aligned} \mathcal{D}^\omega[k](x_0, x) &= \omega^{-\frac{d-1}{2}} f(x_0, x) P[k\rho](x_0, x) + \varepsilon(k)(x_0, x) \\ \|\varepsilon(k)\|_{L^\infty(\partial X \times \partial X)} &= o(\omega^{-\frac{d-1}{2}}), \end{aligned} \tag{2.14}$$

where x_0/x respectively denote the emitter/captor's positions at the boundary, and $Pk(x_0, x)$ is the X-Ray transform of k at the line (x_0, x) when the domain of interest is a ball. This in turn motivates an inversion procedure for k via applying an inverse X-Ray transform to (2.14), which we now present in two dimensions.

2.1.3 Inverse time-harmonic transport in two dimensions

The decomposition (2.14) suggests that one may approximately recover the product $k\rho$ via an inverse X-Ray transform after "preprocessing" the data by dividing pointwise by $\omega^{-\frac{d-1}{2}} f(x_0, x)$ (f is bounded away from zero). The first thing to note is that we need an explicit expression for an inverse X-Ray transform, for which the two-dimensional case has a few well-known reconstruction formulas. In higher dimensions $d \geq 3$ however, since the set of lines is of greater dimension $2(d-1)$ than the dimension of space d, several possible reconstruction formulas exist involving a subset of lines of dimension d (see e.g. Kapralov and Katsevich (2008); Katsevich and Kapralov (2007) for inversion formulas for cone beam transforms in three dimensions). We do not explore this route here.

In Monard and Bal (2012d), we therefore investigate the two-dimensional case where the domain is a centered ball of radius r (call it B_r), in which case the decomposition (2.14) becomes

$$\begin{aligned} &\mathcal{D}^\omega[k](x_0, x) = \omega^{-\frac{1}{2}} f(x_0, x) P[k\rho](x_0, x) + \varepsilon(k)(x_0, x), \qquad \rho(x) := (r^2 - |x|^2)^{-\frac{1}{2}}, \\ &\|\varepsilon(k)\|_{L^\infty(\partial X \times \partial X)} = o(\omega^{-\frac{1}{2}}). \end{aligned} \tag{2.15}$$

Hypothesis 2.1.3 now translates into

$$\mathrm{supp}(k) \subset B_{r-D} \quad \text{for some } D > 0. \tag{2.16}$$

The first thing to do is to parameterize x_0 and x so that the line joining them corresponds to a line $L(s, \theta) = \{s\hat{\theta} + t\hat{\theta}^\perp, t \in \mathbb{R}\}$ for (s, θ) in the truncated cylinder $[-r, r] \times \mathbb{S}^1$. We therefore adopt the *parallel geometry* rather than the *fan-beam* one. This is done using the following parameterization

$$x_0(s,\theta) = s\hat{\theta} - \sqrt{r^2 - s^2}\hat{\theta}^{\perp} \quad \text{and} \quad x(s,\theta) = s\hat{\theta} + \sqrt{r^2 - s^2}\hat{\theta}^{\perp},$$

in particular, $x - x_0 = 2\sqrt{r^2 - s^2}\hat{\theta}$, see Fig. 2.1. In this parameterization, $P[k\rho](x_0, x) =$

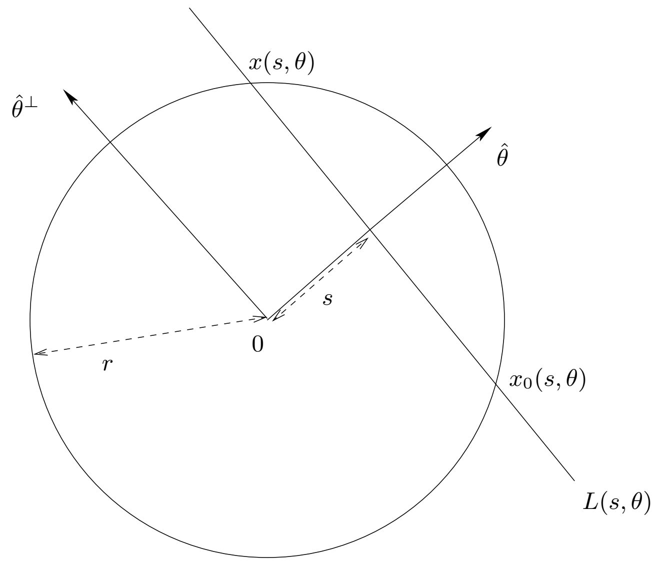

Figure 2.1: Radon parameterization of boundary points.

$P[k\rho](s,\theta)$ is precisely the Radon transform of $k\rho$ at (s,θ).

On to the inversion, we cannot apply directly the inverse Radon transform (iRT), instead we need to regularize the inversion for the following reason: the iRT is a deregularizing operator, and the inversion requires that we apply it to the remainder ε in (2.15), the smoothness properties of whom we do not know (we only have L^∞ estimates). Thus, applying the iRT directly to (2.15) may magnify errors at high frequencies.

We thus introduce a regularized inverse Radon transform $P^{-1,b}$ such that

$$P^{-1,b} \circ P[f] = W_b \star f(x) := f_b(x), \quad \text{where} \quad W_b(x) := \frac{1}{2\pi} \int_{R^2} e^{ix\cdot\xi} \hat{\Phi}\left(\frac{|\xi|}{b}\right) \, d\xi, \tag{2.17}$$

with $\hat{\Phi} : [0, \infty) \to [0, 1]$ a low-pass filter supported inside $[0, 1]$. Note that we have the scaling relation $W_b(x) = b^2 W_1(bx)$. b thus acts as a bandwidth and is such that $\frac{1}{b}$ measures the finest detail size of f_b, the approximation of f. Now the direct inversion scheme reads:

$$P^{-1,b}\left(\frac{\omega^{\frac{1}{2}}}{f}\mathcal{D}^\omega[k]\right) = [k\rho]_b + R^{\omega,b}[k], \quad \text{where} \quad R^{\omega,b}[k] := P^{-1,b}\left(\frac{\omega^{\frac{1}{2}}}{f}\varepsilon(k)\right). \tag{2.18}$$

Again, following consecutively the two strategies of inversion presented in Section 1.2, we first bound the error $R^{\omega,b}[k]$ in $L^\infty(\partial X \times \partial X)$ by assuming regularity on k. Estimates on the regularized inverse $P^{-1,b}$ together with the estimates obtained in Bal et al. (2011a) yield the following result:

Theorem 2.1.7 (Theorem 2.2 in Monard and Bal (2012d), Direct inversion)**.** *Let $\sigma \in \mathcal{C}^2(B_r)$ be known and assume that $k \in \mathcal{C}^2(B_r)$ and that the support hypothesis (2.16) holds. Then for ω large enough and fixed b, knowledge of the data $\mathfrak{D}^\omega[k](s, \theta)$ for $s \in [-r + D, r - D]$ and $\theta \in \mathbb{S}^1$ allows us to reconstruct $[k\rho]_b$ up to an error $R^{\omega,b}[k]$ (2.18) that is bounded in $L^\infty(B_r)$ by*

$$\|R^{\omega,b}[k]\|_{L^\infty} \le C \frac{b}{\sqrt{\omega}}.$$

Here, the constant C depends on $\|k\|_{\mathcal{C}^2}$, $\|\sigma\|_{\mathcal{C}^2}$, D and $\|w_1\|_{L^1}$.

On to the second inversion strategy, in order to obtain convergence of the iterative scheme, we can no longer assume regularity on k since we need the error operator to be a contraction from some L^p space into itself. The main result in the paper Monard and Bal (2012d) establishes the convergence of the following iterative scheme

$$k_0 := P^{-1,b}\left(\frac{\omega^{\frac{1}{2}}}{f}\mathcal{D}^\omega\right), \qquad k_{n+1} = P^{-1,b}\left(\frac{\omega^{\frac{1}{2}}}{f}\mathcal{D}^\omega\right) - R^{\omega,b}[k_n], \quad n \geq 0, \tag{2.19}$$

by showing that the error operator $R^{\omega,b}[k]$ is a contraction in an appropriate space. The main result reads as follows:

Theorem 2.1.8 (Iterative improvement). *Assume that $\sigma \in \mathcal{C}^2(B_r)$ is known and that k satisfies hypothesis (2.16). Suppose further that the function W_1 satisfies $\|W_1\|_{L^1(\mathbb{R}^2)} < \infty$. Then for fixed $b > 0$, there exists $K_1 > 0$ and $\omega_0 > 1$ such that for $\omega \geq \omega_0$ and k such that $\|k\rho\|_\infty \leq K_1$, the iterative scheme (2.19) converges to $k^\star \in B_{K_0}(L^\infty(B_{r-2D}))$. Moreover $k^\star$ satisfies the error estimate*

$$\|k^\star - [k\rho]_b\|_\infty \leq \frac{c_1}{1-c_1}\|[k\rho] - [k\rho]_b\|_\infty, \tag{2.20}$$

where $c_1 \in (0,1)$.

For fixed b, the constant c_1 behaves in ω like

$$c_1 \approx \frac{C_1}{\omega}(b^2 + b^5) + C_2\frac{b^3}{\omega^{\frac{1}{2}}}\log\left(\frac{\omega}{b}\right),$$

where the first and second terms in the right-hand side are respective constants of boundedness in $L^\infty(B_{r-2D})$ for

- the error due to the part of single scattering that is not being inverted for,
- the error due to multiple scattering.

The first point is a relatively straightforward application of the stationary phase formula, in which we analyse the stationary points in (s, θ) of the phase function

$$\varphi_y(s,\theta) = |x_0(s,\theta) - x(s,\theta)| - |x_0(s,\theta) - y| - |x(s,\theta) - y|, \quad y \in B_{r-D}. \tag{2.21}$$

One can show that φ_y has a stationary curve of equation $\{s = y \cdot \hat{\theta}\}$, over which we have $\varphi_y(y \cdot \hat{\theta}, \theta) = 0$. Since the critical points form a one-dimensional manifold, each of them is degenerate. However one can find global coordinates in which the phase is "quadratic" with respect to the first coordinate, so that the behavior of the partial integral with respect to this coordinate is ruled by non-degenerate stationary phase. We can thus write an asymptotic expansion of the first partial integral and analyse the resulting one-dimensional integral more easily. This approach for controlling an oscillatory integral by dimension reduction is summarized in (Monard and Bal, 2012d, Lemma 4.1), a lemma on *stationary phase with parameter* that was tailored for our purposes.

The second point (control of higher order scattering terms) is more technical, and the "least friendly" contribution (i.e. the part that prevents contraction the most) comes from the double scattering term. Controlling this term requires finding the stationary points in (s, θ) of the following phase function

$$\varphi_{y_1,y_2}(s,\theta) = |x_0(s,\theta) - x(s,\theta)| - |x_0(s,\theta) - y_1| - |y_2 - x(s,\theta)|, \quad y_1, y_2 \in B_{r-D}. \tag{2.22}$$

If $y_1 = y_2 = y$, they are clearly reduced to the previous case $\{s = y \cdot \hat{\theta}\}$. Now, if $y_1 \neq y_2$, we show that there still exists a smooth stationary curve $s = \sigma(\theta)$, but whose expression we believe cannot be obtained analytically. The restriction of φ_{y_1,y_2} to this stationary curve has different behaviors according to whether $y_1 = y_2$ or not, and the corresponding oscillatory integral in (s, θ) becomes a singular integral kernel in $|y_1 - y_2|$. Again here, we find coordinates that are adapted to the geometry and that allow us to bound the oscillatory integral properly. Here, the techniques that we used for bounding the geometric scattering series were a combination of techniques similar to the time-independent case as well as stationary phase with parameter.

Remark 2.1.9. *Note that, unlike in Bal et al. (2011a), one can no longer study the stationary behavior of phase functions* (2.21) *and* (2.22) *with respect to* variables that are inside the domain *(y in* (2.21) *and y_1, y_2 in* (2.22)*), because k depends on them and we can no longer differentiate k in order to show a contraction property over L^p-spaces. In so doing, the differentiations that must occur in the stationary phase expansions are applied to the regularized kernel of the iRT instead of k, and this results in the creation of a factor b every time this kernel is differentiated.*

An important conclusion of Theorem 2.1.8 is that, there is a trade-off between the modulation frequency ω and the cutoff bandwidth b in order keep a constant error level on the reconstruction. Moreover, the estimates of the bounds on the error operators, polynomial in b, imply that the reconstruction is now Hölder-stable, which is a great improvement compared to the logarithmic stability (i.e. exponential behavior in b) first displayed in Bal et al. (2008) in the time-independent case.

2.2 Numerical methods for two-dimensional transport

Aside from theoretical aspects of inverse transport problems, numerical methods for transport in two dimensions of space were developed (see work Bal and Monard (2010) with G. Bal), in order to solve forward transport and, in the end, inverse transport problems that required iterated schemes using a forward solver. The main purpose of the code was to solve the transport equation accurately in the transport regime, i.e. when scattering is not high enough to consider the diffusion model, yet scattering is not small enough to restrict the model to free transport, in short, when the domain size equals a few transport mean free paths. The main challenge in this regime is to propagate singularities accurately and comply with the theoretical fact that reconstructions of both optical parameters $(\sigma, k\phi)$ from fully resolved measurements are stable. Our code consisted in implementing the well-known *source iteration* method, i.e. a numerical version of the computation of the Neumann series (1.15) by computing a finite number of terms of the series

$$u = \sum_{p=0}^{\infty} u_p, \quad \text{where } u_0 = \mathcal{J} f_0 \quad \text{and} \quad u_{p+1} = \mathcal{K} u_p, \quad p \geq 0,$$

combining this with a discretization of the space of velocities $\mathbb{S}^1_N = \{\theta_1, \cdots, \theta_N\}$, a method referred to as the method of *discrete ordinates* (DO). Numerically, the discretized operators $\mathcal{J}$ and $\mathcal{K}$ require solving ordinary differential equations (ODE's) along various directions (one for each velocity), as well as angular averaging (weighted sums over directions at each point in physical space). The main novelty of our code was that, by means of *image rotations*, the propagations by ODE were performed on a grid that was always aligned with the direction of propagation, therefore controlling perfectly the transverse effects that usually appear when these directions are not aligned. Image rotations were implemented either using four-point approximations with linear complexity, or using a Fast Fourier Transform based rotation method with spectral accuracy and complexity $\mathcal{O}(n \log n)$, n being the image size.

We also took additional advantage of the grid alignment with the direction of propagation as follows: as in the Fermi pencil-beam approximation, the part of the scattering kernel that is peaked forward can generate *transverse diffusion* effects, and we could implement this effect thanks to the correct grid alignment, using a transverse second derivative term. Implementing this was a way for us to not only (i) tackle the well-known *ray effect* by widening beams (therefore filling the gaps in space where usual DO based codes fail at propagating information), but also (ii) introduce physics-based noise in the data. The latter was done to assess the robustness of some iterative reconstruction schemes that were inspired from explicit reconstructions first derived in Choulli and Stefanov (1999, 1996b), in the context of inverse stationary transport problems with knowledge of the full albedo operator.

Part II

An inverse problem with internal functionals in ultrasound-modulated tomography

Chapter 3

Introduction and preliminaries

3.1 Introduction

3.1.1 Hybrid medical imaging methods

As explained in the general introduction, medical imaging methods may be split into two classes of modalities:

High-contrast modalities: These modalities aim at imaging electrical, optical or elastic properties of tissues. We call them high contrast because all of the properties we mentioned vary greatly between healthy and unhealthy (e.g. cancerous) tissues. The models involved in all these modalities are usually elliptic partial differential equations, for which inverse problems are severely ill-posed. This lack of stability is due to the fact that the singularities of the unknown parameters are not being passed on to the data in the forward process. In short, the forward operator for such models is highly regularizing, so inverting it is a highly de-regularizing process. As a characteristic feature of a de-regularizing process, when measurements are noisy, as is always the case in practice, the inversion procedure magnifies this noise at high frequencies, and exponentially so as frequency increases. This means that small scales can not be reconstructed, and the only way for the overall reconstruction to take maximum advantage of the data present in the measurements while taking out the noise, is to regularize highly the inversion process. The outcome of such regularization is that these modalities give up on small scales of the unknown and thus suffer from low resolution, e.g. centimetric while one desires millimetric and lower for proper tumor detection at early stages.

High-resolution modalities: On the other hand, modalities based on waves or isometric (unitary) transforms, such as magnetic resonance imaging, echography, or modalities involving geometric integral transforms (e.g. X-ray computerized tomography), can be inverted more stably, i.e. with better resolution. However, the quantity they measure is not sensitive enough to whether a tissue is healthy or not, a shortcoming referred to as low contrast.

In the past decade and a half, *hybrid* (a.k.a. coupled physics, multi-wave) medical imaging methods have gained considerable interest by the research community. These techniques aim at combining one modality of each subclass above, the purpose being to benefit from the advantages of both modalities thanks to the existence of a physical coupling between them. In such settings, one has at hand two physical models of different natures, and assumes a coupling between them. Inverse problems arising in hybrid methods may thus be inverted in a two-step process, in which the high-resolution modality may be inverted first, in turn providing *internal* functionals for the high-contrast, low-res problem. See Bal (2012c); Kuchment (2012) for recent topical reviews. Examples of such coupling methods include: Photo-Acoustic Tomography (PAT), Thermo-Acoustic Tomography (TAT) and Ultrasound-Modulated Optical Tomography (UMOT) (also known as Acousto-Optic Tomography), coupling optics or electromagnetism with ultrasound; Ultrasound-Modulated Electrical Impedance Tomography (UMEIT), also called Electro-Acoustic Tomography (EAT) or Impedance-Acoustic Computerized Tomography (ImpACT), resulting from the coupling of electrical currents with ultrasound; Magnetic Resonance EIT (MREIT) and Current Density Impedance Imaging (CDII) couple magnetic resonance with electrical currents; finally, Elastography may be coupled with ultrasound or magnetic resonance to give rise to Transient Elastography (TE) and Magnetic Resonance Elastography (MRE), respectively. We now describe some of them in more detail.

Photo-acoustic and thermo-acoustic tomography (PAT/TAT): Electromagnetic waves (modelled either by photon transport or a diffusion equation for PAT, and Maxwell's equations or Helmholtz for TAT) are being cast through a body and results in energy being absorbed by tissues. One aims at reconstructing absorption, diffusion or conductivity coefficients, depending on the model. By thermoelastic phenomena, the tissue expands in response to this absorption and heats in the milli-Kelvin range, in turn generates pressure waves which are then measured by transducers at the boundary of the domain. Such boundary measurements allow one to stably reconstruct the initial energy absorbed, which in turn represents internal information that may be used to invert for the constitutive parameters of the first PDE model. The first step is an inverse wave problem[1] whose main inversion techniques are summarized in Kuchment (2012) for the constant sound speed case, and in Stefanov and Uhlmann (2012) for the non-constant case. The second step, also called *Quantitative* PAT or TAT, involves the inversion of constitutive parameters of one of the PDE's previously mentioned, from internal functionals depending on (i) the solution of the PDE at order zero, linearly or quadratically, (ii) the unknown parameter, linearly, multiplied by (iii) a coefficient Γ that measures the coupling between both physical models, usually unknown as well but sometimes set to $\Gamma \equiv 1$ as a first simplification

[1] i.e. the inverse problem of reconstructing the initial condition of a wave equation from knowledge of boundary data over a certain time interval

(see Cox et al. (2009) for a discussion on the limitations of such an assumption). These problems have been analyzed in e.g. Bal and Ren (2012); Bal and Uhlmann (2012a, 2010); Zemp (2010); Bal et al. (2011b); Ammari et al. (2012), see also the summary of explicit reconstruction formulas Bal (2012b). We briefly recall the formulations of such problems, so as to familiarize the reader with the type of problems one is left with, at the quantitative step of such hybrid methods. In PAT, radiation is typically near-infra-red light, which we present with a diffusion model here as

$$-\nabla \cdot (\gamma \nabla u_j) + \sigma u_j \quad (X), \quad u_j|_{\partial X} = f_j, \quad 1 \le j \le J, \tag{3.1}$$

where the f_j's are prescribed at the boundary. The measurements acquired, which model the emitted acoustic signal (proportional to the density of energy absorbed), take the form

$$H_j(x) = \Gamma(x)\sigma(x)u_j(x), \quad x \in X, \quad 1 \le j \le J. \tag{3.2}$$

The unknowns in this problem are (γ, σ, Γ), respectively the diffusion tensor, the absorption coefficient and the Grüneisen coefficient. In TAT, radiation is in the low-frequency range (on the order of hundreds of MHz) and is best described using time-harmonic Maxwell's equations for the electric field,

$$-\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) + (\varepsilon\omega^2 - i\omega\sigma)\mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = 0, \tag{3.3}$$

where $\mathbf{E}$ is the vector-valued time-harmonic electric field and $(\mu, \varepsilon, \sigma)$ respectively denote the relative permeability and permittivity and the conductivity. The measurement data in this case are modelled by

$$H(x) = \Gamma(x)\sigma(x)|\mathbf{E}|^2(x), \quad x \in X, \tag{3.4}$$

where Γ denotes again the Grüneisen coefficient.

Assuming that only one scalar component of $\mathbf{E}$ is non-negligible (call it u), equation (3.3) simplifies into the following Helmholtz equation

$$\nabla \cdot (\mu^{-1}\nabla u_j) + (\varepsilon\omega^2 - i\omega\sigma)u_j = 0 \quad (X), \quad u_j|_{\partial X} = f_j, \tag{3.5}$$

where the f_j's are again prescribed at the boundary and ω is also a parameter that can be tuned. In this case, the measurements look like

$$H_j(x) = \Gamma(x)\sigma(x)|u_j|^2(x), \quad x \in X, \quad 1 \le j \le J. \tag{3.6}$$

The unknowns in the last two problems are $(\mu^{-1}, \varepsilon, \sigma, \Gamma)$.

Transient Elastography (TE) and Magnetic Resonance Elastography (MRE): In these modalities, one prescribes again Dirichlet or Neumann conditions at the boundary and solves an elasticity problem in the domain (X) with unknown the displacement field. Either by ultrasound tomography for TE or by magnetic resonance imaging for MRE, one assumes to be able to stably recover the displacement field $H = u$ inside the domain, from which one wants to recover the elastic parameters of the medium. The model may be given by the time-harmonic linear elasticity equation

$$\nabla \cdot (\mathcal{C} : (\nabla \mathbf{u} + \nabla \mathbf{u}^T)) + \omega^2 \mathbf{u} = 0, \tag{3.7}$$

augmented with known boundary conditions, and where $\mathcal{C}$ is a fourth-order elasticity tensor, which can be described by up to 21 parameters (e.g. in the isotropic case, $\mathcal{C}$ is characterized by the well-known Lamé parameters (λ, μ)). The question is then to infer the reconstructibility of $\mathcal{C}$ from knowledge of solutions $\mathbf{u}$ of (3.7) inside the domain. Examples of Elastography problems with internal data may be found in e.g. Albocher et al. (2009); McLaughlin et al. (2010) in reconstruction problems of shear modulus μ.

Current Density Imaging (CDII): Magnetic Resonance EIT (MREIT) and CDII are two modalities that aim to reconstruct the conductivity tensor of an elliptic equation from internal measurements provided by magnetic resonance imaging. The initial model is again given by the conductivity equation for the electrical potential u

$$-\nabla \cdot (\gamma \nabla u) = 0 \quad (X), \quad u|_{\partial X} = f, \tag{3.8}$$

with f prescribed at the boundary. The electrical current density $\mathbf{J} = -\gamma \nabla u$ satisfies the equations

$$\nabla \cdot \mathbf{J} = 0 \quad \text{and} \quad \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}, \quad x \in X,$$

where μ_0 is a constant, known magnetic permeability. The main assumption is that the MRI measurements allows to access the magnitude of the current density $|\mathbf{J}|$ inside the domain, so that the CDII problem consists in recovering the conductivity γ in (3.8) from knowledge of measurements of the form $|\gamma \nabla u|$ inside the domain. See Nachman et al. (2010, 2011) and the recent review paper Seo and Woo (2011) for a thorough account on this problem and related ones.

Hybrid methods involving ultrasound modulations: Our last example regards the work presented hereafter. It deals with Electrical Impedance Tomography or Optical Tomography in an acoustically perturbed context. Differences between acoustically perturbed and unperturbed boundary measurements allows to derive internal functional under some coupling assumptions between acoustic waves and tissue conductivities (or optical properties). In this setting (or, equivalently, the one of Impedance-Acoustic Computerized Tomography, a modality that assumes a coupling similar in spirit to PAT and TAT), one is then left assessing the reconstructibility of the tensor γ in equation (3.8) from, this time, the knowledge of the *power density* $\nabla u \cdot \gamma \nabla u$ inside the domain, where u solves (3.8).

As one might wonder, internal functionals are still being generated in a non-invasive manner. The internal information one obtains, when considered as data for inverse elliptic problems for instance, is usually a much richer set of information than the usual boundary Cauchy data, and thus the corresponding inverse problem is much less ill-posed ("mildly" instead of "severely", in the sense of the Hilbert scale for instance). In this sense, one has found a way of replacing a severely ill-posed problem with a more stable one by enriching the physical setup and coupling two models.

Heuristically, the fact that inverse problems based on PDE's with internal functionals that depend on the solution and the constitutive parameters are relatively stable should not come as a great surprise, since the theory of PDE's (especially microlocal analysis and pseudo-differential operators) describes precisely the interplay between the singularities of the solutions and those of the parameters of that PDE. In particular. Except for rare cases where cancellations occur, internal functionals that depend on solutions, on their gradients and on the constitutive parameters *do* contain the singularities of the unknown constitutive parameters. The mild ill-posedness thus only comes from how much these singularities have been damped. To some extent, the degree of this damping is not always far from being directly intuited from (i) the relative number of derivatives that the PDE involves with respect to the parameters and the solutions, and (ii) how many times the data functionals differentiate the solutions and the unknown parameters.

Mathematical techniques used for tackling such problems are reaching broadly in the realm of applied mathematics, including non-linear PDE's, geometric transforms, differential geometry, pseudo-differential calculus to name a few. Although the problems presented above are all motivated by hybrid methods, the resolution techniques used for solving them may also vary greatly according to the nature of the internal functionals considered. Functionals that depend on the solutions of elliptic PDEs (as in PAT, TAT, TE and MTE) will be approached differently as those that depend on their *gradients* (as in e.g. CDII, MREIT, UMEIT, UMOT, ImpACT). For a given problem, the resolution methods may also vary with the number of internal functionals considered. For instance, CDII with one functional becomes a degenerate elliptic 1-Laplacian equation (see next section), while UMOT with one functional becomes a hyperbolic 0-Laplacian equation. Both of the above equations have very different behaviors, but if both modalities are treated with a large enough number of functionals, then the unknown in both problems, whenever a scalar tensor, may be recovered via a gradient equation, see Monard and Bal (2012c). While the list of theoretical tools for resolution is long, that of numerical approaches for implementation is even longer, leaving room for new algorithmic developments as these problems are fairly new and sometimes, unexpected.

3.1.2 Hybrid methods involving ultrasound modulations

The present work focuses on the study of an inverse problem arising in some hybrid medical imaging methods that involve a coupling between ultrasounds and conductivity or diffusion phenomena. Namely, one starts from the following elliptic PDE over an open bounded domain

$X \subset \mathbb{R}^n$ with $n \geq 2$

$$-\nabla \cdot (\gamma \nabla u) \equiv - \sum_{i,j=1}^{n} \partial_i (\gamma^{ij} \partial_j u) = 0, \quad (X), \quad u|_{\partial X} = g, \tag{3.9}$$

where γ is a real-valued symmetric positive definite tensor with bounded coefficients, satisfying a *uniform ellipticity* condition for some $\kappa \geq 1$:

$$\kappa^{-1} |\xi|^2 \leq \xi \cdot \gamma(x) \xi \leq \kappa |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad x \in X. \tag{3.10}$$

Equation (3.9) may model either one of the following phenomena:

- The *conductivity* equation, which rules the equilibrium distribution of the electrostatic potential u inside the domain X, in response to a prescribed boundary voltage g. Such an equation is the model for the problem of Electrical Impedance Tomography (EIT), where one aims at reconstructing the conductivity tensor γ. Historically, such reconstructions were sought after by prescribing voltage at the boundary and measuring boundary currents of solutions $\gamma \nabla u \cdot \nu|_{\partial X}$, a problem referred to as *Calderón's problem*, which first appeared in the literature in the 1980's (see Calderón (1980)) and received a lot of attention ever since.

- The *stationary diffusion* equation, which rules the equilibrium density u of e.g. photons in human tissues, subject to diffusion and absorption (modelled by an additive term $\sigma_a u$ with $\sigma_a \geq 0$ in the left-hand side of equation (3.9)) phenomena, in response to a stationary boundary illumination g. As explained in Part I, low-energy photons (e.g. at optical wavelengths) scatter a lot in human tissues, thus the Boltzmann transport model initially used for describing photon transport can be accurately described by a diffusion phenomenon such as (3.9) in the limit of high scattering. Such an equation models the medical imaging method of Optical Tomography (OT), the purpose of which is to reconstruct the diffusion tensor γ and the absorption term σ_a mentioned above. Such quantities, also referred to as *optical coefficients* contain valuable information for tumor detection, see e.g. Arridge (1999). Note that the techniques of this chapter have not yet been generalized to the case $\sigma_a \neq 0$, and given the practical relevance of this parameter, this issue should be dealt with in future work.

In both models above, the classical way of taking measurements is two prescribe the Dirichlet boundary condition in (3.9) and to measure the Neumann condition $\gamma\nabla u\cdot\nu|_{\partial X}$, thus giving rise to the *Dirichlet-to-Neumann* map, also known as the *voltage-to-current* mapping, or the *Poincaré-Steklov* operator. Equivalently, one may also work with the converse setting, that is, prescribing fluxes and measuring boundary traces.

In either case, as we mentioned earlier, such inverse problems, whenever injective at all, are severely ill-posed and the above imaging modalities suffer from very low resolution as stand-alone modalities. We will make these statements more precise in Section 3.2.2 by recalling the main features of the Calderón inverse conductivity problem.

The problem we consider here uses a different, internal kind of information in order to reconstruct the tensor γ. Namely, based on two examples of derivations that will be recalled in Section 3.3, we assume that we are able to generate the so-called *power density* of the solution u of (3.9), defined as

$$\mathcal{H}_\gamma[g](x) := \nabla u(x)\cdot\gamma(x)\nabla u(x), \quad x \in X. \tag{3.11}$$

In the isotropic (or scalar) case $\gamma := \sigma\mathbb{I}_n$ with σ a scalar function, this power density takes the form $\sigma|\nabla u|^2(x)$ for $x \in X$. $\mathcal{H}_\gamma$ may thus be seen as, in the largest "physical" spaces, a continuous functional from $H^{\frac{1}{2}}(\partial X)$ to $L^1(X)$ (the mapping properties of $\mathcal{H}_\gamma$, based on standard results in forward elliptic theory, will be studied in Section 3.2). From two solutions u and v of (3.9) with respective Dirichlet boundary values g and h, one may also construct the *mutual power density* via the polarization relation

$$\mathcal{H}_\gamma[g,h] := \nabla u \cdot \gamma \nabla v = \frac{1}{4}\left(\mathcal{H}_\gamma[g+h] - \mathcal{H}_\gamma[g-h]\right). \tag{3.12}$$

Although the generation of functionals of the form (3.11) or (3.12) in practice requires processing that may noisify these functionals (a topic for potential future work not covered here), we formulate the inverse conductivity problem with power density functionals assuming that these functionals are known *exactly* for now. To this end we recall that a tensor γ belongs to $\Sigma(X)$ if there exists a constant $\kappa \ge 1$ such that

$$\kappa^{-1}|\xi|^2 \le \xi \cdot \gamma(x)\xi \le \kappa|\xi|^2, \quad x \in X, \quad \xi \in \mathbb{R}^n,$$

in particular we call C_γ the smallest such κ. The problem we are interested in is

Problem 3.1.1 (Inverse conductivity from power density functionals)**.** *For γ in $\Sigma(X)$ or any subset of it, does the power density measurement operator $\mathcal{H}_\gamma$ uniquely characterize γ ? If yes, what stability estimates do we have ?*

In this area of research, as power densities (3.11) seem hard and expensive to generate in practice, one may consider the same questions of uniqueness and stability with a finite number of such functionals of the form $\{\mathcal{H}_\gamma[g_i]\}_{1\le i\le m}$. The work presented in the next sections attempts to answer Problem 3.1.1 without limitation on the number of functionals (though finite and much preferably the smallest possible).

As we mentioned, being able to generate power density relies on the assumption of a physical coupling between acoustic waves and conductivity/diffusion phenomena. Namely, one probes a conductive (or diffusive) medium with acoustic waves and perturbs the constitutive parameter (e.g. the distribution of light scatterers in the optical case) of (3.9). As it is explained in Ammari et al. (2008), the fact that acoustic waves affect the conductivity of tissues was noticed as far back as the 1900's (Körber, 1909), and the idea of exploiting this property for biomedical imaging purposes came in the 1970's (Geselowitz, 1971). However, focusing acoustic signals physically to perturb the conductivity *locally* is a much more recent story, and was established experimentally in Jossinet et al. (1999, 2005). With these new capabilities at hand, the first experimental approach to generate power densities is to perturb the medium acoustically while making boundary measurements of the solution of (3.9). One can show that, taking difference between acoustically perturbed and unperturbed measurements allows to obtain the power density of the solution at the point of focus, see Ammari et al. (2008). With similar techniques, it was suggested in Kuchment and Kunyansky (2010) that, using a rich enough family of delocalized waves (instead of focused ones) and via appropriate post processing (e.g. inverse Fourier Transform), one could also construct power densities. This approach gave rise to the methods so-called EIT by Elastic Deformation, or Ultrasound-Modulated EIT (UMEIT) or Ultrasound-Modulated OT (UMOT).

Another experimental approach justifying that one considers Problem 3.1.1 was established in Gebauer and Scherzer (2009), assuming a thermoelastic effect as in TAT/PAT presented above: the power density that is absorbed by tissues causes heating which in turn generates pressure waves that propagate throughout the domain. Measuring these waves at the boundary allows one to stably reconstruct the absorbed power density, in turn feeding an inverse conductivity problem with internal functionals. This derivation gave rise to the so-called Impedance Acoustic Computerized Tomography (ImpACT) method.

The experimental settings we have just presented therefore justify that Problem 3.1.1 be studied, and indeed, it has received attention in the last couple of years. The first inversion formula for Problem 3.1.1 was given in Capdeboscq et al. (2009) in the isotropic, two-dimensional setting. There, a constructive algorithm as well as an optimal control approach for numerical reconstruction were presented. Kuchment and Kunyansky (2011) then studied a linearized, isotropic version of Problem 3.1.1 in dimensions two and three with numerical implementation.

Problem 3.1.1 may also be studied under constraints of limitations on the number of power densities available. In particular, one may consider the problem of reconstructing an isotropic tensor $\gamma = \sigma \mathbb{I}_n$ in (3.9) from only one measurement $H = \sigma |\nabla u|^2$. In this case, σ may be expressed as $H/|\nabla u|^2$ and, upon plugging this expression back into (3.9), we obtain the following non-linear partial differential equation of the form

$$\nabla \cdot \left(\frac{H}{|\nabla u|^2} \nabla u \right) = 0 \quad (X), \quad u|_{\partial X} = g.$$

In spite of its elliptic looks, one may show that this PDE is of hyperbolic nature. It is sometimes referred to as the zero-Laplacian, in reference to the p-Laplacian equation

$$\nabla \cdot (H |\nabla u|^{p-2} \nabla u) = 0 \quad (X), \quad u|_{\partial X} = g,$$

an equation that is of elliptic nature for $p > 1$, degenerate elliptic for $p = 1$, and hyperbolic for $p < 1$. Newton-based methods were proposed in Gebauer and Scherzer (2009) in order to reconstruct both u and σ, and the problem was studied theoretically in Bal (2012a).

In search for explicit reconstruction formulas using larger numbers of functionals, the author first extended the reconstruction result from Capdeboscq et al. (2009) to the three-dimensional, isotropic case in Bal et al. (2012a) with Bal, Bonnetier and Triki. This result was then generalized in Monard and Bal (2012c) to n-dimensional, isotropic tensors with more general types of measurements of the form $\sigma^{2\alpha} |\nabla u|^2$ with α not necessarily $\frac{1}{2}$. This allowed to treat the case of CDII presented in the last section, for which $\alpha = 1$. Finally, the same authors derived reconstruction formulas for the fully anisotropic two-dimensional problem and validated them numerically in Monard and Bal (2012a).

In the last three papers presented, the authors derived explicit reconstruction algorithms in the case where the power densities belong to $W^{1,\infty}(X)$, and assuming some qualitative properties satisfied by the solutions. In particular, the reconstruction algorithm for the isotropic case (or, equivalently, of a scalar function in front of a known anisotropic tensor) strongly relies on the existence of n solutions of (3.9) whose gradients form a basis of $\mathbb{R}^n$ at every point of the domain. Under such assumptions, stability estimates were derived for the reconstruction schemes proposed, of Lipschitz type for the determinant of the conductivity tensor under knowledge of the anisotropic structure $\tilde{\gamma} := (\det \gamma)^{-\frac{1}{n}}\gamma$, and of (less stable) Hölder type for the anisotropic structure $\tilde{\gamma}$. Finally, it was shown for certain types of tensors γ that the assumption of linear independence made on the solutions could be guaranteed *a priori* by choosing appropriate boundary conditions, so that all the reconstruction procedures previously established could be properly implemented.

Studying a linearized version of Problem 3.1.1 from the pseudo-differential calculus standpoint, the Lipschitz stability mentioned above was also pointed out in Kuchment and Steinhauer (2011) in an isotropic setting. There, the authors showed that from three measurements of the form (3.11), the linearized power density operator is an elliptic functional of σ. They also studied in more detail the "stabilizing" nature of internal functionals of certain kinds that have arisen in hybrid medical imaging methods such as CDII, MREIT, UMEIT, UMOT, ImpACT. An extension of this result to the anisotropic case is presently investigated by the author, together with Bal and Guo, see Bal et al. (2012b).

The present work aims at summarizing, unifying and extending the work done in Bal et al. (2012a); Monard and Bal (2012c,a), by treating in full extent the general anisotropic, n-dimensional case of Problem 3.1.1. An outline of the next sections is presented below.

3.1.3 Outline

Section 3.2 first recalls a few results of forward elliptic theory, specifically existence, uniqueness and well-posedness of solutions in appropriate spaces. We also review the existing results for the so-called Calderón problem, in particular how ill-posed it is.

In section 3.3, we then propose a derivation that justifies the obtention of power density measurements, and recall some settings that exist in the literature.

In section 4.1, we introduce a frame[2] of interest, and we add to the initial problem an intermediate step toward inversion for the tensor γ, that consists in reconstructing that frame. We also discuss the interest and constraints of replacing the previous frame by an unknown, orthonormalized one.

On to the inversion procedures, we then derive explicit local reconstruction procedures first in the two-dimensional case in Section 4.2, then in general dimension in Section 4.3. In both cases, we first express $\nabla \log |\gamma|$ in terms of the frame previously mentioned, after which we show how to reconstruct the frame locally, so that $\log |\gamma|$ may be reconstructed afterwards. The reconstruction of the anisotropy tensor is also studied by considering larger sets of power densities, and it is obtained under certain conditions on solutions, by means of exact, algebraic relations. The stability of each reconstruction formula is also assessed.

Once the local reconstruction algorithms are derived, we present in section 5.1 how to patch these local reconstructions together so as to obtain a global reconstruction scheme for the quantities of interest. Finally, since these reconstruction formulas require that the solutions of (3.9) generated by boundary conditions satisfy certain qualitative properties, we study in Section 5.2

[2]Over a domain $\Omega \subset \mathbb{R}^n$, a *frame* is a family of n vector fields of maximal rank at each point $x \in \Omega$. See Appendix C for more detail.

what conductivities possess such families of solutions. In our understanding, conductivities that are, in the end, reconstructible from their power density functional will be those that possess solutions satisfying these qualitative properties.

3.2 Recalls on elliptic theory and Calderón's inverse conductivity problem

In this section, we recall elements of forward elliptic theory in various functional settings, and we deduce from the main theorems mapping properties of the power density operator. We then recall the main features of the well-known Calderón problem.

3.2.1 Forward elliptic theory and properties of the power density operator

For a tensor γ satisfying the uniform ellipticity condition (3.10), we may define C_γ as being the smallest constant $\kappa \geq 1$ such that (3.10) holds. Let us first mention that, as pointed out in Astala et al. (2005) and references therein, it is necessary to require $C_\gamma < \infty$ for the injectivity results of the Calderón problem presented hereafter to hold, so we will consider this case as a "natural" setting for our own problems. Borrowing notation from Astala et al. (2005), we thus define

$$\Sigma(X) = \{\gamma \in L^\infty(X, S_n(\mathbb{R})) \mid C_\gamma < \infty\}, \tag{3.13}$$

where $S_n(\mathbb{R})$ denotes the space of real-valued $n \times n$ symmetric matrices. The conductivity equation (3.9) is a particular case of an elliptic PDE of the form

$$\nabla \cdot (\gamma \nabla u) + b \cdot \nabla u + cu = 0 \quad (X), \quad u|_{\partial X} = g, \tag{3.14}$$

with $b \equiv \vec{0}$ and $c \equiv 0$, in which case existence and uniqueness of a solution is straightforward under minimal regularity assumptions, i.e. as soon as γ belongs to $\Sigma(X)$. The following theorem is obtained by writing a variational formulation for (3.9) and applying Lax-Milgram's theorem. See Evans (1998) for instance.

Theorem 3.2.1. *Let ∂X be Lipschitz and assume that $g \in H^{\frac{1}{2}}(\partial X)$ and $\gamma \in \Sigma(X)$. Then equation (3.9) has a unique solution in $H^1(X)$, which moreover satisfies the estimate*

$$\|u\|_{H^1(X)} \le C\|g\|_{H^{\frac{1}{2}}(\partial X)}, \tag{3.15}$$

where the constant C only depends on X and C_γ.

From this we deduce the natural functional setting for the power density operator:

Corollary 3.2.2. *Let ∂X be Lipschitz and $\gamma \in \Sigma(X)$. Then the power density operator $\mathcal{H}_\gamma$ defined in (3.11) is well-defined and continuous in the functional setting*

$$\mathcal{H}_\gamma : H^{\frac{1}{2}}(\partial X) \to L^2(X). \tag{3.16}$$

As we will see later, we will usually need more regularity on the measurements, in particular the power densities considered should be bounded with bounded gradients. In order to obtain such regularity, we will use a more regular estimate than (3.15), given for instance in (Isakov, 2006, Theorem 4.1) (initially given for more general equations of the type (3.14)).

Theorem 3.2.3. *Let $\alpha \ge 0$ and assume that $\partial X \in \mathcal{C}^{2+\alpha}$, $\gamma \in \Sigma(X)$ with components in $C^\alpha(\overline{X})$, $f_j \in \mathcal{C}^\alpha(\overline{X})$ for $0 \le j \le n$ and $g \in \mathcal{C}^{1+\alpha}(\partial X)$. Then the unique solution u of the PDE*

$$-\nabla \cdot (\gamma \nabla u) = f_0 + \sum_{j=1}^n \partial_j f_j \quad (X), \quad u|_{\partial X} = g$$

belongs to $\mathcal{C}^{1+\alpha}(X)$. Moreover, there is a constant C depending only on X and $\|\gamma\|_{\mathcal{C}^\alpha(\overline{X})}$ such *that the following estimate holds*

$$\|u\|_{\mathcal{C}^{1+\alpha}(X)} \leq C \left(\|g\|_{\mathcal{C}^{1+\alpha}(\partial X)} + \sum_{j=0}^{n} \|f_j\|_{\mathcal{C}^{\alpha}(\overline{X})} \right). \tag{3.17}$$

Note that $\partial_j f_j$ has to be understood as a weak derivative if f_j does not have sufficient regularity (see Isakov (2006)), however we will work in settings where this is not the case.

Corollary 3.2.4. *Let $\alpha \geq 0$ and assume that $\partial X \in \mathcal{C}^{2+\alpha}$ and $\gamma \in \Sigma(X)$ with components in $C^{\alpha}(\overline{X})$. Then the power density operator $\mathcal{H}_\gamma$ defined in (3.11) is well-defined and continuous in the functional setting*

$$\mathcal{H}_\gamma : \mathcal{C}^{1+\alpha}(\partial X) \to \mathcal{C}^{\alpha}(X). \tag{3.18}$$

In the sequel, we will see that stability statements and the reconstruction algorithms usually require that the power densities be in $W^{1,\infty}(X)$, and using Corollary 3.2.4 with $\alpha = 1$ and $f \equiv 0$ provides (possibly not sharp) conditions for power densities to be in $W^{1,\infty}(X)$. Here we use the obvious continuity of the injection $\mathcal{C}^1(\overline{X}) \to W^{1,\infty}(X)$.

Corollary 3.2.5. *Assume that $\partial X \in \mathcal{C}^3$ and $\gamma \in \Sigma(X)$ with components in $C^1(\overline{X})$. Then the power density operator $\mathcal{H}_\gamma$ defined in (3.11) is well-defined and continuous in the functional setting*

$$\mathcal{H}_\gamma : \mathcal{C}^2(\partial X) \to W^{1,\infty}(X). \tag{3.19}$$

When ∂X is of class $\mathcal{C}^1$ (which is the case if one is in the setting of Corollary 3.2.5), we recall (see e.g. (Evans, 1998, Theorem 4 p294)) that functions in $W^{1,\infty}(X)$ are precisely the Lipschitz continuous functions over X.

Theorem 3.2.6 (Characterization of $W^{1,\infty}$)**.** *Let X be open and bounded with ∂X of class $\mathcal{C}^1$. Then $u : X \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(X)$.*

On to the continuity of power density operators with respect to changes in the conductivity tensor, we now establish the following

Proposition 3.2.7. *Let $\gamma, \gamma' \in \Sigma(X)$ have their components in $\mathcal{C}^1(X)$, then there exists a constant C such that for any $g \in \mathcal{C}^2(\partial X)$, the following estimate holds*

$$\|\mathcal{H}_\gamma[g] - \mathcal{H}_{\gamma'}[g]\|_{W^{1,\infty}(X)} \le C\|g\|^2_{\mathcal{C}^2(\partial X)}\|\gamma - \gamma'\|_{\mathcal{C}^1(\overline{X})}. \tag{3.20}$$

Proof. If u solves $\nabla \cdot (\gamma \nabla u) = 0$ and u' solves $\nabla \cdot (\gamma' \nabla u') = 0$ over X with Dirichlet boundary condition equal to g for both, then, up to taking the maximum constant, Theorem 3.2.3 implies the following estimates

$$\max(\|u\|_{C^2(X)}, \|u'\|_{C^2(X)}) \le C\|g\|_{\mathcal{C}^2(\partial X)}.$$

Additionally, it is straightforward to show that $u - u'$ satisfies the following PDE

$$-\nabla \cdot (\gamma \nabla (u - u')) = \sum_{j=1}^n \partial_j f_j, \quad f_j := \mathbf{e}_j \cdot (\gamma - \gamma')\nabla u' \quad (X), \quad u - u'|_{\partial X} = 0,$$

where the functions f_j belong to $\mathcal{C}^1(\overline{X})$ with an estimate of the form

$$\|f_j\|_{\mathcal{C}^1(\overline{X})} \le C'\|g\|_{\mathcal{C}^2(\partial X)}\|\gamma - \gamma'\|_{\mathcal{C}^1(\overline{X})}.$$

Since $\gamma \in \mathcal{C}^1(\overline{X})$, applying Theorem 3.2.3 to $u - u'$ this time, we obtain

$$\|u - u'\|_{\mathcal{C}^2(X)} \le C \sum_{j=1}^{n} \|f_j\|_{\mathcal{C}^1(\overline{X})} \le CC'n\|g\|_{\mathcal{C}^2(\partial X)}\|\gamma - \gamma'\|_{\mathcal{C}^1(\overline{X})}.$$

Thus estimate (3.20) follows by noticing the identity

$$\mathcal{H}_\gamma[g] - \mathcal{H}_{\gamma'}[g] = \nabla u \cdot (\gamma - \gamma')\nabla u + (\nabla u - \nabla u') \cdot \gamma'(\nabla u + \nabla u'),$$

and bounding each term and their derivatives appropriately. Proposition 3.2.7 is proved. $\square$

3.2.2 The classical Calderón problem

We now review in short the classical Calderón problem. The unique solution u to (3.9) with $g \in H^{\frac{1}{2}}(\partial X)$ has a natural outgoing flux $\gamma \nabla u \cdot \nu$ at the boundary that is well-defined in the space $H^{-\frac{1}{2}}(\partial X)$, so that the so-called *Dirichlet-to-Neumann* (DN) operator $\Lambda_\gamma : H^{\frac{1}{2}}(\partial X) \to H^{-\frac{1}{2}}(\partial X)$, defined by

$$\Lambda_\gamma g = \gamma \nabla u \cdot \nu|_{\partial X}, \quad g \in H^{\frac{1}{2}}(\partial X), \quad u \text{ solves } (4.51), \tag{3.21}$$

is a bounded linear operator.

The classical Calderón problem, first pointed out by Calderón with application in geophysical prospection in Calderón (1980), may be formulated as follows.

Problem 3.2.1. *Does the operator $\Lambda_\gamma \in \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$ characterize the conductivity γ uniquely ? Stably ?*

This problem has received a lot of attention from the inverse problems community since the publication of Calderón (1980) in the 1980's, we briefly review its main injectivity and stability properties in the case of full boundary data. Topical reviews on this problem and the electrical impedance tomography problem may be found in e.g. Borcea (2002); Uhlmann (2007), and the topical review to come Guillarmou and Tzou (2012).

The isotropic case:

In the isotropic case $\gamma \equiv \sigma \mathbb{I}_n$ with σ a scalar function, the injectivity of Problem 3.2.1 has been obtained throughout the years, progressively dropping the regularity required on σ.

The main idea, first initiated in Calderón (1980), started from the following observation that

$$\int_X \sigma |\nabla u|^2 \; dx = \langle \Lambda_\sigma g, g \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}},$$

where the above right-hand side is being measured. The idea was to find enough solutions of (4.51) in $H^1(X)$ such that $|\nabla u|^2$ is dense in an appropriate topology in order to reconstruct σ in X. Calderón showed uniqueness for the linearized conductivity problem near a constant tensor, in particular by using complex geometrical optics solutions of the form $e^{\boldsymbol{\rho}\cdot x}$, where $\boldsymbol{\rho} \in \mathbb{C}^n$ satisfies $\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$.

As regards the fully non-linear problem, the first injectivity result was obtained for $\mathcal{C}^\infty$-smooth conductivities in Sylvester and Uhlmann (1987). There, the authors started from the observation that a solution u of

$$\nabla \cdot (\sigma \nabla u) = 0 \quad (X), \quad u|_{\partial X} = g,$$

gave rise to a solution of a particular Schrödinger equation. Namely, if one sets $v := \sigma^{\frac{1}{2}} u$, then

v satisfies

$$(-\Delta + q)v = 0 \quad (X), \quad v|_{\partial X} = \sigma^{\frac{1}{2}} g, \quad \text{where} \quad q := \sigma^{-\frac{1}{2}} \Delta \sigma^{\frac{1}{2}}. \tag{3.22}$$

Defining in a similar fashion a Dirichlet-to-Neumann operator for the above Schrödinger equation (call it Λ_q), they showed that Λ_σ determined Λ_q, which in turn determined q which determined σ. Focusing on the second (and hardest) step, the following relation is easy to derive

$$\int_X (q_1 - q_2) v_1 v_2 \; dx = \int_{\partial X} g(\Lambda_{q_1} - \Lambda_{q_2})(g) \; ds,$$

where for $i = 1, 2$, v_i solves (3.22) with potential q_i and same boundary condition g. If Λ_{q_1} and Λ_{q_2} coincide, then the latter right-hand side vanishes, thus the main technical point is to show that the product $v_1 v_2$ are dense in L^2 so that the vanishing of the left-hand side implies $q_1 = q_2$. This point was precisely proved by Sylvester and Uhlmann, as they generalized the notion of complex geometrical optics solutions to Schrödinger-type operators with bounded potential, i.e. solutions of the form $u_{\boldsymbol{\rho}} = e^{\boldsymbol{\rho}\cdot x}(1 + \psi_{\boldsymbol{\rho}})$, with $\psi_{\boldsymbol{\rho}} = \mathcal{O}(|\boldsymbol{\rho}|^{-1})$ in some sense, see Appendix D for more detail.

In two dimensions, the regularity required for injectivity was dropped to C^2 in Nachman (1996), and the use of quasiconformal mappings finally led to uniqueness for bounded conductivities $\sigma \in L^\infty(X)$ in Astala and Päivärinta (2006).

In higher dimensions $n \geq 3$, uniqueness with the lowest regularity is obtained for conductivities in the Sobolev space $W^{\frac{3}{2},\infty}(X)$ by Päivärinta, Panchenko and Uhlmann using CGO solutions in Päivärinta et al. (2003). Uniqueness of globally $\mathcal{C}^{1+\epsilon}$-smooth conductivities having only co-normal singularities is established by Greenleaf, Lassas and Uhlmann in Greenleaf et al. (2003), and also more recently in Haberman and Tataru (2011), who worked with Schrödinger potentials as above with "negative regularity" (i.e. in Besov spaces of negative order). Whether one can drop these regularities lower remains an open question.

As regards stability, it was established in Alessandrini (1988) the Lipschitz stability of the boundary value of the conductivity with respect to the DN map, and a logarithmic stability of the interior values, see more references in Astala et al. (2005). As established more recently in Santacesaria (2011), the stability estimate at the domain's interior takes the form

$$\|\sigma_2 - \sigma_1\|_{L^\infty(X)} \le C \log(3 + \|\Lambda_{\sigma_2} - \Lambda_{\sigma_1}\|^{-1})^{-\alpha}, \tag{3.23}$$

where $\|\Lambda_{\sigma_2} - \Lambda_{\sigma_1}\|$ denotes the operator norm in $\mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$, and where it is assumed that $\sigma_j^{-\frac{1}{2}} \Delta \sigma_j^{\frac{1}{2}} \in W^{m,1}(X')$ for some $X' \Subset X$ and $m > 2$. α depends on m and is such that $\lim_{m\to\infty} \sigma(m) = \infty$, so that the estimate improves with the regularity of σ. In any case, this stability is logarithmic, which means that *Calderón's problem is severely ill-posed*, whenever injective at all. This stability can of course be improved if one makes prior assumptions on the form of the conductivity (thus reducing the dimensionality of the unknown function), we do not explore this route here.

The anisotropic case:

In the anisotropic case, there is an important obstruction to uniqueness which is the following: for $\Psi : \overline{X} \to \overline{X}$ a diffeomorphism onto $\overline{X}$ fixing the boundary ∂X, define the push-forward of γ by Ψ (see Appendix B for more detail) as

$$\Psi_\star \gamma(x) := \left[\frac{D\Psi \; \gamma \; D\Psi^T}{|\det D\Psi|} \right] \circ \Psi^{-1}(x), \quad x \in X. \tag{3.24}$$

Then it is straightforward to check that u is a solution of (4.51) if and only if $v := u \circ \Psi^{-1}$ solves the PDE

$$\nabla \cdot ([\Psi_\star \gamma] \nabla v) = 0 \quad (X), \quad v|_{\partial X} = g,$$

and one can actually show that $\Lambda_\gamma = \Lambda_{\Psi^\star \gamma}$. Therefore, in the anisotropic case, the Dirichlet-to-Neumann map can determine at most the equivalence class of a given conductivity tensor with respect to the equivalence relation "$\gamma \sim \gamma'$ if $\gamma = \Psi_\star \gamma'$ for some diffeomorphism Ψ fixing ∂X". The question now is whether this is the only obstruction to unique identifiability of γ.

In two dimensions of space, the answer is yes for tensors in $L^\infty(X, \mathcal{M}_\kappa^s)$, a result showed by Astala, Lassas and Päivärinta in Astala et al. (2005), following work by Sylvester (1990) for $\mathcal{C}^3$ conductivities and by Sun and Uhlmann (2003) for $W^{1,p}$-conductivities. In Astala et al. (2005), the proof of uniqueness modulo push-forwards is obtained by constructing precisely which representative of the class is uniquely determined by Λ_γ. This is done using isothermal coordinates and the theory of quasiconformal mappings.

In dimensions $n \geq 3$, the above result no longer holds. Lee and Uhlmann (1989) pointed out that this problem was of geometrical nature and that it made sense for general compact Riemannian manifold with boundary: in three dimensions and higher, there is a one-to-one correspondence between conductivity equations and Laplace-Beltrami equations. This is not the case for $n = 2$, where conductivity equations are more numerous because Laplace-Beltrami equations are conformally invariant. Lassas and Uhlmann (2001) proved that unique identifiability up to diffeomorphism held for anisotropic metric tensors with piecewise analytic coefficients. Finally, the most recent and thorough result is due to dos Santos Ferreira, Kenig, Salo and Uhlmann (Dos Santos Fereira et al., 2009), where they define admissible manifolds as those admitting a *limiting Carleman weight*, a crucial tool to prove injectivity in Calderón's problem and other similar inverse geometric problems. After characterizing such admissibility conditions, (Dos Santos Fereira et al., 2009, Theorem 5) states unique identifiability of admissible manifolds with same conformal class from their DN map.

3.3 Derivation of power density internal functionals

3.3.1 By acoustic perturbation

Perturbation of a second-order elliptic equation

A methodology to couple high contrast with high resolution consists of perturbing the diffusion coefficient acoustically. Let an acoustic signal propagate through the domain and assume that there exists a coupling between conductivity (or diffusion) and the acoustic signal. Specifically, the conductivity takes the form

$$\gamma_\varepsilon = \gamma_0 + \varepsilon\gamma_1,$$

where γ_0 is the unperturbed conductivity and ε is a small parameter. The expressions of γ_1 and ε will be clarified later. Let u_ε solve

$$\nabla\cdot(\gamma_\varepsilon \nabla u_\varepsilon) = 0 \quad (X), \quad u_\varepsilon|_{\partial X} = g, \tag{3.25}$$

where the Dirichlet condition g is fixed with or without perturbation. By the continuity of the solution u_ε with respect to changes in the coefficients as explained in Section 3.2.1, we find that $u_\varepsilon = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$. Assuming that we measure the outgoing flux of u_ε at the boundary, that flux admits the following expansion:

$$\mathcal{J}^u_\varepsilon := \gamma_\varepsilon \nabla u_\varepsilon \cdot \nu|_{\partial X} = \gamma_0 \nabla u_0 \cdot \nu|_{\partial X} + \varepsilon\left(\gamma_0\nabla u_1 + \gamma_1 \nabla u_0\right)\cdot\nu|_{\partial X} + \mathcal{O}(\varepsilon^2), \tag{3.26}$$

write it $\mathcal{J}_\varepsilon^u = \mathcal{J}_0^u + \varepsilon \mathcal{J}_1^u + \mathcal{O}(\varepsilon^2)$ with the obvious definitions. Multiplying (3.25) by u_ε and integrating by part, we obtain the relation

$$\int_X \nabla u_\varepsilon \cdot \gamma_\varepsilon \nabla u_\varepsilon \; dx = \int_{\partial X} g \mathcal{J}_\varepsilon^v \; ds. \tag{3.27}$$

Expanding left- and right-hand sides of last equation with respect to ε and equating terms like powers of ε, we arrive at the relations

$$\begin{aligned} \mathcal{O}(\varepsilon^0): &\quad \int_X \nabla u_0 \cdot \gamma_0 \nabla u_0 \; dx = \int_{\partial X} g \mathcal{J}_0^u \; ds \\ \mathcal{O}(\varepsilon^1): &\quad \int_X (2\nabla u_1 \cdot \gamma_0 \nabla u_0 + \nabla u_0 \cdot \gamma_1 \nabla u_0) \; dx = \int_{\partial X} g J_1^u \; ds. \end{aligned} \tag{3.28}$$

Doing the same with the PDE (3.25), we have the PDEs

$$\begin{aligned} \mathcal{O}(\varepsilon^0): &\quad \nabla \cdot (\gamma_0 \nabla u_0) = 0 \quad (X), \quad u_0|_{\partial X} = g \\ \mathcal{O}(\varepsilon^1): &\quad \nabla \cdot (\gamma_0 \nabla u_1 + \gamma_1 \nabla u_0) = 0 \quad (X), \quad u_1|_{\partial X} = 0. \end{aligned}$$

Now, multiplying the PDE for u_0 by u_1 and integrating by parts, we obtain the relation

$$\int_X \nabla u_1 \cdot \gamma_0 \nabla u_0 \; dx = \int_{\partial X} u_1 \gamma_0 \nabla u_0 \cdot \nu \; ds = 0,$$

since $u_1|_{\partial X} = 0$. Thus equation (3.28) becomes

$$\int_X \nabla u_0 \cdot \gamma_1 \nabla u_0 \; dx = \int_{\partial X} g \mathcal{J}_1^u \; ds. \tag{3.29}$$

If one considers a second solution $v_\varepsilon = v_0 + \varepsilon v_1$ of (3.25) with boundary condition $v_\varepsilon|_{\partial X} = h$

and outgoing flux $\mathcal{J}^v_\varepsilon = \mathcal{J}^v_0 + \varepsilon \mathcal{J}^v_1 + \mathcal{O}(\varepsilon^2)$, then similar integrations by parts yield

$$\int_X \nabla u_0 \cdot \gamma_1 \nabla v_0 \; dx = \int_{\partial X} g \mathcal{J}^v_1 \; ds = \int_{\partial X} h \mathcal{J}^u_1 \; ds. \tag{3.30}$$

Since g, h are prescribed and $\mathcal{J}^u_1, \mathcal{J}^v_1$ may be approximately obtained by taking the difference between acoustically perturbed and unperturbed boundary flux measurements (and dividing by $\varepsilon > 0$), the right-hand sides of (3.29)-(3.30) can be measured and thus give us knowledge of the left-hand sides. The subsequent derivations will rely heavily on this principle.

Remark 3.3.1. *One could also very well think of prescribing the current $\mathcal{J}^u$ and measure the boundary value $g^u \equiv u|_{\partial X}$, rather than the other way around. In this case, the perturbation arguments would be identical, except that $\mathcal{J}$ would now be fixed and g^u would admit an expansion of the form*

$$g^u_\varepsilon = g^u_1 + \varepsilon g^u_2 + \mathcal{O}(\varepsilon^2).$$

Then, relations similar to (3.29) and (3.30) could be also derived, where the right-hand sides would respectively look like $-\int_{\partial X} g^u_1 \mathcal{J}^u \; ds$ and $-\int_{\partial X} g^v_1 \mathcal{J}^u \; ds$.

Model for acoustic perturbations

The main assumptions about the coupling between acoustic waves and conductivity (or diffusion) are that the perturbation γ_1 is (i) locally proportional to the shape of the acoustic wave passing through the medium, and (ii) locally proportional to the initial conductivity γ_0. Specifically, we write

$$\gamma_1(x) = \zeta(x)\mathfrak{c}(x)\gamma_0(x), \qquad x \in X,$$

where ζ is a correction factor depending on the application and $\mathfrak{c}$ is the shape of the wave. The small parameter ε defined earlier is of the form $\varepsilon = p\Gamma$, product of the acoustic amplitude $p \in \mathbb{R}$ and a measure $\Gamma > 0$ of the coupling between the acoustic signal and the modulations in conductivity, see Ammari et al. (2008); Kuchment and Kunyansky (2010) and references therein.

The limitations of the above assumptions are not discussed further in the present work.

Derivation of power densities

In the literature, two possible experiments are described in order to obtain power density functionals from acoustic perturbations. These experiments mainly differ in the shape of the waves that are used.

Physical focusing: The first approach, described in Ammari et al. (2008), consists in perturbing the medium with a spherical wave that focuses at a given point $x_f \in X$, so that, as a first approximation, the shape $\mathfrak{c}$ is roughly a Dirac mass supported at x_f, i.e. $\mathfrak{c}(x) = \delta(x - x_f)$. Such practical settings were established experimentally in Bercoff et al. (2004); Jossinet et al. (2005). Applying the relation (3.29) to this shape function yields the relation

$$\int_{\partial X} g\mathcal{J}_1^u \, ds = \zeta(x_f)[\nabla u_0 \cdot \gamma_0 \nabla u_0](x_f) \quad \text{and} \quad \int_{\partial X} h\mathcal{J}_1^u \, ds = \zeta(x_f)[\nabla u_0 \cdot \gamma_0 \nabla v_0](x_f).$$

Repeating the experiment by changing the point of focus, and assuming that one knows the coefficient ζ, one eventually has access to the power densities $\mathcal{H}[g] = \gamma_0 \nabla u_0 \cdot \nabla u_0$ and $\mathcal{H}[g, h] = \gamma_0 \nabla u_0 \cdot \nabla v_0$ throughout the domain. In practice of course, this perturbation is NOT a Dirac mass, rather, a blurred spot. The resolution that becomes accessible should then be the size of such spots, which still may be an improvement compared to the resolution currently available in EIT from boundary measurements.

Synthetic focusing: The second approach, based on the premise that physical focusing is actually hard to achieve, is justified and described in Bal and Schotland (2010); Bal et al. (2012a); Kuchment and Kunyansky (2010) and consists in perturbing the domain with a *delocalized* (plane) wave whose shape is given by $\mathfrak{c}(x) = \cos(k \cdot x + \varphi)$ with $k \in \mathbb{R}^d$ and $\varphi \in \{0, \frac{\pi}{2}\}$, so that the relations (3.29)-(3.30) become

$$\int_{\partial X} g \mathcal{J}_1^u \; ds = \int_X \zeta(x) \cos(k \cdot x + \varphi) [\nabla u_0 \cdot \gamma_0 \nabla u_0](x) \; dx,$$
$$\int_{\partial X} g \mathcal{J}_1^v \; ds = \int_X \zeta(x) \cos(k \cdot x + \varphi) [\nabla u_0 \cdot \gamma_0 \nabla v_0](x) \; dx.$$

Repeating the experiment for many wavenumbers k and phases φ generates the Fourier transforms of $\zeta \mathcal{H}[g]$ and $\zeta \mathcal{H}[g, h]$, which upon inverse Fourier transformation and division by known ζ gives again the power densities over the domain X.

Clearly, similar approaches could be derived for other families of wave shapes $\mathfrak{c}$ that give rise to an isometric transformation of $L^2(\mathbb{R}^n)$.

Remark 3.3.2. *Note that if one has measured $\mathcal{J}_1^u$ for enough wave shapes by any of the above protocols, one can generate mutual power densities of the form $\gamma_0 \nabla u_0 \cdot \nabla v_0$ without repeating any experiment, since according to equation (3.30), $\gamma_0 \nabla u_0 \cdot \nabla v_0$ can be computed from knowledge of $\mathcal{J}_1^u$ and the prescribed boundary condition for v, i.e. h.*

The power densities that require new experiments are those of the form $\gamma_0 \nabla u \cdot \nabla u$, where one changes the Dirichlet boundary condition for u each time.

3.3.2 By thermoelastic effects

A second approach that leads to the inverse conductivity problem with power density functionals is described in Gebauer and Scherzer (2009) in the context of *Impedance-Acoustic Computerized*

Tomography (ImpACT). The coupling model is similar in spirit to the derivations that yield to the hybrid methods of thermo- and photo-acoustic tomography. Here, it is assumed that the conductivity equation (4.51) is solved with a time-dependent boundary condition of the form $g(x)\sqrt{f(t)}$, where f varies slowly in time compared to the speed at which the conductivity equation reaches the equilibrium. In this case, the now time-dependent solution $U(x,t)$ of (4.51) may be written as $U(x,t) = u(x)\sqrt{f(t)}$, where u solves (4.51) with $u|_{\partial X} = g$. Throughout the experiment, the electrical power density $\partial_t Q(x,t)$ absorbed by the tissue may be expressed as

$$\partial_t Q(x,t) = \nabla U(x,t) \cdot \gamma(x) \nabla U(x,t) = (\nabla u \cdot \gamma \nabla u)(x) f(t).$$

If the voltage is applied only for a short time so that thermal diffusion effects can be neglected, the absorbed energy gives rise to a change in temperature

$$\partial_t T(x,t) = \frac{1}{\rho(x,t)c(x)} \partial_t Q(x,t),$$

where c is the specific heat capacity and ρ is the mass density, whose variations are neglected. By thermoelastic expansion (characterized by a coefficient $\beta(x)$), pressure waves are generated, they propagate through the domain and are measured at the boundary of a domain $B \subset \mathbb{R}^n$ (with $X \subset B$) with ultrasound transducers. Assuming that the electric energy is applied only for a very short time compared to the speed of sound, one may replace $f(t)$ by a δ-peak, in which case the pressure field solves the wave equation

$$\begin{aligned} &\frac{1}{v_s^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0, \quad (\mathbb{R}^n) \times \mathbb{R}_+, \quad \text{with initial conditions} \\ &\qquad p(x,0) = p_0(x) := \frac{\beta}{\rho c}(\nabla u \cdot \gamma \nabla u)(x)\chi_X(x) \quad \text{and} \quad \partial_t p(x,0) = 0, \quad (\mathbb{R}^n), \end{aligned} \tag{3.31}$$

where χ_X is the characteristic function of the domain X and v_s is the sound speed. Assuming that one collects measurements of the form $\Lambda p_0 := p|_{\partial B\times[0,T]}$ with T large enough, the ImpACT reconstruction method consists in

1. First, an inverse wave problem : that is, the reconstruction of the initial condition in (3.31) from the boundary measurements Λp_0. This problem is now well-studied in the literature, many stable inversion formulas have been provided using the spherical mean transform in the constant sound speed case, the time-reversal method for non-constant speeds, all of which are reviewed in Kuchment (2012) and references therein, and more recently, explicit inversion formulas for non-constant, non-trapping speeds when T is large enough, see Stefanov and Uhlmann (2009, 2011); Qian et al. (2011) and the review to come Stefanov and Uhlmann (2012).

2. Second, an inverse conductivity problem with power density functionals: indeed, assuming that the product $\frac{\beta}{\rho c}$ is known, the reconstructed p_0 provides us with the power density $\nabla u \cdot \gamma \nabla u$.

Thus this derivation indeed presents an alternative setting where one needs to reconstruct a conductivity tensor from power density functionals.

Remark 3.3.3 (On the computation of mutual power densities)**.** *In this derivation, the power densities are directly computed through the inversion of a wave equation. In this case, if u, v are solutions of* (4.51) *with boundary conditions g, h respectively, the mutual power density $\nabla u \cdot \gamma \nabla v$ may be computed using the polarization identity*

$$4\nabla u \cdot \gamma \nabla v = \nabla(u+v)\cdot\gamma\nabla(u+v) - \nabla(u-v)\cdot\gamma\nabla(u-v),$$

where the terms in the right-hand side may be computed with the ImpACT procedure, using *consecutive boundary conditions* $g + h$ *and* $g - h$.

Chapter 4

Local resolution techniques

This chapter is devoted to establishing local inversion procedures for the unknown γ, obtained by manipulating the power densities as well as certain frames that are of particular interest in this problem. After defining certain vector fields of interest in Section 4.1, we then derive local reconstruction algorithms for (i) the determinant of γ when its anisotropic structure is known and (ii) the anisotropic structure $\tilde{\gamma} := (\det \gamma)^{-\frac{1}{n}} \gamma$. This is first done in two dimensions in Section 4.2, where vector calculus identities are the basic tools for resolution. We then generalize the reconstruction formulas to any dimension $n \ge 3$ in Section 4.3 using differential geometric tools such as tensors, forms, exterior derivatives and connections. The validity of the algorithms derived relies on qualitative properties that must be satisfied by certain families of solutions of the conductivity equation. Whether these properties may be satisfied will be investigated in more detail in Section 5.2.

4.1 Redefining the problem

4.1.1 Reformulation in terms of a frame

When the number of solutions of (3.9) considered equals the dimension of space, i.e. $m = n$, we obtain new tools in order to derive inversion formulas. Indeed, the data $H := \{H_{ij}\}_{1\le i,j\le n} = \{\nabla u_i \cdot \gamma \nabla u_j\}_{1\le i,j\le n}$ may be seen as the Gram matrix for different families of vector fields with respect to different metrics defined on X: it may be seen as the Gram matrix of $(\nabla u_1, \cdots, \nabla u_n)$ for the metric $\gamma_{ij}\, dx^i\, dx^j$ or, equivalently, defining $A = \gamma^{\frac{1}{2}}$ to be the positive squareroot of γ at every point (see Appendix A.4 for details), the matrix-valued data H may be seen as the Gram matrix of $(S_1, \cdots, S_n)$ with respect to the Euclidean metric, where we have defined

$$S_i := A\nabla u_i, \quad 1 \le i \le n, \quad A := \gamma^{\frac{1}{2}}. \tag{4.1}$$

When H is uniformly invertible over some $\Omega \subset X$, which we formulate as the following more restrictive statement

$$\inf_{x\in\Omega} \det H(x) \ge c_0^2 > 0, \tag{4.2}$$

the families $(\nabla u_1, \cdots, \nabla u_n)$ and $(S_1, \cdots, S_n)$ both define *frames* over Ω, i.e. bases of $T_x\Omega$ at every point $x \in \Omega$, over which one can decompose any vector field, in particular the unknown ones. While both frames should be equivalent, we will work in the sequel with the frame $(S_1, \cdots, S_n)$ (call it the S frame), so as to work with the Euclidean structure.

Working in the S frame, it is straightforward that $H_{ij} = S_i \cdot S_j$, where $\cdot$ denotes the Euclidean standard inner product. Over a set $\Omega \subset X$ where (4.2) holds, we have the following decomposition formula, true for any vector field over Ω:

$$V = H^{pq}(V \cdot S_p)S_q, \qquad H^{ij} = [H^{-1}]_{ij}, \tag{4.3}$$

where we use the Einstein summation convention. For any invertible symmetric matrix M, applying (4.3) to MV and multiplying by M^{-1} yields also the more general formula

$$V = H^{pq}(V \cdot MS_p)M^{-1}S_q. \tag{4.4}$$

While the frame is also unknown, we will see that this trick will allow us to express certain unknowns in this basis, after which we will obtain a closed system that will reconstruct the unknown frame. Note that if $S := [S_1|\cdots|S_n]$ is the matrix-valued function whose columns are the vector fields $S_1, \cdots, S_n$, we have the identity $S^T S = H$, so that $\det H = (\det S)^2$ everywhere. If H satisfies (4.2), then $\det S$ has constant sign over Ω and we assume that, up to reindexing the solutions, that *the determinant of S is assumed to be positive whenever condition* (4.2) *holds.*

The partial differential equations satisfied by vector fields of the form $S = A\nabla u$ with u a solution of $\nabla \cdot (\gamma \nabla u) = 0$ are

$$\nabla \cdot (AS) = 0 \quad \text{ and } \quad d(A^{-1}S)^\flat = 0. \tag{4.5}$$

The first equation is nothing but the conductivity equation using $A^2 = \gamma$, and the second equation expresses the fact that the vector field $(A^{-1}S)^\flat = du$ is an exact form therefore it is closed. In coordinates, this equation amounts to $n(n-1)/2$ scalar equations, each of which expresses the equality between commuting second partial derivatives.

4.1.2 The orthonormal frame

In the past section, we introduced an unknown frame S as an intermediate step toward reconstruction of the tensor γ. While S has n^2 components, we claim that the number of unknown scalar functions that represent S is $n(n-1)/2$. In practice, it may therefore be cheaper numerically to go only after the smallest number of unknowns by "stripping off" from the problem the data that we already know. This is done as follows.

Since the inner products $S_i \cdot S_j$ are known, the frame S is known up to an orthonormal frame. In order to see this, it suffices to notice that the linear coefficients in the (orientation-preserving) Gram-Schmidt process only depend on the inner products of the initial basis to orthonormalize, see appendix A.1. After orthonormalizing S via coefficients that are known from the data, it remains an orthonormal frame $R := (R_1, \cdots, R_n)$, i.e. an $SO_n(\mathbb{R})$-valued function over X, where $\dim SO_n(\mathbb{R}) = n(n-1)/2$.

Now, the Gram-Schmidt process is not the only way to orthonormalize a basis, for instance one can also show that the matrix $R := SH^{-\frac{1}{2}}$ is an $SO_n(\mathbb{R})$-valued function as well. Any orthonormalization procedure can be deduced from another one via a rotation matrix, for if we have $R_i = ST_i^T$ for $i = 1, 2$ with $R_1, R_2 \in SO_n(\mathbb{R})$ and T_1, T_2 "orthonormalizing" matrices, we must have $T_1^T = T_2^T R_2^T R_1$, where $R_2^T R_1 \in SO_n(\mathbb{R})$. However, for stability purposes, the orthonormalizing matrices that we deem acceptable here must satisfy a certain continuity property, see below.

An orthonormalizing matrix will be denoted $T = \{t_{ij}\}_{1\le i,j\le n}$ such that $R = ST^T$ or, equivalently $S = RT^{-T}$, i.e.

$$R_i = t_{ij} S_j \quad \text{and} \quad S_i = t^{ij} R_j, \quad 1 \le i \le n. \tag{4.6}$$

From this we immediately have the relation

$$H = S^T S = T^{-1} R^T R T^{-T} = (T^T T)^{-1}. \tag{4.7}$$

Using the PDEs (4.5), we are able to derive PDE's for the elements of the frame R: for instance, starting from the divergence equation, we get

$$\nabla \cdot (AR_i) = \nabla \cdot (t_{ij} AS_j) = \nabla t_{ij} \cdot AS_j + t_{ij} \underbrace{\nabla \cdot AS_j}_{=0} = \nabla t_{ij} \cdot t^{jk} AR_k = V_{ik} \cdot AR_k,$$

where we have defined

$$V_{ik} := (\nabla t_{ij}) t^{jk} = -t_{ij} (\nabla t^{jk}), \quad 1 \le i, k \le n. \tag{4.8}$$

Proceeding similarly for the exterior derivative equation and using the identity $d(f\omega) = df \wedge \omega + f d\omega$ for f a function and ω a one-form, we deduce the following system of PDE's for the R frame

$$\nabla \cdot (AR_i) = V_{ik} \cdot AR_k \quad \text{and} \quad d(A^{-1} R_i)^\flat = V_{ij}^\flat \wedge (A^{-1} R_j)^\flat. \tag{4.9}$$

As mentioned before, the matrix T (or more specifically, the vector fields V_{ij}) must satisfy a continuity condition for stability purposes.

Definition 4.1.1 (Stability condition for an orthonormalizing matrix)**.** *The orthonormalizing matrix T is said to satisfy the stability condition if for any two sets of measurements H, H'*

jointly satisfying (4.2) *over some* $\Omega \subset X$*, we have the following stability estimate*

$$\max_{1\le i,j\le n} \|t_{ij} - t'_{ij}\|_{W^{1,\infty}(\Omega)} \le C_T \|H - H'\|_{W^{1,\infty}(\Omega)}. \tag{4.10}$$

Remark 4.1.2. *Over some* $\Omega \subset X$ *where* H, H' *jointly satisfy* (4.2)*, the matrices* T^{-1} *and* T'^{-1} *automatically have their determinants bounded away from zero from* (4.7)*. As a result, if* T, T' *satisfy estimate* (4.10)*, the components of* T^{-1} *and* T'^{-1} *satisfy a similar estimate, and by product and summation, we deduce that the corresponding vector fields* V_{ij} *defined in* (4.8) *satisfy an estimate of the form*

$$\max_{1\le i,j\le n} \|V_{ij} - V'_{ij}\|_{L^{\infty}(\Omega)} \le C_V \|H - H'\|_{W^{1,\infty}(\Omega)}, \tag{4.11}$$

for some constant C_V*.*

Remark 4.1.3 (Which frame to choose ?)**.** *As one will see in the following sections, the* S *frame will be more interesting from a theoretical viewpoint, as the quantities that will appear in compatibility conditions will be more physically understandable, and some proofs done in the* S *frame may not require additional unnecessary stability conditions. In the* R *frame however, calculations are somewhat more direct, since the metric tensor is the identity and the frame is self-dual. Moreover, numerical implementation will greatly benefit from the* R *frame formulation, as one will reconstruct* $n(n-1)/2$ *scalar functions instead of the* n^2 *functions that the* S *frame represents.*

The last point of this section is to show that the Gram-Schmidt process and the “matrix squareroot” actually satisfy the stability condition.

Proposition 4.1.4. *The Gram-Schmidt process satisfies the stability condition.*

The proof has been given in Bal et al. (2012a) in the cases $n = 2, 3$, which we now generalize to dimension n. The proof relies on the so-called *Fischer's inequality* (A.4) from matrix analysis, a proof of which is given for completeness in appendix A.2.

Proof. The result of the Gram-Schmidt process may be expressed in a non-recursive formula using formal determinants, i.e. one expands the following determinant with respect to the last row

$$R_i = \frac{1}{\sqrt{D_{i-1}D_i}} \begin{vmatrix} S_1 \cdot S_1 & \dots & S_1 \cdot S_i \\ \vdots & \ddots & \vdots \\ S_{i-1} \cdot S_1 & \dots & S_{i-1} \cdot S_i \\ S_1 & \dots & S_i \end{vmatrix} = \frac{1}{\sqrt{D_{i-1}D_i}} \begin{vmatrix} H_{11} & \dots & H_{i1} \\ \vdots & \ddots & \vdots \\ H_{i-1,1} & \dots & H_{i-1,i} \\ S_1 & \dots & S_i \end{vmatrix},$$

where D_i satisfies $D_0 = 1$ by convention and for $1 \le i \le n$,

$$D_i = \det \{H_{pq}\}_{p,q=1}^{i}.$$

This means that the coefficient t_{ij} is given by 0 if $j > i$, and by

$$t_{ij} = \frac{1}{\sqrt{D_{i-1}D_i}} (-1)^{i+j} \det \{H_{pq}, 1 \le p \le i-1, 1 \le q \le i, q \ne j\}. \tag{4.12}$$

In particular when $i = j$, $t_{ii} = \frac{\sqrt{D_{i-1}}}{\sqrt{D_i}}$. We now concentrate on showing that the determinants D_i are bounded away from zero, in which case the t_{ij}'s are smooth functions of the H_{ij}'s with no blowups and estimate (4.10) follows automatically. Fix $1 \le i \le n$ and consider the pinching of H by the $n-i+1$ projections

$$P_i = \sum_{j=1}^{i} \mathbf{e}_j \otimes \mathbf{e}_j \quad \text{and} \quad P_k = \mathbf{e}_k \otimes \mathbf{e}_k, \quad i+1 \le k \le n.$$

Then one may check that $\mathcal{C}(H) = \sum_{k=i}^{n} P_k H P_k$ is nothing but

$$\mathcal{C}(H) = \begin{bmatrix} H_{11} & \cdots & H_{1i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ H_{i1} & \cdots & H_{ii} & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & H_{i+1,i+1} & 0 & \vdots \\ \vdots & \cdots & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & H_{nn} \end{bmatrix},$$

so that Fischer's inequality (A.4) reads

$$\det H \le \det \mathcal{C}(H) = D_i \prod_{k=i+1}^{n} H_{kk} \le D_i \|H\|_\infty^{n-i}.$$

With $\det H$ bounded below by $c_0 > 0$ and $\|H\|_\infty < \infty$, we deduce that D_i is bounded below by $c_0 \|H\|_\infty^{-(n-i)} > 0$. This concludes the proof. □

Proposition 4.1.5. *The orthonormalizing matrix $T = H^{-\frac{1}{2}}$ satisfies the stability property.*

Proof. See Appendix A.4. Whenever the function $x \mapsto H^{-1}(x)$ is uniformly elliptic over some domain Ω, the construction (A.14) is a smooth function of the entries of H. □

4.2 Local reconstruction formulas in two dimensions

The two-dimensional case is somewhat simpler, as many tools or useful properties are present:

- The theory of quasi-conformal mappings (see Astala et al. (2009)) allows to build global coordinate systems over a subdomain from two scalar solutions of a σ-harmonic equation (Alessandrini and Nesi, 2001).
- As regards the Calderón problem from boundary measurements, the lack of injectivity is completely characterized and the uniqueness results are existing with minimal regularity on the conductivity tensor (i.e. γ with bounded components) (Astala et al., 2005).
- The group $SO_2(\mathbb{R})$ is abelian. We will see that this will make the resolution techniques more tractable than in higher dimensions.

A natural decomposition for the conductivity tensor in two dimensions is given by

$$\gamma = |\gamma|^{\frac{1}{2}}\tilde{\gamma}, \quad \text{where} \quad \tilde{\gamma} := |\gamma|^{-\frac{1}{2}}\gamma \quad \text{such that} \quad \det\tilde{\gamma} = 1. \tag{4.13}$$

The unimodular tensor $\tilde{\gamma}$ is also referred to as a *conformal structure* (see Astala et al. (2009)). It is straightforward to see that, when $\gamma \in \Sigma(X)$, then $|\gamma|^{\frac{1}{2}}$ satisfies the uniform estimates $C_\gamma^{-1} \le |\gamma|^{\frac{1}{2}} \le C_\gamma$ over X, and $\tilde{\gamma} \in \Sigma(X)$ with $C_{\tilde{\gamma}} = C_\gamma^2$.

In the context of power density measurements, we will see in the sequel that power densities allows to recover all of the anisotropic tensor, and that the function $|\gamma|^{\frac{1}{2}}$ will be reconstructed with better stability than the tensor $\tilde{\gamma}$.

4.2.1 Preliminary properties and vector calculus identities

For any 2×2 symmetric matrix M, we have the following property that

$$MJM = (\det M)J, \quad \text{where} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.14}$$

The following identity may be checked component by component for any two vector fields U, V

$$[U, V] = (U \cdot \nabla)V - (V \cdot \nabla)U = \nabla \cdot (V \otimes U - U \otimes V) - (\nabla \cdot U)V + (\nabla \cdot V)U, \tag{4.15}$$

where $\nabla \cdot M := \partial_i M_{ji} \mathbf{e}_j$ for M a 2×2 matrix.

4.2.2 Local reconstruction formulas when $\tilde{\gamma}$ is known

We assume that $\nabla u_1, \nabla u_2$ are everywhere linearly independent on a simply connected subset $\Omega \subset X$, that is, the following condition holds:

$$\inf_{x \in \Omega} \det(\nabla u_1, \nabla u_2) \geq c_0 > 0. \tag{4.16}$$

We use the decomposition (4.13) and define $A = \gamma^{\frac{1}{2}}$ as well as $\widetilde{A} = |A|^{-\frac{1}{2}} A = \tilde{\gamma}^{\frac{1}{2}}$. Clearly, $A \in \Sigma(X)$ with $C_A = \sqrt{C_\gamma}$ and $\widetilde{A} \in \Sigma(X)$ with $C_{\widetilde{A}} = \sqrt{C_{\tilde{\gamma}}} = C_\gamma$. For $i = 1, 2$, we now denote $S_i := A \nabla u_i$, as well as the matrix $S := [S_1 | S_2]$. The data H_{ij} now becomes

$$H_{ij} = \gamma \nabla u_i \cdot \nabla u_j = A \nabla u_i \cdot A \nabla u_j = S_i \cdot S_j, \quad 1 \leq i, j \leq 2,$$

so $H = S^T S$ is the Gram matrix of the basis (S_1, S_2), in particular, $\det(S_1, S_2) = |H|^{\frac{1}{2}}$. Since A is uniformly invertible, the positivity condition (4.16) implies that

$$\inf_{x\in\Omega} \det H \geq C_\gamma^{-2} c_0^2 > 0.$$

Now, as explained in Section 4.1.1, any vector field V defined on Ω may be represented in the frame $A^r S$ for any $r \in \mathbb{R}$, via the following representation formula (apply (4.4) with $M = A^r$)

$$V = H^{ij}(V \cdot A^r S_i) A^{-r} S_j, \quad r \in \mathbb{R}. \tag{4.17}$$

The vector fields S_1, S_2 satisfy the following PDE's

$$\nabla \cdot (AS_i) = 0, \quad \text{and} \quad \nabla^\perp \cdot A^{-1} S_i = 0, \quad i = 1, 2, \quad \nabla^\perp := J\nabla = [-\partial_y, \partial_x]^T, \tag{4.18}$$

where the first PDE comes from rewriting the conductivity equation and the second expresses the fact that $A^{-1} S_i = \nabla u_i$ must be curl-free. Decomposition A into a conformal factor $|A|^{\frac{1}{2}}$ and a conformal structure $\widetilde{A}$, we can use vector calculus identities to rewrite equations (4.18) into

$$\nabla \cdot (\widetilde{A} S_i) + \nabla a \cdot \widetilde{A} S_i = 0, \quad \text{and} \quad \nabla^\perp \cdot \widetilde{A}^{-1} S_i - \nabla^\perp a \cdot \widetilde{A}^{-1} S_i = 0, \quad i = 1, 2, \tag{4.19}$$

$$\text{where} \quad a := \log |A|^{\frac{1}{2}}. \tag{4.20}$$

Gradient equation for $|\gamma|^{\frac{1}{2}}$

From the above PDE's, we are now able to derive the following relation

Lemma 4.2.1. *Let a simply connected sub domain $\Omega \subset X$ be such that condition (4.16) holds.*

Then the following relation holds throughout Ω:

$$\boxed{\nabla \log |A| = |H|^{-\frac{1}{2}} \left(\nabla \left(|H|^{\frac{1}{2}} H^{jl} \right) \cdot \widetilde{A} S_l \right) \widetilde{A}^{-1} S_j = \frac{1}{2} \nabla \log |H| + (\nabla H^{jl} \cdot \widetilde{A} S_l) \widetilde{A}^{-1} S_j.} \tag{4.21}$$

Proof. We proceed by analyzing the properties of the vector fields $JA^{-1}S_i$ for $i = 1, 2$. First notice that since J is skew-symmetric, we have

$$JA^{-1}S_i \cdot A^{-1}S_i = 0, \quad i = 1, 2,$$

Then, using the relation (4.14) with $M = A^{-1}$ and the fact that $JS_1 \cdot S_2 = \det(S_1, S_2) =: |H|^{\frac{1}{2}}$,

$$JA^{-1}S_1 \cdot A^{-1}S_2 = -JA^{-1}S_2 \cdot A^{-1}S_1 = (A^{-1}JA^{-1}S_1) \cdot S_2 = |A|^{-1} JS_1 \cdot S_2 = |A|^{-1} |H|^{\frac{1}{2}}.$$

This means that the vector fields $JA^{-1}S_i$ can be expressed using the representation (4.17) with $r = -1$:

$$\begin{aligned} JA^{-1}S_1 &= H^{pq} (JA^{-1}S_1 \cdot A^{-1}S_p) AS_q = H^{2q} |A|^{-1} |H|^{\frac{1}{2}} AS_q, \\ JA^{-1}S_2 &= H^{pq} (JA^{-1}S_2 \cdot A^{-1}S_p) AS_q = -H^{1q} |A|^{-1} |H|^{\frac{1}{2}} AS_q. \end{aligned} \tag{4.22}$$

We now apply the divergence operator to (4.22). Together with the fact that $\nabla \cdot (JA^{-1}S_i) = -\nabla^{\perp} \cdot A^{-1}S_i = 0$ and equation (4.18), and using the identity $\nabla (fV) = \nabla f \cdot V + f \nabla \cdot V$, we derive

$$\nabla |A|^{-1} \cdot |H|^{\frac{1}{2}} H^{qp} AS_p + |A|^{-1} \nabla (|H|^{\frac{1}{2}} H^{qp}) \cdot AS_p = 0, \quad q = 1, 2.$$

Multiplying the last equation by $A^{-1}S_q$, summing over q and dividing by $|A|^{-1}|H|^{\frac{1}{2}}$, we obtain

$$H^{qp}(-\nabla \log |A| \cdot AS_p)A^{-1}S_q + |H|^{-\frac{1}{2}}(\nabla(|H|^{\frac{1}{2}}H^{qp}) \cdot AS_p)A^{-1}S_q = 0.$$

The first term is of the form (4.17) with $r = 1$ and $V = -\nabla \log |A|$ and thus equals $-\nabla \log |A|$. We obtain the first right-hand side of (4.21). The second right-hand side of (4.21) is obtained from the first one by expanding the term $\nabla(|H|^{\frac{1}{2}}H^{pq})$ and using the product rule and identity (4.17) to obtain the $\frac{1}{2}\nabla \log |H|$ term. Finally, the fact that both A and A^{-1} are present in (4.21) cancels out the determinants, and thus A may be replaced by its scaled version $\widetilde{A}$. □

The result of lemma 4.2.1 allows to completely remove $\nabla \log |A| = \nabla \log |\gamma|^{\frac{1}{2}}$ from the problem until everything else has been reconstructed (or is already known). This also shows that the reconstruction of $|\gamma|$ requires the knowledge of S_1, S_2 and $\widetilde{A}$ (or, equivalently $\tilde{\gamma}$) of course, as may be seen from the right-hand sides of (4.21).

On to the reconstruction of (S_1, S_2), we orthonormalize (S_1, S_2) into an orthonormal basis R_1, R_2 via a transfer matrix T such that $R_i = t_{ij}S_j$ (or, in matrix notation, $R = ST^T$ with $T = \{t_{ij}\}_{1\le i,j\le 2}$). For further use, we denote $T^{-1} = \{t^{ij}\}$ and define the vector fields V_{ij} as

$$V_{ij} := \nabla(t_{ik})t^{kj}, \quad V_{ij}^a := \frac{1}{2}(V_{ij} - V_{ji}), \quad V_{ij}^s := \frac{1}{2}(V_{ij} + V_{ji}), \quad 1 \le i, j \le 2. \tag{4.23}$$

The transfer matrix T satisfies $T^TT = H^{-1}$, also written as $t_{ki}t_{kj} = H^{ij}$ for $1 \le i, j \le 2$. It can be constructed by the Gram-Schmidt procedure or by writing $T = H^{-\frac{1}{2}}$. With the V_{ij}'s defined in (4.23), the following important identity holds

$$(\nabla H^{ij})t^{ik}t^{jl} = (\nabla(t_{pi}t_{pj}))t^{ik}t^{jl} = \delta_{pk}(\nabla t_{pj})t^{jl} + \delta_{pl}(\nabla t_{pi})t^{ik} = V_{kl} + V_{lk}, \quad 1 \le l, k \le 2. \tag{4.24}$$

Therefore, starting from (4.21), we have

$$\begin{aligned}\nabla \log|A| &= N + (\nabla H^{jl} \cdot \widetilde{A} S_l)\widetilde{A}^{-1} S_j = N + ((\nabla H^{jl}) t^{lp} t^{jq} \cdot \widetilde{A} R_p)\widetilde{A}^{-1} R_q \\ &= N + ((V_{pq} + V_{qp}) \cdot \widetilde{A} R_p)\widetilde{A}^{-1} R_q, \quad \text{where} \quad N = \frac{1}{2}\nabla \log|H|, \end{aligned} \tag{4.25}$$

and where we have used (4.24) in the last equality.

Since the matrix $R = [R_1|R_2]$ is $SO(2)$-valued, there exists a function $\theta : \Omega \to \mathbb{S}^1$ such that $R_1 = (\cos\theta, \sin\theta)^T$ and $R_2 = (-\sin\theta, \cos\theta)^T$ throughout Ω. For simplification purposes, we will now express (4.25) in terms of θ in the simplest way possible. To this end, we define $\Phi_{ij}(\theta) = R_i \otimes R_j$ for $1 \le i, j \le 2$ and compute

$$\begin{aligned}\nabla \log|A| &= \frac{1}{2}\nabla \log|H| + 2\sum_{p,q=1}^{2} \widetilde{A}^{-1}\Phi_{pq}\widetilde{A} V_{pq}^s \\ &= -V_{11} - V_{22} + 2\widetilde{A}^{-1}\Phi_{11}\widetilde{A} V_{11} + 2\widetilde{A}^{-1}\Phi_{22}\widetilde{A} V_{22} + \widetilde{A}^{-1}(\Phi_{12} + \Phi_{21})\widetilde{A}(V_{12} + V_{21}) \\ &= \widetilde{A}^{-1}(\Phi_{11} - \Phi_{22})\widetilde{A}(V_{11} - V_{22}) + \widetilde{A}^{-1}(\Phi_{12} + \Phi_{21})\widetilde{A}(V_{12} + V_{21}),\end{aligned}$$

where we have used the facts that $\widetilde{A}^{-1}(\Phi_{11} + \Phi_{22})\widetilde{A} = \mathbb{I}_2$ and $-V_{11} - V_{22} = \nabla \log|H|^{\frac{1}{2}}$. The matrices $\Phi_{11} - \Phi_{22}$ and $\Phi_{12} + \Phi_{21}$ are symmetric matrices that can be expressed in the following manner

$$\begin{aligned} &\Phi_{11} - \Phi_{22} = c\mathbb{U} + sJ\mathbb{U} \quad \text{and} \quad \Phi_{12} + \Phi_{21} = -s\mathbb{U} + cJ\mathbb{U}, \quad \text{where} \\ &(c, s) := (\cos(2\theta), \sin(2\theta)), \quad \mathbb{U} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{4.26}$$

From this we deduce the final expression of $\nabla \log |A|$:

$$\boxed{\begin{gathered}\nabla \log |A| = \widetilde{A}^{-1} \left(\cos(2\theta) F_c + \sin(2\theta) J F_c \right), \qquad \text{where} \\ F_c := \mathbb{U}\widetilde{A}(V_{11} - V_{22}) + J\mathbb{U}\widetilde{A}(V_{12} + V_{21}).\end{gathered}} \tag{4.27}$$

Gradient equation for θ

The following lemma gives a second relation which allows to reconstruct the function θ whenever $\tilde{\gamma}$ is known:

Lemma 4.2.2. *Let a simply connected sub domain $\Omega \subset X$ be such that condition (4.16) holds. Upon orthonormalizing the basis S_1, S_2 and defining the unknown function θ as above, the following relation holds throughout Ω:*

$$\boxed{\widetilde{A}^2 \nabla \theta + [\widetilde{A}_2, \widetilde{A}_1] = \widetilde{A}^2 V_{12}^a - \frac{1}{2} J N,} \tag{4.28}$$

where $N := \frac{1}{2}\nabla \log |H|$ and $[\widetilde{A}_2, \widetilde{A}_1] := (\widetilde{A}_2 \cdot \nabla)\widetilde{A}_1 - (\widetilde{A}_1 \cdot \nabla)\widetilde{A}_2$, where $\widetilde{A}_j$ denotes the j-th column of $\widetilde{A}$.

Proof. Equation (4.28) is derived by writing the Lie bracket $[\widetilde{A}R_2, \widetilde{A}R_1]$ in two different manners. On the one hand, writing $[\widetilde{A}R_2, \widetilde{A}R_1]$ in the canonical basis $(\mathbf{e}_1, \mathbf{e}_2)$ and using the following identity (true for smooth functions a, b and vector fields X, Y)

$$[aX, bY] = a(X \cdot \nabla)(b)Y - b(Y \cdot \nabla)(a)X + ab[X, Y],$$

we have that

$$\begin{aligned}[\widetilde{A}R_2, \widetilde{A}R_1] &= [\widetilde{A}_{ij}R_2^j\mathbf{e}_i, \widetilde{A}_{kl}R_1^l\mathbf{e}_k] \\ &= \left(\widetilde{A}_{ij}\widetilde{A}_{kl}(R_2^j\partial_i R_1^l - R_1^j\partial_i R_2^l) + \widetilde{A}_{ij}\partial_i\widetilde{A}_{kl}(R_2^jR_1^l - R_1^jR_2^l)\right)\mathbf{e}_k,\end{aligned}$$

after renumbering indices properly. Moreover, in the parameterization $R(\theta)$ we have

$$R_2^j\partial_i R_1^l - R_1^j\partial_i R_2^l = \begin{cases} \partial_i\theta & \text{if } j = l \\ 0 & \text{if } j \neq l \end{cases} \quad \text{and} \quad R_2^jR_1^l - R_1^jR_2^l = \begin{cases} 0 & \text{if } j = l \\ 1 & \text{if } (j,l) = (2,1) \\ -1 & \text{if } (j,l) = (1,2) \end{cases},$$

thus we obtain

$$[\widetilde{A}R_2, \widetilde{A}R_1] = \left((\widetilde{A}_{i1}\widetilde{A}_{k1} + \widetilde{A}_{i2}\widetilde{A}_{k2})\partial_i\theta + \widetilde{A}_{i2}\partial_i\widetilde{A}_{k1} - \widetilde{A}_{i1}\partial_i\widetilde{A}_{k2}\right)\mathbf{e}_k = \widetilde{A}^2\nabla\theta + [\widetilde{A}_2, \widetilde{A}_1]. \quad (4.29)$$

On the other hand, we compute $[\widetilde{A}R_2, \widetilde{A}R_1]$ using (4.18). First, deriving a divergence equation for $\widetilde{A}R_i, i = 1, 2$, and using the rewritten version of the divergence equation (4.19), we have

$$\begin{aligned}\nabla\cdot(\widetilde{A}R_i) &= \nabla\cdot(\widetilde{A}t_{ij}S_j) = \nabla t_{ij}\cdot\widetilde{A}S_j + t_{ij}\nabla\cdot(\widetilde{A}S_j) \\ &= \nabla t_{ij}\cdot\widetilde{A}t^{jk}R_k - t_{ij}\frac{1}{2}\nabla\log|A|\cdot\widetilde{A}S_j = V_{ik}\cdot\widetilde{A}R_k - \frac{1}{2}\nabla\log|A|\cdot\widetilde{A}R_i \\ &= -\frac{1}{2}N\cdot\widetilde{A}R_i + V_{ik}^a\cdot\widetilde{A}R_k, \end{aligned} \quad (4.30)$$

where we have used (4.25) in the last equality. We now use the vector calculus identity (4.15) with $U = \widetilde{A}R_2$ and $V = \widetilde{A}R_1$, and get

$$\widetilde{A}R_1 \otimes \widetilde{A}R_2 - \widetilde{A}R_2 \otimes \widetilde{A}R_1 = \widetilde{A}(R_1 \otimes R_2 - R_2 \otimes R_1)\widetilde{A} = -\widetilde{A}J\widetilde{A} = -(\det \widetilde{A})J = -J,$$

so the first term in the right-hand side of (4.15) is zero. Thus we have

$$\begin{aligned}
[\widetilde{A}R_2, \widetilde{A}R_1] &= (\nabla \cdot \widetilde{A}R_1)\widetilde{A}R_2 - (\nabla \cdot \widetilde{A}R_2)\widetilde{A}R_1 \\
&= -\frac{1}{2}(N \cdot \widetilde{A}R_1)\widetilde{A}R_2 + \frac{1}{2}(N \cdot \widetilde{A}R_2)\widetilde{A}R_1 + (V_{12}^a \cdot \widetilde{A}R_2)\widetilde{A}R_2 + (V_{12}^a \cdot \widetilde{A}R_1)\widetilde{A}R_1 \\
&= \widetilde{A}(R_1 \otimes R_1 + R_2 \otimes R_2)\widetilde{A}V_{12}^a - \frac{1}{2}\widetilde{A}(R_2 \otimes R_1 - R_1 \otimes R_2)\widetilde{A}N \\
&= \widetilde{A}^2 V_{12}^a - \frac{1}{2}JN,
\end{aligned}$$

where we have used the properties $R_1 \otimes R_1 + R_2 \otimes R_2 = \mathbb{I}_2$ and $R_2 \otimes R_1 - R_1 \otimes R_2 = J$. Combining (4.29) with the last right-hand side yields (4.28). $\square$

Reconstruction strategies and stability

In the case where the anisotropy $\tilde{\gamma}$ is known, equations (4.27) and (4.28) allow us to reconstruct successively θ and $|A| = |\gamma|^{\frac{1}{2}}$. One may first recast equation (4.28) as a gradient equation for θ, under the form

$$\nabla\theta = V_{12}^a - \widetilde{A}^{-2}\left(\frac{1}{2}JN + [\widetilde{A}_2, \widetilde{A}_1]\right), \tag{4.31}$$

where the right-hand side is completely known. Once θ is reconstructed up to a constant over Ω, $\log|A| = \log|\gamma|^{\frac{1}{2}}$ may be reconstructed by solving the gradient equation (4.27) over Ω. The gradient equations may be solved in two possible manners:

The ODE-based method: Pick a point $x_0 \in \Omega$, and for any $x \in \Omega$, pick a $\mathcal{C}^1$ curve joining

x_0 to x, i.e.

$$c_{x_0,x} : [0,1] \ni t \mapsto c_{x_0,x}(t) \in \Omega, \quad c_{x_0,x}(0) = x_0, \quad c_{x_0,x}(1) = x.$$

Integrating the equation (4.31) over the curve $c_{x_0,x}$, we obtain the relation

$$\begin{aligned} \theta(x) &= \theta(x_0) + \int_0^1 \dot{c}_{x_0,x}(t) \cdot \nabla\theta(c_{x_0,x}(t))\ dt \\ &= \theta(x_0) + \int_0^1 \dot{c}_{x_0,x}(t) \cdot \left(V_{12}^a - \widetilde{A}^{-2} \left(\frac{1}{2} JN + [\widetilde{A}_2, \widetilde{A}_1] \right) \right) (c_{x_0,x}(t))\ dt, \end{aligned} \tag{4.32}$$

where $\dot{c}$ denotes $\frac{dc}{dt}$. Since this is true for every $x \in \Omega$ we can reconstruct θ up to the constant $\theta(x_0)$ over Ω. One may call (4.32) the reconstruction by ODE integration. In the case where the data functionals are not corrupted by noise, equation (4.31) must hold true exactly, and thus its right-hand side must be curl-free. Together with the fact that Ω is simply connected, this implies that the reconstruction strategy (4.32) does not depend on the choice of the curve joining x_0 to x. Issues are encountered however when the data functionals become noisy, in which case we can no longer guarantee that the ODE-based reconstruction strategy does not depend on the choice of curve.

The Laplace-based method: A strategy to tackle the problem of dependence on the integration curve would be to project the right-hand side of (4.31) onto curl-free fields. Assuming that we are solving $\nabla\theta = V$ and we are looking for a decomposition of the form

$$V = \nabla\phi + J\nabla\psi, \quad x \in \Omega. \tag{4.33}$$

Then it is straightforward to see that ϕ solves the equation $\Delta\phi = \nabla \cdot V$ over Ω with Neumann boundary condition $\partial_n \phi = V \cdot n$ over $\partial\Omega$. Such an equation admits a unique solution up to a constant, in particular, $\nabla\phi$ is unique and the vector field $V - \nabla\phi$ is divergence-free therefore there is a function ψ (determined up to a constant) such that (4.33) holds. Now, projecting (4.33) onto curl-free fields amounts to setting the $J\nabla\psi$ term to zero. In this case, we are solving $\nabla\theta = \nabla\phi$, thus $\theta = \theta(x_0) + \phi - \phi(x_0)$. We thus see that projecting onto curl-free fields amounts to solving the following equation for θ

$$\begin{aligned} \Delta\theta &= \nabla\cdot V_{12}^a - \nabla\cdot\left(\frac{1}{2}\widetilde{A}^{-2}JN + \widetilde{A}^{-2}[\widetilde{A}_2, \widetilde{A}_1]\right) \qquad (\Omega), \\ \partial_\nu\theta &= \left[V_{12}^a - \widetilde{A}^{-2}\left(\frac{1}{2}JN + [\widetilde{A}_2, \widetilde{A}_1]\right)\right]\cdot\nu \qquad (\partial\Omega). \end{aligned} \tag{4.34}$$

Once this is done, there is no longer interest in solving ODE's since θ has actually been computed during the process of projecting V along curl-free fields. Heuristically, one could consider the Laplace equation to be the normal equation associated to a gradient equation. Thus, solving the Laplace equation allows to project the right-hand side onto the range of the gradient operator (i.e. the curl-free fields).

Over any set $\Omega \subset X$ where the positivity condition (4.16) holds, the reconstruction strategies for θ and $\log|\gamma|^{\frac{1}{2}}$ based on the gradient equations (4.31) and (4.27) display Lipschitz stability in the space $W^{1,\infty}(\Omega)$, as is summarized in the following theorem.

Theorem 4.2.3 (Stability in two dimensions). *Let $\Omega \subset X$ bounded (i.e. $\Delta(\Omega) < \infty$) and simply connected such that the positivity condition (4.16) holds for two sets of measurements H and H'. Assuming that the orthonormalization procedure is stable in the sense of (4.11), then for any given $x_0 \in \Omega$ we have the following stability estimates*

$$\|\theta - \theta'\|_{W^{1,\infty}(\Omega)} \le e_0 + C_\theta \|H - H'\|_{W^{1,\infty}(\Omega)}, \tag{4.35}$$

$$\|\log|\gamma|^{\frac{1}{2}} - \log|\gamma'|^{\frac{1}{2}}\|_{W^{1,\infty}(\Omega)} \le \varepsilon_0 + C_\gamma \|H - H'\|_{W^{1,\infty}(\Omega)}, \tag{4.36}$$

where $e_0 := |\theta(x_0) - \theta'(x_0)|$ *and* ε_0 *is a function of* e_0 *and* $|\log|\gamma|^{\frac{1}{2}}(x_0) - \log|\gamma'|^{\frac{1}{2}}(x_0)|$.

Proof. On to the stability of θ, we take the difference of (4.31) for both sets of data to make appear:

$$\nabla(\theta - \theta') = V_{12}^{a} - V_{12}^{a'} - \frac{1}{2}\widetilde{A}^{-2}J(N - N').$$

Using the stability (4.11), it is clear that we get

$$\|\nabla(\theta - \theta')\|_{L^\infty(\Omega)} \leq C_V \left(1 + \frac{C_\gamma}{2}\right) \|H - H'\|_{W^{1,\infty}(\Omega)}.$$

Equation (4.35) is thus deduced from the previous equation and by applying the following estimate to $f \equiv \theta - \theta'$

$$|f(x)| \leq |f(x_0)| + \Delta(\Omega)\|\nabla f\|_{L^\infty(\Omega)}, \quad x \in \Omega, \quad f \in W^{1,\infty}(\Omega). \tag{4.37}$$

On to the stability of $\log|\gamma|^{\frac{1}{2}}$, we take the difference of equation (4.27) for both sets of data, and obtain

$$\nabla(\log|\gamma|^{\frac{1}{2}} - \log|\gamma'|^{\frac{1}{2}}) = \widetilde{A}^{-1}\big((\cos(2\theta) - \cos(2\theta'))F_c + (\sin(2\theta) - \sin(2\theta'))JF_c +$$
$$+ \cos(2\theta')(F_c - F_c') + \sin(2\theta')J(F_c - F_c')\big).$$

From the expression (4.27) of F_c and the stability condition (4.11), we have

$$\|F_c - F_c'\|_{L^\infty(\Omega)} \leq C\|H - H'\|_{W^{1,\infty}(\Omega)},$$

for some constant C. Using the estimates $|\cos(s) - \cos(t)| \leq |s - t|$ and $|\sin(s) - \sin(t)| \leq |s - t|$, and using (4.35), we deduce

$$\|\nabla(\log|\gamma|^{\frac{1}{2}} - \log|\gamma'|^{\frac{1}{2}})\|_{L^\infty(\Omega)} \le Ce_0 + C'\|H - H'\|_{W^{1,\infty}(\Omega)},$$

and thus (4.36) holds after applying estimate (4.37) to $f \equiv \log|\gamma|^{\frac{1}{2}} - \log|\gamma'|^{\frac{1}{2}}$. □

Remark 4.2.4. *The stability condition (4.11) is not necessary in order to reconstruct* $\log|\gamma|^{\frac{1}{2}}$*, as one can derive a dynamical system for the basis* (S_1, S_2) *that is stable with respect to the functionals* H_{ij} *without having to orthonormalize. In practice however, it is more convenient to orthonormalize as the number of unknown functions (and hence the computations) reduces significantly.*

4.2.3 Reconstruction of the anisotropy with additional functionals

Redundancies in data

In some inverse problems associated to hybrid methods, one is sometimes able to show that, aside from noise reduction purposes, adding more data functionals does not always add more information about the unknowns, see for instance Bal and Ren (2011). It is thus worthwhile investigating those redundancies in the present problem.

If the anisotropy $\widetilde{A}$ is known, then the functional H_{11}, H_{12} should be enough to determine the unknown $|\gamma|^{\frac{1}{2}}$ as pointed out in Kuchment and Steinhauer (2011). A geometric justification of this fact is that the data $\{H_{ij}\}_{1\le i,j\le 2}$ represents the (Euclidean) metric tensor expressed in the frame (S_1, S_2) and as such must satisfy a *flatness* criterion, that is, the sectional curvature in this frame must be zero. Following appendix C, this condition reads, up to a nonzero scalar constant, $\mathcal{R}(S_1, S_2)S_1 \cdot S_2 = 0$, or, using the definition of the curvature tensor,

$$\left(\overline{\nabla}_{S_1}\overline{\nabla}_{S_2}S_1 - \overline{\nabla}_{S_2}\overline{\nabla}_{S_1}S_1 - \overline{\nabla}_{[S_1,S_2]}S_1\right) \cdot S_2 = 0.$$

Isotropic case: In the isotropic case $\widetilde{A} = \mathbb{I}_2$, it has been shown in Monard and Bal (2012c) that this condition is exactly equivalent to $\nabla^\perp \cdot \nabla\theta = 0$, which reads, using (4.28)

$$\nabla^\perp \cdot V_{12}^a + \frac{1}{2}\Delta \log d = 0, \quad d = (H_{11}H_{22} - H_{12}^2)^{\frac{1}{2}}. \tag{4.38}$$

In the Gram-Schmidt case, where $V_{12}^a = \frac{H_{11}}{2d}\nabla\frac{H_{12}}{H_{11}}$, this condition becomes

$$J\nabla\frac{H_{11}}{d} \cdot \nabla\frac{H_{12}}{H_{11}} + \Delta \log d = 0.$$

Anisotropic case: In the anisotropic case, assume now that we have 3 solutions, that is, 6 dimensions of data for their Gram matrix. Then we are able to find two redundancies in these pieces of data, so that the number of effective dimensions is 4. The way we obtain the redundancies is as follows:

- the first relation comes from the fact that S_1, S_2, S_3 are necessarily linearly dependent in 2D, and thus their Gram matrix must be singular, i.e. $\det\{H_{ij}\}_{1\le i,j\le 3} = 0$, or, in its expanded version:

$$H_{11}H_{22}H_{33} + 2H_{12}H_{23}H_{31} - H_{12}^2H_{33} - H_{31}^2H_{22} - H_{23}^2H_{11} = 0 \tag{4.39}$$

- second, expressing S_2 and S_3 in the basis (S_1, JS_1), we have

$$H_{11}S_2 = H_{12}S_1 + d_1 JS_1, \quad d_1 = (H_{11}H_{22} - H_{12}^2)^{\frac{1}{2}},$$
$$H_{11}S_3 = H_{13}S_1 + d_2 JS_1, \quad d_2 = (H_{11}H_{33} - H_{13}^2)^{\frac{1}{2}}.$$

Dotting these two equations together, we obtain the relation

$$H_{11}^2 H_{23} = H_{12}H_{13}H_{11} + d_1 d_2 H_{11}, \quad \text{that is,} \quad d_1 d_2 = H_{11}H_{23} - H_{12}H_{13}. \tag{4.40}$$

Reconstruction formula

In the next two sections, we will derive in two different manners the fact that, if enough data functionals are available, one is able to derive an algebraic equation for $\widetilde{A}$ (pointwise in $x \in X$) of the form

$$J\widetilde{A}X = -\widetilde{A}Y, \quad \text{or, equivalently} \quad \widetilde{A}^2 X = JY, \tag{4.41}$$

where the data vector fields X and Y are known from the data functionals. Once such an equation is derived, and since $\widetilde{A}$ is really described by two scalar functions, the system (4.41) is enough to determine $\widetilde{A}$ completely, at those points wherever X and/or Y do not vanish. While this compatibility condition will be examined in Section 5.2, we now present how to reconstruct $\widetilde{A}$ from equation (4.41).

Multiplying (4.41) by $\widetilde{A}J$ and using the property (4.14), we deduce $\widetilde{A}^2 X = JY$. In order to do so, we also multiply (4.41) by $\widetilde{A}$ to make appear $\widetilde{A}^2 Y = -JX$. To conclude, we obtain

$\widetilde{A}^2[X|Y] = [JY| - JX]$, and post-multiplying by $[X|Y]^{-1} = (JX \cdot Y)^{-1} \left[\frac{-(JY)^T}{(JX)^T} \right]$, we arrive at

$$\boxed{\widetilde{A}^2 = (JX \cdot Y)^{-1} J \left(X \otimes X + Y \otimes Y \right) J.} \tag{4.42}$$

Note that the denominator $JX \cdot Y = -\widetilde{A}^2 Y \cdot Y = -X \cdot JY = -X \cdot \widetilde{A}^2 X$ vanishes precisely wherever $X = Y = 0$. Furthermore, pre- and post-multiplying both sides of (4.42) by J and using $\widetilde{A} J = J \widetilde{A}^{-1}$ and $\widetilde{A}^{-1} J = J \widetilde{A}$, we obtain $\widetilde{A}^{-2}$ for free

$$\widetilde{A}^{-2} = -(JX \cdot Y)^{-1} \left(X \otimes X + Y \otimes Y \right). \tag{4.43}$$

The R frame approach

Using equation (4.28), it is fairly straightforward to deduce an algebraic reconstruction equation for $\widetilde{A}$ of the form $\widetilde{A}^2 X = Y$ for two vector fields X and Y known from the data, provided that one has 3 or more solutions. Here, we assume that we have four illuminations $g_1^{(1)}, g_2^{(1)}, g_1^{(2)}, g_2^{(2)} \in H^{\frac{1}{2}}(\partial X)$, two of which may or may not be equal. To each pair of boundary conditions corresponds an angle function $\theta^{(i)}$ and data vector fields $V_{12}^{a(i)}$ and $N^{(i)}$ as in equation (4.28). Then, taking difference of (4.28) for each system, we obtain the following equation:

$$\widetilde{A}^2 \left(\nabla(\theta^{(2)} - \theta^{(1)}) - V_{12}^{a(2)} + V_{12}^{a(1)} \right) = -\frac{1}{2} J \left(N^{(2)} - N^{(1)} \right). \tag{4.44}$$

Now we claim that the vector field $\nabla(\theta^{(1)} - \theta^{(2)})$ is known from the data, indeed we have

$$\nabla(\theta^{(1)} - \theta^{(2)}) = \cos(\theta^{(1)} - \theta^{(2)}) \nabla \left(\sin(\theta^{(1)} - \theta^{(2)}) \right) - \sin(\theta^{(1)} - \theta^{(2)}) \nabla \left(\cos(\theta^{(1)} - \theta^{(2)}) \right),$$

and then, by definition

$$\cos(\theta^{(2)} - \theta^{(1)}) = R_1^{(1)} \cdot R_1^{(2)} = t_{1i}^{(1)} t_{1j}^{(2)} S_i^{(1)} \cdot S_j^{(2)},$$
$$\sin(\theta^{(2)} - \theta^{(1)}) = R_2^{(1)} \cdot R_1^{(2)} = t_{2i}^{(1)} t_{1j}^{(2)} S_i^{(1)} \cdot S_j^{(2)},$$

where both right-hand sides are known from the data. As a result, we obtain an equation of the form (4.41), where the vector fields

$$\boxed{X := \nabla(\theta^{(2)} - \theta^{(1)}) - V_{12}^{a(2)} + V_{12}^{a(1)} \quad \text{and} \quad Y := \frac{1}{2}\left(N^{(2)} - N^{(1)}\right)} \tag{4.45}$$

are known from the data. Investigating the validity of the reconstruction algorithm (4.42) in this case, we find that, since for a couple of solutions S_1, S_2, we have

$$N = \nabla \log(H_{11}H_{22} - H_{12}^2)^{\frac{1}{2}} = \nabla \log \det(S_1, S_2),$$

the above vector field Y becomes zero wherever both couples of illuminations $(g_1^{(i)}, g_2^{(i)})_{i=1,2}$ are such that the corresponding quantities $N^{(i)} = \nabla \log \det(S_1^{(i)}, S_2^{(i)})$ or, equivalently, the quantities $\nabla \log \det(\nabla u_1^{(i)}, \nabla u_2^{(i)})$, match. Heuristically, if the quantity $\det(\nabla u_1^{(i)}, \nabla u_2^{(i)})$ defines a "volume" spanned by gradients of conductivity solutions, one can say that **what makes the anisotropy visible by the power densities at any given point x is the fact they are generated by two couples of solutions whose relative variations in volume differ from one another at x.**

The S frame approach

Going back to the S_i solutions, we now assume that we have at hand the dotproducts of three solutions S_1, S_2, S_3. Assume further that $\nabla u_1, \nabla u_2$ satisfy a positivity condition (4.16) over some subset Ω, and express the fact that S_1, S_2, S_3 are necessarily linearly dependent as follows:

$$\mu_1 S_1 + \mu_2 S_2 + \mu_3 S_3 = 0, \tag{4.46}$$

where, according to lemma A.3.1, one can express the coefficients μ_i in terms of the H functionals as follows:

$$\mu_1 = H_{12}H_{23} - H_{22}H_{13}, \quad \mu_2 = H_{13}H_{21} - H_{11}H_{23}, \quad \text{and} \quad \mu_3 = H_{11}H_{22} - H_{12}^2. \tag{4.47}$$

We will use (4.46) to derive algebraic equations involving the anisotropy matrix. In order to do so, we write

$$\begin{aligned}
&0 = \nabla \cdot (\mu_i \widetilde{A} S_i) = \nabla \mu_i \cdot \widetilde{A} S_i + \mu_i \nabla \cdot (\widetilde{A} S_i) = \nabla \mu_i \cdot \widetilde{A} S_i - \nabla a \cdot \mu_i \widetilde{A} S_i = \nabla \mu_i \cdot \widetilde{A} S_i. \\
&0 = \nabla^\perp \cdot (\mu_i \widetilde{A}^{-1} S_i) = \nabla^\perp \mu_i \cdot \widetilde{A}^{-1} S_i + \mu_i \nabla^\perp \cdot (\widetilde{A}^{-1} S_i) = \nabla^\perp \mu_i \cdot \widetilde{A}^{-1} S_i + \nabla^\perp a \cdot (\mu_i \widetilde{A}^{-1} S_i) \\
&\quad = \nabla^\perp \mu_i \cdot \widetilde{A}^{-1} S_i.
\end{aligned}$$

In these two equations, we replace S_3 by its expression $S_3 = -\mu_3^{-1}(\mu_1 S_1 + \mu_2 S_2)$, and, rewriting terms of the form

$$\nabla \mu_i - \mu_i \mu_3^{-1} \nabla \mu_3 = \mu_3 \nabla \left(\frac{\mu_i}{\mu_3} \right) =: \mu_3 Z_i, \quad i = 1, 2,$$

we arrive at the following equations

$$Z_1 \cdot \widetilde{A} S_1 + Z_2 \cdot \widetilde{A} S_2 = 0 \quad \text{and} \quad J Z_1 \cdot \widetilde{A}^{-1} S_1 + J Z_2 \cdot \widetilde{A}^{-1} S_2 = 0.$$

Using the fact that $\widetilde{A}J\widetilde{A} = J$, i.e. $\widetilde{A}^{-1}J = J\widetilde{A}$, and the symmetry of $\widetilde{A}$, the previous equations may be rewritten as

$$\widetilde{A}Z_1 \cdot S_1 + \widetilde{A}Z_2 \cdot S_2 = 0 \quad \text{and} \quad J\widetilde{A}Z_1 \cdot S_1 + J\widetilde{A}Z_2 \cdot S_2 = 0. \tag{4.48}$$

We now express in S_2 in the orthogonal basis (S_1, JS_1):

$$S_2 = \frac{1}{H_{11}}\left((S_2 \cdot S_1)S_1 + (S_2 \cdot JS_1)JS_1\right) = \frac{H_{12}}{H_{11}}S_1 + \frac{|H|^{\frac{1}{2}}}{H_{11}}JS_1.$$

Plugging this expression into (4.48) and rearranging terms, we obtain

$$0 = \widetilde{A}\,(H_{11}Z_1 + H_{12}Z_2) \cdot S_1 + \widetilde{A}\left(|H|^{\frac{1}{2}}Z_2\right) \cdot JS_1 = \left(J\widetilde{A}\,(H_{11}Z_1 + H_{12}Z_2) + \widetilde{A}\left(|H|^{\frac{1}{2}}Z_2\right)\right) \cdot JS_1.$$
$$0 = J\widetilde{A}\,(H_{11}Z_1 + H_{12}Z_2) \cdot S_1 + J\widetilde{A}\left(|H|^{\frac{1}{2}}Z_2\right) \cdot JS_1 = \left(J\widetilde{A}\,(H_{11}Z_1 + H_{12}Z_2) + \widetilde{A}\left(|H|^{\frac{1}{2}}Z_2\right)\right) \cdot S_1.$$

Since (S_1, JS_1) is a basis, we deduce the following relation

$$\boxed{J\widetilde{A}X = -\widetilde{A}Y, \quad \text{where} \quad X := H_{11}Z_1 + H_{12}Z_2, \quad Y := |H|^{\frac{1}{2}}Z_2.} \tag{4.49}$$

Considering the validity of reconstruction formula (4.42) applied to (4.49), the question is now to find out when the vectors X and/or Y vanish. Y vanishes precisely where $\nabla\frac{\mu_2}{\mu_3} = \nabla\frac{\det(S_1,S_3)}{\det(S_1,S_2)}$ vanishes, since we assume that $|H|^{\frac{1}{2}}$ does not vanish.

Therefore, in order to make the anisotropy *visible from the functionals* H_{ij}, one must generates three solutions such that the quantity $\nabla\frac{\det(S_1,S_3)}{\det(S_1,S_2)}$ vanishes as rarely as possible. This is in accordance with the analysis of the previous section, since we have the relation

$$\nabla \log\det(S_1, S_3) - \nabla \log\det(S_1, S_2) = \frac{\det(S_1, S_2)}{\det(S_1, S_3)}\nabla\frac{\det(S_1, S_3)}{\det(S_1, S_2)}.$$

We will see in Section 5.2 that Y cannot vanish too often as soon as the boundary conditions (g_1, g_2, g_3) are linearly independent.

4.3 Local reconstruction formulas in dimension $n \geq 3$

In order to extend the previous results to general dimension $n \geq 3$, the derivations must use differential geometric concepts, especially connection calculus. The case $n = 3$ can still treated with three-dimensional vector calculus identities involving the curl operator, as was done in Bal et al. (2012a), although we do not reproduce these calculations here.

Similarly to the two-dimensional case, the conductivity tensor may be decomposed into a scalar part $|\gamma|^{\frac{1}{n}}$ times a rescaled, unimodular anisotropy matrix $\tilde{\gamma} := |\gamma|^{-\frac{1}{n}}\gamma$. We first show that, under knowledge of $\tilde{\gamma}$, one is able to reconstruct both a frame as well as the scalar function $|\gamma|^{\frac{1}{n}}$ locally and Lipschitz-stably in $W^{1,\infty}(\Omega) \to W^{1,\infty}(\Omega)$, where Ω is any subdomain over which n solutions have linearly independent gradients.

Let us introduce $A = \gamma^{\frac{1}{2}}$ and decompose A similarly into

$$A = |A|^{\frac{1}{n}}\widetilde{A} = e^a\widetilde{A}, \quad \det\widetilde{A} = 1, \quad a := \log|A|^{\frac{1}{n}} = \log|\gamma|^{\frac{1}{2n}}. \tag{4.50}$$

Again, it is clear that when $\gamma \in \Sigma(X)$, then $A \in \Sigma(X)$ with $C_A = \sqrt{C_\gamma}$, $\widetilde{A} \in \Sigma(X)$ with $C_{\widetilde{A}} = C_\gamma$ and e^a satisfies the uniform bounds $C_\gamma^{-\frac{1}{2}} \leq e^a \leq C_\gamma^{\frac{1}{2}}$ over X.

Secondly, if the anisotropy is unknown, then considering additional solutions to the n first ones gives us algebraic equations that impose orthogonality constraints on a matrix from which we can deduce the anisotropy. This process will be stable provided that certain conditions are fulfilled, which we make more explicit in the following sections.

4.3.1 Local reconstruction formulas when the anisotropy is known

Let us assume that n solutions $u_1, \cdots, u_n$ of

$$\nabla \cdot (\gamma \nabla u_i) = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad 1 \le i \le n, \tag{4.51}$$

have their gradients linearly independent over an open, simply connected subset $\Omega \subset X$. Up to reindexing, this assumption may be restated as

$$\inf_{x \in \Omega} \det(\nabla u_1, \cdots, \nabla u_n) \ge c_0 > 0. \tag{4.52}$$

We will now derive first-order PDE systems that rely on this fact. We recall the PDE's satisfied by the S_i's defined in (4.1):

$$\nabla \cdot (A S_i) = 0, \quad 1 \le i \le n, \tag{4.53}$$

$$d(A^{-1} S_i)^\flat = 0, \quad 1 \le i \le n. \tag{4.54}$$

Gradient equation for $|\gamma|^{\frac{1}{n}}$

Similar to the two-dimensional case, using the above PDE's (4.60)-(4.61), we are able to derive the following local reconstruction formula

Lemma 4.3.1. *Let a simply connected sub domain $\Omega \subset X$ be such that condition (4.2) holds. Then the following relation holds throughout Ω with the function a defined in (4.50):*

$$\boxed{n \nabla a = |H|^{-\frac{1}{2}} \left(\nabla (|H|^{\frac{1}{2}} H^{jl}) \cdot \widetilde{A} S_l \right) \widetilde{A}^{-1} S_j = \frac{1}{2} \nabla \log |H| + (\nabla H^{jl} \cdot \widetilde{A} S_l) \widetilde{A}^{-1} S_j.} \tag{4.55}$$

The proof essentially relies on the study of the behavior of the *dual coframe*[1] of the frame $(A^{-1}S_1, \cdots, A^{-1}S_n)$.

Proof. For $1 \leq j \leq n$, let us define the vector field X_j by

$$X_j^\flat = \sigma_j \star \left[(A^{-1}S_1)^\flat \wedge \cdots \wedge (A^{-1}S_{\hat{j}})^\flat \wedge \cdots \wedge (A^{-1}S_n)^\flat \right], \quad \sigma_j := (-1)^{j-1}, \tag{4.56}$$

where the hat over an index indicates its omission. X_j is the unique vector field such that at every $x \in \Omega$ and for every vector $V \in T_x\Omega$, we have

$$X_j(x) \cdot V = \det(A^{-1}S_1, \ldots, A^{-1}S_{j-1}, \overbrace{V}^{j}, A^{-1}S_{j+1}, \ldots, A^{-1}S_n).$$

In particular, we have that for any $S_n^+(\mathbb{R})$-valued function M and any vector field V,

$$\begin{aligned} MX_j \cdot V = X_j \cdot MV &= \det(A^{-1}S_1, \ldots, MV, \ldots, A^{-1}S_n) \\ &= \det M \det((M^{-1}A^{-1})S_1, \ldots, V, \ldots, (M^{-1}A^{-1})S_n), \end{aligned}$$

that is, we have that

$$(MX_j)^\flat = \sigma_j \det M \star \left[(M^{-1}A^{-1}S_1)^\flat \wedge \cdots \wedge (M^{-1}A^{-1}S_{\hat{j}})^\flat \wedge \cdots \wedge (M^{-1}A^{-1}S_n)^\flat \right]. \tag{4.57}$$

$(X_1, \cdots, X_n)$ is, up to some scalar factor, the dual basis to $(A^{-1}S_1, \cdots, A^{-1}S_n)$ since we have, for $i \neq j$

$$X_j \cdot A^{-1}S_i = \det(A^{-1}S_1, \ldots, \underbrace{A^{-1}S_i}_{i}, \ldots, \underbrace{A^{-1}S_i}_{j}, \ldots, A^{-1}S_n) = 0,$$

[1]For $(E_1, \cdots, E_n)$ a frame, $(\omega_1, \cdots, \omega_n)$ is called the dual coframe of E if $\omega_i(E_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.

since the determinant contains twice the vector $A^{-1}S_i$. Moreover, when $i = j$, we have

$$X_j \cdot A^{-1}S_j = \det(A^{-1}S_1, \ldots, A^{-1}S_n) = \det A^{-1} \det(S_1, \ldots, S_n) = \det\left(A^{-1}H^{\frac{1}{2}}\right),$$

where we used the fact that $S^TS = H$ therefore $\det S = \det H^{\frac{1}{2}}$. Therefore we can use formula (4.4) with $M \equiv A^{-1}$ to represent X_j as

$$X_j = \sum_{k,l=1}^{n} H^{kl}(X_j \cdot A^{-1}S_k)AS_l = \sum_{l=1}^{n} H^{jl} \det(A^{-1}H^{\frac{1}{2}})AS_l. \tag{4.58}$$

We now show that X_j is divergence-free, that is $\nabla \cdot X_j = 0$ for $1 \le j \le n$. Indeed, we write

$$\nabla \cdot X_j = \star d \star X_j^\flat = \star d\left[(A^{-1}S_1)^\flat \wedge \cdots \wedge (A^{-1}S_{\hat{j}})^\flat \wedge \cdots \wedge (A^{-1}S_n)^\flat\right] = 0,$$

since an exterior product of closed forms is always closed, thus we have

$$\nabla \cdot X_j = 0, \quad 1 \le j \le n. \tag{4.59}$$

Combining equations (4.58) together with (4.59), and using the fact that $\nabla \cdot (fV) = f\nabla \cdot V + \nabla f \cdot V$, we obtain

$$\begin{aligned} 0 = \nabla \cdot X_j &= \nabla \cdot (H^{jl} \det(A^{-1}H^{\frac{1}{2}})AS_l) \\ &= \det(A^{-1}H^{\frac{1}{2}})\nabla H^{jl} \cdot AS_l + H^{jl}\nabla \det(A^{-1}H^{\frac{1}{2}}) \cdot AS_l + \det(A^{-1}H^{\frac{1}{2}})H^{jl}\nabla \cdot (AS_l). \end{aligned}$$

The last term is zero by virtue of (4.60) and the second term expresses the dotproducts of $\nabla \det(A^{-1}H^{\frac{1}{2}})$ with the frame $A^{-1}S$. Thus we use the representation formula (4.4) with $M \equiv A$

and divide by $\det(A^{-1}H^{\frac{1}{2}})$ to obtain

$$\nabla \log\det(A^{-1}H^{\frac{1}{2}}) = H^{jl}(\nabla \log\det(A^{-1}H^{\frac{1}{2}}) \cdot AS_l)A^{-1}S_j = -(\nabla H^{jl} \cdot AS_l)A^{-1}S_j,$$

which upon writing $\log\det(A^{-1}H^{\frac{1}{2}}) = -\log\det A + \frac{1}{2}\log\det H$ yields

$$\nabla \log\det A = \frac{1}{2}\nabla \log\det H + (\nabla H^{jl} \cdot AS_l)A^{-1}S_j.$$

We now plug in the rescaling $A = |A|^{\frac{1}{n}}\widetilde{A}$, which implies $A^{-1} = |A|^{-\frac{1}{n}}\widetilde{A}^{-1}$, and notice that the terms involving $|A|$ cancel out in the right-hand side of the last equation, thus (4.55) is proved. □

Total covariant derivative system for the S frame

The previous section showed that we can express the quantity ∇a in terms of the data H and the unknown frame S. Let us write $F = \nabla a$, then plugging the decomposition (4.50) into (4.60)-(4.61), we obtain the following system

$$\nabla \cdot (\widetilde{A}S_i) = -F \cdot \widetilde{A}S_i, \tag{4.60}$$

$$d(\widetilde{A}^{-1}S_i)^\flat = F^\flat \wedge (\widetilde{A}^{-1}S_i)^\flat. \tag{4.61}$$

Equation (4.60) is no longer useful, as we used it to express F in terms of the S frame. Since F only depends on the data and the S_i's without differentiating them, the above system is thus closed for the frame S. From these equations, we now show how to derive a total covariant derivative system for each of the S_i's.

The isotropic case $\widetilde{A} = \mathbb{I}_n$***:*** If the anisotropy is the identity tensor, then the knowledge of the data $H_{ij} = S_i \cdot S_j$ as well as of PDEs (4.60)-(4.61) allows us to derive directly, using Koszul's formula (C.4), a total covariant derivative system for the frame S (or, equivalently, for the frame R). The corresponding calculations were worked out in Monard and Bal (2012c), and we now present an accelerated version of it.

Recall the Koszul formula for three vector fields X, Y, Z, derived by using the compatibility of the connection with the metric and the torsion-freeness of the connection:

$$2\overline{\nabla}_X Y \cdot Z = \overline{\nabla}_X(Y \cdot Z) + \overline{\nabla}_Y(X \cdot Z) - \overline{\nabla}_Z(X \cdot Y) - Y \cdot [X, Z] - Z \cdot [Y, X] + X \cdot [Z, Y].$$

Using this formula together with the identity

$$dZ^\flat(X, Y) = \overline{\nabla}_X(Z \cdot Y) - \overline{\nabla}_Y(Z \cdot X) - Z \cdot [X, Y],$$

we arrive at a slightly modified Koszul formula, expressed in terms of exterior derivatives instead of Lie brackets

$$2\overline{\nabla}_X Y \cdot Z = \overline{\nabla}_X(Y \cdot Z) - \overline{\nabla}_Y(X \cdot Z) + \overline{\nabla}_Z(X \cdot Y) + dY^\flat(X, Z) + dZ^\flat(Y, X) - dX^\flat(Z, Y). \tag{4.62}$$

Finally, if all three vector fields satisfy the equation $dX^\flat = F^\flat \wedge X^\flat$ for a fixed vector field F, we finally obtain

$$2\overline{\nabla}_X Y \cdot Z = \overline{\nabla}_X(Y \cdot Z) - \overline{\nabla}_Y(X \cdot Z) + \overline{\nabla}_Z(X \cdot Y) + 2(F \cdot Y)(Z \cdot X) - 2(F \cdot Z)(X \cdot Y). \tag{4.63}$$

On to the resolution of our problem, in the isotropic case, we summarize the equations already derived:

$$\nabla \cdot S_i = -F \cdot S_i \quad \text{and} \quad dS_i^\flat = -F^\flat \wedge S_i^\flat, \quad 1 \le i \le n, \tag{4.64}$$
$$F := \frac{1}{n}|H|^{-\frac{1}{2}} \left(\nabla(|H|^{\frac{1}{2}} H^{pq}) \cdot S_p\right) S_q. \tag{4.65}$$

We now combine (C.6) with $E \equiv S$ together with the formula (4.63), and obtain

$$\begin{aligned} \overline{\nabla} S_i &= H^{qk} H^{jp} (\overline{\nabla}_{S_q} S_i \cdot S_p) S_j \otimes S_k^\flat \\ &= \frac{1}{2} H^{qk} H^{jp} \left[\overline{\nabla}_{S_q} H_{ip} - \overline{\nabla}_{S_i} H_{qp} + \overline{\nabla}_{S_p} H_{iq} + 2(F \cdot S_i) H_{pq} - 2(F \cdot S_p) H_{qi}\right] S_j \otimes S_k^\flat \end{aligned}$$

We now define the data vector fields

$$U_{jk} := (\nabla H_{jp}) H^{pk} = -H_{jp} (\nabla H^{pk}), \quad 1 \le j, k \le n, \tag{4.66}$$

and, together with the simplifications $H^{jk} S_j \otimes S_k^\flat = \mathbb{I}_n = \mathbf{e}_i \otimes \mathbf{e}^i$ and (4.3), we obtain

$$\boxed{\overline{\nabla} S_i = \frac{1}{2}\left(U_{ik} \otimes S_k^\flat + S_k \otimes U_{ik}^\flat + (\nabla H^{jk} \cdot S_i) S_j \otimes S_k^\flat\right) + (F \cdot S_i)\mathbb{I}_n - F \otimes S_i^\flat, \quad 1 \le i \le n,} \tag{4.67}$$

with F given in (4.65). The first, second and fourth terms of (4.67) are linear in S while the third and fifth are cubic.

The anisotropic case: In the past section, we have seen that the Koszul formula (C.4) expresses the fact that knowledge of the dotproducts and Lie Brackets (or, equivalently, exterior derivatives) of the elements of a frame allows to setup a total covariant derivative system for this frame. By "knowledge" here, we mean that the Lie Brackets or exterior derivatives may be expressed as functionals of the same frame that do not differentiate it (i.e. the derivatives have been moved on to the data H_{ij}).

The additional difficulty here is that there is no frame of which we easily "know" both the dotproducts and the Lie brackets. Indeed, we know the dotproducts $S_i \cdot S_j$ and the Lie brackets $[\widetilde{A}S_i, \widetilde{A}S_j]$ with $\widetilde{A} \neq \mathbb{I}_n$ this time. Said Lie brackets are expressed as zero-th order functionals of the frame $\widetilde{A}S$ by using the identity (C.2) with $(X, Y, Z) \equiv (\widetilde{A}S_i, \widetilde{A}S_j, \widetilde{A}^{-1}S_k)$ for some $1 \leq i, j, k \leq n$:

$$\begin{aligned}
\widetilde{A}^{-1}S_k \cdot [\widetilde{A}S_i, \widetilde{A}S_j] &= \widetilde{A}S_i \cdot \nabla H_{jk} - \widetilde{A}S_j \cdot \nabla H_{ik} - d(\widetilde{A}^{-1}S_k)^\flat(\widetilde{A}S_i, \widetilde{A}S_j) \\
&= \widetilde{A}S_i \cdot \nabla H_{jk} - \widetilde{A}S_j \cdot \nabla H_{ik} - F^\flat \wedge (\widetilde{A}^{-1}S_k)^\flat(\widetilde{A}S_i, \widetilde{A}S_j) \\
&= \widetilde{A}S_i \cdot \nabla H_{jk} - \widetilde{A}S_j \cdot \nabla H_{ik} - H_{kj}F \cdot \widetilde{A}S_i + H_{ki}F \cdot \widetilde{A}S_j,
\end{aligned}$$

where the choice of the appropriate tensor in front of X, Y or Z was made so that any dotproduct resulting from the calculations only involves the vectors X, Y, Z with no tensor in front.

A way to circumvent the addition of a $(1, 1)$-tensor is to derive a formula that is similar in spirit to Koszul's, yet that exploits Lie brackets of the form $[\widetilde{A}S_i, \widetilde{A}S_j]$ instead of $[S_i, S_j]$. Doing this in the following will make the reconstruction of the frame S via a system of first-order PDEs still possible.

This will require some calculations, which we now break up into a few identities. For M a $(1, 1)$-tensor and X, Y smooth vector fields, we define the M-commutator $[X, Y]^M$ by

$$[X, Y]^M := \overline{\nabla}_{MX}Y - \overline{\nabla}_{MY}X. \tag{4.68}$$

In the same way that one derives Koszul's formula using the compatibility of the connection with the metric and its torsion-freeness, we have the following

Lemma 4.3.2. *For M a $(1,1)$-tensor and X, Y, Z three smooth vector fields, we have the identity*

$$\begin{aligned} 2(\overline{\nabla}_{MX}Y) \cdot Z = \overline{\nabla}_{MX}(Y \cdot Z) + \overline{\nabla}_{MY}(X \cdot Z) - \overline{\nabla}_{MZ}(X \cdot Y) \\ - [X, Z]^M \cdot Y - [Y, Z]^M \cdot X + [X, Y]^M \cdot Z. \end{aligned} \tag{4.69}$$

Proof. Using the compatibility of the connection with the metric and definition (4.68), we can write

$$\begin{aligned} \overline{\nabla}_{MX}(Y \cdot Z) &= \overline{\nabla}_{MX}Y \cdot Z + Y \cdot \overline{\nabla}_{MX}Z = \overline{\nabla}_{MX}Y \cdot Z + Y \cdot [X, Z]^M + Y \cdot \overline{\nabla}_{MZ}X, \\ \overline{\nabla}_{MY}(X \cdot Z) &= \overline{\nabla}_{MY}X \cdot Z + X \cdot \overline{\nabla}_{MY}Z = -[X, Y]^M \cdot Z + \overline{\nabla}_{MX}Y \cdot Z + X \cdot \overline{\nabla}_{MY}Z, \\ \overline{\nabla}_{MZ}(X \cdot Y) &= \overline{\nabla}_{MZ}X \cdot Y + X \cdot \overline{\nabla}_{MZ}Y = \overline{\nabla}_{MZ}X \cdot Y - X \cdot [Y, Z]^M + X \cdot \overline{\nabla}_{MY}Z. \end{aligned}$$

Subtracting the third equation from the sum of the first two yields the result. □

We now connect the commutator $[X, Y]^M$ to commutators of the form $M^{-1}[MX, MY]$. In particular, their difference is of order zero in the vector fields:

Lemma 4.3.3. *For M a given smooth, uniformly invertible $(1,1)$-tensor, the bilinear anti-symmetric operator*

$$\mathcal{A}_M : X, Y \mapsto \mathcal{A}_M(X, Y) := M^{-1}([MX, MY] - M[X, Y]^M) \tag{4.70}$$

defines a vector-valued $(2,0)$-tensor. In local coordinates it is expressed as

$$\mathcal{A}_M(X, Y) = \frac{1}{2}(X^l Y^q - Y^l X^q) M^{-1}[M_l, M_q], \tag{4.71}$$

where $M_l := M(\cdot, \partial_l) = M(dx^l, \cdot)^\sharp$. Moreover, when M is constant in a given coordinate system, then the tensor $\mathcal{A}_M$ is identically zero.

Proof. We first check that $\mathcal{A}_M$ is a well-defined tensor. Bilinearity and anti-symmetry are obvious. By virtue of anti-symmetry, it is enough to check linearity over smooth functions in the first argument. For f a smooth function, we have

$$\begin{aligned}[MfX, MY] = [fMX, MY] &= \overline{\nabla}_{fMX} MY - \overline{\nabla}_{MY} fMX \\ &= f\overline{\nabla}_{MX} MY - (\overline{\nabla}_{MY} f)MX - f\overline{\nabla}_{MY} MX \\ &= f[MX, MY] - (\overline{\nabla}_{MY} f)MX,\end{aligned}$$

On the other hand,

$$\begin{aligned}M[fX, Y]^M &= M\overline{\nabla}_{MfX} Y - M\overline{\nabla}_{MY}(fX) \\ &= M\overline{\nabla}_{fMX} Y - M\overline{\nabla}_{MY}(fX) \\ &= fM\overline{\nabla}_X Y - fM\overline{\nabla}_{MY} X - \underbrace{M(\overline{\nabla}_{MY} f)X}_{=(\overline{\nabla}_{MY} f)MX} \\ &= fM[X, Y]^M - (\overline{\nabla}_{MY} f)MX.\end{aligned}$$

Taking the difference of both calculations, the terms $(\overline{\nabla}_{MY} f)MX$ cancel out, so $\mathcal{A}_M$ fulfills the tensor property. Now, in local coordinates, we write $X = X^l \partial_l$, $Y = Y^m \partial_m$ with $[\partial_p, \partial_q] = 0$ and $M = M_{pq}\partial_p \otimes dx^q$ with $M_{pq} = M_{qp}$, then

$$\begin{aligned}[MX, MY] &= [M_{pq}X^q\partial_p, M_{ml}Y^l\partial_m] \\ &= \Big(M_{pq}X^q\partial_p(M_{ml}Y^l) - M_{pl}Y^l\partial_p(M_{mq}X^q)\Big) \\ &= \frac{1}{2}(X^qY^l - Y^qX^l)[M_q, M_l] + M(\overline{\nabla}_{MX} Y - \overline{\nabla}_{MY} X),\end{aligned}$$

hence the result. $\square$

Now, combining the definition (4.70) with the modified Koszul formula (4.69) and assuming that M is uniformly invertible, we arrive at

$$\begin{aligned}2(\overline{\nabla}_{MX}Y)\cdot Z &= \overline{\nabla}_{MX}(Y\cdot Z)+\overline{\nabla}_{MY}(X\cdot Z)-\overline{\nabla}_{MZ}(X\cdot Y)\\ &\quad - M[X,Z]^M\cdot M^{-1}Y - M[Y,Z]^M\cdot M^{-1}X + M[X,Y]^M\cdot M^{-1}Z\\ &= \overline{\nabla}_{MX}(Y\cdot Z)+\overline{\nabla}_{MY}(X\cdot Z)-\overline{\nabla}_{MZ}(X\cdot Y)\\ &\quad - [MX,MZ]\cdot M^{-1}Y - [MY,MZ]\cdot M^{-1}X + [MX,MY]\cdot M^{-1}Z\\ &\quad - \mathcal{A}_M(X,Z)\cdot Y - \mathcal{A}_M(Y,Z)\cdot X + \mathcal{A}_M(X,Y)\cdot Z.\end{aligned} \tag{4.72}$$

Since the above left-hand side is, by definition, $2\overline{\nabla}Y(Z^\flat, MX)$, we may obtain an inversion formula for the total covariant derivative whenever the tensor M is invertible:

Lemma 4.3.4. *For a given frame $(E_1,\cdots,E_n)$ with dotproducts $g_{ij}=E_i\cdot E_j$ and M a symmetric, uniformly invertible $(1,1)$-tensor, we have the following expression for the total covariant derivative:*

$$\overline{\nabla}E_i = g^{kq}g^{jp}(\overline{\nabla}_{ME_q}E_i\cdot E_p)E_j\otimes(M^{-1}E_k)^\flat \tag{4.73}$$

Proof. Since M is invertible, the family $\{E_j\otimes(M^{-1}E_k)^\flat\}_{1\le j,k\le n}$ provides a basis for all $(1,1)$-tensors, along which we decompose $\overline{\nabla}E_i$ by writing

$$\overline{\nabla}E_i = a_{ijk}E_j\otimes(M^{-1}E_k)^\flat,$$

for some coefficients a_{ijk}. We can determine these coefficients by computing

$$\overline{\nabla}E_i(E_p^\flat, ME_q) = \overline{\nabla}_{ME_q}E_i\cdot E_p = a_{ijk}g_{jp}g_{kq},$$

therefore we obtain

$$a_{ijk} = g^{kq} g^{jp} (\overline{\nabla}_{ME_q} E_i \cdot E_p),$$

hence the result. $\square$

We have now everything in hand in order to tackle our problem. Recalling the useful formulas, we have the Lie bracket identity

$$[\widetilde{A}S_i, \widetilde{A}S_j] \cdot \widetilde{A}^{-1} S_k = \widetilde{A}S_i \cdot \nabla H_{jk} - \widetilde{A}S_j \cdot \nabla H_{ik} - H_{kj} F \cdot \widetilde{A}S_i + H_{ki} F \cdot \widetilde{A}S_j, \tag{4.74}$$

where F takes the expression

$$F := \frac{1}{n} |H|^{-\frac{1}{2}} \left(\nabla(|H|^{\frac{1}{2}} H^{pq}) \cdot \widetilde{A}S_p \right) \widetilde{A}^{-1} S_q. \tag{4.75}$$

Therefore, formula (4.72), written with $M \equiv \widetilde{A}$ and $(X, Y, Z) = (S_q, S_i, S_p)$, reads

$$\begin{aligned} 2\overline{\nabla}_{\widetilde{A}S_q} S_i \cdot S_p = {} & \overline{\nabla}_{\widetilde{A}S_q} H_{ip} + \overline{\nabla}_{\widetilde{A}S_i} H_{pq} - \overline{\nabla}_{\widetilde{A}S_p} H_{iq} \\ & - [\widetilde{A}S_q, \widetilde{A}S_p] \cdot \widetilde{A}^{-1} S_i - [\widetilde{A}S_i, \widetilde{A}S_p] \cdot \widetilde{A}^{-1} S_q + [\widetilde{A}S_q, \widetilde{A}S_i] \cdot \widetilde{A}^{-1} S_p \\ & - \mathcal{A}_{\widetilde{A}}(S_q, S_p) \cdot S_i - \mathcal{A}_{\widetilde{A}}(S_i, S_p) \cdot S_q + \mathcal{A}_{\widetilde{A}}(S_q, S_i) \cdot S_p. \end{aligned}$$

Plugging the Lie bracket (4.74) into last equation yields

$$\begin{aligned} 2\overline{\nabla}_{\widetilde{A}S_q} S_i \cdot S_p = {} & \widetilde{A}S_q \cdot \nabla H_{ip} + \widetilde{A}S_p \cdot \nabla H_{iq} - \widetilde{A}S_i \cdot \nabla H_{pq} + 2H_{pq} F \cdot \widetilde{A}S_i - 2H_{qi} F \cdot \widetilde{A}S_p \\ & - \mathcal{A}_{\widetilde{A}}(S_q, S_p) \cdot S_i - \mathcal{A}_{\widetilde{A}}(S_i, S_p) \cdot S_q + \mathcal{A}_{\widetilde{A}}(S_q, S_i) \cdot S_p. \end{aligned}$$

Finally, using lemma 4.3.4 for the frame S with $M \equiv \widetilde{A}$, we obtain

$$\overline{\nabla} S_i = H^{kq} H^{jp} (\overline{\nabla}_{\widetilde{A} S_q} S_i \cdot S_p) S_j \otimes (\widetilde{A}^{-1} S_k)^\flat$$

For clarity, we will split the result into two parts: $\overline{\nabla} S_i = \overline{\nabla}^0 S_i + \overline{\nabla}^{\mathcal{A}} S_i$, where $\overline{\nabla}^0$ accounts for terms that do not differentiate the anisotropic tensor $\widetilde{A}$, and $\overline{\nabla}^{\mathcal{A}}$ involves the $\mathcal{A}_{\widetilde{A}}$ tensor. In so doing, we obtain

$$\begin{aligned}\overline{\nabla}^0 S_i = \frac{1}{2} H^{kq} H^{jp} \big(&\widetilde{A} S_q \cdot \nabla H_{ip} + \widetilde{A} S_p \cdot \nabla H_{iq} - \widetilde{A} S_i \cdot \nabla H_{pq} \\ &+ 2 H_{pq} F \cdot \widetilde{A} S_i - 2 H_{qi} F \cdot \widetilde{A} S_p \big) S_j \otimes (\widetilde{A}^{-1} S_k)^\flat.\end{aligned}$$

After using the representation formula (4.4) and the fact that $H^{kj} S_j \otimes (\widetilde{A}^{-1} S_k)^\flat = \widetilde{A}^{-1}$, we arrive at

$$\boxed{\begin{aligned}\overline{\nabla}^0 S_i = \frac{1}{2} \Big(&S_k \otimes U_{ik}^\flat + \widetilde{A} U_{ik} \otimes (\widetilde{A}^{-1} S_k)^\flat + \widetilde{A} S_i \cdot \nabla H^{jk} S_j \otimes (\widetilde{A}^{-1} S_k)^\flat \Big) \\ &+ (F \cdot \widetilde{A} S_i) \widetilde{A}^{-1} - \widetilde{A} F \otimes (\widetilde{A}^{-1} S_i)^\flat.\end{aligned}} \tag{4.76}$$

The part $\overline{\nabla}^{\mathcal{A}}$ due to variations of anisotropy is given by

$$\boxed{\overline{\nabla}^{\mathcal{A}} S_i = \frac{1}{2} H^{kq} H^{jp} (-\mathcal{A}_{\widetilde{A}}(S_q, S_p) \cdot S_i - \mathcal{A}_{\widetilde{A}}(S_i, S_p) \cdot S_q + \mathcal{A}_{\widetilde{A}}(S_q, S_i) \cdot S_p) S_j \otimes (\widetilde{A}^{-1} S_k)^\flat.} \tag{4.77}$$

As a conclusion, we have

$$\boxed{\overline{\nabla} S_i = \overline{\nabla}^0 S_i + \overline{\nabla}^{\mathcal{A}} S_i, \quad 1 \le i \le n,} \tag{4.78}$$

where $\overline{\nabla}^0 S_i$ and $\overline{\nabla}^{\mathcal{A}} S_i$ are given in (4.76) and (4.77), respectively. One may check that when $\widetilde{A} = \mathbb{I}_n$, we clearly have $\mathcal{A}_{\widetilde{A}} \equiv 0$ so that $\nabla^{\mathcal{A}} S_i = 0$, and the expression (4.76) matches that of

(4.67). With the presence of anisotropy, it seems that the right-hand sides of (4.77) may be quintic polynomials in the components of S (instead of just cubic in the isotropic case).

Local reconstruction: In the isotropic case, equations (4.67) forms a collection of coupled first-order PDE's, whose right-hand-sides depend polynomially on the unknowns. In the anisotropic case, the same is true for equations (4.78) if the anisotropy structure $\widetilde{A}$ is known. Together with the *a priori* bounds $\|S_i\|^2 = H_{ii} \in L^\infty$, this guarantees that the right-hand sides are Lipschitz functions of the unknowns over Ω, so that local solvability is possible. In particular, say we want to reconstruct $(S_1, \cdots, S_n)$ at $x \in \Omega$ from their value at some fixed $x_0 \in \Omega$. Picking a smooth simple curve $c : [0,1] \to \Omega$ with $c(0) = x_0$ and $c(1) = x$, it is straightforward to see that the function $t \mapsto \{S_i(c(t))\}_{i=1}^n$ satisfies the quasilinear ordinary differential equation (ODE)

$$\frac{d}{dt}(S_i(c(t))) = \overline{\nabla}_{\dot{c}(t)} S_i(c(t)) = \overline{\nabla} S_i(\cdot, \dot{c}(t))(c(t)), \quad 1 \le i \le n, \tag{4.79}$$

where one may plug the right-hand side of (4.67) or (4.78) into the right-hand side of last equation. Since local uniqueness holds because the right-hand sides are Lipschitz functions of the S_i's, and a solution exists (precisely $S_i(c(t))$), we are able to extend the solution of the system of ODE's (4.79) up to $t = 1$ and thus to obtain the values $S_i(c(1)) = S_i(x)$ for $1 \le i \le n$.

Remark 4.3.5 (On compatibility conditions)**.** *The fact that this resolution method does not depend on the choice of the integration curve c relies on higher-order compatibility conditions, namely, that the curvature in the S frame must match the local curvature. In this (Euclidean) case, this curvature must be zero and one may show that these (rather nasty) conditions are hard to make sense of (see Monard and Bal (2012c) in the isotropic case), let alone to enforce* *during resolution. In local coordinates, these conditions also express the necessity of commuting second-order partial derivatives.*

Numerically, the resolution of (4.79) may be done by regular ODE solving methods, discretizing $t \in [0, 1]$ and marching forward in t up to $t = 1$. Although this method has not yet been implemented, we expect it to be sensitive to the choice of curve in practice, not only due to numerical inaccuracies, but also due to noise in the measurements. Furthermore, in the light of the two-dimensional case treated in section 4.2.2, it is a challenge to find ODE solving methods that take into account hard-wired integrability conditions. Indeed, in the two-dimensional case, we have seen that when the right-hand side does not depend on the unknown, the integrability conditions are naturally enforced by solving a Laplace equation rather than a gradient equation. The question remains as to how to generalize this approach to the present case, where the right-hand sides depend on the unknowns.

Total covariant derivative system for the R frame

Still working over a subset $\Omega \subset X$ where the positivity condition (4.2) holds, we now want to reproduce the same analysis for the R frame, that is, an orthonormal frame obtain from S after an orthonormalizing process that only requires knowledge of the available data. As explained in Section 4.1.2, we therefore write

$$R_i = t_{ij} S_j \quad \text{and} \quad S_i = t^{ij} R_j, \quad 1 \le i \le n, \tag{4.80}$$

and, since (4.2) holds and we assume the orthonormalization process orientation-preserving, the frame R takes values in $SO_n(\mathbb{R})$. The R frame satisfies the PDE's

$$\nabla \cdot (AR_i) = V_{ik} \cdot AR_k \quad \text{and} \quad d(A^{-1}R_i)^\flat = V_{ik}^\flat \wedge (A^{-1}R_k)^\flat, \quad 1 \le i \le n.$$

After plugging the decomposition $A = e^a \widetilde{A}$ in the previous PDE's and using notation $F = \nabla a$, we arrive at the following PDE's

$$\nabla \cdot (\widetilde{A} R_i) = V_{ik} \cdot \widetilde{A} R_k - F \cdot \widetilde{A} R_i \quad \text{and} \quad d(\widetilde{A}^{-1} R_i)^\flat = V_{ik}^\flat \wedge (A^{-1} R_k)^\flat + F^\flat \wedge (\widetilde{A}^{-1} R_i)^\flat.$$

As in the S frame, the first step is to express the vector field F in the R frame. One could follow an approach along the same lines as in Lemma 4.3.1, although we could use the result of equation (4.55) to obtain the expression directly. All we need is the following formula

$$(\nabla H^{ij}) t^{ik} t^{jl} = (\nabla (t_{pi} t_{pj})) t^{ik} t^{jl} = \delta_{pk} (\nabla t_{pj}) t^{jl} + \delta_{pl} (\nabla t_{pi}) t^{ik} = V_{kl} + V_{lk},$$

with V_{ij} defined in (4.8). Then, equation (4.55) becomes

$$nF = \nabla \log |H|^{\frac{1}{2}} + (\nabla H^{pq} \cdot \widetilde{A} S_p) \widetilde{A}^{-1} S_q = \nabla \log |H|^{\frac{1}{2}} + ((\nabla H^{pq}) \cdot t^{pk} \widetilde{A} R_k) t^{ql} \widetilde{A}^{-1} R_l,$$

that is,

$$\boxed{nF = \nabla \log |H|^{\frac{1}{2}} + ((V_{ij} + V_{ji}) \cdot \widetilde{A} R_i) \widetilde{A}^{-1} R_j.} \tag{4.81}$$

Also note that, as a straightforward consequence of Liouville's formula (see e.g. (Bal et al., 2012a, Sec. 4.1.3)) we have

$$\nabla \log |H|^{\frac{1}{2}} = - \sum_{i=1}^{n} V_{ii}. \tag{4.82}$$

Next, we may compute a differential system for the R frame. As in the S frame case, we give formulas for both isotropic and anisotropic case.

The isotropic case $\widetilde{A} = \mathbb{I}_n$**:** The vector fields in an orthonormal frame are such that their dotproducts are constants, in which case the modified Koszul formula (4.62) simplifies into

$$2\overline{\nabla}_X Y \cdot Z = dY^\flat(X, Z) + dZ^\flat(Y, X) - dX^\flat(Z, Y).$$

Combining this with the equation $dR_i^\flat = V_{ik}^\flat \wedge R_k^\flat + F^\flat \wedge R_i^\flat$ in the isotropic case, and using the formula (C.6) in a frame where $g_{ij} = \delta_{ij}$, we arrive at the total covariant derivative for R_i:

$$\boxed{\overline{\nabla} R_i = R_k \otimes V_{ik}^{a\flat} - V_{ik}^s \otimes R_k^\flat + (V_{jk}^s \cdot R_i) R_j \otimes R_k^\flat + (F \cdot R_i)\mathbb{I}_n - F \otimes R_i^\flat, \quad 1 \le i \le n,} \tag{4.83}$$

where we have defined

$$V_{ij}^a := \frac{1}{2}(V_{ij} - V_{ji}) \quad \text{and} \quad V_{ij}^s := \frac{1}{2}(V_{ij} + V_{ji}). \tag{4.84}$$

The anisotropic case: Starting from equation (4.72) with $M \equiv \widetilde{A}$ and $(X, Y, Z) = (R_1, R_i, R_p)$, and using the fact that $\nabla(R_k \cdot R_l) = \nabla \delta_{kl} = 0$ for all $1 \le k, l \le n$, we arrive at

$$\begin{aligned} 2\overline{\nabla}_{\widetilde{A}R_q} R_i \cdot R_p = & -[\widetilde{A}R_q, \widetilde{A}R_p] \cdot \widetilde{A}^{-1} R_i - [\widetilde{A}R_i, \widetilde{A}R_p] \cdot \widetilde{A}^{-1} R_q + [\widetilde{A}R_q, \widetilde{A}R_i] \cdot \widetilde{A}^{-1} R_p \\ & - \mathcal{A}_{\widetilde{A}}(R_q, R_p) \cdot R_i - \mathcal{A}_{\widetilde{A}}(R_i, R_p) \cdot R_q + \mathcal{A}_{\widetilde{A}}(R_q, R_i) \cdot R_p. \end{aligned}$$

Moreover, we can derive for any $1 \le i, j, k \le n$,

$$\begin{aligned} [\widetilde{A}R_i, \widetilde{A}R_j] \cdot \widetilde{A}^{-1} R_k &= -d(\widetilde{A}^{-1} R_k)^\flat(\widetilde{A}R_i, \widetilde{A}R_j) \\ &= -(V_{kl}^\flat \wedge (\widetilde{A}^{-1} R_l)^\flat + F^\flat \wedge (\widetilde{A}^{-1} R_k)^\flat)(\widetilde{A}R_i, \widetilde{A}R_j) \\ &= -V_{kj} \cdot \widetilde{A}R_i + V_{ki} \cdot \widetilde{A}R_j - \delta_{kj} F \cdot \widetilde{A}R_i + \delta_{ki} F \cdot \widetilde{A}R_j. \end{aligned}$$

Plugging this result in the previous equation and using the fact that (4.73) reads, when the frame is orthonormal

$$\overline{\nabla} R_i = (\overline{\nabla}_{\widetilde{A} R_q} R_i \cdot R_p) R_p \otimes (\widetilde{A}^{-1} R_q)^\flat, \quad 1 \leq i \leq n,$$

we arrive at the total covariant derivative for R_i:

$$\boxed{\begin{aligned}
\overline{\nabla} R_i &= \overline{\nabla}^0 R_i + \overline{\nabla}^{\mathcal{A}} R_i, \qquad \text{where} && (4.85)\\
\overline{\nabla}^0 R_i &= R_k \otimes V_{ik}^{a\flat} - \widetilde{A} V_{ik}^s \otimes (\widetilde{A}^{-1} R_k)^\flat + (V_{jk}^s \cdot \widetilde{A} R_i) R_j \otimes (\widetilde{A}^{-1} R_k)^\flat \\
&\quad + (F \cdot \widetilde{A} R_i) \widetilde{A}^{-1} - \widetilde{A} F \otimes (\widetilde{A}^{-1} R_i)^\flat, && (4.86)\\
\overline{\nabla}^{\mathcal{A}} R_i &= \frac{1}{2} \left(-\mathcal{A}_{\widetilde{A}}(R_q, R_p) \cdot R_i - \mathcal{A}_{\widetilde{A}}(R_i, R_p) \cdot R_q + \mathcal{A}_{\widetilde{A}}(R_q, R_i) \cdot R_p \right) R_p \otimes (\widetilde{A}^{-1} R_q)^\flat. && (4.87)
\end{aligned}}$$

As for the S frame, the right-hand sides of (4.83) and (4.85) depend polynomially on R and on the data. This system can thus be solved up to a constant for the vectors R_i via ODE integration along any curve in a simply connected domain. In practice, this system is less expensive to integrate than (4.78) since the R frame can be locally parameterized with $n(n-1)/2$ scalar functions (such as the Euler angles) whereas the S frame requires n^2 scalar functions. In fact, for stability purposes, one may pick slightly more than $n(n-1)/2$ scalar functions, since for certain space dimensions, there may be no non-singular charts on $SO_n(\mathbb{R})$ with exactly $n(n-1)/2$ parameters, in which case reconstructions become unstable as the reconstructed function approaches the singularity. This is not the case in two dimensions, but it is the case in three dimensions for instance (see the thorough discussion in Stuelpnagel (1964) on this matter), where the seemingly optimal parameterization is the four-dimensional *quaternion* parameterization, i.e. one more scalar parameter than the dimension of $SO_3(\mathbb{R})$.

Local stability

We now state the local stability of the ODE-based local reconstruction procedures for the S and R frames as well as $|\gamma|^{\frac{1}{n}}$, under knowledge of the anisotropy. All these results are straightforward applications of Gronwall's lemma. It is important to note that when the anisotropy is not constant, the Lipschitz continuity of the right-hand-sides of (4.78) and (4.85) with respect to the components of S and R respectively, will require that the tensor $\mathcal{A}_{\widetilde{A}}$ defined in (4.70) has bounded component, that is, that the anisotropy $\widetilde{A}$ has bounded derivatives.

Proposition 4.3.6 (Local stability for the frame reconstruction)**.** *Consider two tensors $\gamma, \gamma' \in \Sigma(X)$ such that their rescaled anisotropies $\widetilde{A}$ and $\widetilde{A}'$ are known and have their components in $W^{1,\infty}(X)$. Let $\Omega \subset X$ such that the positivity (4.52) holds for two sets of conductivity solutions $(u_1, \cdots, u_n)$ and $(u_1', \cdots, u_n')$ with respective conductivities γ and γ', call their corresponding data sets $\{H_{ij}\}$ and $\{H_{ij}'\}$ and assume that their components are in $W^{1,\infty}(X)$. Then the corresponding frames S and S' satisfy the following stability estimate for some positive constants C, C'*

$$|S(x) - S'(x)| \le C|S(y) - S'(y)| + C' \left(\|\widetilde{A} - \widetilde{A}'\|_{W^{1,\infty}(X)} + \|H - H'\|_{W^{1,\infty}(\Omega)} \right), \quad x, y \in \Omega, \tag{4.88}$$

where the norm $|\cdot|$ above means $|S|^2 := \sum_{i=1}^n S_i \cdot S_i$. Further, if R and R' are built from S and S' using the same orthonormalization procedure, satisfying the stability property, then we also have the stability estimate

$$|R(x) - R'(x)| \le C''|R(y) - R'(y)| + C''' \left(\|\widetilde{A} - \widetilde{A}'\|_{W^{1,\infty}(X)} + \|H - H'\|_{W^{1,\infty}(\Omega)} \right), \quad x, y \in \Omega, \tag{4.89}$$

for some positive constants C'', C'''. Moreover, these constants may be made independent of Ω.

Proof. Without loss of generality, assume Ω convex. For $x, y \in \Omega$ let $c_{x,y} : [0,1] \to \Omega$ be the curve $c_{x,y}(t) = tx + (1-t)y$ for $t \in [0,1]$. Let us denote

$$M := \max_{1 \le i,j \le n} \left(\|H_{ij}\|_{L^\infty(\Omega)}, \|H'_{ij}\|_{L^\infty(\Omega)} \right). \tag{4.90}$$

The total covariant derivative systems (4.78), written on the curve $c_{x,y}$ (drop the indices $_{x,y}$ for clarity) may be written as, for $1 \le i \le n$,

$$\frac{d}{dt} S_i(c(t)) = \mathcal{F}_i(S(c(t)), H(c(t)), \widetilde{A}(c(t))), \quad \text{and} \quad \frac{d}{dt} S'_i(c(t)) = \mathcal{F}_i(S'(c(t)), H'(c(t)), \widetilde{A}'(c(t))),$$

where the functionals $\mathcal{F}_i$ are polynomials of the components of S (with no constant term), the H_{ij}'s and their derivatives, as well as the anisotropy tensor $\widetilde{A}$ and its derivatives. Taking the difference of both equations above, dotting with $S_i(c(t)) - S'_i(c(t))$ and summing over i, we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |S(c(t)) - S'(c(t))|^2 = \sum_{i=1}^n (S_i - S'_i) \cdot (&\mathcal{F}_i(S(c(t)), H(c(t)), \widetilde{A}(c(t))) \\ &- \mathcal{F}_i(S'(c(t)), H'(c(t)), \widetilde{A}'(c(t)))). \end{aligned}$$

Using Cauchy-Schwarz and the triangle inequality, we obtain

$$\frac{d}{dt} |S(c(t)) - S'(c(t))| \le \sum_{i=1}^n |\mathcal{F}_i(S(c(t)), H(c(t)), \widetilde{A}(c(t))) - \mathcal{F}_i(S'(c(t)), H'(c(t)), \widetilde{A}'(c(t)))|.$$

Since S and S' are constrained to live in the bounded set $\{|S(x)|^2 \le nM, x \in \Omega\}$ with M defined in (4.90), the functionals $\mathcal{F}_i$ are Lipschitz in the S variable, say with modulus $L > 0$. Given the form of $\mathcal{F}_i$, we can obtain a positive constant C depending on M and the maximum $W^{1,\infty}$-norms of $H, H', \widetilde{A}, \widetilde{A}'$ (in particular, C is independent on Ω) such that

$$|\mathcal{F}_i(S,H,\widetilde{A}) - \mathcal{F}_i(S',H',\widetilde{A}')| \le L|S-S'| + C_2\left(\|H-H'\|_{W^{1,\infty}(X)} + \|\widetilde{A}-\widetilde{A}'\|_{W^{1,\infty}(X)}\right). \tag{4.91}$$

Thus, defining for $t \in [0,1]$ the function $\phi(t) := |S(c(t)) - S'(c(t))|$, we see that the function $\phi(t)$ satisfies the estimate

$$\frac{d}{dt}\phi(t) \le L\phi(t) + C\left(\|H-H'\|_{W^{1,\infty}(X)} + \|\widetilde{A}-\widetilde{A}'\|_{W^{1,\infty}(X)}\right), \quad t \in [0,1].$$

As a straightforward application of Gronwall's lemma, we obtain

$$\phi(1) \le \left(\phi(0) + C\left(\|H-H'\|_{W^{1,\infty}(X)} + \|\widetilde{A}-\widetilde{A}'\|_{W^{1,\infty}(X)}\right)\right)e^{L\Delta(\Omega)}. \tag{4.92}$$

With $\phi(1) = |S(x) - S'(x)|$ and $\phi(0) = |S(y) - S'(y)|$, we obtain (4.88), where the constants in the right-hand side of (4.92) do not depend on x or y. They can be further made independent of Ω by bounding $\Delta(\Omega)$ above by $\Delta(X)$.

The derivation of (4.89) is quite similar. In this case, R and R' are automatically bounded with norms equal to n and, in order to obtain a bound similar to (4.91), we also requires the fact the orthonormalization procedure satisfies the stability property.

This conclude the proof of the proposition. □

On to the local stability for the reconstruction of $\log|\gamma|^{\frac{1}{n}}$, we have

Proposition 4.3.7 (Local stability for $\log|\gamma|^{\frac{1}{n}}$)**.** *Under the definitions and conditions of Propo-*

sition 4.3.6, we have the following stability estimate on $\log|\gamma|^{\frac{1}{n}}$

$$\|\log|\gamma|^{\frac{1}{n}} - \log|\gamma'|^{\frac{1}{n}}\|_{W^{1,\infty}(\Omega)} \le \varepsilon_0 + C\left(\|H-H'\|_{W^{1,\infty}(X)} + \|\widetilde{A}-\widetilde{A}'\|_{W^{1,\infty}(X)}\right),$$

where the constant C *does not depend on* Ω *and* ε_0 *is the error committed at some* $x_0 \in \Omega$.

Proof. Pick $x_0 \in \Omega$. Taking difference of (4.55) for γ and γ', we have, at every $x \in \Omega$,

$$\nabla(\log|\gamma|^{\frac{1}{n}} - \log|\gamma|^{\frac{1}{n}}) = \frac{1}{2}\nabla(\log|H| - \log|H'|) + (\nabla H^{jl}\cdot \widetilde{A}S_l)\widetilde{A}^{-1}S_j - (\nabla H'^{jl}\cdot \widetilde{A}'S_l')\widetilde{A}'^{-1}S_j'.$$

Using the trick

$$a_1\cdots a_n - b_1\cdots b_n = \sum_{i=1}^n b_1\cdots b_{i-1}(a_i-b_i)a_{i+1}\cdots a_n,$$

the last difference may be written into five terms involving differences of $H-H'$, $\widetilde{A}-\widetilde{A}'$, and $S_i - S_i'$. In order to bound the latter differences, we use the result from Proposition 4.3.6, i.e. the estimates (4.88) with $y = x_0$. We arrive at

$$\|\nabla(\log|\gamma|^{\frac{1}{n}} - \log|\gamma|^{\frac{1}{n}})\|_{L^\infty(\Omega)} \le e_0 + C\left(\|H-H'\|_{W^{1,\infty}(X)} + \|\widetilde{A}-\widetilde{A}'\|_{W^{1,\infty}(X)}\right), \qquad (4.93)$$

with C some positive constant independent of Ω. We conclude by applying the estimate

$$|f(x)| \le |f(x_0)| + \Delta(X)\|\nabla f\|_{L^\infty(\Omega)}, \quad f \in W^{1,\infty}(\Omega),$$

to $f \equiv \log|\gamma|^{\frac{1}{n}} - \log|\gamma|^{\frac{1}{n}}$, combining it with estimate (4.93). □

In particular, the reconstruction of the determinant of the conductivity is a well-posed problem in $W^{1,\infty}$ with respect to the measurements.

4.3.2 Reconstruction of the anisotropy with additional functionals

As in the two-dimensional case, we will see that more functionals will provide algebraic equations, a certain number of whom may allow us to reconstruct the anisotropy matrix $\tilde{\gamma}$.

Let us start with n solutions $(u_1, \cdots, u_n)$ whose gradients form a basis over some $\Omega \subset X$, more specifically, they satisfy the positivity condition (4.2). We will call $(\nabla u_1, \cdots, \nabla u_n)$ the *support basis*. Let us add an additional solution $v \equiv u_{n+1}$ and consider the mutual power densities of these $n+1$ solutions $\{H_{ij}\}_{1 \le i,j \le n+1}$. To keep notation consistent, let us denote $S_v := A\nabla v$.

By linear dependence of $n+1$ vectors, lemma A.3.1 gives an expression for $n+1$ functions $\mu_1, \cdots, \mu_n, \mu$ such that

$$\sum_{i=1}^{n} \mu_i S_i + \mu S_v = 0, \tag{4.94}$$

where, by virtue of assumption (4.2) and expression (A.12), the function μ does not depend on the choice of u, and takes the expression

$$\mu = \det\{H_{ij}\}_{1 \le i,j \le n} = \det \gamma (\det(\nabla u_1, \cdots, \nabla u_n))^2,$$

thus it never vanishes over Ω. In particular, the following expression is well-defined over Ω

$$S_v = -\mu^{-1} \sum_{i=1}^{n} \mu_i S_i, \quad \text{i.e.} \quad \nabla v = -\mu^{-1} \sum_{i=1}^{n} \mu_i \nabla u_i. \tag{4.95}$$

Using the PDE (4.53), we now compute

$$0 = \nabla \cdot (\mu_i A S_i + \mu A S_v) = \nabla \mu_i \cdot A S_i + \mu_i \overbrace{\nabla \cdot (A S_i)}^{=0} + \nabla \mu \cdot A S_v + \mu \overbrace{\nabla \cdot (A S_v)}^{=0}$$
$$= \nabla \mu_i \cdot A S_i + \nabla \mu \cdot A S_v,$$

which after dividing by $|A|^{\frac{1}{n}}$, yields

$$\nabla \mu_i \cdot \widetilde{A} S_i + \nabla \mu \cdot \widetilde{A} S_v = 0. \tag{4.96}$$

Similarly, using (4.54), we have

$$0 = d(\mu_i A^{-1} S_i + \mu A^{-1} S_v)^\flat = d\mu_i \wedge A^{-1} S_i + \mu_i \overbrace{d(A^{-1} S_i)^\flat}^{=0} + d\mu \wedge (A^{-1} S_v)^\flat + \mu \overbrace{d(A^{-1} S_v)^\flat}^{=0}$$
$$= d\mu_i \wedge (A^{-1} S_i)^\flat + d\mu \wedge (A^{-1} S_v)^\flat$$

which after multiplying by $|A|^{\frac{1}{n}}$, yields

$$d\mu_i \wedge (\widetilde{A}^{-1} S_i)^\flat + d\mu \wedge (\widetilde{A}^{-1} S_v)^\flat = 0. \tag{4.97}$$

Plugging expression (4.95) into (4.96)-(4.97), and using the following relation

$$d\mu_i - \mu_i \mu^{-1} \, d\mu = \mu \, d \left(\frac{\mu_i}{\mu} \right),$$

we arrive at the following set of equations

$$\boxed{\sum_{i=1}^{n} Z_i \cdot \widetilde{A} S_i = 0, \qquad \sum_{i=1}^{n} Z_i^\flat \wedge (\widetilde{A}^{-1} S_i)^\flat = 0, \qquad \text{where} \qquad Z_i := \nabla \frac{\mu_i}{\mu}.} \tag{4.98}$$

The equation involving exterior (wedge) products yields $n(n-1)/2$ scalar relations, written by applying this two-form to vector fields $\widetilde{A}S_p, \widetilde{A}S_q$ for $1 \le p < q \le n$:

$$H_{iq}Z_i \cdot \widetilde{A}S_p - H_{ip}Z_i \cdot \widetilde{A}S_q = 0, \quad 1 \le p < q \le n.$$

Put in other terms and defining $Z := [Z_1|\dots|Z_n]$ and $S := [S_1|\dots|S_n]$, these relations express the facts that *the matrix* $Z^T\widetilde{A}S$ *is traceless*, and that *the matrix* $HZ^T\widetilde{A}S$ *is symmetric*. Using the inner product $\mathcal{M}_n(\mathbb{R})^2 \ni (A,B) \mapsto \mathrm{tr}\,(AB^T)$, the above facts mean that the matrix $\widetilde{A}S$ is orthogonal to Z, and also to any matrix of the form $ZH\Omega$ with $\Omega \in A_n(\mathbb{R})$, the subspace of $n \times n$ anti-symmetric matrices, a basis of whom may be given by $\Omega_{pq} := \mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_p$ for $1 \le p < q \le n$ and $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the canonical basis of $\mathbb{R}^n$. More precisely, defining the space

$$P_Z := \{Z(\lambda\mathbb{I}_n + H\Omega), \quad (\lambda, \Omega) \in \mathbb{R} \times A_n(\mathbb{R})\}, \tag{4.99}$$

equations (4.98) precisely mean that

$$\widetilde{A}S \perp P_Z.$$

Therefore, the additional solution v allows to locate the matrix $\widetilde{A}S$ in the orthogonal of a certain subspace of $\mathcal{M}_n(\mathbb{R})$. In order to know precisely how many scalar equations we get from this to characterize $\widetilde{A}S$, we establish the following result.

Proposition 4.3.8. *For $M \in M_n(\mathbb{R})$ with rank$(M) = r$, $1 \le r \le n$, we have the following*

$$\dim(MA_n(\mathbb{R})) = r\left(n - \frac{r+1}{2}\right), \quad \textit{where} \quad MA_n(\mathbb{R}) := \{M\Omega, \Omega \in A_n(\mathbb{R})\}. \tag{4.100}$$

Proof. Without loss of generality, we assume that in the canonical basis,

$$M = [M_1|\cdots|M_r|\underbrace{0|\cdots|0}_{n-r}], \quad M_p := M\mathbf{e}_p, \quad 1 \le p \le r,$$

with $M_1, \cdots, M_r$ a family of r linearly independent vectors and $M\mathbf{e}_p = 0$ for $r < p \le n$. A basis for $A_n(\mathbb{R})$ is given by

$$\Omega_{pq} := \mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_p, \quad 1 \le p < q \le n.$$

Considering products $M\Omega_{pq}$, we have

$$M\Omega_{pq} = M\mathbf{e}_p \otimes \mathbf{e}_q - M\mathbf{e}_q \otimes \mathbf{e}_p = \begin{cases} M_p \otimes \mathbf{e}_q - M_q \otimes \mathbf{e}_p & \text{if } 1 \le p < q \le r, \\ M_p \otimes \mathbf{e}_q & \text{if } 1 \le p \le r < q \le n, \\ 0 & \text{if } r < p < q \le n. \end{cases}$$

We now show that the remaining non-zero matrices are linearly independent, so that their cardinal gives the dimension of $MA_n(\mathbb{R})$. Consider a linear combination such that

$$M' := \sum_{1 \le p < q \le r} \lambda_{pq}(M_p \otimes \mathbf{e}_q - M_q \otimes \mathbf{e}_p) + \sum_{1 \le p \le r < q \le n} \lambda_{pq} M_p \otimes \mathbf{e}_q = 0$$

for some coefficients $\lambda_{pq}, 1 \le p \le r, p < q \le n$. This implies $M'\mathbf{e}_i = 0$ for $1 \le i \le n$, where

$$\begin{aligned} M'\mathbf{e}_i &= \sum_{1 \le p < q \le r} \lambda_{pq}(\delta_{qi} M_p - \delta_{pi} M_q) + \sum_{1 \le p \le r < q \le n} \lambda_{pq} \delta_{qi} M_p \\ &= \begin{cases} \sum_{1 \le p < i} \lambda_{pi} M_p - \sum_{i < q \le r} \lambda_{iq} M_q & \text{if } i \le r, \\ \sum_{1 \le p \le r} \lambda_{pi} M_p & \text{if } i > r. \end{cases} \end{aligned}$$

Since $M'\mathbf{e}_i$ must be zero and $(M_1, \cdots, M_r)$ is free, the above equations imply that all coefficients λ_{pq} must be zero. As a result, the cardinality of this family is given by

$$\#\{(p,q) : 1 \leq p \leq r, p < q \leq n\} = n - 1 + \cdots + n - r = nr - \frac{r(r+1)}{2} = r\left(n - \frac{r+1}{2}\right).$$

and hence the result. □

Remark 4.3.9. *Note that the quantity $r\left(n - \frac{r+1}{2}\right)$ is maximal (i.e. equals $n(n-1)/2$) for both values $rank(M) = n$ and $rank(M) = n - 1$.*

Corollary 4.3.10. *For $Z \in M_n(\mathbb{R})$, the space P_Z defined in (4.99) is such that*

$$\dim P_Z = r\left(n - \frac{r+1}{2}\right) + 1, \quad r = rank(Z). \tag{4.101}$$

Proof. We first show that Z does not belong to $ZHA_n(\mathbb{R})$, which explains the $+1$ term in (4.101). The quickest way to see that is to notice that ZT^{-1} is orthogonal to $ZHA_n(\mathbb{R})T^{-1}$ for any matrix T such that $T^{-1}T^{-T} = H$ (these matrices are the orthonormalizing matrices of S). Indeed, for any $\Omega \in A_n(\mathbb{R})$, we compute, using $H^T = H$,

$$\text{tr}\,(ZH\Omega T^{-1}(ZT^{-1})^T) = \text{tr}\,(ZH\Omega \underbrace{T^{-1}T^{-T}}_{H} Z^T) = \text{tr}\,(ZH\Omega(ZH)^T) = 0,$$

since $ZH\Omega(ZH)^T$ is anti-symmetric, thus the claim is proved. It remains to show that $\dim ZHA_n(\mathbb{R}) = r\left(n - \frac{r+1}{2}\right)$, which follows from Proposition 4.3.8 and the fact that $\text{rank}(ZH) = \text{rank}(Z)$ over sets where H is assumed uniformly invertible. □

Reconstruction of $\widetilde{A}$:

From the previous results, we deduce that, once a support basis (giving rise to a S frame) is given, each additional solution v provides us with a matrix Z, whose subspace $P_Z \subset M_n(\mathbb{R})$ defined in (4.99), imposes *pointwise algebraic constraints* on the matrix $\widetilde{A}S$. Furthermore, the matrix Z can not only be built from the power densities via the formula $Z = [Z_1|\cdots|Z_n]$ where using (A.12), we have

$$Z_i = \nabla\frac{\mu_i}{\mu} = (-1)^{i+n+1}\nabla\frac{\det\{H_{pq}\}_{1\le p\le n, 1\le q\le n+1, q\ne i}}{\det\{H_{pq}\}_{1\le p,q\le n}}, \quad 1 \le i \le n, \tag{4.102}$$

but the vector fields Z_i also have the more “physical” expression

$$Z_i = \nabla\frac{\det(\nabla u_1, \cdots, \overbrace{\nabla v}^{i}, \cdots, \nabla u_n)}{\det(\nabla u_1, \cdots, \nabla u_n)}, \quad 1 \le i \le n. \tag{4.103}$$

The question now is whether we can combine a few such solutions $(v_1, \cdots, v_l)$ for some $l \ge 1$ such that the corresponding matrices $Z_{v_1} \equiv Z_{(1)}, \cdots, Z_{v_l} \equiv Z_{(l)}$ are such that

$$\dim\sum_{k=1}^{l} P_{Z_{(k)}}(x) = n^2 - 1, \quad x \in \Omega, \tag{4.104}$$

i.e. the space $\sum_{k=1}^{l} P_{Z_{(k)}}(x)$ is a hyperplane of $M_n(\mathbb{R})$ for each $x \in \Omega$, so that the direction of $\widetilde{A}S$, since it must be orthogonal to this hyperplane, is determined. That enough solutions exist is not straightforward at first glance, but the example of constant γ treated in Section 5.2 will give a case where this actually works.

If (4.104) is satisfied at a given $x \in \Omega$, the direction of $\widetilde{A}S$ is the normal to this hyperplane of $M_n(\mathbb{R})$, thus $\widetilde{A}S$ is known up to a constant. This constant is known completely in odd dimensions and up to sign in even dimensions, this is due to the fact that $\det(\widetilde{A}S) = \det S = \sqrt{\det H}$. Once

$\widetilde{A}S =: B$ is known, one may write $S = RT^{-T}$, with T function of H (e.g. $T = H^{-1/2}$), and therefore we have $\widetilde{A}R = BT^T$. As a result, we obtain

$$\boxed{\widetilde{A}^2 = \widetilde{A}RR^T\widetilde{A}^T = BT^TTB^T = BH^{-1}B^T,} \tag{4.105}$$

which is an algebraic reconstruction procedure for $\widetilde{A}^2 = \tilde{\gamma}$. Now, $\widetilde{A}$ may also be deduced from $\widetilde{A}^2$ by computing the fully positive squareroot of $\widetilde{A}^2$, which can be computed explicitly.

Remark 4.3.11. *Once $\widetilde{A}^2$ is reconstructed, one may actually gain access to algebraic reconstruction of S or R, since we could reconstruct $\widetilde{A}$ from $\widetilde{A}^2$, and then compute $S = \widetilde{A}^{-1}B$ or $R = \widetilde{A}^{-1}BT^T$. Making use of this observation for alternative reconstruction approaches will appear in future work (Monard and Bal, 2012b).*

Condition for reconstructibility:

In order to reconstruct $\widetilde{A}^2$ over the whole set Ω above, we need to ensure the condition (4.104) at every point of Ω. A way of writing this in terms of continuous functionals is as follows: for $1 \le k \le l$, let $r_k = \text{rank}(Z_{(k)})$ so that, by Corollary 4.3.10, $d_k := \dim P_{Z_{(k)}} = r_k\left(n - \frac{r_k+1}{2}\right) + 1$ and let $\{M_{k,p}\}_{1\le p\le d_k}$ be a basis of $P_{Z_{(k)}}$, obtained for instance by picking $Z_{(k)}$ together with a subset of maximal rank (i.e. $d_k - 1$) of the family

$$Z_{(k)}H\Omega_{pq}, \quad \Omega_{pq} = \mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_p, \quad 1 \le p < q \le n,$$

and consider the collection

$$\mathcal{M} := \{M_{k,p},\ 1 \le k \le l,\ 1 \le p \le d_k\} \equiv \{M_i,\ 1 \le i \le \#\mathcal{M}\}. \tag{4.106}$$

Assuming at the very least that

$$\#\mathcal{M} = \sum_{k=1}^{K} \dim P_{Z_{(k)}} \geq n^2 - 1,$$

condition (4.104) is ensured throughout Ω if at every point, (i) a (n^2-1)-subfamily of $\mathcal{M}$ is linearly independent, so that it forms a hyperplane, and (ii) the normal to that hyperplane is non-singular. Condition (ii) should always be ensured since the normal, when it exists, must be proportional to $\widetilde{A}S$, whose determinant is known to be $|H|^{\frac{1}{2}} \neq 0$. However, we will still encode it in the compatibility condition. Before we can do so, let us quickly recall the generalization of the *cross-product* whose purpose is to complete a basis.

A generalization of the cross-product: Let us consider a N-dimensional inner product space $(\mathcal{V}, \langle , \rangle)$ with an orthonormal (canonical) basis $(\mathbf{e}_1, \cdots, \mathbf{e}_N)$. Given a linearly independent family of $N-1$ vectors $(V_1, \cdots, V_{N-1})$ in $\mathcal{V}$, a (non-normalized) *normal* to the hyperplane spanned by $(V_1, \cdots, V_{N-1})$ is given by computing the formal $\mathcal{V}$-valued determinant

$$\mathcal{N}(V_1, \cdots, V_{N-1}) := \begin{vmatrix} \langle V_1, \mathbf{e}_1 \rangle & \cdots & \langle V_1, \mathbf{e}_N \rangle \\ \vdots & \ddots & \vdots \\ \langle V_{N-1}, \mathbf{e}_1 \rangle & \cdots & \langle V_{N-1}, \mathbf{e}_N \rangle \\ \mathbf{e}_1 & \cdots & \mathbf{e}_N \end{vmatrix}, \tag{4.107}$$

to be expanded along the last row. The function $\mathcal{N}$ can be easily seen to be $N-1$-linear and alternating. Moreover, $\mathcal{N}$ satisfies the orthogonality property

$$\langle \mathcal{N}(V_1, \cdots, V_{N-1}), V_j \rangle = 0, \quad 1 \leq j \leq N-1,$$

as such dotproducts take the form of determinants with identical j-th and N-th rows. Furthermore, one can also show that

$$\langle \mathcal{N}, \mathcal{N} \rangle = \det\{\langle V_i, V_j \rangle\}_{1\le i,j\le N-1}, \tag{4.108}$$

which precisely states that the computed vector is non-zero as soon as $(V_1, \cdots, V_{N-1})$ are linearly independent.

Local solvability condition and reconstruction algorithm: Back to our application, we would like to apply the cross-product (4.107) to a subfamily of $\mathcal{M}$ defined in (4.106) that spans a hyperplane. For the definition of the cross-product, one may pick a relabelling of the canonical basis $\{E_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j\}_{1\le i,j\le n}$ of $\mathcal{M}_n(\mathbb{R})$, where $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ is the canonical basis of $\mathbb{R}^n$. Since we do not know *a priori* which subfamily of $\mathcal{M}$ is the one that spans a hyperplane (and this family may depend on x), the conditions for having at least a spanned hyperplane among the elements of $\mathcal{M}$ may be formulated as a sum of non-negative functions for each n^2-1-subfamily of $\mathcal{M}$. Each function is zero whenever the corresponding subfamily is either linearly dependent or if its orthogonal is a singular matrix (again, this last condition cannot happen when the data are in the range of the measurement operator, however, we will se that it comes up rather naturally in the formulation). We thus formulate the condition as

$$\begin{aligned} &\inf_{x\in\Omega} \mathcal{F}(x) \ge C_M > 0, \qquad \text{where} \\ &\qquad \mathcal{F}(x) := \sum_{1\le i_1<\cdots<i_{n^2-1}\le \#\mathcal{M}} \left(\det\left(\mathcal{N}_{(i_1,\cdots,i_{n^2-1})} H^{-1} \mathcal{N}^T_{(i_1,\cdots,i_{n^2-1})}\right)\right)^{\frac{1}{n}}, \\ &\mathcal{N}_{(i_1,\cdots,i_{n^2-1})} := \mathcal{N}(M_{i_1}, \cdots, M_{i_{n^2-1}}). \end{aligned} \tag{4.109}$$

On to the reconstruction formula, we deduce that for a given n^2-1-subfamily of $\mathcal{M}$ denoted by $(M_{i_1}, \cdots, M_{i_{n^2-1}})$, the normal $\mathcal{N}_{(i_1,\cdots,i_{n^2-1})}$ is either zero or proportional to $\widetilde{A}S$, and in the latter case, the reconstruction formula (4.105) translates into

$$\mathcal{N}_{(i_1,\cdots,i_{n^2-1})} H^{-1} \mathcal{N}^T_{(i_1,\cdots,i_{n^2-1})} = \left(\det\left(\mathcal{N}_{(i_1,\cdots,i_{n^2-1})} H^{-1} \mathcal{N}^T_{(i_1,\cdots,i_{n^2-1})}\right)\right)^{\frac{1}{n}} \widetilde{A}^2.$$

Summing the last equality over all possible subfamilies of $\mathcal{M}$, we deduce the following relation

$$\boxed{\widetilde{A}^2 = \frac{1}{\mathcal{F}(x)} \sum_{1 \le i_1 < \cdots < i_{n^2-1} \le \#\mathcal{M}} \mathcal{N}_{(i_1,\cdots,i_{n^2-1})} H^{-1} \mathcal{N}^T_{(i_1,\cdots,i_{n^2-1})}, \quad x \in \Omega,} \tag{4.110}$$

with $\mathcal{F}(x)$ defined in (4.109). Condition (4.109) precisely ensures that the function $\mathcal{F}(x)$ is uniformly bounded away from zero by $C_M > 0$, so that one may stably divide by it, and thus obtain a reconstruction formula for $\widetilde{A}^2$ that does not require to know precisely which subfamily of $\mathcal{M}$ spans a hyperplane. We also claim that, for future purposes, the reconstruction formula (4.110) should handle noise in the measurements relatively well, as it averages over several potential reconstructions formulas, some of which are supposed to give the same output when measurements are not corrupted.

The local uniqueness and stability result for such a reconstruction formula may be formulated as follows:

Theorem 4.3.12 (Uniqueness and local stability of anisotropy reconstruction)**.** *For $\Omega \subset X$ where $(\nabla u_1, \cdots, \nabla u_n)$ forms a frame and additional solutions $(v_1, \cdots, v_l)$ are such that the hyperplane condition (4.109) is satisfied, then the anisotropy $\tilde{\gamma} = \widetilde{A}^2$ is* uniquely *determined from the power densities via formula (4.110).*

Furthermore, for H, H' two sets of power densities (with components in $W^{1,\infty}(\Omega)$) coming from families of solutions $(u_1, \ldots, u_n, v_1, \ldots, v_l)$ and $(u'_1, \ldots, u'_n, v'_1, \ldots, v'_l)$ that jointly satisfy *the aforementioned conditions over $\Omega \subset X$, we have the* stability *estimate*

$$\|\tilde{\gamma} - \tilde{\gamma}'\|_{L^\infty(\Omega)} \le C\|H - H'\|_{W^{1,\infty}(\Omega)}. \tag{4.111}$$

Proof. Uniqueness is clear, since the inversion is explicit and without ambiguity. Stability is also straightforward due to the following facts:

- The function $\mathcal{F}$ defined in (4.109) and the numerator in the right-hand side of (4.110) are continuous functions of the power densities and their derivatives.
- $\mathcal{F}$ is bounded away from zero by assumption.

Hence the result. $\square$

Chapter 5

Conditions and algorithms for global reconstruction

5.1 Global reconstruction schemes

So far, we have derived reconstruction algorithms that held locally, i.e. over some open, simply connected subset $\Omega \subset X$, under the following assumtions.

- In the case of known anisotropy, the local reconstruction of $|\gamma|$ up to a constant requires to have n solutions with linearly independent gradients. This statement was summarized as a positivity condition (4.2), i.e.

$$\inf_{x \in \Omega} \det(\nabla u_1, \cdots, \nabla u_n) \geq c_0 > 0. \tag{5.1}$$

- Additionally, if the anisotropy is not known, on top of condition (5.1), we need a certain number of additional solutions l, each of which generates an $n \times n$ matrix $Z_{(k)}$ (depending on the data) that in turn characterizes a linear space $P_{Z_{(k)}} = \mathbb{R} Z_{(k)} + Z_{(k)} HA_n(\mathbb{R})$, where

H is the data matrix coming for the n first solutions. The condition for reconstructibility of the anisotropy $\tilde{\gamma}$ is then given by

$$\dim \sum_{k=1}^{l} P_{Z_{(k)}}(x) = n^2 - 1, \qquad x \in \Omega, \tag{5.2}$$

so that, in a pointwise fashion, $\tilde{\gamma} = M H^{-1} M^T$ where M can be explicitely computed using the conditions

$$M(x) \perp \sum_{k=1}^{l} P_{Z_{(k)}} \quad \text{and} \quad \det M = \sqrt{\det H}.$$

In two dimensions of space, condition (5.2) reduced to

$$\nabla \log \det(\nabla u_1, \nabla u_2) \neq \nabla \log \det(\nabla u_3, \nabla u_4), \quad x \in \Omega,$$

for some solutions (u_1, u_2, u_3, u_4).

Based on these local statements, and knowing that we cannot always expect statements such as (5.1) to hold *globally* over X (see section 5.2), we may hope to arrive at global reconstruction algorithms by picking a greater family of solutions (i.e. measurements) and an open cover of X such that, on each open set of the cover, some of the solutions (not always the same subfamily) satisfy the conditions (5.1)-(5.2) that allow the recontruction procedure to hold. Then, patching these reconstructions together carefully allows to obtain global reconstructions of conductivity tensors. The two-dimensional case here may come as "the odd case" where one can actually pick systematically exactly 2 boundary values (g_1, g_2) such that their corresponding solutions (u_1, u_2) satisfy $\det(\nabla u_1, \nabla u_2) > 0$ throughout X, thanks to (Alessandrini and Nesi, 2001, Theorem 4) repeated here as Theorem 5.2.6. That this theorem can not be extended to higher dimensions is justified by counterexamples that may be found in Briane et al. (2004) and Laugesen (1996). We now need to make these global admissibility conditions more explicit before anything else.

5.1.1 The admissibility sets $\mathcal{G}_\gamma^m$, $m \ge n$

For a conductivity tensor $\gamma \in \Sigma(X)$ and an integer $m \ge n$, we now define the following set of *m-tuples of admissible boundary conditions* $\mathcal{G}_\gamma^m$ as follows:

Definition 5.1.1 (Admissibility set $\mathcal{G}_\gamma^m,\ m \ge n$)**.** *Let $\gamma \in \Sigma(X)$ be a given conductivity tensor. For $m \ge n$, an m-tuple $\mathbf{g} = (g_1, .., g_m) \in \left(H^{\frac{1}{2}}(\partial X)\right)^m$ belongs to $\mathcal{G}_\gamma^m$ if the following conditions are satisfied:*

1. *The power densities $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j$ belong to $W^{1,\infty}(X)$ for $1 \le i, j \le m$, where u_i solves (4.51) with boundary condition $u_i|_{\partial X} = g_i$.*
2. *There exists a constant $C_H > 0$ such that*

$$\inf_{x\in X} \mathcal{D}_\gamma^m[\mathbf{g}](x) \ge C_H > 0, \qquad \textit{where} \tag{5.3}$$

$$\mathcal{D}_\gamma^m[\mathbf{g}](x) := \sum_{1\le i_1<\cdots<i_n\le m} D_{i_1,\cdots,i_n}(x), \qquad \textit{where} \qquad D_{i_1,\cdots,i_n} := \det\{H_{i_p i_q}\}_{p,q=1}^n. \tag{5.4}$$

The positivity condition (5.3) ensures that, at every point, at least one subfamily of n elements of $(u_1, \cdots, u_m)$, the *support basis*, satisfies $\det(\nabla u_{i_1}, \cdots, \nabla u_{i_n}) \ne 0$, so that the local reconstruction algorithms derived in Section 4.3 may be applied. The condition that H_{ij} belong to $W^{1,\infty}$ allows to make these statements local, in the sense that we have a finite open cover on X such that on each open set, the same support basis may be used.

Proposition 5.1.2 (Existence of open covers with adapted support bases)**.** *Let $m \ge n$ and $(g_1, \cdots, g_m) \in \mathcal{G}_\gamma^m$, and let C_H as in (5.3). Then there exists a finite open cover $\mathcal{O} = \{\Omega_p\}_{1\le p\le N}$*

with $X \subset \bigcup_{p=1}^N \Omega_p$, *an indexing function* $\tau : [1, N] \to [1, m]^n$ *and a constant* $C > 0$ *such that the following condition holds*

$$\min_{1 \le p \le N} \inf_{x \in \Omega_p} \det(\nabla u_{\tau(p)_1}(x), \cdots, \nabla u_{\tau(p)_n}(x)) \ge C > 0, \tag{5.5}$$

where for $1 \le i \le m$, u_i *solves* (4.51) *with boundary condition* $u_i|_{\partial X} = g_i$. *By default, the open sets may be chosen as open balls.*

Proof. Denote by

$$N(n, m) = \#\{(i_1, \cdots, i_n) : \ 1 \le i_1 < \cdots < i_n \le m\}. \tag{5.6}$$

Due to the assumption that all power densities belong to $W^{1,\infty}(X)$, all the functions $\mathcal{D}_{i_1,\cdots,i_n}$ defined in (5.3) are uniformly Lipschitz with constant, say L. Set $\epsilon := \frac{C_H}{2LN(n,m)}$. Since X has compact closure, there exists a finite family $\{x_p \in X, \ 1 \le p \le N\}$ such that X is covered by the balls $\{\Omega_p := B(X_p, \epsilon), \ 1 \le p \le N\}$. Now, for each $1 \le p \le N$, there exists $1 \le i_1(p) < \cdots < i_n(p) \le m$ such that

$$D_{i_1(p),\cdots,i_n(p)}(x_p) \ge \frac{C_H}{N(n, m)}. \tag{5.7}$$

Call $\tau(p) = (i_1(p), \cdots, i_n(p))$. Since $D_{i_1(p),\cdots,i_n(p)}$ is L-Lipschitz, we have, for every $y \in B(x_p, \epsilon)$, that

$$D_{i_1(p),\cdots,i_n(p)}(y) \ge D_{i_1(p),\cdots,i_n(p)}(x_p) - L\epsilon = \frac{C_H}{2LN(n, m)}.$$

By virtue of the relation

$$D_{i_1(p),\cdots,i_n(p)} = (\det\gamma)(\det(\nabla u_{\tau(p)_1},\cdots,\nabla u_{\tau(p)_n}))^2,$$

we deduce that $\det(\nabla u_{\tau(p)_1},\cdots,\nabla u_{\tau(p)_n})$ has constant sign throughout Ω_p, which we assume positive up to flipping two components of $\tau(p)$. Therefore, we deduce that

$$\det(\nabla u_{\tau(p)_1},\cdots,\nabla u_{\tau(p)_n}) \ge [(\det\gamma)^{-1}\mathcal{D}_{i_1(p),\cdots,i_n(p)}]^{\frac{1}{2}} \ge \left[\frac{C_H C_\gamma^{-n}}{2LN(n,m)}\right]^{\frac{1}{2}}, \quad x\in\Omega_p.$$

Since the above estimate is uniform in $1 \le p \le N$, we obtain estimate (5.5) with $C := \left[\frac{C_H C_\gamma^{-n}}{2LN(n,m)}\right]^{\frac{1}{2}} > 0$, and the proposition is proved. $\square$

Remark 5.1.3. *One could use the existence result of Proposition 5.1.2 as the defining property of an admissible set of boundary values, so that one can drop the $W^{1,\infty}$ assumption on the power densities in Definition 5.1.1. However, the $W^{1,\infty}$ assumption is necessary in order to ensure local stability, hence our decision to incorporate it into the very definition of $\mathcal{G}_\gamma^m$.*

For practical purposes, we would like m to be as small as possible. While the purpose of section 5.2 will be to identify for what tensors γ the sets $\mathcal{G}_\gamma^m$ are non-empty, we now explain how to derive a global reconstruction procedure starting from a given $\mathbf{g}\in\mathcal{G}_\gamma^m$.

5.1.2 Patching together local reconstructions of $|\gamma|$ under knowledge of the anisotropy

The first global reconstruction algorithm was first presented in Bal et al. (2012a) for the isotropic case in three dimensions. Given $\mathbf{g} = (g_1,\cdots,g_m)\in\mathcal{G}_\gamma^m$ for some $m\ge n$, with an open cover $\mathcal{O} = \{\Omega_p\}_{p=1}^N$ and an indexing function τ as in Proposition 5.1.2, the main idea was to solve ODE's locally as in Section 4.3 over each open set Ω_p using the power densities associated with the support basis. Going from one open set to its neighbor would require switching support basis which, from the algorithmic standpoint, was a stable process. The main weakness of this algorithm is that it was point-oriented, rather than open set-oriented, so that a lot of calculations would be repeated several times in practice. It also relied on the somewhat cumbersome definition of an integer K and two functions

$$\begin{aligned} &Y : X \ni x \mapsto Y(x) = (y_1(x) = x_0, y_2(x), \dots, y_K(x), y_{K+1}(x) = x) \in X^{K+1}, \\ &\psi : X \times [1, K] \ni (x, i) \mapsto \psi(x, i) \in [1, N], \end{aligned} \tag{5.8}$$

such that for every $x \in X$,

$$[x_0, x] = \cup_{i=1}^K [y_i(x), y_{i+1}(x)] \quad \text{and} \quad [y_i, y_{i+1}] \subset \Omega_{\psi(x,i)}, \quad 1 \le i \le K, \tag{5.9}$$

where x_0 was fixed a priori, so that, for each $x \in X$, the reconstruction of $|\gamma|(X)$ was done by integrating ODE's along the segments $[y_i(x), y_{i+1}(x)]$ consecutively.

While generalizing this algorithm to anisotropic tensors with known anisotropy would be fairly straightforward, we present here a slightly modified version of this algorithm, which we believe is cheaper computationally and simpler to describe. The algorithm goes as follows.

Global ODE based reconstruction algorithm

Let again $\mathbf{g} = (g_1, \cdots, g_m) \in \mathcal{G}_\gamma^m$ for some $m \ge n$, with an open cover $\mathcal{O} = \{\Omega_p\}_{p=1}^N$ and an indexing function τ as in Proposition 5.1.2. We also assume that the open sets are convex[1]. Up to relabelling them, we assume that the open sets Ω_p are numbered so that for any $2 \le p \le N$,

[1]recall that, in Proposition 5.1.2, we can choose the Ω_p's to be balls, by default.

there exists $1 \leq p'(p) < p$ such that

$$\Omega_{p'(p)} \cap \Omega_p \neq \emptyset. \tag{5.10}$$

In the sequel, in order to define $p'(p)$ uniquely, we may assume that $p'(p)$ is the smallest such index. Condition (5.10) somehow means that the open sets $\Omega_{p'(p)}$ and Ω_p are neighbors. Thus, if one defines a graph with N vertices such that vertices p and q are neighbors if $\Omega_p \cap \Omega_q \neq \emptyset$, then the *Cuthil McKee* algorithm generates a relabelling of the vertices such that condition (5.10) will be satisfied[2].

Once the open cover is numbered properly, the main idea is to reconstruct $|\gamma|$ on one open set after another, and the condition (5.10) guarantees that, when looping over p from 1 to N, the reconstruction over Ω_p can use values of $|\gamma|$ and of a frame that have been priorly solved for on the neighboring set $\Omega_{p'(p)}$.

The frame used can either be the S frame or the orthonormal R frame, and the global stability of the reconstruction algorithm, established in Theorem 5.1.4 below, may not depend on the choice of frame so long as the orthonormalization process satisfies the stability condition (4.10).

The algorithm is summarized in Algorithms 1 (using the S frame) and 2 (using the R frame), and it makes use of the curves $c_{x,y}$ defined as follows

$$c_{x,y} : [0,1] \ni t \mapsto c_{x,y}(t) = ty + (1-t)x, \quad (x,y) \in X \times X. \tag{5.11}$$

Assuming that the open sets Ω_p are convex makes sure that for x, y in Ω_p, the whole curve $c_{x,y}$ is included in Ω_p, so that it makes sense to use the same support basis throughout this curve.

[2]Note that, although it is of no use here, the main purpose of the Cuthil McKee algorithm is to reduce the bandwidth of the adjacency matrix of the graph previously mentioned, that is to say, to minimize the quantity $|p-q|$ over all couples of vertices p and q that are neighbors.

Introducing the necessary notation, for $1 \le p \le N$, we denote $S^{(p)} := [S_{\tau(p)_1}|\cdots|S_{\tau(p)_n}]$, as well as the corresponding data matrix $H^{(p)} = S^{(p)T} S^{(p)}$. Here and below, the superscript $^{-T}$ will denote the matrix inverse transpose. As we need to transfer values from a set $\Omega_{p'}$ to Ω_p by switching support basis, it is also useful to define over $\Omega_{p'} \cap \Omega_p$

$$H^{p',p} = \{H_{\tau(p')_i \tau(p)_j}\}_{i,j=1}^n = S^{(p')T} S^{(p)}.$$

Algorithm 1 S frame based reconstruction procedure

Pick $x_1 \in \Omega_1$ and let $|\gamma|$ and $S^{(1)}$ be given at x_1.
for $x \in \Omega_1$, $x \neq x_1$ **do** {Reconstruction over Ω_1}
  Reconstruct $S^{(1)}(x)$ by integrating the system (4.78) over the curve $c_{x_1,x}$ as defined in (5.11), with initial condition at x_1.
  Reconstruct $\log|\gamma|(y)$ for $y \in (x_1, x]$ by integrating (4.55) over $c_{x_1,x}$ with initial condition at x_1.
end for
for $p = 2$ to N **do** {Reconstruction over Ω_p}
  Pick $x_p \in \Omega_p \cap \Omega_{p'(p)}$ and compute $S^{(p)}(x_p)$ from $S^{p'(p)}(x_p)$ using the relation

$$S^{(p)}(x_p) = S^{(p')-T}(x_p) H^{p',p}(x_p), \quad p' = p'(p). \tag{5.12}$$

  for $x \in \Omega_p - \cup_{j=1}^{p-1} \Omega_j$ **do**
    Reconstruct $S^{(p)}(x)$ by integrating the system (4.78) over the curve $c_{x_p,x}$ as defined in (5.11), with initial condition at x_p.
    Reconstruct $\log|\gamma|(y)$ for $y \in (x_p, x]$ by integrating (4.55) over $c_{x_p,x}$, with initial condition at x_p.
  end for
end for

When using orthonormalized frames, we define by $T^{(p)}$ the orthonormalizing matrix obtained from $H^{(p)}$ and over Ω_p, it then makes sense to define $R^{(p)} = S^{(p)} T^{(p)T}$.

The global stability of such algorithms holds in the sense of the following theorem.

Theorem 5.1.4 (Global uniqueness and stability in n dimensions)**.** *Let $X \subset \mathbb{R}^n$ be an open*

Algorithm 2 R frame based reconstruction procedure

Pick $x_1 \in \Omega_1$ and let $|\gamma|$ and $S^{(1)}$ be given at x_1.
for $x \in \Omega_1$, $x \neq x_1$ **do** {Reconstruction over Ω_1}
  Reconstruct $R^{(1)}(x)$ by integrating the system (4.85) over the curve $c_{x_1,x}$ as defined in (5.11), with initial condition at x_1.
  Reconstruct $\log|\gamma|(y)$ for $y \in (x_1, x]$ by integrating (4.81) over $c_{x_1,x}$ with initial condition at x_1.
end for
for $p = 2$ to N **do** {Reconstruction over Ω_p}
  Pick $x_p \in \Omega_p \cap \Omega_{p'(p)}$ and compute $R^{(p)}(x_p)$ from $R^{p'(p)}(x_p)$ using the relation

$$R^{(p)}(x_p) = R^{(p')}(x_p)T^{(p')}(x_p)H^{p',p}(x_p)T^{(p)T}(x_p), \quad p' = p'(p). \tag{5.13}$$

  for $x \in \Omega_p - \cup_{j=1}^{p-1}\Omega_j$ **do**
    Reconstruct $R^{(p)}(x)$ by integrating the system (4.85) over the curve $c_{x_p,x}$ as defined in (5.11), with initial condition at x_p.
    Reconstruct $\log|\gamma|(y)$ for $y \in (x_p, x]$ by integrating (4.81) over $c_{x_p,x}$, with initial condition at x_p.
  end for
end for

---

convex bounded domain, and let two conductivity tensors γ, γ' be of the respective forms $\gamma = \sigma\tilde{\gamma}$ and $\gamma' = \sigma'\tilde{\gamma}'$ for some σ, σ' positive scalar functions such that there exists $\mathbf{g} \in \mathcal{G}_\gamma^m \cap \mathcal{G}_{\gamma'}^m \neq \emptyset$ for some $m \geq n$ with the same open cover. The known anisotropy tensors $\tilde{\gamma}, \tilde{\gamma}'$ have components in $W^{1,\infty}(X)$. Pick $x_1 \in \overline{\Omega_1} \subset \overline{X}$ and let σ, σ', $S^{(1)}$ and $S'^{(1)}$ be given at x_1. Then we have the following stability estimate on σ and σ'

$$\|\log\sigma - \log\sigma'\|_{W^{1,\infty}(X)} \leq C\big(\varepsilon + \|H - H'\|_{W^{1,\infty}(X)} + \|\widetilde{A} - \widetilde{A}'\|_{W^{1,\infty}(X)}\big), \tag{5.14}$$

where ε is the error at the initial point x_1

$$\varepsilon = |\log\sigma(x_1) - \log\sigma'(x_1)| + |S^{(1)}(x_1) - S'^{(1)}(x_1)|,$$

and where $\widetilde{A} = \tilde{\gamma}^{\frac{1}{2}}$ and $\widetilde{A}' = \tilde{\gamma}'^{\frac{1}{2}}$.

The proof of global stability mainly relies on two ingredients:

- the local stability over a fixed open set Ω_p, as established in Propositions 4.3.6 and 4.3.7,
- a stability statement as one passes from $\Omega_{p'}$ Ω_p, which expresses the fact that equalities (5.12) and (5.13) are stable processes with respect to the data H_{ij}.

Proof. Let us first adjust the notation of Proposition 4.3.6 to the present global setting. Over Ω_p, $1 \leq p \leq N$, the stability estimates (4.88) and (4.89) are written using the restricted set of measurements $H^{(p)}$. Here, if we write

$$\|\Delta H\| := \|H - H'\|_{W^{1,\infty}(X)} := \max_{1\leq i,j\leq m} \|H_{ij} - H'_{ij}\|_{W^{1,\infty}(X)} \geq \max_{1\leq p\leq N} \|H^{(p)} - H'^{(p)}\|_{W^{1,\infty}(X)},$$

as well as $\|\Delta\widetilde{A}\| := \|\widetilde{A} - \widetilde{A}'\|_{W^{1,\infty}(X)}$, then the stability estimates (4.88) and (4.89) may be rewritten as

$$|S^{(p)}(x) - S'^{(p)}(x)| \le C_{1,S}|S^{(p)}(y) - S'^{(p)}(y)| + C_{2,S}\left(\|\Delta\widetilde{A}\| + \|\Delta H\|\right), \tag{5.15}$$

$$|R^{(p)}(x) - R'^{(p)}(x)| \le C_{1,R}|R^{(p)}(y) - R'^{(p)}(y)| + C_{2,R}\left(\|\Delta\widetilde{A}\| + \|\Delta H\|\right), \tag{5.16}$$

for any $x, y \in \Omega_p$ and $1 \le p \le N$, and where the constants and $W^{1,\infty}$-norms above are *global* constants.

Secondly, it is clear that the equations (5.12) and (5.13) are stable in the following sense

$$|S^{(p)}(x_p) - S'^{(p)}(x_p)| \le C_{3,S}|S^{(p'(p))}(x_p) - S'^{(p'(p))}(x_p)| + C_{4,S}\|\Delta H\|, \quad 2 \le p \le N, \tag{5.17}$$

$$|R^{(p)}(x_p) - R'^{(p)}(x_p)| \le C_{3,R}|R^{(p'(p))}(x_p) - R'^{(p'(p))}(x_p)| + C_{4,R}\|\Delta H\|, \quad 2 \le p \le N, \tag{5.18}$$

where the constants are also *global*. Establishing (5.18) clearly requires that the orthonormalization process that is chosen satisfies the stability condition (4.10).

Completion of the argument: Let $y \in X$, then there exists $1 \le p_y \le N$ such that $y \in \Omega_{p_y}$. The sequence $p_y, p_y^1 = p'(p_y), p_y^2 = p'(p_y^1), \dots$ converges to 1 in less than $N-1$ steps, and up to padding with 1's at the end, we will use the sequence of points $x_{p_y}, x_{p_y^1}, \cdots, x_{p_y^{N-1}} = x_1$ as reference points for successive estimations. Following this strategy allows to bound the error $|S^{(p_y)}(y) - S'^{(p_y)}(y)|$ by $|S^{(p_y)}(x_{p_y}) - S'^{(p_x)}(x_{p_y})|$, then by $|S^{(p_y^1)}(x_{p_y}) - S'^{(p_x^1)}(x_{p_y})|$, all the way to $|S^{(1)}(x_1) - S'^{(1)}(x_1)|$ by alternatively using estimates (5.15) and (5.17) at most $N-1$ times. At the end of the process, one obtains the global estimate

$$|S^{(p_y)}(y) - S'^{(p_y)}(y)| \le (C_{1,S}C_{3,S})^{N-1}|S^{(1)}(x_1) - S'^{(1)}(x_1)| + C_{A,S}\|\Delta\widetilde{A}\| + C_{H,S}\|\Delta H\|, \tag{5.19}$$

where the constants $C_{A,S}$ and $C_{H,S}$ are polynomials of $\{C_{i,S}\}_{i=1}^4$ of degree at most $2(N-1)$ and may be made uniform in $y \in X$. The same technique may be applied to the R frame.

On to the stability of σ, we can establish as in Proposition 4.3.7 that, studying the difference of equations (4.55) for σ and σ' over Ω_p, we can write

$$\|\nabla(\log\sigma - \log\sigma)\|_{L^\infty(\Omega_p)} \le C_{1,\sigma}|S^{(p)}(x_p) - S^{'(p)}(x_p)| + C_{2,\sigma}(\|\Delta A\| + \|\Delta H\|), \tag{5.20}$$

for any $1 \le p \le N$. Combining this with the uniform estimate (5.19), we obtain the following estimate, uniform in p

$$\|\nabla(\log\sigma - \log\sigma')\|_{L^\infty(X)} \le C_{3,\sigma}|S^{(1)}(x_1) - S^{'(1)}(x_1)| + C_{4,\sigma}(\|\Delta A\| + \|\Delta H\|).$$

Finally, we arrive at (5.14) by combining the above inequality with the following estimate, uniform in $y \in X$,

$$|\log\sigma - \log\sigma|(y) \le |\log\sigma - \log\sigma|(x_1) + \Delta(X)\|\nabla(\log\sigma - \log\sigma')\|_{L^\infty(X)},$$

and Theorem 5.1.4 is proved. □

5.1.3 The strongly coupled elliptic approach when anisotropy is known

When the anisotropy is known, an alternative approach to the reconstruction of $|\gamma|^{\frac{1}{n}}$ is obtained by setting up a *strongly coupled elliptic system.* This approach may yield a more direct reconstruction procedure, as it does not involve the somewhat tedious reconstruction of an intermediate frame, rather, of the solutions of (4.51) themselves. On the other hand, this reconstruction must be global at once for solutions $(u_1, \cdots, u_m)$ via the resolution of an elliptic system, so this approach is less adapted if one only has data on a strict sub domain of X.

Again, there is a separation in dimension here, as solvability of the elliptic system can be straightforwardly established in two dimensions while the problem is only Fredholm in higher dimensions.

The derivation goes as follows. Let us start from $\mathbf{g} = (g_1, \cdots, g_m) \in \mathcal{G}_\gamma^m$ for some $m \geq n$, and let $\mathcal{O} = \{\Omega_p\}_{p=1}^N$ an open cover, τ an indexing function and C a constant as in Proposition 5.1.2. Let $\{\varphi_p\}_{p=1}^N$ be a smooth partition of unity subordinate to the open cover $\mathcal{O}$. By definition, for each open set Ω_p, we have the relation

$$\nabla \log |A| = \frac{|A|^2}{|H^{\tau(p)}|^{\frac{1}{2}}} \left(\nabla \left(|H^{\tau(p)}|^{\frac{1}{2}} H_{lj}^{\tau(p),-1} \right) \cdot \widetilde{A}^2 \nabla u_{\tau(p)_l} \right) \nabla u_{\tau(p)_j}, \tag{5.21}$$

obtained after writing equation (4.55) on Ω_p and using the relation $S_i = A\nabla u_i$. Now, noticing that the conductivity equation (4.51) can be rewritten as

$$\nabla \cdot (\widetilde{A}^2 \nabla u_i) + \frac{2}{n} \nabla \log |A| \cdot \widetilde{A}^2 \nabla u_i = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad 1 \leq i \leq m,$$

and plugging (5.21) into it, we obtain, for $1 \leq i \leq m$

$$\nabla \cdot (\widetilde{A}^2 \nabla u_i) + \frac{2}{n} \frac{H_{i\tau(p)_j}}{|H^{\tau(p)}|^{\frac{1}{2}}} \nabla \left(|H^{\tau(p)}|^{\frac{1}{2}} H_{lj}^{\tau(p),-1} \right) \cdot \widetilde{A}^2 \nabla u_{\tau(p)_l} = 0, \quad x \in \Omega_p.$$

We now sum the last equation over p with weights φ_p, and arrive at the system

$$\boxed{\nabla \cdot (\widetilde{A}^2 \nabla u_i) + \frac{2}{n} W_{ij} \cdot \widetilde{A}^2 \nabla u_j = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad 1 \leq i \leq m,} \tag{5.22}$$

where the vector fields $\{W_{ij}\}_{i,j=1}^m$ are now defined globally over X by

$$W_{ij} = \sum_{p=1}^K \varphi_p W_{ij}|_{\Omega_p}, \qquad \text{where} \tag{5.23}$$

$$W_{ij}|_{\Omega_p} := \begin{cases} 0 & \text{if } j \notin \tau(p) \\ \sum_{b=1}^n \frac{H_{i\tau(p)_b}}{|H^{\tau(p)}|^{\frac{1}{2}}} \nabla \left(|H^{\tau(p)}|^{\frac{1}{2}} H_{bl}^{\tau(p),-1} \right) & \text{if } j = \tau(p)_l, \ 1 \le l \le n. \end{cases} \tag{5.24}$$

With W_{ij} defined as above, and since all denominators above are bounded away from zero whenever (5.3) is ensured, one can show that there exists a constant $C_W > 0$ such that the following stability statement holds

$$\max_{1\le i,j\le m} \|W_{ij} - W'_{ij}\|_{L^\infty(X)} \le C_W \|H - H'\|_{W^{1,\infty}(X)}, \tag{5.25}$$

whenever two sets of measurements H and H' jointly satisfy (5.3), and when one can construct a triple $(\mathcal{O}, \tau, C)$ as in Proposition 5.1.2 that works for both sets.

On to the solvability of (5.22), let v_i be a H^1-lifting of g_i for $1 \le i \le m$. We now change the unknown into $w_i := u_i - v_i$ and set up the following system with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \nabla \cdot (\widetilde{A}^2 \nabla w_i) + \frac{2}{n} W_{ij} \cdot \widetilde{A}^2 \nabla w_j = f_i \quad (X), \quad w_i|_{\partial X} = 0, \qquad \text{where} \\ f_i := -\nabla \cdot (\widetilde{A}^2 \nabla v_i) - \frac{2}{n} W_{ij} \cdot \widetilde{A}^2 \nabla v_j, \quad 1 \le i \le m. \end{aligned} \tag{5.26}$$

We then define an operator $L^{-1} : f \mapsto u$ such that u is the unique solution to the problem

$$Lu := -\nabla \cdot (\widetilde{A}^2 \nabla u) = f \quad (X), \quad u|_{\partial X} = 0. \tag{5.27}$$

Since $\widetilde{A}^2$ is a uniformly elliptic tensor, the operator $L^{-1} : L^2(X) \to H^2(X)$ is bounded (see Evans

(1998) for instance). Therefore, applying L^{-1} to (5.26), we obtain the integro-differential system

$$w_i + \frac{2}{n} L^{-1}(-W_{ij} \cdot A'^2 \nabla w_j) = L^{-1} f_i, \quad 1 \le i \le m, \tag{5.28}$$

which in vector notation may be recast as

$$\boxed{(I + \mathbf{P}_W)\mathbf{w} = \mathbf{h} := \{L^{-1} f_i\}_{i=1}^m, \qquad \text{where} \qquad \mathbf{P}_W \mathbf{w} := \frac{2}{n} \left[L^{-1}(-W_{ij} \cdot A'^2 \nabla w_j) \right] \mathbf{e}_i.} \tag{5.29}$$

In a similar fashion to (Monard and Bal, 2012c, Lemma 5.1), the operator $\mathbf{P}_W : (H_0^1)^m \to (H_0^1)^m$ is compact and its operator norm satisfies

$$\|\mathbf{P}_W\| \le C \|W\|_\infty, \quad \|W\|_\infty := \max_{1 \le i,j \le m} \|W_{ij}\|_{L^\infty(X)}, \tag{5.30}$$

see Monard and Bal (2012c) for details. Therefore, equation (5.29) is a Fredholm equation whose solvability and the continuity of its inverse with respect to W hold so long as -1 is not an eigenvalue of $\mathbf{P}_W$.

The stability of the solution $\mathbf{u} = \mathbf{w} + \mathbf{v}$ with $\mathbf{w}$ solution of (5.29), with respect to the vector fields W_{ij}, was established in (Monard and Bal, 2012c, Proposition 2.6) in the isotropic case. The only difference in the presence of anisotropy $\widetilde{A}$ is in the elliptic scalar operator L. Restricting ourselves to the case where $\widetilde{A}$ is kept fixed in the stability process, we only restate the stability here.

Proposition 5.1.5 (Stability of the strongly coupled elliptic system). *Let $\{W_{ij}, W'_{ij}\}_{1 \le i,j \le m}$ belong to $L^\infty(X)$ and such that -1 is eigenvalue of neither $\mathbf{P}_W$ not $\mathbf{P}_{W'}$. Let $\mathbf{u}, \mathbf{u}'$ be the unique solutions of (5.29) with same boundary conditions $\mathbf{g}$ and anisotropy $\widetilde{A}$, and respective drift* *terms W, W'. Then we have that $\mathbf{u}-\mathbf{u}' \in (H_0^1)^m$ and satisfies the stability estimate*

$$\|\mathbf{u}-\mathbf{u}'\|_{(H_0^1)^m} \le C\|W-W'\|_{L^\infty(X)}. \tag{5.31}$$

On to the reconstruction of $|\gamma|$, we can sum equation (5.21) with respect to the partition of unity $\{\varphi_p\}_{p=1}^K$ to obtain

$$\nabla|A|^{-\frac{2}{n}} = \nabla|\gamma|^{-\frac{1}{n}} = -\frac{2}{n}\sum_{p=1}^K \frac{\varphi_p}{|H^{\tau(p)}|^{\frac12}} \left(\nabla\left(|H^{\tau(p)}|^{\frac12} H_{lj}^{\tau(p),-1}\right)\cdot A'^2\nabla u_{\tau(p)_l}\right)\nabla u_{\tau(p)_j}, \quad x\in X, \tag{5.32}$$

the right-hand side of which is now completely known and never singular. One may thus choose to solve this equation either solving ODE's along curves or taking divergence on both sides and solving a Poisson equation. The stability of such a reconstruction scheme was first proved in (Monard and Bal, 2012c, Theorem 2.8) and the proof generalizes to the anisotropic case.

Theorem 5.1.6 (Global stability, the elliptic approach)**.** *Let $\{W_{ij}, W'_{ij}\}_{1\le i,j\le m}$ belong to $L^\infty(X)$ and such that -1 is eigenvalue of neither $\mathbf{P}_W$ not $\mathbf{P}_{W'}$. Then the corresponding determinants $|\gamma|, |\gamma|'$ satisfy the following estimate*

$$\||\gamma|^{-\frac1n} - |\gamma|^{-\frac1n}\|_{H^1(X)} \le C\|H-H'\|_{W^{1,\infty}(X)}. \tag{5.33}$$

Injectivity of $I+\mathbf{P}_W$ in the case $m=n=2$

In the two-dimensional case with $m=n=2$ illuminations, we now show that -1 is never an eigenvalue of $\mathbf{P}_W$. Not only will we see in the next section that 2 illuminations are enough to ensure (4.2) with $\Omega = X$, but the elliptic system (5.29) has a nice divergence form.

Equation (5.22) reads, with $m = n = 2$,

$$\nabla \cdot (\widetilde{A}^2 \nabla u_i) + \frac{H_{il}}{|H|^{\frac{1}{2}}} \nabla(|H|^{\frac{1}{2}} H^{lj}) \cdot \widetilde{A}^2 \nabla u_j = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad i = 1, 2.$$

Upon multiplying by $|H|^{\frac{1}{2}} H^{ki}$ and summing over j, we obtain the following system in divergence form:

$$-\nabla \cdot (|H|^{\frac{1}{2}} \widetilde{A}^2 H^{ki} \nabla u_i) = 0, \quad x \in X, \quad u_k|_{\partial X} = g_k, \quad k = 1, 2. \tag{5.34}$$

As in the previous subsection, we define v_i to be a H^1-lifting of $g_i \in H^{\frac{1}{2}}(\partial X)$ inside X. System (5.34) becomes a new system for the unknown $\mathbf{w} = (w_1, w_2) := (u_1 - v_1, u_2 - v_2)$

$$\begin{gathered} -\nabla \cdot (|H|^{\frac{1}{2}} \widetilde{A}^2 H^{ki} \nabla w_i) = f_k, \quad (X), \quad w_k|_{\partial X} = 0, \quad k = 1, 2, \quad \text{where} \\ f_k := -\nabla \cdot (|H|^{\frac{1}{2}} \widetilde{A}^2 H^{ki} \nabla v_i). \end{gathered} \tag{5.35}$$

This new system can now be analyzed via variational theory, as it admits a weak formulation in the space $(H_0^1)^2$, of the form $B(\mathbf{w}, \mathbf{w}') = F(\mathbf{w}')$, where the bilinear form B and the linear form F are obtained from (5.35):

$$B(\mathbf{w}, \mathbf{w}') = \int_X |H|^{\frac{1}{2}} H^{ki} (\widetilde{A}^2 \nabla w_i) \cdot \nabla w_k' \; dx \quad \text{and} \qquad F(\mathbf{w}) = \int_X w_k f_k.$$

Because both matrices $\widetilde{A}^2$ and H^{-1} are uniformly elliptic over X, the bilinear form B is continuous and coercive over the space $(H_0^1(X))^2$. Moreover, the linear functional F is continuous in $(H_0^1(X))^2$. Therefore by virtue of Lax-Milgram's theorem, there exists a unique solution $\mathbf{w} \in (H_0^1(X))^2$ to (5.35), and by extension, a unique solution $(u_1, u_2) \in (H^1(X))^2$ to (5.34). Since the system (5.35) is strictly equivalent to (5.29) with $n = m = 2$, it is thus clear that the uniqueness of the solution to (5.35) guarantees that -1 is not an eigenvalue of $\mathbf{P}_W$. Therefore in this case, one obtains the results of proposition 5.1.5 and theorem 5.1.6 without requiring $I + \mathbf{P}_W$ to be injective.

5.2 How to enforce the reconstructibility conditions

So far, we have derived local and global reconstruction algorithms for the conductivity tensor that relied on a few crucial assumptions:

On the one hand, the reconstruction of the determinant $|\gamma|^{\frac{1}{2}}$ relied on the positivity condition (4.2) satisfied by n solutions locally or, globally, via a family of possibly greater size $m \ge n$ whose gradients would form a family of maximal rank everywhere. This condition was summarized via the definition of the sets of admissibility $\mathcal{G}_\gamma^m$ in 5.1.1. On the other hand, the reconstruction of the rescaled anisotropy $\tilde{\gamma}$ relied on supplementary solutions (on top of the "support basis") that would generate matrices spanning a hyperplane in $\mathcal{M}_n(\mathbb{R})$, a condition summarized in (4.109).

In order to formalize the latter condition, we define for dimension $n \ge 3$[3] the set of *admissibility conditions for anisotropy reconstruction*, whose elements rely on elements of the first set of admissibility $\mathcal{G}_\gamma^m$. Namely, for $\mathbf{g} \in \mathcal{G}_\gamma^m$ and $(u_1, \cdots, u_m)$ the corresponding solutions, we have an open cover $\{\Omega_p\}_{p=1}^N$ and an indexing function τ such that $(\nabla u_{\tau(p)_1}, \cdots, \nabla u_{\tau(p)_n})$ is a frame over Ω_p for all p. Now considering $l \ge 1$ additional solutions $(v_1, \cdots, v_l)$, each additional solution v_j defines a subspace $P_{Z_{(j)}}$ of $\mathcal{M}_n(\mathbb{R})$, to which $\widetilde{A}S$ is orthogonal, where S here is understood as the "local frame" and $H_f = S^T S$. We then define a collection of matrices

$$\mathcal{M} = \{M_{j,q}, \quad 1 \le j \le l, \ 1 \le q \le \dim P_{Z_{(j)}}\},$$

where $\{M_{j,q}\}_{q=1}^{\dim P_{Z_{(j)}}}$ is a basis for $P_{Z_{(j)}}$. We renumber the family $\mathcal{M}$ as $M_1, \cdots, M_{\#\mathcal{M}}$. Note that these matrices are explicit functions of the power densities and their first derivatives. As

[3] The two-dimensional case will be treated separately.

we have seen in Section 4.3, it is customary to introduce the following function

$$
\begin{aligned}
\mathcal{F}_\gamma^{m,l}[\mathbf{g},\mathbf{h}](x) &= \sum_{1\le i_1<\cdots<i_{n^2-1}\le \#\mathcal{M}} \left(\det\left(\mathcal{N}_{(i_1,\cdots,i_{n^2-1})} H_f^{-1} \mathcal{N}^T_{(i_1,\cdots,i_{n^2-1})}\right)\right)^{\frac{1}{n}}, \\
\mathcal{N}_{(i_1,\cdots,i_{n^2-1})} &:= \mathcal{N}(M_{i_1},\cdots,M_{i_{n^2-1}}),
\end{aligned}
\tag{5.36}
$$

initially defined as a function $\mathcal{F}$ in equation (4.109), and where $\mathcal{N}$ is the generalization of the cross-product, defined in (4.107). The condition for $\tilde{\gamma}$ being reconstructible over X is precisely that $\mathcal{F}_\gamma^{m,l}[\mathbf{g},\mathbf{h}]$ is bounded away from zero, a condition that can be checked *directly from the power densities.*

Definition 5.2.1 (Admissibility set $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ for $\mathbf{g}\in\mathcal{G}_\gamma^m$)**.** *For $m\ge n$, let us assume that $\mathbf{g}=(g_1,\cdots,g_m)\in\mathcal{G}_\gamma^m$, and let $\mathcal{O}=\{\Omega_i\}_{i=1}^N$ and τ an open cover and an indexing function associated to it. For $l\ge 1$, we say that l additional boundary conditions $\mathbf{h}=(h_1,\cdots,h_l)\in\left(H^{\frac12}(\partial X)\right)^l$ belong to the set of admissibility $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ if there exists a constant $C_M>0$ such that the following condition holds*

$$
\inf_{x\in X}\mathcal{F}_\gamma^{m,l}[\mathbf{g},\mathbf{h}](x)\ge C_M, \tag{5.37}
$$

where $\mathcal{F}_\gamma^{m,l}[\mathbf{g},\mathbf{h}]$ is defined in (5.36).

If a tensor γ is such that $\mathcal{G}_\gamma^m$ is non-empty for some $m\ge n$ and knowing the anisotropy $\tilde{\gamma}$, then $\det\gamma$ is reconstructible from power densities with Lipschitz stability in $W^{1,\infty}(X)$. Furthermore, if $\mathcal{G}_\gamma^m$ is non-empty for some $m\ge n$, and additionally, $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ is non-empty for some $\mathbf{g}\in\mathcal{G}_\gamma^m$, then the anisotropy can also be reconstructed as well with stability of the form

$$
\|\tilde{\gamma}-\tilde{\gamma}'\|_{L^\infty(X)}\le C\|H-H'\|_{W^{1,\infty}(X)}.
$$

The reconstructibility of a given conductivity tensor thus boils down to showing that such sets of admissibility are not empty, so that one can generate from boundary conditions solutions with appropriate qualitative properties, ensuring in the end that our reconstruction algorithms derived in the previous sections work.

The goal of the present section is thus to find in what cases such admissibility sets are not empty. We first study some properties of the admissibility sets, in particular that the property of "being reconstructible" can be transmitted via diffeomorphisms. We then focus on the two-dimensional case, before tackling dimensions $n \geq 3$. We will see that results vary highly with the space dimension and the smoothness of the conductivity. In some instances, particular solutions such as the solutions of Complex Geometrical Optics and the harmonic polynomials will come in handy to fulfill the conditions of reconstructibility.

5.2.1 Properties of the admissibility sets

Openness with respect to topologies of smooth boundary conditions

The first property to show is that, when γ and ∂X are smooth enough, then the sets $\mathcal{G}_\gamma^m$ and $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ are open for certain topologies.

Lemma 5.2.2. *Assume that $\gamma \in \Sigma(X)$ has coefficients in $\mathcal{C}^1(\overline{X})$ and ∂X is of class $\mathcal{C}^3$, then for $m \geq n$ and $l \geq 1$, the admissibility sets $\mathcal{G}_\gamma^m$ and $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ are open for the topology of $\left(\mathcal{C}^2(\partial X)\right)^m$.*

Proof. We first cover the openness of $\mathcal{G}_\gamma^m$ in detail. By Corollary 3.2.5, the power density operator $\mathcal{H}_\gamma : \mathcal{C}^2(\partial X) \to W^{1,\infty}(X)$ is continuous. Therefore, this means that the operator $\mathcal{D}_\gamma^m$ defined in (5.4), as a homogeneous polynomial of power densities of degree n, is continuous from $[\mathcal{C}^2(\partial X)]^m$ to $W^{1,\infty}(X)$ (thus, to $L^\infty(X)$ as well). Now, if $\mathbf{g} \in \mathcal{G}_\gamma^m$, then there exists $C_H > 0$ as in (5.3), i.e. such that $\inf_{x\in X} \mathcal{D}_\gamma^m[\mathbf{g}](x) \geq C_H$. By continuity of $\mathcal{D}_\gamma^m$, if $\mathbf{g}'$ is such that $\mathbf{g}' - \mathbf{g}$ is small enough in $\mathcal{C}^2$-norm so that $\|\mathcal{D}_\gamma^m[\mathbf{g}] - \mathcal{D}_\gamma^m[\mathbf{g}']\|_{L^\infty(X)} \le \frac{C_H}{2}$, then we have that

$$\inf_{x\in X} \mathcal{D}_\gamma^m[\mathbf{g}'](x) \ge \inf_{x\in X} \mathcal{D}_\gamma^m[\mathbf{g}](x) - \|\mathcal{D}_\gamma^m[\mathbf{g}] - \mathcal{D}_\gamma^m[\mathbf{g}']\|_{L^\infty(X)} \ge \frac{C_H}{2} > 0,$$

so that $\mathbf{g}' \in \mathcal{G}_\gamma^m$. Thus, there exists a small $[\mathcal{C}^2(\partial X)]^m$-neighborhood of $\mathbf{g}$ included in $\mathcal{G}_\gamma^m$.

On to the openness of $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$, the proof is very similar since the defining condition of $\mathbf{h}$ belonging to $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ is a condition on the functional $\mathcal{F}_\gamma^{m,l}[\mathbf{g},\mathbf{h}]$ in (5.36) being bounded away from zero. Again, this functional is a polynomial of power densities and their derivatives (with denominators bounded away from zero), all of which are continuous with respect to $\mathcal{C}^2$-smooth perturbations of boundary conditions. Thus if $\mathbf{h} \in [\mathcal{C}^2(\partial X)]^l$ is such that condition (5.37) holds, it is clear that for $\mathbf{h}'$ close enough to $\mathbf{h}$ in $\mathcal{C}^2(\partial X)$-norm condition (5.37) still holds with a possibly smaller constant C_M. Thus $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ is open for the topology of $[\mathcal{C}^2(\partial X)]^l$. Lemma 5.2.2 is proved. $\square$

In practice, this means that the global reconstruction algorithm is stable under $\mathcal{C}^2$-smooth perturbations of the boundary conditions, whenever γ and ∂X have the regularity as stated above.

Behavior of sets $\mathcal{G}_\gamma^m$ and $\mathcal{A}_\gamma^{m,l}$ with respect to $\mathcal{C}^1$-smooth perturbations of γ

The second fundamental property of the admissibility sets is that a tensor that is $\mathcal{C}^1$-close to a reconstructible tensor, is itself reconstructible using the same boundary conditions.

Lemma 5.2.3. *Assume that $\gamma \in \Sigma(X)$ has coefficients in $\mathcal{C}^1(\overline{X})$ and is such that $\mathbf{g} \in \mathcal{G}_\gamma^m \neq \emptyset$ and $\mathbf{h} \in \mathcal{A}_\gamma^{m,l}(\mathbf{g})$ for some $m \ge n$ and $l \ge 1$. Then for $\gamma' \in \Sigma(X)$ close enough to γ in $\mathcal{C}^1(\overline{X})$-norm, we have that $\mathbf{g} \in \mathcal{G}_{\gamma'}^m$ and $\mathbf{h} \in \mathcal{A}_{\gamma'}^{m,l}(\mathbf{g})$.*

Proof. Using Proposition 3.2.7 on the continuity of $\mathcal{H}_\gamma[g]$ with respect to $\mathcal{C}^1$-smooth perturbations of γ, the proof is very similar to that of Lemma 5.2.2 above: for γ' close enough to γ, the functionals $\mathcal{D}_\gamma^m[\mathbf{g}]$ and $\mathcal{D}_{\gamma'}^m[\mathbf{g}]$ are close enough so that $\mathcal{D}_{\gamma'}^m[\mathbf{g}]$ remains bounded away from zero, whence $\mathbf{g} \in \mathcal{G}_{\gamma'}^m$. Moreover, the functions $\mathcal{F}_\gamma^{m,l}[\mathbf{g}, \mathbf{h}]$ and $\mathcal{F}_{\gamma'}^{m,l}[\mathbf{g}, \mathbf{h}]$ in (5.36) are close to one another if γ and γ' are close enough, so that condition (5.37) holds for γ' with a constant C_M slightly smaller than that of γ, whence $\mathbf{h} \in \mathcal{A}_{\gamma'}^{m,l}(\mathbf{g})$. Thus Lemma 5.2.3 is proved. $\square$

Behavior of sets $\mathcal{G}_\gamma^m$ and $\mathcal{A}_\gamma^{m,l}$ with respect to push-forwards:

The third fundamental property of the admissibility sets is that they "commute with diffeomorphisms". Namely, if $\mathbf{g} \in \mathcal{G}_\gamma^m$ for some given $\gamma \in \Sigma(X)$ and $m \ge n$, then $\mathbf{g} \circ \Psi^{-1} \in \mathcal{G}_{\Psi_\star\gamma}^m$ and vice-versa. In particular, the property of "being reconstructible from power densities", when translated into a statement about non-emptiness of admissibility sets, only depends on the class of equivalence of a tensor for the equivalence relation "$(\gamma, X) \sim (\gamma', X')$ if there exists $\Psi : X \to X'$ a diffeomorphism onto X' such that $\gamma' = \Psi_\star\gamma$". The diffeomorphisms we will consider here are those Ψ whose jacobian determinant is bounded above and below i.e.

$$C_\Psi^{-1} \le |J\Psi| \le C_\Psi \quad \text{for some constant } C_\Psi \ge 1. \tag{5.38}$$

Note that if $\Psi : X \to \Psi(X)$ satisfies (5.38), then $\Psi^{-1} : \Psi(X) \to X$ satisfies (5.38) with the same constant. If we have two diffeomorphisms $\Psi_1 : X_1 \to X_2$ and $\Psi_2 : X_2 \to X_3$ both satisfying (5.38), then $\Psi_2 \circ \Psi_1$ satisfies (5.38) with constant $C_{\Psi_1} C_{\Psi_2}$.

Proposition 5.2.4. *For $\gamma \in \Sigma(X)$ and $\Psi : X \to \Psi(X)$ a $W^{1,2}$-diffeomorphism satisfying (5.38), we have for any $m \ge n$,*

$$\mathcal{G}_{\Psi_\star\gamma}^m = \{\mathbf{g} \circ \Psi^{-1} : \ \mathbf{g} \in \mathcal{G}_\gamma^m\}. \tag{5.39}$$

Proof. Let $\mathbf{g} \in \mathcal{G}_\gamma^m$, so that condition (5.3) holds for some $C_H > 0$. Let H_{ij} be the power densities defined on X and H'_{ij} those defined on $\Psi(X)$. Then, using (B.5) and the estimate (5.38), we have,

$$\sum_{1\le i_1<\cdots<i_n\le m} \det\{H_{i_p i_q}\}_{p,q=1}^n(x) = |J_\Psi(x)|^n \sum_{1\le i_1<\cdots<i_n\le m} \det\{H'_{i_p i_q}\}_{p,q=1}^n(\Psi(x)), \quad x \in X.$$

Thus, if the left-hand side is uniformly bounded below by some $C_H > 0$, then the right-hand is uniformly bounded below by $C_\Psi^{-n} C_H > 0$, so that $\mathbf{g}\circ\Psi^{-1} \in \mathcal{G}^m_{\Psi_\star\gamma}$. Therefore the inclusion $\supset$ holds in (5.39). The converse inclusion holds as well by considering the diffeomorphism Ψ^{-1} and the tensors $\Psi_\star\gamma$ and $\Psi_\star^{-1}\Psi_\star\gamma = \gamma$, thus the proposition is proved. □

As regards Proposition 5.1.2, it is fairly clear that an open cover $\mathcal{O} = \{\Omega_p\}_{p=1}^N$ of X and an indexing function τ adapted to some $\mathbf{g} \in \mathcal{G}_\gamma^m$ will map naturally to an open cover $\Psi(\mathcal{O}) := \{\Psi(\Omega_p)\}_{p=1}^N$ of $\Psi(X)$, with the same indexing function τ being adapted to $\mathbf{g}\circ\Psi^{-1} \in \mathcal{G}^m_{\Psi_\star\gamma}$. In the light of this comment, reconducting the property of being reconstructible via diffeomorphisms also works for the admissibility sets $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$, as the following proposition shows.

Proposition 5.2.5. *For $\gamma \in \Sigma(X)$ and $\Psi : X \to \Psi(X)$ a $W^{1,2}$-diffeomorphism satisfying (5.38), we have for any $\mathbf{g} \in \mathcal{G}_\gamma^m$, $m \ge n$ and any $l \ge 1$,*

$$\mathcal{A}^{m,l}_{\Psi_\star\gamma}(\mathbf{g}\circ\Psi^{-1}) = \{\mathbf{h}\circ\Psi^{-1} :\ \mathbf{h} \in \mathcal{A}^{m,l}_\gamma(\mathbf{g})\}. \tag{5.40}$$

Proof. Applying Proposition 5.1.2 to $\mathbf{g}$, we let $\mathcal{O} = \{\Omega_p\}_{p=1}^N$ an open cover of X and τ an indexing function adapted to $\mathbf{g}$. Then $(\Psi(\mathcal{O}), \tau)$ is also adapted to $\mathbf{g}\circ\Psi^{-1}$. Denote by $(u_1, \cdots, u_m, v_1, \cdots, v_l)$ and $(u'_1, \cdots, u'_m, v'_1, \cdots, v'_l)$ the solutions of (4.51) and their push-forwards by Ψ, respectively. For $1 \le k \le l$, the transformation law for the space $P_{Z_{(k)}}$ is computed in (B.7) and reads

$$P_{Z_{(k)}}(x) = \{D\Psi^T(x) Z'_{(k)}(\Psi(X))(\mathbb{R}\mathbb{I}_n + |J_\Psi(x)|^n H'(\Psi(x)) A_n(\mathbb{R}))\},$$

so in particular, when condition (5.38) is satisfied, $|J_\Psi(x)|^n A_n(\mathbb{R}) = A_n(\mathbb{R})$, so that

$$P_{Z_{(k)}}(x) = D\Psi^T(x) P_{Z'_{(k)}}(\Psi(x)), \quad x \in X.$$

By virtue of (5.38), $D\Psi$ is uniformly invertible, so that, in the end,

$$\dim P_{Z_{(k)}}(x) = \dim P_{Z'_{(k)}}(\Psi(x)), \quad x \in X,$$

thus for $\{M_i(x),\ 1 \le i \le \dim P_{Z_{(k)}}(x)\}$ a basis of $P_{Z_{(k)}}(x)$, we pick $M'_i(\Psi(x)) := D\Psi^T(x) M_i(x)$ as a basis for $P_{Z'_{(k)}}(\Psi(x))$. Moreover, at any point of $\Psi(X)$, the bilinear form

$$A, B \mapsto \langle A, B\rangle_\Psi := \mathrm{tr}\ \left(([D\Psi(\Psi^{-1}(x))]^{-T} A)^T ([D\Psi(\Psi^{-1}(x))]^{-T} B)\right),$$

defines an inner product on $M_n(\mathbb{R})$, which we use to define the cross-product $\mathcal{N}$ as in (4.107) locally on $\mathcal{M}_n(\mathbb{R})$ over $\Psi(X)$. Constructing $\mathcal{F}^{m,l}_{\Psi_\star\gamma}[\mathbf{g}\circ\Psi^{-1}, \mathbf{h}\circ\Psi^{-1}]$ using this cross-product, we obtain the relation

$$\mathcal{F}^{m,l}_\gamma[\mathbf{g}, \mathbf{h}](\Psi(x)) = |J\Psi|^{-1}(x) \mathcal{F}^{m,l}_{\Psi_\star\gamma}[\mathbf{g}\circ\Psi^{-1}, \mathbf{h}\circ\Psi^{-1}](\Psi(x)), \quad x \in X.$$

By virtue of (5.38), $\mathcal{F}^{m,l}_\gamma[\mathbf{g}, \mathbf{h}]$ is bounded away from zero over X if and only if $\mathcal{F}^{m,l}_{\Psi_\star\gamma}[\mathbf{g}\circ\Psi^{-1}, \mathbf{h}\circ \Psi^{-1}]$ is over $\Psi(X)$. Therefore the inclusion $\supset$ holds in (5.40). The converse inclusion holds as well by considering the inverse diffeomorphism Ψ^{-1}, and the proposition is proved. □

5.2.2 The two-dimensional case

The frame condition

The obtention of two solutions with linearly independent gradients throughout X was justified in Alessandrini and Nesi in the context of homogenization. In Alessandrini and Nesi (2001), they prove the following theorem, rewritten with the notation of the present manuscript.

Theorem 5.2.6 (Alessandrini and Nesi (2001), Theorem 4)**.** *Let $X \subset \mathbb{R}^2$ be a bounded simply connected open set, whose boundary ∂X is a simple closed curve. Let $\mathbf{g} = (g_1, g_2)$, $\mathbf{g} : \partial X \to \mathbb{R}^2$ be a homomorphism of ∂X onto a closed curve Γ and let D be the bounded convex domain bounded by D. Let $\gamma \in \Sigma(X)$, and let $U \in W^{1,2}_{loc}(X, \mathbb{R}^2) \cap \mathcal{C}(\overline{X}, \mathbb{R}^2)$ be the γ-harmonic mapping whose components are the solutions of the Dirichlet problems*

$$\nabla \cdot (\gamma \nabla u_i) = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad i = 1, 2.$$

Then U is a homeomorphism of $\overline{X}$ onto $\overline{D}$. In particular, there exists $\sigma = \pm 1$ such that

$$\forall \Omega \Subset X, \quad \sigma \det(\nabla u_1, \nabla u_2) > 0, \quad a.e.\ in\ \Omega.$$

Morally speaking, with very few restrictions on the conductivity tensor γ, the boundary conditions $g_1(x, y) = x$ and $g_2(x, y) = y$, for $(x, y) \in \partial X$ generate solutions that satisfy the positivity assumption (4.16) throughout the domain.

Corollary 5.2.7. *For any $\gamma \in \Sigma(X)$, the set $\mathcal{G}^2_\gamma$ is not empty.*

The anisotropy condition

As was first analyzed in Monard and Bal (2012a) and pointed out in Section 4.2.3, the anisotropy may be reconstructed if we can find three or four solutions (u_1, u_2, u_3, u_4) of (4.51) (with possibly $u_2 = u_3$) such that

$$\nabla \frac{\det(\nabla u_1, \nabla u_2)}{\det(\nabla u_3, \nabla u_4)}(x) \neq 0, \quad x \in X. \tag{5.41}$$

Therefore, we may define the sets $\mathcal{A}^m_{2,\gamma} \subset H^{\frac{1}{2}}(\partial X)$ for $m = 3, 4$ where $\mathbf{g} = (g_1, \cdots, g_m)$ belongs to $\mathcal{A}^m_{2,\gamma}$ if the corresponding solutions $(u_1, \cdots, u_m)$ satisfy (5.41). Numerical experiments in Monard and Bal (2012a) have showed that three solutions are usually enough to satisfy (5.41) almost everywhere, although the case $m = 4$ is useful for the theoretical proof of Proposition 5.2.8 below.

As in the previous section, the sets $\mathcal{A}^m_{2,\gamma}$ are open for the topology of $\mathcal{C}^2(\partial X)$, and the property of non-emptiness is stable under $\mathcal{C}^1$ perturbations of conductivity tensors. Moreover, if we have $\gamma' = \Psi_\star \gamma$ for some diffeomorphism $\Psi : \overline{X} \to \Psi(\overline{X})$, then it is straightforward to check that

$$\begin{aligned} &\nabla \frac{\det(\nabla u_1, \nabla u_2)}{\det(\nabla u_3, \nabla u_4)}(x) = D\Psi^t(x) \nabla \frac{\det(\nabla v_1, \nabla v_2)}{\det(\nabla v_3, \nabla v_4)}(\Psi(x)), \quad x \in X, \qquad \text{where} \\ &\nabla \cdot (\gamma \nabla u_i) = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad v_i = u_i \circ \Psi^{-1}, \quad 1 \le i \le 4. \end{aligned}$$

Since $D\Psi$ is invertible everywhere, it is clear from the previous equation that $\mathbf{g} \in \mathcal{A}^m_{2,\gamma}$ if and only if $\mathbf{g} \circ \Psi^{-1} \in \mathcal{A}^m_{2,\gamma'}$, in particular, $\mathcal{A}^m_{2,\gamma} \neq \emptyset$ if and only if $\mathcal{A}^m_{2,\gamma'} \neq \emptyset$. Therefore, in two dimensions, since every tensor is the push-forward of an isotropic tensor, the reconstructibility question of finding three or four solutions satisfying (5.41) may be reduced to isotropic tensors. Using Appendix D on CGO solutions, we will see that if $\gamma = \sigma \mathbb{I}_2$ with σ smooth enough, then these solutions are almost perfectly tailored to satisfy condition (5.41).

Proposition 5.2.8. *Assuming that $\gamma \equiv \sigma \mathbb{I}_2$ for some scalar function $\sigma \in H^{4+\epsilon}(X)$ for some $\epsilon > 0$, then $\mathcal{A}^4_{2,\gamma}$ is not empty.*

Proof. Under the assumed regularity, pick $\mathbf{k}_1, \mathbf{k}_2$ in $\mathbb{S}^1$ with $\mathbf{k}_1 \neq \mathbf{k}_2$, and construct for ρ large enough CGO solutions $u_{\boldsymbol{\rho}_1}$ and $u_{\boldsymbol{\rho}_2}$ using Lemma D.0.2, where for $j = 1, 2$, $\boldsymbol{\rho}_j := \rho(\mathbf{k}_j + i\mathbf{k}_j^\perp)$ and $^\perp$ designates the direct orthogonal vector in this case. Now we pick $(u_1, u_2, u_3, u_4) = (u^{\Re}_{\boldsymbol{\rho}_1}, u^{\Im}_{\boldsymbol{\rho}_1}, u^{\Re}_{\boldsymbol{\rho}_2}, u^{\Im}_{\boldsymbol{\rho}_2})$ and notice that

$$\nabla \frac{\det(\nabla u_1, \nabla u_2)}{\det(\nabla u_3, \nabla u_4)} = 2\rho(\mathbf{k}_1 - \mathbf{k}_2) + f(x), \quad f \in L^\infty(X),$$

so that, for ρ large enough, the above right-hand side cannot vanish. As a conclusion, for ρ large enough, $(g^{\Re}_{\boldsymbol{\rho}_1}, g^{\Im}_{\boldsymbol{\rho}_1}, g^{\Re}_{\boldsymbol{\rho}_2}, g^{\Im}_{\boldsymbol{\rho}_2}) \in \mathcal{A}^4_{2,\gamma} \neq \emptyset$, hence the result. □

Towards proving the non-emptiness of $\mathcal{A}^3_{2,\gamma}$ for generic tensors γ, we suggest the following proposition:

Proposition 5.2.9. *Assume that three solutions (u_1, u_2, u_3) satisfy the conductivity equation over X, and such that there exists a nonempty open set $\Omega \subset X$ where the vector field $\nabla \frac{\det(\nabla u_1, \nabla u_3)}{\det(\nabla u_1, \nabla u_2)}$ vanishes. Then there exist constants a, b, c such that*

$$u_3 = au_1 + bu_2 + c, \quad x \in X.$$

Remark 5.2.10. *By continuity, this implies that the boundary conditions must satisfy the same relation $g_3 = ag_1 + bg_2 + c$. By contrapositive, if g_1, g_2, g_3 are chosen to be linearly independent in $H^{\frac{1}{2}}(\partial X)$, then we are sure that the set over which $\nabla \frac{\det(\nabla u_1, \nabla u_3)}{\det(\nabla u_1, \nabla u_2)}$ vanishes has empty interior.*

Proof of Proposition 5.2.9: Assume σ isotropic with the uniform estimate $\kappa^{-1} \le \sigma \le \kappa$. Then there is a constant b such that,

$$\det(\nabla u_1, \nabla u_3) = b \det(\nabla u_1, \nabla u_2) \quad \Leftrightarrow \quad \det(\nabla u_1, \nabla u_3 - b\nabla u_2) = 0, \quad x \in \Omega. \tag{5.42}$$

Since b is constant throughout Ω, this implies $J\nabla u_1 \cdot \nabla(u_3 - bu_2) = 0$ throughout Ω. Let us define $\tilde{u}_1$ to be the *dual function* to u_1, that is, the one that satisfies $J\sigma\nabla u_1 = \nabla\tilde{u}_1$ and that satisfies the elliptic equation $\nabla\cdot(\sigma^{-1}\nabla\tilde{u}_1) = 0$. In particular, the complex mapping $f := u_1 + i\tilde{u}_1$ is a κ-quasiconformal mapping that is locally univalent, see Alessandrini and Nesi (2001). Fix $x_0 \in \Omega$ and a neighborhood of x_0, say $\Omega' \subset \Omega$, such that $f|_{\Omega'}$ is univalent, and define the functions $v_i := u_i \circ (f|_{\Omega'})^{-1}$ for $1 \le i \le 3$. The functions v_j depend on the local coordinates $(\xi, \eta) \equiv (u_1, \tilde{u}_1)$ and we clearly have $v_1(\xi, \eta) = \xi$ for $(\xi, \eta) \in f(\Omega')$. Second, using the change of variable f, we can show that (v_1, v_2, v_3) satisfy the elliptic equation

$$\partial_\xi^2 v_j + \partial_\eta(c\partial_\eta v_j) = 0, \quad (\xi, \eta) \in f(\Omega'), \quad 1 \le j \le 3, \quad \text{where} \quad c := (\sigma \circ f^{-1})^2. \tag{5.43}$$

Now, we look at the function $v_3 - b\ v_2$, and compute

$$\partial_\eta(v_3 - b\ v_2)(\xi, \eta) = \left(\frac{\partial x}{\partial\eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\eta}\frac{\partial}{\partial y}\right)(u_3 - b\ u_2)(x(\xi, \eta), y(\xi, \eta)),$$

where the latter differential operator is proportional to $J\nabla u_1 \cdot \nabla$, thus by virtue of equation (5.42), we deduce that $\partial_\eta(v_3 - b\ v_2) = 0$ throughout $f(\Omega')$ and thus the function $v_3 - b\ v_2 = c_1(\xi)$ for some function c_1. Plugging c_1 into equation (5.43) tells us that $\partial_\xi^2 c_1(\xi) = 0$, so that $c_1(\xi) = a\xi + c = av_1 + c$ over $f(\Omega')$. As a result, we have $v_3 = av_1 + bv_2 + c$ over $f(\Omega')$, i.e. $u_3 = au_1 + bu_2 + c$ over Ω'. We now conclude by saying that the function $w := u_3 - au_1 - bu_2 - c$ satisfies $\nabla \cdot (\sigma\nabla w) = 0$ throughout X and is identically zero over Ω', then by the unique continuation principle for elliptic equations, $w = 0$ throughout X.

$\square$

5.2.3 The n-dimensional case, $n \geq 3$

In two dimensions, we usually proved that some illuminations worked for isotropic case. Then for any isotropic metric, one would use its *isotropic representative* (or equivalently, isothermal coordinates), find Dirichlet conditions that would work and show that push-forwarding the coordinates by a diffeomorphism would not change the qualitative properties that needed to be satisfied.

In higher dimensions, not all conductivity tensors may be push-forwarded to isotropic ones, so we have to use different techniques. Furthermore, Theorem 5.2.6 could not be extended to higher dimensions in a systematic fashion, as counterexamples found in Briane et al. (2004) and Laugesen (1996) invalidate the idea that one could systematically construct n boundary conditions such that (4.2) holds throughout X.

The positivity condition

If the conductivity is isotropic with enough regularity, then the CGO solutions come to the rescue, as may be seen from the following theorem, established in Bal et al. (2012a) in dimension three and generalized to $n \geq 3$ in Monard and Bal (2012c).

Proposition 5.2.11. *Assuming that $\gamma \equiv \sigma \mathbb{I}_n$ for some scalar function $\sigma \in H^{\frac{n}{2}+3+\epsilon}(X)$ for some $\epsilon > 0$, then $\mathcal{G}_\gamma^n$ is not empty if n is even and $\mathcal{G}_\gamma^{n+1}$ is not empty if n is odd.*

Proof. We use lemma D.0.2 to construct CGO solutions of the form $u_{\boldsymbol{\rho}}$ whose gradients behave properly. For $u_{\boldsymbol{\rho}}$ a CGO solution as in Appendix D, we will use the real and imaginary parts of

$u_{\boldsymbol{\rho}}$, respectively denoted $u_{\boldsymbol{\rho}}^{\Re}$ and $u_{\boldsymbol{\rho}}^{\Im}$, as real-valued conductivity solutions. In this case, equation (D.3) holds with $|\boldsymbol{\varphi}_{\boldsymbol{\rho}}| \in L^\infty(X)$.

Case n even: Set $n = 2p$, define $\boldsymbol{\rho}_l = \rho(\mathbf{e}_{2l} + i\mathbf{e}_{2l-1})$ for $1 \le l \le p$, and construct

$$S_{2l-1} = \sqrt{\sigma}\nabla u_{\boldsymbol{\rho}_l}^{\Re} \quad \text{and} \quad S_{2l} = \sqrt{\sigma}\nabla u_{\boldsymbol{\rho}_l}^{\Im}, \quad 1 \le l \le p.$$

Using (D.3), we obtain that

$$\det(S_1, \dots, S_n) = \rho^n e^{2\rho \sum_{l=1}^p x_{2l}}(1 + f(x)), \quad \text{where} \quad \lim_{\rho\to\infty} \sup_{\overline{X}} |f| = 0.$$

Letting ρ so large that $\sup_{\overline{X}} |f| \le \frac{1}{2}$ and denoting $\gamma_0 := \min_{x\in\overline{X}}(\rho^n e^{2\rho\sum_{l=1}^p x_{2l}}) > 0$, we have $\inf_{x\in\overline{X}} \det(S_1, \dots, S_n) \ge \frac{\gamma_0}{2} > 0$. Concluding, we have that for ρ large enough,

$$(g_{\boldsymbol{\rho}_1}^{\Re}, g_{\boldsymbol{\rho}_1}^{\Im}, \cdots, g_{\boldsymbol{\rho}_p}^{\Re}, g_{\boldsymbol{\rho}_p}^{\Im}) \in \mathcal{G}_\gamma^n \neq \emptyset,$$

hence the result.

Case n odd: Set $n = 2p-1$, define $\boldsymbol{\rho}_l = \rho(\mathbf{e}_{2l} + i\mathbf{e}_{2l-1})$ for $1 \le l \le p-1$, and $\boldsymbol{\rho}_p = \rho(\mathbf{e}_n + i\mathbf{e}_1)$ and construct

$$S_{2l-1} = \sqrt{\sigma}\nabla u_{\boldsymbol{\rho}_l}^{\Re} \quad \text{and} \quad S_{2l} = \sqrt{\sigma}\nabla u_{\boldsymbol{\rho}_l}^{\Im}, \quad 1 \le l \le p.$$

Using (D.3), we obtain that

$$\det(S_1, \dots, S_{n-1}, S_n) = \rho^n e^{\rho(x_n + 2\sum_{l=1}^{p-1} x_{2l})}\left(-\cos(\rho x_1) + f_1(x)\right), \quad \lim_{\rho\to\infty} \sup_{\overline{X}} |f_1| = 0,$$
$$\det(S_1, \dots, S_{n-1}, S_{n+1}) = \rho^n e^{\rho(x_n + 2\sum_{l=1}^{p-1} x_{2l})}\left(-\sin(\rho x_1) + f_2(x)\right), \quad \lim_{\rho\to\infty} \sup_{\overline{X}} |f_2| = 0.$$

Letting ρ so large that $\sup_{\overline{X}}(|f_1|, |f_2|) \le \frac{1}{4}$ and denoting $\gamma_1 := \min_{x\in\overline{X}}(\rho^n e^{\rho(x_n+2\sum_{l=1}^{p-1} x_{2l})}) > 0$, we have that $|\det(S_1, \dots, S_{n-1}, S_n)| \ge \frac{\gamma_1}{4}$ on sets of the form $X \cap \{\rho x_1 \in]\frac{-\pi}{3}, \frac{\pi}{3}[+m\pi\}$ and $|\det(S_1, \dots, S_{n-1}, S_{n+1})| \ge \frac{\gamma_1}{4}$ on sets of the form $X \cap \{\rho x_1 \in]\frac{\pi}{6}, \frac{5\pi}{6}[+m\pi\}$, where m is a signed integer. Since the previous sets are open and a finite number of them covers X (because X is bounded and ρ is finite), we therefore have fulfilled the desired requirements of the construction. Upon changing the sign of S_n or S_{n+1} on each of these sets if necessary, we can assume that the determinants are all positive. Concluding, we have that for ρ large enough,

$$(g^{\Re}_{\boldsymbol{\rho}_1}, g^{\Im}_{\boldsymbol{\rho}_1}, \cdots, g^{\Re}_{\boldsymbol{\rho}_p}, g^{\Im}_{\boldsymbol{\rho}_p}) \in \mathcal{G}^{n+1}_\gamma \ne \emptyset,$$

hence the result. □

Corollary 5.2.12. *If $\gamma \in \Sigma(X)$ is defined as in Proposition 5.2.11 then we have the following properties*

1. *for $\Psi : X \to \Psi(X)$ a $W^{1,2}$-diffeomorphism satisfying (5.38), the tensor $\Psi_\star\gamma \in \Sigma(\Psi(X))$ is such that $\mathcal{G}^n_{\Psi_\star\gamma}$ is not empty if n is even and $\mathcal{G}^{n+1}_{\Psi_\star\gamma}$ is not empty if n is odd.*
2. *there exists a $\mathcal{C}^1(\overline{X})$ neighborhood G of γ such that for any $\gamma' \in G$, $\mathcal{G}^n_{\gamma'}$ is not empty if n is even and $\mathcal{G}^{n+1}_{\gamma'}$ is not empty if n is odd.*

Proof. The first point follows from combining Propositions 5.2.11 and 5.2.4, and the second follows from combining Proposition 5.2.11 and Lemma 5.2.3. □

The anisotropy condition

As of now, there is not much to say about this problem. The case of a constant tensor $\gamma = \tilde{\gamma}_0$ is instructive on its own right, and since these cases works, the approach will still hold true for small enough perturbations of these tensors.

Constant tensor:

Identity tensor: Assume the case $\gamma = \mathbb{I}_n$ over any bounded domain X, i.e. the harmonic equation. In this case, we can pick as "support basis" $u_i = x_i$ for $1 \le i \le n$, so that $\nabla u_i = \mathbf{e}_i$, for which the matrix that we must characterize is $\widetilde{A}S = \mathbb{I}_n$. For u an additional harmonic solution, the vector fields Z_i generated by the linear dependence of $(u_1, \cdots, u_n, u)$ are nothing but

$$Z_i = \nabla \frac{\det(\mathbf{e}_1, \cdots, \overbrace{\nabla u}^{i}, \cdots, \mathbf{e}_n)}{\det(\mathbf{e}_1, \cdots, \mathbf{e}_n)} = \nabla(\nabla u \cdot \mathbf{e}_i) = [\nabla^2 u]\mathbf{e}_i.$$

In other words, we have that the matrix $Z = [Z_1|\cdots|Z_n] = \nabla^2 u$. Now we want to prove that if we pick enough functions u, then the sum of the corresponding spaces P_Z (4.99) completely characterizes the orthogonal of $\widetilde{A}S \equiv \mathbb{I}_n$ in $\mathcal{M}_n(\mathbb{R})$, that is, all traceless matrices. It is clear that we can do so by picking a basis for harmonic homogeneous quadratic polynomials, i.e. for instance

$$v_{ij}(x) = x_i x_j, \quad 1 \le i < j \le n \quad \text{and} \quad w_i(x) = \frac{1}{2}(x_i^2 - x_{i+1}^2), \quad 1 \le i < n, \quad x \in X, \qquad (5.44)$$

so that the Hessians $\nabla^2 v_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i$ and $\nabla^2 w_i = \mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}$ span all symmetric traceless matrices. Further, since the spaces P_Z also contain matrices of the form $[\nabla^2 u]\Omega$ with u any of the functions in (5.44) and $\Omega \in A_n(\mathbb{R})$, we can construct a basis for traceless matrices.

<u>In three dimensions.</u> Now this is far too many functions, as we will see for instance in three dimensions that $w_1 = x_1^2 - x_2^2$ and $w_2 = x_2^2 - x_3^3$ are such that $P_{Z_1} + P_{Z_2}$ spans all traceless matrices. Indeed, we have

$$\nabla^2 w_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \quad \text{and} \quad \nabla^2 w_2 = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3.$$

Picking a basis for $A_3(\mathbb{R})$ given by $\Omega_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i$ for $(i,j) \in \{(1,2),(2,3),(3,1)\}$, we compute

$$[\nabla^2 w_1]\Omega_{12} = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \quad [\nabla^2 w_1]\Omega_{23} = -\mathbf{e}_2 \otimes \mathbf{e}_3, \quad [\nabla^2 w_1]\Omega_{31} = -\mathbf{e}_1 \otimes \mathbf{e}_3,$$
$$[\nabla^2 w_2]\Omega_{12} = -\mathbf{e}_2 \otimes \mathbf{e}_1, \quad [\nabla^2 w_2]\Omega_{23} = \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2, \quad [\nabla^2 w_2]\Omega_{31} = -\mathbf{e}_3 \otimes \mathbf{e}_1.$$

The eight matrices from the last three equations span all traceless matrices, thus the claim is proved.

<u>In n dimensions.</u> In three dimensions, the above result is sharp because we are able to generate a hyperplane of $\mathcal{M}_3(\mathbb{R})$ with 2 additional solutions, while according to Corollary 4.3.10, one solution can generate at most a 4-dimensional subspace of $\mathcal{M}_3(\mathbb{R})$. The construction above suggests that in general dimensions, the functions $w_i(x) = \frac{1}{2}(x_i^2 - x_{i+1}^2)$ for $1 \le i \le n-1$ should be enough to span a hyperplane of $\mathcal{M}_n(\mathbb{R})$, and we now check that this is indeed the case. Each function w_i is such that $\nabla^2 w_i = \mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{e}_{i+1} \otimes \mathbf{e}_{i+1}$ has rank 2, so that each of them generates a subspace of dimension $2n-2$ according to Corollary 4.3.10, and we have $n-1$ of them. One may check that for $n \ge 3$, we have $(n-1)(2n-2) \ge n^2 - 1$, so that this family is a good candidate. We conclude the argument by writing the following calculation

$$[\nabla^2 w_i]\Omega_{p,q} = \begin{cases} \mathbf{e}_i \otimes \mathbf{e}_{i+1} + \mathbf{e}_{i+1} \otimes \mathbf{e}_i & \text{if } (p,q) = (i,i+1), \\ \mathbf{e}_i \otimes \mathbf{e}_l & \text{if } (p,q) = (i,l), l \notin \{i,i+1\}, \\ -\mathbf{e}_{i+1} \otimes \mathbf{e}_l & \text{if } (p,q) = (i+1,l), l \notin \{i,i+1\}. \end{cases}, \quad 1 \le i \le n-1.$$

The last two cases cover all possible extra diagonal traceless matrices, which together with the fact that $\{\nabla^2 w_i\}_{i=1}^{n-1}$ span all diagonal traceless matrices, ensures that the space

$$\sum_{i=1}^{n-1}[\nabla^2 w_i]\,(\mathbb{R}\mathbb{I}_n + A_n(\mathbb{R}))$$

is precisely that of traceless matrices, the orthogonal of which cannot but be the identity tensor. We conclude that **in general dimension and constant anisotropy case,** $n-1$ **additional solutions allow to characterize the anisotropy.** Unlike the three-dimensional case, it is not clear whether $n-1$ is the smallest number one can add, as the rank of the Hessians of the solutions we have chosen is not maximal.

General constant tensor: The general constant case γ with $\det\gamma = 1$ can be proved by seeing that, if $\widetilde{A}$ is the positive squareroot of γ then the mapping $\Psi : \mathbb{R}^n \ni x \mapsto \widetilde{A}x \in \mathbb{R}^n$ is a global diffeomorphism and such that, straightforwardly, $\Psi_\star\mathbb{I}_n = \gamma$. Thus the results from the harmonic case may be transported.

Again, after proving this for constant tensors, the property of reconstructibility may be reconducted to push-forwards of constant tensors by diffeomorphisms by virtue of Proposition 5.2.5, and to $\mathcal{C}^1$-smooth perturbations of constant tensors by virtue of Lemma 5.2.3.

We summarize these results into the following proposition.

Proposition 5.2.13. *If $\gamma \in \Sigma(X)$ is the identity tensor, then there exists $1 \le l \le n-1$ such that $\mathcal{A}_\gamma^{n,l}(\mathbf{g})$ is non-empty, where we can choose $\mathbf{g} = Id|_{\partial X}$. Furthermore we have the following properties:*

1. *for $\Psi : X \to \Psi(X)$ a $W^{1,2}$-diffeomorphism satisfying (5.38), the tensor $\Psi_\star\gamma \in \Sigma(\Psi(X))$ is such that $\mathbf{g}\circ\Psi^{-1} \in \mathcal{G}^n_{\Psi_\star\gamma} \neq \emptyset$ and $\mathbf{h}\circ\Psi^{-1} \in \mathcal{A}^{n,l}_{\Psi_\star\gamma}(\mathbf{g}\circ\Psi^{-1})$ for the same l as above. This covers in particular all constant tensors.*

2. *there exists a* $\mathcal{C}^1(\overline{X})$ *neighborhood* G *of* γ *such that for any* $\gamma' \in G$, $\mathbf{g} \in \mathcal{G}^n_{\gamma'}$ *and* $\mathbf{h} \in \mathcal{A}^{n,l}_{\gamma'}(\mathbf{g})$ *for the same* l *as above.*

Conclusion and perspectives

As a conclusion, we have shown in the present manuscript two different methods for improving resolution (i.e. stability) in some severly ill-posed problems by enriching the set of measurements used. In both cases, the initial inverse problem was replaced by another inverse problem with the same unknown and a less regularizing, more informative measurement operator.

On inverse transport theory

In Part I, the stability improvement was achieved in the context of inverse transport by going from the stationary regime to the time-harmonic regime. Although the measurements remained supported at the boundary, this change in regime has allowed the them to depend on an additional modulation frequency ω. This in turn allowed them to stably "see" finer scales (i.e. details) of the unknown inside the domain. Mathematically, the inverse problem that was considered (reconstruction of the spatial part of the scattering kernel) went from being a severely ill-posed problem to a mildly ill-posed problem, which in practice guarantees improvements in resolution.

Potential extensions: As potential improvements and further topics in this area, we will raise the following points:

- In the paper Bal et al. (2008), the reconstruction theorems assume that $\|\sigma\|_\infty$ is small, due to the fact that the half-adjoint A of the single scattering measurements $\mathcal{M}_1$ was replaced by the double layer potential. I believe that this smallness assumption can be dropped, as more attenuation should make things more favorable. In particular, the estimates obtained in theorems 2.1.1 and 2.1.2 should not depend on $\|\sigma\|_\infty$.

- The results in Monard and Bal (2012d) are established in a two-dimensional setting. This is because the inversion formulas for the inverse X-Ray transform are explicit and thus the composition of operators that was analyzed there was also quite explicit to write down and work on. Toward an extension to the (practically more interesting) three-dimensional case, one must first pick an *injectivity set* for the X-Ray of cone-beam transform before deriving an inversion formula, after which one must carry similar stationary phase calculations in order to see if the results in Monard and Bal (2012d) reconduct to higher dimensions. It is not clear as of yet that they do, as the space dimension works against us in the asymptotic behavior of time-harmonic forward transport solutions (Bal et al., 2011a).

- Improving the models used here would of course require taking into account the finiteness of the number of measuring captors, as well as their small but non-negligible extension. These aspects would also impact negatively the true stability and resolution.

On Ultrasound-modulated techniques

In Part II, the stability improvement was achieved in the context of inverse conductivity (or diffusion), by replacing the initial boundary measurements by internal measurements, the obtention of which was justified by an argument of coupling between two physical models, in the context of hybrid medical imaging methods. The initial problem with boundary measurements (so-called

Calderón's problem) was a very nasty inverse problem, since it was only injective in very few cases (scalar tensors), and whenever it was injective, was severely unstable. Using internal power densities instead of boundary measurements, this inverse problem was made injective for a larger class of tensors, with explicit reconstruction formulas that displayed either well-posed stability or at the most mildly ill-posed instability. The validity of these reconstruction algorithms relied on qualitative assumptions on families of solutions of the associated elliptic problem, and the question of whether a conductivity tensor is characterized by its power density operator was reduced to a statement of non-emptiness for the admissibility sets $\mathcal{G}_\gamma^m$ and $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$.

- The non-emptiness of $\mathcal{G}_\gamma^m$ implied that one could reconstruct a scalar factor of that tensor, provided that the remaining anisotropy was known. Non-emptiness of $\mathcal{G}_\gamma^n$ was shown for smooth enough scalar tensors, $\mathcal{C}^1$-smooth perturbations of them and their push-forwards by diffeomorphisms.

- Further, whenever a set $\mathcal{G}_\gamma^m$ was non-empty, one could study the non-emptiness of the set $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ with $\mathbf{g} \in \mathcal{G}_\gamma^m$ which, whenever true, guaranteed the existence of boundary conditions such that the corresponding solutions would satisfies the appropriate properties for recovering the anisotropic structure of the tensor γ. Non-emptiness of $\mathcal{A}_\gamma^{m,l}(\mathbf{g})$ was shown for constant tensors, $\mathcal{C}^1$-smooth perturbations of them and their push-forwards by diffeomorphisms.

Future work: We now outline some points regarding the completion of this work and potential additional research topics:

- The first next target result we have in mind is to show that these admissibility sets are not empty when m or l are large enough, provided that the coefficients of γ are smooth. Such a result would be based on the Runge approximation for elliptic equations as soon as the tensor γ has smooth enough components, in a similar fashion to Bal and Uhlmann (2012a,b).

- The second most important improvement for this model is to take into account the absorption term in the model for Optical Tomography, i.e. the missing term of the form $\sigma_a u$ (so that the equation becomes $-\nabla\cdot(\gamma u)+\sigma_a u=0$). In this case, the power density would look more like

$$\mathcal{H}_{\gamma,\sigma_a}[g] = \nabla u\cdot\gamma\nabla u + \alpha\sigma_a u^2, \tag{5.45}$$

  where α is a constant of physical coupling that one may assume to be known as a first simplified model. The corresponding inverse problem would now be the recovery of (γ,σ_a) from knowledge of power densities of the form (5.45). It is worth asking whether the frame-based approach previously presented could adapt to such reconstruction problems.

- In the same order of ideas, one could consider adding a drift term of the form $\vec{\beta}\cdot\nabla u$ to the conductivity equation (3.9) and assess the reconstructibility of the vector field $\vec{\beta}$ from power densities.

- Another important area of investigation is the use of Dirichlet boundary data on a restricted part of the boundary instead of the whole. In particular, can we still control the qualitative behavior of solutions of (3.9) from a restricted part of the boundary ?

- Limited number of functionals. In particular, the main philosophy of this work postulated that one can generate arbitrary amounts of data functionals. While this is certainly not the case in practice, the main purpose here is to exhaust the potentialities of power densities as measurement functionals. In particular, as an extension of this work, it would be interesting to consider data coming from less functionals to see what and how we can reconstruct them. Idea in 3D: can we foliate the domain with graphs of 2 solutions, so that the isotropy is reconstructible along each folium separately. Is it ever possible to construct families of solutions such that their gradients form integrable distributions ?

- Among possible generalizations of the approach presented here, the present calculations may generalize to Riemannian manifolds on which the metric and its corresponding Levi-Civita connection are known. The derivations based on connection calculus appear to work quite similarly with little adaptation.

- On to taking into account noise in the measurements, let us mention that, although explicit reconstruction procedures present the advantage of being straightforward, they may lead to catastrophic reconstructions if one applies them directly to noisy data. Therefore, taking into account noise in the measurements is a great challenge that may not only use explicit inversions, but also regularization techniques. In this case, making optimal use of the joint collaboration between both methods mentioned remains to be analyzed in detail.

- Noise models that take into account the hybrid nature of the problem should be studied and taken into account in the inversions as well. Noisy power densities that is. For instance, in the case of ImpACT, how to model the type of noise that comes from the wave inversion.

- Finally, taking into account the unknown Grüneisen coefficient, i.e. the coefficient that accounts for coupling in hybrid imaging methods, remains to be done. In other words, the true functionals are $\Gamma(x)H_{ij}(x)$, where Γ accounts for a coupling coefficient that is not always properly known. The question remains as to power densities offer enough information so that Γ may be reconstructed as well from knowledge of enough power densities.

References

Albocher, U., Oberai, A. A., Barbone, P. E., and Harari, I. (2009). Adjoint-weighted equation for inverse problems of incompressible plane-stress elasticity. *Comput. Methods Appl. Mech. Engrg.*, 198:2412–2420. 51

Alessandrini, G. (1988). Stable determination of conductivity by boundary measurements. *Appl. Anal.*, 27:153–172. 69

Alessandrini, G. and Nesi, V. (2001). Univalent e^{σ}-harmonic mappings. *Arch. Rat. Mech. Anal.*, 158:155–171. 88, 142, 166, 169

Ammari, H., Bonnetier, E., Capdeboscq, Y., Tanter, M., and Fink, M. (2008). Electrical Impedance Tomography by elastic deformation. *SIAM J. Appl. Math.*, 68:1557–1573. 57, 75

Ammari, H., Garnier, J., Jing, W., and Nguyen, H. L. (2012). Quantitative thermo-acoustic imaging: an exact reconstruction formula. submitted. 49

Ammari, H. and Kang, H. (2004). *Reconstruction of Small Inhomogeneities from Boundary Measurements*. Lecture Notes in Mathematics. Springer, Berlin. 29

Arridge, S. (1999). Optical tomography in medical imaging (topical review). *Inverse Problems*, 15:R41–R93. 16, 55

Astala, K., Iwaniec, T., and Martin, G. (2009). *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*. Princeton University Press. 88

Astala, K., Lassas, M., and Päivärinta, L. (2005). Calderón's inverse problem for anisotropic conductivity in the plane. *Comm. in Partial Diff. Eq.*, 30:207–224. 3, 62, 69, 70, 88, 203

Astala, K. and Päivärinta, L. (2006). Calderón's inverse conductivity problem in the plane. *Annals of Math.*, 163:265–299. 68

Bal, G. (2004). Lecture notes, introduction to inverse problems. 2

Bal, G. (2008). Inverse transport from angularly averaged measurements and time-harmonic isotropic sources. In Censor, Y., Jiang, M., and Louis, A., editors, *Mathematical Methods in Biomedical Imaging and Intensity-Modulated Radiation Therapy*, CRM, pages 19–35, Scuola Normale Superiore Pisa, Italy. 19

Bal, G. (2009). Inverse transport theory and applications (review paper). *Inverse Problems*, 25:053001. 15, 16

Bal, G. (2012a). Cauchy problem for ultrasound modulated EIT. *submitted*. 58

Bal, G. (2012b). Explicit reconstructions in QPAT, QTAT, TE, and MRE. submitted. 49

Bal, G. (2012c). *Inside Out*, chapter Hybrid inverse problems and internal functionals. Cambridge University Press. 47

Bal, G. (2012d). *Introduction to Inverse Problems*. to be announced. 2, 4

Bal, G., Bonnetier, E., Monard, F., and Triki, F. (2012a). Inverse diffusion from knowledge of power densities. *to appear in Inverse Problems and Imaging*. arXiv:1110.4577. 58, 59, 76, 86, 108, 123, 145, 170

Bal, G., Guo, C., and Monard, F. (2012b). The linearized inverse anisotropic conductivity problem with power densities: a pseudo-differential approach. *In preparation.* 59

Bal, G. and Jollivet, A. (2008). Stability estimates in stationary inverse transport. *Inverse Probl. Imaging*, 2(4):427–454. 15, 19

Bal, G. and Jollivet, A. (2009). Time-dependent angularly averaged inverse transport. *Inverse problems*, 25:075010. 19

Bal, G. and Jollivet, A. (2010a). Approximate stability estimates in inverse transport theory. In Censor, Y., Jiang, M., and Wang, G., editors, *Biomedical Mathematics*. 19

Bal, G. and Jollivet, A. (2010b). Stability for time dependent inverse transport. *SIAM J. Math. Anal.*, 42(2):679–700. 15, 19

Bal, G., Jollivet, A., Langmore, I., and Monard, F. (2011a). Angular average of time-harmonic transport solutions. *Comm. PDE*, 36(6):1044–1070. 19, 33, 34, 35, 39, 42, 178

Bal, G., Langmore, I., and Monard, F. (2008). Inverse transport with isotropic sources and angularly-averaged measurements. *Inverse Probl. Imaging*, 2(1). 16, 19, 24, 26, 27, 28, 29, 30, 31, 32, 42, 178

Bal, G. and Monard, F. (2010). An accurate solver for forward and inverse transport. *Journal of Comp. Phys.*, 229(13). 42

Bal, G. and Ren, K. (2011). Multiple-source quantitative photoacoustic tomography. *Inverse Problems*, 27(7). 100

Bal, G. and Ren, K. (2012). On multi-spectral quantitative photoacoustic tomography. *Inverse Problems*, 28:025010. 49

Bal, G., Ren, K., Uhlmann, G., and Zhou, T. (2011b). Quantitative thermo-acoustics and related problems. *Inverse Problems*, 27(5):055007. 49

Bal, G. and Schotland, J. C. (2010). Inverse scattering and acousto-optic imaging. *Phys. Rev. Letters*, 104:043902. 76

Bal, G. and Tamasan, A. (2007). Inverse source problem in transport equations. *SIAM J. Math. Anal.*, 39(1):57–76. 26

Bal, G. and Uhlmann, G. (2010). Inverse diffusion theory of photoacoustics. *Inverse Problems*, 26(8). 49, 210

Bal, G. and Uhlmann, G. (2012a). Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions. *submitted*. 3, 49, 180

Bal, G. and Uhlmann, G. (2012b). Reconstructions for some coupled-physics inverse problems. *Applied Mathematics Letters (to appear)*. 180

Bercoff, J., Tanter, M., and Fink, M. (2004). Supersonic shear imaging: a new technique for soft tissue elasticity mapping. *IEEE Trans. Ultrasonics Ferro. Freq. Control*, 51:396–409. 75

Bhatia, R. (1996). *Matrix Analysis*. Springer, New York. 191, 192, 193, 197

Borcea, L. (2002). Electrical Impedance Tomography (topical review). *Inverse Problems*, 18:R99–R136. 67

Briane, Milton, and Nesi, V. (2004). Change of sign of the correctors determinant for homogenization in three-dimensional conductivity. *Arch. Rat. Mech. Anal.*, 173(1):133–150. 143, 170

Calderón, A. (1980). On an inverse boundary value problem. *Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matematica, Rio de Janeiro*, pages 65–73. 4, 28, 54, 66, 67, 209, 210

Capdeboscq, Y., Fehrenbach, J., de Gournay, F., and Kavian, O. (2009). Imaging by modification: Numerical reconstruction of local conductivities from corresponding power density measurements. *SIAM Journal on Imaging Sciences*, 2:1003–1030. 58

Choulli, M. and Stefanov, P. (1996a). Inverse scattering and inverse boundary value problems for the linear Boltzmann equation. *Comm. Partial Differential Equations*, 21(5-6):763–785. 15

Choulli, M. and Stefanov, P. (1996b). Reconstruction of the coefficients of the stationary transport equation from boundary measurements. *Inverse Problems*, 12(5):L19–L23. 15, 19, 44

Choulli, M. and Stefanov, P. (1999). An inverse boundary value problem for the stationary transport equation. *Osaka J. Math.*, 36(1):87–104. 15, 17, 18, 19, 44

Cox, B. T., Laufer, J. G., and Beard, P. C. (2009). The challenges for quantitative photoacoustic imaging. *Proc .of SPIE*, 7177:717713. 49

Dautray, R. and Lions, J.-L. (1993). *Mathematical Analysis and Numerical Methods for Science and Technology*, volume 6. Springer Verlag, Berlin. 17, 18, 21

Dos Santos Fereira, D., Kenig, C. E., Salo, M., and Uhlmann, G. (2009). Limiting Carleman weights and anisotropic inverse problems. *Inventiones Mathematicae*, 178:119–171. 70, 71

Evans, L. (1998). *Partial Differential Equations*. Graduate Studies in Mathematics Vol.19, AMS. 62, 64, 154

Folland, G. B. (1995). *Introduction to Partial Differential Equations*. Princeton University Press, Princeton New Jersey. 29

Gebauer, B. and Scherzer, O. (2009). Impedance-Acoustic Tomography. *SIAM J. Applied Math.*, 69(2):565–576. 57, 58, 76

Geselowitz, D. (1971). An application of electrocardiographic lead theory to impedance plethysmography. *IEEE Trans. Biomed. Eng.*, 18:38–41. 57

Greenleaf, A., Lassas, M., and Uhlmann, G. (2003). The Calderón problem for conormal potentials, i: Global uniqueness and reconstruction. *Comm. Pure Appl. Math.*, 56:328–352. 68

Guillarmou, C. and Tzou, L. (2012). *Inside Out*, chapter Survey on Calderón inverse problem in dimension 2. Cambridge University Press. 67

Haberman, B. and Tataru, D. (2011). Uniqueness in Calderón's problem with Lipschitz conductivities. *arXiv:1108.6068*. 68

Hörmander, L. (1971). Fourier integral operators. i. *Acta Mathematica*, 127(1):79–183. 6

Isakov, V. (2006). *Inverse Problems for Partial Differential Equations*. Number 127 in Applied Mathematical Science. Springer, 2 edition. 2, 63, 64

Jossinet, J., Lavandier, B., and Cathignol, D. (1999). Impedance modulation by pulsed ultrasound. *Ann. New York Acad. Sci.*, 873:396407. 57

Jossinet, J., Trillaud, C., and Chesnais, S. (2005). Impedance changes in liver tissue exposed in vitro to high-energy ultrasound. *Physiol. Meas.*, 26:4958. 57, 75

Kaipio, J. and Somersalo, E. (2004). *Statistical and computational inverse problems*, volume 160 of *Applied Mathematical Sciences*. Springer. 2

Kapralov, M. and Katsevich, A. (2008). A study of 1PI algorithms for a general class of curves. *SIAM J. Imaging Sci.*, 1(4):418–459. 37

Katsevich, A. and Kapralov, M. (2007). Filtered backprojection inversion of the cone beam transform for a general class of curves. *SIAM J. Appl. Math.*, 68(2):334–353. 37

Körber, F. (1909). Über den Einfluss des Druckes auf das elektrolytische Leitvermögen von Lösungen. *Z. Phys. Chem.*, 67:212–248. 57

Kuchment, P. (2012). Mathematics of hybrid imaging. a brief review. *to appear in The Mathematical Legacy of Leon Ehrenpreis*. 47, 48, 78

Kuchment, P. and Kunyansky, L. (2010). Synthetic focusing in ultrasound modulated tomography. *Inverse Probl. Imaging*, 4:665–673. 57, 75, 76

Kuchment, P. and Kunyansky, L. (2011). 2d and 3d reconstructions in acousto-electric tomography. *Inverse Problems*, 27. 58

Kuchment, P. and Steinhauer, D. (2011). Stabilizing inverse problems by internal data. arXiv:1110.1819. 59, 100

Kühnel, W. (2006). *Differential Geometry, Curves-Surfaces-Manifolds*. Number 16 in Student Mathematical Library. American Mathematical Society, 2 edition. 206, 208

Langmore, I. (2008). The stationary transport problem with angularly averaged measurements. *Inverse Problems*, 24:015024. 19

Langmore, I. and McDowall, S. (2008). Optical tomography for variable refractive index with angularly averaged measurements. *Comm. PDE*, 33:2180–2207. 19

Lassas, M. and Uhlmann, G. (2001). On determining a riemannian manifold from the dirichlet-to-neumann map. *Annales scientifiques de l'École Normale Suprieure*, 34:771–787. 70

Laugesen, R. (1996). Injectivity can fail for higher-dimensional harmonic extensions. *Complex Var. Theory Appl.*, 28:357–369. 143, 170

Leakeas, C. and Larsen, E. W. (2001). Generalized Fokker-Planck approximations of particle transport with highly forward-peaked scattering. *Nucl. Sci. Eng.*, 137:236–250. 16, 19

Lee, J. (1997). *Riemannian Manifolds, An Introduction to Curvature.*, volume 176 of *Graduate Texts in Mathematics*. Springer. 205, 206

Lee, J. M. and Uhlmann, G. (1989). Determining anisotropic real-analytic conductivities by boundary measurements. *Communications on Pure and Applied Mathematics*, 42(8):1097–1112. 70

McDowall, S., Stefanov, P., and Tamasan, A. (2010). Gauge equivalence in stationary radiative transport through media with varying index of refraction. *Inverse Probl. Imaging*, 4(1):151–167. 3

McLaughlin, J., Zhang, N., and Manduca, A. (2010). Calculating tissue shear modules and pressure by 2d log-elastographic methods. *Inverse Problems*, 26:085007, 25. 51

Mokhtar-Kharroubi, M. (1997). *Mathematical Topics in Neutron Transport Theory.* World Scientific, Singapore. 21

Monard, F. and Bal, G. (2012a). Inverse anisotropic diffusion from power density measurements in two dimensions. *to appear in Inverse Problems*. arXiv:1110.4606. 59, 167

Monard, F. and Bal, G. (2012b). Inverse conductivity from power density functionals, the general case. in preparation. 136

Monard, F. and Bal, G. (2012c). Inverse diffusion problems with redundant internal information. *to appear in Inverse Problems and Imaging*. arXiv:1106.4277. 53, 58, 59, 101, 113, 121, 155, 156, 170

Monard, F. and Bal, G. (2012d). Inverse transport with isotropic time-harmonic sources. *SIAM J. Math. Anal.*, 44(1):134–161. 19, 24, 26, 37, 39, 41, 178

Nachman, A. (1996). Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math.*, 143(2):71–96. 68

Nachman, A., Tamasan, A., and Timonov, A. (2010). Reconstruction of planar conductivities in subdomains from incomplete data. *SIAM J. Appl. Math.*, 70(8):3342–3362. 51

Nachman, A., Tamasan, A., and Timonov, A. (2011). Current density impedance imaging. *AMS series in Contemporary Mathematics*, 559. 51

Natterer, F. (2001). *The Mathematics of Computerized Tomography.* SIAM. 2, 6, 7

Päivärinta, L., Panchenko, A., and Uhlmann, G. (2003). Complex geometrical optics solutions for Lipschitz conductivities. *Revista Matematica Iberoamericana*, 19:57–72. 68

Qian, J., Stefanov, P., Uhlmann, G., and Zhao, H.-K. (2011). An efficient Neumann-series based algorithm for the thermoacoustic and photoacoustic tomography with variable sound speed. *SIAM J. Imaging Sciences*, 4:850–883. 26, 78

Reed, M. and Simon, B. (1981). *Functional Analysis (Methods of Modern Mathematical Physics)*, volume 1. Academic Press. 199

Santacesaria, M. (2011). New global stability estimates for the Calderón problem in two dimensions. *arXiv:1110.0335*. 8, 69

Scherzer, O. (2011). *Handbook of mathematical methods in imaging*. Springer Verlag, New York. 2

Seo, J. K. and Woo, E. J. (2011). Magnetic resonance electrical impedance tomography (MREIT). *SIAM Review*, 53(1):40–68. 51

Spivak, M. (1999). *A comprehensive introduction to Differential Geometry*, volume 2. Publish or Perish. 208

Stefanov, P. and Uhlmann, G. (2003). Optical tomography in two dimensions. *Methods and Applications of Analysis*, 10(1):001–010. 19

Stefanov, P. and Uhlmann, G. (2009). Thermoacoustic tomography with variable sound speed. *Inverse Problems*, 25:075011. 26, 78

Stefanov, P. and Uhlmann, G. (2011). Thermoacoustic tomography arising in brain imaging. *Inverse Problems*, 27:045004. 26, 78

Stefanov, P. and Uhlmann, G. (2012). *Inside Out*, chapter Multi-wave methods by ultrasounds. Cambridge University Press. 48, 78

Stuelpnagel, J. (1964). On the parametrization of the three-dimensional rotation group. *SIAM Review*, 6. 125

Sun, Z. and Uhlmann, G. (2003). Anisotropic inverse problems in two dimensions. *Inverse Problems*, 19:1001–1010. 70

Sylvester, J. (1990). An anisotropic boundary value problem. *Comm. Pure Appl. Math.*, 43:201–232. 70

Sylvester, J. and Uhlmann, G. (1987). A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169. 67, 209, 210

Uhlmann, G. (2007). Commentary on Calderón's paper 28, On an Inverse Boundary Value Problem. 67

Wang, L. and Wu, S.-I. (2007). *Biomedical optics. Principles and Imaging*. Wiley-Interscience, Hoboken, NJ. 17

Zemp, R. J. (2010). Quantitative photoacoustic tomography with multiple optical sources. *Applied Optics*, 49:3566–3572. 49

Appendix A

Linear algebraic matters

A.1 The Gram-Schmidt process

For an invertible n-by-n matrix $S = (S_1, \ldots, S_n)$ with row vectors $S_1, \ldots, S_n$, we recall that the *Gram-Schmidt* orthonormalization process produces an orthonormal basis $R = (R_1, \ldots, R_n)$ from S, in particular $R \in O(n, \mathbb{R})$. R is unique if we further assume that $S_i \cdot R_i \geq 0$ for all $1 \leq i \leq n$. The algorithm leading from S to R may be defined recursively as

$$R_1 := \frac{X_1}{\|X_1\|}, \quad \text{and} \quad R_{k+1} = \frac{S_{k+1} - \sum_{j=1}^{k}(S_{k+1} \cdot R_j)R_j}{\|S_{k+1} - \sum_{j=1}^{k}(S_{k+1} \cdot R_j)R_j\|}, \quad 1 \leq k < n.$$

It is clear from the construction that we have the property

$$\operatorname{span}\{S_1, \ldots, S_j\} = \operatorname{span}\{R_1, \ldots, R_j\}, \quad 1 \leq j \leq n,$$

and in particular we have, for any $1 \leq j \leq n$, that $S_j \in \operatorname{span}\{R_1, \ldots, R_j\} = \{R_{j+1}, \ldots, R_n\}^{\perp}$ and thus $S_j \cdot R_l = 0$ if $l > j$. Furthermore, in matrix notation, this procedure builds the *QR-decomposition* (here it would be more appropriate to call it RT^{-T}) of S as $S = RT^{-T}$

with T lower triangular and R orthogonal. T also has the expression $T^{-1} = S^T R$, that is $t^{ij} = S_i \cdot R_j$ whence $t^{ij} = 0$ if $j > i$. Taking determinants in the relation $S = RT^T$ and using the Cauchy-Schwarz inequality and the fact that $|\det R| = 1$, we arrive at

$$|\det S| = |\det R||\det T^{-1}| = \prod_{i=1}^{n} t^{ii} = \prod_{i=1}^{n} S_i \cdot R_i \leq \prod_{i=1}^{n} \|S_i\|,$$

which is also referred to as *Hadamard's inequality*. When H is now a Gram matrix of the form $H_{ij} = S_i \cdot S_j$, i.e. $H = S^T S$, then we have that

$$\det H = (\det S)^2 \leq \left(\prod_{i=1}^{n} \|S_i\|\right)^2 = \prod_{i=1}^{n} \|S_i\|^2 = \prod_{i=1}^{n} H_{ii}, \tag{A.1}$$

where we have used Hadamard's inequality in the first inequality. The result of equation (A.1) is also referred to as *Hadamard's determinant theorem*, see (Bhatia, 1996, Theorem II.3.17).

Let us give the expressions of the matrix T and the vector fields $V_{ij} := (\nabla t_{ik}) t^{kj}$ in two and three dimensions as an example. The transition matrix T such that $R = ST^T$ is given by:

$$T = \{t_{ij}\}_{1 \leq i,j \leq 3} = \begin{bmatrix} H_{11}^{-\frac{1}{2}} & 0 & 0 \\ -H_{12} H_{11}^{-\frac{1}{2}} d^{-1} & H_{11}^{\frac{1}{2}} d^{-1} & 0 \\ (H_{12}H_{23} - H_{22}H_{13})(dD)^{-1} & (H_{12}H_{13} - H_{11}H_{23})(dD)^{-1} & dD^{-1} \end{bmatrix}, \tag{A.2}$$

$$\text{with} \quad d := (H_{11}H_{22} - H_{12}^2)^{\frac{1}{2}} \quad \text{and} \quad D = (\det H)^{\frac{1}{2}}.$$

The vector fields V_{ij} defined in (4.8) take the following expression

$$\{V_{ij}\}_{1\leq i,j\leq 3} = \begin{bmatrix} \nabla \log t_{11} & 0 & 0 \\ \frac{t_{22}}{t_{11}}\nabla\frac{t_{21}}{t_{22}} & \nabla \log t_{22} & 0 \\ \frac{t_{33}}{t_{11}}\nabla\frac{t_{31}}{t_{33}} - \frac{t_{21}t_{33}}{t_{11}t_{22}}\nabla\frac{t_{32}}{t_{33}} & \frac{t_{33}}{t_{22}}\nabla\frac{t_{32}}{t_{33}} & \nabla \log t_{33} \end{bmatrix}. \tag{A.3}$$

When $n = 2$, T and V_{ij} are given by the top-left 2×2 blocs of (A.2) and (A.3), respectively.

A.2 Fischer's inequality

A generalization of inequality (A.1) is the following: for a given operator A and $P_1, \ldots, P_r$ a family of orthogonal projections satisfying $\oplus_{i=1}^r P_i = \mathbb{I}_n$, we call *pinching* of A the action of taking A to $\mathcal{C}(A) := \sum_{i=1}^r P_i A P_i$. When A is positive, the so-called *Fischer's inequality* gives an inequality between $\det \mathcal{C}(A)$ and $\det A$ (see (Bhatia, 1996, Problem II.5.6)):

Theorem A.2.1 (Fischer's inequality)**.** *For A a positive operator and $\mathcal{C}(A)$ any pinching of A, then the following inequality holds:*

$$\det A \leq \det \mathcal{C}(A). \tag{A.4}$$

Remark A.2.2. *Note that Hadamard's determinant theorem can be a consequence of that by considering the pinching $\mathcal{C}(A) = \sum_{i=1}^n \mathbf{e}_i \otimes \mathbf{e}_i \ A \ \mathbf{e}_i \otimes \mathbf{e}_i$, with $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ the canonical basis.*

In order to prove Fischer's inequality, we will need to define a little bit of machinery which is treated in greater detail in Bhatia (1996). Here it is enough to restrict to symmetric real-valued matrices, although most of the results presented hold in the hermitian context.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote by $x^\downarrow$ the vector obtained by rearranging the components of x in decreasing order, i.e. $x_1^\downarrow \geq \cdots \geq x_n^\downarrow$. For $x, y \in \mathbb{R}^n$, we say that x is *majorised* by

y, in symbols $x \prec y$, if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}, \quad 1 \leq j \leq n, \qquad \text{and} \qquad \sum_{j=1}^{n} x_j^{\downarrow} = \sum_{j=1}^{n} y_j^{\downarrow}. \tag{A.5}$$

Now, an $n \times n$ real-valued matrix $A = \{a_{ij}\}$ is called *doubly stochastic* if it has non-negative entries and if each of its rows and columns add up to 1, that is,

$$\sum_{i=1}^{n} a_{ij} = 1,\ 1 \leq j \leq n, \qquad \text{and} \qquad \sum_{j=1}^{n} a_{ij} = 1,\ 1 \leq i \leq n. \tag{A.6}$$

A good characterization of doubly stochastic matrices is given in the following theorem, whose proof may be found in Bhatia (1996).

Theorem A.2.3 (Bhatia (1996), Theorem II.1.9)**.** *A matrix A is doubly stochastic if and only if it satisfies $Ax \prec x$ for all vector $x \in \mathbb{R}^n$.*

Using this fact, we are able to prove the following result

Theorem A.2.4 (Schur's theorem)**.** *Let A be an $n \times n$ symmetric matrix. Let $\operatorname{diag}(A)$ denote the vector whose diagonal entries are the diagonal entries of A and $\lambda(A)$ the vector whose coordinates are the eigenvalues of A specified in any order. Then the following holds*

$$\operatorname{diag}(A) \prec \lambda(A). \tag{A.7}$$

Proof. Since A is symmetric, it can be written under the form $A = P^T DP$ with $P = \{p_{ij}\} \in SO(n, \mathbb{R})$ and $D = \{d_{ij}\}$ a diagonal matrix containing the eigenvalues of A, say $d_{ij} = \delta_{ij}\lambda_j$. This implies that $a_{ij} = \sum_{k,l} p_{ki} d_{kl} p_{lj} = \sum_k p_{ki}\lambda_k p_{kj}$, in particular $a_{ii} = \sum_k (p_{ki})^2 \lambda_k$ for $1 \leq i \leq n$. Since $P \in SO(n, \mathbb{R})$, it is easy to check that the matrix $P \circ P := \{p_{ij}^2\}_{1 \leq i,j \leq n}$ is doubly stochastic, and that we have the relation

$$\operatorname{diag}(A) = (P \circ P)\lambda(A),$$

thus the result follows from theorem A.2.3. $\square$

With this result in mind, we are able to prove more interesting properties of symmetric matrices.

Proposition A.2.5 (Ky Fan's maximum principle)**.** *For A a given $n \times n$ symmetric matrix, the following holds for $1 \leq k \leq n$*

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \max \sum_{j=1}^{k} x_j \cdot Ax_j, \tag{A.8}$$

where the maximum is taken over all orthonormal k-tuples of vectors $\{x_1, \ldots, x_k\}$ in $\mathbb{R}^n$.

Proof. Fix $1 \leq k \leq n$ and pick $x_1, \ldots, x_k$ orthonormal vectors, and complete them into an orthonormal basis $x_1, \ldots, x_n$. Call X the matrix whose k-th column is x_j. Then the matrix $X^T AX$ is symmetric and such that, up to reindexing the spectra, $\lambda(X^T AX) = \lambda(A)$, this is because (λ, v) is an eigenpair of A if and only if $(\lambda, X^T v)$ is an eigenpair of $X^T AX$. Using Schur's theorem, we deduce that

$$\operatorname{diag}(X^T AX) \prec \lambda(X^T AX) = \lambda(A).$$

In particular, at the k-th sum,

$$\sum_{j=1}^{k} x_j \cdot Ax_j \leq \sum_{j=1}^{k} (\operatorname{diag}(X^T AX))_j^{\downarrow} \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}(A).$$

Since this is true for any orthonormal $\{x_1, \ldots, x_k\}$, the set $\{\sum_{j=1}^k x_j \cdot Ax_j | x_1, \ldots, x_k \text{ orthonormal}\}$ has a supremum, smaller than $\sum_{j=1}^k \lambda_j^{\downarrow}(A)$. Moreover, this supremum is achieved and equal to $\sum_{j=1}^k \lambda_j^{\downarrow}(A)$ when $x_1, \ldots, x_k$ is an orthonormal collection of eigenvectors associated to the k largest eigenvalues of A. Hence the result. □

We have the straightforward corollary

Corollary A.2.6. *For A, B two symmetric $n \times n$ matrices, and $1 \leq k \leq n$, we have*

$$\sum_{j=1}^k \lambda_j^{\downarrow}(A+B) \leq \sum_{j=1}^k \lambda_j^{\downarrow}(A) + \sum_{j=1}^k \lambda_j^{\downarrow}(B). \tag{A.9}$$

Proof. Fix $1 \leq k \leq n$ and pick an orthonormal family $x_1, \ldots, x_k$. Then, using Proposition A.2.5, we have

$$\sum_{j=1}^k x_j \cdot (A+B)x_j = \sum_{j=1}^k x_j \cdot Ax_j + \sum_{j=1}^k x_j \cdot Bx_j \leq \sum_{j=1}^k \lambda_j^{\downarrow}(A) + \sum_{j=1}^k \lambda_j^{\downarrow}(B).$$

Passing to the maximum over all orthonormal $\{x_1, \ldots, x_k\}$ in the left-hand side and using Ky Fan's maximum principle again yields the result. □

We are now ready to show the following proposition, of which Fischer's inequality is a corollary

Proposition A.2.7. *For A an $n \times n$ symmetric matrix, let $\mathcal{C}(A)$ be a pinching of type 2×2, that is, $\mathcal{C}(A) = PAP + (\mathbb{I}_n - P)A(\mathbb{I}_n - P)$ for some projection P. Then we have the following*

$$\lambda(\mathcal{C}(A)) \prec \lambda(A). \tag{A.10}$$

Proof. Let $r = \text{rank}(P)$ and, without loss of generality, assume that A is represented in a basis

$\mathbf{x}_1, \ldots, \mathbf{x}_n$ in which $P = \sum_{j=1}^{r} \mathbf{x}_j \otimes \mathbf{x}_j$. Writing A under the block form $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ with A_{11} of size $r \times r$, we have that $\mathcal{C}(A) = \left(\begin{smallmatrix} A_{11} & 0 \\ 0 & A_{22} \end{smallmatrix}\right)$. If $U := \left(\begin{smallmatrix} \mathbb{I}_r & 0 \\ 0 & -\mathbb{I}_{n-r} \end{smallmatrix}\right)$, then we also have the relation

$$\mathcal{C}(A) = \frac{1}{2}(A + UAU),$$

where both A and UAU are symmetric, and satisfy, up to reindexing, $\lambda(A) = \lambda(UAU)$, since (λ, v) is an eigenpair of A if and only if (λ, Uv) is an eigenpair of UAU. We now use the inequality (A.9) with $B \equiv UAU$ and obtain

$$2\sum_{j=1}^{k} \lambda_j^{\downarrow}(\mathcal{C}(A)) \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}(A) + \sum_{j=1}^{k} \lambda_j^{\downarrow}(UAU) = 2\sum_{j=1}^{k} \lambda_j^{\downarrow}(A).$$

It remains to check that $\sum_{j=1}^{n} \lambda_j^{\downarrow}(\mathcal{C}(A)) = \sum_{j=1}^{n} \lambda_j^{\downarrow}(A)$, i.e. tr $(\mathcal{C}(A)) =$ tr (A), which comes from the fact that pinching leaves the diagonal unchanged. This concludes the proof. $\square$

The following corollary generalizes the previous proposition to any pinching, since each pinching can be written as the composition of $r-1$ pinchings of 2×2 type.

Corollary A.2.8. *Inequality (A.10) remains true when the pinching $\mathcal{C}(A)$ comes from $r \geq 2$ orthogonal projections $P_1, \ldots, P_r$ such that $\oplus_{j=1}^{r} P_j = \mathbb{I}_n$.*

Corollary A.2.8 states that for A an $n \times n$ symmetric matrix and $\mathcal{C}(A)$ any pinching of A, we have

$$\lambda(\mathcal{C}(A)) \prec \lambda(A). \tag{A.11}$$

To obtain Fischer's inequality, we need one last definition, that of *Schur-convexity*: $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *Schur-convex* if for any $x, y \in \mathbb{R}^n$ we have that $x \prec y$ implies $\varphi(x) \leq \varphi(y)$. φ is *Schur-concave* if $-\varphi$ is Schur-convex.

One can show (see e.g. (Bhatia, 1996, Example II.3.16)) that the function $\mathbb{R}^n \ni x \mapsto S_n(x) = \prod_{j=1}^n x_j$ is Schur-concave over $\mathbb{R}_+^n$. Therefore, when A is a nonnegative symmetric $n \times n$ matrix, we have that $\lambda(A)$ and $\lambda(\mathcal{C}(A))$ belong to $\mathbb{R}_+^n$, and combining (A.11) with the Schur-concavity of S_n defined above yields Fischer's inequality.

A.3 Establishing relations of linear dependence using inner products

Lemma A.3.1. *Let $(V_1, \dots, V_{n+1})$ be $n+1$ vectors in $\mathbb{R}^n$, and denote $H_{ij} = V_i \cdot V_j$ for $1 \le i, j \le n$. Then the following linear dependence relation $\sum_{i=1}^{n+1} \mu_i V_i = 0$ holds with coefficients*

$$\begin{aligned} \mu_i &= -\det(V_1, \dots, V_n) \cdot \det(V_1, \dots, \underbrace{V_{n+1}}_{i}, \dots, V_n), \\ &= (-1)^{i+n+1} \det\{H_{pq}\}_{1 \le p \le n, 1 \le q \le \le n+1, q \ne i}, \quad 1 \le i \le n, \\ \textit{and} \quad \mu_{n+1} &= \det(V_1, \dots, V_n)^2 = \det\{H_{ij}\}_{1 \le i,j \le n}. \end{aligned} \tag{A.12}$$

Proof. Define the μ_i's as in the statement of the function and let us show that $\sum_{i=1}^{n+1} \mu_i V_i = 0$. Consider the vector field defined by the following formal $(n+1) \times (n+1)$ determinant

$$V = \det \begin{pmatrix} V_1 \cdot V_1 & \cdots & V_1 \cdot V_n & V_1 \cdot V_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ V_n \cdot V_1 & \cdots & V_n \cdot V_n & V_n \cdot V_{n+1} \\ V_1 & \cdots & V_n & V_{n+1} \end{pmatrix},$$

i.e. computed by expanding along the last row. Then we have

$$
\begin{aligned}
V &= \sum_{i=1}^{n+1} (-1)^{i+n+1} \det\left(\{H_{pq}\}_{1\le p\le n, 1\le q\le n+1, q\neq i}\right) V_i \\
&= \sum_{i=1}^{n+1} (-1)^{i+n+1} \det(V_1,\dots,V_n)\cdot\det(V_1,\dots,V_{\hat{i}},\dots,V_{n+1})V_i \\
&= -\sum_{i=1}^{n} \det(V_1,\dots,V_n)\cdot\det(V_1,\dots,\underbrace{V_{n+1}}_{i},\dots,V_n)V_i + \det(V_1,\dots,V_n)^2 V_n = \sum_{i=1}^{n+1}\mu_i V_i,
\end{aligned}
$$

where moving V_{n+1} back to the i-th position in the i-th requires $n-i$ sign flips. We now show that $V = 0$. For $1 \le i \le n$, the dotproduct $V\cdot V_i$ becomes a determinant of a matrix whose rows of indices i and $n+1$ are equal, therefore $V\cdot S_i = 0$. Moreover, $V\cdot S_{n+1}$ is nothing but the determinant of the Gramian matrix of $(V_1,\dots,V_{n+1})$, which is zero since $n+1$ vectors are necessarily linearly dependent. Concluding, we have

$$
V\cdot V = \sum_{i=1}^{n+1}\mu_i V\cdot V_i = 0,
$$

thus $V = 0$, hence the lemma. □

A.4 The positive squareroot of a matrix

For H a n-by-n positive definite symmetric matrix, there exists a unique positive definite symmetric matrix $H^{\frac{1}{2}}$ such that $H^{\frac{1}{2}}H^{\frac{1}{2}} = H$. Via diagonalization, i.e. writing $H = PDP^T$ with D a diagonal matrix with diagonal elements the eigenvalues of H and P orthogonal such that its, vector columns form a basis of eigenvectors of H, it is easy to define $H^{\frac{1}{2}} = PD^{\frac{1}{2}}P^T$, where $D^{\frac{1}{2}}$ is the diagonal matrix whose entries are the positive squareroots of the entries of D. However, in order to show that the entries of $H^{\frac{1}{2}}$ are smooth functions of the entries of H, we need more explicit expressions of the entries of $H^{\frac{1}{2}}$.

A.4.1 Construction for general n

We start from the following lemma, see e.g. (Reed and Simon, 1981, Sec. VI.4 p. 195).

Lemma A.4.1. *The power series for* $\sqrt{1-z} \overset{z=0}{=} \sum_{p=0}^{\infty} c_p z^p$ *about zero converges absolutely for all complex numbers satisfying* $|z| \leq 1$.

On symmetric non-negative matrices $S_n^+(\mathbb{R})$, the following quantity is a norm

$$\|A\| := \sup_{X \neq 0} \frac{AX \cdot X}{X \cdot X}, \tag{A.13}$$

equal to the maximum eigenvalue of A and such that, for any integer power k of A, we clearly have $\|A^k\| = \|A\|^k$. Moreover, if $A \neq 0$, then we claim that $\mathbb{I}_n - \frac{A}{\|A\|}$ has spectral radius less than 1. Indeed, any $A \in S_n^+(\mathbb{R})$ may be written as

$$A = \sum_{\lambda \in \lambda(A)} \lambda P_\lambda,$$

where P_λ is an orthogonal projection such that $\sum_{\lambda \in \lambda(A)} P_\lambda = \mathbb{I}_n$. Now it is straightforward to derive the relation

$$\mathbb{I}_n - \frac{A}{\|A\|} = \sum_{0 \leq \lambda < \|A\|} \left(1 - \frac{\lambda}{\|A\|}\right) P_\lambda =: K,$$

where the right-hand side is a symmetric nonnegative matrix with $\|K\| < 1$, thus the power series $\sum_{p=0}^{\infty} c_p K^p$ makes sense (i.e. is absolutely convergent), and, from the relation $A = \|A\|(\mathbb{I}_n - K)$, one may define the squareroot of A as

$$A^{\frac{1}{2}} := \sqrt{\|A\|} \sum_{p=1}^{\infty} c_p K^p = \sqrt{\|A\|} \sum_{p=0}^{\infty} c_p \left(\mathbb{I}_n - \frac{A}{\|A\|} \right)^p . \tag{A.14}$$

As the spectral radius is a smooth function of the entries of A away from $A = 0$ and the series $\sum_{p\geq 0} c_p z^p$ is analytic in its convergence disk, the squareroot function is thus a smooth function of the entries of A.

A.4.2 Example: the case $n = 2$

Recall Cayley-Hamilton's theorem for 2-by-2 matrices

$$H^2 - (\text{tr } H)H + (\det H)\mathbb{I}_2 = 0. \tag{A.15}$$

Judging by the construction (A.14) and the fact that any integer power of H may be expressed as a linear combination of $\mathbb{I}_2$ and H via the relation (A.15), $H^{\frac{1}{2}}$ may be sought as a first-order polynomial of H. We thus write

$$H^{\frac{1}{2}} = \alpha \mathbb{I}_2 + \beta H, \quad \alpha, \beta \in \mathbb{R},$$

with (α, β) to be determined. If $H = \lambda \mathbb{I}_2$, then clearly $H^{\frac{1}{2}} = \sqrt{\lambda}\mathbb{I}_2$. If $\mathbb{I}_2$ and H are linearly independent, writing the relation $H^{\frac{1}{2}} H^{\frac{1}{2}} = H$ and equating terms in front of $\mathbb{I}_2$ and H yields the relations

$$\alpha^2 = \beta^2 \det H \quad \text{ and } \quad 2\alpha\beta + \beta^2 \text{tr } H = 1,$$

which, after choosing $\alpha \geq 0$ and $\beta \geq 0$, yields

$$\alpha = \beta\sqrt{\det H}, \quad \beta = \left(2\sqrt{\det H} + \mathrm{tr}\ H\right)^{-\frac{1}{2}},$$

that is,

$$H^{\frac{1}{2}} = \left(2\sqrt{\det H} + \mathrm{tr}\ H\right)^{-\frac{1}{2}} \left(\sqrt{\det H}\mathbb{I}_2 + H\right). \tag{A.16}$$

With tr H and det H bounded away from zero, we clearly have that the entries of $H^{\frac{1}{2}}$ are smooth functions of the entries of H.

For $n \geq 3$, a similar process may be done by searching for a linear combination of $\mathbb{I}_n, H, \cdots, H^{n-1}$, although this becomes a tedious process.

Appendix B

Push-forwards by diffeomorphisms

Diffeomorphisms are useful for transporting differential structures from one domain[1] to another. Here and below, we call X an open bounded domain in $\mathbb{R}^n$.

The first thing that one may push-forward are functions, that is, for $\Psi : X \to \Psi(X)$ a diffeomorphism, then the *push-forward of a function* $u : X \to \mathbb{R}$ by Ψ is nothing but the function $v : \Psi(x) \to \mathbb{R}$ defined by $v := u \circ \Psi^{-1}$. At any given point $x \in X$, if $\Psi(x) = (\psi_1(x), \cdots, \psi_n(x))$ with the ψ_i's scalar functions, the *jacobian matrix* of Ψ at x, denoted $D\Psi(x)$, is the $n \times n$ matrix with entries

$$[D\Psi(x)]_{ij} = \frac{\partial \psi_i}{\partial x^j}(x), \quad 1 \le i, j \le n.$$

With the above notation, if we have $u = v \circ \Psi$ with u, v smooth, then we have the chain rule

$$\nabla_x u(x) = D\Psi^T(x) \nabla_y v(\Psi(x)), \tag{B.1}$$

[1]or more generally, manifold

where y denotes coordinates in $\Psi(X)$.

We now consider push-forwards of conductivity tensors by $W^{1,2}$-diffeomorphisms. For $\gamma \in \Sigma(X)$, let us consider the conductivity equation (4.51) over X. Then if $\Psi : X \to \Psi(X)$ is a $W^{1,2}$-diffeomorphism, one may show that the push-forward $v = u \circ \Psi^{-1}$ of any solution u of (4.51) solves another conductivity equation over $\Psi(X)$, given by

$$-\nabla \cdot (\Psi_\star\gamma)\nabla v = 0 \quad (\Psi(X)), \quad v|_{\partial(\Psi(X))} = u|_{\partial X} \circ \Psi^{-1}, \tag{B.2}$$

where we have defined $\Psi_\star\gamma$, the *push-forward* of γ by Ψ, by the relation

$$\Psi_\star\gamma(x) = \left[\frac{D\Psi\ \gamma\ D\Psi^T}{|\det D\Psi|}\right] \circ \Psi^{-1}(x), \quad x \in \Psi(X). \tag{B.3}$$

As it is explained in Astala et al. (2005), $W^{1,2}$-diffeomorphisms are precisely those that conserve the property $C_\gamma < \infty$ among conductivity tensors. Namely, if $\gamma \in \Sigma(X)$ and $\Psi : X \to \Psi(X)$ a $W^{1,2}$-diffeomorphism, then $\Psi_\star\gamma \in \Sigma(\Psi(X))$.

The fact that u solves (4.51) if and only if $v = u \circ \Psi^{-1}$ solves (B.2) can easily be obtained by noticing the identity

$$\int_X \nabla u \cdot \gamma \nabla u\ dx = \int_{\Psi(X)} \nabla v \cdot \Psi_\star\gamma\nabla v\ dx', \quad u = v \circ \Psi, \tag{B.4}$$

so that u minimizes the left-hand side whenever v minimizes the right-hand side.

Since we expect these diffeomorphisms to transport the natural settings in which both conductivity equations are defined, it is a fact that $W^{1,2}$-diffeomorphisms induce isomorphisms between $H^1(X)$ and $H^1(\Psi(X))$ and between $H^{\frac{1}{2}}(\partial X)$ and $H^{\frac{1}{2}}(\partial(\Psi(X)))$, see (Astala et al., 2005, p. 13).

Push-forward of power densities and related quantities

Regarding the power densities, if $u = v \circ \Psi$, using the chain rule, we see that

$$\begin{aligned}\nabla u(x) \cdot \gamma(x) \nabla u(x) &= \nabla v(\Psi(x)) \cdot D\Psi(x)\ \gamma(x)\ D\Psi^T(x) \nabla v(\Psi(x)) \\ &= |J\Psi(x)| \left(\nabla v \cdot \Psi_\star \gamma \nabla v\right)(\Psi(x)),\end{aligned}$$

where $J\Psi = \det D\Psi$. Thus, under diffeomorphism, the power densities transform as follows

$$\mathcal{H}_\gamma[g](x) = |J\Psi(x)| \mathcal{H}_{\Psi_\star \gamma}[g \circ \Psi^{-1}](\Psi(x)), \quad x \in X. \tag{B.5}$$

Now considering the matrices Z defined by considering families of $n+1$ solutions $(u_1, \cdots, u_n, v)$ of (4.51), i.e. $Z = [Z_1|\cdots|Z_n]$ where for $1 \le i \le n$, the vector field Z_i is defined by (4.103), and considering the corresponding solutions $(u_1', \cdots, u_n', v') := (u_1, \cdots, u_n, v) \circ \Psi^{-1}$ and defining Z' similarly, then using the chain rule, it is straightforward to establish the relation

$$Z_i(x) = D\Psi^T(x) Z_i'(\Psi(x)), \quad 1 \le i \le n, \quad \text{i.e.} \quad Z(x) = D\Psi^T(x) Z'(\Psi(x)), \quad x \in X. \tag{B.6}$$

Since the matrix H of power densities corresponding to the support basis $(u_1, \cdots, u_n)$ follows the transformation law (B.5) componentwise, the space $P_Z(x) := \{Z(x)(\mathbb{R}\mathbb{I}_n + H(x)A_n(\mathbb{R}))\}$ defined in (4.99) admits the transformation law

$$P_Z(x) = \{D\Psi^T(x) Z'(\Psi(X))(\mathbb{R}\mathbb{I}_n + |J_\Psi(x)|^n H'(\Psi(x)) A_n(\mathbb{R})\}. \tag{B.7}$$

Appendix C

Elements of Riemannian geometry

Restricting to the purposes of the present manuscript, we work on a convex set $\Omega \subset \mathbb{R}^n$ with the Euclidean metric $g(X, Y) \equiv X \cdot Y = \delta_{ij} X^i Y^j$ on $\mathbb{R}^n$. This metric induces a natural isomorphism between the tangent and cotangent bundles, given by $T\Omega \ni (x, X) \mapsto (x, X^\flat) \in T^\star\Omega$ where $X^\flat(Y) = g(X, Y)$. Its inverse is thus expressed as $T^\star\Omega \ni (x, \omega) \mapsto (x, \omega^\sharp) \in T\Omega$ with $g(\omega^\sharp, X) = \omega(X)$.

Following Lee (1997), we denote by $\overline{\nabla}$ the Euclidean connection, i.e. the unique[1] connection that is torsion-free, and compatible with the Euclidean metric in the sense that

$$\overline{\nabla}_X(Y \cdot Z) = (\overline{\nabla}_X Y) \cdot Z + Y \cdot (\overline{\nabla}_X Z),$$

for smooth vector fields X, Y, Z. On zero- and one-forms, this connection takes the expression:

$$\overline{\nabla}_X f = X \cdot \nabla f = X^i \partial_i f, \quad \text{and} \quad \overline{\nabla}_X Y = (X \cdot \nabla Y^j)\mathbf{e}_j = X^i(\partial_i Y^j)\mathbf{e}_j,$$

[1] On a general manifold, any metric gives rise to a *unique* connection satisfying the conditions stated above. This particular connection is called the *Levi-Civita* connection.

for given vector fields $X = X^i\mathbf{e}_i$ and $Y = Y^i\mathbf{e}_i$. An important identity is the following characterization of the exterior derivative of a one-form ω

$$d\omega(X, Y) = \overline{\nabla}_X(\omega(Y)) - \overline{\nabla}_Y(\omega(X)) - \omega([X, Y]), \tag{C.1}$$

or equivalently in the Euclidean metric, writing $\omega = Z^\flat$ for some vector field Z,

$$Z \cdot [X, Y] = \overline{\nabla}_X(Z \cdot Y) - \overline{\nabla}_Y(Z \cdot X) - dZ^\flat(X, Y), \tag{C.2}$$

where the Lie bracket (commutator) of X and Y coincides with (and thus may be "defined" here as) $[X, Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X$ by virtue of the torsion-free property.

Frames and Koszul identity: A *frame* refers to an oriented family $E = (E_1, \dots, E_n)$ of n vector fields over Ω such that for every $x \in \Omega$, $(E_1(x), \dots E_n(x))$ is a basis of $T_x\Omega \equiv \mathbb{R}^n$. For a given frame E and $1 \leq i, j \leq n$, $\overline{\nabla}_{E_i}E_j$ is a vector field, and as such decomposes into the frame E. This motivates the definition of the *Christoffel symbols* (of the second kind) with respect to this frame, by the relations

$$\overline{\nabla}_{E_i}E_j = \Gamma^k_{ij}E_k, \quad \text{i.e.} \quad \Gamma^q_{ij} = g^{pq}\overline{\nabla}_{E_i}E_j \cdot E_p, \qquad g_{ij} = E_i \cdot E_j, \quad g^{pq} := (g^{-1})_{pq}. \tag{C.3}$$

The following identity (also referred to as the *Koszul formula* in Kühnel (2006)) allows us to compute the Christoffel symbols from inner products and Lie brackets of a given frame (see e.g. (Lee, 1997, Eq. 5.1 p. 69)):

$$2(\overline{\nabla}_X Y) \cdot Z = \overline{\nabla}_X(Y \cdot Z) + \overline{\nabla}_Y(Z \cdot X) - \overline{\nabla}_Z(X \cdot Y) - Y \cdot [X, Z] - Z \cdot [Y, X] + X \cdot [Z, Y], \tag{C.4}$$

where X, Y, Z are smooth vector fields.

Total covariant derivative of a vector field: For a vector $X = X^j \mathbf{e}_j$, we want to form the matrix of partial derivatives $(\partial_j X^i)_{ij}$. Geometrically, gradients generalize to tensors via *the total covariant derivative*, which maps for instance a vector field X to a tensor of type $(1,1)$ defined by

$$\overline{\nabla} X(\omega, Y) = \omega(\overline{\nabla}_Y X). \tag{C.5}$$

In a given frame E, we may express $\overline{\nabla} E_i$ in the basis $\{E_j \otimes E_k^\flat\}_{j,k=1}^n$ of such tensors by writing $\overline{\nabla} E_i = a_{ijk} E_j \otimes E_k^\flat$ and identifying the coefficients a_{ijk} by writing, on one hand,

$$\overline{\nabla} E_i(E_p^\flat, E_q) = E_p^\flat(\overline{\nabla}_{E_q} E_i) = \overline{\nabla}_{E_q} E_i \cdot E_p = g_{pr} \Gamma^r_{qi},$$

and on the other,

$$\overline{\nabla} E_i(E_p^\flat, E_q) = a_{ijk} E_j \otimes E_k^\flat(E_p^\flat, E_q) = a_{ijk} g_{jp} g_{kq}.$$

Equating the two, we obtain the representation

$$\overline{\nabla} E_i = g^{qk} \Gamma^j_{qi} E_j \otimes E_k^\flat = g^{qk} g^{jp} (\overline{\nabla}_{E_q} E_i \cdot E_p) E_j \otimes E_k^\flat. \tag{C.6}$$

Curvature: The *curvature tensor* $\mathcal{R}$ can be defined in terms of the connection by the following formula for three vector fields X, Y, Z

$$\mathcal{R}(X,Y)Z := \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z. \tag{C.7}$$

Expanding the covariant derivatives using the Christoffel symbols, we obtain the formula

$$\mathcal{R}(E_i, E_j)E_k = [E_i \cdot \nabla\Gamma^p_{jk} - E_j \cdot \nabla\Gamma^p_{ik} + \Gamma^l_{jk}\Gamma^p_{il} - \Gamma^l_{ik}\Gamma^p_{jl} - (\Gamma^l_{ij} - \Gamma^l_{ji})\Gamma^p_{lk}]E_p.$$

As one can see from (Spivak, 1999, Prop. 10 and 12 pp 196-197), the symmetries of the curvature tensor are such that $\mathcal{R}$ is identically zero if and only if it satisfies the $n(n-1)/2$ scalar conditions in a given frame $(E_1, \cdots, E_n)$

$$\frac{\mathcal{R}(E_i, E_j)E_i \cdot E_j}{(E_i \cdot E_i)(E_j \cdot E_j) - (E_i \cdot E_j)^2} = 0, \quad 1 \leq i < j \leq n. \tag{C.8}$$

The left-hand side in (C.8) is called the *sectional curvature* and does not depend on the choice of basis to represent the plane spanned by E_i, E_j at any given point. Furthermore, if we consider the biquadratic form $\kappa(X, Y) = \mathcal{R}(X, Y)Y \cdot X$, one may find in (Kühnel, 2006, Theorem 6.5 p253) that the curvature $\mathcal{R}(X, Y)Z \cdot W$ is an explicit linear combination of terms of the form $\kappa(U, V)$ for U, V various linear combinations of (X, Y, Z, W).

The Euclidean connection satisfies the zero curvature condition. This condition is important to enforce whenever one needs to solve overdetermined PDE's of the form $\overline{\nabla}S = M$, where S is a vector field and M is a tensor of type $(1, 1)$.

Appendix D

On complex geometrical optics solutions

The Complex Geometrical Optics solutions were first introduced in Calderón (1980) in the context of linearized inverse problems, then extended in Sylvester and Uhlmann (1987) in the context of non-linear inverse problems.

In some sense, they are a generalization of the harmonic complex plane waves of the form $x \mapsto e^{\boldsymbol{\rho} \cdot x}$ with $\boldsymbol{\rho} \in \mathbb{C}^n$ such that $\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$, so that $\Delta e^{\boldsymbol{\rho} \cdot x} = \boldsymbol{\rho} \cdot \boldsymbol{\rho} e^{\boldsymbol{\rho} \cdot x} = 0$.

Their main use is the fact that their asymptotic leading term has remarkable features which may be used for proving several results in inverse problems, specifically injectivity of Dirichlet-to-Neumann map arising from the conductivity equation and from the Schrödinger equation with potential as well (in particular, when the conductivity is isotropic $\gamma = \sigma \mathbb{I}_n$ and smooth enough, the former equation may be recast as the latter with potential $-\Delta \sigma^{\frac{1}{2}} / \sigma^{\frac{1}{2}}$).

At several times, this leading behavior comes in handy for our purposes, in particular to control the direction of the gradient of solutions.

Construction of CGO solutions : Consider $\gamma = \sigma \mathbb{I}_n$ an isotropic conductivity tensor. The purpose is to construct solutions of $\nabla \cdot (\sigma \nabla u) = 0$ with leading behavior that is similar in spirit to harmonic complex plane waves. The construction below can be made for any $n \geq 2$.

We first extend the conductivity equation $\nabla \cdot (\sigma(x) \nabla u) = 0$ to $\mathbb{R}^n$, where $\sigma(x)$ is extended in a continuous manner outside of X and such that $\sigma \equiv 1$ outside of a large ball. Assuming that $\sigma|_X \in H^{\frac{n}{2}+3+\varepsilon}(X)$ for some $\varepsilon > 0$ (this implies $\sigma \in \mathcal{C}^3(\overline{X})$ by Sobolev imbedding), it is shown in (Bal and Uhlmann, 2010, Corollary 3.2) following works in Calderón (1980); Sylvester and Uhlmann (1987), that there exist complex-valued solutions of the above full-space diffusion equation of the form

$$u_{\boldsymbol{\rho}} = \frac{1}{\sqrt{\sigma}} e^{\boldsymbol{\rho} \cdot x} (1 + \psi_{\boldsymbol{\rho}}), \tag{D.1}$$

where $\boldsymbol{\rho} \in \mathbb{C}^n$ is a complex frequency satisfying $\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$, which is equivalent to taking $\boldsymbol{\rho} = \rho(\mathbf{k} + i\mathbf{k}^\perp)$ for some $\mathbf{k}, \mathbf{k}^\perp \in \mathbb{S}^{n-1}$ such that $\mathbf{k} \cdot \mathbf{k}^\perp = 0$ and $\rho = |\boldsymbol{\rho}|/\sqrt{2} > 0$. Moreover, the remainder $\psi_{\boldsymbol{\rho}}$ satisfies the PDE

$$\Delta \psi_{\boldsymbol{\rho}} + 2\boldsymbol{\rho} \cdot \nabla \psi_{\boldsymbol{\rho}} = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} (1 + \psi_{\boldsymbol{\rho}}). \tag{D.2}$$

Energy estimates on (D.2) done in Bal and Uhlmann (2010) use the regularity assumed on σ to show that $\rho \psi_{\boldsymbol{\rho}}|_X = O(1)$ in $\mathcal{C}^1(\overline{X})$. Using this estimate, computing the gradient of (D.1) and rearranging term, we arrive at

$$\sqrt{\sigma} \nabla u_{\boldsymbol{\rho}} = e^{\boldsymbol{\rho} \cdot x} (\boldsymbol{\rho} + \boldsymbol{\varphi}_{\boldsymbol{\rho}}), \quad \text{with} \quad \boldsymbol{\varphi}_{\boldsymbol{\rho}} := \nabla \psi_{\boldsymbol{\rho}} + \psi_{\boldsymbol{\rho}} \boldsymbol{\rho} - (1 + \psi_{\boldsymbol{\rho}}) \nabla \sqrt{\sigma}.$$

Because $\nabla \sqrt{\sigma}$ is bounded and $\rho \psi_{\boldsymbol{\rho}}|_X = O(1)$ in $\mathcal{C}^1(\overline{X})$, the $\mathbb{C}^n$-valued function $\boldsymbol{\varphi}_{\boldsymbol{\rho}}$ satisfies $\sup_{\overline{X}} |\boldsymbol{\varphi}_{\boldsymbol{\rho}}| \leq C$ independent of $\boldsymbol{\rho}$. Moreover, the constant C is in fact independent of σ provided that the norm of the latter is bounded by a uniform constant in $H^{\frac{n}{2}+3+\epsilon}(X)$. The properties we will need for our purposes may be summarized into the following lemma.

Lemma D.0.2 (On the existence of CGO's)**.** *Assuming that the isotropic conductivity tensor σ belongs to $H^{\frac{n}{2}+k+\epsilon}(X)$ for some $\epsilon > 0$ and k a positive integer, for any $\boldsymbol{\rho}(\rho, \mathbf{k}, \mathbf{k}^\perp) = \rho(\mathbf{k} + i\mathbf{k}^\perp)$ with $\rho > 0, \mathbf{k}, \mathbf{k}^\perp \in \mathbb{S}^{n-1}$ with $\mathbf{k}\cdot\mathbf{k}^\perp = 0$, there exists a complex-valued $g_{\boldsymbol{\rho}} \in H^{\frac{1}{2}}(\partial X)$ such that the corresponding complex-valued solution $u_{\boldsymbol{\rho}}$ of*

$$\nabla\cdot(\sigma\nabla u_{\boldsymbol{\rho}}) = 0 \quad (X), \quad u_{\boldsymbol{\rho}}|_{\partial X} = g_{\boldsymbol{\rho}},$$

takes the form

$$u_{\boldsymbol{\rho}}(x) = \frac{1}{\sqrt{\sigma}} e^{\boldsymbol{\rho}\cdot x}(1 + \psi_{\boldsymbol{\rho}}),$$

where the function $\psi_{\boldsymbol{\rho}}$ satisfies the estimate

$$\rho\|\psi_{\boldsymbol{\rho}}\|_{\mathcal{C}^k(\overline{X})} + \|\psi_{\boldsymbol{\rho}}\|_{\mathcal{C}^{k+1}(\overline{X})} \le C\|q\|_{H^{\frac{n}{2}+k+\epsilon}(X)}, \quad q := -\Delta\sigma^{\frac{1}{2}}/\sigma^{\frac{1}{2}}.$$

Since we use at several times in the manuscript gradients of CGO solutions, we now write down the real and imaginary parts of $\sqrt{\sigma}\nabla u_{\boldsymbol{\rho}}$:

$$\begin{aligned}
\sqrt{\sigma}\nabla u_{\boldsymbol{\rho}}^{\Re} &= \rho e^{\rho\mathbf{k}\cdot x}\left((\mathbf{k} + \rho^{-1}\boldsymbol{\varphi}_{\boldsymbol{\rho}}^{\Re})\cos(\rho\mathbf{k}^\perp\cdot x) - (\mathbf{k}^\perp + \rho^{-1}\boldsymbol{\varphi}_{\boldsymbol{\rho}}^{\Im})\sin(\rho\mathbf{k}^\perp x)\right), \\
\sqrt{\sigma}\nabla u_{\boldsymbol{\rho}}^{\Im} &= \rho e^{\rho\mathbf{k}\cdot x}\left((\mathbf{k}^\perp + \rho^{-1}\boldsymbol{\varphi}_{\boldsymbol{\rho}}^{\Im})\cos(\rho\mathbf{k}^\perp\cdot x) + (\mathbf{k} + \rho^{-1}\boldsymbol{\varphi}_{\boldsymbol{\rho}}^{\Re})\sin(\rho\mathbf{k}^\perp x)\right).
\end{aligned} \tag{D.3}$$

Index

admissibility sets

$\mathcal{A}_{\gamma}^{m,l}$, 160

$\mathcal{G}_{\gamma}^{m}$, 143

properties, 161, 163

Albedo decomposition, 20

anisotropic tensor

conformal structure, 88

reconstruction, 100

decomposition, 88, 90, 108

reconstruction, 130

Boltzmann transport, 11

numerical methods, 42

stationary, 27

time-harmonic, 32, 37

Calderón's problem, 54, 66

anisotropic, 69

isotropic, 67

Complex geometrical optics solutions, 209

existence, 211

Conductivity equation, 67

connection

Euclidean, 205

cross-product

generalization, 137

curvature, 207

Dirichlet-to-Neumann operator, 4, 66

double stochasticity, 193

Electrical Impedance Tomography, 54

Elliptic PDE, 62

perturbation, 72

Fischer's inequality, 192

frame, 81, 104, 112, 206

orthonormal, 83, 103, 122

Fredholm equation, 155

Gauge transform, 3

Gram matrix, 90

Gram-Schmidt process, 83, 190

Hybrid methods, 11

hyperplane condition, 135, 142

enforcing, 167, 172

Impedance-Acoustic Computerized Tomography, 76

Inverse problems

ill-posedness, 3

injectivity, 3

resolution, 7

stability, 3

Inverse wave problem, 78

Koszul formula, 206

modified, 115

Ky Fan's maximum principle, 194

linear dependence

relations of, 105, 197

majorization, 193

Optical Tomography, 55

ordinary differential equation, 121

orthonormalizing matrix, 83

stability condition, *see* Stability condition

Physical focusing, 75

pinching, 195

positivity condition, 89, 109, 141

enforcing, 166, 170

Power density

measurements, 72

operator

continuity, 64, 65

push-forward, 204

Push-forward, 202

of a conductivity tensor, 69, 163

of a function, 202

of power densities, 204

QR decomposition, 190

Schrödinger equation, 68

Schur's theorem, 193

Squareroot of a SPD matrix, 87, 198

Stability

condition, 85

strongly coupled elliptic system, 152, 154, 157

Synthetic focusing, 76

total covariant derivative, 207

system, 112, 122

$W^{1,\infty}$ space, 65